Heuristic Sequential Model Selection Strategies for R-Vine Copulas: Comparison and Applications

Diplomarbeit
von
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Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

Garching, den 03. April 2012
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At this point I would like to thank all those people who supported me while writing the thesis.

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Contents
1 Introduction

Copulas and their applications are one of the fastest emerging research fields in the statistics of the past few years. With good reason since copulas extend the modelling possibilities enormously. Bivariate distributions in the classical statistics had often the problem, that the univariate margins had to be from the same family. Only in some special cases, which had a high practical use like the negative binomial distribution for insurance companies, bivariate distributions with different marginal distributions were investigated in detail. However, ? showed that every bivariate distribution can be written as a function of the marginal distributions. Vice versa it is possible to combine arbitrary marginal distributions to a bivariate distribution. This functional relationship respectively this bivariate distribution is called copula. Thus an enormous restriction had been lifted since marginal distributions and the functional relationship (the copula) can be modelled separately. Later, ? and ? introduced vines as a graphical model to extend the bivariate theory on multivariate cases. The result is a pair-copula construction (PCC) method, which uses the bivariate copulas as building blocks to describe a multivariate distribution function. A good description of this method can be found in ?. For a complete overview to the history of copulas, see the Introduction in ?.

In the economy, especially in the financial sector it is crucial to model dependencies optimal and we now have the possibility to model dependencies much more accurately than it has been possible so far. Therefore, the investigations have been intensified in this area over the last years. The different types of vines (C-, D- and R-vine) were studied and different weights for the selection of the variable pairs in the PCC were introduced. We will therefore intensively make reference to the predecessor theses of ?, ?, and ?.

In this thesis we will take the results of the predecessor theses and use them to compare different weights for the R-vine tree structure selection. We use the regular (R-) vine since we like to make general statements and C- and D-vines are only special cases. We will consider 5 different weights: The empirical Kendall’s $\tau$, the p-value, the partial correlations, the AIC and a mixture of the Kendall’s $\tau$- and p-values. A detailed explanation to these weights can be found in Section 2.2. We will examine the quality of the found models as well as the costs of the corresponding algorithms.

First we will introduce some basic terms and definitions in Chapter 2 in order to facilitate the understanding of later chapters. We will introduce the different vines and define the weights. Additionally we will describe the various tests and goodness of fit indicators that we need to determine the quality of the found models later.

In Chapter 3 the different strategies and their corresponding algorithms are introduced. We will also discuss the running times in this context.
Then we will set up a simulation study in Chapter 4. Here we will examine the performance of the various strategies under known conditions. We will set up 20 different scenarios which differ in the strength of correlations, the copula family choices and the dependency pattern in the higher trees (simplification, truncation). We will simulate data sets from the given R-vine copula specifications and use our strategies to fit models. After that we will determine the overall goodness of fit of the found models and compare them to each other as well as to the true model.

In the Chapters 5 and 6 we will then apply the strategies to real life data sets and compare the results to those of the simulation study. We will use an exchange rates data set as already introduced in ? and a bond returns data set of European indices (both 9-dimensional).

Finally we will summarize the overall results in Chapter 7 and give an outlook on the topics which may have to be further examined.
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Before we start the investigation to the heuristic sequential model selection strategies for R-vine copulas, we need to introduce some terms and basic concepts. But since many definitions have already been described in detail in the previous investigations of ?, ?, and ?, we will restrict ourselves to the most essential. For additional background ? and ? are recommended.

2.1 Vines

In this thesis we consider heuristic sequential model selection strategies for R-vines. Therefore it is time to explain, what a vine actually is and to introduce some terms. Furthermore we will introduce the matrix notation of a vine in this section.

2.1.1 C-, D-, R-vines

The regular vine (⇒ R-vine) was introduced in ? and further developed in ? as a graphical model for dependencies in multivariate data sets. They stand out due to particularly clear presentations. Vines have been vividly described in ? as follows: A vine on \( n \) elements \( \nu = (T_1, \ldots, T_{n-1}) \) is a nested set of trees where the edges of tree \( j \) are nodes of the tree \( j + 1 \) and each tree has the maximum number of edges. A regular vine on \( n \) elements is one in which two edges in tree \( j \) are joined by an edge in tree \( j + 1 \) only if these edges share a common node (proximity condition).

The formal definition follows below.

**Definition 2.1 (Vine, R-vine (?))**

\( \nu = (T_1, \ldots, T_{n-1}) \) is a vine on \( n \) elements if

(i) \( T_1 \) is a tree with nodes \( N_1 = 1, \ldots, n \) and a set of edges denoted \( E_1 \).

(ii) For \( i = 2, \ldots, m \), \( T_i \) is a tree with nodes \( N_i = E_{i-1} \) and edge set \( E_i \).

A vine \( \nu \) is a R-Vine on \( n \) elements if:

(iii) **(proximity condition)** For \( i = 2, \ldots, n-1 \) and \( \{a,b\} \in E_i \) with \( a = \{a_1, a_2\} \) and \( b = \{b_1, b_2\} \) it must hold that \( \#(a \cap b) = 1 \).

We furthermore define two special vines: The D-vine and the canonical vine (C-vine).
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Definition 2.2 (D-vine, C-Vine (\(?\)))

An R-Vine is called a

(i) **D-vine** if each node in \(T_1\) has degree of at most 2.

(ii) **C-vine** if each tree \(T_i\), \(i=1,\ldots,n-1\), has one unique node of degree \(n-1\).

Low dimensional examples to different vines can be found in the Figures 2.1-2.3. A large example for an R-vine can be found in Figure 4.1 on page 36.

![Diagram of vines](image)

Figure 2.1: Simple 4-dimensional D-Vine for illustration of some definitions

![Diagram of vines](image)

Figure 2.2: Simple 4-dimensional C-Vine
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Figure 2.3: Simple 5-dimensional R-vine

We have chosen a 5-dimensional R-vine here since every vine up to 4 dimensions is either a C- or a D-vine.

Now we will introduce some basic terms that we need to describe the vines and illustrate them with the aid of the simple 4-dimensional D-vine shown in Figure 2.1.

**Definition 2.3**

1. For $e \in E_i, i \leq n - 1$, the constraint set associated with $e$ is the complete union $U_e^*$ of $e$, that is, the subset of $\{1, \ldots, n\}$ reachable from $e$ by the membership relation.

2. For $i=1, \ldots, n-1$, $e \in E_i$, if $e = \{j, k\}$ then the conditioning set associated with $e$ is

$$D_e = U_j^* \cap U_k^*$$

and the conditioned set associated with $e$ is

$$C_e = \{C_{e,j}, C_{e,k}\} = \{U_j^* \setminus D_e, U_k^* \setminus D_e\}.$$

**Notation:** $\{C_e \mid D_e\}$ denotes the constraint set of an edge $e$, where $C_e$ and $D_e$ denote the conditioned and the conditioning sets of $e$.

3. $B_x = \{x\} \cup A_x$ where $x \in C_e$ and $A_x = D_e$ for some edge $e \in T_j$. 
Example 1 (Illustration of Definition 2.3)
Now we illustrate Definition 2.3 on the basis of our example D-Vine in Figure 2.1:

We first look at the edge \( a = \{1, 2\} \) in \( T_1 \).
1. Only the nodes 1 and 2 are reachable from \( a \), so \( U^*_a = \{1, 2\} \), which therefore is the constraint set of edge \( a \).
2. Since \( a = \{1, 2\} \) the conditioning set is given by \( D_a = U^*_a \cap U^*_b = \{1\} \cap \{2\} = \{\} \) and the conditioned set is given by \( C_a = \{1\} \cap \{2\} = \{\} \).

In the same way we get \( U^*_b = \{2, 4\} \), \( D_b = \{\} \), \( C_b = \{2, 3\} \) and \( B_3 = \{3\} \).

Example 2 (Illustration of Definition 2.4)
We consider our sample D-Vine in Figure 2.1 again:
\( f = \{14 | 2, 3\} \) is the \( m \)-parent of \( d = \{13 | 2\} \) and \( e = \{2, 4 | 3\} \), which are in turn the \( m \)-children of \( f \).

Definition 2.4
Let \( a = \{x, u | A \setminus \{u\}\}, b = \{y, v | A \setminus \{v\}\} \) be two adjacent edges in \( T_i \) and \( c = \{x, y | A\} \) their connecting edge in \( T_{i+1} \) via the proximity condition. Then \( a \) and \( b \) are called \( m \)-children of \( c \) and \( c \) is called the \( m \)-parent of \( a \) and \( b \).

2.1.2 The R-vine matrix
To make the vine usable for algorithms, we have to find a way to store the structure and other properties like the copula family choices and the parameter. Therefore a matrix notation was introduced. We will in this thesis use the matrix notation as presented in . At first we have to introduce some variables and sets:

- \( M = (m_{i,j})_{i,j=1,...,d} \in \{1, ..., d\}^{d \times d} \) a lower triangular matrix.
- \( L_M(i) = \{m_{i,1}, ..., m_{i,d}\} \) the set of non-zero entries in the \( i \)th column of \( M \).
- \( B_M(i) = \{(m_{i,k}, D) : k = i + 1, ..., d, D = \{m_{k,1}, ..., m_{k,i}\}\} \)
- \( BM = B_M(1) \cup ... \cup B_M(n-1) \)
• \( \tilde{B}_M(i) = \{(m_{k,i}, D) : k = i + 1, \ldots, d, \ D = \{m_{i,i}\} \cup \{m_{k+1,i}, \ldots, m_{d,i}\}\)\

\[\tilde{B}M = \tilde{B}_M(1) \cup \ldots \cup \tilde{B}_M(n-1)\]

Now we can formally define the R-vine matrix.

**Definition 2.5 (R-vine matrix (\(?)\))**

Let \( M \in \{1, \ldots, n\}^{n \times n} \) be a lower triangular matrix which satisfies the following conditions

(i) \( L_M(i) \subset L_M(j) \) for \( 1 \leq j \leq i \leq n \),

(ii) \( m_{i,i} \notin L_M(i+1) \) for \( i=1, \ldots, n-1 \),

(iii) for \( i=1, \ldots, n-1 \) and for all \( k=i+1, \ldots, n-1 \)

\[ (m_{k,i}, \{m_{k+1,i}, \ldots, m_{n,i}\}) \in B_M(i+1) \cup \ldots \cup B_M(n-1) \cup \tilde{B}_M(i+1) \cup \ldots \cup \tilde{B}_M(n-1) \]

Then \( M \) is called an R-vine matrix.

**Example 3 (R-vine matrix)**

We consider our sample R-Vine from Figure 2.3. The corresponding R-vine matrix \( M_4 \) is shown below. We will not explain in detail, how the R-vine matrix is computed from the R-vine or vice versa. These procedures can be found in (\(?)\, Appendix A). Instead we will just show how to interpret the R-vine matrix.

\[
M_4 = \begin{bmatrix}
1 & 4 & 5 \\
3 & 2 & 3 \\
2 & 3 & 2 \\
\end{bmatrix}
\]

As we can see the single edge \((15 | 234)\) of the 4th tree from our R-vine is already highlighted. The conditioned set is marked red and the conditioning set is marked green.

In the next matrix the two edges of the 3rd tree, \((14 | 23)\) and \((45 | 23)\), are highlighted:

\[
M_3 = \begin{bmatrix}
1 & 4 & 5 \\
3 & 2 & 3 \\
2 & 3 & 2 \\
\end{bmatrix}
\]

The edges of the trees 2 and 1 can be found in the same way:

\[
M_2 = \begin{bmatrix}
1 & 4 & 5 \\
3 & 2 & 3 \\
2 & 3 & 2 \\
\end{bmatrix} \quad M_1 = \begin{bmatrix}
1 & 4 & 5 \\
3 & 2 & 3 \\
2 & 3 & 2 \\
\end{bmatrix}
\]
In the matrix $M$ only the vine structure is stored so far. But also the copula families and parameters can be expressed in matrix form:

**Example 4 (Copula type and parameter matrices)**

We extend Example 3 by the fictive copula type matrix $T$ and the parameter matrix $P$ as shown below (In the case of copulae with more than one parameter this theory of course also holds for several parameter matrices $P_1, P_2, \ldots$). $N$ denotes the Gauss copula, $G$ a Gumbel, $RG$ a rotated Gumbel ($180^\circ$) and $I$ an independence copula.

$$M = \begin{bmatrix} 1 & 5 & 4 \\ 4 & 5 & 5 \\ 3 & 2 & 3 \\ 2 & 3 & 2 & 3 \end{bmatrix}, \quad T = \begin{bmatrix} RG \\ N & I \\ G & RG & N \\ N & G & N & RG \end{bmatrix}, \quad P = \begin{bmatrix} 1.1 \\ 0.4 & 0 \\ 0.5 & 1.0 & 0.6 \\ 0.8 & 0.9 & 0.4 & 1.1 \end{bmatrix}$$

As we already know from Example 3 the highlighted variables in the matrix $M$ illustrate the edge $(13 | 2)$ in the 2nd tree of our sample R-vine. The corresponding entries in the copula type and parameter matrices are highlighted blue. This means the variable pair $(13 | 2)$ is modelled by a Gumbel copula with parameter 0.5. The specifications of the other edges can be found similarly.

### 2.2 Weights

#### 2.2.1 Kendall’s $\tau$

The Kendall rank correlation coefficient as introduced in is a rank correlation measure. This means that the Kendall’s $\tau$ value does not depend on the exact data points, but only on the ordering of these points. Therefore it is especially useful for copulas since the ranks are invariant under non-linear strictly increasing transformations and independent of the margins of $X_1$ and $X_2$.

**Definition 2.6 (Kendall’s $\tau$)\footnote{\textit{\cite{Kendall}}}

Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be two independent pairs of random variables with a joint distribution $F$ and marginal distributions $F_X$ and $F_Y$. Then the Kendall’s $\tau$ is defined by

$$\tau = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0) = E[\text{sgn}(X_1 - X_2)\text{sgn}(Y_1 - Y_2)],$$

where sgn is the sign-function.

Since all of the algorithms we will introduce in the following work with the empirical Kendall’s $\tau$ values we will also explain here, how these values are calculated. Therefore we need some additional terms. Let $(x_i, y_i)$ and $(x_j, y_j)$ be two data points. The pair of points is called
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- **concordant**, if \( x_i < x_j \) and \( y_i < y_j \)
  The number of concordant pairs in a data set is denoted by \( C \).

- **discordant**, if \( x_i < x_j \) and \( y_i > y_j \)
  The number of discordant pairs in a data set is denoted by \( D \).

- **tied in the \( x \)s**, if \( x_i = x_j \) and \( y_i \neq y_j \)
  The number of tied pairs in the \( x \)s in a data set is denoted by \( T_X \).

- **tied in the \( y \)s**, if \( y_i = y_j \) and \( x_i \neq x_j \)
  The number of tied pairs in the \( y \)s in a data set is denoted by \( T_Y \).

**Empirical Kendall’s \( \tau \)**

Now we can define the empirical Kendall’s \( \tau \) as follows:

\[
\hat{\tau}(X,Y) = \frac{C - D}{\sqrt{C + D} \times \sqrt{C + D + T_X + T_Y}}
\]

The denominator ensures that \(-1 \leq \hat{\tau} \leq 1\). If there are more concordant than discordant pairs in a data set \( \hat{\tau} \) is positive and we therefore speak of positive correlated variables. If there are more discordant than concordant pairs \( \hat{\tau} \) is negative and we therefore speak of negative correlated variables. If, however, \( \hat{\tau} = 0 \) the two variables are uncorrelated.

**Theoretical Kendall’s \( \tau \)**

The theoretical Kendall’s \( \tau \) is based on the functional relationship between the Kendall’s \( \tau \) values and the first parameters of the fitted copulas. This means that the theoretical Kendall’s \( \tau \) values can be computed on the basis of the first copula parameters found by our strategies for the respective copulas. The formulas for this computations are shown in Table 2.2.1, where the first parameters are denoted as \( \rho \).

<table>
<thead>
<tr>
<th>Copula Family</th>
<th>Kendall’s ( \tau )</th>
<th>( \tau \in \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian copula</td>
<td>( \tau = \frac{2}{\pi} \arcsin(\rho) )</td>
<td>[-1,1]</td>
</tr>
<tr>
<td>Student t copula</td>
<td>( \tau = \frac{2}{\pi} \arcsin(\rho) )</td>
<td>[-1,1]</td>
</tr>
<tr>
<td>Gumbel copula</td>
<td>( 1 - \frac{1}{\rho} )</td>
<td>[0,1]</td>
</tr>
<tr>
<td>rotated Gumbel copula (180°)</td>
<td>( 1 - \frac{1}{\rho} )</td>
<td>[0,1]</td>
</tr>
<tr>
<td>rotated Gumbel copula (90°)</td>
<td>( -1 - \frac{1}{\rho} )</td>
<td>[-1,0]</td>
</tr>
<tr>
<td>rotated Gumbel copula (270°)</td>
<td>( -1 - \frac{1}{\rho} )</td>
<td>[-1,0]</td>
</tr>
</tbody>
</table>

Table 2.1: Formulas for the calculation of the theoretical Kendall’s \( \tau \) values on the basis of the first parameters of the respective copula family. The first parameters are denoted by \( \rho \).

The Kendall’s \( \tau \) values for the independence copula are obviously 0. For the proofs of the formulas given in Table 2.2.1 see ?, ?, or ?.
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2.2.2 Akaike Information Criterion (AIC)

The AIC was first introduced in ?. In general, the AIC is defined as follows:

\[ AIC = -2\ln(L) + 2k; \]

where \( k \) is the number of parameters and \( L \) denotes the likelihood of the estimated model.

The AIC value is composed of 2 parts: On the one hand the loglikelihood as a measure of the goodness of fit and on the other hand a penalty term (2\( k \)) for the model complexity. An increasing number of parameters always leads to a higher loglikelihood. With this penalty term overfitting shall be avoided. If the number of parameters reaches a critical value, the AIC even goes up again.

The AIC value is not designed for hypothesis testing or to tell how well the estimated model fits a data set. The use of the AIC value is in the model selection by creating a ranking of the models according to their AIC values. The model with the lowest AIC value is preferred.

Another penalty term is used in the Bayesian Information Criterion (BIC) as introduced in ?. The BIC is defined as

\[ BIC = -2\ln(L) + k \times \ln(n), \]

where \( k \) and \( L \) are defined as above and \( n \) is the sample size.

The advantage of the BIC compared to the AIC is that it takes the sample size into account. A bigger sample size leads generally to a higher loglikelihood value. With the adjusted penalty term of the BIC it is now possible to compare the goodness of fit of models fitted on the basis of different sample sizes.

For further informations and a detailed comparison of AIC and BIC, see ?.

2.2.3 P-value corresponding to the Cramer-von-Mises goodness of fit statistics

Two of the weights we will use for the tree structure selection are the "p-value" and a mixture of the weights "p-value" and "Kendall’s \( \tau \)". Of course there are plenty of p-values and we have to define here, which one we will use in this thesis. For our purposes we will use the p-value of the Cramér von Mises statistic as described in ? and ?. We are interested in testing the null hypothesis

\[ H_0 : F = F_0, \]
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where $F$ denotes the cumulative distribution function.
The Cramér von Mises statistic is defined as follows:

$$T_n = n \int_{-\infty}^{\infty} \left[ \hat{F}_n(x) - F_0(x) \right]^2 dF_0(x),$$

where $\hat{F}$ denotes the empirical distribution function. Thus the Cramér von Mises measures the goodness of fit of the cumulative distribution function $F$ compared to a given empirical distribution function $\hat{F}$. When we compare the value $T$ with the corresponding tabular values, we obtain the p-value. If the value $T$ is larger than the tabulated values the null hypothesis has to be rejected. However, in practice it is not feasible to look up each value in a table, therefore in \( ? \) and \( ? \) the so called fast multiplier approach was developed to obtain approximate p-values. The corresponding implementation in the statistical software R is described in \( ? \).

2.2.4 Partial Correlations

Partial correlations are a method to describe the linear dependency of two variables $X$ and $Y$, where the linear influences of other variables $Z=\{Z_1, Z_2, \ldots, Z_n\}$ are eliminated. We will illustrate this with a small example in the case where $Z$ is a single variable. First we eliminate the linear influence of $Z$ on $X$ and $Y$. Therefore we write $X$ and $Y$ as linear functions of $Z$:

- $\hat{X} = a + bZ$
- $\hat{Y} = c + dZ$

The resulting residuals are then independent of $Z$:

- $X^* = X - \hat{X}$
- $Y^* = Y - \hat{Y}$

The partial correlation coefficient is then defined as:

$$\rho_{XY,Z} = \rho_{X^*Y^*} = \frac{Cov(X^*, Y^*)}{\sqrt{Var(X^*)} \times \sqrt{Var(Y^*)}}$$

So technically the partial correlations of two variables $X$ and $Y$ are the correlations between the residuals $X^*$ and $Y^*$ resulting from the linear regression from $X$ and $Y$ with respect to other variables $Z=\{Z_1, Z_2, \ldots, Z_n\}$. Graphically interpreted the partial variance or covariance given $Z$ can be considered as the variance or covariance between residuals of the projections of $X$ and $Y$ on the linear space spanned by $Z$ (see \( ? \)).

However, it can be very expensive to solve linear regression problems. One approach is therefore to use matrix inversion:

Let $V=\{X_1, X_2, \ldots, X_n\}$ be a set of variables and $\Omega = (\omega_{ij})$ its correlation matrix, where
ω_{ij} = \rho_{X_i X_j}, and let N be the normalized inverse matrix of \Omega (see Definition 2.7 and Example 5 below), then

\[ M = (\rho_{X_i X_j} \setminus \{X_i, X_j\}) \]

is the matrix of partial correlation values. For further information see ? and ?.

**Definition 2.7 (Normalized Matrix)**

There are many different definitions for normalized matrices. However, since we want to use normalized inverse matrices as conditioned correlation matrices later, they have to fulfill 2 conditions:

(i) The diagonal entries have to be all 1

(ii) The matrix has to be symmetric

We therefore define the normalized matrix as follows: Let A be a m×n matrix with entries (i,j),i\in[1,m],j\in[1,n]. Then the m×n matrix A^* with entries

\[(i,j)^* = (i,j)/\sqrt{(i,i)^*(j,j)}\]

is called the **normalized matrix**.

**Example 5 (Calculation of the normalized inverse matrix)**

To illustrate Definition 2.7 we go through the calculation of the normalized inverse matrix step by step. As initial matrix we use the matrix A.

\[
A = \begin{bmatrix}
1 & 0.8951 & 0.5392 & 0.7059 & 0.4281 \\
0.8951 & 1 & 0.4985 & 0.6809 & 0.4435 \\
0.5392 & 0.4985 & 1 & 0.8653 & 0.2796 \\
0.7059 & 0.6809 & 0.8653 & 1 & 0.4439 \\
0.4281 & 0.4435 & 0.2796 & 0.4439 & 1 \\
\end{bmatrix}
\]

**Step 1:**

As the first step we have to invert the matrix and obtain:

\[
A_1 = \begin{bmatrix}
5.5123 & -4.2455 & 0.0229 & -1.0032 & -0.0382 \\
-4.2455 & 5.3127 & 0.6039 & -1.0327 & -0.2489 \\
0.0229 & 0.6039 & 4.3962 & -4.4368 & 0.4629 \\
-1.0032 & -1.0327 & -4.4368 & 6.6080 & -0.8056 \\
-0.0382 & -0.2489 & 0.4629 & -0.8056 & 1.3549 \\
\end{bmatrix}
\]

We will not go into the algorithm of the inversion of a matrix here, since that is a completely other topic. If interested detailed informations can be found in ? and ?.

However, it is obvious that the matrix A_1 cannot be used as a correlation matrix yet.
Although the matrix $A_1$ is already symmetric, the diagonal entries are still unequal 1. Therefore we have to normalize the matrix.

**Step 2:**
In the second step (and first step of the normalization) we compute the square roots of the diagonal entries of $A_1$ and get:

$$A_2 = \begin{bmatrix} 2.3478 & 2.3049 \\ & 2.0967 \\ & & 2.5706 \\ & & & 1.1640 \end{bmatrix}$$

**Step 3:**
In the final step we now apply the normalization formula:

$$(i,j)^* = (i,j) / \sqrt{(i,i)(j,j)}$$

This means we divide each element $(i,j)$ of the matrix $A_1$ by the diagonal elements in row (=column) $i$ and $j$ of the matrix $A_2$. The entry $(2,3)$ of the matrix $A_1$ (=0.6039) for example gets divided by $2.3049$ (the 2nd diagonal entry of $A_2$) and $2.0967$ (the 3rd diagonal entry of $A_2$):

$$(2,3)^* = (2,3) / \sqrt{(2,2)(3,3)} = 0.6039 / (2.3049 * 2.0967) = 0.1250$$

We obtain the full normalized inverse matrix $A^*$:

$$A^* = \begin{bmatrix} 1 & -0.7845 & 0.0047 & -0.1662 & -0.0140 \\ -0.7845 & 1 & 0.1250 & -0.1743 & -0.0928 \\ 0.0047 & 0.1250 & 1 & -0.8232 & 0.1897 \\ -0.1662 & -0.1743 & -0.8232 & 1 & -0.2692 \\ -0.0140 & -0.0928 & 0.1897 & -0.2692 & 1 \end{bmatrix}$$

The matrix $A^*$ is symmetric and the diagonal entries are all equal 1 as desired. Therefore both required conditions are fulfilled and the matrix can be used for our purposes.

**2.3 Tail Dependence**

We have already introduced Kendall’s $\tau$ and partial correlations as dependence measures. These take the whole space $[0, 1]^2$ into account. In some cases, however, we are only interested in the tail dependencies or graphically spoken in the dependencies in the upper right and the lower left quadrants of $[0, 1]^2$. We are interested in the extreme
values and try to examine, if large (small) values of one random variable cause large (small) values of the other. So we define the lower and upper tail dependence as in ?.

**Definition 2.8 (Lower and Upper Tail Dependence (?))**

Let \( X \) and \( Y \) be continuous random variables with distribution functions \( F \) and \( G \), respectively. The upper tail dependence parameter \( \lambda_U \) is the limit (if it exists) of the conditional probability that \( Y \) is greater than the 100\(t\)-th percentile of \( G \) given that \( X \) is greater than the 100\(t\)-th percentile of \( F \) as \( t \) approaches 1, i.e.

\[
\lambda_U = \lim_{t \to 1^-} P \left[ Y > G^{(-1)}(t) \mid X > F^{(-1)}(t) \right].
\]

Similarly, the lower tail dependence parameter \( \lambda_L \) is the limit (if it exists) of the conditional probability that \( Y \) is less than or equal to the 100\(t\)-th percentile of \( G \) given that \( X \) is less or equal to the 100\(t\)-th percentile of \( F \) as \( t \) approaches 0, i.e.

\[
\lambda_L = \lim_{t \to 0^+} P \left[ Y \leq G^{(-1)}(t) \mid X \leq F^{(-1)}(t) \right].
\]

The calculation of these values is, however, often very difficult and only with a great effort feasible. Therefore other measures of tail dependence were developed in the course of time and we will introduce two of them in the following.

**2.3.1 Tail Cumulations**

We start with the tail cumulations, a very vivid graphical way to determine the tail dependencies.

**Definition 2.9 (Tail cumulation, s.? )**

Let \( \alpha \in [0,1] \), \( c_l \) and \( c_u \) such that \( P(U_1 \leq c_l, U_2 \leq c_l) = \alpha \) and \( P(U_1 > c_u, U_2 > c_u) = \alpha \) for two independent and uniformly on \([0,1]\) distributed random variables \( U_1 \) and \( U_2 \), then the lower and upper tail cumulation is defined as:

\[
\tilde{\gamma}_{\text{lower}} = \frac{\# \{\text{observations in } [0,c_l(\alpha)]^2\} - \alpha}{n} = \frac{1}{n} \sum_{i=1}^{n} 1_{U_{1i} \leq c_l(\alpha), U_{2i} \leq c_l(\alpha)} - \alpha
\]

\[
\tilde{\gamma}_{\text{upper}} = \frac{\# \{\text{observations in } [c_u(\alpha), 1]^2\} - \alpha}{n} = \frac{1}{n} \sum_{i=1}^{n} 1_{U_{1i} > c_u(\alpha), U_{2i} > c_u(\alpha)} - \alpha
\]

where \( n \) is the number of observations.

(Note: \( \alpha = P(U_1 \leq c_l, U_2 \leq c_l) = P(U_1 \leq c_l)P(U_2 \leq c_l) = c_l^2 \iff c_l = \sqrt{\alpha} \) and \( c_u = 1 - \sqrt{\alpha} \))
Graphically seen, we only compare the plot of our observations with those of two independent and uniformly on $[0,1]$ distributed random variables $U_1$ and $U_2$. If we have more observations in the bottom left corner than $U_1$ and $U_2$, this is a sign for lower tail dependence. If we have more observations in the top right corner of our plot than $U_1$ and $U_2$, this is a sign of upper tail dependence. For later applications we will set $\alpha = 0.1$. Furthermore, it is often useful to have one single value as a tail dependence measure instead of one value for the lower and one value for the upper tail dependence. We will therefore use the maximum of both values. For further information see (? , pp.42-25).

### 2.3.2 Tail Exceedance Dependence

Another approach to model the tail dependencies are the tail exceedance dependencies. These were were introduced in ? and further developed in ?.

? defined the exceedance correlation as follows:

$$corr(X_1, X_2 \mid X_1 \leq \delta_1, X_2 \leq \delta_2).$$

In this definition the classical Pearson’s correlation product-moment correlation coefficient is used. But this is just a measure of linear dependence and is not independent of the margins of $X_1$ and $X_2$. ? therefore used Kendall’s $\tau$ as the correlation measure instead which is invariant under non-linear strictly increasing transformations and independent of the margins of $X_1$ and $X_2$. The lower and upper exceedance Kendall’s $\tau$ are defined as follows:

**Definition 2.10 (Lower and upper exceedance Kendall’s $\tau$)**

$$\tau^{\text{lower}}(X_1, X_2 \mid X_1 \leq \delta_1, X_2 \leq \delta_2),$$

$$\tau^{\text{upper}}(X_1, X_2 \mid X_1 > \delta_1, X_2 > \delta_2).$$

### 2.4 Basic sequential selection algorithms for R-vines

? already introduced an algorithm, which constructs the vine-structure in a bottom-up fashion from the unconditioned $T_1$ to the maximum conditioned $T_n$ with regard to the proximity condition. However in Kurowicka & Joe (2011) another approach was introduced: An algorithm which constructs the vine-structure in a top-down fashion from $T_n$(maximum conditioned) to $T_1$(unconditioned). Therefore two conditions were implemented which shall ensure that the proximity condition is satisfied in this direction, too.

**Condition 1:**
Suppose $B_x = B_y$ and $x \neq y$, then there exists an edge $f$ of $T_{j-1}$ such that $C_f = \{x, y\}$
and \( D_f = A_x \setminus \{y\} \).

(Remember: \( B_x = \{x\} \cup A_x \) where \( x \in C_e \) and \( A_x = D_e \) for some edge \( e \in T_j \))

**Example 6 (Illustration of Condition 1)**

Consider the two edges \( a = \{12 \mid 35\} \) and \( b = \{43 \mid 25\} \) in \( T_i \) which provide us the sets \( B_1 = \{1,3,5\} \), \( B_2 = \{2,3,5\} \), \( B_3 = \{3,2,5\} \) and \( B_4 = \{4,2,5\} \). Since \( B_2 \) is just a rearrangement of \( B_3 \) Condition 1 is satisfied, what provides us the edge \( c = \{23 \mid 5\} \) in \( T_{i-1} \).

**Condition 2:**

For all \( B_{i_1}, ..., B_{i_n} \) such that \(| B_{i_p} \triangle B_{i_q} | = 2\), \( B_{i_p} = \{i_p, s \mid A_{i_p} \setminus \{s\}\} \) or \( B_{i_p} = \{i_p, t \mid A_{i_p} \setminus \{t\}\}, s,t \in A_{i_p}, \)

where \( U \triangle V \) denotes the symmetric difference of sets \( U \) and \( V \) \( (U \triangle V = (U \cup V) \setminus (U \cap V)) \).

**Example 7 (Illustration of Condition 2)**

Consider the two edges \( a = \{12 \mid 345\} \) and \( b = \{26 \mid 345\} \) in \( T_i \) which provide us the sets \( B_1 = \{1,3,4,5\} \), \( B_2 = \{2,3,4,5\} \) and \( B_6 = \{6,3,4,5\} \). Since \( A_x = \{3,4,5\} \) for all \( x \in \{1,2,6\} \), \(| B_{i_p} \triangle B_{i_q} | = 2 \) for all \( x,y \in \{1,2,6\} \). Therefore Condition 2 states that there may be up to a maximum of 2 different partners for 1, 2 and 6, but not more. The sets of possible partners are \( \{3\}, \{4\}, \{5\}, \{3,4\}, \{3,5\} \) or \( \{4,5\} \). What happens if more than 2 different partners are chosen for variables which are affected by Condition 2 is shown in detail in Example 9.

We are now ready to introduce the algorithm published in \[ \] which determines the vine-structure in a top-down fashion:

**Algorithm 2.1 Vine construction starting at the highest tree**

1: Choose two variables, say \( x,y \in \{1,\ldots,n\} = I \), to be in the conditioned set of the top edge; constraint sets of its m-children are \( B_x = \{x\} \cup \{I \setminus \{x,y\}\} \) and \( B_y = \{y\} \cup \{I \setminus \{x,y\}\} \); choose partners of \( x \) and \( y \) in \( T_{n-2} \) from the set \( I \setminus \{x,y\} \). Then there are two edges in \( T_{n-2} \):

\[
E_{n-2} = \{\{x,pt(x) \mid I \setminus \{x,y,pt(x)\}\}, \{y,pt(y) \mid I \setminus \{x,y,pt(y)\}\}\}.
\]

For all \( j = n-2,...,1 \)
2: Set \( B_x = \{x\} \cup A_x \) such that \( x \in C_e \) and \( A_x = D_e \) for each edge \( e \) of tree \( T_j \);
3: Remove all sets for which \( x_i = x_k \) for \( i \neq k \);
4: Apply Condition 1;
5: Choose partners of \( x \) such that Condition 2 is satisfied;

Edge set of \( T_{j-1} \) is the set containing elements of the form \( x,pt(x) \mid A_x \setminus \{pt(x)\} \).
2 Mathematical Necessities

Example 8
We will now show how the algorithm works with an 7 dimensional example. The variables are 1,2,3,4,5,6,7

Tree T6 and T7:
Step 1:
We choose $f = \{4, 7 \mid 1, 2, 3, 5, 6\}$ as the top node to get the 1-elemental tree 7 and get the two sets $B_4 = \{4\} \cup A_4 = \{4, 1, 2, 3, 5, 6\}$ and $B_7 = \{7\} \cup A_7 = \{7, 1, 2, 3, 5, 6\}$. Since $4 \notin B_7$ and $7 \notin B_4$ condition 1 is not applicable. As $A_4 = A_7 = \{1, 2, 3, 5, 6\}$ and $x=4 \neq 7=y$, the symmetric difference of $B_4$ and $B_7$ is 2 and therefore the prerequisite of condition 2 is satisfied. But as $x$ and $y$ are only 2 variables, obviously no more than two partners can be chosen and condition 2 has no effect. We obtain tree 6 by choosing 6 as partner of 4 (notation $p(4)=6$) and $p(7)=5$. Remember: The nodes of $T_i$ are the edges of $T_{i−1}$. In our case: $N_6 = \{\{4, p(4) \mid A_4 \setminus p(4)\}, \{7, p(7) \mid A_7 \setminus p(7)\}\}$

\[
\begin{bmatrix}
47 & 12356 \\
46 & 12356 \\
57 & 1236 \\
\end{bmatrix}
\]

Tree T5:
Step 2 & 3:
We get the sets $B_4 = \{4\} \cup A_4 = \{4, 1, 2, 3, 5\}$, $B_6 = \{6\} \cup A_6 = \{6, 1, 2, 3, 5\}$, $B_5 = \{5\} \cup A_5 = \{5, 1, 2, 3, 6\}$ and $B_7 = \{7\} \cup A_7 = \{7, 1, 2, 3, 6\}$. We summarize these sets in the matrix

\[
\begin{bmatrix}
4 & 6 & 5 & 7 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
5 & 5 & 6 & 6 \\
\end{bmatrix}
\]

Since all $x \neq y$ Step 3 does not change the matrix $B$.

Step 4:
$B_5 = B_6$ but $5 \neq 6$ therefore Condition 1 provides us node $f = \{5, 6 \mid 1, 2, 3\} \in T_5$ respectively edge $f = \{5, 6 \mid 1, 2, 3\} \in T_4$. This gives $p(5)=6$ and $p(6)=5$.

Step 5:
We still have to find partners for 4 and 7 but since we only have 2 variables left like in Step 1, there cannot be more than 2 partners $\rightarrow$ Condition 2 is not applicable $\rightarrow p(4)$
and \( p(7) \) are freely selectable from their particular conditioning sets 
\( A_4 = \{1, 2, 3, 5\} \) and \( A_7 = \{1, 2, 3, 6\} \).
We choose \( p(4) = 5 \) and \( p(7) = 1 \).

\[
\begin{array}{cccccc}
45 & | & 123 & & 46 & | & 1235 \\
56 & | & 123 & & 57 & | & 1236 \\
71 & | & 236 & & & &
\end{array}
\]

**Tree 4:**

**Step 2 & 3:**

We obtain the new matrix of \( B_x \) sets arising from nodes 
\{45 | 123\}, \{56 | 123\} and \{71 | 236\} in \( T_5 \).

\[
B = \begin{pmatrix}
B_4 & B_5 & B_6 & B_7 & B_1 \\
4 & 5 & 5 & 6 & 7 & 1 \\
1 & 1 & 1 & 2 & 2 & 1 \\
2 & 2 & 2 & 3 & 3 & 2 \\
3 & 3 & 3 & 6 & 6 & 3
\end{pmatrix}
\]

\[\text{Step 3} \rightarrow \begin{pmatrix} 4 & 5 & 6 & 7 & 1 \\
1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 6 & 6 \end{pmatrix} \]

(1)

Since the second and third column of \( B \) equal each other the third column is deleted in 
Step 3.

**Step 4:**

Similarly to \( T_6 \) \( B_6 = B_1 \) but \( 6 \neq 1 \) therefore Condition 1 provides us edge 
\( f = \{61 | 23\} \in T_4 \implies p(6) = 1 \) and \( p(1) = 6 \).

**Step 5:**

We now need to choose partners for 4, 5 and 7. Since \( 4 \neq 5 \neq 6 \) but 
\( A_4 = A_5 = A_6 = \{1, 2, 3\} \), the symmetric differences \( | B_i \triangle B_j | \) are all 2 
for \( i, j \in \{4, 5, 6\}, i \neq j \) (e.g. \( B_4 \triangle B_5 = \{4, 1, 2, 3\} \)) and 
\( B_5 = \{5, 1, 2, 3\} \implies | B_4 \triangle B_5 | = \{4, 5\} = 2 \).
By Condition 2 we know that there cannot 
be more than 2 different partners for 4, 5 and 6. We already know \( p(6) = 1 \) by Step 4.

Thus the possible sets of partners for 4 and 5 (out of the set of available partners 
\{1, 2, 3\}) are \{1, 1\}, \{2, 1\}, \{1, 2\}, \{1, 3\} and \{3, 1\}.

We choose \{1, 1\}. \( p(7) = 2 \) is again freely chosen.

\[
\begin{array}{cccccc}
41 & | & 23 & & 45 & | & 123 \\
51 & | & 23 & & 56 & | & 123 \\
61 & | & 23 & & 71 & | & 236 \\
72 & | & 36 & & & &
\end{array}
\]

**Tree 3:**

**Step 2 & 3:**

We obtain the new matrix of \( B_x \) sets arising from the nodes

\[\text{Step 3} \rightarrow \begin{pmatrix} 5 & 1 & 6 & 7 & 1 \\
1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 6 & 6 \end{pmatrix} \]
2 Mathematical Necessities

\{41 \mid 23\}, \{51 \mid 23\}, \{61 \mid 23\} and \{72 \mid 36\} in \(T_4\):

\[
B = x \begin{pmatrix}
B_4 & B_1 & B_5 & B_1 & B_6 & B_1 & B_7 & B_2 \\
4 & 1 & 5 & 1 & 6 & 1 & 2 & 7 \\
2 & 2 & 3 & 2 & 3 & 3 & 6 & 6 \\
3 & 3 & 3 & 3 & 3 & 6 & 6 & 3
\end{pmatrix}
\xrightarrow{\text{Step 3}}
\begin{pmatrix}
4 & 1 & 5 & 6 & 2 & 7 \\
2 & 2 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 3 & 6 & 6
\end{pmatrix}
\]

Since the 2nd, 4th and 6th column equal each other the 4th and 6th column are deleted in Step 3.

**Step 4:**
\(B_6 = B_2\) but \(6 \neq 2\) therefore Condition 1 provides us edge \(f = \{6, 2 \mid 3\} \in T_3 \Rightarrow p(6) = 2\) and \(p(2) = 6\).

**Step 5:**
We need to choose partners for 1, 4, 5 and 7. As \(4 \neq 1 \neq 5 \neq 6\) but \(A_4 = A_1 = A_5 = A_6 = \{2, 3\}\) we know that the symmetric differences \(|B_i \triangle B_j|\) are all 2 for \(i, j \in \{1, 4, 5, 6\}, i \neq j\). But since \(A_4(= A_1 = A_5 = A_6) = \{2, 3\}\), which is the set of available partners for 1, 4 and 5, is only a 2-element set there obviously cannot be more than 2 different partners for 1, 4 and 5. The partners are again freely selectable. For the same reason we can choose \(p(7)\) free.

We choose \(p(1) = 3, p(4) = 2, p(5) = 3, p(6) = 2\) and \(p(7) = 3\).

\[
\[
\begin{array}{c}
24 \mid 3 \\
41 \mid 23 \\
13 \mid 2 \\
16 \mid 23 \\
26 \mid 3 \\
72 \mid 36 \\
37 \mid 6
\end{array}
\]

**Tree 2:**

**Step 2 & 3:**
The new matrix arising from the nodes \{24 \mid 3\}, \{13 \mid 2\}, \{26 \mid 3\}, \{37 \mid 6\} and \{35 \mid 2\} in \(T_4\) is:

\[
B = x \begin{pmatrix}
B_4 & B_1 & B_5 & B_1 & B_6 & B_1 & B_7 & B_2 & B_7 & B_2 \\
1 & 3 & 2 & 4 & 2 & 6 & 3 & 5 & 3 & 7 \\
2 & 2 & 3 & 3 & 3 & 2 & 6 & 6 & 3
\end{pmatrix}
\xrightarrow{\text{Step 3}}
\begin{pmatrix}
1 & 3 & 2 & 4 & 6 & 5 & 3 & 7 \\
2 & 2 & 3 & 3 & 3 & 2 & 6 & 6
\end{pmatrix}
\]
Step 4:
We get by Condition 1: \( \{ f_1 = \{36\}, f_2 = \{23\} \} \in T_2 \)

Step 5:
The partners for the remaining variables 1, 4, 5 and 7 are obviously \( p(1)=2, p(4)=3, p(5)=2 \) and \( p(7)=6 \).

Example 9 (Illustration of Condition 2)
To make clear what causes Condition 2, we will go through the above example again without consideration of Condition 2. We start at \( T_5 \) where Condition 2 comes into effect the first time.
Tree 4:  
**Step 2 & 3 & 4:** We get the matrix $B$ by Step 3 & 4 and $f = \{61 \mid 23\} \in T_4$ by Step 4 (s.1)).

$$B = \begin{pmatrix} 4 & 5 & 6 & 7 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 6 & 6 \end{pmatrix}$$

**Step 5:**
$4 \neq 5 \neq 6$ but $A_4 = A_5 = A_6$. By Condition 2 we know that there cannot be more than 2 different partners for 4, 5 and 6. We already know $p(6)=1$ by Step 4. Without consideration of Condition 2 we now choose: $p(4)=3$, $p(5)=2$ and $p(7)=2$ and obtain a alternative $T_4$:

$$T_4 = \begin{array}{ccccccccc} 43 & 12 & 45 & 123 & 52 & 13 & 56 & 123 & 61 & 23 & 71 & 236 & 76 & 23 \end{array}$$

**Step 2 & 3:**
The new matrix is now:

$$\begin{pmatrix} 4 & 3 & 5 & 2 & 1 & 6 & 7 & 6 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \end{pmatrix} \xrightarrow{\text{Step 3}} \begin{pmatrix} 4 & 3 & 5 & 2 & 1 & 6 & 7 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

**Step 4:**
We apply Condition 1 and get: $3 \neq 2 \neq 1$ but $B_3 = B_2 = B_1$. Therefore we have $\{f_1 = \{12 \mid 3\}, f_2 = \{23 \mid 1\}, f_3 = \{13 \mid 2\}\} \in T_3$. But we still have to choose partners for 4, 5, 6 and 7. So we end up with 7 nodes for $T_3$ what is not possible $\implies$ no tree construction possible!

We see: If we ignore Condition 2 in $T_j$ there is no tree construction possible for $T_j-1$.

### 2.5 The Vuong Test

To be able to compare two models with each other and determine which of them fits a data set best, we need a test statistic. For our purposes we will use the Vuong test which was introduced in ?.

**Definition 2.11 (The Vuong Test)**
Let \( m := (m_1, ..., m_n)^t \) with \( m_i := \log f_1(Y_i | x_{i,1}, \hat{\beta}_1) \frac{f_2(Y_i | x_{i,2}, \hat{\beta}_2)}{f_2(Y_i | x_{i,1}, \hat{\beta}_1)} \) and \( E_0(m) := (\mu_1^m, ..., \mu_n^m)^t \) its expected value under the null-hypothesis, whereas \( f_1(Y_i | x_{i,1}, \hat{\beta}_1) \) and \( f_2(Y_i | x_{i,2}, \hat{\beta}_2) \) are the likelihoods of the models, we want to compare. The null hypothesis is:
\[
H_0 : \mu_0^m = 0 \text{ versus } H_1 : \mu_0^m \neq 0
\]

Further let
\[
\nu := \frac{\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n m_i \right)}{\sqrt{\frac{1}{n} \sum_{i=1}^n (m_i - \bar{m})^2}}
\]

where \( \bar{m} = \frac{1}{n} \sum_{i=1}^n m_i \).

Vuong showed that under the null-hypothesis \( \nu \) converges in distribution to a standard normal distribution.
\[
\nu \xrightarrow{D} N(0,1)
\]

Hence we can reject the null-hypothesis at the significance level of \( \alpha \% \) if \( |\nu| \geq \Phi^{-1}(1 - \alpha/2) \), where \( \Phi^{-1} \) denotes the quantile function of the standard normal distribution. If \( \nu > -\Phi^{-1}(1 - \alpha/2) \) we prefer model 1 to model 2, if the other way round \( \nu < -\Phi^{-1}(1 - \alpha/2) \) we choose model 2. Unless otherwise explicitly stated we will work with \( \alpha = 5\% \) in the following.

So far we considered the basic form of the Vuong test. However, the number of parameters in the models are not taken into account yet. Vuong suggested to use either the

Schwarz’s Bayesian information criteria : \( \frac{p}{2n} \log n - \frac{q}{2n} \log n \)

or

Akaike’s information criteria : \( p - q \)

The resulting loglikelihood ratios are:

\[
\sum_{i=1}^n m_i - p - q \quad (\text{Akaike})
\]

\[
\sum_{i=1}^n m_i - \frac{p}{2} \ln(n) - \frac{q}{2} \ln(n) \quad (\text{Schwarz})
\]

where \( p \) denotes the number of parameters in model 1, \( q \) the number of parameters in model 2 and \( n \) the number of observations. For further information see ? or ?.
3 Five strategies for sequential R-vine model selection and estimation

In this chapter we will now introduce 5 strategies to estimate the tree-structure, the pair-copula families and their corresponding parameters of an R-vine specification for a given data set. Here we proceed in sequential way. As we can see in \( \text{?} \) different tasks are required:

(i) Selection of the R-Vine (structure), i.e. selecting which conditioned and unconditioned pairs to use for the PCC.

(ii) Choice of a pair-copula family for each pair selected in (a).

(iii) Estimation of the corresponding parameters for each copula.

From these 3 steps, we can now derive 3 distinctive features of our strategies:

(i) Is the vine constructed in a top-down fashion, i.e. from \( T_n \) (maximum conditioned) to \( T_1 \) (unconditioned), or bottom-up (\( T_1 \) to \( T_n \))?  

(ii) Is the tree structure determined first and then the copula families or vice versa?  

(iii) What characteristics are used to determine the tree structure: The empirical partial correlations or a goodness-of-fit indicator of the fitted copulae?

We start with the most commonly used method so far:

3.1 Strategy 1: R-vine model selection and estimation based on empirical Kendall’s \( \tau \)

In this strategy we start at \( T_1 \) and sequentially make our way to \( T_n \) (bottom-up). For this purpose we calculate the corresponding Kendall’s \( \tau \) value for each pair, which satisfies the proximity condition, and choose the tree, which maximizes the sum of absolute Kendall’s \( \tau \)’s. After the choice of the tree-structure we have to find a suitable copula family for every edge. Thus we fit every admitted copula type for every edge of our tree by determination of their parameter(s) and calculate the corresponding AIC-values. We choose the copula type with the lowest AIC-value for every edge. Then
we transform the copula observations using the previously determined copula family choices and parameters to obtain the pseudo-observations for the next tree. The aim of this strategy is to model the dependencies of the highly correlated variables in the first trees. In this approach, the transformed variables tend to be independent in the higher trees what could be further used to truncate the vine for example. For more detailed explanations see (??, pp.49-51). For a better overview, the single steps are summarized in the following algorithm:

**Algorithm 3.1** R-vine model selection and estimation based on empirical Kendall’s $\tau$

1: Calculate the empirical Kendall’s $\tau$ for all possible variable pairs from the copula data.
2: Select the tree that maximizes the sum of absolute Kendall’s $\tau$’s.
3: For all these pairs select a copula by choosing the one with the lowest AIC value and estimate the corresponding parameter(s).
4: Transform the copula observations using the copula family choices and parameters from Step 3 to obtain the pseudo observations for the next tree.
5: Use these pseudo observations to calculate empirical Kendall’s $\tau$ for all pairs which satisfy the proximity condition.
6: Proceed with Step 2 selecting a tree among all pairs which are allowed by Step 5 until the R-Vine is fully specified.

Now we will have a closer look at the running time of this algorithm for the first tree:

We consider a multivariate data set of n different variables on $[0,1]^n$ and want to fit an appropriate R-vine copula specification. We admit copulae of the types 1,...,k.

| Tree-structure selection by Kendall’s $\tau$ (Step 1 & Step 2) | Determine the empirical Kendall’s $\tau$ values for all different pairs $(i,j)$ $i,j \in \{1,...,n\}$, which satisfy the proximity-condition. $\Rightarrow$Apply a maximum-spanning-tree algorithm such as the algorithm of Prim (see ??) Cost: $\frac{n* (n-1)}{2} * (\text{Kendall’s } \tau - \text{estimation}) + (\text{algorithm of Prim})$
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Copula type by AIC (fast) (Step 3 - Part 1)</td>
<td>For each edge and each copula type, estimate the corresponding parameters by maximum likelihood estimation. $\Rightarrow$Choose copula type with the smallest AIC Cost: $(n-1) * k * (\text{Fitting of a copula parameter for a fixed copula family}) + (n - 1) * k * (\text{AIC} - \text{estimation})$</td>
</tr>
</tbody>
</table>
3.5 Strategies

Determine the p-value (Cramer-von Mises statistic as described in Section 2.2.3) for every edge of our tree.
If the p-value < a predetermined value, use a non-parametric copula.
If the p-value ≥ the predetermined value, use copula type and parameter(s) as chosen in Step 3 - Part 1.
Cost: \((n - 1) \times (p - value - estimation) + at most\)
\((n - 1) \times (Fitting of a non-parametric copula)\)

The obvious advantage of this strategy is that the expensive p-value calculation is performed only once per edge. Thus the overall algorithm is very fast.

3.2 Strategy 2: R-vine model selection and estimation based on p-values

In this strategy we also start at the unconditioned \(T_1\) but the tree-structure selection is not based on a correlation measure like Kendall’s \(\tau\) (strategy 1) but on a goodness-of-fit indicator: the p-values. To obtain these values we first have to fit every admitted copula type for every possible pair of variables, which satisfy the proximity condition, by determination of their parameter(s) and calculate the corresponding AIC-values. We choose the copula type with the lowest AIC-value for every edge. Then we calculate the p-value for every edge and the just determined copula type and choose the tree which maximizes the sum of p-values. Like in strategy 1 we then have to transform the copula observations using the previously determined copula family choices and parameters to obtain the pseudo-observations for the next tree. Aim of this strategy is to get best fits in the most important lower trees. Compared with the first strategy the ordering of the steps was changed, what also becomes apparent in the algorithm:

**Algorithm 3.2 R-vine model selection and estimation based on p-values**

1: Fit the corresponding parameter(s) for all possible pairs and for all admitted copula types.
2: Select the copula type with the smallest AICs.
3: Select the tree that maximizes the sum of p-values.
4: Transform the observations using the copulae and parameters from Step 1 & 2 to obtain the transformed pseudo observations.
5: Use these transformed pseudo observations to fit the corresponding parameter(s) for all pairs, which satisfy the proximity condition, and for all allowed copula types.
6: Proceed with Step 2 using all pairs allowed by Step 4 until the R-vine is fully specified.

As for the first strategy we will again take a look at the running time of this algorithm for the first tree:
3.5 Strategies

The scenario of n variables and k admitted copula types remains valid.

<table>
<thead>
<tr>
<th>Copula type by AIC</th>
<th>Calculate the copula-parameters for every edge ((i, j)), (i, j \in {1, ..., n}), and for every copula type 1,...,k and determine the corresponding AIC.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(fast) (Step 1 &amp; 2)</td>
<td>(\Rightarrow)Choose copula type with the smallest AIC  (\text{Cost:} \frac{n \times (n-1)}{2} \times k \times (AIC - \text{estimation}))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Tree structure selection by p-value</th>
<th>Determine the p-value for every pair ((i, j)), (i, j \in {1, ..., n}), with the copula type specified in Step 2.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(expensive) (Step 3)</td>
<td>(\Rightarrow)Apply a maximum-spanning-tree algorithm (Algorithm of Prim)  (\text{Cost:} \frac{n \times (n-1)}{2} \times (p - \text{value} - \text{estimation}) + (\text{Algorithm of Prim}))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GoF by p-value (expensive)</th>
<th>Check all edges for a lack of fit:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>If the p-value &lt; a predetermined value, use a non-parametric copula.</td>
</tr>
<tr>
<td></td>
<td>If the p-value (\geq) the predetermined value, use copula type and parameter(s) as chosen in Step 1 &amp; 2.</td>
</tr>
<tr>
<td></td>
<td>(\text{Cost:} \text{At most} (n - 1) \times (\text{Fitting of a non} - \text{parametric copula}))</td>
</tr>
</tbody>
</table>

The disadvantage of this strategy compared with the first one is that the expensive p-value analysis needs to be performed \(\frac{n \times (n-1)}{2}\)-times now, compared to the \((n - 1)\)-times in Strategy 1. We will have to see whether the costs are justified by better results.

3.3 Strategy 3: R-vine model selection and estimation based on partial correlations

This is the only strategy in our selection which starts at the maximum conditioned \(T_n\) and ends with \(T_1\) (top-down). This approach was developed by ?. Its aim is to produce lowest possible correlations in the higher copulas to assume independence later. Therefore the algorithm works with the partial correlations matrix, which are equal to the conditioned correlations for elliptical distributions. So even though zero partial correlations do not have to indicate conditional independence in general, we choose partners in this algorithm such that the absolute values of partial correlations are the smallest ((?, pp.242)). Another important difference to the other approaches is that the strategy can be divided in two sequential parts: In the first part we obtain the whole vine structure by application of the sequential Algorithm 2.1 (see pp.16), where the partial correlations are used as weights, i.e. in each step the edges with the lowest
3.5 Strategies

Partial correlations are chosen. A detailed description of this method is offered in Example 10. In the second part we sequentially determine the copula families and parameters on the basis of the AIC-values like in the Strategies 1 and 2. This part is performed in a bottom-up fashion again. Like for the previous strategies, we summarize the single steps in the corresponding algorithm:

**Algorithm 3.3** R-vine model selection and estimation based on partial correlations

1. Use Algorithm 2.1, as described on page 16, under the additional condition that always pairs with the lowest partial correlation are elected to obtain the R-vine-structure.
2. Select a copula and fit the corresponding parameter for all edges in the first tree.
3. Transform the observations using the copulae and parameter from Step 2 to obtain the pseudo observations.
4. Use these pseudo observations to select the copula type and fit the corresponding parameter for all edges in the next tree.
5. Proceed with Step 3 until the R-vine is fully specified.

Before we look at the running time, we will explain how this modified Algorithm 3.3 works.

**Example 10 (Illustration of Algorithm 3.3)**

For the illustration of Algorithm 3.3 we will use a data set of daily returns of financial indices. The original data set, presented in ?, consists of 16 different indices: 5 equity, 9 fixed income and 2 commodity indices. Since we just want to use it for illustration purposes here, we will restrict ourselves to the 5 equity indices, which are displayed in Table 3.1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Index Name</th>
<th>Currency</th>
<th>Token</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>DAX30 PERFORMANCE</td>
<td>EUR</td>
<td>Dax</td>
</tr>
<tr>
<td>2</td>
<td>DJ STOXX 50</td>
<td>EUR</td>
<td>STOXX50</td>
</tr>
<tr>
<td>3</td>
<td>S&amp;P 500 COMPOSITE</td>
<td>USD</td>
<td>S&amp;P500</td>
</tr>
<tr>
<td>4</td>
<td>MSCI WORLD USD</td>
<td>USD</td>
<td>MSCI-WORLD</td>
</tr>
<tr>
<td>5</td>
<td>MSCI EM EASTERN EUROPE USD</td>
<td>USD</td>
<td>MSCI-EE</td>
</tr>
</tbody>
</table>

Table 3.1: Overview to the 5 equities of the example

For each equity index, the data were collected from 01/01/2001 to 12/15/2009 what provides us a total of 2,337 data points for each index. The corresponding graphs can be seen in Figure 3.1. For further information to the data set, see ?.

**Tree 5:**
We start with the 5-dimensional empirical correlation matrix:
Figure 3.1: Equity indices from 01/01/2001 to 12/15/2009
The normalized inverse matrix $M^*$ (see Definition 2.7) of $M$, which provides the fully conditioned correlations is given below. The complete calculation of this step can also be found in Example 5 on page 13. In this example the calculation of the normalized inverse was shown on the basis of this matrix $M$.

Fully conditioned means that the union of conditioning set and conditioned set is the entire variable set. To explain and illustrate: The matrix entry $-0.7845$ at the position $(1,2)$ of the matrix $M^*$ is the correlation of variable 1 (DAX) and variable 2 (STOXX50), conditioned on all other variables of our example (S.P500, MSCI.WORLD, MSCI.EE).

$$M^* = \begin{bmatrix}
0.0047 & -0.1662 & -0.0140 \\
-0.1662 & 0.1250 & -0.8232 & 0.1897 \\
-0.0140 & -0.1743 & -0.8232 & 1 & -0.2692 \\
0.0047 & -0.1250 & -0.8232 & 0.1897 & -0.2692 & 1
\end{bmatrix}$$

The smallest absolute value is 0.0047 at position $M^*[1,3]$ respectively $M^*[3,1]$. This provides us $[1,3|2,4,5]$ as the top node.

**Tree 4:**

Now we have to find partners for 1 and 3 in $T_4$. We start with variable 1 and delete variable 3 from our correlation matrix by deleting the third row and column. We get the 4-dimensional correlation matrix $M_1$:

$$M_1 = \begin{bmatrix}
1 & 2 & 4 & 5 \\
1 & 0.8951 & 0.7059 & 0.4281 \\
2 & 0.8951 & 1 & 0.6809 & 0.4435 \\
4 & 0.7059 & 0.6809 & 1 & 0.4439 \\
5 & 0.4281 & 0.4435 & 0.4439 & 1
\end{bmatrix}$$

The normalized inverse

$$M^*_{\text{normalized}} = \begin{bmatrix}
1 & -0.7913 & -0.2860 & \mathbf{-0.0151} \\
-0.7913 & 1 & -0.1268 & -0.1196 \\
-0.2860 & -0.1268 & 1 & -0.2029 \\
\mathbf{-0.0151} & -0.1196 & -0.2029 & 1
\end{bmatrix}$$

To find the partner for 1 we have to find the smallest absolute value in the first row(or
column), which is 0.0151 at position $M_{1}[1,4]$. Hence we get the node $[1,5|2,4]$ in $T_4$, since the fourth column corresponds to the 5th variable after the third column is deleted.

To get a partner for variable 3 we delete variable 1 from our correlation matrix and obtain the 4-dimensional adjusted correlation matrix $M_3$:

$$M_3 = \begin{bmatrix}
2 & 3 & 4 & 5 \\
1 & 0.4985 & 0.6809 & 0.4435 \\
3 & 0.4985 & 1 & 0.8653 & 0.2796 \\
4 & 0.6809 & 0.8653 & 1 & 0.4439 \\
5 & 0.4435 & 0.2796 & 0.4439 & 1
\end{bmatrix}$$

The normalized inverse is:

$$\begin{bmatrix}
2 & 3 & 4 & 5 \\
1 & 0.2074 & -0.4983 & -0.1673 \\
3 & 0.2074 & 1 & -0.8340 & 0.1898 \\
4 & -0.4983 & -0.8340 & 1 & -0.2754 \\
5 & -0.1673 & 0.1898 & -0.2754 & 1
\end{bmatrix}$$

To find the partner for 3 we have to find the smallest absolute value in the 2nd row (corresponding to the 3rd variable now), which is 0.1898 at position $M_4[2,4]$. Hence we get the node $[3,5|2,4]$ as the second node in $T_4$.

**Tree 1-3:**

Similarly we get $T_3$ with the nodes $[1,4|2]$, $[5,2|4]$ and $[3,2|4]$. From now on we have no options any more and the whole vine structure is specified. The vine structure is shown in Figure 3.2.

![Figure 3.2: R-vine structure found by Algorithm 3.3 in Example 10.](image-url)
Now we are ready to take a look at the running time. Since we calculate the whole Vine-Structure in one single step we will now also consider the running time for the overall algorithm:

| Tree structure selection by Algorithm C (Step 1) | Calculate the normalized inverse of the correlation matrix to get the top-node
|------------------------------------------------|-------------------|
|                                                | \( \implies \text{Calculate the normalized inverse matrices of the adjusted correlation matrices for all variables out of the conditioned sets of the higher tree} \)
|                                                | Cost: \( \text{Inversion and Normalization of an } (n \times n) - \text{Matrix} + \sum_{i=1}^{n-3} (2i) \times (\text{Inversion and Normalization of an } ((n - i) \times (n - i)) - \text{Matrix}) \)

| Copula type by AIC (fast) (Step 2 - Part 1) | Calculate the copula-parameters for every edge, which has been chosen in the first step, and for every copula type 1,...,k and determine the corresponding AIC.
|---------------------------------------------|-------------------|
|                                              | \( \implies \text{Choose copula type with the smallest AIC} \)
|                                              | Cost: \( \frac{n \times (n + 1)}{2} \ast k \ast (\text{Fitting of a copula}) + \frac{n \times (n + 1)}{2} \ast k \ast (\text{AIC - estimation}) \)

| GoF by p-value (expensive) (Step 2 - Part 2) | Determine the p-value for every edge of our tree.
|---------------------------------------------|-------------------|
|                                              | If the p-value < a predetermined value, use a non-parametric copula.
|                                              | If the p-value \( \geq \) the predetermined value, use copula type and parameter(s) as chosen in Step 2 - Part 1.
|                                              | Cost: \( \frac{n \times (n + 1)}{2} \ast (p - \text{value - estimation}) + \text{at most} \frac{n \times (n + 1)}{2} \ast (\text{Fitting of a non-parametric copula}) \)

Obviously the determination of the copula types and parameter happens in the same way like in strategy 1. Since the normalization and inversion of a matrix can also be done very fast by a computer, this is also a fast algorithm. Nevertheless the question remains, how well this strategy performs for non-elliptical distributions.

### 3.4 Strategy 4: R-vine model selection and estimation based on AIC-values

This algorithm is only a slightly modified form of the second strategy. In this strategy we also start at the unconditioned \( T_1 \) and the tree-structure selection is based on a goodness-of-fit indicator. But instead of determining the structure using the expensive
3.5 Strategies

p-values, the much cheaper AIC-values are used here. To obtain these values we have to fit every allowed copula type for every possible pair of variables, which satisfy the proximity condition, by determination of their parameter(s) and calculate the corresponding AIC-values. We choose the copula type with the lowest AIC-value for every edge. In contrast to Strategy 5 we do not have to determine the p-values now, but can use the just computed AIC-values to choose the tree which minimizes the sum of AIC-values. As always we then have to transform the copula observations using the previously determined copula family choices and parameters to obtain the pseudo-observations for the next tree. The aim of this strategy is a compromise between best fits in the lower trees and cost efficiency.

Algorithm 3.4 R-vine model selection and estimation based on AIC-values

1: Fit the corresponding parameter(s) for all possible pairs and for all admitted copula types.
2: Select the copula type with the smallest AICs.
3: Select the tree that minimizes the sum of AIC-Values.
4: Transform the observations using the copula type choices and parameters from Step 1 & 2 to obtain the pseudo observations.
5: Use these pseudo observations to fit the corresponding parameter(s) for all pairs, which satisfy the proximity condition, and for all admitted copula types.
6: Proceed with Step 2 using all pairs which are allowed by Step 5 until the R-vine is fully specified.

Since only Step 3 was changed compared to Algorithm 2, we will start our investigation of the running time with the structure-selection:

| Tree structure selection by AIC-Value (very fast) (Step 3) | Since all AIC-values have already been computed in the first step, only the structure selection itself is left to be done:  
| ➞ Apply a maximum-spanning-tree algorithm (Algorithm of Prim)  
| As we seek a minimum-spanning-tree we must reverse the signs of our weights (AIC-values), of course.  
| Cost: (Algorithm of Prim) |
| GoF by p-value (expensive) | Calculate the p-values of all edges of the current tree and check all edges for a lack of fit:  
| If the p-value < a predetermined value, use a non-parametric copula.  
| If the p-value ≥ the predetermined value, use copula type and parameter(s) as chosen in Step 1 & 2.  
| Cost: (n – 1) * (p-value – estimation) + at most  
| (n – 1) * (Fitting of a non-parametric copula) |

As in strategy 1 the choice of the tree-structure is based on a feature, which is inexpensive to determine (Kendall’s-Tau-values in strategy 1, AIC-values in strategy 4),
while the expensive p-value evaluation has just to be done for the n-1 edges of the determined tree. Therefore, the running time compares to that of strategy 1.

### 3.5 Strategy 5: R-vine model selection and estimation based on Kendall’s $\tau$ & p-values

The last strategy in our selection is a mixture between Strategy 1 and 2. Therefore it is of course also bottom-up fashioned but the tree structure selection criterion is the product of p-value times Kendall’s-$\tau$-value, the structure selection criterion in Strategy 1 and 2. After the tree structure is determined using a maximum spanning tree algorithm, we determine the copula families as in the previous strategies using the AIC-values and transform the variables for the next tree. Aim of this strategy is obviously to use both kinds of weights: the partial correlations as well as a goodness-of-fit indicator. It remains to verify whether such a combination leads to better outcomes than the pure weights, or whether the weights even contradict each other. While the tree-structure is determined first in strategy 1, we will follow Strategy 3 for the tree structure selection and therefore adjust its algorithm:

**Algorithm 3.5** R-vine model selection and estimation based on Kendall’s $\tau$ & p-values

1. Fit the corresponding parameter(s) for all possible pairs and for all admitted copula types.
2. Select the copulae with the smallest AICs.
3. Calculate the empirical Kendall’s $\tau$ for all possible variable pairs.
4. Select the tree that maximizes the sum of (p-value)*(absolute Kendall’s-$\tau$).
5. Transform the observations using the copulae and parameters from Step 1 & 2 to obtain the pseudo values.
6. Use these pseudo observations to fit the corresponding parameter(s) for all possible pairs , which satisfy the proximity condition, and for all admitted copula types on the one hand and to calculate empirical Kendall’s $\tau$ for all possible pairs on the other hand.
7. Proceed with Step 4 selecting a tree among all pairs which are allowed by Step 6 until the R-vine is fully specified.

Finally we will now consider the running time of strategy 5 for the first tree, too:

The scenario of n variables and k admitted copula types still remains valid.
3.5 Strategies

**Copula type by AIC**

(Step 1 & 2)

- Calculate the copula-parameters for every edge \((i,j)\), \(i,j \in \{1, ..., n\}\), and for every copula type 1,...,k and determine the corresponding AIC.

\[ \Rightarrow \text{Choose copula type with the smallest AIC} \]

Cost: \(\frac{n(n-1)}{2} \ast k \ast (AIC - \text{estimation})\)

**Tree structure selection by p-value*Kendall’s \(\tau\)**

(Step 3 & 4)

- Determine the p-value for every pair \((i,j)\), \(i,j \in \{1, ..., n\}\), with the copula type specified in Step 2. Determine the empirical Kendall’s Tau for all possible variable pairs.

\[ \Rightarrow \text{Apply a maximum-spanning-tree algorithm (Algorithm of Prim)} \]

Cost: \(\frac{n(n-1)}{2} \ast (Kendall’s - \tau - \text{estimation}) + \frac{n(n-1)}{2} \ast (p-value - \text{estimation}) + (\text{Algorithm of Prim})\)

**GoF by p-value**

(Step 3 & 4)

- Check all edges for a lack of fit:
  - If the p-value < a predetermined value, use a non-parametric copula.
  - If the p-value \(\geq\) the predetermined value, use copula type and parameter(s) as chosen in Step 1 & 2.

Cost: At most \((n - 1) \ast (Fitting \ of \ a \ non-\ parametric \ copula)\)

Since we consider the combination of the most expensive feature with a second, it is easy to see that this is the most expensive strategy. We will have to examine in more detail below, whether such a combination performs significantly better than the individual strategies.

We conclude this section with a summary of the various strategies based on their distinctive features in Table 3.2.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>weight</th>
<th>direction</th>
<th>tree structure first/copula choice first</th>
<th>costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy 1</td>
<td>Kendall’s (\tau)</td>
<td>bottom-up</td>
<td>tree structure first</td>
<td>cheap</td>
</tr>
<tr>
<td>Strategy 2</td>
<td>p-Value</td>
<td>bottom-up</td>
<td>copula choice first</td>
<td>expensive</td>
</tr>
<tr>
<td>Strategy 3</td>
<td>partial correlations</td>
<td>top-down</td>
<td>tree structure first</td>
<td>cheap</td>
</tr>
<tr>
<td>Strategy 4</td>
<td>AIC-value</td>
<td>bottom-up</td>
<td>tree structure first</td>
<td>cheap</td>
</tr>
<tr>
<td>Strategy 5</td>
<td>p-value*Kendall’s (\tau)</td>
<td>bottom-up</td>
<td>copula choice first</td>
<td>expensive</td>
</tr>
</tbody>
</table>

Table 3.2: Overview of the 5 strategies
4 Simulation study

4.1 Overview

Now that we have presented our five strategies, and investigated the proposed algorithms, we will now examine the strategies in terms of their goodness of fit. We therefore set up a simulation study. We construct 20 different R-vine copula specifications (different in terms of dependencies, strength of correlations and copula types) and simulate a data set from them. Then we apply our 5 strategies on this data set and check which of the found models fits the data set best in terms of loglikelihood and AIC-values and comes closest to the original R-vine specification. In this way we can figure out which strategy performs best in which scenarios since we know the original tree-structure, copula families and parameters. This will help us to decide for a strategy when the specification of the original model is not fully known. Especially for large data sets it is important to know in advance which strategy will perform best and not to have to run all algorithms of our strategies. For each of the 20 scenarios we will simulate 100 data sets, each containing 1000 data points of an 8-dimensional R-vine specification. To make a better comparability to the original model, we will in addition to the five models, that were found by our strategies, now introduce three new models:

- **TTT (True-True-True):** This is simply the true model. Tree structure, copula types and parameters are given. This model serves as a benchmark for our comparison criteria.
- **TTE (True-True-Estimated):** Tree structure and copula types are taken from the true model but the corresponding parameters are estimated.
- **TEE (True-Estimated-Estimated):** Only the tree structure is from the true model. Copula families and parameters are estimated sequentially.

A further advantage of this subdivision of the original model is also to see which factor has the greatest influence on the goodness of fit: Tree structure selection, copula family choice or parameter estimation?

We will use an 8-dimensional R-vine for our simulation study, which is shown in Figure 4.1. Now we will introduce our scenarios. As already mentioned we will different the scenarios by the following 3 criteria:

**Dependencies** We will distinguish three different cases:

- all trees are dependent
- T4-T7 simplified (all Gauss-copula)
Figure 4.1: 8-dimensional R-vine tree structure for the simulation study. The corresponding R-vine matrix can be found on page ??.
4 Simulation study

- T4-T7 independent

**Strength of correlations** The four different cases are:

- Correlations monotonically decreasing
- Correlations constant weak
- Correlations constant strong
- Correlations monotonically increasing

**Copula types** Two different cases:

- mixed (all copula families are usable)
- T1-T3 all t-copula; T4-T7 all Gauss copula

We will denote these scenarios as elliptical in the following.

This gives us a total of 24 scenarios. However, in the Scenarios "T1-T3:t-copula; T4-T7:Gauss copula" (Copula type differentiation) the two cases "all trees dependent" and "T4-T7 simplified" (Dependency differentiation) coincide and we therefore have only 20 Scenarios left. This also becomes evident in Table 4.1, in which the different scenarios are summarized again. A detailed overview of copula families and parameters for the various scenarios follows in Appendix A.1. For a better overview the R-vine structure matrix $M$ and the copula family matrices for the mixed scenarios $F_{\text{mixed}}$ and for the elliptical scenarios $F_{\text{elliptical}}$ are shown below.

$$
M = \begin{pmatrix}
4 & 7 & 8 \\
7 & 5 & 7 & 5 \\
6 & 5 & 7 & 7 \\
8 & 6 & 6 & 1 & 6 \\
1 & 3 & 1 & 2 & 1 & 1 \\
2 & 1 & 3 & 2 & 3 & 2 \\
3 & 2 & 2 & 6 & 3 & 2 & 3 & 3
\end{pmatrix}
\quad
F_{\text{mixed}} = \begin{pmatrix}
1 & 24 & 14 \\
24 & 4 & 1 & 2 \\
34 & 1 & 14 & 34 \\
1 & 24 & 24 & 2 & 1 \\
2 & 14 & 2 & 34 & 14 & 1 \\
24 & 2 & 4 & 1 & 34 & 4 & 2
\end{pmatrix}
\quad
F_{\text{elliptical}} = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{pmatrix}
$$

As we can see in $F_{\text{mixed}}$ and $F_{\text{elliptical}}$, we allow the following copula families in our studies:

- 0:Independence copula
- 1:Gaussian copula
- 2:Student t copula (t-copula)
- 4:Gumbel copula
- 14:rotated Gumbel copula (180°; "survival Gumbel")
- 24:rotated Gumbel copula (90°)
- 34:rotated Gumbel copula (270°)

This may seem at first glance like a very strong restriction, but previous investigations have shown that most dependency patterns can be represented with the aid of these copula families and thus we have a good basis for our investigations.
4 Simulation study

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Copula typ</th>
<th>Dependencies</th>
<th>Strength of correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>mixed</td>
<td>T4-T7 indep.</td>
<td>mon. decreasing</td>
</tr>
<tr>
<td>2</td>
<td>mixed</td>
<td>T4-T7 indep.</td>
<td>const. weak</td>
</tr>
<tr>
<td>3</td>
<td>mixed</td>
<td>T4-T7 indep.</td>
<td>const. strong</td>
</tr>
<tr>
<td>4</td>
<td>mixed</td>
<td>T4-T7 indep.</td>
<td>mon. increasing</td>
</tr>
<tr>
<td>5</td>
<td>mixed</td>
<td>T4-T7 simplified</td>
<td>mon. decreasing</td>
</tr>
<tr>
<td>6</td>
<td>mixed</td>
<td>T4-T7 simplified</td>
<td>const. weak</td>
</tr>
<tr>
<td>7</td>
<td>mixed</td>
<td>T4-T7 simplified</td>
<td>const. strong</td>
</tr>
<tr>
<td>8</td>
<td>mixed</td>
<td>T4-T7 simplified</td>
<td>mon. increasing</td>
</tr>
<tr>
<td>9</td>
<td>mixed</td>
<td>all dependent</td>
<td>mon. decreasing</td>
</tr>
<tr>
<td>10</td>
<td>mixed</td>
<td>all dependent</td>
<td>const. weak</td>
</tr>
<tr>
<td>11</td>
<td>mixed</td>
<td>all dependent</td>
<td>const. strong</td>
</tr>
<tr>
<td>12</td>
<td>mixed</td>
<td>all dependent</td>
<td>mon. increasing</td>
</tr>
<tr>
<td>13</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>T4-T7 indep.</td>
<td>mon. decreasing</td>
</tr>
<tr>
<td>14</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>T4-T7 indep.</td>
<td>const. weak</td>
</tr>
<tr>
<td>15</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>T4-T7 indep.</td>
<td>const. strong</td>
</tr>
<tr>
<td>16</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>T4-T7 indep.</td>
<td>mon. increasing</td>
</tr>
<tr>
<td>17</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>T4-T7 simplified</td>
<td>mon. decreasing</td>
</tr>
<tr>
<td>18</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>T4-T7 simplified</td>
<td>const. weak</td>
</tr>
<tr>
<td>19</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>T4-T7 simplified</td>
<td>const. strong</td>
</tr>
<tr>
<td>20</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>T4-T7 simplified</td>
<td>mon. increasing</td>
</tr>
<tr>
<td>21=17</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>all dependent</td>
<td>mon. decreasing</td>
</tr>
<tr>
<td>22=18</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>all dependent</td>
<td>const. weak</td>
</tr>
<tr>
<td>23=19</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>all dependent</td>
<td>const. strong</td>
</tr>
<tr>
<td>24=20</td>
<td>T1-T3:t;T4-T7:Gauss</td>
<td>all dependent</td>
<td>mon. increasing</td>
</tr>
</tbody>
</table>

Table 4.1: Summary of the individual scenarios for the simulation study

4.2 LogLikelihood and AIC analysis

For a first overview of the performance of our strategies we start our investigation with
the analysis of the loglikelihood- and AIC-values of our strategies in the single scenarios. Remember: For each strategy we have 100 AIC- respectively loglikelihood-values per scenario since we simulate 100 data sets for each scenario. The corresponding boxplots are shown in Figure 4.2 and 4.3, respectively in Figure 4.4 and 4.5 we take an additional look at the percentage deviations between the individual strategies, where we set the median of TTT as 100 percent. In the percentage representation we can also unify the scales, which allows us to improve comparability between the scenarios.
4 Simulation study

Figure 4.2: Loglikelihoods of the 8 Strategies (including TTT, TTE and TEE) for each of the 20 scenarios
Figure 4.3: AIC-values of the 8 Strategies (including TTT, TTE and TEE) for each of the 20 scenarios
Figure 4.4: Loglikelihoods in % to TTT of the 8 Strategies (including TTT, TTE and TEE) for each of the 20 scenarios
Figure 4.5: AIC-values in % to TTT of the 8 Strategies (including TTT, TTE and TEE) for each of the 20 scenarios
4.2.1 Generall conclusions

**Generally,** it is striking that the tree structure selection has the greatest impact on the loglikelihood and the AICs. This is evident by the large jump in the loglikelihood between TEE, where the true tree structure is still given, and the loglikelihoods of our 5 strategies, which estimate all: tree structure, copula families and parameters. In some scenarios (Scenarios 4, 8 and 12), the tree structure selection leads to a decrease in the loglikelihood of over 30 percent compared to TEE. For the performance of each strategy it will therefore be particularly important, how close their respective selected tree structure comes to the original specification. Especially since the same procedures for choosing the copula family and estimating the parameters are used in all of our 5 strategies (the copula family with the highest AIC value is selected and the parameters are estimated via most likelihood). As the number of parameters compared to the loglikelihood is almost negligible and cause the figures of loglikelihoods and AICs lead to the same conclusions, we will henceforth not discuss the different representations separately.

In the following we will accurate investigate separately the four correlation-groups: scenarios with mon. decreasing correlations, const. weak correlations, const. strong correlations and mon. increasing correlations. Note: The four columns of Figures 4.2 - 4.5 correspond to the correlation-groups in the above order.

4.2.2 Scenarios with monotonically decreasing correlations

Since correlations in real life data sets often show monotonically decreasing behaviour, most strategies are developed to fit this kind of data sets as well as possible. It is therefore not surprising that the scenarios with monotonically decreasing correlations (Scenarios 1, 5, 9, 13 & 17) have the highest loglikelihoods for all strategies. It is remarkable that in scenarios with independent higher trees (Scenario 1 & 13) all strategies perform about equally well. However, these scenarios also show a lower loglikelihood, what indicates, that all strategies have problems to recognize independencies in the data sets. In the scenarios, in witch all trees are dependent, the loglikelihood increases. Strategies 1 & 4 though fall clearly back compared to the others. Strategy 2, 3 and 5 benefit more from the more complex specifications.

4.2.3 Scenarios with constant weak correlations

In the case of scenarios with constant weak correlations (Scenarios 2, 6, 10, 14 & 18) we come to similar conclusions as for the scenarios with monotonically decreasing correlations. However, we now find ourselves at a much lower loglikelihood level: While TTT in Scenario 9 for example had loglikelihoods around 21,000, we now seldom exceed the 5,000 mark. The low correlations mean that the weights in each strategy are very
similar. Therefore the decision criteria for the tree selection algorithms are blurred and thus wrong decision become more and more likely. Again we can see an even further drop in the likelihood of scenarios with independent higher trees. In these scenarios Strategies 2 & 5 are still better than strategies 1 & 4, too. Strategy 3 has though to be considered separately: This strategy shows also in Scenario 10 a very low loglikelihood. This demonstrates that this strategy was originally developed for elliptical distributions and may have problems with non-elliptical copulas. In the purely elliptical Scenarios 14 & 18 on the other hand Strategy 3 shows its strengths again and performs best compared to the other of our five strategies.

### 4.2.4 Scenarios with constant strong correlations

In scenarios with **constant strong correlations** (Scenarios 3, 7, 11, 15 & 19) quite high loglikelihoods are reached by the true model TTT again, though they do not get at the level of scenarios with monotonically decreasing correlations. High correlations, especially in the lower trees, seem to be favoured by our strategies. Even so the percentage loglikelihood drop from the true model TTT to the models that were found by our strategies gets bigger. Scenario 7 and 11 even drop over 40 percent. However, it is striking that in scenarios with dependent higher trees (Scenarios 7, 11 & 19) the parameter estimation seems to have problems as well. This leads to a slump in percentage loglikelihood of about 10 percent in the transition from TTT to TTE. Because this behaviour does not show up in scenarios with independent higher trees, we can conclude that high correlations especially lead to problems in higher trees. Particularly affected is Strategy 3, since the aim of it’s tree structure selection procedure is to find lowest possible correlations in the higher trees, what becomes obviously difficult in scenarios with constant strong correlations. Anyway Strategy 3 still performs better than strategies 1 and 4.

### 4.2.5 Scenarios with monotonically increasing correlations

Finally, we consider the scenarios with **monotonically increasing correlations** (Scenarios 4, 8, 12, 16 & 20). As mentioned before, this constellation is rather unusual in real data sets and therefore we do not expect a particularly good fit. In scenarios with independent higher trees (Scenarios 4 & 16) this becomes not so clear. The loglikelihood of the true model TTT is just under the values of the scenarios with constant strong correlations. The situation is different in scenarios with dependent higher trees (Scenarios 8, 12 & 20): Again in comparison with the scenarios with constant strong correlations, the loglikelihoods slump from a level of about 20,000 on a level of just over 9,000. As expected, the high correlations in the upper trees cause problems. But the scenarios with constant strong correlations have this problem, too. That the loglikelihoods yet fall so strong is also due to the weak correlations in the lower trees. As we have already seen in the scenarios with constant weak correlations these lead to
problems in the tree selection and thus to low loglikelihood values. Also the problems in
the parameter estimation by the high correlations in the upper trees seems to get
aggravated by the problems in the tree selection caused by the weak correlations in the
lower trees. At the transition from TTT to TTE the loglikelihoods drop in scenarios
with dependent higher trees by nearly 20 percent and even in scenarios with
independent higher trees a slight slope is noticeable. Also the overall loglikelihood drop
from TTT to the loglikelihood levels of the models found by our strategies becomes
larger and is in Scenarios 8 and 12 partly even over 50 percent.

4.2.6 Comparison: Elliptical scenarios versus mixed scenarios

Next, we consider the differences between the scenarios with mixed copula families
(all copula families are usable) and the scenarios which go exclusively with elliptical
copulas (Tree 1-3 all t-copula; Tree 4-7 all Gauss copula). In the Figures 4.2 - 4.5 the
mixed scenarios are located in the first three rows and the elliptical scenarios in the
bottom two rows. Considering our additive models TTT, TTE and TEE, we detect no
major differences. The loglikelihood levels remain the same between the mixed and the
elliptical scenarios. In the models found by our five strategies on the other hand, we
determine a slight higher loglikelihood level in the elliptical scenarios compared to the
mixed scenarios. This applies to the scenarios with independent higher trees as well as
the scenarios with dependent higher trees. As a consequence the percentage drop range
of the loglikelihood level from TTT to the loglikelihood levels of the models found by
our strategies becomes smaller. Also the percentage differences between the loglikelihood
levels of the various strategies are lower in the elliptical scenarios. While the values in
the mixed scenarios sometimes even were separated up to 15 percent, they are in the
elliptical scenarios almost always within a 5 percent interval.

In this section we also have to pay particular attention to Strategy 3, since in its tree
selection procedure the partial correlations are used as the conditioned correlations. But
this equality only holds for elliptical distributions. It is therefore particularly interesting
how Strategy 3 performs in the mixed scenarios compared to the pure elliptical
scenarios. In the mixed scenarios Strategy 3 is indeed lagging behind Strategies 2 & 5,
but is in the most mixed scenarios still better than Strategies 1 & 4. So although the
theoretical justification of the tree selection of this strategy is not one hundred percent
correct, the resulting error is nevertheless limited. Since the theoretical justification is
given in the purely elliptic scenarios, it is not surprising that Strategy 3 performs better
here. But it performs not only better, but is even the best strategy. In 6 of the 8 pure
elliptical scenarios the models found by Strategy 3 show up the highest loglikelihood
values. But purely elliptical distributions are rather rare in real life data sets. As already
mentioned before, the loglikelihood levels of the different strategies in the elliptical
scenarios are all close together, what means all strategies perform about equally well.
Thus, although Strategy 3 performs best in the elliptical scenarios, we would rather use
strategy 2 or 5 for an unknown origin model.
4 Simulation study

4.2.7 AIC-ranking for each scenario

At the end of this section we will summarize the AIC-rankings in Table 4.2. For each scenario 100 data sets were simulated and then 5 models were fitted by our strategies. These models can be accommodated in a unique order with the aid of the AIC-values. Thus we can determine for each scenario and each strategy, in how many of the 100 cases the particular strategies get the places 1-5. For clarity, the strategies were highlighted. Green strategies have performed well, yellow moderately and orange bad. But this tells us nothing about the statistical significance of these differences yet. This will be examined in the next section using the Vuong-statistics.
4 Simulation study

Table 4.2: AIC-rankings of each scenario
4 Simulation study

4.3 Non nested model comparison using the Vuong test

4.3.1 Test setup and illustrations

As already announced we will in this section compare the individual strategies, respectively their models, based on the Vuong test as described in Definition 2.11. In contrast to the AIC- and loglikelihood-values, the Vuong statistics tell us nothing about the general goodness of fit, but reveals which of two models fits a simulated dataset better and if this difference is statistical significant. For each of the 20 scenarios we consider again the 100 simulated data sets and perform the Vuong test to all strategy-pairs. Then we plot the statistics-data for a first overview and show in addition the normal Q-Q plot for the verification of the assumption that the Vuong statistic is standard normal distributed. As an example, we see in Figure 4.6 the plots for the Vuong statistics of Strategy 1 vs. Strategy 2 in Scenario 1.

![Figure 4.6: Plot of the 100 Vuong-statistics of Strategy 1 vs Strategy 2 for Scenario 1 and its corresponding normal Q-Q plot](image)

The plots for all scenarios and all strategy-pairs can be found in appendix A.2. Since the Vuong statistics only asymptotically converge to the standard normal distribution and our sample size of 100 is relatively small, it is not surprising that many normal Q-Q plots look nowhere near as good as our example in Figure 4.6. It is therefore clear that we have to work with a sense of caution in the evaluation of the normal Q-Q plots. In Figure 4.7 two examples are given, where we need a lot of goodwill to recognize a standard normal distribution. However, in the majority of cases it is sufficient to turn a blind eye on the normal Q-Q plots and the tests based on the Vuong statistics keep their validity. Therefore we will use the Vuong tests in the following without restrictions.
4 Simulation study

After we took a look at the quality of the tests, we now turn to the obtained results. The results of the Vuong tests are displayed in the Figures 4.8 and 4.9. In Figure 4.8 it is shown which strategy was preferred by the Vuong test how often in the 100 simulations.

**Explanation to Figure 4.8:**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>S1 vs S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>36 23 41</td>
</tr>
</tbody>
</table>

This means in Scenario 1 the model found by Strategy 1 was preferred by the Vuong test in 36 cases versus the model found by strategy 2. The other way round Strategy 2 was preferred in 41 cases. In 23 cases the test statistic was not significant.

This representation is still a bit confusing. Therefore Figure 4.9 shows a simplified representation for a better overview. For every scenario and every strategy-pair Figure 4.9 indicates by a 1 which strategy was preferred by the Vuong test in the majority of the cases. If the test is not significant in more than 50 cases, we rate the 2 strategies in the respective scenario as equally good. In the line 'TOTAL' we can then see, which strategy performed better in how many scenarios.

**Explanation to Figure 4.9:**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>S1 vs S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 0 1</td>
</tr>
</tbody>
</table>

The 1 in the green field indicates that the Vuong test preferred Strategy 2 compared to Strategy 1 in the majority of the cases in Scenario 1 what we have already seen in the explanation to Figure 4.8. Based on this test we can therefore conclude, that Strategy 2 is better than Strategy 1 in the first scenario.
### Figure 4.8: Results of the Vuong test for our simulation study: It is displayed which strategy was preferred by the Vuong test how often in the 100 simulations.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>S1 vs S2</th>
<th>S1 vs S3</th>
<th>S2 vs S3</th>
<th>S2 vs S4</th>
<th>S3 vs S4</th>
<th>S4 vs S5</th>
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<td>23</td>
<td>41</td>
<td>17</td>
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<tr>
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<td>17</td>
<td>36</td>
<td>47</td>
</tr>
<tr>
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<td>58</td>
<td>23</td>
<td>41</td>
<td>17</td>
<td>36</td>
<td>47</td>
</tr>
<tr>
<td>5</td>
<td>58</td>
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</tr>
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<td>58</td>
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<td>41</td>
<td>17</td>
<td>36</td>
<td>47</td>
</tr>
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<td>17</td>
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</tr>
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<td>15</td>
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</tr>
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<td>16</td>
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<td>58</td>
<td>23</td>
<td>41</td>
<td>17</td>
<td>36</td>
<td>47</td>
</tr>
</tbody>
</table>

**TOTAL**: 211 291 1060 216 196 1888 741 845 614 248 125 1412 1107 855 538 1600 240 151 741 691 506 1507 867 120 585 380 1035 190 260 1544

### Figure 4.9: Results of the Vuong test for our simulation study: It is just displayed which strategy was preferred in the majority of the 100 simulations.
4 Simulation study

4.3.2 Initial ranking for each scenario

With the results of Figures 4.8 and 4.9, we can now create an initial ranking of the strategies in the various scenarios:

![Figure 4.10: Ranking of individual strategies based on the Vuong test](image)

In Figure 4.3.2 we can see for each scenario, which place the individual strategies got by the Vuong test. The first place of course means that the strategy has performed best, while the 5th place performed worst. As with the analysis of the AIC- and loglikelihood values it is obvious that the mixed scenarios (all copula families are allowed) and the elliptical scenarios (Tree 1-3 all t-copula; Tree 4-7 all Gauss copula) have to be considered separately.

4.3.3 Ranking for the mixed scenarios

We start with the mixed scenarios and look, how often the strategies got the individual places in these scenarios:
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The picture that we have already received by the loglikelihood- and AIC-analysis gets confirmed. Strategies 2 and 5 are competing for rank 1 and 2 while the last 2 places are almost exclusively occupied by the Strategies 1 and 4, which perform about equally bad. Strategy 3 is assigned into the middle again and is ranked on the 3rd place in 10 of the 12 scenarios.

<table>
<thead>
<tr>
<th></th>
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<th>2.</th>
<th>3.</th>
<th>4.</th>
<th>5.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
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<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>S2</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S3</td>
<td>1</td>
<td>0</td>
<td>10</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>S4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>S5</td>
<td>4</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

4.3.4 Ranking for the elliptical scenarios

In the elliptical scenarios we obtain the following ranking:

The statements of the AIC- and loglikelihood-analysis get confirmed again. In 6 of the 8 scenarios Strategy 3 gets the first place (Note: We have 8 elliptical scenarios but the first place for example is occupied 10 times. This is possible since the Strategies 2, 3 and 5 share the first place in Scenario 18). However, based on the Vuong test we can better distinguish between the Strategies 2 and 5 in the elliptical scenarios. While the strategies 2 and 3 still compete for the first rank, Strategy 5 slipped to the midfield. This became not so clear in the analysis of the AIC- and loglikelihood-values. Strategies 1 and 4 play again no major role in the search for the best model.

<table>
<thead>
<tr>
<th></th>
<th>1.</th>
<th>2.</th>
<th>3.</th>
<th>4.</th>
<th>5.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>S2</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S3</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>2</td>
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<tr>
<td>S5</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

4.3.5 Aggregated ranking

Finally, let’s take a look at the aggregated ranking. But this overview has to be treated with caution since there are 12 mixed scenarios but only 8 pure elliptical.
Without much surprise we note, that the strategies 2, 3 and 5 perform all well while Strategies 1 and 4 can not compete with them. Strategy 3, however, has weaknesses in the mixed scenarios and Strategy 5 does not perform that well in the pure elliptical scenarios. The overall winner on the basis of the Vuong test is definitely Strategy 2, which provides very good results in all scenarios. As we have seen, the algorithm of Strategy 2 was very expensive. Nevertheless, that is a worthwhile expense, because the end result is top in all scenarios. Especially since the original model is unknown in real life data sets we prefer an algorithm which can handle all kinds of correlations, copula families and dependencies.

4.4 Summary

After we have analysed the AIC- and loglikelihood values as well as the Vuong statistics in the last sections it is now time to compile the results. While we partly remained quite technically so far, we now try to work out the key messages.

- The tree selection has the greatest influence on the goodness of fit. Copula family choice and parameter estimation play a subordinate role.

- Independent higher trees may be a desirable goal for the constructed models. If these, however, already occur in the original model, this leads to a worse goodness of fit of the found models.

- In the scenarios with monotonically increasing correlations, the loglikelihood values of the models found by the five strategies are the greatest. This is not surprising. Since many real life data sets show these correlation structure, most strategies were developed for this case.

- In the mixed scenarios Strategies 2 and 5 perform best by far.

- all strategies perform slightly better in the pure elliptical scenarios. However, the differences between the various strategies become smaller.

- The best strategy in the pure elliptical scenarios is Strategy 3, closely followed by Strategy 2. This is to be expected since partial correlations are only interpretable in elliptical settings.

- Strategies 1 and 4 play no major role in the search for the best model. Only in a few scenarios they can compete with the other strategies.
4 Simulation study

- Main problem are weak correlations in the lower trees. These lead to problems in the tree selection, which has the greatest influence on the goodness of fit as we have already mentioned. Vice versa high correlations in the lower trees almost surely lead to a high goodness of fit.

- High correlations in the higher trees on the other hand lead to problems in the parameter estimation. However, their influence on the overall goodness of fit is limited.
5 Application 1: Exchange Rates Dataset

In this chapter we will apply our strategies on an exchange rates data set to determine how well the different strategies perform in this record. This data set was introduced in ? and already used in ? and ?. The advantage of using this already well known data set is obviously that we can compare our results with those found in previous studies. Once we have adapted the individual strategies to the data set, we will examine how well the found models match the exchange rates data set. Now, of course, the true model is in contrast to the simulation study unknown. Afterwards we will compare the results with those of the simulation study and make reference to models found in earlier investigations.

5.1 Introduction to the data set

The data set consists of the exchange rates of 9 different countries respectively regions with respect to the US-Dollar. An overview to the states and the used notations in the following can be found in Table 5.1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Region</th>
<th>Currency</th>
<th>Token</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Europe</td>
<td>Euro</td>
<td>EUR</td>
</tr>
<tr>
<td>2</td>
<td>United Kingdom</td>
<td>British Pound</td>
<td>UK</td>
</tr>
<tr>
<td>3</td>
<td>Canada</td>
<td>Canadian Dollar</td>
<td>CAN</td>
</tr>
<tr>
<td>4</td>
<td>Australia</td>
<td>Australian Dollar</td>
<td>AUS</td>
</tr>
<tr>
<td>5</td>
<td>Brazil</td>
<td>Brazilian Real</td>
<td>BRA</td>
</tr>
<tr>
<td>6</td>
<td>China</td>
<td>Chinese Yuan</td>
<td>CH</td>
</tr>
<tr>
<td>7</td>
<td>Japan</td>
<td>Japanese Yen</td>
<td>JPN</td>
</tr>
<tr>
<td>8</td>
<td>Swiss</td>
<td>Swiss Franc</td>
<td>SZ</td>
</tr>
<tr>
<td>9</td>
<td>India</td>
<td>Indian Rupee</td>
<td>IN</td>
</tr>
</tbody>
</table>

Table 5.1: Overview to the 9 currencies of the exchange rates data set

For each nation 1007 observations were collected from 7/22/2005 to 7/17/2009. The corresponding plots are shown in Figure 5.1. To obtain the uniformly distributed copula data, ARMA(1,1)-GARCH(1,1)-models were fitted to each time series and the standardized residuals were transformed by the empirical distribution function. The full details with all parameters of the models can be found in ?. For a detailed description of
the theory of ARMA- and GARCH models, see ? and ?.

For a first overview to the data set, the pairsplots are shown in Figure 5.2. Above the diagonal the data pairs of the variables (diagonal entries) are plotted. The corresponding empirical Kendall’s $\tau$ values can be found below the diagonal. As expected some variable pairs show strong correlations. So it is for example not surprising that the British Pound and the Swiss Franc are closely linked to the Euro. But also other pairs like Switzerland-Australia are clearly correlated, what might not have been expected. The reasons for this are not so obvious but also not the subject of this thesis. As mentioned above, Figure 5.1 shall only give a first impression of the dependency structure.
Figure 5.2: Pairs-plot for the exchange rates data set (copula data). The corresponding empirical Kendall’s $\tau$ values are shown below the diagonal.
5.2 Application of the 5 strategies on the exchange rates data set

After we introduced the data set we will now apply our 5 strategies on the exchange rates data set. Since the algorithms of the various strategies have already been discussed in detail, we will not show how the individual models were found step-by-step, but focus on the results. In Figure 5.3 an overview to the models found by our strategies is given. The respective structure matrices and copula family matrices are shown as well as the matrices of theoretical Kendall’s $\tau$ values. An overview of the numbering of the copula families is given in Table 5.2.

$\begin{align*}
0 &= \text{independence copula} \\
1 &= \text{Gaussian copula} \\
2 &= \text{Student t copula} \\
4 &= \text{Gumbel copula} \\
14 &= \text{rotated Gumbel copula (180 degrees; ”survival Gumbel”)} \\
24 &= \text{rotated Gumbel (90 degrees)} \\
34 &= \text{rotated Gumbel (270 degrees)}
\end{align*}$

Table 5.2: Overview of the numbering of the different copula families

In the following we will investigate the differences of the found models with respect to the structure-, copula-family- and Kendall’s-$\tau$matrices.
### Application 1: Exchange Rates Dataset

#### Exchange rates

<table>
<thead>
<tr>
<th>EUR</th>
<th>UK</th>
<th>CAN</th>
<th>AUS</th>
<th>BRA</th>
<th>CH</th>
<th>JPN</th>
<th>SZ</th>
<th>IN</th>
</tr>
</thead>
</table>

#### 5. Application 1: Exchange Rates Dataset

<table>
<thead>
<tr>
<th>Strategy 1</th>
<th>Structure</th>
<th>Family</th>
<th>AIC</th>
<th>LogLik</th>
<th>Vuong</th>
<th>Kendall’s Tau</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 0 0 0 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0 0</td>
<td>-3733 100%</td>
<td>2219 100%</td>
<td>0</td>
<td>0</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>8 0 0 0 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0 0</td>
<td>-3733 100%</td>
<td>2219 100%</td>
<td>0</td>
<td>0</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>2 0 0 0 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0 0</td>
<td>-4372 100%</td>
<td>2219 100%</td>
<td>0</td>
<td>0</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>9 0 6 1 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0 0</td>
<td>-3733 100%</td>
<td>2219 100%</td>
<td>-0.3</td>
<td>0</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>5 5 6 6 5 0 0 0 0</td>
<td>54 0 1 0 0 0 0 0</td>
<td>-3733 100%</td>
<td>2219 100%</td>
<td>-1.6</td>
<td>-0.06</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>3 8 5 9 6 5 0 0 0</td>
<td>1 0 1 0 0 0 0 0</td>
<td>-4772 100%</td>
<td>2209 100%</td>
<td>-2.3</td>
<td>-0.10</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>4 3 5 9 6 5 0 0 0</td>
<td>34 1 0 0 0 0 0 0</td>
<td>-4239 98%</td>
<td>2175 98%</td>
<td>-2.1</td>
<td>-0.06</td>
<td>0.04 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>1 1 6 5 9 6 5 0 0</td>
<td>54 0 1 0 0 0 0 0</td>
<td>-4217 98%</td>
<td>2175 98%</td>
<td>-2.7</td>
<td>-1.59</td>
<td>-0.23 0.31 0.22 0.07 0.03 0 0 0 0</td>
</tr>
<tr>
<td>1 1 4 4 4 9 6 6 6</td>
<td>2 2 2 2 2 2 2 2 2</td>
<td>-3915 90%</td>
<td>1973 89%</td>
<td>0.17</td>
<td>0.52</td>
<td>0.68 0.46 0.42 0.42 0.19 0.10</td>
</tr>
</tbody>
</table>

#### Figure 5.3: Overview to the 5 models found by our strategies for the exchange rates data set and the corresponding theoretical Kendall’s τ values

5 Application 1: Exchange Rates Dataset
5.2.1 Structure Investigation

In this subsection we will take a closer look at the tree structures found by our five strategies. The structure matrices are displayed in Figure 5.3. However, since these representations are not particularly vivid we will draw the first two trees of our vines. In the following Figures 5.4 and 5.5 the first two trees of the R-vine specifications obtained by our five strategies for the exchange-rates data set are drawn. Since in ? an optimal C-vine was already found for this data set, we will show this results, too, to compare them to the first trees found by our strategies. The tree structure selection for the C-vine was based on Kendall’s τ like in strategy 1. For more detailed information see (?), pp. 89 et seq.) For the sake of completeness we will also construct a D-Vine for the exchange-rates data set. We determine this D-vine as follows: We start with the edge with the highest Kendall’s τ value. After that, we always add that node to the tree, that has the highest Kendall’s τ value with one of the leaf nodes of our current tree. We continue until the first tree is completely determined. The higher trees in the D-vine are defined by the first tree, so we already have the whole tree-structure. After that we will determine the copula families by AIC and the parameters by most likelihood just as in the other strategies. This approach guarantees no global maximum of the sum of Kendall’s τ values, but is sufficient for our comparison purposes here.

Now we will now show the first trees of the seven strategies followed by the second ones. For each edge the corresponding Kendall’s τ value and the copula family choice are displayed as well.

Abbreviation explanation:

- $t = \text{Student t copula (t-copula)}$
- Gauss = Gaussian copula
- indep. = independence copula
- $G = \text{Gumbel copula}$
- $G90°/G180°/G270° = \text{rotated Gumbel copulas}$

![Diagram of the first tree of Strategy 1](image-url)
5 Application 1: Exchange Rates Dataset

T1-Strategy 2

T1-Strategy 3

T1-Strategy 4

T1-Strategy 5

T1-DVine

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5 Application 1: Exchange Rates Dataset

Figure 5.4: First trees of the models found by our five strategies for the exchange rates data set. Additionally the first trees for a C- and a D-vine are displayed.
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T2-Strategy 1

G270°
EUR-SZ
-0.19

G180°
EUR-AUS
0.11

CH-IN
0.05

JPN-SZ

G270°
EUR-UK
-0.15

CAN-AUS
0.12

AUS-BRA
0.07

T2-Strategy 2

G
0.11

t
0.12

CAN-AUS

G
0.08

t
0.06

CH-IN
0.09

JPN-SZ

UK-BRA

CAN-AUS
0.28

UK-CH
0.51

indep.
0.51

EUR-UK
0.17

t
0.17

EUR-AUS
0.15

EUR-SZ
0.19

UK-CH
0.00

indep.
0.69

EUR-IN
0.08

SZ-IN
0.06

EUR-IN
0.08

indep. -0.05

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5 Application 1: Exchange Rates Dataset

T2-Strategy 3

EUR-UK $\rightarrow$ EUR-AUS $\rightarrow$ AUS-IN $\rightarrow$ SZ-IN $\rightarrow$ CH-IN

EUR-UK $\rightarrow$ EUR-AUS $\rightarrow$ AUS-JPN $\rightarrow$ AUS-IN $\rightarrow$ CH-IN

T2-Strategy 4

UK-CAN $\rightarrow$ CAN-AUS $\rightarrow$ AUS-BRA $\rightarrow$ BRA-CH $\rightarrow$ CH-JPN $\rightarrow$ JPN-SZ $\rightarrow$ JPN-SZ $\rightarrow$ IN-EUR

T2-Strategy 5

UK-BRA $\rightarrow$ EUR-UK $\rightarrow$ EUR-AUS $\rightarrow$ EUR-SZ $\rightarrow$ EUR-IN $\rightarrow$ EUR-CH

CAN-AUS $\rightarrow$ JPN-SZ
Figure 5.5: Second trees of the models found by our five strategies for the exchange rates data set. Additionally the second trees for a C- and a D-vine are displayed.
5 Application 1: Exchange Rates Dataset

It is apparent that the various strategies lead to very different tree structures. The strategies can be divided into two groups: Strategies, whose tree selection algorithm is based on correlations (Strategies 1, 3, and 5 as well as the C- and D-vine) and strategies, whose tree selection algorithm is based on a goodness of fit measure (Strategies 2 and 5). Despite the similarity in the weights, the small differences in the algorithms lead to completely different structures. Admittedly Strategy 4 leads to a D-vine but with other nodes than the pure D-vine algorithm.

Furthermore, it is eye-catching that some variable pairs like EUR-UK, CAN-AUS or SZ-JPN occur more often in the first trees of the different strategies. If we look at Figure 5.2 we note that these are all pairs with high correlations. It is therefore not surprising that these variable pairs occur in several models: 5 of our 7 strategies aim to produce high correlations in the lower trees. Furthermore, it is generally easier to fit a copula to a dependent variable pair than to two almost independent variables. This leads to a better goodness of fit, what means pairs with high correlations also become interesting for Strategies 2 and 5.

However, the overall conclusion is that the tree selection algorithms of the strategies, which have the greatest impact on the model quality as we have seen in the simulation study, lead to very different results.

5.2.2 Copula family investigation

In this subsection, we will look at the copula family choices of each strategy in detail. Here we see that the Student-t copula is predominant in the first tree, followed by the Gauss-copula. Only in Strategies 2 and 5 (strategies with weights based on goodness of fit indicators) some rotated Gumbel copulas appear in the first tree. A slight exception makes Strategy 4: Two variable pairs in the first tree are modelled by the independence copula. This is according to our previous studies certainly not desirable, but we have already noted in the simulation study that Strategy 4 has the greatest problems in the tree structure selection. By some unfavourable choices in the tree structure selection, these low correlated variable pairs already occur in the first tree.

From the second tree upwards non-elliptical copulas become more frequent, however Strategy 3 has only 1 non-elliptical copula in the trees 2 and 3. This could be particularly useful in this model since Strategy 3 has its strengths especially in non-elliptical scenarios.

Also independent copulas are piling up in the higher trees. Especially Strategies 1 and 5 (based on Kendall’s $\tau$ respectively mixed) model many variable pairs as independent. Strategy 5 considers most variable pairs from the 4th tree upwards as independent, Strategy 1 already from the 3rd tree upwards. This might be useful for a truncation later. Which influence this has on the goodness of fit, we will discuss in the following sections. between parameters and Kendall’s $\tau$.
5.2.3 Investigation of corresponding Kendall's τ estimates using the functional relationship between parameters and Kendall's τ

In this subsection we will investigate the theoretical Kendall's τ matrices of the models found by our strategies. For the explanation of the theoretical Kendall's τ values see Table 2.2.1 on page 9. This representation is important since it allows us to classify the found models in the categorization of our simulation study. Even though we do not know the true model of the data set, we are able to determine to which scenario the individual strategies assign the data record. Here it is worthwhile again to distinguish between strategies with weights based on a correlation measure and strategies with weights based on a goodness of fit indicator. As Strategy 5 shows a behaviour very similar to Strategy 1 we will consider the weights of Strategy 5 as based on a correlation measure (Kendall's τ) here.

The strategies with weights based on a correlation measure (Strategies 1, 3 and 5) provide all models with almost monotonically decreasing correlations. Of course we do not get an absolute accurate scenario as in the simulation study here, but the number of outliers is so small that we can clearly identify a scenario with monotonically decreasing correlations. This is plausible since decreasing correlations are the normal case, as already mentioned in the simulation study. Furthermore, we know that our strategies performed particularly well in these scenarios in terms of LogLikelihood and AIC. Therefore we expect a good performance of the Strategies 1, 3 and 5 here as well.

The models provided by strategies with weights based on a goodness of fit indicator (Strategies 2 and 4) can not be classified that well. Altogether the correlation level is lower and the behaviour more erratic. There are some very weak correlations in the lower trees but some high correlations in the upper trees, whereby no clear trend can be detected. If we really want to force this models in the systematic of the simulation study, we have to consider it as a model for a scenario with constant weak correlations (and many outliers). As we have seen in the simulation study this is not desirable, since these scenarios had the lowest LogLikelihood- respectively highest AIC-levels by far. We therefore do not expect a particularly good fit of the models found by the Strategies 2 and 4 what is a surprise since Strategy 2 performed best in our simulation study in most of the scenarios.

When we parallel consider the copula family choices again, we also note that in many cases high correlated variable pairs have been modelled with a Gauss- or Student-t copula while weak correlated variable pairs are often modelled by a non-elliptical copula.

5.3 Goodness of fit of the found models

After we introduced the models found by our five strategies for the exchange rates data set in the last section we will now examine, how well the models fit the data record. We will therefore start with the consideration of the loglikelihood- and AIC-values. Then we will examine how well the found models reflect some key figures of the original data set.
For this purpose we will consider the empirical Kendall’s $\tau$ values as well as the lower and upper tail cumulations. Finally we will perform the Vuong tests and take a look at the different rankings based on the loglikelihood-values, the AIC-values and the Vuong statistics. A first overview to the goodness of fit of the bivariate copulas, fitted by our five strategies, can be found in Table ?? on page ??, where the p-values of the Cramer-von-Mises statistics are displayed. Already here we can say that there are very high p-values for most of the copulas in all models - especially in the lower trees. We therefore recognize a good fit of all strategies.

### 5.3.1 Loglikelihood results

As usual we use the loglikelihood values for an first overview to the goodness of fit of the models found by our five strategies for the exchange rates data set. The respective values can be found in Figure 5.3. For a better overview the values are also listed here in Table ??.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>based on ...</th>
<th>LogLikelihood</th>
<th># Parameter</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Kendall’s $\tau$</td>
<td>2219.28</td>
<td>33</td>
<td>-4372.56</td>
</tr>
<tr>
<td>2</td>
<td>p-value</td>
<td>2216.29</td>
<td>44</td>
<td>-4344.58</td>
</tr>
<tr>
<td>3</td>
<td>partial correlations</td>
<td>2231.99</td>
<td>51</td>
<td>-4361.97</td>
</tr>
<tr>
<td>4</td>
<td>AIC</td>
<td>2192.74</td>
<td>41</td>
<td>-4303.47</td>
</tr>
<tr>
<td>5</td>
<td>Kendall’s $\tau \times$ p-value</td>
<td>2225.57</td>
<td>35</td>
<td>-4381.15</td>
</tr>
</tbody>
</table>

Table 5.3: Overview to the goodness of fit indicators for the models found by the strategies for the exchange rates data set

As we can see, the values are all close together. When we set the smallest value as basis (2192.74 for Strategy 4), the best value (2231.99 for Strategy 3) is not even 2 percent better. Nevertheless, we can use these values to create an initial ranking, which is displayed in Table ??.

<table>
<thead>
<tr>
<th>LogLik-Ranking</th>
<th>5th Place</th>
<th>4th Place</th>
<th>3rd Place</th>
<th>2nd Place</th>
<th>1st Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy Value</td>
<td>2192.74</td>
<td>2216.29</td>
<td>2219.28</td>
<td>2225.57</td>
<td>2231.99</td>
</tr>
</tbody>
</table>

Table 5.4: Loglikelihood ranking for the exchange rates data set

The model complexity is not taken into account in this ranking yet, but our suspicions from the model analysis in the previous section gets confirmed. Strategies with weights based on a correlation measure get the first places, while Strategies 2 and 4, whose weights are based on goodness of fit indicators, get the places 4 and 5. The best model in terms of the pure loglikelihood is model 3, followed by the models of Strategies 5 and 1. The model found by Strategy 2 reaches only the 4th place while Strategy 4 takes the 5th place as expected.
5.3.2 Number of Parameters and AIC results

Now, that we have given an overview, we turn to the model complexity. For this purpose we will consider the number of parameters required by the different strategies and investigate the corresponding AIC-values on this basis.

As we can see in Table ?? Strategy 3, the winner of the loglikelihood-ranking, needs with 51 the most parameters by far. Then come the Strategies 2 and 4, whose weights are based on a correlation measure, with 44 and 41 parameters. The least parameters of all are used by the Strategies 5 and 1, which require only 35 and 33 parameters. Thus the differences in the number of parameters between the different models are very significant.

In the next step we consider the AIC-ranking, which takes the model complexity into account by the help of the number of parameters. The corresponding values can be found in Table ?? again. Remember:

\[ AIC = 2k - 2 \times \text{loglikelihood} \]

where \( k \) denotes the number of parameters.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>5th Place</th>
<th>4th Place</th>
<th>3rd Place</th>
<th>2nd Place</th>
<th>1st Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>-4303.47</td>
<td>-4344.58</td>
<td>-4361.97</td>
<td>-4372.56</td>
<td>-4381.15</td>
</tr>
</tbody>
</table>

Table 5.5: AIC ranking for the exchange rates data set

As we can see the placement of Strategies 4 and 2 remains unchanged. The first ranks on the other hand get reassigned under consideration of the model complexity. Strategy 3 gets punished for its very large number of required parameters and slips down from rank 1 to rank 3. Strategies 1 and 5 can therefore climb up one place each. Thus the new winner of this AIC ranking is Strategy 5, followed by the Strategies 1 and 3. We see that in this case the number of parameters had a significant influence on the ranking. This shows us that in cases where the loglikelihood values are close together, the role of the number of parameters must not be underestimated. However, it still remains to investigate why Strategy 2, which performed best in the simulation study, can not compete with the first places.

5.3.3 Analysis of pairwise key characteristics

In this subsection we will take a closer look at some key figures of the original exchange rates data set and check, how well the models found by our five strategies can reproduce them. For this purpose we will calculate the empirical Kendall’s \( \tau \) values as well as the lower and the upper tail cumulations for every variable pair of the original data set. Since we have a 9-dimensional data set we get a total of 36 values for each key figure. These will serve us as reference values. To obtain comparative values for our models, we simulate 100 data sets from each of the five models in the original size (1007 data
5 Application 1: Exchange Rates Dataset

points). Afterwards we calculate the empirical Kendall’s τ values and the lower and upper tail cumulations also for these simulated data sets. For each variable pair and each key figure, we therefore have 100 comparative values for each of the 5 models. To make this clear and to be able to interpret the data, we present the results in boxplots. The empirical Kendall’s τ values can be found in the Figures ?? - ?? according to the models in Appendix ???. The lower tail cumulations are displayed in the Figures ?? - ?? and the upper tail cumulations in the Figures ?? - ???. Additionally we show the lower and upper tail exceedance correlations in the Figures ?? - ???. The red lines represent the true values of the original data set. In the ideal case the true value (red line) is near the median (black line) of the 100 comparative data of the simulated data sets. To determine in which tree possible outliers occur, the variable pairs in the x-axis are marked in colour. The associated key can be found in the upper right corner of the figures.

Results

We start the investigation with the representations for the Strategy 1 and see that the values of the simulated data sets are very close to the original ones in the majority of the variable pairs. In no case is the true value outside the range of the simulated values. Especially in the first tree the true values and the medians of the simulated values almost coincide. However, if we look a little bit closer, we recognize some variable pairs that do not look that good: In the pairs EUR-CH, EUR-IN and UK-IN for example, the true values are located at the edge of the intervals, which are generated by the values of the simulated data sets. So we check the specifications of these variable pairs in Figure 5.3 and realize that a large part of the pairs that do not fit ideally are modelled by the independence copula. Although Strategy 1 decides for the independence copula, the variable pairs seem to be not completely independent, what is of course reasonable from the economical point of view. However, the independence copulas are in these cases not possible to reflect the true empirical Kendall’s τ values as well as the lower and upper tail cumulations perfectly. Overall seen the true values are reflected well.

Next we consider Strategy 2. Although we already know that this strategy performed not that well in terms of loglikelihood and AIC, Strategy 2 reflects the true values better than Strategy 1. There are far fewer outliers and most of them can again be traced back to independence copulas. Of course the other strategies benefit from the fact, that Strategy 1 has chosen the most independence copulas.

Particularly Strategy 3 benefits from that circumstance since this has not chosen a single independence copula. So it is not surprising that this strategy reflects the true values best in this representation. Especially since Strategy 3 already performed best in the loglikelihood analysis and fell back behind Strategy 1 and 5 in the AIC-ranking just because of the large number of parameters. But these also remain unconsidered here.

Also in this analysis Strategy 4 can not score. As in the previous investigations as well as in the simulation study, Strategy 4 shows the worst results. To make matters worse its model already shows up independence copulas in the first tree. But there are many
other outliers, that can not be traced back to independence copulas. Thus we have to recognize, that the model performs worst also in this analysis. But worst does still not mean bad. Again the key figures are reflected very well in many variable pairs and the outliers are not extreme. Its just the number of outliers that is greater than in the other strategies.

**Strategy 5** is in this analysis just slightly better than Strategy 2, shows constantly good results and the few outliers are limited. Although some independence copulas appear in the higher trees, the variable pairs in the lower trees seem to be chosen better, so that the problems of Strategy 1 do not come into effect in such a great extent here.

Overall, it may be said that all strategies find models that reflect the key figures of the original data set well. However, if we want to bring them in an order: Strategy 5 wins followed by Strategies 5 and 2. Strategy 1 is only fourth place while Strategy 4 comes in last again.

### 5.3.4 Vuong statistics

In the previous investigations we have given an overview to the general goodness of fit indicators and used them to create rankings. In this subsection we will now determine, if the recognized differences are statistical significant by the help of the Vuong test. Since we want to take the model complexity into account also here, we use the Vuong test with Akaike corrections. For further information see Definition 2.11 on page 22. The test statistics and their corresponding p-values are shown in Table 5.6.

<table>
<thead>
<tr>
<th>Vuong statistics with Akaike corrections</th>
<th>p-values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Strat2</strong></td>
<td><strong>Strat3</strong></td>
</tr>
<tr>
<td>Strat1</td>
<td>-1.10</td>
</tr>
<tr>
<td>Strat2</td>
<td>-0.59</td>
</tr>
<tr>
<td>Strat3</td>
<td>1.90</td>
</tr>
<tr>
<td>Strat4</td>
<td>-2.52</td>
</tr>
</tbody>
</table>

Table 5.6: Vuong statistics and corresponding p-values for the exchange rates data set

Explanation to Table 5.6:
The entry (3,3) of the statistics matrix (1.90) for example shows the statistic of the Vuong test with Akaike corrections from Strategy 3 against Strategy 4. Remember: A positive number (here 1.90) indicates that Strategy 3 is preferred against Strategy 4 by the Vuong test. However, since we work with an 5% confidence interval, all test statistics \( \nu \) with \( |\nu|<\Phi^{-1}(1-\alpha/2) \approx 1.96 \) are not statistical significant. \( \Phi^{-1} \) denotes the quantile function of the standard normal distribution. This also becomes evident since the corresponding p-value is with 6% larger than the allowed 5% confidence level. At a confidence level of 10% the test would be significant.

We recognize that many test are not significant on the 5% confidence level as we already
expected during the previous studies. Only for Strategy 4 we can note once again, that a confidence level of 15% would be enough to declare that Strategy 4 is significantly worse then the others. The other strategies already showed up very close results in the loglikelihood- and AIC-analysis, so it is not surprising that the tests do not significantly vary. Nevertheless we will use the results of the Vuong tests with Akaike corrections to create a new ranking in Table ??.

<table>
<thead>
<tr>
<th>Vuong-Ranking</th>
<th>4th Place</th>
<th>1st Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>4 &amp; 2</td>
<td>3 &amp; 1 &amp; 5</td>
</tr>
</tbody>
</table>

Table 5.7: Vuong test ranking for the exchange rates data set

That ”3 & 1 & 5” are on the first place means that the models are equally good according to the Vuong test. The test statistics indicate that 5 is better than 1 and 1 is better than 3, but these differences are statistically not significant. The same holds for ”4 & 2”.

As we can see, we come to the same conclusions as in the AIC-ranking. But now we know, what we previously only suspected: The differences in the goodness of fit between the various models are present, but only in a few cases they are statistical significant.

### 5.4 Truncation

In this section we will amplify our investigations on the possibility to truncate the found models. For this purpose we consider all truncated models of the various strategies. We will explain the procedure on basis of Strategy 1: We already have the complete model (we call it Model C here) found by Strategy 1 for the exchange rates data set. We now take only the first tree of that model and consider it as an own model. We call it Model T1. Now we calculate the loglikelihood- and AIC-values for Model T1.

In the next step we take the first two trees of Model C and obtain the new model T2. Again we calculate the loglikelihood- and AIC-values for Model T2, but now we also perform the Vuong test between Model T1 and T2 to determine, if the new model performs significantly better.

We extend the model in each step by another tree until we have reached the complete model again with Model T9.

All loglikelihood- and AIC-values as well as the Vuong statistics can be found in Figure 5.3 on page 59. For a better overview we also have a summary of the AIC-values and Vuong statistics in Table ?? below.
5 Application 1: Exchange Rates Dataset

Table 5.8: Overview of the truncation possibilities for the exchange rates data set

<table>
<thead>
<tr>
<th>Tree</th>
<th>Strategy 1</th>
<th>Strategy 2</th>
<th>Strategy 3</th>
<th>Strategy 4</th>
<th>Strategy 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AIC %</td>
<td>Vuong</td>
<td>AIC %</td>
<td>Vuong</td>
<td>AIC %</td>
</tr>
<tr>
<td>9</td>
<td>-4373</td>
<td>100%</td>
<td>4392</td>
<td>100%</td>
<td>-4303</td>
</tr>
<tr>
<td>8</td>
<td>-4373</td>
<td>100%</td>
<td>4392</td>
<td>100%</td>
<td>-4303</td>
</tr>
<tr>
<td>7</td>
<td>-4373</td>
<td>100%</td>
<td>4392</td>
<td>100%</td>
<td>-4303</td>
</tr>
<tr>
<td>6</td>
<td>-4374</td>
<td>100%</td>
<td>4393</td>
<td>100%</td>
<td>-4303</td>
</tr>
<tr>
<td>5</td>
<td>-4374</td>
<td>100%</td>
<td>4393</td>
<td>100%</td>
<td>-4303</td>
</tr>
<tr>
<td>4</td>
<td>-4377</td>
<td>98%</td>
<td>-4381</td>
<td>97%</td>
<td>-4303</td>
</tr>
<tr>
<td>3</td>
<td>-4293</td>
<td>98%</td>
<td>-4381</td>
<td>97%</td>
<td>-4303</td>
</tr>
<tr>
<td>2</td>
<td>-4267</td>
<td>98%</td>
<td>-4381</td>
<td>97%</td>
<td>-4303</td>
</tr>
<tr>
<td>1</td>
<td>-3915</td>
<td>90%</td>
<td>-4195</td>
<td>90%</td>
<td>-4195</td>
</tr>
</tbody>
</table>

The red lines in the Vuong statistics columns indicate, when the statistics
$\nu > -\Phi^{-1}(1 - \alpha/2) \approx -1.96$ and we can conclude that the larger model performs not
significantly better than the smaller one. Based on the Vuong test we would cut off the
models at this lines. In The %-column we can see how many percent of the maximum AIC-value are already reached in the respective truncated model.

We note that the Strategies 1 and 3 are the fastest to provide good models. Both
Models can already be truncated after the fourth tree. Strategy 1 even reaches 90% of
the maximum AIC-value already in the first tree. Strategy 5 achieved with 84% in
the first tree also a very good result, but cuts off only after the fifth tree. However, Strategy
5 provides the best model in terms of AIC also in the truncated models. Strategy 4 can
exhibit a special feature. Based on the Voung test Strategy 4 would cut off after the 6th
tree since including the 7th tree yields no significant improvement. Including the 8th
tree though would be significantly better than the model with just 7 trees. This
indicates again a poor model choice. Strategy 2 cuts off last. Only after the 7th tree the
improvements are no longer significant. Strategy 2 also started weakest. In the first tree
only 21% of the maximum possible AIC-value are achieved. Nevertheless, the found
model is with 2 AIC-points gossamer better than the model found by Strategy 3. This
shows once again how similar the various strategies perform although the models are
very different. The corresponding ranking can be found in Table ??.

Table 5.9: AIC ranking of the truncated models for the exchange rates data set

<table>
<thead>
<tr>
<th>Truncated AIC-Ranking</th>
<th>5th Place</th>
<th>4th Place</th>
<th>3rd Place</th>
<th>2nd Place</th>
<th>1st Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy Value</td>
<td>-4195</td>
<td>-4343</td>
<td>-4345</td>
<td>-4357</td>
<td>-4367</td>
</tr>
</tbody>
</table>

5.5 Special investigations to Strategy 2

In the previous studies we have already seen that Strategy 2 can not fulfill the high
expectations in the exchange rates data set. The good results from the simulation study
were not confirmed. Furthermore we noticed that it is difficult to classify the model
found by Strategy 2 in the systematic of the simulation study. While we had scenarios
with constant and monotonic correlations there, the correlations show a very erratic
behaviour here. The suspicion is therefore obvious that Strategy 2 reacts sensitively to deviations from the smooth structure of the simulation study and the delivered models are not as structured as the models found by the strategies whose weights are based on correlations. The same statements hold for Strategy 4, of course, but since this strategy performed worst in almost every investigation we will focus on Strategy 2 here.

**Introduction and classification of Scenario 101**

To verify this assumption we will examine another scenario which also shows some deviations from the smooth simulation study systematic. We choose the model found by Strategy 1 for the exchange rates data set. The model can still be roughly classified in the systematic of the simulation study but also shows some deviations in the monotonically decreasing correlations. The model can be found in Figure ?? or Figure ???. Since the model also shows a good fit with respect to the exchange rates data set, we assume that it comes close to the unknown true model.

If we want to fit the Scenario in the systematic of the simulation study we get:

- **Copula types:** t-copulas in the first tree, mixed copulas in tree 2 and many independent copulas in the trees 3-9
- **Dependencies:** T4-T7 independent (with many exceptions)
- **Strength of correlations:** Monotonically decreasing correlations (with some exceptions)

Scenario 101 is therefore most likely comparable with Scenario 1 or 13. A final decision is not possible yet.

In the following we will follow the procedure of the simulations study. First we apply all strategies on the data sets simulated from Scenario 101 and get the loglikelihood and AIC overviews in Table ???. For explanations to the procedure and the meaning of the figures, see Chapter 4.
5 Application 1: Exchange Rates Dataset

We note that the images are very similar to those of Scenario 13 and will therefore put Scenario 101 in the same categorization. Remember: The specification of Scenario 13 was

- Copula types: T1-T3 t-copula
- Dependencies: T4-T7 independent
- Strength of correlations: Monotonically decreasing correlations

Though the first 3 trees of Scenario 101 are obviously not only modelled by the t-copula, the first tree seems to be sufficient to obtain a similar pattern. This assessment appears particularly plausible to us, since the goodness of fit indicators of the strategies in Scenario 13 were very close together just as for the exchange rates data set. However, this assessment has still to be treated with caution since the loglikelihood level of Scenario 101 is with 2300 even under the level of the scenarios with constant weak correlations, which had the lowest loglikelihood level in the simulation study. Scenario 13 had a loglikelihood level of about 15.700.
5 Application 1: Exchange Rates Dataset

Results

Now we consider the representations in Table ?? more precisely. Both the loglikelihood as well as the AIC representation show that the models found by the Strategies 1 and 3 perform best, followed by Strategy 5. The models found by the Strategies 2 and 4 are in fact the worst in this scenario. Although the differences are not large as already mentioned, our assumptions regarding Strategy 2 seem to get confirmed. In Figure ?? an overview is given, in how many of the 100 simulated data sets the various strategies get which places how often.

<table>
<thead>
<tr>
<th>Scenario101</th>
<th>1.</th>
<th>2.</th>
<th>3.</th>
<th>4.</th>
<th>5.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy1</td>
<td>65</td>
<td>32</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Strategy2</td>
<td>0</td>
<td>0</td>
<td>19</td>
<td>35</td>
<td>46</td>
</tr>
<tr>
<td>Strategy3</td>
<td>33</td>
<td>54</td>
<td>8</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Strategy4</td>
<td>0</td>
<td>1</td>
<td>14</td>
<td>42</td>
<td>43</td>
</tr>
<tr>
<td>Strategy5</td>
<td>2</td>
<td>13</td>
<td>56</td>
<td>20</td>
<td>9</td>
</tr>
</tbody>
</table>

Figure 5.7: Overview to the AIC values of Scenario 101

<table>
<thead>
<tr>
<th>Scenario 101</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1 vs S2</td>
</tr>
<tr>
<td>S1 vs S3</td>
</tr>
<tr>
<td>S1 vs S4</td>
</tr>
<tr>
<td>S1 vs S5</td>
</tr>
<tr>
<td>S2 vs S3</td>
</tr>
<tr>
<td>S2 vs S4</td>
</tr>
<tr>
<td>S2 vs S5</td>
</tr>
<tr>
<td>S3 vs S4</td>
</tr>
<tr>
<td>S3 vs S5</td>
</tr>
<tr>
<td>S4 vs S5</td>
</tr>
</tbody>
</table>

This results are also statistically significant as the Vuong tests show. The plot oft the statistics and the corresponding normal Q-Q plots can be found in Table ???. The normal Q-Q plots indicate that the assumption of standard normal distributed Vuong statistics is plausible in Scenario 101. An overview to the Vuong test is given in Figure ???. It is displayed in how many of the 100 cases which strategy was preferred by the Vuong test how often.

Conclusion

It seems to come true that Strategy 2 exhibits weaknesses in scenarios that differ from the pure correlation structures of the simulation study. Pure is meant in the sense of
constant and monotone correlation structures here. The loglikelihood- and AIC-analyses have shown that Strategy 2 drops behind the Strategies 1 and 3 in Scenario 101 and with the Vuong tests it was shown that these differences are also statistically significant. Of course this is only the examination of one individual scenario, but our initial suspicion got corroborated. The good performance of Strategy 2 has therefore to be treated with caution since pure correlation structures are very rare in real life data sets.
5 Application 1: Exchange Rates Dataset

\[ \text{hmargin}=2\text{cm}, \text{top}=2\text{cm}, \text{bottom}=4\text{cm} \]
| Table 5.11: Plotted Vuong statistics and the corresponding normal Q-Q plots for Scenario 101 |

<table>
<thead>
<tr>
<th>S1 vs S2</th>
<th>Normal Q-Q Plot</th>
<th>S1 vs S3</th>
<th>Normal Q-Q Plot</th>
<th>S1 vs S4</th>
<th>Normal Q-Q Plot</th>
<th>S1 vs S5</th>
<th>Normal Q-Q Plot</th>
<th>S2 vs S3</th>
<th>Normal Q-Q Plot</th>
<th>S2 vs S4</th>
<th>Normal Q-Q Plot</th>
<th>S2 vs S5</th>
<th>Normal Q-Q Plot</th>
<th>S3 vs S4</th>
<th>Normal Q-Q Plot</th>
<th>S3 vs S5</th>
<th>Normal Q-Q Plot</th>
<th>S4 vs S5</th>
<th>Normal Q-Q Plot</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tr>
</tbody>
</table>
5.6 Summary and conclusions

In this section we will summarize the results of the previous investigations and work out the central points:

- **All strategies are able to find a suitable model**
  In terms of loglikelihood the models do not even differ by 2%. Also key figures of the original data set like the empirical Kendall’s $\tau$ values as well as the upper and lower tail cumulations are reproduced by the models very well. Only Strategy 4 lags slightly behind the other strategies in all analyses. How similar the various strategies perform also becomes evident in the Vuong test: Most of the Vuong test are statistical not significant.

- **The strategies provide very different models**
  This may not be surprising at the first glance: According to there are $\frac{n!}{2} \times 2^{(n-2)}$ possible regular vines on n nodes. However, the algorithms of our strategies differ only in details. Nevertheless, the selected vine structures are nearly completely different.

- **Strategies with weights based on correlations provide models with almost monotonically decreasing correlations**
  The Strategies 1, 3 & 5 all find models, which have nearly monotonically decreasing theoretical Kendall’s $\tau$ values. This is not only desirable as the found models can be classified in the categorization of the simulation study, the scenarios with monotonically decreasing correlations also showed the best goodness of fit in the simulation study. This is a further reason for the good performance of the Strategies 1, 3 & 5 in the exchange rates data set.

- **Strategies with weights based on a goodness of fit indicator provide unstructured models**
  Although Strategy 2 in particular comes off very well in the simulation study, in the exchange rates data set it falls slightly behind the strategies with weights based on correlations. We notice that the Strategies 1 and 4 provide models with an erratic correlation pattern and also a lower correlation level, what leads to a worse goodness of fit. In the simulation study we have considered scenarios with monotone and constant correlations. However, strategies with weights based on a goodness of fit indicator seem to be more sensitive to deviations in this steady structure and the results are also not that structured.

- **Number of parameters: Strategy 3 delivers best loglikelihood at the highest parameter costs**
  The least parameters are needed by the Strategies 1 and 5, followed by the Strategies 2 and 4. The most parameters by far are needed by Strategy 3. However, Strategy 3 also provides the best model in terms of the loglikelihood. Only in terms of AIC it falls back to rank 3. Nevertheless, a high number of parameters does not inevitably lead to a high calculation time and the running time of the algorithm is still one of the fastest. Therefore it is a question of faith if
someone prefers Strategy 3 with a large number of parameters and a cheap algorithm or for example Strategy 5 with a low number of parameters but a very expensive algorithm.

- **Strategy 5 delivers best model in terms of AIC**
  When we take the model complexity into account by the help of the AIC, we recognize that Strategy 5 performs best, followed by the Strategies 1 and 3. However, if we consider the Vuong test with Akaike corrections, we see that the differences between these models are statistically not significant.

- **Truncation: Strategies 1 and 3 cut off first**
  The Strategies 1 and 3 cut off the model based on the Vuong test already after the 4th tree since they have already nearly reached their maximum AIC- respectively loglikelihood-level at this point. Strategy 1 even reaches 90% percent of the maximum AIC-level after the first tree. For very large data sets it is often not desirable or even not possible to determine the whole model. In this case the Strategies 1 and 3 would be recommended.

- **Strong dependencies are modelled by the Student-t copula**
  This became evident in the analysis of the copula family choices of the various strategies. This is noteworthy since we have seen in the simulation study that all strategies profit from elliptical copulas in the lower trees. In the case of the exchange rates data set especially strategies with weights based on a correlation measure (Strategies 1, 3 & 5) profit from this fact since they aim to achieve strong correlations in the lower trees.
6 Application 2: Bond Returns Dataset

As a second application we consider the daily returns of 9 German and European bonds. These are part of an 16-dimensional data set of financial indices which was introduced in ?. The whole data set consists of 5 stocks, 9 bonds and 2 commodities. The stocks were already used in Example 10 on page 30 for illustration of Algorithm 3.3. We consider only a 9-dimensional sub-dataset since thereby the comparability to the exchange rates data set is maintained.

6.1 Introduction to the data set

As already mentioned the data set consists of 9 variables representing the daily returns of 9 bonds from Germany and the Eurozone. All bonds are priced in EURO. An overview to the different Bonds can be found in Table ??.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Bond Name</th>
<th>Currency</th>
<th>Region</th>
<th>Token</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>IBOXX EURO SOV. GERMANY 3-5 YRS</td>
<td>Euro</td>
<td>Germany</td>
<td>IBOXX-G-3-5</td>
</tr>
<tr>
<td>2</td>
<td>IBOXX EURO SOV. GERMANY 7-10 YRS</td>
<td>Euro</td>
<td>Germany</td>
<td>IBOXX-G-7-10</td>
</tr>
<tr>
<td>3</td>
<td>IBOXX EURO SOV. EZONE 1-3 YRS</td>
<td>Euro</td>
<td>Eurozone</td>
<td>IBOXX-E-1-3</td>
</tr>
<tr>
<td>4</td>
<td>IBOXX EURO SOV. EZONE 5-7 YRS</td>
<td>Euro</td>
<td>Eurozone</td>
<td>IBOXX-E-5-7</td>
</tr>
<tr>
<td>5</td>
<td>IBOXX EURO SOV. EZONE 10+ YRS</td>
<td>Euro</td>
<td>Eurozone</td>
<td>IBOXX-E-10+</td>
</tr>
<tr>
<td>6</td>
<td>IBOXX EURO CORP. A RATED ALL MATS.</td>
<td>Euro</td>
<td>Eurozone</td>
<td>IBOXX-E-A</td>
</tr>
<tr>
<td>7</td>
<td>IBOXX EURO CORP. AA RATED ALL MATS.</td>
<td>Euro</td>
<td>Eurozone</td>
<td>IBOXX-E-AA</td>
</tr>
<tr>
<td>8</td>
<td>IBOXX EURO CORP. AAA RATED ALL MATS.</td>
<td>Euro</td>
<td>Eurozone</td>
<td>IBOXX-E-AAA</td>
</tr>
<tr>
<td>9</td>
<td>IBOXX EURO CORP. BBB RATED ALL MATS.</td>
<td>Euro</td>
<td>Eurozone</td>
<td>IBOXX-E-BBB</td>
</tr>
</tbody>
</table>

Table 6.1: Overview to the 9 bonds

The market prices were recorded from 12/29/2000 to 12/14/2009, what gives us 2,336 data points. As for the exchange rates data set an ARMA(1,1)-GARCH(1,1) model was fitted to each of the time series to obtain the independent uniformly distributed copula data. For more details, see ?. The time series are displayed in Figure ??.

At first glance it is striking that the graphs show similar patterns. This is of course not surprising, given that European states are economically very closely linked. How strong the correlations of the different bonds really are becomes even more clear in the pairsplot in Figure ?? . The empirical Kendall’s τ values are for all pairs over 0.5, in the most cases even significantly higher.
Figure 6.1: Fixed income indices from 12/29/2000 to 12/14/2009.
Figure 6.2: Pairs-plot for the Bond Returns data set (copula data). The corresponding empirical Kendall’s $\tau$ values are shown below the diagonal.
6.2 Application of the 5 strategies on the bond returns data set

Now we will examine how well our 5 strategies perform for this highly dependent real life data set. An overview to the models found by our strategies for the bond returns data set are shown in Figure ??, The complete models including all parameters can be found in Figure ?? on page ??, For explanation of the numbering of the copula families and the calculation of the theoretical Kendall’s $\tau$ values, see Tables 5.2 and 2.2.1 on page 9, In the following analyses we will follow the procedure of application 1. We will consider the found vine structures, the chosen copula families and the theoretical Kendall’s $\tau$ values.
### Bonds

### Application 2: Bond Returns Dataset

#### Figure 6.3: Overview to the 5 models found by our strategies for the bond returns data set and the corresponding theoretical Kendall’s $\tau$ values

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Structure</th>
<th>Truncation: AIC</th>
<th>LogLik</th>
<th>Vuong</th>
<th>Kendall’s $\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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86 Application 2: Bond Returns Dataset

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6 Application 2: Bond Returns Dataset
6 Application 2: Bond Returns Dataset

6.2.1 Structure Investigation

The first two trees of the models found by our five strategies are drawn in the Figures ?? and ?? . The whole vine structure matrices can be found in Figure ?? . As for the exchange rates data set, the respective copula family choice and the empirical Kendall’s \( \tau \) value are also shown for each edge. ”t” denotes the Student-t copula, ”indep” the independence copula. Other copula family choices are not made by our strategies for the bond returns data set in the first two trees. For a better overview we will not write out the bond names in full, but use the variable numeration instead. Remember:

1 = IBOXX-G-3-5  6 = IBOXX-E-A
2 = IBOXX-G-7-10  7 = IBOXX-E-AA
3 = IBOXX-E-1-3  8 = IBOXX-E-AAA
4 = IBOXX-E-5-7  9 = IBOXX-E-BBB
5 = IBOXX-E-10+

T1-Strategy 1

T1-Strategy 2

T1-Strategy 3
Figure 6.4: First trees of the models found by our five strategies for the bond returns data set.
T2-Strategy 1

13 \rightarrow t -0.35 \rightarrow 14 \rightarrow t -0.39 \rightarrow 24 \rightarrow t -0.21 \rightarrow 47 \rightarrow t 0.27 \rightarrow 78 \rightarrow t 0.05 \rightarrow 67 \rightarrow t -0.16 \rightarrow 69

T2-Strategy 2

12 \rightarrow t 0.72 \rightarrow 23 \rightarrow t 0.69 \rightarrow 35 \rightarrow t 0.62 \rightarrow 34 \rightarrow indep. 0 \rightarrow 47 \rightarrow indep. 0 \rightarrow 67 \rightarrow t -0.16 \rightarrow 69 \rightarrow t 0.54 \rightarrow 89

T2-Strategy 3

24 \rightarrow t 0.58 \rightarrow 23 \rightarrow t 0.53 \rightarrow 36 \rightarrow t 0.48 \rightarrow 35 \rightarrow t 0.61 \rightarrow 57 \rightarrow t -0.21 \rightarrow 17 \rightarrow t 0.86 \rightarrow 78
Figure 6.5: Second trees of the models found by our five strategies for the bond returns data set.
6 Application 2: Bond Returns Dataset

Again we note that in some cases very different vine structures were found. On the other hand the same first tree was found by the Strategies 1 and 4. This is surprising at first sight since Strategy 4 showed severe problems in the tree structure selection in the exchange rates data set as well as in the most scenarios of the simulation study. But we have already seen in the simulation study that high correlations lead to a better goodness of fit. Therefore the weights of strategies based on correlations and the weights of strategies based on goodness of fit indicators seem to lead to similar conclusions in highly correlated data sets. However, the models found by the Strategies 1 and 4 are only equal in the first tree. But we have already seen that the first tree has the biggest influence on the goodness of fit.

Also the structures of the second trees seem to be quite similar. The second trees of the Strategies 1, 3, 4 and 5 have all the same shape and the second tree of Strategy 2 has also only 1 different edge. However, the nodes represent different variable pairs, whereby the trees are different again, even if they appear similar at first glance. The second trees of the models found by the Strategies 1 and 4 are an exception of course. Since the first trees are even, the second trees obviously also show up the same variable pairs in the most cases.

Overall it can be determined that the strategies lead to different vine structures also in the bond returns data set. However, the results of the strategies with weights based on correlations and the strategies with weights based on goodness of fit indicators seem to come closer.

6.2.2 Copula family investigation

In this subsection we will take a closer look at the copula family choices of the various strategies in the bond returns data set. As already mentioned in the previous subsection, in the first two trees are exclusively t-copulas and independence copulas chosen. This continues in the higher trees. Only very occasionally Gumbel copulas or its rotated versions occur in the copula family matrices. One Gumbel copula is chosen by the Strategies 4 and 5 each, and two rotated Gumbel copulas are chosen by Strategy 3. All the other variable pairs are modelled by a t- or an independence copula.

This also confirms another observation from the first application: High correlations are in the most cases modelled by the Student-t copula. Therefore is it easy to understand that many t-copulas occur in this highly dependent data set.

Even the appearance of some independence copulas already in the lower trees can be traced back to the high dependencies. This sounds contradictory, but if for example 3 variables are highly dependent, it is quite possible that 2 variables conditioned on the third variable are almost independent.

Caused by the very strong dependencies there are therefore almost exclusively Student-t and independence copulas chosen in the bond returns data set.
6.2.3 Investigation of corresponding Kendall’s $\tau$ estimates using the functional relationship between parameters and Kendall’s $\tau$

In this subsection we will consider the theoretical Kendall’s $\tau$ values of the found models to check if the results are structured and can be classified in the categorization of the simulation study. In contrast to the exchange rates data set it is here not possible to divide into bad models found by strategies with weights based on a goodness of fit indicator and good models found by strategies with weights based on a correlation measure.

Of all the models the ones found by strategies with weights based on a goodness of fit indicator show quite structured patterns here. Of course there are some higher correlations in the upper trees and some weak correlations in the lower trees, but some outliers always occur. Especially Strategy 4 surprises with a well structured monotonically decreasing correlation structure. This could not have been expected due to the bad results in the simulation study and the exchange rates data set. On the other hand this shows, that the Strategies 2 and 4 particularly benefit from the very high correlations in the bond results data set. As already seen before high correlations lead to a higher goodness of fit whereby strategies with weights based on a goodness of fit indicator can turn this into their advantage.

The models found by the strategies with weights based on correlations do not show such a consistent behaviour. The best structured model is again found by Strategy 1: Almost monotonically decreasing correlations with nearly no outliers. This confirms the good results of Strategy 1 in the exchange rates data set. Even if this strategy lagged a bit behind in the simulation study, it shows its strengths in real life data sets. The models found by the Strategies 3 and 5 on the other hand are not that good. An monotonically decreasing correlation structure is noticeable also here, but the number of deviating values is much higher. Strategy 5 shows even in the 8th tree a theoretical Kendall’s $\tau$ value of 45.

Unlike in the exchange rates data set we have a rough imagination of the correlation structure of the true model here. As we have seen in Figure ?? all variable pairs are strongly correlated and in the previous section we have seen that almost all pairs are modelled by Student-t copulas. We can therefore classify the bond results data set as an elliptical scenario with constant strong correlations (Scenarios 15 and 19 in the simulation study). It is interesting to note that all strategies are trying to construct a model with monotone decreasing correlations. Consequently, the models of all strategies will underestimate the correlations of many variable pairs, but this will be considered in a later investigation. What also becomes clear, however, is that the results for real life data sets do not always match the outcomes of the simulation study. In the Scenarios 15 and 19 of the simulation study, the Strategies 3 and 2 were preferred, while the Strategies 1, 4 and 5 did not perform that well. For the bond results data set though, the Strategies 1, 2 and 4 delivered the best structured correlation matrices as we have seen above. So as in the exchange rates data set it turns out that the simulation study is
not able to cover everything and take all factors into account. Overall, however, Strategy 1 seems to deal best with the irregular behaviour of real life data sets.

6.3 Goodness of fit of the found models

After we described the models found by the five strategies for the bond returns data set we will now take a look at the goodness of fit of the models. Like in the previous sections, we will follow the procedure of the exchange rates data set and start with the loglikelihoods. However, a first impression of the goodness of fit of the fitted copulas is given in Table ?? on page ??, which shows the p-values of the Cramer-von-Mises tests. As for the exchange rates data set, the p-value are very high in most of the cases and we can therefore notice a good fit of all strategies also here.

6.3.1 Loglikelihood results

The loglikelihood values are the most suitable starting point since they give us a vivid overview to the general goodness of fit level. The values are displayed in Table ??.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>based on ...</th>
<th>LogLikelihood</th>
<th># Parameter</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Kendall’s (\tau)</td>
<td>30084.00</td>
<td>59</td>
<td>-60050.01</td>
</tr>
<tr>
<td>2</td>
<td>p-value</td>
<td>29915.80</td>
<td>58</td>
<td>-59715.60</td>
</tr>
<tr>
<td>3</td>
<td>partial correlations</td>
<td>29626.99</td>
<td>70</td>
<td>-59113.98</td>
</tr>
<tr>
<td>4</td>
<td>AIC</td>
<td>29882.29</td>
<td>56</td>
<td>-59652.57</td>
</tr>
<tr>
<td>5</td>
<td>Kendall’s (\tau) × p-value</td>
<td>29807.72</td>
<td>61</td>
<td>-59493.44</td>
</tr>
</tbody>
</table>

Table 6.2: Overview to the goodness of fit indicators for the models found by the strategies for the bond results data set

It is striking that the loglikelihood level is way higher than for the exchange rates data set. While the values in those models had a level of around 2,200 we now reach the 30,000. This shows once again how important high correlations for the strategies are. Furthermore the percentage values are again very close together. As in the exchange rates data set the smallest value (29626.99 for Strategy 3) and the highest value (300084.00 for Strategy 1) are not even 2 percent apart. This confirms the result that all strategies are able to find suitable models and only littlenesses decide the ranking which is shown in Table ??.

<table>
<thead>
<tr>
<th>LogLik-Ranking</th>
<th>5th Place</th>
<th>4th Place</th>
<th>3rd Place</th>
<th>2nd Place</th>
<th>1st Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy Value</td>
<td>29626.99</td>
<td>29807.72</td>
<td>29882.29</td>
<td>29915.80</td>
<td>30084.00</td>
</tr>
</tbody>
</table>

Table 6.3: Loglikelihood ranking for the exchange rates data set

We notice that the ranking follows the conclusions of the theoretical Kendall’s \(\tau\) investigation of the last section. The Strategies 1, 2 and 4, which delivered the best
structured models, are on the first places. The Strategies 3 and 5, which had quite unstructured models, take the 4th and 5th place.

6.3.2 Number of Parameters and AIC results

In this section we will additionally take the model complexity into account. Therefore we consider the number of parameters, that the various strategies need, and the related AIC values. The definition of the AIC can be found on page 10.

As we can see in Figure ?? not only the loglikelihood level rises compared to the models in the exchange rates data set, also the number of parameters goes up. Largely this is due to the large number of Student-t copulas which have 2 parameters. Strategy 1 nearly doubles its number of parameters from 33 in the exchange rates data set to 59 (+26) in the bond returns data set. Overall, however, the differences in the number of parameters between the various strategies becomes smaller. They lie between 56 and 61. Only Strategy 3 needs with 70 significantly more parameters like in the exchange rates data set. We use the AIC values now to create a ranking which takes the model complexity into account.

<table>
<thead>
<tr>
<th>AIC-Ranking</th>
<th>5th Place</th>
<th>4th Place</th>
<th>3rd Place</th>
<th>2nd Place</th>
<th>1st Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Value</td>
<td>-59113.98</td>
<td>-59493.44</td>
<td>-59652.57</td>
<td>-59715.60</td>
<td>-60050.01</td>
</tr>
</tbody>
</table>

Table 6.4: AIC ranking for the bond returns data set

We note that the order of the AIC ranking did not change compared to the loglikelihood ranking. This is because the number of parameters is negligible in comparison with the huge loglikelihood values. Accordingly the statements of the loglikelihood analysis regarding the goodness of fit of the found models remain unchanged.

6.3.3 Analysis of pairwise key characteristics

In this subsection we will examine if the found models are able to reflect some key figures of the original bond returns data set. We consider the empirical Kendall’s $\tau$ values as well as the lower and upper tail cumulations. We therefore calculate these values for the original bond returns data set to obtain the comparative values. Then we simulate 100 data sets from each of the models found by our strategies in the size of the original data set (2337 data points), calculate the respective key figures for each of them and present the results in boxplots. For further descriptions to the procedure, see Subsection ?? on page ??.. Additionally the lower and upper tail exceedance correlations are displayed in the Figures ?? - ?? in Appendix ?? as for the exchange rates data set.

We start with the investigations of the empirical Kendall’s $\tau$ values in the Figures ?? - ??.. We note that the original values are not so well reproduced as for the exchange rates data set. All models seem to underestimate the Kendall’s $\tau$ values and the higher
the tree, the more apparent this phenomenon occurs. Already in the investigation of the theoretical Kendall’s τ values in the last section we noticed that all models show a monotonically decreasing correlation structure while the original data set clearly exhibits constant strong correlations. Now the suspicion gets confirmed that the parameters were not adjusted properly. In almost all variable pairs of all strategies the true empirical Kendall’s τ value (red line in the boxplots) is over the corresponding values of the simulated data sets. This is particular evident in Figure ?? for Strategy 4. An overview to the theoretical Kendall’s τ values and the corresponding empirical values of the corresponding variable pairs of the original bond returns data set is given in Figure ???. Of course you can not compare these values one by one since the theoretical correlations are conditioned, but it gives a first impression.

This also confirms a result from the simulation study. We already recognized there that strong correlations in the higher trees lead to problems in the parameter estimation. This became evident due to the significant loglikelihood drop from the true model to the model found by the ”Strategy” TTE (Tree structure and copula family choices from the true model and estimated parameters). Now we see, that the problems are caused by underestimating the correlations in the higher trees. Nevertheless the overall loglikelihood values in scenarios with strong correlations were very high.

Considering the lower tail cumulations in the Figures ?? - ?? reveals a similar picture. The original values are distinctly higher than the values of the simulated data sets. This indicates that the original data set has a clearly heavier lower tail than it is modelled by our strategies. This is apparent for all strategies. These differences may occur due to the copula family choices. As we have already seen nearly all variable pairs are modelled by the Student-t copula. They were chosen since the correlations are very high but it is an elliptical copula and therefore not that suitable to model heavy or light tails. It might be useful to admit more copula families to have more options to model the strong dependencies as well as the tail behaviours. In the case of the lower tail cumulations, our copula family selection seems not to be sufficient.

When we consider the upper tail cumulations in the Figures ?? - ?? on the other hand, we obtain the reverse picture. The upper tail cumulations seems to get overestimated by our models in most of the cases, though not to the same extent. But not only the weight of the upper tail gets overestimated here. The figures of the upper tail exceedance correlations show that also the strength of the correlations in the upper tail get significantly overestimated by all strategies. Again, we can assume that the weaknesses in the modelling are due to the Student-t copulas.

### 6.3.4 Vuong statistics

Finally we consider the statistical significance of the differences in the goodness of fit between the models. As for the exchange rates data set, we will use the Vuong test to determine the statistical significance of the differences and take the model complexity into account with the aid of the Akaike corrections. The test statistics and their
Figure 6.6: Comparison of the theoretical Kendall’s $\tau$ values of the models found by our strategies and the corresponding empirical values of the original bond results data set.
corresponding p-values are shown in Table ??.

\[
\begin{array}{cccc}
\text{Vuong statistics with Akaike corrections} \\
\begin{array}{cccc}
\text{Strat1} & \text{Strat2} & \text{Strat3} & \text{Strat4} \\
4.82 & 9.63 & 7.20 & 6.89 \\
5.51 & 0.87 & 3.03 & 3.03 \\
-5.03 & -3.39 & 6.89 & 6.89 \\
2.09 & 2.09 & 2.09 & 2.09
\end{array}
\end{array}
\]

\[
\begin{array}{cccc}
\text{p-values} \\
\begin{array}{cccc}
\text{Strat1} & \text{Strat2} & \text{Strat3} & \text{Strat4} \\
1.45E-06** & 3.50E-08** & 4.80E-07** & 6.46E-12** \\
0** & 0.3856** & 0.0007* & 0.0007* \\
5.88E-13** & 0.0025** & 0.0363** & 0.0363**
\end{array}
\end{array}
\]

Table 6.5: Vuong statistics and corresponding p-values for the bond results data set

The percentage differences of the loglikelihood and AIC values between the various models are low, but because of the very high level of these values, the absolute differences are large. The loglikelihood values of Strategy 1 (best in terms of loglikelihood and AIC) and Strategy 3 differ by over 450 points. It is therefore not surprising that the test are all statistically significant with the exception of the test "Strategy 1 vs. Strategy 4". In the most cases the test are even highly significant. Remember: If the absolute test statistics $|\nu| > \Phi^{-1}(1 - \alpha/2) \approx 1.96$ (respectively the p-value is smaller than 0.05) the test is statistically significant to the 5% confidence level. For more detailed informations see Definition 2.11 on page 22 or the explanations in the corresponding subsection for the exchange rates data set on page ??.

We create a ranking on the basis of the Vuong tests with Akaike corrections which is displayed in Table ??.

<table>
<thead>
<tr>
<th>Vuong-Ranking</th>
<th>5th Place</th>
<th>4th Place</th>
<th>2nd Place</th>
<th>1st Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>3</td>
<td>5</td>
<td>4 &amp; 2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 6.6: Vuong test ranking for the bond returns data set

That "4 & 2" are on the second place means that the models are equally good. The Vuong statistics of the test "Strategy 2 vs Strategy 4" is positive, what indicates that Strategy 2 is better, but the difference is not statistically significant.

As expected, this ranking confirms the order of the loglikelihood- and AIC-rankings. So now we know that that differences are statistically significant. But since the percentage differences are very low and the absolute values of the goodness of fit indicators are very high, this does not change the fact that all strategies provide similar good results.
6.4 Truncation

In this section we consider the possibility to truncate the found models for the bond results data set. An overview to the sequentially truncated models is given in Table 6.7.

For explanation of the procedure see Section ?? - the corresponding investigation for the exchange rates data set.

As we can see the models are truncated later than for the exchange rates data set. The models found by the Strategies 2, 3 and 5 cut off not until the second last tree based on the Vuong test. All models get not truncated until the maximum AIC- respectively loglikelihood-value is reached. This is the case when all higher trees are independent, i.e. all variable pairs in the higher trees are modelled by the independence copula.

This very late truncation may also be traced back to the high AIC-value level of the models. Even if the AIC-values have reached almost 100% in terms of percentage in lower trees, the absolute values are still so large that the Vuong test detects the differences to be significant. We see for example in the model found by Strategy 2 that the rounded percentage AIC-value already reaches 100% of the maximum possible value after the 5th tree, but the model is not truncated until the 8th tree.

A special feature reappears for Strategy 4. Its model is truncated at the base Vuong tests after the 5th tree, although the addition of the trees 7 and 8 would lead to significant improvements. So even though Strategy 4 offers better results in the lower trees for the bond returns data set, the AIC-values exhibit weaknesses as weights in the tree structure selection.

It is also striking that the models of strategies with weights based on goodness of fit indicators start with way higher percentage AIC-values than their models for the exchange rates data set. The model of Strategy 2 reaches already 65% of the maximum AIC-value in the first tree, while its first tree of the model for the exchange rates data set had only 21%. The largest improvement, however, makes Strategy 4: Its model starts with 90% of the maximum AIC-value in the first tree compared to 30% of its model for the exchange rates data set. But of course we expected a good performance of Strategy 4 in this investigation since it has the same first tree like Strategy 1. Nevertheless, it gets confirmed once again that strategies with weights based on goodness of fit indicators benefit most from the constant strong correlations in the tree structure selection.

---

Table 6.7: Overview of the truncation possibilities for the bond results data set

<table>
<thead>
<tr>
<th>Tree</th>
<th>Strategy 1</th>
<th></th>
<th>Strategy 2</th>
<th></th>
<th>Strategy 3</th>
<th></th>
<th>Strategy 4</th>
<th></th>
<th>Strategy 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AIC</td>
<td>% Vuong</td>
<td></td>
<td>AIC</td>
<td>% Vuong</td>
<td></td>
<td>AIC</td>
<td>% Vuong</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-60056</td>
<td>100%</td>
<td>0</td>
<td>-59716</td>
<td>100%</td>
<td>0</td>
<td>-59114</td>
<td>100%</td>
<td>-2.8</td>
</tr>
<tr>
<td>8</td>
<td>-60056</td>
<td>100%</td>
<td>0</td>
<td>-59598</td>
<td>-3.6</td>
<td>-59671</td>
<td>100%</td>
<td>-2.7</td>
<td>-59662</td>
</tr>
<tr>
<td>7</td>
<td>-60056</td>
<td>100%</td>
<td>0</td>
<td>-59618</td>
<td>100%</td>
<td>-3.7</td>
<td>-59626</td>
<td>100%</td>
<td>-3.1</td>
</tr>
<tr>
<td>6</td>
<td>-59968</td>
<td>100%</td>
<td>-5.2</td>
<td>-59540</td>
<td>-8.6</td>
<td>-57695</td>
<td>98%</td>
<td>-15.1</td>
<td>-59220</td>
</tr>
<tr>
<td>5</td>
<td>-59910</td>
<td>100%</td>
<td>-10.4</td>
<td>-59431</td>
<td>-11.8</td>
<td>-59418</td>
<td>98%</td>
<td>-13.7</td>
<td>-59151</td>
</tr>
<tr>
<td>4</td>
<td>-59223</td>
<td>99%</td>
<td>-13.3</td>
<td>-59332</td>
<td>-20.9</td>
<td>-58482</td>
<td>98%</td>
<td>-24.9</td>
<td>-58583</td>
</tr>
<tr>
<td>3</td>
<td>-57801</td>
<td>96%</td>
<td>-20.4</td>
<td>-57136</td>
<td>-87%</td>
<td>-54915</td>
<td>81%</td>
<td>-25.2</td>
<td>-57333</td>
</tr>
<tr>
<td>2</td>
<td>-53564</td>
<td>89%</td>
<td>-38954</td>
<td>69%</td>
<td>-38294</td>
<td>69%</td>
<td>-33964</td>
<td>90%</td>
<td>-38569</td>
</tr>
</tbody>
</table>

---

98
Strategy 1 starts with 89% as well as for the exchange rates data set (90%), while Strategy 3 has also a better start (48% for the exchange rates data set \(\Rightarrow\) 65% for the bond returns data set). The only Strategy that loses percentage points in the first tree is Strategy 5: Its first tree drops from 84% to 66%.

### 6.5 Summary and conclusions

In this section we will summarize the results of the previous investigations for the bond results data set:

- **All models show extremely high loglikelihood (respectively low AIC) levels**
  
  Caused by the very high correlations in the bond results data set, all models show a extraordinary good fit. As we have already seen in the simulation study, all strategies benefit from strong correlations and the bond returns data set shows constant strong correlations as can be seen in Figure ?? on page ???. The loglikelihood level is around 30,000 while we had only a level of around 2,200 for the exchange rates data set.

- **All strategies show similar good fit**
  
  As for the exchange rates data set the values of the goodness of fit indicators do not even differ by 2%. Nevertheless, the Vuong tests recognizes these differences to be significant, but this is due to the very high loglikelihood level. Even if the percentage differences are small, the absolute differences in the loglikelihood values are large and thus the Vuong tests consider these differences to be statistically significant.

- **Correlation measures and goodness of fit indicators as weights lead to similar conclusions in the tree structure selection for highly dependent data sets**
  
  We have already seen that high correlations lead to a better goodness of fit of the fitted copula. Therefore both types of weights lead to the same conclusions and Strategies with weights based on goodness of fit indicators and Strategies with weights based on correlations get similar results. Strategy 1 (based on Kendall’s \(\tau\)) and Strategy 4 (based on AIC) have even the same first tree for the bond returns data set.

- **Strategies with weights based on goodness of fit indicators benefit most from the very high correlations in the bond returns data set**
  
  While strategies with weights based on goodness of fit indicators lagged clearly behind the strategies with weights based on correlations in the exchange rates data set, they can close the gap in this high correlated data set. As we have seen in the last point, high correlations lead to a better goodness of fit. Therefore clearer decisions can be made in the tree structure selection due to the heavier weights.

- **Best structured model delivered by Strategy 1 again**
  
  The model found by Strategy 1 shows almost monotonically decreasing
correlations with nearly no outliers. But also the models found by the Strategies 2
and 4 (strategies with weights based on goodness of fit indicators) show quite
structured models in this highly dependent data set. Strategies 3 and 5 on the
other exhibits some weaknesses here. An monotonically decreasing correlation
structure is noticeable, but the number of deviating values is much higher.

- **Model structures decide the rankings**
The more structured the correlation structure of a model is, the better is the
goodness of fit. Both in the exchange rates data set as well as for the bond returns
data set, the Strategies with the best structured models, get the first places in the
rankings. For the bond returns data set this means: Strategy 1 delivers the best
structured models and is on the first place, followed by the Strategies 2 and 4. The
Strategies 3 and 5, which delivered not that structured models, are on the 5th and
4th place.

- **High correlations get underestimated by all strategies**
Already in the simulation study we have seen that the strategies have problems
with the parameter estimation in scenarios with strong dependencies in the higher
trees. In the bond returns data set we recognized that all strategies underestimate
the strong dependencies especially in the higher trees. So although the goodness of
fit of the found models is very good, some key figures are not reproduced perfectly.

- **All strategies have problems to reproduce the tail behaviour**
Other key figures that are not reproduced perfectly are the lower and upper tail
cumulations. We noticed that strong correlations are often modelled by the
Student-t copula whereby many of them occur in the highly dependent bond
returns data set. However, the Student-t copula is an elliptical copula and therefore
perhaps not able to model the tail behaviour appropriate. The original data set
has a clearly heavier lower tail than it is modelled by our strategies and the upper
tail cumulations get overestimated on the other hand. Perhaps it might be helpful
to allow more copula types here.

- **High parameter cost for all strategies**
Since all models exhibit a lot of Student-t copulas, which have two parameters,
and just a few independence copulas or other copula types, the number of
parameters is much higher than for the exchange rates data set. Some strategies
nearly double their number of parameters. The numbers lie close together for the
strategies with values between 56 and 61. Only Strategy 3 needs significantly more
(70) as in the exchange rates data set.
7 Conclusions and Outlook

The aim of this thesis was to compare 5 different weights for the tree structure selection and to find out, what their respective strengths and weaknesses are. The investigation of the tree structure selection is particularly important since it has the greatest influence on the goodness of fit as we have found out. Copula family choices and parameter estimation play a subordinate role. Especially the tree structure of the lower trees have an enormous influence on the overall goodness of fit of the model. However, we had to notice that the simulation study and both applications came to different results. In the simulation study, Strategies 2 (p-value) and 5 (Kendall’s $\tau \times$ p-value) were preferred in the mixed scenarios and Strategies 3 (partial correlations) and 2 were preferred in the elliptical scenarios. In the applications on the other hand, Strategy 1 (Kendall’s $\tau$) performed best, followed by the Strategies 3 and 5 in the exchange rates data set and the Strategies 2 and 4 (AIC) in the bond returns data set. One weak point of the simulation study certainly was that the correlations structure was always pure, i.e. constant weak/strong correlations or monotonically increasing/decreasing correlations. Real life data sets obviously do not exhibit such pure structures. Strategy 1, however, seems to be particularly stable against deviations in the pure structure.

But it was also eye-catching that all strategies performed about equally well. In the applications the percentage AIC-values of the different strategies were never more than 2% apart. Nevertheless, the found R-vine specifications are completely different in most of the cases. This may not be surprising at the first glance since 7 already showed that there are $\frac{n!}{2^n} \times 2^{\left(2^{n-2}-2\right)}$ possible regular vines on n nodes. But it is surprising that these completely different models are so close together in terms of goodness of fit. We also found out that high correlations for a variable pair lead to a better goodness of fit of the fitted copula. Therefore correlation measures and goodness of fit indicators as weights lead to similar conclusions in the tree structure selection for highly dependent data sets. In the bond returns data set, which showed constant strong correlations, the Strategies 1 and 4 even chose the same first tree.

So overall, we can present no clear winner among the strategies. It seems that for each strategy there are scenarios in which it performs particularly well. No strategy revealed any real bad results and since we do not know the specification of the true model in real life data sets it is particularly important to know, that all strategies are able to find suitable models. All strategies found models that were able to reproduce some key figures of the original data set like the empirical Kendall’s $\tau$ values, the lower and upper tail cumulations and the lower and upper exceedance Kendall’s $\tau$. If we really had to emphasize one strategy, it would be Strategy 1. It seems to be quite stable against deviations in the pure correlation structure as mentioned above, it performed best in the real life data sets and it has one of the cheapest algorithms.

Strategy 3 has to be briefly considered separately: The special feature of this strategy is
that the partial correlations as its weights are theoretically only justified for elliptical distributions, but we also admitted non-elliptical copulas. We therefore had to check if this "error" has an noticeable influence. We recognized in the simulation study, that Strategy 3 indeed performed better, and even best, in the elliptical scenarios. In the mixed scenarios on the other hand Strategy 3 was ranked on the middle places. Also in the applications Strategy 3 showed us two different pictures. In the exchange rates data set it performed quite well compared to the other strategies and got the second place in the rankings. In the bond returns data set, though, it was ranked on the last place. However, we can say that Strategy 3 can compete with the other strategies also without theoretical justification of its weight.

Let’s turn to the question now, how these studies could be continued. One possibility is certainly to extend the settings of this thesis. We saw in the applications that the 20 scenarios of the simulation study were not able to cover all possible data set structures. Also the number of admitted copula types could be increased to have more options to improve the goodness of fit of the models and to be able to reproduce key figures of the original data set better. Furthermore we always used the whole original data sets for the model selection. Another approach is to use just a part of the data set for the model selection and the whole data set for the goodness of fit determination. This approach has the benefit that we can also make better statements to the predictive capability of the found models. Of course it is also possible to move away from the vines and to use other graphical models instead. So this thesis may be a corner stone for further investigations in many directions.
A Simulation Study Data

A.1 General Data

Figure A.1: RVine-matrix for the simulation study

On the following pages the copula family-, Kendall’s \( \tau \)- and 2nd parameter-matrices for each of the 20 scenarios of our simulation study are shown.

Overview of the numbering of the copula families:

- \( 0 \) = independence copula
- \( 1 \) = Gaussian copula
- \( 2 \) = Student t copula
- \( 4 \) = Gumbel copula
- \( 14 \) = rotated Gumbel copula (180 degrees; ”survival Gumbel”)
- \( 24 \) = rotated Gumbel (90 degrees)
- \( 34 \) = rotated Gumbel (270 degrees)
### Scenario 1:

<table>
<thead>
<tr>
<th>U</th>
<th>V</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
</tbody>
</table>

<table>
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A Simulation Study Data
## Scenarios:

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## Scenario: 6

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## Scenario: 7

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### Scenario 9

<table>
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<th>21424134</th>
<th>24243442</th>
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</thead>
<tbody>
<tr>
<td>Kendall's Rank Correlation</td>
<td>0.2</td>
<td>-0.25</td>
<td>0.3</td>
<td>-0.43</td>
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<td>-0.9</td>
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### Scenario 10

<table>
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<th>1242421</th>
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<th>24243442</th>
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</thead>
<tbody>
<tr>
<td>Kendall's Rank Correlation</td>
<td>0.35</td>
<td>-0.4</td>
<td>0.31</td>
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<td>0.35</td>
<td>-0.3</td>
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### Scenario 11

<table>
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<tr>
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### Scenario 12

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<th>24243442</th>
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<td>Kendall's Rank Correlation</td>
<td>0.9</td>
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<td>0.61</td>
<td>0.58</td>
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### P2

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A Simulation Study Data
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<td>10 6 7 8 4</td>
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<td>0 0 0 0 0</td>
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<td>10 6 7 8 4</td>
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<td>9 7 8 9 6 7</td>
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### Scenario 17:

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<td>1 1 1</td>
<td>0.32, 0.31, 0.3</td>
</tr>
<tr>
<td>1 1 1 1</td>
<td>0.43, 0.39, 0.45, 0.41</td>
</tr>
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<td>2 2 2 2 2 2</td>
<td>0.53, 0.5, 0.54, 0.49, 0.48</td>
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<td>2 2 2 2 2 2 2</td>
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</tr>
<tr>
<td>2 2 2 2 2 2</td>
<td>0.9, -0.85, 0.92, 0.87, 0.9, 0.88, 0.91</td>
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### Scenario 18:

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<td>2 2 2 2 2 2</td>
<td>0.3, -0.36, 0.35, 0.35, 0.29, 0.41, 0.33</td>
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### Scenario 19:

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<td>2 2 2 2 2 2</td>
<td>0.59, 0.69, 0.65, -0.66, 0.67, 0.68</td>
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<td>2 2 2 2 2</td>
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### Scenario 20:

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<td>1 1 1</td>
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<td>1 1 1 1</td>
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<td>2 2 2 2 2 2</td>
<td>0.28, 0.31, 0.33, -0.29, 0.26, 0.32</td>
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### Scenario 21:

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### Scenario 22:

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<tr>
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<tr>
<td>2 2 2 2 2 2</td>
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### Scenario 23:

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<td>1 1 1</td>
<td>0.61, 0.64, 0.67</td>
</tr>
<tr>
<td>1 1 1 1</td>
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<td>0.65, 0.66, 0.63, 0.67</td>
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<tr>
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<td>0.59, 0.69, 0.65, -0.66, 0.67, 0.68</td>
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<tr>
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### Scenario 24:

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</tr>
<tr>
<td>1 1 1 1</td>
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</tr>
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<tr>
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<td>0.28, 0.31, 0.33, -0.29, 0.26, 0.32</td>
</tr>
<tr>
<td>2 2 2 2 2 2</td>
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### Scenario 25:

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</tr>
<tr>
<td>1 1 1</td>
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</tr>
<tr>
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A Simulation Study Data

A.2 Vuong statistics for simulation study
A Simulation Study Data

Scenario 1
A Simulation Study Data

![Simulation Study Data](image-url)
A Simulation Study Data

Scenario 2
A Simulation Study Data
A Simulation Study Data

Scenario 3
A Simulation Study Data

Scenario 4
A Simulation Study Data
A Simulation Study Data

Scenario 5
A Simulation Study Data
A Simulation Study Data

Scenario 6
A Simulation Study Data
A Simulation Study Data

Scenario 7
A Simulation Study Data
A Simulation Study Data

Scenario8
A Simulation Study Data

![Simulation Study Data Plots]

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A Simulation Study Data
A Simulation Study Data

Scenario 10
A Simulation Study Data
A Simulation Study Data

Scenario 11
A Simulation Study Data
Scenario 12
A Simulation Study Data
A Simulation Study Data

Scenario 13
A Simulation Study Data

Scenario 14
A Simulation Study Data
A Simulation Study Data
A Simulation Study Data

Scenario 16
A Simulation Study Data

Scenario 17
A Simulation Study Data
A Simulation Study Data

Scenario 18
A Simulation Study Data
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Scenario 19
A Simulation Study Data
A Simulation Study Data

Scenario 20
A Simulation Study Data
## B Applications

### B.1 Exchange rates

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Table B.1: P-values of the Cramer-von-Mises tests for the bivariate copulas of the models found by the five strategies for the exchange rates data set. For independence copulas no p-values are displayed. Remember: A p-value $\leq 0.05$ indicates a lack of fit of the bivariate copula.

Remark to Table ??: These p-values are not exact, since it is only a approximate solution of the fast multiplier approach developed by ? (see also Definition 2.2.3). Since the computation time grows very fast with the number of iterations of the fast multiplier approach, the number is kept very small. Therefore these values can just give a first overview to the goodness of fit of the bivariate copulas. When we repeat the computation-algorithm, the results can differ significantly from the above values.
### Applications

#### Exchange Rates

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#### Strategies 2

| Figure B.1: Overview to the 5 models found by our strategies for the exchange rates data set including the parameters | }

**Figure B.1:** Overview to the 5 models found by our strategies for the exchange rates data set including the parameters
Exchange-Rates Dataset: Kendall’s τ values

Figure B.2: Kendall’s τ values of 100 data sets simulated from the R-Vine Matrix found by Strategy 1 for the Exchange-Rates data set. The red lines represent the true values of the original data set.

Figure B.3: Kendall’s τ values of 100 data sets simulated from the R-Vine Matrix found by Strategy 2 for the Exchange-Rates data set. The red lines represent the true values of the original data set.
Figure B.4: Kendall’s $\tau$ values of 100 data sets simulated from the R-Vine Matrix found by **Strategy 3** for the Exchange-Rates data set. The red lines represent the true values of the original data set.

Figure B.5: Kendall’s $\tau$ values of 100 data sets simulated from the R-Vine Matrix found by **Strategy 4** for the Exchange-Rates data set. The red lines represent the true values of the original data set.
Figure B.6: Kendall's $\tau$ values of 100 data sets simulated from the R-Vine Matrix found by Strategy 5 for the Exchange-Rates data set. The red lines represent the true values of the original data set.
Exchange-Rates Dataset: Lower Tail cumulations

Figure B.7: Lower tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 1 for the Exchange-Rates data set. The red lines represent the true values of the original data set.

Figure B.8: Lower tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 2 for the Exchange-Rates data set. The red lines represent the true values of the original data set.
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Figure B.10: Lower tail cumulations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 4** for the Exchange-Rates data set. The red lines represent the true values of the original data set.
Figure B.11: Lower tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 5 for the Exchange-Rates data set. The red lines represent the true values of the original data set.
Figure B.12: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 1** for the Exchange-Rates data set. The red lines represent the true values of the original data set.

Figure B.13: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 2** for the Exchange-Rates data set. The red lines represent the true values of the original data set.
Figure B.14: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 3 for the Exchange-Rates data set. The red lines represent the true values of the original data set.

Figure B.15: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 4 for the Exchange-Rates data set. The red lines represent the true values of the original data set.
Figure B.16: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 5 for the Exchange-Rates data set. The red lines represent the true values of the original data set.
Exchange Rates Dataset: Lower tail exceedance correlations

Figure B.17: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 1** for the exchange rates data set. The red lines represent the true values of the original data set.

Figure B.18: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 2** for the exchange rates data set. The red lines represent the true values of the original data set.
Figure B.19: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 3 for the exchange rates data set. The red lines represent the true values of the original data set.

Figure B.20: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 4 for the exchange rates data set. The red lines represent the true values of the original data set.
Figure B.21: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 5 for the exchange rates data set. The red lines represent the true values of the original data set.
Exchange Rates Dataset: Upper tail exceedance correlations

Figure B.22: Upper tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 1 for the exchange rates data set. The red lines represent the true values of the original data set.

Figure B.23: Upper tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 2 for the exchange rates data set. The red lines represent the true values of the original data set.
B Applications

Figure B.24: Upper tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 3 for the exchange rates data set. The red lines represent the true values of the original data set.

Figure B.25: Upper tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 4 for the exchange rates data set. The red lines represent the true values of the original data set.
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### Bonds

#### Figure B.27: Overview to the 5 models found by our strategies for the bond returns data set including the parameters

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Figure B.28: Kendall’s $\tau$ values of 100 data sets simulated from the R-Vine Matrix found by **Strategy 1** for the bond returns data set. The red lines represent the true values of the original data set.

Figure B.29: Kendall’s $\tau$ values of 100 data sets simulated from the R-Vine Matrix found by **Strategy 2** for the bond returns data set. The red lines represent the true values of the original data set.
Figure B.30: Kendall’s $\tau$ values of 100 data sets simulated from the R-Vine Matrix found by **Strategy 3** for the bond returns data set. The red lines represent the true values of the original data set.

Figure B.31: Kendall’s $\tau$ values of 100 data sets simulated from the R-Vine Matrix found by **Strategy 4** for the bond returns data set. The red lines represent the true values of the original data set.
Figure B.32: Kendall’s $\tau$ values of 100 data sets simulated from the R-Vine Matrix found by **Strategy 5** for the bond returns data set. The red lines represent the true values of the original data set.
Figure B.33: Lower tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 1 for the bond returns data set. The red lines represent the true values of the original data set.

Figure B.34: Lower tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 2 for the bond returns data set. The red lines represent the true values of the original data set.
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Figure B.36: Lower tail cumlations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 4** for the bond returns data set. The red lines represent the true values of the original data set.
Figure B.37: Lower tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 5 for the bond returns data set. The red lines represent the true values of the original data set.
Bond Returns Dataset: Upper tail cumulations

Figure B.38: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 1 for the bond returns data set. The red lines represent the true values of the original data set.

Figure B.39: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 2 for the bond returns data set. The red lines represent the true values of the original data set.
Figure B.40: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 3 for the bond returns data set. The red lines represent the true values of the original data set.

Figure B.41: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by Strategy 4 for the bond returns data set. The red lines represent the true values of the original data set.
Figure B.42: Upper tail cumulations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 5** for the bond returns data set. The red lines represent the true values of the original data set.
Bond Returns Dataset: Lower tail exceedance correlations

Figure B.43: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 1 for the bond returns data set. The red lines represent the true values of the original data set.

Figure B.44: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 2 for the bond returns data set. The red lines represent the true values of the original data set.
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Figure B.46: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 4 for the bond returns data set. The red lines represent the true values of the original data set.
Figure B.47: Lower tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 5 for the bond returns data set. The red lines represent the true values of the original data set.
Bond Returns Dataset: Upper tail exceedance correlations

Figure B.48: Upper tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 1** for the bond returns data set. The red lines represent the true values of the original data set.

Figure B.49: Upper tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by **Strategy 2** for the bond returns data set. The red lines represent the true values of the original data set.
Figure B.50: Upper tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 3 for the bond returns data set. The red lines represent the true values of the original data set.

Figure B.51: Upper tail exceedance correlations of 100 data sets simulated from the R-Vine Matrix found by Strategy 4 for the bond returns data set. The red lines represent the true values of the original data set.
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