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Volatility of Volatility

Masterarbeit

von

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Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

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Contents

1 Introduction

Throughout the last fifteen years, new technological breakthroughs in high frequency data processing as well as the accompanying introduction of new mathematical techniques have accelerated the development of the realm of high frequency financial econometrics and statistics to a rapid pace. This allowed for a great deal deeper insight into the valuable econometric quantities such as jumps, volatility or even volatility of volatility. The increasingly important role of these quantities in the nowadays financial world could be readily demonstrated by the following recent observation. On April 16, 2013, the CBOE's (Chicago Board Options Exchange®) Market Summary indicated an all-time, single-day trading volume record of VIX (CBOE Volatility Index®) Options, amounting to 1,399,863 contracts over 673,970 for S&P 500®, which means that over one day the number of option contracts changing hands for (implied, forward looking) index volatility was two times higher than the amount of the option contracts for the index itself.

The contribution of this master thesis primarily consists in the derivation of nonparametric high frequency estimator for the volatility of volatility in the presence of jumps for the (log) price process. However, substantial amount of time was also devoted first to the the comprehensive study of the most frequently utilized underlying processes in financial econometrics, namely Itô semimartingales, as well as in the acquisition of detailed knowledge about the related estimates and relevant limit theorems. Furthermore, the formulation of the problem of the thesis as well as the taste of its addressing was developed and greatly influenced by various scientific papers, among which [1] and [2] deserve a special mention. Volatility of volatility (or strictly speaking, in this context quadratic variation of the squared volatility process) could be regarded as a mathematical measure of the variability of volatility or the speed of its change, which is of paramount importance from an economic point of view. The first estimators for volatility of volatility appeared independently in the paper [3] of M. Vetter (in the form of sum of nontruncated unnormalized returns) as well as in the paper [1] of P. A. Mykland, N. Shephard, and K. Sheppard (in the form of unnormalised multipower variation and edge effect corrected realised variance), where both research groups did not allow for jumps for the price process (assuming it a Brownian Itô semimartingale). The new (truncated unnormalized) estimator in the thesis allows for the presence of jumps for the price process (assuming it a general Itô semimartingale) as well as at the same time proves to be consistent with a certain convergence rate.

Furthermore, in the master thesis proper consideration is given to the following inherent characteristics of financial data. Firstly, the price process is observed at discrete times, since rarely is its complete path available. Secondly, intervals between consecutive observation times tend to zero, i.e the data is supposed to be sufficiently frequent.

Finally, a single path of the price process is observed on a finite time horizon with the volatility process being latent. It has to be pointed out that throughout the thesis the data is not assumed to be perturbed by any kind of noise.

The thesis is structured in the following way. Chapter Preliminaries introduces specific theoretical concepts prerequisite for proper understanding of the research topic of the master thesis. Chapter Consistency and CLT firstly outlines the underlying model framework and subsequently states the main theoretical results as well as provides their proofs. Furthermore, chapter Complementary Estimates and Lemmas supports the proofs of the previous chapter with more detailed insight into some utilized thematic concepts. Finally, chapter Conclusion and Potential Further Developments summarizes briefly the results and highlights some further promising research directions.

2 Preliminaries

In this chapter specific theoretical concepts are introduced for proper understanding of the research topic of the master thesis. These concepts primarily comprise quadratic variation of a semimartingale, Itô semimartingale with Grigelionis representation as well as some limit theorems. For this purpose both books [4] and [5] are followed sometimes very closely and sometimes one-to-one. For a more detailed study the reader is encouraged to read [6] or [7]. It assumed here that the reader is already familiar with Wiener, Lévy, and semimartingale processes as well as with integration with respect to random measures.

2.1 Quadratic Variation of a Semimartingale

Definition 2.1 *A real-valued process X on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \underline{P})$ is called a semimartingale if it can be written as*

$$X = A + M, \tag{2.1}$$

where M is a local martingale and A is an adapted càdlàg process "with finite variation", which means that the total variation of each path $t \longrightarrow A_t(\omega)$ is bounded over each finite interval $[0, t]$.

For defining the quadratic variation, one needs to recall some properties. The first one is that a local martingale M can always be written as $M_t = M_0 + M_t^c + M_t^d$, where $M_0^c = M_0^d = 0$ and M^c is a continuous local martingale, and M^d is a local martingale orthogonal to each continuous (local) martingale. The second one is that a local martingale starting from 0, which has bounded variation in the context of (2.1), and which is continuous, is almost surely vanishing everywhere.

Therefore, if we consider two decompositions $X = M + A = M' + A'$ as (2.1), then necessarily $M^c = M'^c$ a.s. In other words, we can write the semimartingale X as

$$X_t = X_0 + X_t^c + M_t + A_t,$$

where $A_0 = M_0 = 0$ and where A is of finite variation and M is a local martingale orthogonal to all continuous martingales, and X^c is a continuous local martingale starting at 0. In this decomposition the two processes M and A are still not unique, but the process X^c is unique (up to null sets), and it is called the continuous martingale part of X (although it usually is a local martingale only). When X is d-dimensional, so are X^c, M and A.

At this point, we can introduce the quadratic variation of a one-dimensional semimartingale X as being

$$[X,X]_t = \langle X^c, X^c \rangle_t + \sum_{s \le t} (\Delta X_s)^2. \tag{2.2}$$

The sum above makes sense, since it is a sum of positive numbers on the countable set $\{s : \Delta X_s \neq 0\} \bigcap [0,t]$. What is not immediately obvious is that it is a.s. finite, but this fact is one of the main properties of semimartingales. Hence the process $[X,X]$ is increasing and càdlàg, and also adapted (another intuitive but not mathematically obvious property). Another name for $[X,X]$ is the "square bracket". Note that $[X,X] = \langle X,X \rangle$ when X is a continuous local martingale, and in general $[X^c,X^c] = \langle X^c,X^c \rangle$ is the "continuous part" of the increasing process $[X,X]$ (not to be confused with its "continuous martingale part", which is identically 0).

For example, if $X_t = \sigma W_t$, where W is Brownian motion, then $[X,X]_t = \sigma^2 t$. So $[X,X]_t$ is not random, and coincides with the variance of X_t. This is not the case in general. $[X,X]_t$, unlike the variance, is a random variable. It is not defined by taking expectations. For example, for a Poisson process, since N jumps by 1 whenever it does, $[N,N]t = N_t$ is the number of jumps of the process between 0 and t, and we also have $[X,X]_t = N_t$ for the martingale $X_t = N_t - \lambda t$ if λ is the parameter of the Poisson process N. Moreover, $[X,X]_t$ is well defined for all semimartingales, including those with infinite variance.

2.2 Itô Semimartingale

Definition 2.2 *A d-dimensional semimartingale X is an Itô semimartingale if its characteristics (B,C,ν) are absolutely continuous with respect to Lebesgue measure, in the sense that*

$$B_t = \int_0^t b_s ds, \qquad C_t = \int_0^t c_s ds, \qquad \nu(dt,dx) = dt F_t(dx), \tag{2.3}$$

where $b = (b_t)$ an $\mathbb{R}^d$-valued process, $c = (c_t)$ is a process with values in the set of all $d \times d$ symmetric non-negative matrices, and $F_t = F_t(\omega, dx)$is for each (ω, t) a measure on $\mathbb{R}^d$.

These b_t, c_t and F_t necessarily have some additional measurability properties, so that (2.3) makes sense: we may choose b_t and c_t predictable (or simply progressively measurable, this makes no difference, and does not change the class of Itô semimartingales), and F_t is such that $F_t(\mathcal{A})$ is a predictable process for all $\mathcal{A} \in \mathbb{R}^d$ (or progressively measurable, again this makes no difference). This automatically fulfills $(\| x \|^2 \wedge 1) \star \nu_t(\omega) < \infty$ (where $\star$ denotes integration with respect to a random measure) or equivalently $\sum_{s \le t} \| \Delta X_s \|^2 < \infty$, and we can and will choose a version of F_t which satisfies identically

$$\int (\| x \|^2 \wedge 1) F_t(\omega, dx) < \infty \quad \text{and} \quad \int_0^t ds \int (\| x \|^2 \wedge 1) F_s(\omega, dx) < \infty. \tag{2.4}$$

For the full use of the notion Itô semimartingale, particular extension of a probability space, namely *very good* extension, is prerequisite.

Let the space $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$ be fixed and let $(\Omega',\mathcal{F}')$ be another measurable space, and $\mathbb{Q}(\omega,d\omega')$ be a transition probability from $(\Omega,\mathcal{F})$ into $(\Omega',\mathcal{F}')$. We can define the products

$$\widetilde{\Omega}=\Omega\times\Omega', \qquad \widetilde{\mathcal{F}}=\mathcal{F}\otimes\mathcal{F}', \qquad \widetilde{\mathbb{P}}(d\omega,d\omega')=\mathbb{P}(d\omega)\mathbb{Q}(\omega,d\omega'). \tag{2.5}$$

The probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ is called an extension of $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$. Any variable or process which is defined on either Ω or Ω' is extended in the usual way to $\widetilde{\Omega}$, with the same symbol: for example $X_t(\omega,\omega')=X_t(\omega)$ if X_t is defined on Ω. In the same way, a set $\mathcal{A}\in\Omega$ is identified with the set $\mathcal{A}\times\Omega'\in\Omega$, and we can thus identify $\mathcal{F}_t$ with $\mathcal{F}_t\otimes\{\emptyset,\Omega'\}$ so $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_{t\geq 0},\widetilde{\mathbb{P}})$ is a filtered space.

The filtration $(\mathcal{F}_t)$ on the extended space does not incorporate any information about the second factor Ω'. To bridge this gap we consider a bigger filtration $\widetilde{\mathcal{F}}_{t\geq 0}$ on $(\widetilde{\Omega},\widetilde{\mathcal{F}})$, that is with the inclusion property $\mathcal{F}_t\subset\widetilde{\mathcal{F}}_t,\forall t\geq 0$. The filtered space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_{t\geq 0},\widetilde{\mathbb{P}})$ is then called a *filtered extension* of $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$.

A filtered extension is called *very good* if it satisfies

$$\omega\longrightarrow\int 1_{\mathcal{A}}(\omega,\omega')\mathbb{Q}(\omega,d\omega')\text{is } \mathcal{F}_t\text{-measurable for all}\mathcal{A}\in\widetilde{\mathcal{F}}_t,\text{all } t\geq 0. \tag{2.6}$$

A very good filtered extension is very good because it has the following nice properties:

- any martingale, local martingale, submartingale, supermartingale on $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$ is also a martingale, local martingale, submartingale, supermartingale on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_{t\geq 0},\widetilde{\mathbb{P}})$;
- a semimartingale on $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$ a semimartingale on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_{t\geq 0},\tilde{\mathbb{P}})$ with the same characteristics.

(2.6) is equivalent to saying that any bounded martingale on $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$ is a martingale on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_{t\geq 0},\widetilde{\mathbb{P}})$. For example a Brownian motion on $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$ is also a Brownian motion on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_{t\geq 0},\widetilde{\mathbb{P}})$ if the extension is very good, and the same for Poisson random measures.

At this point it is possible to give representation theorem for Itô semimartingale. The difficult part comes from the jumps of the semimartingale, and it is fundamentally a representation theorem for integer-valued random measure in terms of a Poisson random measure. The representation below will be called here the *Grigelionis* form of the semimartingale X.

Let d-dimensional Itô semimartingale X with characteristics (B,C,ν) be given. Moreover, d' is an arbitrary integer with $d'\geq d$, and E is an arbitrary Polish space with a σ-finite and infinite measure λ having no atom, and $\overline{q}(dt,dx)=dt\otimes\lambda(dx)$. Then one can construct a very good filtered extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_{t\geq 0}, \widetilde{\mathbb{P}})$, on which there are defined a d'-dimensional Brownian motion W and a Poisson random measure p on $\mathbb{R}_+ \times E$ with intensity measure λ, such that

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s^{(1)} + (\delta 1_{\{\|\delta\| \leq 1\}}) \star (\underline{p} - \underline{q})_t + (\delta 1_{\{\|\delta\| > 1\}}) \star \underline{p}_t, \tag{2.7}$$

where σ_t is an $\mathbb{R}^d \otimes \mathbb{R}^{d'}$-valued process on $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ which is predictable (or only progressively measurable), and δ is a predictable $\mathbb{R}^d$-valued function on $\Omega \times \mathbb{R}_+ \times E$, both being such that the integrals in (2.7) make sense.

The process b_t is the same here and in (2.3), and we have close connections between $(\sigma_t, \delta(t,z))$ and (c_t, F_t). Namely, a version of the spot characteristics $_t$ and F_t is given by the following:

- $c_t(\omega) = \sigma_t(\omega)\sigma_t^\star(\omega)$
- $F_t(\omega, .)$ = the image of the measure λ restricted to the set $\{x : \delta(\omega, t, x) \neq 0\}$ by the map $x \to \delta(\omega, t, x)$.

Conversely, any process of the form (2.7) (with possibly b, σ and δ defined on the extension instead of $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ is an Itô semimartingale on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_{t\geq 0}, \widetilde{\mathbb{P}})$, and on $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ as well if it is further adapted to $\mathcal{F}_t$. Therefore, the formula (2.7) may serve as the definition of Itô semimartingales, if extension of the space is permitted, and for practical applications it is indeed! Therefore in the thesis the Grigelionis form above will be utilized freely, pretending that it is defined on our original filtered space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$.

2.3 Limit Theorems

2.3.1 Stable Convergence in Law

First of all, let E denote a Polish (that is, metric complete and separable) space, with metric δ and Borel σ-field $\mathcal{E}$.

Definition 2.3 *We say that Z_n stably converges in law if there is a probability measure η on the product $(\Omega \times E, \mathcal{F} \otimes \mathcal{E})$, such that $\eta(\mathcal{A} \times E) = \mathbb{P}(\mathcal{A})$ for all $\mathcal{A} \in \mathcal{F}$ and*

$$\mathbb{E}(Y f(Z_n)) \longrightarrow \int Y(\omega) f(x) \eta(d\omega, dx) \tag{2.8}$$

for all bounded continuous functions f on E and all bounded random variables Y on $(\Omega, \mathcal{F})$.

Since, in contrast with convergence in law, all Z_n here are defined on the same space $(\Omega, \mathcal{F}, \mathbb{P})$, it is natural to realize Z on an (arbitrary) extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, as defined by (2.5). Recalling that every variable defined on Ω is automatically extended as a variable on $\widetilde{\Omega}$, with the same symbol, for example $Z_n(\omega, \omega') = Z_n(\omega)$. Letting Z be an

E-valued random variable defined on this extension, (2.8) is equivalent to saying (with $\widetilde{\mathbb{E}}$ denoting expectation w.r.t. $\widetilde{\mathbb{P}}$)

$$\mathbb{E}(Yf(Z_n)) \longrightarrow \widetilde{\mathbb{E}}(Yf(Z)) \tag{2.9}$$

for all f and Y as above, as soon as $\widetilde{\mathbb{P}}(\mathcal{A} \bigcap \{Z \in B\}) = \eta(\mathcal{A} \times B)$ for $\mathcal{A} \in \mathcal{F}$ and $B \in \mathcal{E}$. It is possible then to say that Z_n converges stably to Z, and this convergence is denoted by $Zn \stackrel{\mathcal{L}-s}{\longrightarrow} Z$. Note that the stable convergence in law holds as soon as (2.9) holds for all Y as above and all functions f which are bounded and Lipschitz.

Stable convergence in law obviously implies convergence in law. But it implies much more, and in particular the following crucial result: if Y_n and Y are variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in the same Polish space E, then

$$Z_n \stackrel{\mathcal{L}-s}{\longrightarrow} Z, Y_n \stackrel{\mathbb{P}}{\longrightarrow} Y \Rightarrow (Y_n, Z_n) \stackrel{\mathcal{L}-s}{\longrightarrow} (Y, Z). \tag{2.10}$$

Moreover, when Z is defined *on the same space* Ω as all Z_n:

$$Z_n \stackrel{\mathcal{L}-s}{\longrightarrow} Z \Longleftrightarrow Z_n \stackrel{\mathbb{P}}{\longrightarrow} Z. \tag{2.11}$$

Moreover, a simple necessary and sfficient condition for the stable convergence in law is:

the sequence Z_n converges stably in law if and only if, for any $q \geq 1$ and any q-dimensional variable Y on $(\Omega, \mathcal{F})$,the sequence (Z_n, Y) converges in law. (2.12)

2.3.2 Convergence for Stochastic Processes

Finite-dimensional convergence is a weak form of convergence, for example if Y^n converges to Y in this sense, the suprema $\sup_{s \leq t} \| Y_s^n \|$ do not converge to the supremum of the limit, in general. To remedy this, a stronger form of convergence is necessary, called "functional" convergence. This means that we consider each process Y^n as taking its values in a functional space (i.e., a space of functions from $\mathbb{R}_+$ into $\mathbb{R}^q$), and this functional space is endowed with a suitable topology: as seen before, this functional space has to be a Polish space.

Basically, two functional spaces are going to be of interest here. One is the space $\mathbb{C}^q = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^q)$ of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^q$, endowed with the local uniform topology corresponding for example to the metric $\delta_U(x, y) = \sum_{n \geq 1} 2^{-n}(1 \wedge \sup_{s \leq t} \| x(s) - y(s) \|)$. The Borel σ-field for this topology is $\sigma(x(s) : s \geq 0)$, and with this topology the space $\mathbb{C}^q$ is a Polish space.

However, although the limiting processes Y are continuous in the master thesis, this is rarely the case of the pre-limiting processes Y_n, which typically are based upon the discrete observations $X_{T(n,i)}$: they often come up as partial sums $\sum_{i \geq 1} f(\Delta_i^n X) 1_{\{T(n,i) \leq t\}}$. Such a process has discontinuous, although càdlàg, paths. Therefore, the other functional space of interest for us is the *Skorokhod space*: this is the set $\mathbb{D}^q = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^q)$ all càdlàg functions from from $\mathbb{R}_+$ into $\mathbb{R}^q$.

One possible metric on $\mathbb{D}^q$ is δ_U, which makes $\mathbb{D}^q$ a Banach space, but under which it is unfortunately not separable (hence, not Polish). This prompted the development of the Skorokhod topology. There is a metric δ_S compatible with this topology, such that $\mathbb{D}^q$ is a Polish space, and again the Borel σ-field is $\sigma(x(s) : s \geq 0)$. This metric is not needed here and the reader is referred to [6] for a more detailed study.

Denote by $x_n \xrightarrow{u.c.p.} x$ if $\delta_U(x_n, x) \to 0$ (this makes sense for any functions x_n, x), and $x_n \xrightarrow{Sk} x$ if $\delta_S(x_n, x) \to 0$ (this makes sense for $x_n, x \in \mathbb{D}^q$). The following properties, for $x_n, y_n, x, y \in \mathbb{D}^q$, are worth stating:

$$x_n \xrightarrow{u.c.p.} x \qquad \Rightarrow \qquad x_n \xrightarrow{Sk} x \tag{2.13}$$

$$x_n \xrightarrow{Sk} x, \quad x \in \mathbb{C}^q \qquad \Rightarrow \qquad x_n \xrightarrow{u.c.p.} x \tag{2.14}$$

$$x_n \xrightarrow{Sk} x, \quad y_n \xrightarrow{u.c.p} y \qquad \Rightarrow \qquad x_n + y_n \xrightarrow{Sk} x + y. \tag{2.15}$$

However, it has to be pointed out that the Skorokhod topology also suffers from some drawbacks, of which the reader should be aware:

- $x_n \xrightarrow{Sk} x$ and $y_n \xrightarrow{Sk} y$ does not necessarily means that $x_n + y_n \xrightarrow{Sk} x + y$
- The mapping $x \to x(t)$ is not continuous for the Skorokhod topology, although it is continuous at each point x such that $x(t) = x(t-)$ where $x(t-)$ denotes the left limit of x at time t. Given that x is càdlàg, $x(t) = x(t-)$ means that x is continuous at time t.

Therefore, this topology is the one to be used when dealing with càdlàg functions or processes, but a lot of care is necessary when utilizing it.

Returning to the sequence Y_n of $\mathbb{R}^q$-valued càdlàg processes, and its potential limit Y, $\mathbb{R}^q$-valued càdlàg process, both processes can be considered as random variables with values in the space $\mathbb{D}^q$, and thus the notions of convergence in law, or stably in law, or in probability, of Y_n towards Y are obtained. In the first case, Y is defined on an arbitrary probability space, in the second case it is defined on an extension, and in the third case it is defined on the same space as are all the Y_n's. The "local uniform convergence" refers to the metric δ_U above on $\mathbb{D}^q$, and we write:

$$Y^n \overset{u.c.p.}{\Longrightarrow} Y \text{ if } \delta_U(Y^n, Y) \xrightarrow{\mathbb{P}} 0, \text{ or equivalently if, for all } T, \sup_{t \leq T} \| Y_t^n - Y_t \| \xrightarrow{\mathbb{P}} 0. \tag{2.16}$$

When dealing with the Skorokhod topology, and implicitly using the metric δ_S, write for convergence in probability

$$Y^n \overset{\mathbb{P}}{\Longrightarrow} Y \text{ if } \delta_S(Y^n, Y) \xrightarrow{\mathbb{P}} 0.$$

Ssimilarly define $Y^n \overset{\mathcal{L}}{\Longrightarrow} Y$ and $Y^n \overset{\mathcal{L}-s}{\Longrightarrow} Y$ for convergence in law and stable convergence in law, using the Skorokhod topology. Therefore, a double arrow *always means functional convergence*.

3 Consistency and CLT

In this chapter firstly the underlying model framework is outlined and subsequently the main theoretical results are stated as well as proved.

3.1 Model Framework

Define on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ X as a 1-dimensional Itô semimartingale with jump measure μ and characteristics (B, C, ν), where

$$B_t = \int_0^t b_s ds, \qquad C_t = \int_0^t c_s ds, \qquad \nu(dt, dx) = dt F_t(dx), \tag{3.1}$$

with $b = (b_t)$ an $\mathbb{R}$-valued process, $c = (c_t)$ a an $\mathbb{R}$-valued non-negative process, and $F_t = F_t(\omega, dx)$ for each (ω, t) a measure on $\mathbb{R}$, all those being progressively measurable in (ω, t).
Then the Grigelionis form of X is:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \tilde{\sigma}_s dW_s + (\delta 1_{\{|\delta|\leq 1\}}) \star (\underline{p} - \underline{q})_t + (\delta 1_{\{|\delta|>1\}}) \star \underline{p}_t.$$

W here is a d'-dimensional Brownian motion and it should hold that $d' \geq 2$ for a genuine stochastic volatility, as there should be at least two independent Brownian motions to drive the pair processes $(X, \tilde{\sigma})$. For illustrative purposes and without loss of generality, we shall assume that $d' = 2$, which results in $W = (W^{(1)}, W^{(2)})$ and $\tilde{\sigma} = (\sigma, 0)$, therefore leading to:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s^{(1)} + (\delta 1_{\{|\delta|\leq 1\}}) \star (\underline{p} - \underline{q})_t + (\delta 1_{\{|\delta|>1\}}) \star \underline{p}_t.$$

$\underline{p}$ in both settings above stands for a Poisson random measure on $\mathbb{R}_+ \times E$ with $(E, \mathcal{E})$ an auxiliary Polish space. Moreover, b as well as σ are 1-dimensional progressively measurable processes and δ is a predictable function on $\Omega \times \mathbb{R}_+ \times E$. Finally, it has to be pointed out:

- $c_t(\omega) = \sigma_t(\omega)^2$
- $F_t(\omega, .)$ = the image of the measure λ restricted to the set $\{x : \delta(\omega, t, x) \neq 0\}$ by the map $x \to \delta(\omega, t, x)$.

Let σ be Brownian Itô semimartingale with the same W:

$$\sigma_t = \sigma_0 + \int_0^t b_s^{(\sigma)} ds + \int_0^t \sigma_s^{(\sigma)} dW_s = \sigma_0 + \int_0^t b_s^{(\sigma)} ds + \int_0^t \sigma_s^{(\sigma,1)} dW_s^{(1)} + \int_0^t \sigma_s^{(\sigma,2)} dW_s^{(2)}.$$

This automatically leads to

$$c_t = c_0 + \int_0^t b_s^{(c)} ds + \int_0^t \sigma_s^{(c)} dW_s = c_0 + \int_0^t b_s^{(c)} ds + \int_0^t \sigma_s^{(c,1)} dW_s^{(1)} + \int_0^t \sigma_s^{(c,2)} dW_s^{(2)},$$

where the coefficients of c can be explicitly computed utilizing Itô formula:

$$b_s^{(c)} = 2\sigma_s b_s^{(\sigma)} + (\sigma_s^{(\sigma,1)})^2 + (\sigma_s^{(\sigma,2)})^2, \sigma_s^{(c,1)} = 2\sigma_s \sigma_s^{(\sigma,1)}, \sigma_s^{(c,2)} = 2\sigma_s \sigma_s^{(\sigma,2)}.$$

In the model framework, the process X, being Itô semimartingale in its full generality, which makes it highly appropriate to model (log) asset prices, stock market indices, and exchange or interest rates in a fair market without arbitrage. Therefore, for the purpose of complying with the real data, the price process X is observed at the discrete times $i\Delta_n$ over a finite time interval $[0,T]$, where $\Delta_n \to 0$ in n and $\Delta_n^i X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$ stands for the observed (log) returns. The volatility process c, or strictly speaking σ, is latent with $\Delta_n^i c = c_{i\Delta_n} - c_{(i-1)\Delta_n}$ representing its increments. For convenience define also $\mathcal{F}_i^n := \mathcal{F}_{i\Delta_n}$. Finally, in the thesis the constant K may differ from line to line and depend on an extra parameter p denoted then by K_p.

Throughout the master thesis, the quantity of particular interest for estimation is $[c,c]$ - the quadratic variation of the latent volatility process c, which can be explicitly computed in the model framework above:

$$[c,c]_t = \int_0^t c_s^{(c)} ds = \int_0^t (\sigma_s^{(c,1)})^2 + (\sigma_s^{(c,2)})^2 ds.$$

However, before introducing the estimator for the $[c,c]$ and proving its consistency as well as the associated central limit theorem, it is sensible to start first imposing some additional mild assumptions on the pair process (X,c).

Assumption (H-r). In the Grigelionis representation of the Itô semimartingale:

(i) the process b is locally bounded;

(ii) the process σ is càdlàg;

(iii) there is a sequence (τ_n) of stopping times increasing to ∞ and, for each n, a deterministic nonnegative function J_n on E satisfying $\int J_n(z)\lambda(dz) < \infty$ and such that $|\delta(\omega,t,z)|^r \wedge 1 \leq J_n(z)$ for all (ω,t,z) with $t \leq \tau_n(\omega)$.

Under (H-1) (and consequently for any $r < 1$) define the genuine drift process

$$b'_t = b_t - \int_{\{|\delta(t,z)|\leq 1\}} \delta(t,z)\lambda(dz),$$

which will then be well defined at all times, as $|\int_{\{|\delta(t,z)|\leq 1\}} \delta(t,z)\lambda(dz)| \leq \int J_n(z)\lambda(dz)$ for $t \leq \tau_n$ with (τ_n) increasing to ∞. By the same token, $|\delta| \star \underline{q}_t < \infty$ for all times, which allows to rewrite X as:

$$X_t = X_0 + \int_0^t b'_s ds + \int_0^t \sigma_s dW_s + \delta \star \underline{p}_t = X'_t + \delta \star \underline{p}_t.$$

Note that both σ as well as X' are continuous.

Assumption (PCC-r). For $r \leq 1$ the process X satisfies (H-r) and σ - (H-0) correspondingly (with $\delta^{(\sigma)} \equiv 0$) and:

(i) the process b' is càdlàg or càglàd;

(ii) the process $b^{(\sigma)}$ is càdlàg or càglàd.

There are the following strengthened assumptions of the previous ones above, which are prerequisites for reasonable estimates for the increments of the processes X and σ (hence also for the process c when applying Itô formula to σ) due to the boundedness conditions. The connection between the weak and strong assumptions is established by the powerful Localization Lemma from [4] on page 118, which states that if any property under (SH-r) or (SPCC-r) holds, it will hold also under (H-r) or (PCC-r) correspondingly.

Assumption (SH-r). The assumption (H-r) is fulfilled and furthermore:

(i) the processes b and σ are bounded;

(ii) there is a a deterministic non-negative bounded function J on E satisfying $\int J(z)\lambda(dz) < \infty$ such that $|\delta(\omega,t,z)|^r \leq J(z)$ for all (ω,t,z).

Assumption (SPCC-r). The assumption (PCC-r) is fulfilled and furthermore:

(i) the process X satisfies (SH-r), for $r \leq 1$;

(ii) the process σ satisfies (SH-0);

3.2 Main Theorem

Relying upon the notation from the previous subsection, define for some $\alpha, \beta \in (0,\infty)$, $\varpi \in (0, \frac{1}{2})$, $u_n \sim \alpha \Delta_n^{\varpi}$ and $k_n \sim \frac{\beta}{\sqrt{\Delta_n}}$

$$c(k_n,u_n)_i^n = \frac{1}{k_n\Delta_n}\sum_{m=0}^{k_n-1}(\Delta_{i+m}^n X)^2 1_{\{|\Delta_{i+m}^n X|\le u_n\}}$$

$$c'(k_n)_i^n = \frac{1}{k_n\Delta_n}\sum_{m=0}^{k_n-1}(\Delta_{i+m}^n X')^2$$

Then for each $T>0$ define the estimator for volatility of volatility (or strictly speaking, the estimator for quadratic variation of squared volatility):

$$[c,c]_T^n = \frac{3}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}((c(k_n,u_n)_{i+k_n}^n - c(k_n,u_n)_i^n)^2 - \frac{4}{k_n}(c(k_n,u_n)_i^n)^2)$$

as well as the estimator for volatility of volatility if the process X' were directly observable:

$$[c,c]'^n_T = \frac{3}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}((c'(k_n)_{i+k_n}^n - c'(k_n)_i^n)^2 - \frac{4}{k_n}(c'(k_n)_i^n)^2)$$

Note that $\frac{4}{k_n}(c(k_n)_i^n)^2$ plays a role of a de-biasing term.

Theorem 3.1 *For each $T>0$ under (PCC-r) and for $\varpi < \frac{7}{4(4-r)} \wedge \frac{1}{2}$ both consistency and corresponding central limit theorem hold:*

$$[c,c]_T^n \xrightarrow{\mathbb{P}} [c,c]_T \text{ for } r<1$$

$$\frac{1}{\Delta_n^{1/4}}([c,c]_T^n - [c,c]_T) \xrightarrow{\mathcal{L}-s} \mathcal{U}_T \text{ for } r<0.5,$$

where $\mathcal{U}_T$ is a random variable defined on a very good extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}_{t\ge0},\widetilde{\mathbb{P}})$ of $(\Omega,\mathcal{F},\mathcal{F}_{t\ge0},\mathbb{P})$ and which is, conditionally on $\mathcal{F}$, a continuous centered Gaussian martingale with variance

$$\mathbb{E}(\mathcal{U}_T^2|\mathcal{F}) = \int_0^T (\frac{48}{\beta^3}(c_s)^4 + \frac{12}{\beta}(c_s)^2 c_s^{(c)} + \frac{151}{70}(c_s^{(c)})^2)ds. \tag{3.2}$$

The Proof.

The scheme of the proof comprises four steps. In step 1, the preliminary decomposition is performed, which results in 3 terms or 3 successive steps correspondingly, in which two terms prove to be asymptotically negligible and one converging to the required random variable presented above. The novelty of the proof consists in the extension of the step 1 from the book [5] on the page 451 and the performance of the step 2. Two other steps are taken one-to-one from the book [5] on the pages 454 and 457 and are further commented on for a reader less familiar with the research topic. Throughout the proof various estimates and lemmas are used which are all placed in chapter Complementary Estimates and Lemmas.

Step 1: the Decomposition

As mentioned before in the model framework for some $\alpha, \beta \in (0, \infty)$, $\varpi < \frac{7}{4(4-r)} \wedge \frac{1}{2}$, $u_n \sim \alpha \Delta_n^{\varpi}$ and $k_n \sim \frac{\beta}{\sqrt{\Delta_n}}$ as well with $m \in \{0, ..., 2k_n - 1\}$ and $j, l \in \mathbb{Z}$ and $u, v, u', v' \in \{1, 2\}$ we set

$$\varepsilon(1)_n^m = \begin{cases} -1 & \text{if } 0 \leq m < k_n \\ 1 & \text{if } k_n \leq m < 2k_n, \end{cases} \qquad \varepsilon(2)_n^m = (m+1) \wedge (2k_n - m - 1), \qquad \varepsilon(3)_n^m = 1,$$

$$z_{u,v}^n = \begin{cases} 1/\Delta_n & \text{if } u = v = 1 \\ 1 & \text{otherwise}, \end{cases} \qquad \gamma(u,v;m)_{j,l}^n = \frac{3}{2k_n^3} \sum_{q=0 \vee (j-m)}^{(l-m-1) \wedge (2k_n - m - 1)} \varepsilon(u)_q^n \varepsilon(v)_{q+m}^n,$$

$$\Gamma(u,v)_m^n = \gamma(u,v;m)_{0,2k_n}^n, \qquad H(u,v;u'v')^n = z_{u,u'}^n z_{v,v'}^n \sum_{m=1}^{2k_n - 1} \Gamma(u,v)_m^n \Gamma(u',v')_m^n,$$

which clearly satisfy

$$\gamma_{u,v}^{\tilde{n}} = sup_{j,m,l} |\gamma(u,v;m)_{j,l}^n| \leq \begin{cases} K & \text{if } (u,v) = (2,2) \\ K/k_n & \text{if } (u,v) = (1,2), (2,1) \\ K/k_n^2 & \text{if } (u,v) = (1,1). \end{cases} \tag{3.3}$$

We also need, for $m \in \{0, ..., k_n - 1\}$ and $j, l \in \mathbb{Z}$ and $u, v \in \{1, 2\}$, the numbers

$$\overline{\epsilon}(1)_m^n = 1, \qquad \overline{\epsilon}(2)_m^n = k_n - m - 1, \qquad \gamma(u,v;m)_{j,l}^n = \frac{6}{k_n^4} \sum_{q=0 \wedge dd(j-1)}^{(l-m-1) \wedge (k_n - m - 1)} \overline{\epsilon}(u)_q^n \overline{\epsilon}(v)_{q+m}^n,$$

which satisfy

$$|\overline{\gamma}(u,v;m)_{j,l}^n| \leq \begin{cases} K/k_n & \text{if } (u,v) = (2,2) \\ K/k_n^2 & \text{if } (u,v) = (1,2), (2,1) \\ K/k_n^3 & \text{if } (u,v) = (1,1). \end{cases} \tag{3.4}$$

We need to compute the numbers $\Gamma(u,v)^n_m$: a tedious (approximately 15 pages in the same format) but elementary calculation shows that they are as follows

$$\Gamma(1,1)^n_m = \begin{cases} \frac{6k_n-9m}{2k_n^3} & \text{if } m \le k_n-1 \\ -\frac{6k_n-3m}{2k_n^3} & \text{if } m \ge k_n, \end{cases} \quad \Gamma(1,2)^n_m = \begin{cases} -\frac{12k_nm-9m^2+6k_n-9m}{4k_n^3} & \text{if } m \le k_n-1 \\ -\frac{3(2k_n-m)(2k_n-m-1)}{4k_n^3} & \text{if } m \ge k_n, \end{cases}$$
$$\Gamma(2,1)^n_m = \begin{cases} \frac{12k_nm-9m^2-6k_n+9m}{4k_n^3} & \text{if } m \le k_n-1 \\ \frac{3(2k_n-m)(2k_n-m+1)}{4k_n^3} & \text{if } m \ge k_n, \end{cases} \quad \Gamma(2,2)^n_m = \begin{cases} \frac{4k_n^3-6k_nm^2+3m^3+2k_n-3m}{4k_n^3} & \text{if } m \le k_n-1 \\ \frac{(2k_n-m)^3-2k_n+m}{4k_n^3} & \text{if } m \ge k_n, \end{cases} \tag{3.5}$$

This yields the following behavior of $H(u,v;u'v')^n$ as $n \longrightarrow \infty$, stated for $(u,v) \le (u',v')$ only because of the obvious symmetry $H(u,v;u',v')^n = H(u',v';u,v)^n$:

$$\sqrt{\Delta_n}H(u,v;u',v')^n \longrightarrow \begin{cases} \frac{3}{\beta^3} & \text{if } (u,v,u',v') = (1,1,1,1) \\ \frac{3}{4\beta} & \text{if } (u,v,u',v') = (1,2,1,2),(2,1,2,1) \\ \frac{151\beta}{280} & \text{if } (u,v,u',v') = (2,2,2,2) \\ 0 & \text{if } otherwise. \end{cases} \tag{3.6}$$

At this stage it is possible to introduce the required decomposition terms.

$$c'(k_n)^n_i = c_{(i-1)\Delta_n} + \frac{1}{k_n}\sum_{j=0}^{k_n-1}\sum_{u=1}^{2}\overline{\epsilon}(u)^n_j\zeta(u)^n_{i+j}$$
$$c'(k_n)^n_{i+k_n} - c'(k_n)^n_i = \frac{1}{k_n}\sum_{j=0}^{2k_n-1}\sum_{u=1}^{2}\epsilon(u)^n_j\zeta(u)^n_{i+j}.$$

Thus

$$(c'(k_n)^n_i)^2 = (c_{(i-1)\Delta_n})^2 + \frac{2c_{(i-1)\Delta_n}}{k_n}\sum_{u=1}^{2}\sum_{j=0}^{k_n-1}\overline{\epsilon}(u)^n_j\zeta(u)^n_{i+j}$$
$$+ \frac{1}{k_n^2}\sum_{u,v=1}^{2}\Big(\sum_{j=0}^{k_n-1}\overline{\epsilon}(u)^n_j\overline{\epsilon}(v)^n_j\zeta(u)^n_{i+j}\zeta(v)^n_{i+j} + 2\sum_{j=0}^{k_n-2}\sum_{l=j+1}^{k_n-1}\overline{\epsilon}(u)^n_j\overline{\epsilon}(v)^n_l\zeta(u)^n_{i+j}\zeta(v)^n_{i+l}\Big)$$

$$(c'(k_n)^n_{i+k_n})^2 - c'(k_n)^n_i)^2)^2 = \frac{1}{k_n^2}\sum_{u,v=1}^{2}\sum_{j=0}^{2k_n-1}\epsilon(u)^n_j\epsilon(v)^n_j\zeta(u)^n_{i+j}\zeta(v)^n_{i+j}$$
$$+ \frac{2}{k_n^2}\sum_{u,v=1}^{2}\sum_{j=0}^{2k_n-2}\sum_{l=j+1}^{2k_n-1}\epsilon(u)^n_j\epsilon(v)^n_l\zeta(u)^n_{i+j}\zeta(v)^n_{i+l}.$$

Then with the convention $\sum_{i=a}^{a'} = 0$ when $a > a'$:

$$\overline{A}(0)^n_T = \frac{6}{k_n^2} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} (c_{(i-1)\Delta_n})^2, \quad \overline{A}(1;u)^n_T = \frac{12}{k_n^3} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} c^2_{(i-1)\Delta_n} \sum_{j=0}^{k_n-1} \overline{\epsilon}(u)^n_j \zeta(u)^n_{i+j}$$

$$\overline{A}(2;u,v)^n_T = \sum_{i=1}^{[T/\Delta_n]-k_n} \overline{\gamma}(u,v;0)_{i-1-[t/\Delta_n],i} \zeta(u)^n_i \zeta(v)^n_i$$

$$\overline{A}(3;u,v)^n_T = \sum_{i=2}^{[T]/\Delta_n-k_n} (\sum_{m=1}^{(i-1)\wedge(k_n-1)} \gamma(u,v;m)^n_{i-1-[T/\Delta_n],i} \zeta(u)^n_{i-m}) \zeta(v)^n_i$$

and

$$\rho(u,v)^n_i = \sum_{m=1}^{2k_n-1} \Gamma(u,v)^n_m \zeta(u)^n_{i-m}, \quad Z(u,v)^n_T = \sum_{i=2k_n}^{[T/\Delta_n]} \rho(u,v)^n_i \zeta''(v)^n_i$$

$$A(1;u,v)^n_T = \Gamma(u,v)^n_0 \sum_{i=1}^{[T/\Delta_n]} \zeta(u)^n_i \zeta(v)^n_i$$

$$A(2;u,v)^n_T = \sum_{i=1}^{[T]/\Delta_n} (\gamma(u,v;0)^n_{i+2k_n-1-[T/\Delta_n],i} - \Gamma(u,v)^n_0) \zeta(u)^n_i \zeta(v)^n_i$$

$$A(3;u,v)^n_T = \sum_{i=2}^{[T]/\Delta_n} (\sum_{m=1}^{(i-1)\wedge(2k_n-1)} \gamma(u,v;m)^n_{i+2k_n-1-[T/\Delta_n],i} \zeta(u)^n_{i-m} - \rho(u,v)^n_i 1_{\{i\geq 2k_n\}}) \zeta(v)^n_i$$

$$A(4;u,v)^n_T = \sum_{i=2k_n}^{[T/\Delta_n]} \rho(u,v)^n_i \zeta'(v)^n_i$$

Having done appropriate changes of order of summations as well as after some tedious computations (approximately 10 pages in the same format), it is possible to derive the required decomposition for some

$$\tilde{\varepsilon} > 0 \text{ arbitrarily small enough and } q(\varpi,r) = \begin{cases} 1/4 & \text{if } r < 1/2 \\ (4-r)\varpi - 3/2 - \tilde{\varepsilon} & \text{if } 1/2 \leq r < 1 \end{cases} :$$

$$\begin{aligned} &\frac{1}{\Delta_n^{q(\varpi,r)}}([c,c]^n_T - [c,c]_T) = \frac{1}{\Delta_n^{q(\varpi,r)}}([c,c]^n_T - [c,c]'^n_T) + \frac{1}{\Delta_n^{q(\varpi,r)}}([c,c]'^n_T - [c,c]_T) = \\ &\frac{1}{\Delta_n^{q(\varpi,r)}} Dif^n_T + \frac{1}{\Delta_n^{q(\varpi,r)}}(A^n_T - \overline{A}^n_T - [c,c]_T) + \frac{1}{\Delta_n^{q(\varpi,r)}} U^n_T, \end{aligned} \tag{3.7}$$

where $Dif^n_T = [c,c]^n_T - [c,c]'^n_T$, $\quad [c,c]'^n_T = A^n_T - \overline{A}^n_T + U^n_T$

$$A^n_T = \sum_{u,v=1}^{2} (A(1;u,v)^n_T + A(2;u,v)^n_T + 2A(3;u,v)^n_T + 2A(4;u,v)^n_T)$$

$$\overline{A}^n_T = \overline{A}(0)^n_T + \sum_{u=1}^{2} \overline{A}(1;u)^n_T + \sum_{u,v=1}^{2} (\overline{A}(2;u,v)^n_T + 2\overline{A}(3;u,v)^n_T)$$

$$U^n_T = 2\sum_{u,v=1}^{2} Z(u,v)^n_T.$$

In addition, the following simplified notation will be used throughout the proof:

$$\zeta(1)^n_i = \frac{1}{\Delta^n}(\Delta^n_i X')^2 - c_{(i-1)\Delta_n}, \qquad \zeta(2)^n_i = \Delta^n_i c,$$

$$\zeta'(r)^n_i = \mathbb{E}(\zeta(r)^n_i \mid \mathcal{F}^n_{i-1}), \qquad \zeta''(r)^n_i = \zeta(r)^n_i - \zeta'(r)^n_i,$$

$$\alpha^n_i = \frac{1}{k_n}\sum_{j=0}^{k_n-1} \zeta(1)^n_{i+j}, \qquad \beta^n_i = \frac{1}{k_n}\sum_{j=0}^{k_n-1} (1)_{(i+j-1)\Delta_n}$$

$$\overline{\beta}^n_i = \beta^n_i - c_{(i-1)\Delta_n} = \frac{1}{k_n}\sum_{j=0}^{k_n-1}(c_{(i+j-1)\Delta_n} - c(_{i-1)\Delta_n}) = \frac{1}{k_n}\sum_{j=0}^{k_n-2}(k_n - j - 1)\zeta(2)^n_{i+j}.$$

Step 2: the Asymptotic Negligibility 1

Since for each $T > 0$ the $\mathcal{L}-s$ convergence to the random variable defined on the same probability space is equivalent to the convergence in $\mathbb{P}$ to the same random variable, in this step it is proved that $\frac{1}{\Delta_n^{q(\varpi,r)}} Dif^n_T \xrightarrow{\mathbb{P}} 0$.

$$\begin{aligned}
Dif^n_T &= [c,c]^n_T - [c,c]'^n_T \\
&= \frac{3}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1} (c(k_n,u_n)^n_{i+k_n} - c(k_n,u_n)^n_i)^2 - (c'(k_n)^n_{i+k_n} - c'(k_n)^n_i)^2 \\
&\quad + \frac{3}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1} \frac{4}{k_n}(c'(k_n)^n_i)^2 - \frac{4}{k_n}(c(k_n,u_n)^n_i)^2 = \\
&= \frac{3}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1} (W_i + V_i),
\end{aligned}$$

where

$$W_i = (c(k_n,u_n)^n_{i+k_n} - c(k_n,u_n)^n_i)^2 - (c'(k_n)^n_{i+k_n} - c'(k_n)^n_i)^2$$

$$V_i = \frac{4}{k_n}(c'(k_n)^n_i)^2 - \frac{4}{k_n}(c(k_n,u_n)^n_i)^2$$

It is obvious that if $\frac{3}{2\Delta_n^{q(\varpi,r)}k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1} W_i$ is asymptotically negligible, than $\frac{3}{2\Delta_n^{q(\varpi,r)}k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}(W_i+V_i)$ will also be negligible. Therefore, consider equality valid for any reals $a^2-b^2=(a-b)(a-b+2b)$ and for W_i:

$$
\begin{aligned}
W_i &= (c(k_n,u_n)^n_{i+k_n} - c(k_n,u_n)^n_i)^2 - (c'(k_n)^n_{i+k_n} - c'(k_n)^n_i)^2 \\
&= (\frac{1}{k_n\Delta_n}\sum_{m=0}^{k_n-1}(\Delta^n_{i+k_n+m}X)^2 1_{\{|\Delta^n_{i+k_n+m}X|\le u_n\}} - (\Delta^n_{i+m}X)^2 1_{\{|\Delta^n_{i+m}X|\le u_n\}})^2 \\
&\quad - (\frac{1}{k_n\Delta_n}\sum_{m=0}^{k_n-1}(\Delta^n_{i+k_n+m}X')^2 - (\Delta^n_{i+m}X')^2)^2 \\
&= \frac{1}{k_n^2\Delta_n^2}\Bigg(\sum_{m=0}^{k_n-1}(\Delta^n_{i+k_n+m}X)^2 1_{\{|\Delta^n_{i+k_n+m}X|\le u_n\}} - (\Delta^n_{i+k_n+m}X')^2 \\
&\quad + (\Delta^n_{i+m}X')^2 - (\Delta^n_{i+m}X)^2 1_{\{|\Delta^n_{i+m}X|\le u_n\}}\Bigg) \\
&\quad \Bigg(\sum_{m=0}^{k_n-1}(\Delta^n_{i+k_n+m}X)^2 1_{\{|\Delta^n_{i+k_n+m}X|\le u_n\}} - (\Delta^n_{i+k_n+m}X')^2 + (\Delta^n_{i+m}X')^2 - (\Delta^n_{i+m}X)^2 1_{\{|\Delta^n_{i+m}X|\le u_n\}} \\
&\quad + 2\Big((\Delta^n_{i+k_n+m}X')^2 - (\Delta^n_{i+m}X')^2\Big)\Bigg).
\end{aligned}
$$

Taking into account Markov inequality for some $\varepsilon>0$:

$$
\mathbb{P}\left(\left|\frac{3}{2\Delta_n^{q(\varpi,r)}k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1} W_i\right| > \varepsilon\right) \le \frac{\mathbb{E}\left[\left|\frac{3}{2\Delta_n^{q(r)}k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1} W_i\right|\right]}{\varepsilon} \tag{3.8}
$$

and utilizing triangular inequality, it is sufficient to consider only two kinds of cross-products of sums:

$$
KT\Delta_n^{-(5/2+q(\varpi,r))}\mathbb{E}\left[\left|\left((\Delta^n_i X)^2 1_{\{|\Delta^n_i X|\le u_n\}} - (\Delta^n_i X')^2\right)\left((\Delta^n_j X)^2 1_{\{|\Delta^n_j X|\le u_n\}} - (\Delta^n_j X')^2\right)\right|\right] \tag{3.9}
$$

$$
KT\Delta_n^{-(5/2+q(\varpi,r))}\mathbb{E}\left[\left|\left((\Delta^n_{i+k_n} X')^2 - (\Delta^n_i X')^2\right)\left((\Delta^n_j X)^2 1_{\{|\Delta^n_j X|\le u_n\}} - (\Delta^n_j X')^2\right)\right|\right]. \tag{3.10}
$$

Utilization of Cauchy-Schwarz inequality bounds the expression (3.9) by:

$$KT\Delta_n^{-(5/2+q(\varpi,r))}\mathbb{E}\left[\left|(\Delta_i^n X)^2 1_{\{|\Delta_i^n X|\le u_n\}} - (\Delta_i^n X')^2\right|^2\right] \tag{3.11}$$

After some tedious but elementary calculation (approximately 2 pages in the same format, generalized version of the inequality can be found in [4] in the proof of Theorem 9.3.2), it is possible to show that for any reals $x, y \in \mathbb{R}$ and $u, v, q > 0$:

$$\left|(x+y)^2 1_{\{|x+y|\le u\}} - x^2\right|^q \le K_{q,v}\left(u^{2q}(\frac{|y|}{u}\wedge 1)^{2q} + |x|^q u^q(\frac{|y|}{u}\wedge 1)^q + \frac{|x|^{q(2+v)}}{u^{qv}}\right) \tag{3.12}$$

Now setting $x = \Delta_i^n X'$ as well as $y = \Delta_i^n(X - X')$ and applying (3.12) yields with triangular inequality and conditioning on $\mathcal{F}_{\Delta_n(i-1)}$:

$$\begin{aligned}
&KT\Delta_n^{-(5/2+q(\varpi,r))}\mathbb{E}\left[\left|(\Delta_i^n X)^2 1_{\{|\Delta_i^n X|\le u_n\}} - (\Delta_i^n X')^2\right|^2\right] \le \\
&K_v T\Delta_n^{-(5/2+q(\varpi,r))}\mathbb{E}\left(u_n^4\mathbb{E}\left(\sup_{s\le t}(\frac{|\Delta_i^n(\delta\star \underline{p})|}{u_n}\wedge 1)^4|\mathcal{F}_{\Delta_n(i-1)}\right)\right.\\
&\qquad + \mathbb{E}\left(|\Delta_i^n X'|^2 u_n^2 \sup_{s\le t}(\frac{|\Delta_i^n(\delta\star \underline{p})|}{u_n}\wedge 1)^2|\mathcal{F}_{\Delta_n(i-1)}\right)\\
&\qquad \left. + \mathbb{E}\left(\frac{|\Delta_i^n X'|^{2(2+v)}}{u_n^{2v}}\right)\right).
\end{aligned}$$

After applying Hölder inequality, for some $\varepsilon^{(1)} > 0$ sufficiently small, (4.10) as well as (4.11), for some ϕ^n vanishing for n large, the expression above becomes bounded by

$$K_v T\Delta_n^{-(5/2+q(\varpi,r))}(\phi^n\Delta_n^{(4-r)\varpi+1} + \Delta_n^{1+2\varpi+1-r\varpi-\varepsilon^{(1)}} + \Delta_n^{2+v(1-2\varpi)})$$

Recalling that $\varpi < 1/2$ the last summand above with v large enough becomes negligible in front of the other two summands. The second summand is also negligible in front of the first one when $\varepsilon^{(1)} > 0$ is sufficiently small. Thus, for $\Delta_n^{-5/2-q(\varpi,r)+(4-r)\varpi+1}$ the corresponding power $(4-r)\varpi - 3/2 - q(\varpi, r) > 0$ for ϖ chosen sufficiently close to 1/2 or $\frac{7}{4(4-r)}$.

Now, utilization of triangular and subsequently Hölder inequality, for some $\varepsilon^{(2)} > 0$ sufficiently small, bounds the expression (3.10) by:

$$K T \Delta_n^{-(5/2+q(\varpi,r))} \left(\mathbb{E}\left[\left| (\Delta_i^n X')^2 \right|^{1/\varepsilon^{(2)}} \right] \right)^{\varepsilon^{(2)}} \left(\mathbb{E}\left[\left| (\Delta_j^n X)^2 1_{\{|\Delta_j^n X| \le u_n\}} - (\Delta_j^n X')^2 \right|^{1/(1-\varepsilon^{(2)})} \right] \right)^{1-\varepsilon^{(2)}} \tag{3.13}$$

Utilizing (4.10) for $\mathbb{E}\left[\left| (\Delta_i^n X')^2 \right|^{1/\varepsilon^{(2)}} \right]$ and by analogy with (3.11) for $\mathbb{E}\left[\left| (\Delta_j^n X)^2 1_{\{|\Delta_j^n X| \le u_n\}} - (\Delta_j^n X')^2 \right|^{1/(1-\varepsilon^{(2)})} \right]$, bounds the expression above by:

$$\begin{aligned} K_v T \Delta_n^{1-(5/2+q(\varpi,r))} \Bigg(\mathbb{E}\Bigg(u_n^{2/(1-\varepsilon^{(2)})} \mathbb{E}\Bigg(\sup_{s \le t} \Big(\frac{|\Delta_i^n (\delta \star \underline{p})|}{u_n} \wedge 1 \Big)^{2/(1-\varepsilon^{(2)})} |\mathcal{F}_{\Delta_n(i-1)} \Bigg) \\ + \mathbb{E}\Bigg(|\Delta_i^n X'|^{1/(1-\varepsilon^{(2)})} u_n^{1/(1-\varepsilon^{(2)})} \sup_{s \le t} \Big(\frac{|\Delta_i^n (\delta \star \underline{p})|}{u_n} \wedge 1 \Big)^{1/(1-\varepsilon^{(2)})} |\mathcal{F}_{\Delta_n(i-1)} \Bigg) \\ + \mathbb{E}\Bigg(\frac{|\Delta_i^n X'|^{(2+v)/(1-\varepsilon^{(2)})}}{u_n^{v/(1-\varepsilon^{(2)})}} \Bigg) \Bigg) \Bigg)^{1-\varepsilon^{(2)}}. \end{aligned}$$

After applying Hölder inequality, for some $\varepsilon^{(3)} > 0$ sufficiently small, (4.10) as well as (4.11), for some ϕ^n vanishing for n large, the expression above becomes bounded by

$$\begin{aligned} & K_v T \Delta_n^{1-(5/2+q(\varpi,r))} \Big(\phi^n \Delta_n^{(2/(1-\varepsilon^{(2)})-r)\varpi+1} + \Delta_n^{1/(2(1-\varepsilon^{(2)}))+\varpi/(1-\varepsilon^{(2)})+1-r\varpi-\varepsilon^{(3)}} \\ & + \Delta_n^{(1+\frac{v}{2}(1-2\varpi))/(1-\varepsilon^{(2)})} \Big)^{1-\varepsilon^{(2)}}. \end{aligned}$$

Recalling that $\varpi < 1/2$ the last summand above with v large enough becomes negligible in front of the other two summands. The second summand is also negligible in front of the first one when $\varepsilon^{(3)} > 0$ is sufficiently small. Hence, the expression above is bounded for some $\varepsilon^{(4)} > 0$ arbitrarily small by:

$$K_v T \Delta_n^{1-(5/2+q(\varpi,r))} \Delta_n^{(2-r)\varpi+1-\varepsilon^{(4)}} = K_v T \Delta_n^{(2-r)\varpi-1/2-q(\varpi,r)-\varepsilon^{(4)}}.$$

Hence, for $\Delta_n^{(2-r)\varpi-1/2-q(\varpi,r)-\varepsilon^{(4)}}$ the corresponding power $(2-r)\varpi-1/2-q(\varpi,r)-\varepsilon^{(4)} > 0$ for ϖ chosen sufficiently close to $\frac{7}{4(4-r)} \wedge \frac{1}{2}$.

Finally, taking into account the utilization of Markov inequality (3.8) as well as boundedness of (3.9) and (3.10) above, it is possible to conclude that for each $T > 0$ $\frac{1}{\Delta_n^{q(\varpi,r)}} Dif_T^n \xrightarrow{\mathbb{P}} 0$ and *Step 2* is completed.

Step 3: the Asymptotic Negligibility 2

For the same reasons as in the previous step, at this stage it is proved that for all $T > 0$ $\frac{1}{\Delta_n^{1/4}}(A_T^n - \overline{A}_T^n - [c,c]_T^n)$ goes to 0 in probability, implying convergence for $\frac{1}{\Delta_n^{q(\varpi,r)}}(A_T^n - \overline{A}_T^n - [c,c]_T^n)$, since $q(\varpi,r) \leq \frac{1}{4}$.

From Lemma 4.2 and (4.14) it is possible to conclude for the reals a_i^n's, all bounded by some constant L, and $q \geq 2$ (recall $k_n\Delta_n \leq K$):

$$\mathbb{E}\left(|\sum_{j=0}^{2k_n-1} a_j^n \zeta(u)_{i+j}^n|^q\right) \leq \begin{cases} K_q L^q k_n^{q/2} & \text{if } u = 1 \\ K_q L^q / k_n^{q/2} & \text{if } u = 2. \end{cases} \tag{3.14}$$

Firstly, recalling that

$$A_T^n = \sum_{u,v=1}^{2} (A(1;u,v)_T^n + A(2;u,v)_T^n + 2A(3;u,v)_T^n + 2A(4;u,v)_T^n)$$
$$\overline{A}_T^n = \overline{A}(0)_T^n + \sum_{u=1}^{2} \overline{A}(1;u)_T^n + \sum_{u,v=1}^{2} (\overline{A}(2;u,v)_T^n + 2\overline{A}(3;u,v)_T^n),$$

the "non-trivial" terms $A(1;1,1)^n, A(1;2,2)^n$ and $\overline{A}(0)^n$ are considered. According to (3.5) $\Gamma(2,2)_0^n = 1 + O(1/k_n^2)$, hence the CLT for the approximate quadratic variation of the process (X',c), see Theorem 5.4.2 of [4], yields firstly stable convergence in law to some process and results with subsequent multiplication with $\Delta_n^{1/4}$ in u.c.p convergence to 0 as follows

$$\frac{1}{\Delta_n^{1/4}}(A(1;2,2)^n - [c,c]_T^n) \stackrel{u.c.p.}{\Longrightarrow} 0$$

Next, Theorem 10.3.2 of [4] for the function $\overline{F}((x,y),(x',y')) = (x'^2 - y)^2$ and the process (X',c), plus $\Gamma(1,1)_0^n = 3/k_n^2$, yield (with $C(4)_T = \int_0^T (c_s)^2 ds$ being the quarticity) by analogy with the explanation above:

$$\frac{1}{\Delta_n^{1/4}}(k_n^2\Delta_n A(1;1,1)^n - 6C(4)) \stackrel{u.c.p.}{\Longrightarrow} 0.$$

Finally, since c satisfies (SH-0), Theorem 6.1.2 of [JP] yields similarly

$$\frac{1}{\Delta_n^{1/4}}(k_n^2\Delta_n \overline{A}(0)^n - 6C(4)) \stackrel{u.c.p.}{\Longrightarrow} 0.$$

In view of these and of the definition of A^n and $\overline{A}^n$, it remains to prove

$$\frac{1}{\Delta_n^{1/4}} B_T^n \xrightarrow{\mathbb{P}} 0 \text{if} \begin{cases} (a) B^n = \overline{A}(1;u)^n, & u = 1,2 \\ (b) B^n = \overline{A}(j;u,v)^n, & j = 2,3, u = 1,2, v = 1,2 \\ (c) B^n = A(1;u,v)^n, & (u,v) = (1,2),(2,1) \\ (d) B^n = A(j;u,v)^n, & j = 2,3,4, all(u,v). \end{cases} \tag{3.15}$$

Here Case (a) of (3.15) is considered. The variable $\chi_i^n = \sum_{j=0}^{k_n-1} \overline{\varepsilon}(u)_j^n \zeta(u)_{i+j}^n$ is $\mathcal{F}_{i+k_n}^n$-measurable, and by (4.14) and (3.14) it satisfies for both $u = 1, 2$:

$$|\mathbb{E}(\chi_i^n | \mathcal{F}_{i-1}^n)| \leq K, \qquad \mathbb{E}(|\chi_i^n|^2 | \mathcal{F}_{i-1}^n) \leq K k_n.$$

The result for Case (a) follows from (4.8) applied to the array $\xi_i^n = \frac{12}{k_n^3 \Delta_n^{1/4}} c_{(i-1)\Delta_n} \chi_i^n$.

In Case (b) for $j = 2$, upon using (4.14) and (3.4), we see that the variable $\xi_i^n = \overline{\gamma}(u,v;0)_{i+k_n-1-[T/\Delta_n],i} \zeta(u)_i^n \zeta(v)_i^n$ has $(|\xi_i^n|) \leq K/k_n^3$ for all $u, v = 1, 2$, and $(|B_T^n|) \leq KT\sqrt{k_n}$ follows, implying Case (b) for $j = 2$.

Next, supposing $j = 3$ in Case (b) and denoting by χ_i^n the ith summand in the sum defining $\overline{A}(3; u, v)_T^n$, which is $\mathcal{F}_i^n$-measurable. By (4.14), (3.4) and successive conditioning one obtains for all $u, v = 1, 2$:

$$|\mathbb{E}(\chi_i^n | \mathcal{F}_{i-1}^n)| \leq K\Delta_n^{3/2}, \qquad \mathbb{E}(|\chi_i^n|^2|) \leq K\Delta_n^{5/2},$$

and Case (b) for $j = 3$ follows from (4.8) applied to the array $\xi_i^n = \chi_i^n / \Delta_n^{1/4}$.

Here, for instance, Case (c) is considered for (u,v)=(1,2). Then Theorem 10.3.2 of [JP] applied to the process (X', c) and the function $\overline{F}((x,y),(x',y')) = (x'^2 - y)y'$ implies that $\sqrt{\Delta_n} \sum_{i=1}^{[T/\Delta_n]} \zeta(1)_i^n \zeta(2)_i^n$ converges stably in law to some limiting process. Since $\Gamma(1,2)_0^n = -3/2k_n^2$, it is possible to deduce that B_n satisfies Case (c). A similar argument shows the result for $(u, v) = (2, 1)$.

Here Case (d) for $j = 2$ is proved. By the first part of (3.3) all summands in B^n vanish, except for $4k_n$-2 of them, namely those for i between 1, and $2k_n$-1, and between $[T/\Delta_n] - 2k_n + 2$ and $[T/\Delta_n]$, and for those the coefficient in front of $\zeta(u)_i^n \zeta(v)_i^n$ is smaller than $\overline{\gamma}_{u,v}^n$. In view of (3.3) and (4.14), it follows (using Cauchy-Schwarz inequality when $u \neq v$) that in all cases $\mathbb{E}(|B_T^n|) \leq K sqrt \Delta_n$, and Case (d) follows for $j = 2$.

Next Case (d) is proved for $j = 3$. As above, all summands in B^n vanish, except for $4k_n - 2$ values of i. Below we treat only the firts $2k_n - 1$ summands (for simplicity of notation), but the last $2k_n - 2$ are treated analogously. It is possible to rewrite the sum of these first summands as

$$B_T^{n,(1)} = \sum_{i=1}^{(2k_n-1)\wedge[T/\Delta_n]} \chi_i^n, \qquad \chi_i^n = \delta_i^n \zeta(v)_i^n, \qquad \delta_i^n = a_{i,m}^n \zeta(u)_{i-m}^n,$$

where the $a_{i,m}^n$'s are reals such that $|a_{i,m}^n| \leq 2\overline{\gamma}_{u,v}^n$, and of course depend on (u, v). It is then possible to apply (3.14) with $L = 2\overline{\gamma}_{u,v}^n$ and (3.3) to obtain

$$\mathbb{E}(|\delta_i^n|^p) \leq \begin{cases} K_p/k_n^{3p/2} & \text{if } v = 1 \\ K_p/k_n^{p/2} & \text{if } v = 2. \end{cases} \tag{3.16}$$

Moreover, δ_i^n is $\mathcal{F}_{\Delta_n(i-1)}$-measurable, so (4.14) yields

$$|\mathbb{E}(\chi_i^n | \mathcal{F}_{i-1}^n)| \leq K\Delta_n^{5/4}, \qquad \mathbb{E}(|\chi_i^n|^2)| \leq K\Delta_n^{3/2},$$

and $B_T^{n,(1)} \xrightarrow{\mathbb{P}} 0$ follows from (4.8) applied with $m_n = 2k_n - 1$ and $l_n = 1$ and $\xi_i^n = \chi_i^n / \Delta_n^{1/4}$.

In conclusion, Case (d) for $j = 4$ is proved. Similarly to (3.16):

$$\mathbb{E}(|\rho(u,v)_i^n|^p | \mathcal{F}_{\Delta_n(i-2k_n)}) \leq \begin{cases} K_p / k_n^{3p/2} & \text{if } v = 1 \\ K_p / k_n^{p/2} & \text{if } v = 2. \end{cases} \tag{3.17}$$

In view of (4.14) for $u = 1, 2$:

$$\mathbb{E}(|A(4;u,1)_T^n|) \leq K\Delta_n^{5/4} \mathbb{E}\left(\sum_{i=1}^{[T/\Delta_n]} (\sqrt{\Delta_n} + \eta_i^n) \right).$$

By Lemma 4.1, this implies $A(4;u,1)_T^n / \Delta_n^{1/4} \xrightarrow{\mathbb{P}} 0$.

Now, letting $v = 2$ and $V(2) = b^{(c)}$ as well as utilizing (4.14), the first part of Lemma 4.1 and Lemma 4.2, it is possible to conclude that $\mathbb{E}(|\zeta'(2)_i^n - V(2)_{(i-2k_n)\Delta_n} \Delta_n|^2 | \mathcal{F}_{\Delta_n(i-2k_n)}) \leq K(\Delta_n \eta'^n_{i-2k_n+1})^2$. Then the Cauchy-Schwarz inequality and (3.17) for $p = 2$, plus Lemma 4.1, yield:

$$\mathbb{E}(|\frac{1}{\Delta_n^{1/4}} \sum_{i=2k_n}^{[T/\Delta_n]} \rho(u,2)_i^n (\zeta'(2)_i^n - V(2)_{(i-2k_n)\Delta_n} \Delta_n)|) \leq K\Delta_n \mathbb{E}(\sum_{i=2k_n}^{[T/\Delta_n]} (\sqrt{\Delta_n} + \eta'^n_i)) \to 0.$$

Observe that $\Delta_n^{3/4} \sum_{i=2k_n}^{T/\Delta_n} \rho(u,v)_i^n V(v)_{(i-2k_n)\Delta_n} = G_T^n + M_T^n$, where

$$\xi_{i,T}^n = \sum_{m=(2k_n - i)\vee 1}^{([T/\Delta_n]-i)\wedge(2k_n-1)} \Gamma(u,v)_m^n V(2)_{(i+m-2k_n)\Delta_n}$$

$$G_T^n = \Delta_n^{3/4} \sum_{i=1}^{[T/\Delta_n]-1} \xi_{i,T}^n \zeta'(u)_i^n, M_T^n = \Delta_n^{3/4} \sum_{i=1}^{[T/\Delta_n]-1} \xi_{i,T}^n \zeta''(u)_i^n.$$

Since $\xi_{i,T}^n \leq K k_n \overline{\gamma}_{u,2}^n$, so (4.14) and (3.3) yield $\mathbb{E}(|\xi_{i,T}^n \zeta'(u)_i^n|) \leq K\sqrt{\Delta_n}$ in all cases, and $G_T^n \xrightarrow{\mathbb{P}} 0$ follows. On the other hand, $\xi_{i,t}^n$ is $\mathcal{F}_{\Delta_n(i-1)}$-measurable, hence Doob's inequality and (4.14) and (3.3) again yield $\mathbb{E}(\sup_{s\leq T} |M_s^n|^2) \leq KT\sqrt{\Delta_n} \longrightarrow 0$ in all cases. The proof of *Step 3* is complete.

Step 4: the Required Convergence

In this step, finally, in view of (3.7) and three steps of the proof above, it remains to prove

$$\frac{1}{\Delta_n^{q(\varpi,r)}} U_T^n \begin{cases} \xrightarrow{\mathcal{L}-s} \mathcal{U}_T & \text{if } r < 0.5 \\ \xrightarrow{\mathbb{P}} 0 & \text{if } 0.5 \leq r < 1. \end{cases} \tag{3.18}$$

Assume first that the following (joint) stable convergence in law holds, where (u,v) runs through the set $P = \{1,2\}^2$:

$$\frac{1}{\Delta_n^{1/4}} (Z(u,v)^n)_{(u,v)\in P} \overset{\mathcal{L}-s}{\Longrightarrow} Z(Z(u,v))_{(u,v)\in P}, \tag{3.19}$$

where Z is defined on a very good extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_{t\geq 0}, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ and is, conditionally on $\mathcal{F}$, a continuous centered Gaussian martingale with covariance structure

$$\tilde{\mathbb{E}}(Z(u,v)_T Z(u',v')_T | \mathcal{F}) = G(u,v;u',v')_T = \int_0^T g(u,v;u',v)_s ds, \tag{3.20}$$

where the process $g(u,v;u',v')$ is given in the following display:

$$g(u,v;u',v')_T = \begin{cases} \frac{12}{\beta^3}(c_T)^4 & (u,v;u',v') = (1,1;1,1) \\ \frac{3}{2\beta}(c_T)^2(c_T^{(c)}) & (u,v;u',v') = (1,2;1,2),(2,1;2,1) \\ \frac{151\beta}{280}(c_T^{(c)})^2 & (u,v;u',v') = (2,2;2,2) \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Indeed, suppose that (3.19) holds. Consequently, (3.18) will also hold (recalling that $q(\varpi, r) \leq 1/4$) with

$$U_T = 2 \sum_{u,v=1}^{2} Z(u,v)_T.$$

Then U_T is, conditionally on $\mathcal{F}$, centered Gaussian variables with variances given by (3.2), as a simple calculation shows: Thus Theorem 3.1 is proved.

Hence, it remains to prove (3.19). Recalling

$$\frac{1}{\Delta_n^{1/4}} Z(u,v)_T^n = \sum_{i=2k_n}^{[T/\Delta_n]} \xi(u,v)_i^n, \qquad \xi(u,v)_i^n = \frac{1}{\Delta_n^{1/4}} \rho(u,v)_i^n \zeta''(v)_i^n,$$

and the $\xi(u,v)_i^n$ are martingales increments, relative to the discrete time filtration $(\mathcal{F}_i^n)$. Then, using a standard criterion for the stable convergence of triangular arrays of martingale increments (see e.g. Theorem 2.2.15 of [4]), in order to obtain the convergence (3.19), it suffices to prove the following three properties: for all $t > 0$, all

$(u,v),(u',v') \in \mathbb{R}$, and all martingales N which are either bounded and orthogonal to W, or equal to one component W^j:

$$G(u,v;u',v')^n_T := \sum_{i=2k_n}^{[T/\Delta_n]} \mathbb{E}(\xi(u,v)^n_i \xi(u',v')^n_i | \mathcal{F}^n_{(i-1)}) \xrightarrow{\mathbb{P}} G(u,v;u',v')_T \tag{3.22}$$

$$\sum_{i=2k_n}^{[T/\Delta_n]} \mathbb{E}(|\xi(u,v)^n_i|^4 | \mathcal{F}^n_{(i-1)}) \xrightarrow{\mathbb{P}} 0 \tag{3.23}$$

$$B(N;u,v)^n_T = \sum_{i=2k_n}^{[T/\Delta_n]} \mathbb{E}(\xi(u,v)^n_i \Delta_n N | \mathcal{F}^n_{(i-1)}) \xrightarrow{\mathbb{P}} 0. \tag{3.24}$$

The property (3.23) is simple. If we combine (4.14) and (3.17), by successive conditioning, we see that $\mathbb{E}(|\xi(u,v)^n_i|^4) \le K\Delta_n^2$ in all cases, obviously implying (3.23).

For the proof of (3.24) consider when N is a bounded martingale orthogonal to W, it is possible to apply the estimates (4.16), successive conditioning, and (3.17), to obtain

$$\mathbb{E}(|B(N;u,v)^n_T|) \le K\Delta_n \sum_{i=2k_n}^{[T/\Delta_n]} \mathbb{E}(N^{*n}_i).$$

Doob's inequality (for $N^{*n}_t = (\mathbb{E}\sup_{t\in((i-1)\Delta_n, i\Delta_n)} \left|N_t - N_{(i-1)\Delta_n}\right|^2 \mid \mathcal{F}^n_{i-1})^{1/2}$) yields $(N^{*n}_i)^2 \le 4\mathbb{E}((\Delta^n_i N)^2 | \mathcal{F}^n_{(i-1)})$, hence by the Cauchy-Schwarz inequality

$$\mathbb{E}(|B(N;u,v)^n_T|) \le K\sqrt{T\Delta_n}(\mathbb{E}(\sum_{i=2k_n}^{[T/\Delta_n]} (\Delta^n_i N)^2))^{1/2} = K\sqrt{T\Delta_n}(\mathbb{E}([N,N]_{\Delta_n[T/\Delta_n]}))^{1/2}.$$

Since N is a bounded martingale, $\mathbb{E}([N;N]_T) \le K$ for all T, and (3.24) follows for all martingales N which are bounded and orthogonal to W.

Now turn to the case $N = W^j$ for some $j = 1,\dots,d'$, and essentially reproduce the end of *Step 3* Case (d), with a different meaning for the notation $V(v)$. Namely, set $V(1) = 0$ and $V(2) = \sigma^{(c)}$ and $V(3) = \sigma$ and also

$$B'(N;u,v)^n_T = \Delta_n^{3/4} \sum_{i=2k_n}^{[T/\Delta_n]} \rho(u,v)^n_i V(v)_{(i-2k_n)\Delta_n}.$$

(4.18) and the property $\mathbb{E}((V(v)_{i\Delta_n} - V(v)_{(i-2k_n)\Delta_n})^2 | \mathcal{F}^n_{(i-2k_n)}) \le K((\Delta_n) + (\eta'^n_{i-2k_n+1})^2)$, plus again (3.17), Cauchy-Schwarz inequality and Lemma 4.1, yield

$$\mathbb{E}(|B(N;u,v)^n_T - B'(N;u,v)^n_T|) \le K\Delta_n \mathbb{E}(\sum_{i=1}^{[T/\Delta_n]} \eta'^n_i) \longrightarrow 0.$$

Moreover $B'(N;u,1)^n \equiv 0$, so it remains to show that $B'(N;u,1)^n_T \xrightarrow{\mathbb{P}} 0$ when $v = 2,3$. This is proved as in the end of *Step 3* Case (d), since here $B'(N;u,1)^{n,\sigma}_T$ is exactly $G^n + M^n$ there, with processes $V(v)$ which are different but still bounded. Hence, (3.24) is proved.

For the proof of (3.22) fix the two pairs (u,v) and (u',v') and begin with a reduction of the problem, in the same spirit as in the previous proof. Set

$$V_T = \begin{cases} 2(c_T)^2 & \text{if } (v,v') = (1,1) \\ c_T^{(c)} & \text{if } (v,v') = (2,2) \\ 0 & \text{otherwise} \end{cases} \qquad \overline{V}_T = \begin{cases} 2(c_T)^2 & \text{if } (u,u') = (1,1) \\ c_T^{(c)} & \text{if } (u,u') = (2,2) \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $z^n_{v,v'}$ is $1/\Delta_n$ if $v = v' = 1$ and 1 otherwise. Then, with the notation

$$\overline{G}^n_T = z^n_{v,v'}\sqrt{\Delta_n}\sum_{i=2k_n}^{[T/\Delta_n]} \rho(u,v)^n_i \rho(u',v')^n_i V_{(i-1)\Delta_n},$$

it is possible to deduce from (4.15) and (3.17) that

$$\mathbb{E}(|G(u,v;u',v')^n_T - \overline{G}^n_T|) \leq KT\sqrt{\Delta_n}.$$

So it remains to prove that $\overline{G}^n_T \xrightarrow{\mathbb{P}} G(u,v;u',v')_T$, and the only non-trivial cases are when $(v,v') = (1,1),(2,2)$, since otherwise these processes are identically vanishing.

A further reduction is amenable. Namely, set

$$\overline{G'}^n_T = z^n_{v,v'}\sqrt{\Delta_n}\sum_{i=2k_n}^{[T/\Delta_n]} \rho(u,v)^n_i \rho(u',v')^n_i V_{(i-2k_n)\Delta_n}, \tag{3.25}$$

We have $\mathbb{E}((V_{(i-1)\Delta_n} - V_{(i-2k_n-1)\Delta_n})^2) \leq K(\mathbb{E}(\Delta_n) + (\eta'^n_{i-2k_n})^2)$. Then, Lemma 4.1 and (3.17) and the Cauchy-Schwarz inequality yield

$$\mathbb{E}(|\overline{G}^n_T - \overline{G'}^n_T|) \leq K\sqrt{T}(\Delta_n \sum_{i=1}^{[T/\Delta_n]} \mathbb{E}((\eta'^n_i)^2))^{1/2} \longrightarrow 0.$$

So, it remains to show that, for $(v,v') = (1,1),(2,2)$:

$$\overline{G'}^n_T \xrightarrow{\mathbb{P}} G(u,v;u',v')_T. \tag{3.26}$$

In view of (3.25) it is necessary to express the product $\rho(u,v)^n_i\rho(u',v')^n_i$ in a more tractable way. Then:

$$\overline{G'}^n_T = \sum_{j=1}^{3} \hat{G}(j)^n_T, \qquad \widehat{G}(j)^n_T = z^n_{v,v'}\sqrt{\Delta_n}\sum_{i=2k_n}^{[T/\Delta_n]} \widehat{\rho}(j)^n_i V_{(i-2k_n)\Delta_n}$$

$$\widehat{\rho}(1)^n_i = \sum_{m=1}^{2k_n-1} \Gamma(u,v)^n_m \Gamma(u',v')^n_m \zeta(u)^n_{i-m}\zeta(u')^n_{i-m}$$

$$\widehat{\rho}(2)^n_i = \sum_{m=1}^{2k_n-2} \Gamma(u,v)^n_m \zeta(u)^n_{i-m} \sum_{m'=m+1}^{2k_n-1} \Gamma(u',v')^n_{m'}\zeta(u')^n_{i-m'}$$

$$\widehat{\rho}(2)^n_i = \sum_{m'=1}^{2k_n-2} \Gamma(u',v')^n_{m'} \zeta(u')^n_{i-m'} \sum_{m=m'+1}^{2k_n-1} \Gamma(u,v)^n_{m}\zeta(u)^n_{i-m}$$

Observe that $\widehat{G}(2)^n_T = \sum_{i=2}^{[T/\Delta_n]-1} \xi^n_i \zeta(u)^n_i$, where

$$\xi^n_i = z^n_{v,v'}\sqrt{\Delta_n} \sum_{m=1\vee(2k_n-i)}^{([T/\Delta_n]-i)\wedge(2k_n-2)} \Gamma(u,v)^n_m V_{(i+m-2k_n)\Delta_n} \sum_{m'=m+1}^{2k_n-1} \Gamma(u',v')^n_{m'}\zeta(u')^n_{i-m'}$$

is $\mathcal{F}^n_{(i-1)}$-measurable. Then $\sum_{m'=m+1}^{2k_n-1} \Gamma(u',v')^n_{m'}\zeta(u')^n_{i-m'}$ satisfies (3.14) with $L = \tilde{\gamma}^n_{v,v'}$ and u' instead of u, whereas V_t is bounded, hence we obtain for $p = 1,2$, and with $a = 1/2$ if $u' = 1$ and $a = -1/2$ when $u' \geq 2$, that $\mathbb{E}(|\xi^n_i|^p) \leq K_p(z^n_{v,v'}\tilde{\gamma}^n_{u,v}\tilde{\gamma}^n_{u',v'}k^a_n)$ An examination of all possible cases (recall that $(v,v') = (1,1),(2,2)$) leads us to

$$\mathbb{E}(|\xi^n_i|^p) \begin{cases} K\Delta_n^{3p/4} & \text{if } u = 1 \\ K\Delta_n^{p/4} & \text{if } u = 2. \end{cases}$$

If we combine this with (4.14), plus the martingale increment property of $\zeta''(u)^n_i$, we obtain by the usual argument:

$$\mathbb{E}(|\sum_{i=2}^{[T/\Delta_n]-1} \xi^n_i \zeta'(u)^n_i|) \longrightarrow 0, \qquad \mathbb{E}(|\sum_{i=2}^{[T/\Delta_n]-1} \xi^n_i \zeta''(u)^n_i|^2) \longrightarrow 0$$

Therefore, $\widehat{G}(2)^n_T \xrightarrow{\mathbb{P}} 0$, and the property $\widehat{G}(3)^n_T \xrightarrow{\mathbb{P}} 0$ is obtained in exactly the same way.

At this stage it remains to prove that $\widehat{G}(1)^n_T \xrightarrow{\mathbb{P}} G(u,v;u',v')_T$. Letting now $\xi^n_i = \zeta(u)^n_i\zeta(u')^n_i$ and $\xi'^n_i = \mathbb{E}(\xi^n_i|\mathcal{F}^n_{(i-1)})$ and $\xi''^n_i = \xi^n_i - \xi'^n_i$:

$$\widehat{G}(1)^n = \widehat{G'}^n + \widehat{G''}^n, \qquad \widehat{G'}^n_T = \sum_{i=1}^{[T/\Delta_n]-1} \mu^n_{i,t}\xi'^n_i, \qquad \widehat{G''}^n_T = \sum_{i=1}^{[T/\Delta_n]-1} \mu^n_{i,t}\xi''^n_i$$

$$\text{with}\mu^n_{i,t} = z^n_{v,v'}\sqrt{\Delta_n} \sum_{m=1\vee(2k_n-i)}^{([T/\Delta_n]-i)\wedge(2k_n-1)} \Gamma(u,v)^n_m \Gamma(u',v')^n_m V_{(i+m-2k_n)\Delta_n}.$$

It thus suffices to show that

$$\widehat{G'}^n_T \xrightarrow{\mathbb{P}} G(u,v;u',v')_T, \qquad \widehat{G''}^n_T \xrightarrow{\mathbb{P}} 0. \tag{3.27}$$

We observe that $\mu^n_{i,t}$ is $\mathcal{F}^n_{i-1}$-measurable and

$$|\mu^n_{i,t}| \le K z^n_{v,v'}\tilde{\gamma}^n_{u,v}\tilde{\gamma}^n_{u',v'} \le \begin{cases} K\Delta_n & \text{if } u == u' = 1 \\ K\sqrt{\Delta_n} & \text{if } u \wedge u' = 1 < u \wedge u' \\ K & \text{if } u, u' \ge 2. \end{cases} \tag{3.28}$$

In view of (4.14) and the martingale increment property of ξ''^n_i, we deduce $\mathbb{E}((\widehat{G''}^n_T)^2) \le KT\Delta_n$ in all cases, implying the second part of (3.27).

For the first part of (3.27) we use (4.15) and (3.28) and the usual argument (as above for $\widehat{G}(2)^n_T$ to obtain $\mathbb{E}(|\widehat{G'}^n_T - \widehat{G}^n_T|) \longrightarrow 0$ in all cases, where

$$\widehat{G}^n_T = \Delta_n \sum_{i=1}^{[T/\Delta_n]-1} \overline{\mu}^n_{i,t} V_{(i-1)\Delta_n}\overline{V}_{(i-1)\Delta_n}, \quad \overline{\mu}^n_{i,t} = z^n_{v,v'}z^n_{u,u'}\sqrt{\Delta_n} \sum_{m=1\vee(2k_n-i)}^{([T/\Delta_n]-i)\wedge(2k_n-1)} \Gamma(u,v)^n_m\Gamma(u',v')^n_m.$$

Observe that $|\overline{\mu}_{i,t}| \le K z^n_{v,v'}z^n_{u,u'}\tilde{\gamma}^n_{u,v}\tilde{\gamma}^n_{u',v'}$, which is bounded by (4.16). Moreover, the equality $\overline{\mu}_{i,t} = \sqrt{\Delta_n}H(u,v;u',v')$ except when $i \le 2k_n - 2$ or $i \ge [T/\Delta_n] - 2k_n + 2$. Therefore, in view of (3.6), in which the limit is denoted by $H(u,v;u',v')$, and by Riemann integration, it is possible to obtain (3.27) with $G(u,v;u',v')_T = H(u,v;u',v')\int_0^T V_s\overline{V}_s ds$, and the proof of (3.22) is complete.

4 Complementary Estimates and Lemmas

For the proof of Theorem 3.1 it is sensible to offer more detailed insight into some utilized thematic concepts in the form of the following estimates as well as lemmas. All these concepts are treated in-depth in the books [4] and [5].

4.1 Preliminaries for Complementary Estimates and Lemmas

With any q-dimensional càdlàg bounded process Y we associate the variables

$$\eta_{i,j}^n = \sqrt{\mathbb{E}\left(\sup_{s\in(0,j\Delta_n)} \| Y_{(i-1)\Delta_n+s} - Y_{(i-1)\Delta_n} \|^2 | \mathcal{F}_{i-1}^n\right)}, \quad \eta_i^n = \eta_{n,1}^n, \quad \eta'^n = \eta_{i,2k_n}^n. \tag{4.1}$$

Lemma 4.1 *For all $i \le i' < i' + j \le i + 2k_n$ we have $\mathbb{E}\left(\eta_{i',j}^n \mid \mathcal{F}_{i-1}^n\right) \le K\eta_i'^n$, and for all t we have $\Delta_n \mathbb{F}\left(\sum_i^{[t/\Delta_n]} \eta_i'^n\right) \to 0$ and $\Delta_n \mathbb{F}\left(\sum_i^{[t/\Delta_n]} \eta_i^n\right) \to 0$.*

Proof. The first claim follows from Cauchy-Schwarz inequality. For the second one, setting $\gamma_t^n = \sup_{s\in(0,(2k_n+1)\Delta_n)} \| Y_{t+s} - Y_t \|^2$, we observe that $\mathbb{E}((\eta_i'^n)^2)$ is always smaller than a constant, and smaller than $\frac{1}{\Delta_n}\int_{(i-2)\Delta_n}^{(i-1)\Delta_n} \mathbb{E}(\gamma_s^n)ds$ when $i \ge 2$. Hence

$$\Delta_n \mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} \eta_i'^n\right) \le \sqrt{t}\left(\mathbb{E}\left(\Delta_n \sum_{i=1}^{[t/\Delta_n]} (\eta'^n)^2\right)\right)^{1/2} \le \sqrt{t}\left(K\Delta_n + \mathbb{E}\left(\int_0^t \gamma_s^n ds\right)\right)^{1/2}.$$

We have $\gamma_s^n \le K$ and the càdlàg property of Y yields that $\gamma_s^n(\omega) \to 0$ for all ω, and all s except for countably many strictly positive values (depending on ω). Then, the second claim follows by the dominated convergence theorem, and it clearly implies the third one.

Lemma 4.2 *For any reals a_i^n with $\left|a_i^n\right| \le L$ for all n, i, and any array ξ_i^n of one dimensional variables such that each ξ_i^n and $\mathcal{F}_i{}^n$-measurable and satisfies*

$$\| \mathbb{E}(\xi_i^n \mid \mathcal{F}_{i-1}^n) \| \le L', \qquad \mathbb{E}(\| \xi_i^n \|^q | \mathcal{F}_{i-1}^n) \le L_q,$$

where $q \le 2$ and L, L', L_q are constants, we have

$$\| \mathbb{E}\left(\sum_{j=1}^{2k_n-1} a_j^n \xi_{i+j}^n \mid \mathcal{F}_{i-1}^n\right) \| \le LL'k_n, \quad \mathbb{E}\left(\| \sum_{j=1}^{2k_n-1} a_j^n \xi_{i+j}^n \|^q | \mathcal{F}_{i-1}^n\right) \le K_q L^q (L_q k_n^{q/2}).$$

Proof. Set $\xi_i'^n = \mathbb{E}(\xi_i^n \mid \mathcal{F}_{i-1}^n)$ and $\xi_i''^n = \xi_i^n - \xi' n_i$, and also $A_n' = \sum_{j=1}^{2k_n-1} a_j^n \xi_{i+1}'^n$ and $A_n'' = \sum_{j=1}^{2k_n-1} a_j^n \xi_{i+j}''^n$. We obviously have $\| A_n' \| \leq LL'k_n$, implying the first claim.
The variables $\xi_{i+j}''^n$ are martingale increments for the discrete-time filtration $(\mathcal{F}_{i+j}^n)_{j\geq 0}$. Then Burkholder-Gundy and Hölder inequalities give us

$$\mathbb{E}(\| A_n'' \|^q \mid \mathcal{F}_{i-1}^n) \leq K_q \mathbb{E}\left(\left(\sum_{j=0}^{2k_n-1} \| a_j^n \xi_{i+j}''^n \|\right)^{q/2} \mid \mathcal{F}_{i-1}^n\right) \leq L^q K_q k_n^{q/2-1} \mathbb{E}(\sum_{j=0}^{2k_n-1} \| \xi_{i+j}^n \|^q \mid \mathcal{F}_{i-1}^n),$$

which is smaller than $K_q L^q L_q k_n^{q/2}$. The second claim readily follows.

Finally, we prove some estimates for one dimensional continuous semimartingale of the form

$$Y_t = \int_0^t b_s^Y ds + \int_0^t \sigma_s^Y dW_s.$$

Note that $Y_0 = 0$. Here, W is a q'-dimensional Brownian motion, with q' arbitrary, and $c^Y = \sigma^Y \sigma^{Y\star}$. We assume that for some constant A we have

$$\| b^Y \| \leq A, \qquad \| \sigma^Y \| \leq A. \tag{4.2}$$

In connection with (4.1), we associate with any process Z the variables

$$\eta(Z)_t = \sqrt{\mathbb{E}\left(\sup_{s\leq t} \| Z_s - Z_0 \|^2 \mid \mathcal{F}_0\right)}.$$

Lemma 4.3 *In the previous setting, and with the constant K below only depending on A in (4.2), we have for $t \in [0,1]$:*

$$\begin{aligned} \| \mathbb{E}(Y_t \mid \mathcal{F}_0) - t b_0^Y \| &\leq t\eta(b^Y)_t \leq Kt \\ \left|\mathbb{E}(Y_t^j Y_t^m \mid \mathcal{F}_0) - t c_0^{Y,jm}\right| &\leq Kt(t + \sqrt{t}\eta(b^Y)_t + \eta(c^Y)_t) \leq Kt. \end{aligned} \tag{4.3}$$

If further

$$\| \mathbb{E}(c_t^Y - c_0^Y \mid \mathcal{F}_0) \| + \mathbb{E}(\| c_t^Y - c_0^Y \|^2 \mid \mathcal{F}_0) \leq At \tag{4.4}$$

for all t, we also have

$$\left|\mathbb{E}(Y_t^j Y_t^m \mid \mathcal{F}_0) - t c_0^{Y,jm}\right| \leq Kt^{3/2}(\sqrt{t} + \eta(b^Y)_t) \leq Kt^{3/2}, \tag{4.5}$$

$$\left|\mathbb{E}(Y_t^j Y_t^k Y_t^l Y_t^m \mid \mathcal{F}_0) - t^2(c_0^{Y,jk} x_0^{Y,lm} + c_0^{Y,jl} c_0^{Y,km} + c_0^{Y,jm} c_0^{Y,kl})\right| \leq Kt^{5/2}. \tag{4.6}$$

Proof. The first part of (4.3) follows by taking the $\mathcal{F}_0$-conditional expectation in the decomposition $Y_t = M_t + t b_0^Y + \int_0^t (b_s^Y - b_0^Y) ds$, where M is a one dimensional martingale with $M_0 = 0$. For the second part we deduce from Itô's formula that $Y^j Y^m$ is the sum of martingale vanishing at 0, plus the process

$$b_0^{Y,j} \int_0^t Y_s^m ds + b_0^{Y,m} \int_0^t Y_s^j ds + \int_0^t Y_s^m (b_s^{Y,j} - b_0^{Y,j}) ds$$
$$+ \int_0^t Y_s^j (b_s^{Y,m} - b_0^{Y,m}) ds + c_0^{Y,jm} t + \int_0^t (c_s^{Y,jm} - c_0^{Y,jm}) ds.$$

Since $\mathbb{E}(\| Y_t \| | \mathcal{F}_0) \le K\sqrt{t}$, we deduce both the second part of (4.3) and (4.5) by taking again the conditional expectation and by using Cauchy-Schwarz inequality and the first part.
For and indices $j_1, \cdots, j_4$ Itô's formula yields that, with M a martingale vanishing at 0,

$$\begin{aligned}\prod_{l=1}^{4} Y_t^{j_l} =& M_t + \sum_{l=1}^{p} \int_0^t b_s^{Y,j_l} \prod_{1\le m\le p, m\ne l} Y_s^{j_m} ds \\ &+ \frac{1}{2} \sum_{1\le l,l'\le d, l\ne l'} c_0^{Y,j_l,j_{l'}} \int_0^t \prod_{1\le m\le 4, m\ne l,l'} Y_s^{j_m} ds \\ &+ \frac{1}{2} \sum_{1\le l,l'\le d, l\ne l'} \int_0^t (c_s^{Y,j_l j_{l'}} - c_0^{Y,j_l j_{l'}}) \prod_{1\le m\le 4, m\ne l,l'} Y_s^{j_m} ds.\end{aligned} \tag{4.7}$$

Again, we take the $\mathcal{F}_0$-conditional expectation; using $\mathbb{E}(\| Y_t \|^q | \mathcal{F}_0) \le K t^{q/2}$ for all $q \ge 0$ and a simple calculation yields (4.6).

Lemma 4.4 *If $m_n, l_n \ge 1$ are arbitrary integers, and if for all $n \ge 1$ and $1 \le i \ge m_n$ the variable ξ_i^n is $\mathcal{F}_{i+l_n}^n$-measurable, we have*

$$\left.\begin{aligned} \sum_{i=1}^{m_n} \left|\mathbb{E}(\xi_i^n \mid \mathcal{F}_{i-1}^n)\right| \xrightarrow{\mathbb{P}} 0 \\ l_n \sum_{i=1}^{m_n} \mathbb{E}(\left|\xi_i^n\right|^2) \xrightarrow{\mathbb{P}} 0 \end{aligned}\right\} \Rightarrow \sup_{j\le m_n} \left|\sum_{i=1}^{j} \xi_i^n\right| \xrightarrow{\mathbb{P}} 0 \tag{4.8}$$

Proof. With the convention $\xi_i^n = 0$ when $i > m_n$, we set

$$\xi_i'^n = \mathbb{E}(\xi_i^n \mid \mathcal{F}_{i-1}^n), \xi_i''^n = \xi_i^n - \xi_i'^n,$$

$$A_n = \sum_{i=1}^{m_n} \left|\xi_i'^n\right|, \quad M(k)_i^n = \sum_{j=0}^{i} \xi''^n_{k+l_n j}, \quad \overline{M}(k)_n = \sup_{i \leq (m_n - k)/l_n} \left|M(k)_i^n\right|,$$

so

$$\sup_{j \leq m_n} \left|\sum_{i=1}^{i} \xi_i^n\right| \leq A_n + \sum_{k=1}^{l_n} \overline{M}(k)_n. \tag{4.9}$$

The first condition in (4.8) implies $A_n \xrightarrow{\mathbb{P}} 0$. On the other hand, each sequence $M_n(k)$ is a martingale, relative to the discrete-time filtration $(\mathcal{F}^n_{k+(i+)l_n})_{i \geq 0}$, hence Doob's inequality gives us $\mathbb{E}(|\overline{M}(k)_n|^2) \leq 4\sum_{j=0}^{(m_n-k)/l_n} \mathbb{E}(|\xi''^n_{k+l_n j}|^2)$, which in turn is smaller than $4\sum_{j=0}^{(m_n-k)/l_n} \mathbb{E}(|\xi^n_{k+l_n j}|^2)$. Since $\mathbb{E}(|\overline{M}(k)_n|^2) \leq ł_n \sum_{k=1}^{l_n} \mathbb{E}(|\overline{M}(k)_n|^2)$, the second condition in (4.8) yields that this expectation goes to 0, and this completes the proof.

4.2 Increment Estimates

Under (SPCC-r) for $r < 1$, $T > 0$ and $q \geq 0$ it can be concluded by (2.1.44) of [4]:

$$\begin{aligned} &\mathbb{E}(\sup_{s \leq t} |X_{T+s} - X_T|^q | \mathcal{F}_T) \leq \begin{cases} K_q t^{q/2} & \text{if X is continuous} \\ K_q t^{(q/2)\wedge 1} & \text{otherwise,} \end{cases} \\ &\mathbb{E}(\sup_{s \leq t} |X'_{T+s} - X'_T|^q | \mathcal{F}_T) \leq K_q t^{q/2}, \qquad |\mathbb{E}(X'_{T+t} - X'_T | \mathcal{F}_T)| \leq Kt \\ &\mathbb{E}(\sup_{s \leq t} |c_{T+s} - c_T|^q | \mathcal{F}_T) \leq K_q t^{q/2}, \qquad |\mathbb{E}(c_{T+t} - c_T | \mathcal{F}_T)| \leq Kt. \end{aligned} \tag{4.10}$$

Similarly under (SPCC-r) for $r < 1$, $T > 0$ and $q \geq 1$ it can be concluded by Corollary (2.1.9) of [4] with ϕ a function depending on J, r, q (recalling that J is a a deterministic non-negative bounded function from (SPCC-r) satisfying $\int J(z)\lambda(dz) < \infty$ such that $|\delta(\omega, t, z)|^r \leq J(z)$ for all (ω, t, z)), but not on δ as well as satisfying $\phi(t) \longrightarrow 0$ as $t \longrightarrow 0$ and $Y = \delta \star \underline{p}$:

$$t \leq 1, \; 0 < \chi < \frac{1}{2} \qquad \Rightarrow \qquad \mathbb{E}\left(\sup_{s \leq t} (\frac{|Y_{T+s} - Y_T|}{t^\chi} \wedge 1)^q | \mathcal{F}_T\right) \leq K t^{1-\chi r} \phi(t). \tag{4.11}$$

For simpler notation late on, we define the following 1-dimensional variables

$$
\begin{aligned}
\zeta(1)_i^n &= \frac{1}{\Delta^n}(\Delta_i^n X')^2 - c_{(i-1)\Delta_n}, & \zeta(2)_i^n &= \Delta_i^n c, \\
\zeta'(r)_i^n &= \mathbb{E}(\zeta(r)_i^n \mid \mathcal{F}_{i-1}^n), & \zeta''(r)_i^n &= \zeta(r)_i^n - \zeta'(r)_i^n, \\
\alpha_i^n &= \frac{1}{k_n}\sum_{j=0}^{k_n-1} \zeta(1)_{i+j}^n, & \beta_i^n &= \frac{1}{k_n}\sum_{j=0}^{k_n-1} (1)_{(i+j-1)\Delta_n}
\end{aligned}
$$

$$
\overline{\beta}_i^n = \beta_i^n - c_{(i-1)\Delta_n} = \frac{1}{k_n}\sum_{j=0}^{k_n-1}(c_{(i+j-1)\Delta_n} - c(_{i-1)\Delta_n}) = \frac{1}{k_n}\sum_{j=0}^{k_n-2}(k_n-1-j)\zeta(2)_{i+j}^n. \tag{4.12}
$$

Here, k_n is the sequence of integers used to construct the spot volatility estimators, and it either satisfies $k_n \sim \beta/\sqrt{\Delta_n}$ for some $\beta > 0$, or $k_n\sqrt{\Delta_n} \to 0$.

Estimates under (SPCC-r), for $r < 1$.

$$
\eta_i^n, \eta_i'^n \text{associated by (4.1) with process } Y = (b, b^c, \sigma^c, c^c, c^{X',c}). \tag{4.13}
$$

We apply (4.10) and also Lemma 4.3 to the processes $Y_t = X'_{(i-1)\Delta_n+t} - X'_{(i-1)\Delta_n}$ or $Y_t = c_{(i-1)\Delta_n+t} - c_{(i-1)\Delta_n}$, to obtain

$$
\begin{aligned}
&\| \zeta'(1)_i^n \| \le K\sqrt{\Delta_n}(\sqrt{\Delta_n} + \eta_i^n) \le K\sqrt{\Delta_n}, \quad \mathbb{E}(\| \zeta(1)_i^n \|^q \mid \mathcal{F}_{i-1}^n) \le K_q \\
&\| \zeta'(2)_i^n - b_{(i-1)\Delta_n}^{(c)}\Delta_n \| + \| \zeta'(3)_i^n - b_{(i-1)\Delta_n}\Delta_n \| \le K\Delta_n(\sqrt{\Delta_n} + \eta_i^n) \le K\Delta_n \\
&\mathbb{E}(\| \zeta(2)_i^n \|^q \mid \mathcal{F}_{i-1}^n) + \mathbb{E}(\| \zeta(3)_i^n \|^q \mid \mathcal{F}_{i-1}^n) \le K_q\Delta_n^{q/2},
\end{aligned} \tag{4.14}
$$

and also, with $\overline{\zeta(r)}_i^n$ denoting either $\zeta(r)_i^n$ or $\zeta''(r)_i^n$:

$$
\begin{aligned}
&\Big|\mathbb{E}(\overline{\zeta}(1)_i^{n,jk}\overline{\zeta}(1)_i^{n,lm} \mid \mathcal{F}_{i-1}^n) - (c_{(i-1)\Delta_n}^{jl}c_{(i-1)\Delta_n}^{km} + c_{(i-1)\Delta_n}^{jm}c_{(i-1)\Delta_n}^{kl})\Big| \le K\sqrt{\Delta_n} \\
&\Big|\mathbb{E}(\overline{\zeta}(2)_i^{n,jl}\overline{\zeta}(2)_i^{n,km} \mid \mathcal{F}_{i-1}^n) - c_{(i-1)\Delta_n}^{(c),jl,km}\Delta_n\Big| \le K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_i^n) \le K\Delta_n^{3/2} \\
&\Big|\mathbb{E}(\overline{\zeta}(3)_i^{n,j}\overline{\zeta}(3)_i^{n,k} \mid \mathcal{F}_{i-1}^{(n)}) - c_{(i-1)\Delta_n}^{jk}\Delta_n\Big| \le K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_i^n) \le K\Delta_n^{3/2} \\
&\Big|\mathbb{E}(\overline{\zeta}(2)_i^{n,jk}\overline{\zeta}(3)_i^{n,l} \mid \mathcal{F}_{i-1}^n) - c_{(i-1)\Delta_n}^{(X',c),jk,l}\Delta_n\Big| \le K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_i^n) \le K\Delta_n^{3/2} \\
&\Big|\mathbb{E}(\overline{\zeta}(1)_i^{n,jk}\overline{\zeta}(2)_i^{n,lm} \mid \mathcal{F}_{i-1}^n)\Big| + \Big|\mathbb{E}(\overline{\zeta}(1)_i^{n,jk}\overline{\zeta}(3)_i^{n,l} \mid \mathcal{F}_{i-1}^n)\Big| \le K\Delta_n.
\end{aligned} \tag{4.15}
$$

Finally, for any bounded martingale N which is orthogonal to W, and with the notation $N_t^{*n} = (\mathbb{E}\sup_{t\in((i-1)\Delta_n, i\Delta_n)} \big|N_t - N_{(i-1)\Delta_n}\big|^2 \mid \mathcal{F}_{i-1}^n)^{1/2}$, and upon using Itô's formula, one gets

$$
\begin{aligned}
&\Big|\mathbb{E}(\zeta''(1)_i^{n,jk}\Delta_i^n N \mid \mathcal{F}_{i-1}^n)\Big| \le K\sqrt{\Delta_n}N_i^{*n} \\
&\Big|\mathbb{E}(\zeta''(2)_i^{n,j}\Delta_i^n N \mid \mathcal{F}_{i-1}^n)\Big| + \Big|\mathbb{E}(\zeta''(3)_i^{n,j}\Delta_i^n N \mid \mathcal{F}_{i-1}^n)\Big| \le K\Delta_n N_i^{*n},
\end{aligned} \tag{4.16}
$$

whereas when $N = W^l$ is one of the components of W, we have instead:

$$\left|\mathbb{E}(\zeta''(1)_i^{n,jk}\Delta_i^n W^l \mid \mathcal{F}_{i-1}^n)\right| \leq K\Delta_n$$
$$\left|\mathbb{E}(\zeta''(2)_i^{n,jk}\Delta_i^n W^l \mid \mathcal{F}_{i-1}^n) - \sigma_{(i-1)\Delta_n}^{(c),jk,l}\Delta_n\right| \leq K\Delta_n\eta_i^n \tag{4.17}$$
$$\left|\mathbb{E}(\zeta''(3)_i^{n,j}\Delta_i^n W^l \mid \mathcal{F}_{i-1}^n) - \sigma_{(i-1)\Delta_n}^{jl}\Delta_n\right| \leq K\Delta_n^{3/2}. \tag{4.18}$$

Lemma 4.5 *Under (SPCC-r), for $r<1$, we have for all $q \geq 2$:*

$$\begin{aligned}
&\| \mathbb{E}(\alpha_i^n \mid \mathcal{F}_{i-1}^n) \| \leq K\sqrt{\Delta_n}(\sqrt{\Delta_n} + \overline{\eta}_i'^n) \\
&\left|\mathbb{E}(\alpha_i^{n,jk}\alpha_i^{n,lm} \mid \mathcal{F}_{i-1}^n) - \frac{1}{k_n}(c_{(i-1)\Delta_n}^{jl}c_{(i-1)\Delta_n}^{km} + c_{(i-1)\Delta_n}^{jm}c_{(i-1)\Delta_n}^{kl})\right| \leq K\sqrt{\Delta_n}(\frac{1}{k_n} + \overline{\eta}_i'^n) \\
&\mathbb{E}(\| \alpha_i^n \|^q \mid \mathcal{F}_{i-1}^n) \leq K_q(\Delta_n^{q/2} + k_n^{-q/2}) \\
&\left|\mathbb{E}(\alpha_i^{n,jk}\beta_i^{n,lm} \mid \mathcal{F}_{i-1}^n)\right| \leq Kk_n\Delta_n \\
&\| \mathbb{E}(\overline{\beta}_i^n \mid \mathcal{F}_{i-1}^n) \| \leq Kk_n\Delta_n \\
&\mathbb{E}(\| \overline{\beta}_i^n \|^q \mid \mathcal{F}_{i-1}^n) \leq \begin{cases} K_q(k_n\Delta_n)^{q/2} & \textit{if } c \textit{ is continuous,} \\ K_q k_n\Delta_n & \textit{otherwise.} \end{cases}
\end{aligned}$$

Proof. The first claim above directly follows from (4.14) and (4.15). For the second claim, we set $\xi_i^n = c_{(i-1)\Delta_n}^{jl}c_{(i-1)\Delta_n}^{km} + c_{(i-1)\Delta_n}^{jm}c_{(i-1)\Delta_n}^{kl}$ and write $\alpha_i^{n,jk\alpha_i^{n,lm}}$ as

$$\frac{1}{k_n^2}\sum_{u=0}^{k_n-1}\zeta(1)_{i+u}^{n,jk}\zeta(1)_{i+u}^{n,lm} + \frac{1}{k_n^2}\sum_{u=0}^{k_n-2}\sum_{v=u+1}^{k_n-1}(\zeta(1)_{i+u}^{n,jk}\zeta(1)_{i+v}^{n,lm} + \zeta(1)_{i+u}^{n,lm}\zeta(1)_{i+v}^{n,jk}).$$

By (4.14) and (4.15) and and successive conditioning and the first part of Lemma 4.1, the $\mathcal{F}_{i-1}^n$-conditional expectation of the last term above is smaller than $K\sqrt{\Delta_n}(\sqrt{\Delta_n} + \overline{\eta}_i'^n)$. The conditional expectation of the first term, up to $K\sqrt{\Delta_n}/k_n$, is $\frac{1}{k_n^2}\sum_{u=0}^{k_n-1}\mathbb{E}(\xi_{i+u}^n \mid \mathcal{F}_{i-1}^n)$. Using the boundedness of c_t and (4.10), we easily check that $\left|\mathbb{E}(\xi_{i+u}^n \mid \mathcal{F}_{i-1}^n) - \xi_i^n\right| \leq Kk_n\Delta_n$ when $u \leq k_n$, and the second claim follows. For the third claim, we use (4.14) and (4.15) and Hölder's inequality, plus Burkholder-Gundy inequality $\zeta(1)_{i+j}''^n$. For the fourth claim, we use (4.14) and (4.15) and Hölder's inequality again, plus successive conditioning. The last two claims are obvious consequences of (4.10) applied to the martingale increments.

5 Conclusion and Potential Further Developments

In this chapter main results of the master thesis are briefly summarized and further promising research directions are discussed.

For a bivariate process (X, σ^2), with X enjoying full generality of an Itô semimartingale and σ^2 being a Brownian Itô semimartingale, consistent truncated unnormalized estimator with a certain convergence rate has been derived for data observed on a discrete grid with the grid size Δ_n. The underlying model framework for the estimator makes its utilization particularly appropriate for high-frequency financial data.

Potential further developments are:

- derivation of an analogous estimator of quadratic covariation for the bivariate process (X, σ^2);
- increase of a Blumenthal-Getoor index r in assumption (PCC-r) to its theoretically maximal value 2 or, in other words, allowance for maximal degree of jump activity;
- introduction of full generality of Itô semimartingale for the volatility process σ^2;
- enhancement of estimation robustness with respect to the presence of microstructure noise.

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