A Distributed Controller Approach for Delay-Independent Stability of Networked Control Systems ⋆

Sandra Hirche, Tilemachos Matiakis, Martin Buss

Institute of Automatic Control Engineering, Technische Universität München, D-80290 Munich, Germany

Abstract

This article introduces a novel distributed controller approach for networked control systems (NCS) to achieve finite gain $\mathscr{L}_2$ stability independent of constant time delay. The distributed controller is alternatively interpreted as a linear transformation of the input-output space of the controller and the plant, and in fact represents a generalization of the well-known scattering transformation. The main results of this article are a) a sufficient stability condition for general multi-input-multi-output (MIMO) input-feedforward-output-feedback-passive (IF-OFP) nonlinear systems and b) a necessary and sufficient stability condition for linear time-invariant (LTI) single-input-single-output (SISO) systems. The performance advantages in terms of sensitivity to time delay and steady state error are discussed in comparison with alternative delay-independent small gain type approaches. Simulations verify the superiority of the proposed approach to an LQR controller for zero time delay without the proposed transformation and to a delay-independent small gain based controller.

Key words: networked control system (NCS), delay-independent stability, finite gain $\mathscr{L}_2$ stability, input-feedforward-output-feedback passivity

1 Introduction

In networked control systems (NCS) the spatially separated plant and controller are connected through a communication network. (Tipsuwan & Chow 2003, Hristu-

⋆ This paper was not presented at any IFAC meeting. Corresponding author T. Matiakis. Tel. +49-89-289-23404. Fax +49-89-289-28340.

Email addresses: `s.hirche@ieee.org` (Sandra Hirche), `t.matiakis@tum.de` (Tilemachos Matiakis), `m.buss@ieee.org` (Martin Buss).

Varsakelis & Levine 2005, Antsaklis & Baillieul 2007) provide excellent overviews on the field. The motivation for replacing the classical point-to-point control architecture by an NCS also originates from its flexible reconfiguration capabilities: Plant and controller nodes can be added or removed without additional wiring effort. The number of active nodes sharing the communication channel has an effect on the communication parameters in terms of communication time delay, packet loss, and available communication bandwidth. In consequence, these communication parameters are, in general, not exactly known during the controller design.

In this work the problem of unknown, constant time delay is addressed. It is well-known that time delay in a control loop degrades the performance and can lead to instability. Time delay system approaches are classified into delay-dependent and delay-independent approaches, see (Richard 2003, Gu, Kharitonov & Chen 2003) and the references therein for a concise overview and introduction to the rich literature of time delay systems. While mostly state-space approaches are considered, the input-output approaches in (Georgiou & Smith 1992, Bonnet & Partington 1999) assume the time delay to be known, i.e. they are delay-dependent. Input-output approaches with uncertain time delay are considered in (Hale & Verduyn Lunel 1993, Miller & Davison 2005), however, they are suited only for linear systems. The classical small gain result (Khalil 1996), applicable also to general nonlinear systems, is known to be rather conservative. For example, systems with an integrator in the open loop do not satisfy the small gain condition. In consequence, the steady state tracking performance to a reference input is rather poor.

The main contribution of this work is the analysis and design of a distributed controller to achieve input-output stability of nonlinear systems in the presence of unknown constant time delay. In contrast to most of the literature in NCS we propose to use the limited computational power available at the plant side for the implementation of some low order control action. Specifically, a linear static input-feedforward-output-feedback controller is introduced at the plant side and a similar modification at the controller. The controller is assumed to be designed in advance without considering the time delay. The additional control actions can also be interpreted as a linear transformation of the variables transmitted over the network: Instead of the original plant and controller outputs a linear combination of the respective inputs and outputs is communicated.

The proposed approach can be applied to input-feedforward-output-feedback-passive (IF-OFP) nonlinear plants and controllers. It is based on stability concepts in lines with the seminal works (Zames 1966*a*, Zames 1966*b*) where conditions for the open-loop behavior of feedback components are provided that guarantee input-output stability of the feedback interconnection. The main result is stated as follows: “If the open loop can be factored into two suitably proportioned, conic relations then the closed loop is bounded-input-bounded-output stable.” Here, by the IF-OFP assumption we require that the open loop system consisting of a plant and a controller but *without time delay* can be factored into such suitable conic sectors. The proposed input-output transformation then exactly preserves the IF-OFP property of the plant *with arbitrarily large constant time delay*. As a result every controller-plant pair, which is stable without the network based on the IF-OFP assumption, is also stable with the network *with arbitrarily large constant time delay* and the proposed transformation. The result is constructive with respect to the transformation.

The proposed approach has convincing performance advantages over standard small gain approaches such as zero steady state error and low sensitivity to time delay, which are discussed in this article for LTI SISO systems. Note that the controller design consists of two steps: The original controller for zero time delay is modified by the proposed transformation to stabilize in the presence of arbitrarily large constant time delay. Accordingly, the controller for zero time delay can be tuned rather aggressively compared to the standard small gain approach. Combined with low sensitivity to time delay this results in good tracking performance over a large range of time delay values.

The proposed approach builds on ideas analogous to the scattering transformation (Anderson & Spong 1989, Niemeyer & Slotine 1991), which is frequently used in force feedback telepresence applications. However, within this framework the subsystems are required to be passive. IF-OFP systems represent a substantially larger class, with passive systems as a special case. In this sense, the proposed approach in this article represents a generalization to the scattering transformation. Preliminary work of the same authors apply the scattering transformation for the first time to non-passive LTI NCS with arbitrary constant time delay (Matiakis, Hirche & Buss 2005) and to NCS with IF-OFP subsystems in (Matiakis, Hirche & Buss 2006). A less conservative result is presented here using a general input-output transformation.

The remainder of this article is organized as follows: In Section II the background on IF-OFP systems and finite gain $\mathscr{L}_2$ stability is presented, followed by the problem setting in Section III. The stability conditions together with a geometrical interpretation are presented in Section IV. Performance issues are discussed in Section V and validated through a numerical example in Section VI.

2 Background

Notation. Let $\mathscr{L}_{2e}^m$ denote the extended $\mathscr{L}_2$ space of time functions of dimension m with support on $[0,\infty)$. The notation $\|u\|$ stands for the $\mathscr{L}_2$ norm of a piecewise square-integrable function $u(\cdot):\mathbb{R}_+\to\mathbb{R}^m$ with $\mathbb{R}_+$ being the set of non-negative real numbers and $\mathbb{R}^m$ the Euclidean space of dimension m. The truncation of $u(\cdot)$ up to the time t is denoted by $u_t(\cdot)$. The inner product of the truncated signals u_t, y_t is denoted by $\langle u,y\rangle_t$, hence $\|u_t\|^2=\langle u,u\rangle_t$. The H_∞ norm of a transfer function $G(s)$

is denoted by $\|G\|_\infty$. $M > 0$ means that the matrix M is positive definite; I stands for the unit matrix.

In this article the dynamical systems are considered from an input-output point of view as causal input-output mapping operators $h : \mathscr{U} \to \mathscr{Y}$ with $h(t \leq 0) = 0$ and $\mathscr{U} \subset \mathscr{L}_{2e}^m$ representing the admissible input space and $\mathscr{Y}$ accordingly the output space. The system is supposed to be well defined in the sense that to each element in $\mathscr{U}$ an element in $\mathscr{Y}$ is associated.

2.1 IF-OFP Systems

Input-feedforward-output-feedback-passive systems are a special class of dissipative systems. Recall that a dynamical system $h : \mathscr{U} \to \mathscr{Y}$ is called dissipative with respect to the supply rate $s(u,y)$ if for each admissible $u \in \mathscr{U}$ and each $t \geq 0$

$$\int_0^t s(u,y)\mathrm{d}\tau \geq 0, \tag{1}$$

holds, refer to (Willems 1972*a*, Willems 1972*b*, Hill & Moylan 1976) for more details. Often, e.g. in (Willems 1972*b*), a quadratic supply rate $s(u,y) = z^T P z$ with $z^T = [u\ y]$

$$P = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}, \tag{2}$$

is considered. With the special choice $Q = -\delta I$, $R = -\varepsilon I$, $S = \eta I$, $\delta, \varepsilon \in \mathbb{R}$, $\eta = \frac{1}{2}$ IF-OFP systems are characterized.

Definition 1 *A dynamical system $h : \mathscr{U} \to \mathscr{Y}$ is called input-feedforward-output-feedback-passive (IF-OFP) if for each admissible $u \in \mathscr{U}$ and each $t \geq 0$*

$$\langle u,y \rangle_t \geq \delta \|u_t\|^2 + \varepsilon \|y_t\|^2. \tag{3}$$

Note that the IF-OFP property represents a generalization of the passivity concept. If $\delta = \varepsilon = 0$ the system is called passive, if $\delta = 0$ and $\varepsilon > 0$ the system is called output-feedback strictly passive and if $\delta > 0$ and $\varepsilon = 0$ the system is called input-feedforward strictly passive. If one or both of the values δ, ε are negative there is a shortage of passivity.

2.2 Finite Gain $\mathscr{L}_2$ Stability

Among the variety of stability notions we consider finite gain $\mathscr{L}_2$ stability in this article, which is another special case of quadratic dissipativity with $S = 0, R = I, Q = -\gamma^2 I, \gamma \in \mathbb{R}_+$.

Definition 2 *(Khalil 1996) A dynamical system $h : \mathscr{U} \to \mathscr{Y}$ is called finite gain $\mathscr{L}_2$ stable if there exists a constant $\gamma \in \mathbb{R}_+$ such that for each admissible $u \in \mathscr{U}$ and*

each $t \in [0, \infty)$
$$\|y_t\| \leq \gamma \|u_t\|. \tag{4}$$
Finite gain $\mathscr{L}_2$ stability of a feedback interconnection can be concluded from the IF-OFP properties of its subsystems. Consider two IF-OFP systems h_p and h_c satisfying (3) with δ_i, ε_i, $i \in \{p, c\}$.

Proposition 1 *(Khalil 1996) The negative feedback interconnection of h_p and h_c is finite gain $\mathscr{L}_2$ stable if*
$$\varepsilon_c + \delta_p > 0 \qquad \textit{and} \qquad \varepsilon_p + \delta_c > 0. \tag{5}$$
Clearly, some of the δ_i, ε_i can be negative if compensated by appropriate positive values. Within the passivity formalism this can be interpreted as balancing shortage of passivity with excess of passivity between subsystems.

3 Problem Setting

We consider a system comprising a plant $h_p : \mathscr{U}_p \to \mathscr{Y}_p$ and a controller $h_c : \mathscr{E} \to \mathscr{Y}_c$ as mappings from the plant input $u_p \in \mathscr{U}_p \subset \mathscr{L}_{2e}^m$ to the plant output $y_p \in \mathscr{Y}_p \subset \mathscr{L}_{2e}^m$ and from the control error $e \in \mathscr{E} \subset \mathscr{L}_{2e}^m$ to the controller output $y_c \in \mathscr{Y}_c \subset \mathscr{L}_{2e}^m$. The control error is defined as $e = w - u_c$ where $w \in \mathscr{W} \subset \mathscr{L}_{2e}^m$ is the reference input, see Fig. 1 for visualization. The blocks M and its inverse M^{-1} represent the transformation which is introduced later. Without them the plant is directly connected with the controller through a communication network.

The network is modelled as a forward time delay operator h_{T_1} (controller to plant channel) and backward time delay operator h_{T_2} (plant to controller channel) with time delays T_1 and T_2, respectively. The input-output relations are given by $h_{T_1} : u_r(t) = u_l(t - T_1)$ and $h_{T_2} : v_l(t) = v_r(t - T_2)$. It is assumed that $u_l(t) = 0 \ \forall t \in [-T_1, 0]$ and $v_r(t) = 0 \, \forall t \in [-T_2, 0]$. The time delays $T_1, T_2 \in \mathbb{R}_+$ are assumed to be constant but unknown.

Without any further control measures the closed loop system with time delay can be unstable. This can easily be verified, as shown in (Anderson & Spong 1989) in the example of passive subsystems. In order to address this problem we propose to transmit a linear transformation of the plant input-output vector $z_p^T = [u_p^T \, y_p^T]$ over the plant-to-controller channel instead of directly transmitting the plant output. The righthand side transmitted values $s_r^T = [u_r^T \, v_r^T]$, see Fig. 1, are related to the plant input-output via the transformation matrix $M \in \mathbb{R}^{2m \times 2m}$
$$s_r = M z_p. \tag{6}$$
Equivalently stated, a static output-feedback-input-feedforward controller is inserted at the plant side leading to a distributed controller architecture. To avoid confusion in the remainder of this article we will refer to the static output-feedback-input-feedforward controller as the transformation M. The controller h_c is analogously modified, i.e. the relation between the original controller input-output vec-

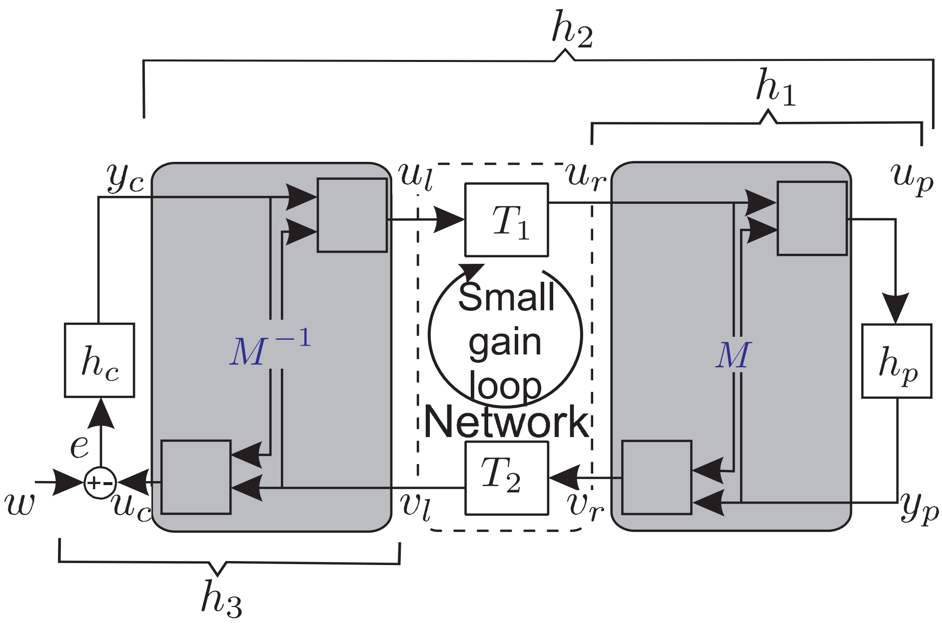

Fig. 1. NCS with input-output transformation.

tor $z_c^T = [y_c^T \; u_c^T]$ with $z_c \in \mathscr{Y}_c \times \mathscr{U}_c$ and the lefthand side transmitted values $s_l^T = [u_l^T \; v_l^T]$ is given by

$$s_l = M z_c. \tag{7}$$

Note that for $M = I$ the standard approach without transformation/ local control at the plant side is recovered. For a specific choice of M, as discussed later, the scattering transformation is recovered guaranteeing stability for passive subsystems h_p and h_c with arbitrarily large constant time delay.

Throughout the article we assume that the closed loop system is well posed, i.e. for each input signal $w \in \mathscr{W}$ there exists a unique solution for the signals $e, u_c, y_c, u_l, v_l, u_r, v_r, u_p, y_p$ that causally depends on w. Note, that this implies the invertibility of the matrix M as otherwise for a solution of u_l, v_l, u_r, v_r there are several equivalent solutions for u_p, y_p, u_c, y_c. For further reference we define the following three subsystems: $v_r = h_1(u_r)$, $u_c = h_2(y_c)$, and $u_l = h_3(v_l, w)$, see Fig. 1.

4 Conditions for Delay-Independent Stability

4.1 Delay-independent stability for IF-OFP systems

Without loss of generality we can assume that the dissipativity parameters of every considered IF-OFP system $\delta, \varepsilon, \eta$ belong to the domain $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \{\delta, \varepsilon, \eta \in \mathbb{R} | \delta\varepsilon - \eta^2 < 0\}$ and $\Omega_2 = \{\delta, \varepsilon, \eta \in \mathbb{R} | \delta\varepsilon - \eta^2 = 0; \delta, \varepsilon > 0\}$, or equivalently that the dissipativity matrix P (2) has either m negative and m positive, or m negative and m zero eigenvalues. For a proof, see Lemma 1 in the appendix. Where it is non-ambiguous, the time argument t is dropped.

Throughout this section we make the following assumption:

A1 Plant h_p and controller h_c are IF-OFP with δ_i, ε_i where $i \in \{p, c\}$ satisfying (5), i.e. the negative feedback interconnection *without time delay* is finite gain $\mathscr{L}_2$ stable.

For subsequent derivations the transformation matrix M is decomposed into a rotation matrix R and matrix B, i.e.

$$M = RB,\ R = \begin{bmatrix} \cos\theta I & \sin\theta I \\ -\sin\theta I & \cos\theta I \end{bmatrix},\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{8}$$

$$B = \begin{bmatrix} b_{11} I & b_{12} I \\ b_{21} I & b_{22} I \end{bmatrix},$$

with $b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{R}$ and $\det B \neq 0$. For the following stability result only the rotation R is crucial. The matrix B gives an additional degree of freedom for performance design aspects. The overall system is decomposed into the feedback-interconnected subsystems h_c and h_2, with the latter defined by $u_c = h_2(y_c)$ and comprises the plant h_p, the forward and backward time delay operators, and the right and left transformations M and M^{-1}, see Fig. 1. The subsystem h_2 can be shown to be IF-OFP. In fact, the following theorem gives necessary and sufficient conditions for the *exact* preservation of the plant IF-OFP properties to the subsystem h_2 independently of the constant time delay. Define the dissipativity matrix P_p (2) with elements $(\delta_p, \varepsilon_p, \eta_p = \frac{1}{2}) \in \Omega$ and furthermore $\delta_B, \varepsilon_B, \eta_B$ as the elements of the matrix P_B

$$P_p = \begin{bmatrix} -\delta_p I & \eta_p I \\ \eta_p I & -\varepsilon_p I \end{bmatrix};\ P_B = B^{-T} P_p B^{-1} = \begin{bmatrix} -\delta_B I & \eta_B I \\ \eta_B I & -\varepsilon_B I \end{bmatrix}. \tag{9}$$

Theorem 1 *Assume that the plant h_p is IF-OFP with δ_p, ε_p, $\eta_p = \frac{1}{2}$. Then the subsystem h_2 is IF-OFP with δ_p, ε_p, $\eta_p = \frac{1}{2}$ if and only if for each B the angle θ is chosen as the one of the two solutions of*

$$\cot 2\theta = \frac{\varepsilon_B - \delta_B}{2\eta_B}, \tag{10}$$

which simultaneously satisfies

$$\alpha(\theta) = 2\eta_B \sin(\theta)\cos(\theta) - \delta_B \cos^2(\theta) - \varepsilon_B \sin^2(\theta) \geq 0. \tag{11}$$

Proof: (*sufficiency*) Rewriting (3) for the plant in matrix form, in terms of the transmitted variables s_r yields

$$\int_0^t s_r^T M^{-T} P_p M^{-1} s_r \mathrm{d}\tau \geq 0 \Leftrightarrow \int_0^t s_r^T R^{-T} P_B R^{-1} s_r \mathrm{d}\tau \geq 0 \tag{12}$$

with P_B given by (9) and

$$R^{-T} P_B R^{-1} = \begin{bmatrix} \alpha(\theta) I & \zeta(\theta) I \\ \zeta(\theta) I & -\beta(\theta) I \end{bmatrix}, \tag{13}$$

parameterized by θ, δ_B, ε_B, η_B through $\alpha(\theta)$ (11),

$$\beta(\theta) = \alpha(\theta) + \delta_B + \varepsilon_B,$$

and

$$\zeta(\theta) = \eta_B \cos 2\theta - \frac{\varepsilon_B - \delta_B}{2} \sin 2\theta. \tag{14}$$

Choosing θ according to (10), it follows that $\zeta(\theta) = 0$ (14), and hence we can rewrite (12)

$$\alpha(\theta)\|u_{r,t}\|^2 - \beta(\theta)\|v_{r,t}\|^2 \geq 0.$$

According to Sylvester's law of inertia, congruence transformations do not change the inertia of the matrix, i.e. the number of positive, negative and zero eigenvalues. Thus $(\delta_p, \varepsilon_p, \eta_p = \frac{1}{2}) \in \Omega \Leftrightarrow (\delta_B, \varepsilon_B, \eta_B) \in \Omega$. For this domain of $(\delta_B, \varepsilon_B, \eta_B)$ we can always choose one of the two solutions to (10) in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, denoted by θ^+ and θ^- respectively, so that $\alpha(\theta^+) \geq 0$ as required by (11) and furthermore $\beta(\theta^+) > 0$, see Lemma 2 in the Appendix. For this choice of θ the subsystem h_1 is finite gain $\mathscr{L}_2$ stable with

$$\|v_{r,t}\| = \|h_1(u_{r,t})_t\| \leq \gamma_{h_1}\|u_{r,t}\| \quad \forall t \; \gamma_{h_1}^2 = \frac{\alpha(\theta^+)}{\beta(\theta^+)}. \tag{15}$$

Considering further that the constant time delay operator has an $\mathscr{L}_2$ gain one, and using the assumption that $u_l(t) = 0 \;\; \forall t \in [-T_1, 0]$ and $v_r(t) = 0 \; \forall t \in [-T_2, 0]$, we may state $\|u_{r,t}\|^2 \leq \|u_{l,t}\|^2, \; \|v_{l,t}\|^2 \leq \|v_{r,t}\|^2, \; \forall t > 0$. It follows that

$$\alpha(\theta^+)\|u_{l,t}\|^2 - \beta(\theta^+)\|v_{l,t}\|^2 \geq 0.$$

Analogously to (12) we may rewrite the latter equation as

$$\int_0^t s_l^T M^{-T} P_p M^{-1} s_l \mathrm{d}\tau \geq 0 \tag{16}$$

which expressed in the variables y_c, u_c becomes

$$\langle y_c, u_c \rangle_t \geq \delta_p \|y_{c,t}\|^2 + \varepsilon_p \|u_{c,t}\|^2.$$

Thus, the subsystem h_2 satisfies (1) with the exact same dissipativity parameters $\delta_p, \varepsilon_p, \eta_p = \frac{1}{2}$ as the plant. For *necessity* it only has to be shown that without setting $\zeta(\theta) = 0$ the time delay alters the IF-OFP property of the subsystem h_2. This can be shown straightforwardly through the counter example $y_p(t) = k \cdot u_p(t)$. ■

Observe that θ^+ exists for each B, i.e. $b_{11}, b_{12}, b_{21}, b_{22}$ can be chosen freely to meet performance requirements. From this result it is straightforward to conclude finite gain $\mathscr{L}_2$ stability.

Corollary 1 *The closed loop system with the input-output transformation* (8) *satisfying Theorem 1 is delay-independently finite gain $\mathscr{L}_2$ stable.*

Proof: We have to show that bounded input $w \in \mathscr{L}_{2e}$ implies bounded output $y_p \in \mathscr{L}_{2e}$. By applying Proposition 1 to the closed loop system decomposed into subsystems h_2 and h_c it is straightforward that also the signals $u_c, y_c, e \in \mathscr{L}_{2e}$. Since u_l, v_l are linear combinations of u_c, y_c we have $u_c, y_c \in \mathscr{L}_{2e} \Rightarrow u_l, v_l \in \mathscr{L}_{2e}$. The forward constant time delay operator is finite gain $\mathscr{L}_2$ stable so $u_l \in \mathscr{L}_{2e} \Rightarrow u_r \in \mathscr{L}_{2e}$. Furthermore h_1 is finite gain $\mathscr{L}_2$ stable thus $u_r \in \mathscr{L}_{2e} \Rightarrow v_r \in \mathscr{L}_{2e}$. Since again u_p, y_p

are a linear transformation of u_r, v_r, we have that $u_r, v_r \in \mathscr{L}_{2e} \Rightarrow u_p, y_p \in \mathscr{L}_{2e}$, i.e. there exists a $\gamma < \infty$ such that $\|y_{p,t}\| \leq \gamma \|w_t\|$ holds $\forall t$. Assuming the plant output to be unbounded, i.e. $y_p \notin \mathscr{L}_{2e}$, results with the same arguments as above in a contradiction to the assumption $w \in \mathscr{L}_{2e}$. ■

In short, the central point of the proposed approach is that the righthand input-output transformation transforms the IF-OFP plant h_p into the finite gain $\mathscr{L}_2$ stable subsystem h_1, see (15). A constant time delay operator does not alter this system's $\mathscr{L}_2$ gain since it has an $\mathscr{L}_2$ gain one, $\gamma_{T_1} = \gamma_{T_2} = 1$. The lefthand transformation M^{-1} is the inverse of the righthand transformation, and therefore the exact IF-OFP plant properties are recovered to the subsystem h_2. Thus, a bounded input $w \in \mathscr{L}_{2e}$ implies that the signals in the feedback interconnection are bounded, i.e. $e, u_c, y_c \in \mathscr{L}_{2e}$. The invertibility of the transformation further implies that all signals are bounded, i.e. $e, u_c, y_c, u_l, v_l, u_r, v_r, u_p, y_p \in \mathscr{L}_{2e}$. As an important result, the feedback interconnection of any controller-plant pair satisfying the finite gain $\mathscr{L}_2$ condition from Proposition 1 *without time delay* is finite gain $\mathscr{L}_2$ stable for *arbitrarily large time delay* by using the proposed input-output transformation.

Remark 1 *In case of unstable plants the proposed approach locally pre-stabilizes by the righthand input-output transformation. This becomes clear from* (15)*, where every IF-OFP plant h_p results in a finite gain $\mathscr{L}_2$ stable system h_1.*

Remark 2 *For passive plants, i.e. with $\delta = \varepsilon = 0$ in* (3)*, the proposed input-output transformation with $\theta = \frac{\pi}{4}$ and the elements of B given by $b_{11} = \sqrt{b}, b_{22} = \frac{1}{\sqrt{b}}, b > 0$ and $b_{12}, b_{21} = 0$, is equivalent to the scattering transformation (Anderson & Spong 1989, Niemeyer & Slotine 1991).*

4.2 Small Gain Interpretation

An interesting viewpoint gives the interpretation of Theorem 1 from a small gain perspective. Therefore, we assume that Theorem 1 is satisfied. For the analysis, the closed loop system is decomposed into the subsystems h_1, h_3, h_{T_1}, and h_{T_2}, where the *transmitted* signals u_l, u_r, v_r, v_l act as inputs and outputs, and the open loop system

$$h_{OL} = h_3 \circ h_{T_1} \circ h_1 \circ h_{T_2} \tag{17}$$

is considered, see Fig. 1. In the next it is shown than h_{OL} is finite gain $\mathscr{L}_2$ stable, i.e.

$$\|h_{OL}(v_{l,t})_t\| \leq \gamma_{OL} \|v_{l,t}\| \quad \forall t, \tag{18}$$

with $\gamma_{OL} < 1$, i.e. the system satisfies the small gain condition in the transformed variables.

Corollary 2 *The open loop system h_{OL} has an $\mathscr{L}_2$ gain $\gamma_{OL} < 1$.*

Proof: For the subsystem h_{OL} it is straightforward to show that $\|h_{OL}(v_{l,t})_t\| \leq \gamma_{OL} \|v_{l,t}\|$ with $\gamma_{h_{OL}} \leq \gamma_{h_3} \gamma_{T_1} \gamma_{h_1} \gamma_{T_2} = \gamma_{h_3} \gamma_{h_1}$ since for the time delay operators $\gamma_{T_1} = \gamma_{T_2} = 1$ holds. It remains to show that $\gamma_{h_3} \gamma_{h_1} < 1$. From (15) the finite $\mathscr{L}_2$ gain stability of h_1 is certified with gain $\gamma_{h_1}^2$. For h_3, consider the dissipativity inequality of the controller expressed in the variables s_l

$$\int_0^t s_l^T M^{-T} P_c M^{-1} s_l \mathrm{d}\tau \geq 0, \text{ with } P_c = \begin{bmatrix} -\varepsilon_c I & -\frac{1}{2} I \\ -\frac{1}{2} I & -\delta_c I \end{bmatrix}, \tag{19}$$

where the negative signs in the off-diagonals of P_c result from the *negative* feedback interconnection. Setting $k = \min[(\varepsilon_p + \delta_c), (\varepsilon_c + \delta_p)] > 0$, where positivity comes from assumption A1, it is straightforward to show that

$$P_c \leq -(P_p + kI). \tag{20}$$

Thus, by substituting (20) in (19) it follows that

$$-\int_0^t s_l^T M^{-T} P_p M^{-1} s_l + k s_l^T M^{-T} M^{-1} s_l \mathrm{d}\tau \geq 0 \Rightarrow$$
$$-\int_0^t s_l^T M^{-T} P_p M^{-1} s_l + k\lambda_{min} s_l^T s_l \mathrm{d}\tau \geq 0$$

with $\lambda_{min} > 0$ the minimum eigenvalue of $M^{-T} M^{-1} > 0$. Following the derivations of the proof of Theorem 1 using (12), (13) and choosing θ^+, the quadratic term above involving P_p is simplified and the inequality can be rewritten as

$$\|u_{l,t}\| = \|h_3(v_{l,t})_t\| \leq \gamma_{h_3} \|v_{l,t}\| \quad \forall t \ \gamma_{h_3}^2 = \frac{\alpha(\theta^+) - k\lambda_{min}}{\beta(\theta^+) + k\lambda_{min}}.$$

Therewith, the subsystem h_3 is certified to be finite $\mathscr{L}_2$ gain stable with gain γ_{h_3}. Accordingly, with (15)

$$\gamma_{h_3}^2 \gamma_{h_1}^2 \leq \frac{\alpha(\theta^+)}{\beta(\theta^+)} \frac{\beta(\theta^+) - k\lambda_{min}}{\alpha(\theta^+) + k\lambda_{min}} < 1,$$

hence $\gamma_{h_3}\gamma_{h_1} < 1$, and thus $\gamma_{h_{OL}} < 1$. ■

Hence, the small gain condition holds in the loop of the communicated variables, see also Fig. 1 for visualization. In fact, with equality in Proposition 1, i.e. marginal stability, also the open loop gain becomes $\gamma_{OL} \leq 1$. The $\mathscr{L}_2$ gains of the subsystems h_1 and h_3 depend on the IF-OFP properties of plant and controller $\gamma_{h_1} = \gamma_{h_1}(\delta_p, \varepsilon_p)$ and $\gamma_{h_3} = \gamma_{h_3}(\delta_c, \varepsilon_c)$. More conservative, i.e. higher values of δ_p, ε_p and δ_c, ε_c in Proposition 1 result in a smaller open loop gain, hence in a higher stability reserve. Note, that the small gain theorem is only satisfied for the mappings with the communicated (transformed) variables u_l, u_r, v_r, v_l as input/output, but not for the mappings with the (original) control variables e, y_c, u_p, y_p, Therefore, less conservative behavior than through the standard small gain approaches can be achieved.

Remark 3 *Observe that for the stability guarantee only the finite $\mathscr{L}_2$ gain $\gamma = 1$ property of the time delay operator is important. Accordingly, stability is guaranteed also for any other norm bounded uncertainty h_* in the loop of the transformed variables, replacing the time delay operators h_{T_1}, h_{T_2}, or being in cascade with them, as long as $\gamma_{h_*} \leq 1$. Many scattering based approaches addressing* *time-varying delay (Lozano, Chopra & Spong 2002, Munir & Book 2002), packet loss (Secchi, Stramigioli & Fantuzzi 2003, Berestesky, Chopra & Spong 2004, Hirche & Buss 2004), and sampled-data systems (Stramigioli 2001) are based on the same argument, introducing control actions to keep the $\mathscr{L}_2$ gain of the corresponding input-output operator $\gamma \leq 1$. These approaches are straightforward to combine with the proposed approach.*

4.3 Conic Sectors Interpretation

Conic sectors in the input-output space give a nice geometrical interpretation of IF-OFP systems behavior, see e.g. (Zames 1966*a*, Zames 1966*b*). Following these lines, the input-output transformation can be interpreted as a rotation of conic sectors. For simplicity a memory-less, SISO, IF-OFP system is considered as plant, even though stability related notions are futile in this case. The IF-OFP inequality (3) holds instantaneously, i.e.

$$u_p y_p \geq \delta_p u_p^2 + \varepsilon_p y_p^2, \quad \forall t, \quad (\delta_p, \varepsilon_p, \eta_p = \frac{1}{2}) \in \Omega. \tag{21}$$

Geometrically, this equation describes a conic sector in the u_p-y_p-plane which is sufficiently described by its center-line angle θ_z and its apex angle $2\theta_{k,p}$. At each time instant t the input and output lies within the conic sector $\theta_p(t) \in [\theta_z - \theta_{k,p}, \theta_z + \theta_{k,p}]$ or its mirrored counterpart, see Fig 2 (a) for a visualization. The center-line angle is straightforwardly derived by parameterizing the plant input and output in polar coordinates $u_p(t) = r_p(t)\cos\theta_p(t), \quad y_p(t) = r_p(t)\sin\theta_p(t)$ in (21), and is implicitly given as the solution of

$$\cot 2\theta_z = \varepsilon_p - \delta_p, \tag{22}$$

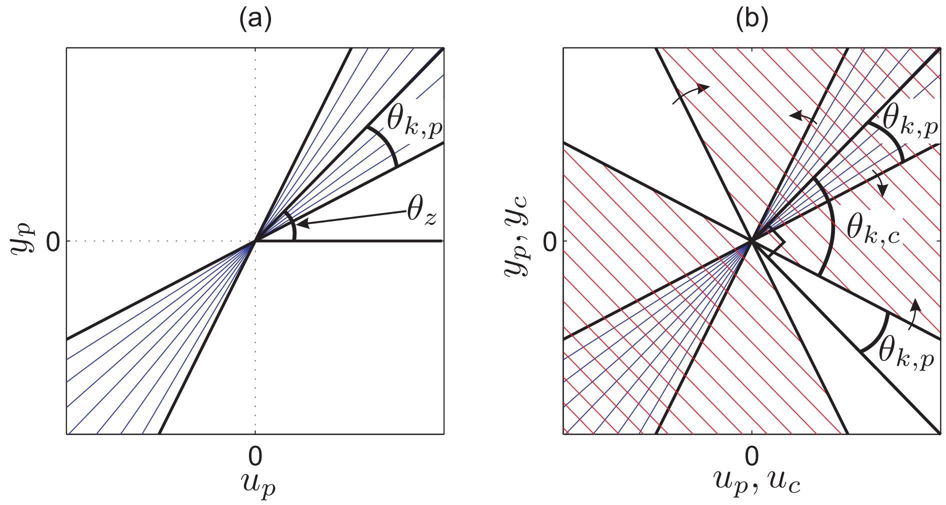

Fig. 2. (a) The conic sector of an IF-OFP plant. (b) The conic sector of the same plant and the corresponding controller satisfying Proposition 1.

in the interval $[0, \frac{\pi}{2}]$. Similarly, the apex angle $2\theta_{k,p}$ is given by the solution of

$$\cos 2\theta_{k,p} = \frac{\varepsilon_p + \delta_p}{\sqrt{(1-4\delta_p\varepsilon_p) + (\varepsilon_p + \delta_p)^2}},$$

with $\theta_{k,p} \in [0, \frac{\pi}{2})$.

4.3.1 Conic Sectors Interpretation of Proposition 1

Given the plant sector by (21) the finite gain $\mathscr{L}_2$ stability condition determines the allowable controller sector. Using a similar technique as in proof of Corollary 2, the allowable controller sector is derived to be $\theta_c(t) \in (\theta_z - \theta_{k,c}, \theta_z + \theta_{k,c})$ where $\theta_{k,c} = \frac{\pi}{2} - \theta_{k,p}$. Note, that due to the strict inequality in Proposition 1 the controller is confined to an open set in a sector with the same center-line as the plant, and complementary angle with respect to 90^o. The larger the sector of the plant is, the smaller is the allowable sector for the controller, as visualized by the arrows in Fig. 2 (b).

4.3.2 Conic Sectors with Input-Output Transformation

As discussed above, only the rotation matrix R of the input-output transformation (8) plays a role for stability. Thus, for clarity of presentation and without loss of generality, we consider in the remainder of this section that $B = I$. By the input-output transformation satisfying Theorem 1 the IF-OFP plant with input u_p and output y_p is transformed into a finite gain $\mathscr{L}_2$ stable subsystem h_1 with input u_r and output v_r. Observe that the center-line angle for the IF-OFP plant given by (22) is equal to the rotation angle θ^+ derived from Theorem 1 for $B = I$. Thus, by the input-output transformation the sector of the plant is rotated such that the sector

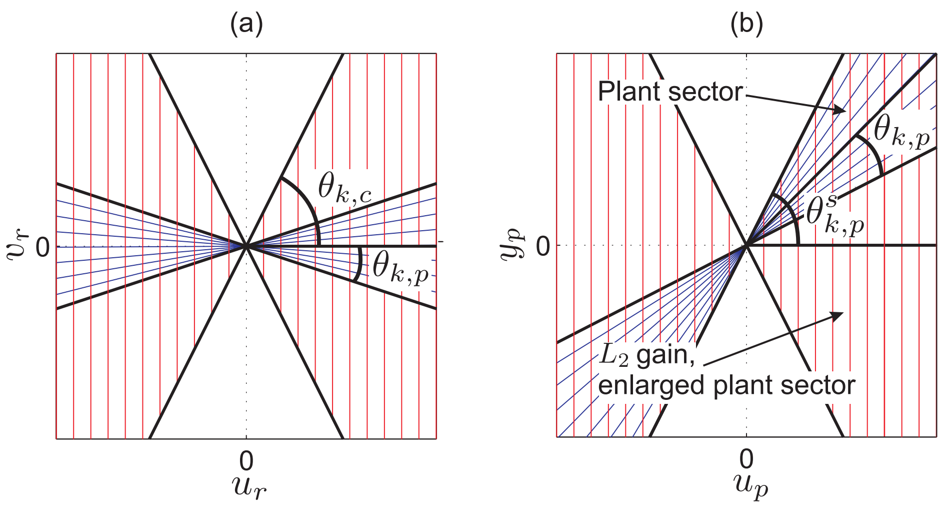

Fig. 3. (a) Finite gain $\mathscr{L}_2$ stable system after applying input-output transformation to the plant and controller from Fig. 2 (b). (b) Equivalent $\mathscr{L}_2$ gain sector of an IF-OFP system.

of the subsystems h_1 has a center-line angle of $\theta_z = 0$. This, however, is exactly the conic sector representation for finite gain $\mathscr{L}_2$ stability, i.e. for the plant side in the transformed coordinates $\|v_{r,t}\| \le \gamma_{h_1}\|u_{r,t}\|$. The apex angles $2\theta_{k,p}$, $2\theta_{k,c}$ of the plant and of the also rotated allowable controller sector, are invariant to the rotation and are related to the $\mathscr{L}_2$ gain by $\tan\theta_{k,p} = \gamma_{h_1}$ and $\tan\theta_{k,c} = 1/\gamma_{h_1}$. The allowable controller area thus expresses the small gain theorem of the open loop system with the "rotated" subsystems h_1 and h_3 as has been shown also in Corollary 2. The rotation of the IF-OFP plant and controller from Fig. 2(b) to a finite gain $\mathscr{L}_2$ stable system is visualized in Fig. 3(a).

For comparison, the classical small gain approach without input-output transformation is discussed. The classical small gain approach can be applied only if the plant is initially finite gain $\mathscr{L}_2$ stable. This means that the plant's sector lies in the first and fourth quadrant. Clearly, in this case the IF-OFP plant sector from Fig. 2 (a) can also be represented by an *enlarged* conic sector symmetric to the u_p axis, as shown in Fig. 3 (b). For the open loop gain $\gamma_p^s \gamma_c^s = \tan(\theta_{k,p}^s)\tan(\theta_{k,c}^s) < 1$ has to hold, where $|2\theta_{k,p}^s| \ge |2\theta_{k,p}|$ is the apex angle of the enlarged conic sector of the plant. Accordingly, the stability allowable controller sector with apex angle $|2\theta_{k,c}^s| \le |2\theta_{k,c}|$ is smaller than with the transformation approach, i.e. is more conservative.

Last, for comparison, the scattering transformation is also discussed. The scattering transformation, representing a rotation of $\theta^+ = 45^o$, is recovered by the proposed approach in case of a passive system. The sector of a passive system is the first quadrant, i.e. $\theta_z = 45^o$, requiring thus, a rotation of exactly 45^o in order to become a finite gain $\mathscr{L}_2$ stable system. Generally, using the scattering transformation in a non-passive system leads to conservatism, as with a rotation of 45^o the center line of the sector does not necessarily coincide with the axis u_p. Stability may be guaranteed in some cases by considering the enlarged, finite gain stability sectors, as in the classical small gain case. Nevertheless, with the proposed parameterization, conservatism is in all cases avoided. Hence, as long as stability is guaranteed for the initial plant and controller without the network, stability is again guaranteed for arbitrarily large constant time delay and the appropriate rotation.

With the intuition of conic sectors, the main idea of the proposed approach can be summarized into rotating the plant and controller conic sectors to achieve a non-conservative $\mathscr{L}_2$ gain representation in the communicated signals compared to the classical small gain approach. Arbitrarily large constant time delay does not alter this argument.

Note, however, that Corollary 1 gives only a sufficient condition for finite gain $\mathscr{L}_2$ stability as it relies on the sufficient stability condition from Proposition 1. This can be expected as only very little knowledge of the plant and controller input-output relation is required. In the following LTI systems with *known* transfer functions are considered as plant and controller, and the necessary and sufficient conditions for delay-independent stability are derived.

4.4 Stronger Stability Condition for Known LTI Systems

The remainder of this article concerns LTI systems. The presented results are restricted to the SISO case. The LTI plant and controller are described by the transfer functions $G_p(s) = \frac{Y_p(s)}{U_p(s)},\ G_c(s) = \frac{Y_c(s)}{E(s)}$ respectively, where $Y_p(s)$ and $U_p(s)$ represent the Laplace transformations of the plant output $y_p(t)$ and input $u_p(t)$, and $Y_c(s)$ and $E(s)$ the Laplace transformations of the controller output $y_c(t)$ and input $e(t)$. Where it is non-ambiguous the Laplace variable s is dropped for convenience of notation. Consider the transfer function

$$G_{OL} = G_1 G_3 = \frac{m_{21} + m_{22} G_p}{m_{11} + m_{12} G_p} \frac{m_{12} - m_{11} G_c}{m_{22} - m_{21} G_c} \tag{23}$$

with G_1 and G_3 being the transfer functions of h_1 and h_3 respectively, and $\{m_{ij} \in \mathbb{R}$, with $i, j \in \{1,2\}\}$ the elements of $M \in \mathbb{R}^{2\times 2}$. The following corollary gives a necessary and sufficient condition for delay-independent stability.

Corollary 3 *The LTI closed loop system consisting of plant G_p, controller G_c and the input-output-transformation $M \in \mathbb{R}^{2\times 2}$ is delay-independently stable if and only if G_1, G_3 are stable and*

$$|G_{OL}| < 1,\ \forall \omega > 0. \tag{24}$$

Proof: For delay-independent stability the closed loop system has to be stable when $T_1 = T_2 = \infty$, i.e. G_1, G_3 must be stable. Consider now the open loop transfer function including the time delay operators, i.e. $G_{OL} e^{-j\omega T}$ with $T = T_1 + T_2$. For stability $|G_{OL} e^{-j\omega T}| < 1$ must hold, when $\arg\{G_{OL} e^{-j\omega T}\} \leq -180^o$. For arbitrary T and $\omega \neq 0$, $e^{-j\omega T}$ defines an arbitrary phase shift. Thus, for all $\omega > 0$, $|G_{OL}| < 1$ must hold. ∎

Observe that Theorem 1 leads to the more conservative stability result $\gamma_{h_1}\gamma_{h_3} = \|G_1\|_\infty \|G_3\|_\infty < 1$. The conservatism comes from the fact that more generally $\max_{\omega>0} |G_1 G_3| \leq \|G_1 G_3\|_\infty \leq \|G_1\|_\infty \|G_3\|_\infty$ holds with strict inequality. Equality is given only if the maximum magnitude of G_1 and G_3 appears at the same frequency $\omega_{max} = \arg\sup_\omega |G_1| = \arg\sup_\omega |G_3|$, which is not equal to zero.

Under the restriction of Corollary 3, the controller and the input-output transformation can be conjointly designed in the LTI case with known transfer functions. Knowledge of the time delay value for the controller design is not required.

5 Performance Issues

In the following some performance issues, i.e. the sensitivity to time delay, the steady state behavior, and the zero time delay case are briefly discussed for LTI systems. Based on the Corollary 3, in the remainder of this section it is considered that $\|G_{OL}\|_\infty < 1$. The closed loop transfer function $G(s) = \frac{Y_p(s)}{W(s)}$, from the reference input W to the plant output Y_p, is computed by (6) (7) to be

$$G(s) = G_0(s)G_{tr}(s)e^{-sT_1},\ G_{tr}(s) = \frac{1-G_{OL}(s)}{1-G_{OL}(s)e^{-sT}}, \tag{25}$$

with $G_0 = (G_pG_c)(1+G_pG_c)^{-1}$ and G_{OL} given from (23). The system can be seen as a series connection of the standard closed loop system G_0 without time delay and without input-output transformation, and of G_{tr} which describes the influence of the time delay and the input-output transformation. Obviously, if G_{tr} is far away from identity, the behavior of the closed loop system with time delay and transformation largely differs from the behavior of the closed loop system without time delay and without transformation.

5.1 Sensitivity to Time Delay

Sensitivity to time delay is an interesting aspect of performance, especially in NCS where the time delay is not exactly known in advance. Low sensitivity to time delay means that a similar input-output behavior is achieved in a large range of time delay values. The sensitivity function with respect to the round trip time delay $T = T_1 + T_2$ is given by the infinite dimensional transfer function

$$S_T^{G^*} = \frac{T}{G^*}\frac{dG^*}{dT} = sTe^{-sT}\frac{G_{OL}}{1-G_{OL}e^{-sT}},$$

where $G^*(s) = G_0(s)G_{tr}(s)$ is the transfer function (25) without the purely time shifting part e^{-sT_1}. For the norm of $S_T^{G^*}$ a frequency-dependent maximum can be computed as stated in the next theorem.

Theorem 2 *When $\|G_{OL}\|_\infty < 1$ holds, the norm of the time delay sensitivity function is for each frequency ω_0 bounded from above by*

$$|S_T^{G^*}(j\omega_0)| \leq \frac{\omega_0 T\|G_{OL}\|_\infty}{1-\|G_{OL}\|_\infty}. \tag{26}$$

Proof: Straightforward computation of the norm of the sensitivity function yields

$$|S_T^{G^*}(j\omega_0)| = \frac{\omega_0 T|G_{OL}|}{|1-G_{OL}e^{-j\omega_0 T}|} \leq \frac{\omega_0 T|G_{OL}|}{1-|G_{OL}|} \leq \frac{\omega_0 T\|G_{OL}\|_\infty}{1-\|G_{OL}\|_\infty},$$

where the dependence on $j\omega_0$ in $|G_{OL}(j\omega_0)|$ is suppressed for convenience of notation. ■

Interestingly, the performance requirement for low sensitivity to time delay is compatible with the demand for large stability reserve; in both cases $\|G_{OL}\|_\infty$ is required to be small, and of course below one. This can be seen by taking the derivative of (26) with respect to $\|G_{OL}\|_\infty$ and showing that the righthand part of (26) is a strictly increasing function of $\|G_{OL}\|_\infty$ when $\|G_{OL}\|_\infty < 1$. Thus, minimizing $\|G_{OL}\|_\infty$ jointly achieves stability and sensitivity goals.

Insensitivity $S_T^{G^*} = 0$ can be achieved by using a proportional controller $G_c(s) = \frac{m_{12}}{m_{11}}$, independently of the plant. This follows straightforwardly from substituting G_c in (24) resulting in $G_{OL} = 0 \Rightarrow S_T^{G^*} = 0 \Rightarrow G_{tr}(s) = 1$. The closed loop transfer function (25) reduces to $G(s) = G_0(s)e^{-sT_1}$ with the time shifting part having no effect on the transient response. This fact reflects the intuition that if a static controller G_c is used in the proposed setup, then it can be implemented at the plant side and no remote control action is required. However, a proportional controller usually does not meet the performance requirements and a compromise should be made between performance and sensitivity to time delay.

Remark 4 *The minimization of $\|G_{OL}\|_\infty$ can be formulated as an optimization problem with bilinear matrix inequality constraints, as shown in (Matiakis, Hirche & Buss 2008). Nevertheless, as this is out of the scope of this article, in the numerical example that follows in Section 6, classical gradient descent optimization is used instead, for the design of M.*

5.2 Zero Time Delay Case

As the time delay reduces to zero, i.e. $T_1 = T_2 = T = 0$, the system reduces to that without input-output transformation, i.e. $G(s) = G_0(s)$ as straightforward computable from (25). The statement holds as well for the general nonlinear case, since $s_l = s_r$ when $T_1 = T_2 = 0$. This is interesting as the controller can be rather aggressively designed, compared to the standard small gain approach, without considering time delay. For zero time delay "nominal" performance is recovered. Together with low sensitivity this means that good performance is achieved in a large range of time delay values.

5.3 Steady State Behavior

The steady state behavior of the system with the input-output transformation and time delay is equivalent to the steady state behavior of the system without the input-output transformation and without time delay as easily derivable by setting $s = 0$ in (25), resulting in $G(0) = G_0(0)$. For the nonlinear case this can be observed from the steady state condition $s_l = s_r$, hence $z_c = M^{-1}s_l = M^{-1}s_r = M^{-1}Mz_p = z_p$.

In terms of steady state error the proposed approach clearly outperforms the standard small gain approach which requires $|G_c(j\omega)G_p(j\omega)| < 1, \omega > 0$, i.e. free integrators in the open loop are not allowed, leading thus to a non-zero steady state error. In the proposed approach free integrators in plant or controller do not necessarily appear as free integrators in G_{OL} (24). As a result delay-independent stability based on Corollary 3 and steady state error zero can be simultaneously guaranteed. This is demonstrated in the following example.

Example: Consider the plant $G_p(s) = \frac{1}{s+1}$ and the controller $G_c(s) = \frac{s+1}{s(s+10)}$. The input-output transformation minimizing $\|G_{OL}\|_\infty$ in numerical optimization is given by $m_{11} = m_{22} = 0.866$, $m_{12} = 0.5$, and $m_{21} = -0.5$. The open loop transfer function $G_c G_p$ violates the small gain condition. With transformation, i.e. distributed control approach, zero steady state error is achieved.

In summary, the proposed distributed control approach indicates significant advantages over the standard small gain approach. In fact, even delay-dependent input-output approaches are outperformed in simulation and experiments as shown in (Matiakis & Hirche 2006). Here we demonstrate its efficacy by a numerical example.

6 Numerical Example

As plant we consider the NN8 example, extracted from the publicly available benchmark collection COMPl$_e$ib (Leibfritz 2004), regarding only its first input and output, resulting in a SISO system. The state space matrices are

$$A_p = \begin{bmatrix} -0.25 & 0.1 & 1 \\ -0.05 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{array}{l} B_p = [0\ 0\ 1]^T \\ C_p = [1\ 0\ 0] \\ D_p = 0. \end{array}$$

Three different controllers are compared. A linear quadratic regulator (LQR), with and without the transformation, and a small gain based controller with state feedback. The exact design procedure is described in the following.

6.1 Linear Quadratic Regulator

The controller h_c is an LQR for zero time delay minimizing the cost function

$$J = \int_0^\infty y^2(\tau) + 0.01u^2(\tau) d\tau. \tag{27}$$

A full state observer is computed with its poles placed at the real axis to [-2 -3 -4]. The overall observer based controller is given by

$$A_c = \begin{bmatrix} -8 & 0.1 & 1 \\ -240 & 0 & 0 \\ -3.122 & -0.339 & -4.387 \end{bmatrix},\ B_c = \begin{bmatrix} -7.8 \\ -239.95 \\ 6 \end{bmatrix},$$
$$C_c = [-9.122 - 0.339 - 3.387], \qquad D_c = 0.$$

6.2 Transformation

The LQR described in Section 6.1 is used as the pre-designed controller. The transformation M is designed by numerical optimization solving $\min_M \|G_{OL}\|_\infty$. The optimization is performed using `fminsearch` of the Matlab optimization toolbox. Note that the optimization problem is not convex. Therefore the optimization algorithm is executed starting from different random initial conditions M_0, and the best achieved (locally optimal) solution after several trials is applied. The computation of $\|G_{OL}\|_\infty$ is done by expressing $\|G_{OL}\|_\infty$ as an optimization problem with linear matrix inequality constraints, and using the YALMIP Matlab toolbox (Löfberg 2004) with the SDPT3 solver (Tutuncu, Toh & M.J. 2003). The best achieved solution is $\|G_{OL}\|_\infty = 0.5533$ for the transformation

$$M = \begin{bmatrix} 0.7778 & 5.2414 \\ 0.0474 & -11.9826 \end{bmatrix}. \tag{28}$$

6.3 Small gain based controller

For the small gain based controller the LQR state feedback problem is solved, formulated in LMIs (Boyd, Ghaoui, Feron & Balakrishnan 1994), with a additional small gain constraint of the open loop transfer function, which ensures delay-independent stability. The problem is described as

$$\textbf{minimize } x_0\mathbf{K_1}x_0 \ \textbf{ subject to}$$
$$\begin{bmatrix} 0.01I & B_p^T\mathbf{K_1} \\ \mathbf{K_1}B_p & A_p^T\mathbf{K_1} + \mathbf{K_1}A_p + C_p^TC_p \end{bmatrix} < 0,$$
$$\begin{bmatrix} Ap^T\mathbf{K_2} + \mathbf{K_2}A_p + \mathbf{K_1}B_pR^{-2}Bp^T\mathbf{K_1} & \mathbf{K_2}B_p \\ B_p^T\mathbf{K_2} & -I \end{bmatrix} > 0,$$
$$\mathbf{K_1} > 0,\ \mathbf{K_2} > 0,$$

where with bold letters the optimization parameters are denoted, and $x_0^T = [1\,1\,1]$ is the initial condition. For the solution the YALMIP Matlab toolbox (Löfberg 2004)

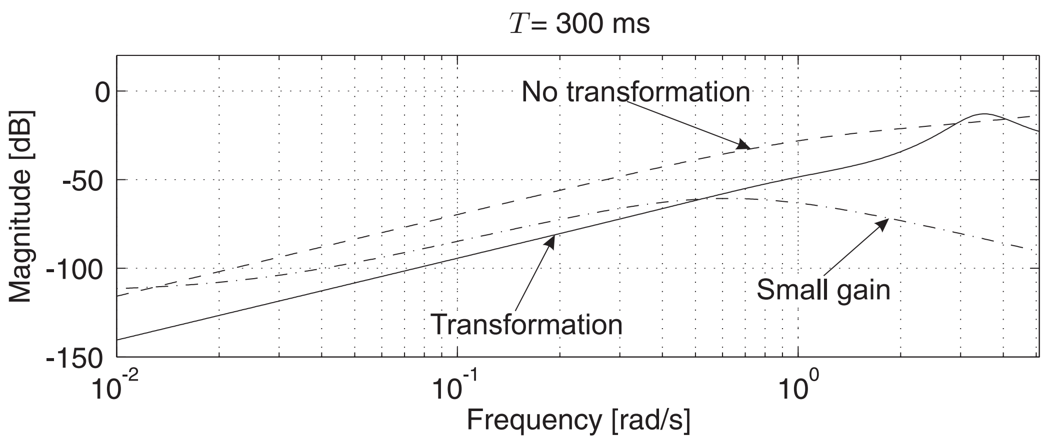

Fig. 4. Norm of the sensitivity to time delay function $|S_T^{G^*}|$ of the systems with the LQR with and without the input-output transformation, and the small gain based controller.

with the local solver PENBMI (Kočara & Stingl 2003) is used, trying several different initial conditions. The obtained state feedback is $F = [183.05\ 92.631\ 5.914]10^{-3}$.

6.4 Simulations

The norm of the sensitivity function with respect to time delay $|S_T^{G^*}|$ is shown in Fig. 4 for the three different approaches for round trip time delay $T = 300$ms, plotted until the maximum cutoff frequency of the three closed loop systems. The sensitivity of the proposed approach is less than the LQR without the transformation, in almost all the considered frequencies, except for a small range $\omega \in [10^{0.2} 10^{0.3}]$rad/s. The state feedback small gain based controller shows lower sensitivity in the higher frequencies, it is however very conservative as explained in the next. The response for the three approaches with initial condition $x_0^T = [1\,1\,1]$ and roundtrip time delay values $T = 0, 150, 300, 450$ms equally divided in the forward and backward channel are presented in Fig. 5. The system with the input-output transformation remains stable in all cases, and its response is only slightly affected by the time delay value. On the contrary, the system without the transformation is sensitive to the time delay, and becomes unstable for T=288ms. The response of the system with the state feedback small gain based controller is also slightly affected by the time delay value, but it is very conservative. The value of the cost function (27) for the simulation time horizon of 10sec, is further presented in Table 1, certifying that the proposed approach shows the best performance for increasing time delay values.

In short, compared to the LQR without the transformation, the proposed approach shows significantly lower sensitivity, and compared to the state feedback small based controller significantly better performance.

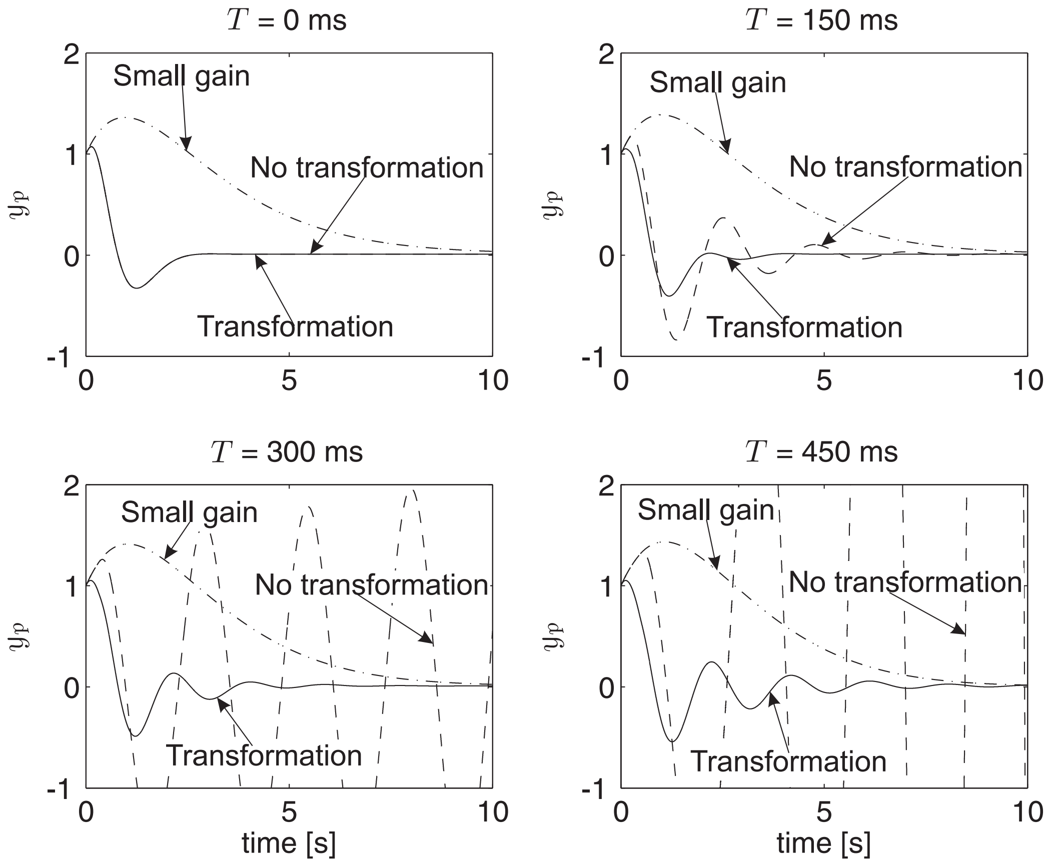

Fig. 5. Impulse response of the systems with the LQR with and without the input-output transformation, and the small gain based controller, for various time delay values.

Table 1
Cost function J for time horizon 10sec.

Time Delay [ms]	**0**	**150**	**300**	**450**
Transformation	0.2731	0.2887	0.3155	0.3529
LQR	0.2731	0.6146	unst.	unst.
Small gain	2.6110	2.6983	2.7876	2.8790

7 Conclusions

This article presents a novel distributed controller approach for delay-independent stability of NCS. The key idea is to use the limited computation power in the plant side to implement a transformation of the transmitted through the network signals. Instead of direct communication, a linear combination of plant and controller input and output is transmitted. In case of non-linear, IF-OFP systems with largely unknown model, delay-independent stability is guaranteed for every plant-controller pair which is stable without time delay based on their dissipativity parameters. A geometrical interpretation in terms of conic sectors is given. In case of LTI systems with known transfer functions, a necessary and sufficient stability condition is given. The proposed approach allows non-conservative controller design without considering time delay in the loop, resulting in a superior tracking performance. Due to the low sensitivity to time delay the performance remains good even for high time delay values. Simulations verify the proposed approach in a comparison with an LQR without the input-output transformation and with a small gain based state feedback controller. Future research addresses the investigation of more general transformations, robustness issues, time-varying delay and packet loss.

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References

Anderson, R. J. & Spong, M. W. (1989), 'Bilateral control of teleoperators with time delay', *IEEE Transactions on Automatic Control* **34**(5), 494–501.

Antsaklis, P. & Baillieul, J. (2007), 'Special Issue on Technology of Networked Control Systems', *Proceedings of the IEEE* **95**(1).

Berestesky, B., Chopra, N. & Spong, M. W. (2004), Discrete Time Passivity in Bilateral Teleoperation over the Internet, *in* 'Proceedings of the IEEE International Conference on Robotics and Automation ICRA'04', New Orleans, US, pp. 4557–4564.

Bonnet, C. & Partington, J. (1999), 'Bezout Factors and L1-Optimal Controllers for Delay Systems using a Two-Parameter Compensator Scheme', *IEEE Transactions on Automatic Control* **44**(8), 1512–1521.

Boyd, S., Ghaoui, L. E., Feron, E. & Balakrishnan, V. (1994), *Linear Matrix Inequalities in System and Control Theory*, 1 edn, SIAM, 3600 University City Science Center, Philadelphia, Pennsylvania 19104-2688.

Georgiou, T. & Smith, M. (1992), 'Robust Stabilization in the Gap Metric: Controller Design for Distributed Plants', *IEEE Transactions on Automatic Control* **37**(8), 1133–1143.

Gu, K., Kharitonov, V. & Chen, J. (2003), *Stability of Time-Delay Systems*, Birkhäuser.

Hale, J. K. & Verduyn Lunel, S. M. (1993), *Introduction to functional-differential equations*.

Hill, D. & Moylan, P. (1976), 'The Stability of Nonlinear Dissipative Systems', *IEEE Transactions on Automatic Control* **21**(5), 708–711.

Hirche, S. & Buss, M. (2004), Packet Loss Effects in Passive Telepresence Systems, *in* 'Proceedings of the 43rd IEEE Conference on Decision and Control', Paradise Island, Bahamas, pp. 4010–4015.

Hristu-Varsakelis, D. & Levine, W. S. (2005), *Handbook of Networked and Embedded Control Systems*, 1 edn, Birkhäuser.

Khalil, H. K. (1996), *Nonlinear Systems*, Pearson Education International, Prentice Hall.

Kočara, M. & Stingl, M. (2003), 'PENNON: a code for convex nonlinear and semidefinite programming', *Optimization Methods and Software* **18**.

Leibfritz, F. (2004), 'Compleib: Constraint matrix-optimization problem library - a collection of test examples for nonlinear semidefinite programs, control system design and related problems', *Technical Report, Department of Mathematics, University of Trier* .
URL: *http://www.complib.de*

Löfberg, J. (2004), YALMIP : A toolbox for modeling and optimization in MATLAB, *in* 'Proceedings of the CACSD Conference', pp. 284–289.
URL: *http://control.ee.ethz.ch/∼joloef/yalmip.php*

Lozano, R., Chopra, N. & Spong, M. (2002), Passivation of Force Reflecting Bilateral Teleoperators with Time Varying Delay, *in* 'Proceedings of the 8. Mechatronics Forum', Enschede, Netherlands, pp. 954–962.

Matiakis, T. & Hirche, S. (2006), Memoryless input-output encoding for networked systems with unknown constant time delay, *in* 'Proceedings of 2006 IEEE Conference on Control Applications', pp. 1707–1712.

Matiakis, T., Hirche, S. & Buss, M. (2005), The scattering transformation for networked control systems, *in* 'Proceedings of 2005 IEEE Conference on Control Applications', pp. 705–710.

Matiakis, T., Hirche, S. & Buss, M. (2006), Independent-of-delay stability of nonlinear networked control systems by scattering transformation, *in* 'Proceedings of the 2006 American Control Conference, ACC2006', pp. 2801–2806.

Matiakis, T., Hirche, S. & Buss, M. (2008), Local and remote control measures for networked control systems, *in* 'Proceedings of the 47th IEEE Conference on Decision and Control, CDC 2008'.

Miller, D. E. & Davison, D. E. (2005), 'Stabilization in the Presence of an Uncertain Arbitrarily Large Delay', *IEEE Transactions on Automatic Control* **50**(8), 1074–1089.

Munir, S. & Book, W. (2002), 'Internet Based Teleoperation using Wave Variable with Prediction', *ASME/IEEE Transactions on Mechatronics* **7**(2), 124–133.

Niemeyer, G. & Slotine, J. (1991), 'Stable adaptive teleoperation', *International Journal of Oceanic Engineering* **16**(1), 152–162.

Richard, J.-P. (2003), 'Time-delay systems: an overview of some recent advances and open problems', *Automatica* **39**, 1667–1694.

Secchi, C., Stramigioli, S. & Fantuzzi, C. (2003), Dealing with Unreliabilities in Digital Passive Geometric Telemanipulation, *in* ‘Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems IROS’, Vol. 3, Las Vegas (NV), US, pp. 2823– 2828.

Stramigioli, S. (2001), *Modeling and IPC Control of Interactive Mechanical Systems: a coordinate free approach*, Springer, London.

Tipsuwan, Y. & Chow, M. Y. (2003), ‘Control methodologies in network control systems’, *Control Engineering Practice* **11**, 1099–1011.

Tutuncu, R., Toh, K. & M.J., T. (2003), ‘Solving semidefinite-quadratic-linear programs using SDPT3’, *Mathematical Programming Ser. B* **95**, 189–217.

Willems, J. C. (1972*a*), ‘Dissipative Dynamical Systems - Part I: General Theory’, *Arch. Rational Mechanics Analysis* **45**, 321–351.

Willems, J. C. (1972*b*), ‘Dissipative Dynamical Systems - Part II: Linear Systems with Quadratic Supply Rates’, *Arch. Rational Mechanics Analysis* **45**, 352–393.

Zames, G. (1966*a*), ‘On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems, Part I: Conditions Derived Using Concepts of Loop Gain, Conicity, and Positivity’, *IEEE Transactions on Automatic Control* **11**(2), 228–238.

Zames, G. (1966*b*), ‘On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems, Part II: Conditions Involving Circles in the Frequency Plane and Sector Non-Linearities’, *IEEE Transactions on Automatic Control* **11**(3), 465–476.

A Appendix

The following lemma imposes restrictions on the eigenvalues of the dissipativity matrix P.

Lemma 1 *The dissipativity parameters $\delta, \varepsilon, \eta$ of all dissipative systems belong to the domain $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \{\delta, \varepsilon, \eta \in \mathbb{R} | \delta\varepsilon - \eta^2 < 0\}$ and $\Omega_2 = \{\delta, \varepsilon, \eta \in \mathbb{R} | \delta\varepsilon - \eta^2 = 0; \delta, \varepsilon > 0\}$.*

Proof: For convenient notation the proof is given for the SISO case. In case of MIMO system the proof is exactly the same, only the multiplicity of the eigenvalues changes accordingly. For $(\delta, \varepsilon, \eta) \in \overline{\Omega} = \Omega_3 \cup \Omega_4$ with $\Omega_3 = \{\delta, \varepsilon, \eta \in \mathbb{R} | \delta\varepsilon - \eta^2 > 0\}$, and $\Omega_4 = \{\delta, \varepsilon \in \mathbb{R} | \delta\varepsilon - \eta^2 = 0; \varepsilon, \delta < 0\}$ degenerate cases occur. The condition $(\delta, \varepsilon, \eta) \in \Omega_3$ is equivalent to positive or negative definiteness of matrix P, i.e. $\det P = \lambda_1\lambda_2 = \delta\varepsilon - \eta^2 > 0$ where λ_1, λ_2 are the two eigenvalues of P. Hence, $\lambda_1, \lambda_2 > 0 \Leftrightarrow P > 0$ or $\lambda_1, \lambda_2 < 0 \Leftrightarrow P < 0$. For $P > 0$ (1) is satisfied for any pair $u(\tau), y(\tau)$ imposing no restriction to the system input-output behavior. Analogously, for $P < 0$ (1) cannot be satisfied for any pair $u(\tau), y(\tau)$. In case

$(\delta, \varepsilon, \eta) \in \Omega_4$ we get $\lambda_1 = 0, \lambda_2 = -\delta - \varepsilon > 0$. Thus, P is positive semidefinite and (1) is again satisfied for any pair $u(\tau), y(\tau)$. ∎

Lemma 1 implies that without loss of generality we can restrict P to have either one positive and one negative, or one zero and one negative eigenvalue. For Ω the next lemma holds.

Lemma 2 *Consider the expressions*

$$\begin{aligned} \alpha(\theta) &= 2\eta \sin(\theta)\cos(\theta) - \delta\cos^2(\theta) - \varepsilon \sin^2(\theta) \\ \beta(\theta) &= \alpha(\theta) + \delta + \varepsilon \end{aligned}$$

where $\theta = \theta^+$ and $\theta = \theta^-$ are the two solutions of

$$\cot(2\theta) = \frac{\varepsilon - \delta}{2\eta}$$

in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and $(\delta, \varepsilon, \eta) \in \Omega$. The following statements are true:

- $(\delta, \varepsilon, \eta) \in \Omega_1$
  $\Rightarrow \alpha(\theta^+) > 0$, $\beta(\theta^+) > 0$, *and* $\alpha(\theta^-) < 0$, $\beta(\theta^-) < 0$
- $(\delta, \varepsilon, \eta) \in \Omega_2$
  $\Rightarrow \alpha(\theta^+) = 0$, $\beta(\theta^+) > 0$, *and* $\beta(\theta^-) = 0$, $\alpha(\theta^-) < 0$

Proof: For the two angles $\theta = \theta^+$ and $\theta = \theta^-$ it can be shown that $\alpha(\theta)\beta(\theta) = \eta^2 - \delta\varepsilon$. Thus for $(\delta, \varepsilon, \eta) \in \Omega_1$, $\alpha(\theta)\beta(\theta) > 0$ meaning that α, β have always the same sign for each angle. Furthermore $\alpha(\theta^-) = -\beta(\theta^+), \beta(\theta^-) = -\alpha(\theta^+)$ meaning that $\alpha(\theta), \beta(\theta)$ have always different signs for the two angle solutions $\theta = \theta^+$ and $\theta = \theta^-$. Combining the above the first part of the lemma is proved. For $(\delta, \varepsilon, \eta) \in \Omega_2$ we get $\alpha(\theta)\beta(\theta) = \eta^2 - \delta\varepsilon = 0$ meaning that $\alpha(\theta)$ and/or $\beta(\theta)$ are zero. Furthermore, $\beta(\theta) = \alpha(\theta) + \delta + \varepsilon \Rightarrow \beta(\theta) > \alpha(\theta)$. If $\alpha(\theta^+) = -\beta(\theta^-) = 0$ then $\beta(\theta^+) > 0, \alpha(\theta^-) < 0$; analogously for the other case $\alpha(\theta^-) = -\beta(\theta^+) = 0$. ∎