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# Equilibrium Analysis of Core-Selecting Auctions and the Impact of Risk Aversion and Allocation Constraints

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# Abstract

Combinatorial auctions are market institutions used for the simultaneous allocation of heterogeneous items to bidders, when items exhibit complementarities or substitabilites. The goals when designing such markets are the maximization of social welfare (efficiency) and the truthful revelation of bidders' preferences (incentive compatibility). Lately, 2008, the importance of core outcomes was also recognized and core-selecting combinatorial auctions were introduced (Day and Milgrom, 2008). Core outcomes address fairness issues and avoid renegotiations among bidders and the seller after the end of the auction procedure. Unfortunately, all three goals can only be satisfied in restricted cases, to which the thesis at hand refers as 'BSM world'. The thesis analyzes pricing rules and optimal bidding strategies in such auctions.

The first part of the thesis deals with allocation constraints and bidding languages, which are essential in most application domains of combinatorial auctions. Two different pricing rules for ascending combinatorial auctions which allow for flexibility in the allocation constraints and bidding languages are analyzed: winning levels, and deadness levels. Firstly, algorithms for these pricing rules are designed and their computational complexity is determined. Subsequently, it is shown that deadness levels actually satisfy an ex-post equilibrium, while winning levels do not allow for a strong game-theoretical solution concept. In a next step, the relationship of deadness levels and the simple price update rules, used in efficient ascending combinatorial auction formats, is investigated. Ascending combinatorial auctions with deadness level pricing rules maintain a strong game theoretical solution concept and reduce the number of bids and rounds required at the expense of higher computational effort. The calculation of exact deadness levels is a $\Pi_2^P$-complete problem. Nevertheless, numerical experiments show that for mid-sized auctions this is a feasible approach.

The second part of the thesis models core-selecting combinatorial auctions as

Bayesian games. In the non-BSM world, free riding incentives lead to outcomes that are further from the core than the VCG outcome and are inefficient. Risk aversion is arguably a significant driver of bidding behavior in high-stakes auctions. Therefore, the thesis examines the impact of risk aversion on equilibrium bidding strategies, efficiency and revenue in both ascending and sealed-bid core-selecting auctions. The examined environment models a threshold problem with one global and several local bidders. First, the necessary and sufficient conditions for the perfect Bayesian equilibria of the ascending core-selecting auction mechanism to have the small bidders to drop at the reserve price are characterized. The first main result is a generalization of the condition for a non-bidding equilibrium from Goeree and Lien (2009) and Sano (2012), which allows for arbitrary concave utility functions. In spite of free-riding opportunities of local bidders, risk-aversion reduces the scope of the non-bidding equilibrium in the sense that dropping at the reserve price ceases to be equilibrium as the bidders become more risk averse. Equilibrium bidding strategies and revenue for sealed-bid core-selecting auctions are also derived. The role of reserve prices is analyzed and a comparison of the efficiency and revenue of the various auction formats is conducted. Despite of the extreme free-riding equilibrium, the ascending auction outperforms the sealed-bid auction in terms of efficiency and revenue, when optimal reserve prices are set.

# Zusammenfassung

Die kombinatorischen Auktionen sind Marketinstitutionen für die gleichzeitige Allokation von heterogenen Gütern an Bieter. Sie werden eingesetzt, wenn die Güter Komplementaritäten oder Substitabilitäten aufweisen. Derer Entwurfsziele sind die Maximierung der sozialen Wohlfahrt (Effizienz) und die wahrheitsgemäße Offenbarung der Präferenzen der Bieter (Anreizkompatibilität). Die letzten Jahre wird auch große Bedeutung den Auktionsergebnissen, die im Core sind, beigemessen. Die core-selecting kombinatorischen Auktionen wurden 2008 von Day und Milgrom eingeführt. Ergebnisse im Core gewährleisten dass keine Nachverhandlungen zwischen den Bietern und dem Auktionator nach der Beendigung des Auktionsverfahrens durchgeführt werden. Alle diese Ziele sind jedoch nur in eingeschränkten Fällen miteinander vereinbar. Diese Fälle werden in der vorliegenden Arbeit als 'BSM world' bezeichnet.

Der erste Teil der Arbeit behandelt Allokationsregeln und Bietsprachen, die unerlässlich in den meisten Anwendungsdomänen von kombinatorischen Auktionen sind. Zwei unterschiedliche Preisregeln für aufsteigende kombinatorische Auktionen, die Flexibilität bzgl. der Allokationsregeln und der Bietsprachen ermöglichen, werden analysiert: Winning Levels and Deadness Levels. Algorithmen werden für diese Preisregeln konzipiert und ihre Zeitkomplexität wird ermittelt. Es wird gezeigt, dass Deadness Levels ein ex-post Equilibrium aufweisen. Im Gegensatz dazu erlauben Winning Levels kein starkes Lösungskonzept. Anschließend wird die Beziehung zwischen Deadness Levels und den simplen Preisregeln, die in etablierten effizienten aufsteigenden kombinatorischen Auktionsformaten verwendet werden, untersucht. Aufsteigende kombinatorische Auktionen mit auf Deadness Levels basierten Preisregeln bewähren ihr starkes Lösungskonzept auf Kosten von erhöhtem Rechenaufwand. Die exakte Berechnung von Deadness Levels ist ein $\Pi_2^P$-vollständiges Problem. Dennoch weisen numerische Experimente da-

rauf hin, dass für mittelgroße Auktionen die Berechnung von Deadness Levels durchführbar ist.

Der zweite Teil der Arbeit modelliert core-selecting kombinatorische Auktionen als bayesianische Spiele. In der 'non-BSM world' führen Free-Riding-Anreize zu Ergebnissen die ferner von dem Core wie das VCG Ergebnis liegen und ineffizient sind. Die Risikoaversion ist ein signifikanter Einflussfaktor des Bietverhaltens in Auktionen mit hohem Ertrag. Aufgrunddessen wird der Einfluss der Risikoaversion auf das Bietverhalten im Equilibrium, auf die Effizienz und auf den Auktionserlös sowohl in aufsteigenden als auch in sealed-bid core-selecting Auktionen analysiert. Die betrachteten Modelle stellen ein Thresholdproblem mit einem globalen und mehreren lokalen Bietern dar. Zuerst werden die notwendigen und hinreichenden Bedingungen charakterisiert, sodass im perfekten bayesianischen Equilibrium des aufsteigenden core-selecting Auktionsmechanismus die lokalen Bieter schon bei den Mindestpreisen aufhören zu bieten. Das erste zentrale Ergebnis ist eine Generalisierung dieser Bedingungen von Goeree and Lien (2009) und Sano (2012), die für alle Nutzenfunktionen gilt. Trotz der Free-Riding-Möglichkeiten der lokalen Bieter reduziert die Risikoaversion den Geltungsbereich des Equilibriums. Optimale Bietstrategien werden auch in sealed-bid core-selecting Auktionen abgeleitet. Die Rolle der Mindestpreisen wird analysiert und die Effizienz sowie der Auktionserlös von unterschiedlichen Auktionsformaten werden verglichen. Trotz des extremen Free-Riding-Equilibriums ist die aufsteigende Auktion der Sealed-Bid Auktion überlegen, wenn optimale Mindestpreise gesetzt werden.

# Acknowledgments

Although I sometimes do it in order to avoid missunderstandings, I do not like to say frequently “thank you” or “I love you”. The inflationary usage causes loss of meaning. I believe that one has to show his acknowledge with actions, with his everyday behavior and not with words.

I feel lucky to have met Martin in August 2006 and develop inspiration for his research topics. His guidance throughout all these years, the freedom he gave me, his trust and support, whenever I had an issue, made this work possible. Thank you Martin.

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I thank my parents for their support, patience and all the sacrifices they have done for me to enjoy the best possible education at our small home town and abroad. Ευχαριστώ Γιώργο και Κωνσταντίνα.

All errors, idiocies and inconsistencies are my own.

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# Introduction

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## 1.1 Auctions

Auctions are market institutions for resource allocation to self-interested agents. They define the space of possible actions of the agents, based on which the resource allocation and monetary payments among the agents are determined. One of the most widespread auction types is the ***English auction***. The auctioneer announces his intention to sell a single item for at least $x$ dollars (***reserve price***) and starting from this price bidders are invited to submit higher bids. A new bid must always exceed the current highest bid. After a certain termination condition is satisfied (no new bids in the last minute or predefined deadline) the auction ends and the bidder with the ultimately highest bid wins the item and pays the amount bid. Another iterative auction is the Dutch auction. The price is set at a high level and drops continuously until a bidder accepts to buy. Alternatives to the iterative auctions constitute the sealed-bid (one-shot) auctions. The bidders here cannot observe the bidding behavior of each other. Each bidder submits a single bid and the highest bidder wins and pays his own bid (***first price auction***) or the highest losing bid (***second price auction*** or ***Vickrey auction***). Even in the seemingly simple task of allocating a single item, a number of auction design issues arise. Should the auction be iterative or sealed-bid, should the winner pay his bid or receive a discount, how the reserve price should be set, should a bidder be allowed to bid at any point during the auction or forced to drop out in certain circumstances, when the auction should terminate, how much higher should a new bid be from the previously highest one? The answer to these questions necessitates firstly to define the desiderata of the auction. In the simplest case, the choice has to be made between maximizing the ***auctioneer's revenue*** and (allocative) ***efficiency***. Another term which is used interchangeably with efficiency is ***social welfare***, in the sense that the item should be allocated to the bidder who desires it at most and can produce the highest economic value out of it. Secondly, for each design choice the bidding behavior of the participating bidders has to be predicted. To this end, having some beliefs about the bidders' ***valuations*** (how much worth is the item to them, what is their maximum willingness to pay) and their ***risk aversion*** can be crucial. Changing the perspective from the mechanism designer to the bidders, the challenging question is to find the optimal bidding strategy. Obviously a bidder in a first price sealed-bid auction should bid less than his valuation otherwise his profit will be guaranteed non-positive. But how much less than his valuation should he bid? Such questions will be addressed in this thesis.

The first deployment of an auction has been reported by Herodotus, a fifth century (BC) Greek historian. In Babylon, once a year in each village the maidens who were about to get married were gathered together, sorted in descending order according to their beauty and then sequentially auctioned to candidate bridegrooms. The money gathered by auctioning the most beautiful maidens was given to the men who were assigned the ugliest ones (Rawlinson, 1885). Nowadays, auctions are being employed for more ethical purposes. Beside selling rare objects like art objects and antiques or perishable objects like fishes and flowers, auctions are heavily used by governments to sell treasury bills, public property or the rights to use it (e.g. drilling rights for oil). Procurement auctions in which bidders compete for the right to sell products or services are commonly used in G2B and B2B sector.

Auctions should be preferred for price formation when the seller is uncertain about the right price to sell the item. According to Paul Milgrom (Milgrom, 1989) auctions perform well when items are not standardized or the market clearing prices are highly unstable. Suppose that instead of auctioning the item, the seller attaches a fixed price to it. If two or more buyers are willing to pay the price, the seller can select one of them to sell. This is suboptimal in terms of both revenue and efficiency since it would be possible for the seller to receive a higher price and also it cannot be ensured that the bidder with the highest valuation gets the item, which impacts negatively the efficiency. If no buyer accepts the price, the item goes unsold. In these cases, the seller could do better by adjusting the price according to the demand. But this is exactly the essence of an iterative auction. A second possibility is to ask every buyer to declare his maximum willingness to pay, which gives us a sealed-bid auction. Another alternative to auctions is individual bargaining. Its shortcomings are that it is time consuming, if possible at all in the presence of numerous buyers, and that if the seller is not the owner of the item, he can manipulate the process to his advantage. Multilateral bargaining can also lead to disagreement and inefficiency and should be avoided when there is enough competition for auctions to be used (Milgrom, 1989). Starting with the work by the Nobel prize winner William Vickrey (Vickrey, 1961), theory on single-item auction has made great progress in the last fifty years. Two excellent surveys are Krishna (2002) and Klemperer (1999).

## 1.2 Combinatorial Auctions

A ***combinatorial auction*** (CA), which is the subject of this thesis, extends single-item auctions to markets comprising more than one item. Consider two adjacent parcels of land $A$ and $B$. Bidder $B1$, a manufacturing company wishing to build a new factory, has valuations $v_{B1}(A) = v_{B1}(B) = 10$ for each single parcel. If acquiring both, it has a higher valuation, $v_{B1}(AB) = 40$ [1], than the sum of the valuations of the single parcels since it can build a large factory and benefit from economies of scale. Bidder $B2$, another company, needs only parcel $A$ and values it at $v_{B2}(A) = 15$. Now suppose that a single-item auction is conducted for $A$ and subsequently another one for $B$. In the first auction, it is very probable that $B1$ will be outbid by $B2$. The reason is that if $B1$ pays more than 10 for $A$ in the first auction, it runs the risk of not winning $B$ in the second auction and thus realizing a negative payoff. The efficiency will be only $(15 + 10)/40 = 62.5\%$ (the efficient allocation is to allocate both parcels to $B1$). This problem of $B1$ is called the ***exposure problem*** and constitutes the main motivation behind CAs. In a CA, bidders

[1] The reader shall pay attention to this notation. The valuation refers to the case that the bidder acquires only the items listed in the argument.

are allowed to explicitly bid on combinations of items called ***packages*** or ***bundles***. Hence in a CA $B1$ can express his preferences by stating that he is willing to pay up to 40 only if acquiring both items. A second example of complementarities is depicted in Figure 1.1. A carrier located in Munich competes for the right to serve two delivery requests, one in Augsburg and one in Landsberg. Winning carriers receive \$1500 for each request. Assuming that his costs are $\$10/km$, his valuation of serving only Augsburg is $v(A) = 1500 - 2 \times 75 \times 10 = \$0$, only Landsberg $v(L) = 1500 - 2 \times 620 = \$260$ and both $v(AL) = 2 \times 1500 - 750 - 620 - 420 = \$1210$. The profit of both requests is much higher than the sum of the profits of each single request.

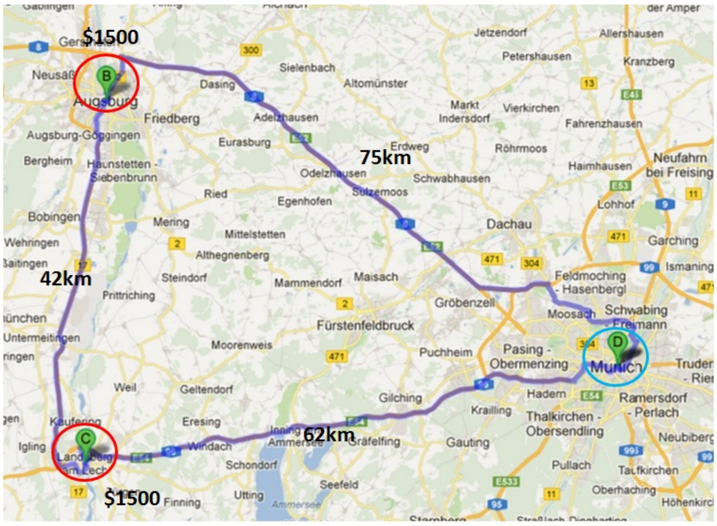


FIGURE 1.1: Superadditive Valuations

As the examples foreshadowed, CAs are pertinent when the bidder valuations are not additive, like the valuations of $B1$. For him, $A$ and $B$ are characterized as ***complements*** since their valuations are ***superadditive***, i.e. $v(AB) > v(A) + v(B)$. When the inequality is reversed, items are ***substitutes*** and the valuations ***subadditive***. For instance, an airline plans exactly one new flight landing at a certain airport and wishes a landing time slot in the evening. If $A$ and $B$ represent such time slots, then they are substitutes for the airline, since the airline has little interest in acquiring both of them. The second slot can have though a small marginal value if it prevents competitors from landing or due to the probability of planning a second flight in the future. It holds $v(AB) < v(A)+v(B)$. In case of additive valuations, i.e. $v(AB) = v(A)+v(B)$, item valuations do not depend on each other. Winning $A$ does not influence the willingness to pay for $B$. Hence separate single-item auctions should be used in this case instead of CAs.

The first proposal of a CA was by Rassenti et al. (1982) for allocating airport time slots. CAs gained prominence in 1994, when the US Federal Communications Commission decided to auction spectrum licenses to telecommunications service providers and it was recognized that licenses admit complementarities. Nowadays, CAs for spectrum are used in various countries (Cramton (2013), Bichler et al. (2013a)). Other applications include industrial procurement (Bichler et al., 2006), truckload transportation (Caplice, 2007), energy markets (Meeus et al., 2009), bus routes (Cantillon and Pesendorfer, 2006) and housing space in a new building (Goossens et al., 2011). Advances in Information Technology contributed decisively to the realization of complex markets such as CAs. Their design has attracted, due to its complexity, academic interest in the disciplines of Computer Science, Economics and Operations Research.

The main source of complexity in CAs is that there are $2^n$ different packages, with $n$ denoting the number of auctioned items, and bidders may have valuations for each one of them. As we will see, the problem of determining the winning bids (and thus the efficient allocation if bids correspond to real valuations), known as the ***Combinatorial Allocation Problem*** (CAP, also known as the Winner Determination Problem), is a $NP$-complete problem. Communication complexity describes the number of messages that have to be exchanged between the auctioneer and the bidders in order to compute the efficient allocation. Nisan and Segal (2006) show that an exponential number of messages are required. Coping with computational and communication issues is not the end of the story. Strategic complexity arises since bidders should decide what is the best bidding strategy for them. Should they bid on all packages they have positive valuations for? How much should they bid on each of them? In an iterative CA, such decisions must be made in every round. And before being able to answer these questions, bidders should have determined their valuations for the $2^n$ packages. This task requires often significant effort since companies have to analyze thoroughly various business cases and scenarios. This type of complexity is called valuation complexity.

The primary objective of CAs is to find an efficient allocation. By doing so, also revenue can be impacted positively since high levels of efficiency induce large bidder participation and bidders have incentives to bid truthfully. Having mentioned this, the revenue maximizing CA even for a market with two items and two bidders is unknown in general [2]. Finding the efficient allocation requires that bidders are incetivized to reveal their valuations truthfully, otherwise the optimization problem is based on false input and the efficient allocation cannot be computed [3]. There are many notions of ***incentive compatibility***, called ***solution concepts*** in the game theoretical literature (auction is regarded as a ***game***, bidders as ***players*** or ***agents***). The strongest solution concept is the ***dominant strategy***. A player following this strategy maximizes his payoff whatever the strategies and the ***types*** (the type of a player encapsulates all his private information and determines his payoff for each outcome of the game; in the context of CAs it determines his valuation of every package) of the other players may be. In other words, he does not need to speculate either about their behavior or their types. This is the case in a second price single item auction and every player reveals his valuation truthfully. The reasoning is based on the fact that the winner cannot influence the price he pays, hence he cannot profit by deviating and shading his true valuation. A weaker concept, applied mostly to iterative (multistage, dynamic, extended form) games is the ***expost Nash equilibrium***. Following the strategy prescribed in this equilibrium is optimal for a player if all other players do the same. This holds for all types the other players may have. Hence players do not need to speculate about types but only about strategies. An expost Nash equilibrium admit iterative CAs in certain cases. Unfortunately, equilibria that take away so much strategic complexity from the bidders do not always exist. In a ***Bayes Nash equilibrium*** the prescribed strategies are optimal exante if they are played by every player and if there is a common prior belief, not necessarily symmetrical, about the players' types. Translated in the context of auctions, a common prior belief could be that the valuations of bidders for a certain package are uniformly distributed between 0 and 1. Exante implies here that the strategy is optimal only in expectation, since the types of the opponents and hence their actions in the game are not known with certainty. The exante optimal strategy could be improved (theoretically, if we could go back in time) after having observed the opponents' bids. A refinement of this equilibrium in iterative games is the ***Perfect Bayes Nash equilibrium***. Another goal when designing CAs, and generally mechanisms, is ***individual rationality***. The strongest form of individual rationality is the expost individual rationality and prescribes that

[2] Progress in this area has been made by T.Sandholm, the interested reader is referred to Sandholm et al. (2012)

[3] In single item auctions truthful revelation is not required as long as the bidding function is an increasing function of bidders' valuation for the item. In such case the bidder with the highest valuation submits the highest bids and wins, hence the allocation is efficient.

players never realize negative payoffs whereas exinterim individual rationality only poses this restriction in expectation, i.e. after a player has learned his type and before the game has been played. Individual rationality is desirable since it induces voluntary participation. ***Weak budget balance*** is achieved by a mechanism if it does not need to be subsidized.

The four theoretical desiderata presented so far are achieved by an extension of the Vickrey (second price) auction to the combinatorial environment. In this auction, called VCG due to Vickrey, Clarkes and Groves (Vickrey (1961), Clarke (1971), Groves (1973)), each bidder pays the externality he exerts on the rest bidders and his payoff is his marginal contribution to the social welfare (difference in the value of efficient solution with and without him). This is why he has a truthful dominant strategy. Despite these astonishing properties (it is far from obvious that such an auction is possible in a so complex environment), the VCG auction is not used often due to numerous shortcomings (Ausubel and Milgrom, 2006b). One of them is that the outcome of the auction, defined as the pair of allocation and prices, is not always in the ***core***. After the end of the auction there may be a ***blocking coalition*** of bidders who are unsatisfied and can negotiate with the seller a different outcome in which every one, including the seller, is better off. Note that in the Vickrey auction, this cannot happen. To see this, suppose three bidders with valuations $v_1 = 100, v_2 = 70, v_3 = 50$. Since they are truthful, the bids are equal to their valuations. The first bidder wins and pays the second price 70. Hence the seller obtains a revenue of 70 and could be better off only if a bidder or bidder coalition offers him more. But there is no losing bidder who can do this, since no one has a valuation greater than 70. Back to the CA environment, the failure of the VCG auction to produce core outcomes, its proneness to shill bidding and empirical observations of low revenue gave rise the last decade to a family of CAs, the ***core-selecting combinatorial auctions*** (Day and Milgrom, 2008). But unfortunately VCG is the only efficient [4] CA in which truthful bidding is a dominant strategy. Hence the following impossibility result: There is no CA which always satisfies incentive compatibility in truthful dominant strategies and produces core outcomes. In contrast to VCG, core-selecting CAs do always produce outcomes within the core. This core is the one computed based on the bids and deviates from the true core if bidders do not bid truthfully on every single package. But at least, even within this core, there is no losing

[4] Unless otherwise stated this term is used in the sequel in a strict sense: An outcome with less than 100% efficiency is not referred as efficient but e.g. as an outcome with a high efficiency level.

bidder (or bidder coalition) who can complain to the seller that he bid more on a package than what the winner of this package paid. The core is defined by a set of constraints that prevent the existence of blocking coalitions and contains a multitude of outcomes. Various payment rules have been proposed that specify how one of the many core outcomes should be selected. All rules select in unison an outcome from the ***bidder-optimal*** frontier of the core while they differ in the selection of one of the many bidder-optimal outcomes. An outcome is bidder-optimal in the Pareto sense: there is no other outcome weakly preferred by every bidder. If and only if a condition called ***bidders are substitutes*** (BAS) concerning bidders' valuation profiles is satisfied, then the VCG outcome is in the core and it is the only bidder-optimal outcome. Thus sealed-bid core-selecting auctions admit a dominant strategy when BAS is common knowledge. For ascending CAs to reach this point, a stronger condition, the ***bidder submodularity*** (BSM), is sufficient. In this case the ***straightforward bidding strategy***, which prescribes to always bid on the package(s) that maximize bidder's payoff given prices, is an expost Nash equilibrium. Figure 1.2 by Bichler et al. (2013a) shows how widespread core-selecting auctions have become the last years (CCA and sealed-bid CA are ascending and sealed-bid core-selecting auction formats respectively, while SMRA represents an alternative, not combinatorial auction).

## 1.3 Research Objectives

Figure 1.3 shows the embedment of the thesis' two main research objectives (RO) in the theoretical framework of core-selecting CAs introduced in Section 1.2. The satisfaction or not of bidder submodularity divides the theory in two worlds and is the point of departure of the thesis, which deals with both worlds. The interesting research questions in the BSM world diverge from the questions in the non-BSM world, since theoretical results in the former are far more advanced than in the latter.

In the BSM world, the dominant strategies in sealed-bid CAs and expost NE in ascending CAs are powerful solution concepts and characterize the auction's outcome which is efficient and in the true core. This allows the exploration of additional real-world elements of CAs. RO1 deals with allocation constraints, which dictate which allocations are feasible and which not. While its theoretical results regarding efficiency and solution concepts can be applied only in the BSM world, the design artefacts developed there can be used in any iterative

| Country | 700/800/900 MHz | 1800 MHz | 2600 MHz |
|---|---|---|---|
| Australia | CCA (2013) | - | CCA (2013) |
| Austria | CCA (2013) | - | CCA (2010) |
| Denmark | - | SMRA (2010) | CCA (2010) |
| Germany | SMRA (2010) | | |
| Finland | - | - | SMRA w. switching (2009) |
| France | Sealed-bid CA (2011) | - | Sealed-bid CA (2011) |
| Hong Kong | - | - | SMRA |
| Netherlands | - | CCA (2012) | CCA (2010) |
| Italy | SMRA w. ranking (2011) | | |
| Ireland | CCA (2012) | | |
| Norway | - | - | SMRA w. switching (2007) |
| Spain | SMRA (2011) | Beauty contest | SMRA (2011) |
| Sweden | SMRA w. switching (2011) | CCA (2011) | SMRA w. switching (2008) |
| Switzerland | CCA (2012) | | |
| UK | CCA (2012) | - | CCA (2012) |

M. Bichler, P. Shabalin, and J. Wolf. Efficiency, auctioneer revenue, and bidding behavior in the combinatorial clock auction. Experimental Economics, 2012.

FIGURE 1.2: Core-Selecting Auctions in Use for Spectrum Sale Worldwide

CA in order to facilitate bidders to submit meaningful bids. On the other side, RO2 develops theory in the non-BSM world. There, no strong solution concept is possible and all that is known about equilibrium behavior concerns simple environments with a restricted number of bidders and packages for which bidders have positive valuations. While the results in these environments cannot completely forejudge the performance of CAs in general, they shed the first rays of light on how the bidders behave, how much they shade their valuations and on the implications to efficiency and revenue. Equilibrium analysis until now, concentrated only on risk-neutral utility functions. RO2 canvasses the impact of risk aversion on the equilibrium bidding strategies in the non-BSM world. In the BSM world, the strong solution concepts are independent of any risk profile.

### 1.3.1 RO1: Allocation Constraints

A multitude of bidding languages and allocation constraints are used in a variety of application domains of CAs such as transportation or industrial pro-

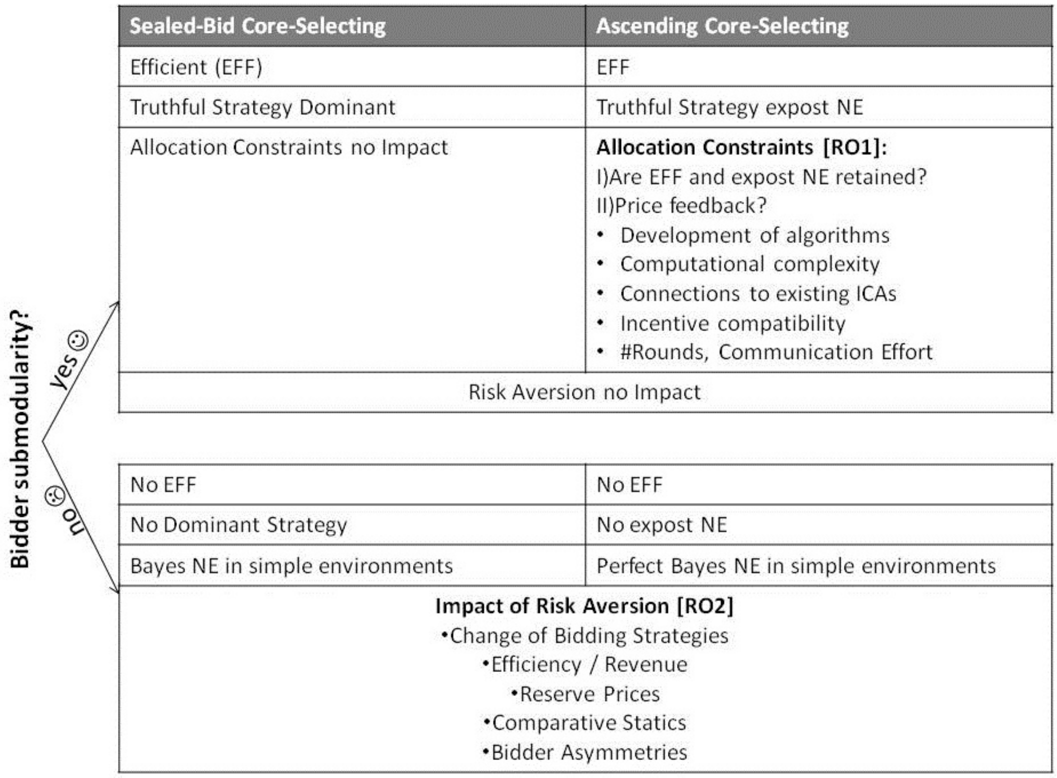


FIGURE 1.3: Research Objectives

curement. For instance in a procurement auction the auctioneer may wish only allocations with at most $x$ different winners in order to avoid contractual and transactional costs. A government selling spectrum licenses will have interest that at least a certain number of companies get licenses allocated, otherwise the outcome may induce undesired oligopolies. A bidder may wish to bid on various two-item packages but specify at the same time that he does not need more than 4 items in total. This flexibility in the bidding languages and the allocation constraints is essential in these domains, but has not been considered in the theoretical literature so far. In RO1, different pricing rules for ascending CAs are analyzed which allow for such flexibility: winning levels, and deadness levels. The computational complexity of these pricing rules is determined and is shown that deadness levels actually satisfy an expost equilibrium, while winning levels do not allow for a strong game-theoretical solution concept. The relationship of deadness levels and the simple price update rules used in efficient ascending combinatorial auction formats is investigated. It is shown that ascending CAs with deadness level pricing rules maintain a strong game theoretical solution concept and reduce the number of bids and rounds required at the expense of higher computational effort. The calculation of exact deadness levels is a $\Pi_2^P$-complete problem. Nevertheless, numerical experiments show that for mid-sized auctions this is a feasible approach. RO1 provides a foundation for allocation constraints in CAs and a theoretical framework for recent Information Systems contributions in this field.

### 1.3.2 RO2: Risk Aversion

Risk aversion is thought an important factor influencing bidder behavior in the non-BSM world and there is strong experimental evidence that bidders are risk averse (Kagel and Levin, 2012). Generally risk aversion seems more plausible to hold in reality than risk-neutrality. When the amounts at stake are relatively large, such as in auctions, humans are risk-averse. For instance they strictly prefer to receive one million dollars with certainty than participating in a lottery which yields ten millions with probability 10% and otherwise zero. The RO2 aims to analyze the impact of risk aversion in both ascending and sealed-bid core-selecting auctions.

In the first part, the necessary and sufficient conditions are derived for the perfect Bayesian equilibria of the ascending core-selecting auction mechanism to have the small bidders to drop at the reserve price for general environments with risk-averse bidders. Small bidders, who either win all together or lose all together, intend in this case to free-ride on each other, i.e. to force other small bidders to carry alone the burden of overbidding the large bidders and therefore winning by paying less than them. The first main result is a generalization of the condition for a non-bidding equilibrium in Sano (2012), which allows for arbitrary concave utility functions. Second, this condition is discussed in the presence of asymmetries, and third, comparative statics are provided. It is shown that risk aversion and bidder asymmetries affect the equilibrium outcomes in ways that can be systematically analyzed. In particular, risk-aversion reduces the scope of non-bidding equilibrium in the sense that dropping at the reserve price ceases to be equilibrium as the bidders become more risk averse. Different wealth levels and stochastic dominance orderings of the valuation distributions and their impact on non-bidding are analyzed as well.

In the second part, the impact of risk aversion on the bidding strategies in the Bayes Nash equilibrium of the sealed-bid core-selecting auction with the closest-to-Vickrey payment rule, which is used by almost all, if not all, sealed-bid CAs of Figure 1.2, is analyzed. Firstly, the existence of an equilibrium in nondecreasing bidding strategies is shown. Secondly, it is shown that the higher the risk-aversion, the lower the incentives to shade the valuations and bid much lower than them. As in first-price single-item auction, bidders bid higher as it were to buy insurance against the possibility of losing (Krishna, 2002). Bidder asymmetries and the role of reserve prices is analyzed. The bidding strategies are given explicitly for parametric cases and efficiency and revenue are evaluated.

## 1.4 Outline

- The present chapter introduced the reader to the main concepts of the thesis, using intentionally textual descriptions and avoiding symbols and mathematical formalism.
- Chapter 2 presents the theoretical background in a more rigorous manner.
- Chapter 3 deals with allocation constraints in iterative CAs.
- Chapter 4 provides game-theoretical treatment of iterative core-selecting CAs with risk-averse bidders.
- Chapter 5 provides game-theoretical treatment of sealed-bid core-selecting CAs with risk-averse bidders.
- Chapter 6 contains concluding thoughts.

Chapter 3 is based on a paper accepted by the INFORMS Information System Research (ISR) journal (Petrakis et al., 2012), is joint work with G. Ziegler and M. Bichler and addresses RO1. Chapter 4 and 5 are joint work with K. Guler and M. Bichler and address RO2. Chapters 3-5, which are the main building blocks of this thesis, repeat intentionally definitions of Chapter 2 to facilitate the reader who wishes to read them independently and without needing to leaf backwards to find relevant definitions. They also contain an independent introduction and conclusion.

# Chapter 2

# Theoretical Background

Jedes überflussige Wort wirkt
seinem Zweck gerade entgegen.

A. Schopenhauer

This chapter presents the theory underlying the research work of the thesis and builds on the introductory chapter.

Section 2.1 presents the equilibrium solution concepts that will be used throughout the thesis.

Section 2.2 introduces formally the environment of CAs. The Combinatorial Allocation Problem has to be solved by any CA. The VCG auction is the only auction where bidders have a truthful dominant bidding strategy. Unfortunately, its outcomes are not always in the core.

In Section 2.3 the core and the sealed-bid core-selecting auctions are presented. If the condition 'bidders are substitutes' is fulfilled, VCG is in the core and bidders have a truthful dominant strategy. Else, the optimal bidding strategies are optimal only in expectation and with the assumption that the distributions of bidders' valuations are common knowledge. These strategies are derived by a Bayes Nash equilibrium analysis.

Section 2.4 revises the theoretical foundations of ascending core-selecting auctions. Efficient auctions in this family, such as iBundle, APA and dVSV are designed to converge to a competitive equilibrium, which implies a core outcome. The design is based on linear programming and duality theory. The competitive equilibrium is of utmost importance since every efficient auction has to compute it. When the bidder submodularity condition is fulfilled, these auctions admit an expost Nash equilibrium, else Bayesian analysis has to be conducted, just as in the sealed-bid case when the 'bidders are substitutes' condition does not hold. The 'bidder are substitutes' condition implies bidder submodularity but not vice versa.

Section 2.5 comments on the limitations of theory and provides references to experimental work which complements the theory.

## 2.1 Game Theory

Many definitions in this section are adopted from Shoham and Leyton-Brown (2009).

### 2.1.1 Games

Static (one-shot) games are represented as normal-form games.

**Definition 2.1.** *A* ***normal-form game*** *is a tuple* $(\mathcal{I}, A, u)$ *where:*

- $\mathcal{I}$ *is the set of* $m$ *players (agents) indexed by* $i$
- $A = A_1 \times ... \times A_m$*, where* $A_i$ *is a set of actions available to player* $i$
- $u = (u_1, ..., u_m)$ *where* $u_i : A \Rightarrow \mathbb{R}$ *is the real-valued utility function of player* $i$

With $a_i \in A_i$ being the action chosen by $i$, the vector $a = (a_1, ..., a_m) \in A$ is called an ***action profile***. Players choose actions in order to maximize their expected utility (payoff). The payoff for each player is determined based on the chosen action profile and is common knowledge to every player. Since every player knows everything about the game (the number of players, their available actions and their payoffs) games in normal-form are called ***complete information*** games. An auction is a complete information game only if every valuation of every player is known to everyone (hence payoffs are known) and this is common knowledge (everyone knows that everyone knows every valuation). A player follows a ***pure strategy*** when it chooses one single action to play. If it randomizes and plays certain actions with certain probability, then he follows a ***mixed strategy***. In other words, a mixed strategy $s_i$ of $i$ is a probability distribution over his available actions $A_i$ and $s_i \in S_i \equiv \Pi(A_i)$, where $\Pi(A_i)$ the set of all probability distributions over $A_i$. The vector $s = (s_1, ..., s_m) \in S_1 \times ... \times S_m$ is called a (mixed) ***strategy profile***.

When the game has a temporal structure, like chess where a move of a player is followed by a move of his opponent, it can be modeled in ***extensive-form*** using a tree. Each tree-node represents the choice of one of the players, each edge a possible action, and the leaves the final outcomes over which each player has a utility function. When the node at each point in time is observable by all players, there is ***perfect information***. An iterative auction where bidders do not fully observe the behavior of others (they may get some feedback via prices) is an extensive form game of imperfect (and incomplete if payoffs unknown) information. Extensive-form games can be converted in normal-form at the expense of redundant, exponential representation.

Often, especially in auctions, payoffs of bidders are private information. In this case, the game played is called Bayesian (incomplete information). Although only uncertainty about payoffs is explicitly modeled, it can be shown that even uncertainty about the number of players or their available auctions can be modeled by reducing it in uncertainty about payoffs.

**Definition 2.2.** *A **Bayesian game** is a tuple* $(\mathcal{I}, A, \Theta, p, u)$ *where:*

- $\mathcal{I}$ *is the set of* $m$ *players (agents) indexed by* $i$
- $A = A_1 \times ... \times A_n$*, where* $A_i$ *is the set of actions available to player* $i$
- $\Theta = \Theta_1 \times ... \times \Theta_m$*, where* $\Theta_i$ *is the type space of player* $i$
- $p : \Theta \Rightarrow [0, 1]$ *is a common prior over types*
- $u = (u_1, ..., u_m)$ *where* $u_i : A \times \Theta \Rightarrow \mathbb{R}$ *is the real-valued utility function of player* $i$

Comparing to Definition 2.1, the first thing to notice are the players' types. A player's type encompasses all his private information. In auctions, this information is typically his valuation profile and possibly his beliefs about the opponents' profiles or their beliefs about his profile. The common prior $p$ encompasses information shared by every bidder and is nothing else than a probability distribution over the players' types. For example, in a single-item auction every bidder may know that every bidder's valuation is drawn from a uniform distribution [0,1]. The distribution is known to everyone whereas the realization of the random variable for each bidder is only known to him and constitutes his type. Another difference to Definition 2.1 is that the utility function not only depends on the action profile but also on the types. The definition of strategies must also be slightly adjusted in Bayesian games to account for the fact that they depend on types. A pure strategy $a_i : \Theta_i \Rightarrow A_i$ maps each type to an action and a mixed strategy $s_i$ is a probability distribution over pure strategies.

### 2.1.2 Solution Concepts

Solutions concepts deal with the strategies that will be played in the games and predict in this way their outcome. Every player $i$ wants to choose a strategy profile that maximizes his payoff. This is a complicated task since it depends on the strategies the other players choose. If he could somehow know these strategies, denoted as $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..s_m)$, then he could compute his best-reply to these strategies. In the rock-paper-scissors game if he knew the opponent will play paper, then his best-reply would be scissors.

**Definition 2.3.** *The **best-reply** of player $i$, given $s_{-i}$ is $s_i^* \in S_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_i \in S_i$.*

Now if in a strategy profile every player's strategy is a best reply to the strategies of the opponents, then the strategy profile is called Nash equilibrium.

**Definition 2.4.** *A strategy profile $s = (s_1, ..., s_m)$ is a **Nash equilibrium** if $s_i$ is best reply to $s_{-i}$ for every player $i$.*

Nash equilibrium is a stable outcome in the sense that unilateral deviations of the chosen strategies are never beneficial. Every game has at least one Nash equilibrium (Shoham and Leyton-Brown, 2009) but it may not be in pure strategies. It can be easily verified that this is the case in the rock-paper-scissors game. A player $i$ knowing the strategy of his opponent $j$, computes his best reply on $j$'s strategy and wins whereas opponent $j$ loses. The strategy of opponent $j$ is thus not best reply to the strategy of $i$. Thus, there is no

Nash equilibrium in pure strategies. In the Nash equilibrium, both players randomize among the three pure strategies with equal probabilities. There is a considerable amount of literature devoted on the conditions under which a pure strategy equilibrium can be guaranteed (Maskin and Dasgupta (1986), Milgrom and Roberts (1990)). In auction games, pure increasing strategies are of interest, i.e. strategies that fulfill the property that the higher the valuation, the higher the bid. Athey (2001) examines such conditions in games of incomplete information and her theory applies also to auction games. Another desirable characteristic of Nash equilibrium is the uniqueness. A multitude of equilibria weakens the predictive power of this solution concept since bidders may be unsure about which equilibrium strategies will be played by opponents and coordination problems between them arise. The complexity of computing a Nash equilibrium has been a recent topic of research, with the main finding being that there exists not always a polynomial algorithm (Daskalakis et al., 2006).

A Bayes Nash equilibrium finds application in games of incomplete information. For ease of notation, the definition of pure equilibria is given below. Symbol $\theta_{-i}$ is defined analogously to $s_{-i}$ and $p(\theta_{-i}|\theta_i)$ are the beliefs of $i$ about opponents' types. Since types may be interdependent, learning the own type $\theta_i$ may update the beliefs about other types and this is why the conditional probability is used.

**Definition 2.5.** *A strategy profile $s = (s_1, ..., s_m)$ is a (pure)* ***Bayes Nash equilibrium*** *if* $s_i(\theta_i) \in \arg\max_{s_i' \in S_i} \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i(s_i', s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \quad \forall i, \theta_i$

Note that players maximize their ***exinterim expected utility***, i.e. the expected utility after learning their own type and before learning the opponents' types. If they knew all types, they would possibly revise their optimal strategy and maximize their ***expost utility***.[1] In extensive form games, where players act sequentially, the Bayes Nash equilibrium is refined and called ***perfect Bayes equilibrium***. At every point in the game, players must act optimally and according to a belief system.

If learning the opponents' types would never lead in revising the optimal strategy, then the equilibrium is much stronger since no assumptions about the

[1] Here it is dispensed with the term "expected" since it is assumed that the opponents' strategies and types are known and the strategies are pure, hence there is no stochastic element anymore.

distribution of these types have to be made. In this case the equilibrium is called expost Nash equilibrium.

**Definition 2.6.** *A strategy profile* $s^* = (s_1, ..., s_m)$ *is an* ***expost Nash equilibrium*** *if* $s_i(\theta_i) \in \arg\max_{s'_i \in S_i} u_i(s'_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \quad \forall i, \theta_i, \theta_{-i}$

The strategies here are optimal for every $\theta_{-i}$, hence no distributional knowledge of the opponents' types is necessary. It has only to be speculated that opponents will play their equilibrium strategies. Sometimes even that is dispensable; if the strategy is optimal whatever the strategies and the types of the opponents are, then this strategy is called dominant.

**Definition 2.7.** *A strategy profile* $s = (s_1, ..., s_m)$ *is a* ***dominant strategy equilibrium*** *if* $s_i(\theta_i) \in \arg\max_{s'_i \in S_i} u_i(s'_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \quad \forall i, s_{-i}, \theta_i, \theta_{-i}$

Obviously a dominant strategy equilibrium is also a Nash equilibrium.

To summarize, the strongest solution concept is dominant strategy. Bidders do not have to speculate neither about opponents' strategies nor types. The second strongest solution concept is the expost Nash equilibrium, where only speculation about opponents' strategies are needed. In the weaker concept of Bayes-Nash equilibrium, bidders speculate about strategies and types.

### 2.1.3 Application in Single-item Auctions

It is instructive to apply the presented solution concepts in single-item auctions. The strategies here will be called bidding strategies and denoted by $\beta(v)$, where $v$ is the type of a bidder, i.e. his valuation for the item. Valuations are supposed to be private, independent and identical distributed and bidders symmetric and risk-neutral.

**Theorem 2.1.** *In the second-price sealed-bid auction,* $\beta(v) = v$ *is a dominant strategy.*

While the complete proof can be found in Krishna (2002), the intuition is the following: The winner cannot influence by his bid the price $p$ he pays, since the price is equal to the bid of the second-highest bidder. Hence a bidder's bid only decides whether he wins or not. By bidding above $v$ he runs the risk that $p > v$ and thus incurring a loss. Since he only wants to win if $p < v$, bidding above $v$ is never beneficial to him compared with bidding $v$. Furthermore, bidding below $v$ instead of $v$ will never reduce his payment but may only turn him from winner to loser. Thus it is optimal to bid $v$. The proof holds for every strategy and valuation of the opponents, hence the derived bidding strategy is dominant. An English auction, where there is an ascending price clock and bidders declare their willingness to accept current prices by e.g. keeping a button pressed, is strategically equivalent to the second-price sealed-bid auction. The clock stops when the second-highest bidder drops (releases the button) and the winner pays not his true valuation, which remains unrevealed, but the second-highest valuation. In an English auction, the difference between expost equilibrium and dominant strategies is subtle: While the just described auction with the clock and the button admits a dominant strategy, an ***English outcry auction*** does not. In this auction, bidders call out their bids and jump bids are allowed. Bidders must not accept prices during the auction and may remain silent until the last moment. This is the most known auction format to the wide public, since by its means famous artworks are usually auctioned.

**Theorem 2.2.** *In the English outcry auction, $\beta(v) = v$ is an expost Nash equilibrium.*

If every opponent behaves truthfully, bidding gradually up to $v$ is expost Nash equilibrium. The striking question is why it is not a dominant strategy. Suppose a somewhat benighted opponent who will bid a huge amount, over his true valuation, if he hears someone else bidding 110, a number he detests. When the current highest bid is at 100, and minimum increment 10, then the strategy of gradually increasing the bid by one increment is worse than another strategy that prescribes a jump bid at 100. Thus, it is not always the optimal strategy to increase the bids truthfully by the minimum amount required. It is only optimal if every opponent follows it and that is why it is not dominant.

Moving to first-price auctions, being truthful yields always a zero payoff. Hence, the optimal amount of ***bid shading*** $v - b(v)$, which in this case represents the targeted payoff, must be determined. The trade off bidders face is that the higher the amount of bid shading, the higher their payoff in case they win but the lower the probability of winning. To determine the optimal amount they need to know the strategies and the valuations of the opponents.

**Theorem 2.3.** *In the first-price sealed-bid auction with $m$ bidders, whose values are uniformly distributed on $[0, 1], \beta(v) = \frac{m-1}{m}v$ is a Bayes Nash equilibrium.*

*Proof.* Only the sketch of the proof is given with the aim being to illustrate that distributional assumptions have to be made in order to derive the equilibrium, which is unique here. It will be sought for a symmetric equilibrium in increasing strategies. Suppose all bidders but bidder 1 follow the symmetric, increasing and differentiable strategy $\beta$. The expected payoff of bidder 1 by bidding b is $(v - b) \times ProbWin(b)$. The probability of winning $ProbWin(b)$ is equal to the probability that $b$ is higher than all other bids or equivalently higher than the highest of them. This highest bid is submitted by the bidder with the highest valuation $Y_1$, since the equilibrium is in increasing strategies. Hence $b > \beta(Y_1)$ is equivalent to $\beta^{-1}(b) > Y_1$. The cumulative distribution of $Y_1$, denoted by $G$, is known and is nothing else than the distribution of the $(m-1)$-th order statistic of $m-1$ uniformly distributed variables on $[0, 1]$. Thus $G(x) = x^{m-1}$ and the payoff can be rewritten as $(v - b) \times G(\beta^{-1}(b))$. Setting the derivative w.r.t. b equal zero, then setting $b = \beta(x)$ since a symmetric equilibrium is sought, and solving the resulting differential equation, yields $b(v) = \frac{m-1}{m} v$. In a next step it can be verified that $b(v)$ is indeed an equilibrium (Krishna, 2002). □

The equilibrium strategy in case of two bidders is that they bid 50% of their true valuations while with ten bidders 90%. The more bidders, the higher the competition and the lower the optimal amount of bid shading. The equilibrium in increasing bidding strategies implies that the auction is fully efficient, since the item is always allocated to the bidder with the highest valuation. Quite surprisingly, under the assumptions of this section, all three formats examined in this subsection yield the same revenue, although the bidding strategies in the three auction formats examined differ. This result is known as the ***revenue equivalence theorem***, introduced by Myerson (1981) and Riley and Samuleson (1981). The theorem does not only hold for these three formats: Any symmetric and increasing equilibrium of any auction format, such that the expected payment of a bidder with value zero is zero, and the bidder with the highest bid wins, yields the same expected revenue. The theorem does not require any specific form for payments, hence even an all-pay auction, by which every bidder pays his bid even if he loses, yields the same expected revenue. Bidders will obviously bid lower amounts than in the first price auction and in expectation their bids sum up to the bid of the winner of a first-price auction.

When bidders are risk-averse, the revenue equivalence theorem does not hold. While risk aversion has no impact on the equilibrium strategies of the English and second-price sealed-bid auction, it leads bidders in first-price sealed-bid auctions to submit higher bids and hence the revenue increases (Maskin and Riley, 1984).

## 2.2 Combinatorial Auctions Basics

The introductory chapter discussed when and why CAs should be used instead of single-item auctions. Here the CA environment is defined formally. There are $n$ indivisible heterogeneous items [2] to be auctioned to $m$ bidders. Symbols $\mathcal{K}$ and $\mathcal{I}$ denote the item and bidder set respectively. Bidders submit bids for packages of items. A package is denoted by $S$ and $S \subseteq \mathcal{K}$. Every bidder $i$ has a valuation $v_i(S)$ for every package $S$. His ***valuation profile*** refers to all of his valuations. A CA collects systematically bids $b_i(S)$ and outputs an allocation $X$ of the items to the bidders and payments $p$ from bidders to the seller. The ***outcome*** of an auction is a pair $(X, p)$[3]. The outcome determines unambiguously the payoff of bidders and the seller. In the environment of study following assumptions are made, unless otherwise specified:

- Utilities or payoffs are quasi-linear and equal to the valuation of the package received minus the price paid, i.e $v_i(S) - p_i(S)$. Note that in case of an OR bidding language where a bidder wins two bids, he effectively gets allocated one single package which is the union of all items won. This $S$ is considered then in the formula for his payoff, since $v_i(S)$ is the valuation of bidder $i$ when he wins only the items in $S$ and no more or less. Quasi-linearity also implies the absence of budget constraints since if $p_i(S) \leq v_i(S)$ but $p_i(S) \geq budget$, then the payoff cannot be positive.

- Valuations satisfy $v_i(S) \geq v_i(T), T \subseteq S$. The economic interpretation is that winning additional items never reduces the valuation of a bidder; if he does not need these items, he can dispose them at no cost (free disposal).

- Every bidder knows only his own valuations and they do not depend on others' valuations (private independent valuations).

[2]Items will be treated as heterogeneous but this does not exclude the possibility that some of them are identical. Identical items means effectively that every bidder assigns the same value to each one of them.

[3]This simple definition is of utmost importance. Readers shall notice the difference between an allocation and an outcome.

- The valuations of the auctioneer for the items are normalized to zero. His payoff is the revenue of the auction, i.e. the sum of bidders' payments.

### 2.2.1 Combinatorial Allocation Problem

The combinatorial allocation problem $CAP$, also known as Winner Determination Problem, is the problem of determining the winning bids and is inherent to almost every CA format. Sealed-bid formats solve $CAP$ at least once whereas iterative formats usually solve it in every round. The objective is to maximize the social welfare which is equal to the sum of the values of the winning bids, provided that bidders bid their true valuations. For notational convenience the symbol for valuations is used to indicate the bids which are the input to $CAP$. The first constraint of $CAP$ is that no item can be allocated more than once (resource constraint). The bidding language specifies how bids should be interpreted by the auctioneer and is modeled by one or more constraints. The XOR bidding language means that a bidder does not wish more than one bid of him to win while OR poses no such constraint and multiple bids of one bidder can win. Allocation constraints further preclude certain bid combinations from winning and can be posed by bidders themselves in case no XOR is used (e.g. "I want to win at most 3 items") or by the auctioneer ("no bidder is allowed to win more than 3 items of a certain category). The following version of $CAP$ is known as $CAP - I$ and models the resource and XOR constraint. Bichler and Kalagnanam (2005) and Klimova (2008) discuss how to model additional allocation and further types of constraints.

$$\begin{aligned}
\max \sum_{S\subseteq\mathcal{K}} \sum_{i\subseteq\mathcal{I}} x_i(S) v_i(S) & \\
\sum_{S\subseteq\mathcal{K}} x_i(S) \leq 1 & \qquad \forall i \subseteq \mathcal{I} \\
\sum_{S:l\in S} \sum_{i\subseteq\mathcal{I}} x_i(S) \leq 1 & \qquad \forall l \in \mathcal{K} \\
x_i(S) \in \{0,1\} & \qquad \forall i \subseteq \mathcal{I}, S \subseteq \mathcal{K}
\end{aligned}$$

$CAP - I$ is a NP-hard problem, with or without the XOR constraint:

**Theorem 2.4.** *(Rothkopf et al. (1998)) The decision version of $CAP - I$ without the XOR constraint is NP-complete even for instances where every bid has a value equal to 1, and every bidder bids only on subsets of size of at most 3.*

**Theorem 2.5.** *(Van Hoesel and Müller (2001)) The decision version of $CAP - I$ with the XOR constraint is NP-complete even for instances where every bid has a value equal to 1, every bidders bids only on subsets of size of at most 2.*

The reductions are conducted from the set-packing problem and the three-dimensional matching problem respectively. Further results regarding computational complexity of $CAP$ can be found in Lehmann et al. (2006) and Mueller (2006). Sandholm (2002) provides optimal algorithms. Sandholm (2012) reports optimal solving of $CAP$ with more than 2.6 million bids and 300.000 constraints.

### 2.2.2 VCG Auction

Solving the $CAP$ to optimality does not yield the efficient allocation if not all bidders' valuations are revealed truthfully. The only auction where truthfully revealing all valuations is a dominant strategy, is the Vickrey-Clarkes-Groves or VCG auction (Green and Laffont, 1977; Holmstrom, 1979). In the VCG auction, winners are determined by solving the $CAP - I$ (with the XOR constraint [4]) and charged the opportunity costs, or equivalently the negative externality they impose to others. In the Vickrey auction, if the three highest bids are \$10, \$7 and \$5, the opportunity costs the winner pays are \$7 because if he were not present, the second highest bidder would win and \$7 in total would be generated from his trade with the seller. The third bidder would not win if the first bidder were not present, hence no negative externalities are imposed to him. The same ideas apply to the VCG auction, with the difference being that due to package bidding it is more complicated to compute the opportunity costs. Following definition is needed:

**Definition 2.8.** *The* ***coalitional value function*** *$V(J)$ with $J \subseteq \mathcal{I} \cup 0$, with $0$ denoting the seller and $\mathbb{C}$ the set of feasible allocations, is defined by:*

$$V(J) = \max_{(S_1,\ldots,S_m)\in\mathbb{C}} \sum_{i\in\mathcal{I}} v_i(S_i) \qquad \text{if } 0 \in J$$

$$V(J) = 0 \qquad \text{otherwise}$$

[4] XOR is the only language permitting full expressiveness in the general case, which is a precondition for full efficiencyNisan (2006).

Symbol $V(J)$ denotes the value that bidders in coalition $J$ can create by trading among themselves (Ausubel and Milgrom, 2006a). The seller has to participate in the coalition, otherwise the value is zero. Symbol $V(J)$ is also the value of $CAP$ when only the bids of bidders in coalition $J$ are considered. The VCG payments are then computed as follows:

**Definition 2.9.** *The* ***VCG payment*** *of bidder $i$ winning $S$ by submitting a bid of value $b_i(S)$ is*

$$p_i^{VCG} = b_i(S) - (V(\mathcal{I}) - V(\mathcal{I}\backslash i))$$

The term $V(\mathcal{I}) - V(\mathcal{I}\backslash i)$ is called ***VCG discount*** and is equal to the ***VCG payoff*** of $i$ since $b_i(S) = v_i(S)$. Hence a bidder in the VCG auction receives a payoff equal to his marginal contribution to social welfare. To see why the payment is equal to opportunity costs, rewrite it as $p_i^{VCG}(S) = V(\mathcal{I}\backslash i) - (V(\mathcal{I}) - b_i(S))$. The first term is the total payoff of all $i$'s opponents and the seller when $i$ is not present and the second term the payoff of all $i$'s opponents and the seller when $i$ is present.

**Example 2.1.** *Two items A, B are auctioned by means of VCG to three single-minded bidders $B1, B2$ and $B3$. Their valuations are $v_{B1}(AB) = 10, v_{B2}(A) = 7, v_{B3}(B) = 7$. Bidders $B2$ and $B3$ win and $p_{B1}^{VCG} = p_{B2}^{VCG} = 7-(14-10) = 3$.*

Example 2.1 shows how VCG payments are computed and at the same time discloses the major drawback of the design (further drawbacks are discussed in Rothkopf (2007) and Ausubel and Milgrom (2006b)). The outcome is not in the core because the losing bidder $B1$ bids more than what the winners pay, which is $3 + 3 = 6 < 10$.

## 2.3 Sealed-Bid Core-Selecting Combinatorial Auctions

### 2.3.1 Core and Payment Rules

**Definition 2.10.** *The* **core** *of a game is defined by a set of payoff vectors, with $\pi = (\pi_0, \pi_1, ..., \pi_m)$ and $\pi_i$ being the payoff of $i$:*

$$Core(\mathcal{I}, V(\cdot)) = \left\{ \pi : V(\mathcal{I}) = \sum_{i\in\mathcal{I}} \pi_i, V(J) \le \sum_{i\in J} \pi_i \quad \forall J \subseteq \mathcal{I} \right\}$$

The definition implies that every core outcome is efficient since $V(\mathcal{I}) = \sum_{i \in \mathcal{I}} \pi_i$. In an outcome out of the core there is at least one coalition $J$ such that $V(J) > \sum_{i \in J} \pi_i$ which is called ***blocking coalition***. Although payoff constraints define the core, they can be translated into price constraints, since payoff is the difference of valuation and price. In Example 2.1 the sum of the prices should be at least 10 so that $B1$ does not block the outcome and no higher than 14 else no bidder could pay them and they would lead to zero payoffs which are out of the core (payoffs must sum to 14). Furthermore, the price of each single item must be at most 7. Figure 2.1 shows the core of that game. The core is the blue triangle defined by the price constraints. Whereas

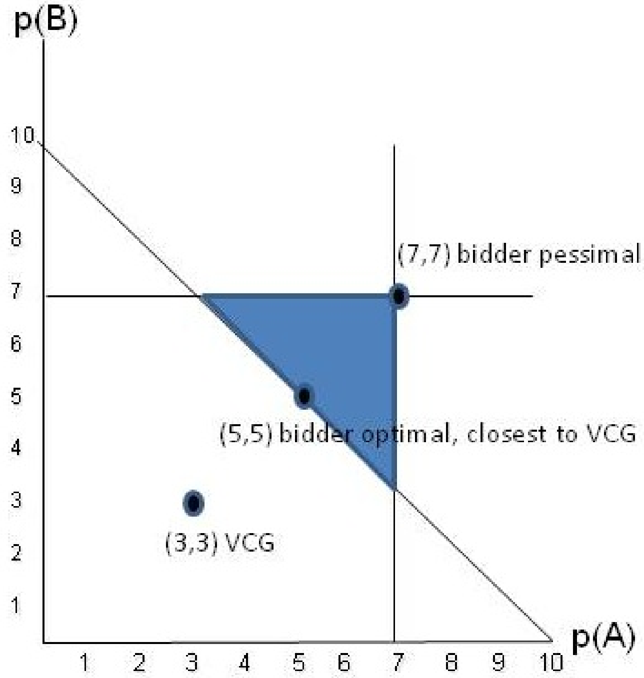


FIGURE 2.1: The core of Example 2.1

the VCG prices (3, 3) are unique, a multitude of prices can be selected from the core. Day and Cramton (2012) and Day and Milgrom (2008) suggest selecting a bidder-optimal outcome in order to minimize the bidders' incentives for bid shading. Specifically, the sum of each bidder's maximum possible gain from unilaterally deviating from bidding truthfully is minimized. Their arguments are based on a complete information equilibrium analysis.

Core-selecting auctions are CAs that produce core outcomes. A subtle but important distinction is between the true and the reported core. Since in core-selecting auctions truthful revealing the whole valuation profile is not an equilibrium strategy in general, the coalitional value function and hence the core is computed based on the reported valuations (bids) which may differ from the real ones. Symbol $\widehat{V}(\cdot)$ denotes the coalitional value function based on the reported valuations and leads to the reported core. The true core is the core based on the true (possibly unrevealed) valuations [5]. The definition of core-selecting CAs makes explicit the fact that the outcomes are in the reported and not the true core:

**Definition 2.11.** *A combinatorial auction resulting in payoff vector $\pi$ is **core-selecting**, if $\pi \in Core(\mathcal{I}, \widehat{V}(\cdot))$*

**Definition 2.12.** *A **bidder-optimal** core outcome is a core outcome with payoff vector $\pi$ if there is no alternative payoff vector $\pi'$ in the core such as $\pi' \geq \pi$ (pointwise) with strict inequality for at least one bidder.*

In Figure 2.1 all outcomes on the hypotenuse of the triangle are bidder-optimal. Day and Cramton (2012) suggest to use the bidder-optimal point which is closest to the VCG point, based on the euclidean distance. This payment rule is called ***closest to Vickrey payment rule*** and currently is preferred unanimously by auction designers in the field. Erdil and Klemperer (2010) propose an alternative payment rule, the ***reference rule***. Instead of focusing on maximum possible gains from bid shading, they argue that marginal incentives for truthful behavior matter at most for the robustness of a mechanism. According to them, bidders in complex environments may not have a clear view of the full space of alternatives, but they may identify how to gain with small deviations from truthful behavior. In Example 2.1, if the first small bidder deviates from 7 to a bid of 6, then the VCG point moves from (3, 3) to (3, 4) and the closest to Vickrey point in the core from (5, 5) to (4.5, 5.5). Thus, by deviating by 1 from truth telling, the first bidder increases his payoff by 0.5. Hence Erdil and Klemperer (2010) examine rules that minimize gains from marginal deviations. In this example, their rule would be to select a bidder-optimal point in the core, which is not closest to the VCG point, but closest to another point of reference payments on the bidder-optimal frontier of the core. This point of reference payments is not computed based on the winners' bids, unlike the

[5] To which core is referred in the following, should be clear from the respective context.

VCG point, hence winners have less marginal incentives to deviate from truth. Further payment rules are described in Ausubel and Baranov (2010).

### 2.3.2 Bidders are Substitutes: Dominant Strategy Equilibrium

Example 2.1 demonstrated that the VCG outcome may not fall into the core. This was not just a coincidence. Thanks to Bikhchandani and Ostroy (2002), it can be completely characterized when VCG is in and when out of the core. As one may expect, since the definitions of core and VCG outcomes are based on valuation functions, whether VCG is in the core depends on the valuation profiles of the bidders:

**Definition 2.13.** *The* ***bidders are substitutes*** *(BAS) if:*

$$V(\mathcal{I}) - V(\mathcal{I}\backslash J) \geq \sum_{i \in J} (V(\mathcal{I}) - V(\mathcal{I}\backslash i))$$

Such a condition underlies, for example, the belief that workers are better off when forming a union (left side of the inequality) rather than when bargaining individually with management (right side of the inequality) (Bikhchandani and Ostroy, 2002). The term substitutes comes from the fact that removing single bidders from the grand coalition $\mathcal{I}$ is not so effective as removing all of them [6].

**Theorem 2.6.** *(Bikhchandani and Ostroy, 2002) The VCG outcome is in the core if and only if BAS is satisfied.*

What is more, in this case the bidder-optimal frontier in the core reduces to one single point and every payment rule which selects an outcome in this frontier, coincides with the VCG payment rule.

**Theorem 2.7.** *(Ausubel and Milgrom, 2006b) If the VCG outcome is in the core, then it is the unique bidder-optimal point in the core and it is* ***bidder-dominant*** *(preferred unanimously by all bidders). If the VCG outcome is not in the core, there is a multiplicity of bidder-optimal points in the core, none of which is bidder-dominant.*

[6] In a lunch buffet with various meat dishes, vegetables and sweets, removing one sweet at a time is almost negligible since there are other sweeties fulfilling similar needs. But removing all of them together, leads in a massive decrease in buffet's quality. Sweets are in this case substitutes as bidders in the BAS condition are.

From a practical point of view, the satisfaction of BAS must be common knowledge else bidders in an auction with core payments may not play the truthful equilibrium strategy since they are not sure whether BAS holds.

In environments with bidders, each of whom desires exactly one package, Sano (2011a) provides a wonderful interpretation of the BAS condition. Before it can be enjoyed, two more simple definitions are required:

**Definition 2.14.** *Bidders are **single-minded**, if for every bidder $i$ there is a package $S_i$ such as: $v_i(S) = v_i(S_i)$ if $S \supseteq S_i$ and $v_i(S) = 0$ otherwise.*

**Definition 2.15.** *Bidder $j$ is **rival** of $i \neq j$ if and only if $S_j \cap S_i \neq \emptyset$. Else they are friends. $R_i$ denotes the set of rivals of $i$.*

Intuitively two bidders are rivals if they compete for overlapping packages (e.g. $AB$ and $BC$) and therefore cannot win together. Sano's triangular condition states that "every rival of my rivals is my rival":

**Definition 2.16.** *The **triangular condition** is satisfied if and only if for every bidder $i$: $\forall j \in R_i, R_j \subset R_i + i$*

**Theorem 2.8.** *(Sano, 2011a) In environments with single-minded bidders, BAS is satisfied if and only if the triangular condition is satisfied.*

In Example 2.1 the condition is not satisfied since the rival of $B2$ is $B1$ and $B1$ is rival of $B3$. But $B2$ and $B3$ are not rivals (Figure 2.2).

### 2.3.3 Bidders are not Substitutes: Bayes Nash Equilibrium

When BAS does not hold, it is not optimal to bid truthfully. While generally setting the optimal bidding behavior remains a mystery, following economically motivated setting has been thoroughly analyzed: There are three single-minded bidders and two items $A$ and $B$. Bidders $B1, B2$ are called small or local bidders and want only $A$ and $B$ respectively. The third bidder $B3$ is the large or global bidder and has a positive valuation only for the package $AB$. One can imagine that the small bidders are telecom providers operating on a local basis, e.g. in the eastern and western part of a country respectively and the global bidder is a provider interested only in covering the whole country. Two allocations are possible: either the two local bidders win together or the

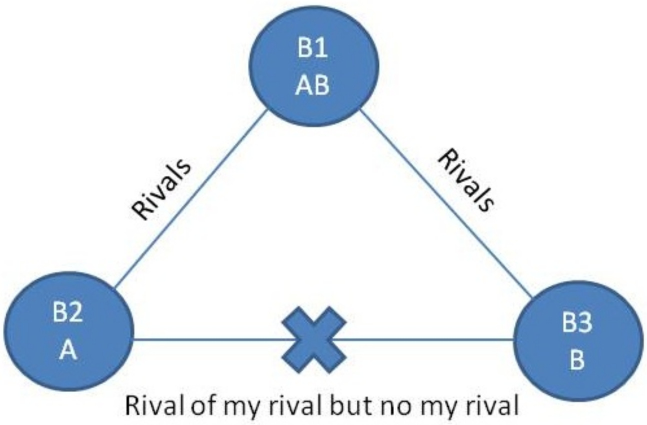


FIGURE 2.2: The triangular condition. Each node represents a single-minded bidder. The first line in each node gives the bidder's name and the second line the package he desires.

global bidder alone. Whenever the efficient allocation is the first one, then the VCG outcome is not in the core and BAS is not satisfied. The violation of BAS can be observed in Example 2.1: Let $J = \{B1, B2\}$ be the coalition of the two small bidders: then $V(B1, B2, B3) - V(B3) = 14 - 10 = 4$ and $V(B1, B2, B3) - V(B1, B3) + V(B1, B2, B3) - V(B2, B3) = 14 - 10 + 14 - 10 = 8 > 4$. Therefore BAS is violated.

The global bidder has always, under any bidder-optimal core payment rule, a truthful dominant strategy. The reason is that when he wins he always pays the sum of the small bidders' bids. His payment does not depend on his bid and therefore its optimal for him to be truthful. The argument is the same as in the single-item second price auction: If he shades his bid, his payment in case of winning does not change; he only decreases his chances of winning. On the other side, small bidders' payments depend on the value of their bids and they have incentives to shade them. Under the closest-to-Vickrey payment rule, the payment of a small bidder who wins is (denote the bids of the three bidders as $b_1, b_2$ and $b_3$) (Goeree and Lien, 2012):

$$p_1 = \max(0, b_3 - b_2) + \frac{1}{2}\{b_3 - \max(0, b_3 - b_2) - \max(0, b_3 - b_1)\}$$

The last term reveals that if his bid is below the global bidder's bid, shading it by $\epsilon$ reduces his payment by $\frac{\epsilon}{2}$. When shading the bid, there is a trade off between decreasing the payment in case of winning and at the same time decreasing the probability of winning. Hence the optimal bid depends on the distributions of the valuations of the second local bidder and the global bidder. When the distributions of the small bidders are uniform in the interval $[0, 1]$ and of the global bidder uniform in $[0, 2]$, there is a closed-form expression for the bidding strategies:

**Theorem 2.9.** *(Goeree and Lien, 2012) The Bayes Nash equilibrium of the closest-to-Vickrey core-selecting auction is given by the following strategies: The global bidder bids* $b_3(v) = v$ *and each small bidder* $b(v) = max(v - (3 - 2\sqrt{2}), 0)$

Thus every small bidder shades his valuation by about 0.17.

**Example 2.2.** *Suppose the three bidders have valuations* $v_1 = v_2 = 0.8, v_3 = 1.5$ *respectively. Their equilibrium bids are* $b_1 = b_2 = 0.63, b_3 = 1.5$ *and since* $b_3 > b_2 + b_1$ *the global bidder wins although the efficient allocation is that the small bidders win. The outcome is in the reported core but not in the true core, since the allocations in the true core are efficient.*

Example 2.2 illustrates a shortcoming pertinent in all core-selecting auctions. Whenever BAS is not satisfied, their outcome is always out of the true core.

**Theorem 2.10.** *(Goeree and Lien, 2012) If BAS is not satisfied, there exists no Bayesian compatible auction with outcomes in the true core.*

This impossibility result raises doubts on the core-selecting auctions because they fail to achieve their primary goal, to be in the (true) core. Nevertheless, they are in the reported core and this is appreciated in the field.

The Bayes Nash Equilibrium of further payment rules in this setting is analyzed by Ausubel and Baranov (2010). Their work sheds light on the case where the valuations are not independent but correlated. Correlation changes considerably the bidding strategies and has different effects on different rules. The authors provide revenue and efficiency rankings of payment rules for various levels of correlation.

## 2.4 Ascending Core-Selecting Combinatorial Auctions

An ***iterative combinatorial auction*** allows bidders to submit multiple bids during the course of the auction and provides information feedback, based on which bidders can revise their bids or submit new ones (Parkes, 2006). An important subclass of them are the ***ascending combinatorial auctions***. The information feedback includes prices (and possibly further information) which do not drop in the course of the auction.

Although the English auction is strategically equivalent to the second-price sealed-bid auction, it is much more popular. Iterative CAs are also often preferred to their sealed-bid pendants. While VCG requires a bid on every package from every bidder, which must be equal to the corresponding valuation, iterative auctions can reach the same outcome as VCG without the need of a bid on every package and even without disclosure of the valuations. Iterative CAs have thus lower preference elicitation costs and privacy is preserved up to a certain extent. They are more transparent and let the bidders understand their valuation profiles better through the iterative process, which is particularly important when the valuations are interdependent.

### 2.4.1 Ascending Auctions as Indirect Mechanisms

Auctions were considered in Section 2.1 as games. The discipline of mechanism design aims to design the rules of the game so that defined desiderata are fulfilled. In this context, auctions are referred as mechanisms. A mechanism defines the available actions or message space of the players and an outcome based on the chosen actions. The outcome consists of a choice from a set of alternatives (allocations in the case of auctions) and payments. Sealed-bid auctions are direct-revelation (henceforth direct) mechanisms due to the fact that the available message space of the players is to direct report their whole valuation function (their type).

**Definition 2.17.** *A (deterministic)* **direct mechanism** *consists of a function* $x : \Theta \Rightarrow \mathcal{X}$*, where* $\mathcal{X}$ *the set of alternatives and a payment function* $t : \Theta \Rightarrow \mathbb{R}^m$*.*

**Definition 2.18.** *A (deterministic)* **indirect mechanism** *consists of a strategy space* $\Sigma = S_1 \times ... \times S_m$*, a function* $x : \Sigma \Rightarrow \mathcal{X}$*, where* $\mathcal{X}$ *the set of alternatives and a payment function* $t : \Sigma \Rightarrow \mathbb{R}^m$*.*

A strategy $s_i(\theta_i) \in S_i$ defines the message(s) that an agent will send to the mechanism for all possible types and information states. An information state delineates a possible state of the indirect mechanism, and a fully specified strategy should define a message to send for all possible information states (Parkes, 2008). Translated to the CA environment, an iterative auction is an indirect mechanism since bidders are not requested to report their valuation function, but rather they are requested to accept prices or submit bids based on some price or other feedback. For instance they answer questions like "which package do you prefer at most given these prices?" [7]. An information state is what bidders observe during the process of the iterative auction such as prices and other bids.

A famous result in mechanism design is the ***revelation principle***.

**Theorem 2.11.** *(Krishna (2002)) Given an indirect mechanism and an equilibrium of it, there exists a direct mechanism in which i)it is a Nash equilibrium when each player reports his true type and ii)the outcomes are the same in both mechanisms.*

As a consequence, for any indirect mechanism which admits an expost Nash equilibrium, a direct mechanism can be constructed which admits a truthful dominant strategy (Parkes, 2008). Thus, if an iterative CA with an expost equilibrium is encountered, it can be concluded that a sealed-bid CA with truthful dominant strategies can be constructed. The opposite is also true. David Parkes refers to it as "***unrevelation principle***":

**Theorem 2.12.** *Given is a direct mechanism $(x, t)$ with truthful dominant strategies. Then strategy profile $s^*$ is an expost Nash equilibrium of an indirect mechanism $M$ if:*
*i)every player $i$ plays $s_i^*$, then $M$ produces the outcome $(x(\theta), t(\theta))$ for all $\theta \in \Theta$*

*ii)given $\theta_i \in \Theta_i, \theta_{-i} \in \Theta_{-i}$ and all players but $i$ follow $s_{-i}^*$, the outcome of $M$ is $(x(\theta_i', \theta_{-i}), t(\theta_i', \theta_{-i}))$ for some $\theta_i' \in \Theta_i$ ($\theta_i'$ may not be the true type of $i$, $\theta_i$)*

*Proof.* Fix all strategies of players other than $i$ and all types (inclusive $i$). The proof is based on the fact that if an alternative strategy $s_i'$ is followed, which leads to reporting type $\theta_i'$, produces an outcome for $i$ better than following $s_i^*$, then $s^*$ is not a dominant strategy equilibrium in the direct mechanism. □

[7] This point of view of auctions gives rise to a strand of literature dealing with preference elicitation via queries (Blumrosen and Nisan, 2005).

The unrevelation principle has a direct application in CAs: The VCG auction admits an equilibrium in truthful dominant strategies. If an iterative CA can be constructed which leads to VCG outcomes for a strategy profile, then it admits an expost Nash equilibrium. This is crucial in designing and analyzing iterative CA formats as it will become obvious in the next subsection.

Iterative CAs differ from sealed-bid CAs in that they request from bidders to submit bids in more than one point in time. The most important family follows the scheme in Figure 2.3. In each round bids of all bidders are collected. Subsequently $CAP$ is solved and price feedback and provisionally winning bids are communicated. When every bidder is 'happy' with the outcome, the auction terminates.

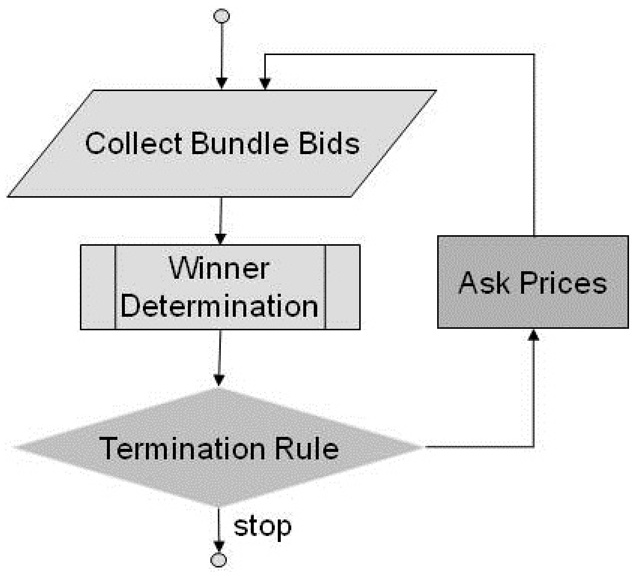


FIGURE 2.3: An Iterative Combinatorial Auction

### 2.4.2 Competitive Equilibrium

Auctions in this family are designed to terminate when a competitive equilibrium (CE) is reached. Competitive equilibrium extends the concept of Walrasian equilibrium to CAs and is not a variation of the Nash equilibrium. Given prices, in a CE every bidder wins the package he desires at most (this may also be the empty package if prices are too high for him), hence he is happy, and the seller is also happy in the sense that he chooses to allocate the items so that his revenue is maximized [8]. Example 2.3 stresses the fact that not only the bidders but also the seller must be happy:

**Example 2.3.** *There are two bidders with valuations $v_{B1}(AB) = 10, v_{B1}(C) = 2, v_{B2}(B) = 5$. The prices are $p(AB) = 10, p(C) = 0, p(B) = 2$. With these prices $B1$ desires package $C$ and $B2$ package $B$. This allocation is feasible since the packages do not overlap but is neither efficient ($B1$ gets $AB$ in the efficient allocation) nor given the prices in a CE (seller prefers the efficient allocation). It will be seen that only efficient allocations can be part of a competitive equilibrium.*

Next, the possible types of prices, the set of packages that bidders desire given prices and finally the CE will be defined. Around the CE are built the theoretical foundations of CAs and especially iterative CAs.

Let $p_i(S)$ be the price of package $S$ for bidder $i$.

**Definition 2.19.** *The prices are* ***anonymous linear prices*** *if $p_i(S) = \sum_{l \in S} p(l)$ where $p(l)$ indicates the anonymous price of item $l$. Thus the prices are described by a vector $p \in \mathbb{R}_+^{|\mathbb{K}|}$.*

**Definition 2.20.** *The prices are* ***personalized linear prices*** *if $p_i(S) = \sum_{l \in S} p_i(l)$ where $p_i(l)$ indicates the price of item $l$ for bidder $i$. Thus the prices are described by a vector $p \in \mathbb{R}_+^{|\mathcal{K}| \times |\mathcal{I}|}$.*

**Definition 2.21.** *The prices are* ***anonymous non-linear prices*** *if $p_i(S) = p(S)$. Thus the prices are described by a vector $p \in \mathbb{R}_+^{2^{|\mathcal{K}|}}$.*

**Definition 2.22.** *The* ***personalized non-linear prices*** *are described by a vector $p \in \mathbb{R}_+^{2^{|\mathcal{K}| \times |\mathcal{I}|}}$.*

**Definition 2.23.** *The* ***demand set*** *of bidder $i$ at price $p$ is the set of packages that maximize his profit given prices $p$:*
$D_i(p) = \{S \subseteq \mathcal{K} : u_i(S) - p_i(S) \geq u_i(T) - p_i(T) \; \forall T \in \subseteq \mathcal{K}\}$

[8]Notice that the seller maximizes his revenue based on the prices and not based on bids, which represent bidders' preferences and are not considered in the CE.

**Definition 2.24.** *A **competitive equilibrium** (CE) is a pair $(X, p)$, where $X = (S_1, ..., S_m)$ is a feasible allocation and $p$ prices such that:*
*i)$S_i \in D_i(p) \forall i$*
*ii)$X \in \underset{X \in C}{argmax} \sum_{S_i \in X} p_i(S)$ $X$ is **supported by prices** $p$ in CE.*

As an example, the two conditions can be written in case of personalized linear prices as follows:
i)$u_i(X_i) - \sum_{l \in X_i} p_i(l) \geq u_i(T) - \sum_{l \in T} p_i(l) \quad \forall i \in \mathcal{I}, T \subseteq \mathcal{K}$
ii)$\sum_i \sum_{l \in X_i} p_i(l) \geq \sum_i \sum_{l \in Y_i} p_i(l) \quad \forall Y = (Y_i, ..., Y_m) \in \mathbb{C}$

**Theorem 2.13.** *(Bikhchandani and Ostroy, 2002) Allocation $X$ is supported in competitive equilibrium if and only if $X$ is efficient.*

Hence, for every efficient allocation there are always prices so that a CE is reached. A CE is also a certificate that the allocation is efficient. The beauty of this statement is that one does not need the whole valuation profiles to derive the efficient allocation. It suffices that bidders and seller declare themselves happy. Nisan and Segal prove an even stronger statement:

**Theorem 2.14.** *(Nisan and Segal, 2006) A combinatorial auction realizes the efficient allocation if and only if the auction also realizes a competitive equilibrium.*

Though, the prices the bidders pay may differ from the CE prices. CE prices are only needed (and are indispensable) as a certificate that the realized allocation is efficient. In Example 2.1 the outcome of the VCG auction is not a CE but the auction has collected enough information to be able to compute (realize) CE prices. The outcome except of not being a CE outcome is also not in the core, as already seen. This is not a coincidence. Let $\pi = (\pi_1, ..., \pi_m)$ be the payoff vector of bidders and $\pi_s$ the seller's payoff:

**Theorem 2.15.** *(Bikhchandani and Ostroy, 2002) If and only if $(\pi_s, \pi) \in Core(\mathcal{I}, V(\cdot))$, there exists a CE $(X, p)$ such that $\pi_s = \max_{X \in C} \sum_{S_i \in X} p_i(S)$ and $\pi_i = \max_{S \in K}(v_i(S) - p_i(S))$ $\forall i$*

Thus there exists an equivalence between core and CE. Every core outcome can be priced in a CE and every CE yields core payoffs. What is more, the core is never empty [9] and a CE always exist. The key question is what kind of prices are needed to support the efficient allocation in the CE. Linear prices are easy to communicate and to understand. Non-linear prices are often prohibitive and bidders cannot cope with them easily. What is more, if they are personalized, their number is even higher and additionally they may be perceived as unfair ("Why must I pay \$10 whereas the other bidder only \$5 for the same thing?"). Unfortunately, linear prices are only in special cases possible.

**Theorem 2.16.** *(Bikhchandani and Ostroy, 2002) A competitive equilibrium with non-linear personalized prices always exists. A competitive equilibrium with other types of prices does not exist always.*

For the first part, observe that prices $p_i(S) = v_i(S)$ support trivially the efficient allocation in CE. For other types of prices, counterexamples show that they do not suffice.

**Example 2.4.** *There are two items and two bidders with valuations* $v_{B1}(AB) = 10, v_{B2}(A) = v_{B2}(B) = v_{B2}(AB) = 8$, *other valuations zero. The efficient allocation is that* $B1$ *gets* $AB$ *and* $B2$ *nothing. For a CE with anonymous non-linear prices, it must hold* $p(A) \geq 8$ *and* $p(B) \geq 8$ *otherwise* $B2$ *would be unhappy with the empty package. But with these prices the seller does not desire the efficient allocation but the allocation in which he sells the two items separately and gets a revenue of 16. Here it becomes clear that the seller decides the allocation he desires based on prices and he is ignorant of bids or valuations of bidders. Anonymous prices are not powerful enough to prevent him from desiring the allocation of separately selling the items. With personalized prices this would be possible, e.g.* $p_{B1}(AB) = 9, p_{B2}(A) = p_{B1}(A) = p_{B1}(B) = 0, p_{B2}(A) = p_{B2}(B) = p_{B2}(AB) = 9$. *Here the allocation which yielded a revenue of 16 before, now yields* $p_{B1}(A) + p_{B2}(B) = 9$ *and is not selected by him.*

A counterexample for anonymous linear prices can also be easily constructed, but Example 2.4 suffices to prove that they do not exist always since they are a subset of anonymous personalized prices. As fas as personalized linear prices are concerned, the thesis at hand provides a novel proof that they only exist if and only if anonymous linear prices exist.

**Theorem 2.17.** *A CE with personalized linear prices exist if and only if an equilibrium with anonymous linear prices exist.*

[9] The core may be empty if more than one seller are present, in this case the environment is a combinatorial exchange and not CA.

*Proof.* Only the direction personalized linear prices ⇒ anonymous linear prices is nontrivial. To construct an anonymous linear prices CE from a personalized, let $p = p_i(l)$ be personalized linear prices that support $X$. Let $p^*(l) = \max_i p_i(l)$. Then it can be shown that $p' = p^*(l)$ also support $X$. Intuitively, in the personalized CE, the seller sells each item to the bidder with the highest price for it. Then by setting the prices of all bidders equal to this maximum price, only the prices of bidders who do not receive the item, are increased. Hence bidders who initially did not desire and did not get allocated a package, continue not to desiring it since its price increased whereas the price for the package they initially desired and won (possibly the empty package) remained stable (maximum). The seller also continues to allocate each item at maximum price. Hence everyone continues to be happy and $(X, p')$ is also a CE. □

The fact that only non-linear personalized priced CE always exist raises the question whether CE with other types of prices can be guaranteed under certain conditions. Following theorems provide sufficient conditions. Since they are not necessary, this is an open issue that future research has to address. Gul and Stacchetti (1999) show that if the items are substitutes for every bidder, linear prices suffice.

**Definition 2.25.** *The* ***goods are substitutes*** *(GAS) or equivalently* ***gross substitutes*** *or* ***substitutes*** *if and only if:*
*For any bidder $i$ and any two anonymous linear price vectors $p$ and $q$ such as $p \geq q$ (component-wise) and any $S \in D_i(p)$, there exists $T \in D_i(q)$ such that $\{l \in S | p_l = q_l\} \subset T$.*

In words, if a bidder demands the items of package $S$ at prices $p$ and then the prices of other items increases, he continues to demand all items in $S$. GAS precludes any complementarities and Example 2.5 shows a case where GAS is violated due to complementarities:

**Example 2.5.** *Suppose valuations of a bidder $v(A) = v(B) = 3, v(AB) = 10$. At prices $p = (4, 4)$ the bidder demands $AB$. The price of $B$ increases and at the new prices $q = (4, 7)$ the bidder demands the empty package. Although the price of $A$ did not change, he does not demand it anymore. $A$ and $B$ are complements.*

**Theorem 2.18.** *(Gul and Stacchetti, 1999) If GAS is satisfied, a CE with linear prices always exists. GAS is the largest set containing unit demand* *preferences* [10] *for which the existence can be guaranteed.*

Regarding the existence of CE with anonymous non-linear prices, Parkes provides three sufficient conditions.

**Definition 2.26.** *Valuations are* ***supermodular valuations*** *if and only if for every i:*
$v_i(S \cap T) + v_i(S \cap T) \geq v_i(T) + v_i(S) \quad \forall, S, T \subseteq \mathcal{K}$

**Definition 2.27.** *Valuations are* ***safe valuations*** *if and only if for every i:*
$v_i(S) > 0, v_i(T) > 0 \Rightarrow S \cap T \neq \emptyset \quad \forall, S, T \subseteq \mathcal{K}$

**Theorem 2.19.** *(Parkes, 2001) If valuations are supermodular or safe or bidders are single-minded, then CE with anonymous non-linear prices always exist.*

Bikhchandani and Ostroy (2002) connect the existence of CE with the integrality of the linear relaxation of formulations of the $CAP$, called $CAP - I$, $CAP - II$ and $CAP - III$ ($CAP - I$ has already been introduced). Linear CE exists if and only if (the linear relaxation of) $CAP - I$ admits an integral optimal solution. Anonymous non-linear CE exists if and only if $CAP - II$ admits an integral optimal solution. Non-linear CE exists if and only if $CAP - III$ admits an integral optimal solution. The arguments are based on duality theory. The variables of the dual programs are interpreted as prices and payoffs. This has enabled the design of iterative CAs that implement these programs and thus converge to CE outcomes. This is the reason why these auctions are known as primal-dual auctions.

**Theorem 2.20.** *(Bikhchandani and Ostroy, 2002)* $(X, p)$ *is a CE with linear prices if and only if* $X$ *is an optimal solution to (the linear relaxation of the) primal* $CAP - I$ *and* $(\pi, \pi_s, p)$ *is an optimal solution to the dual* $CAP - I$*, where* $\pi_s = \max_{X \in C} \sum_{S_i \in X} p_i(S)$ *and* $\pi_i = \max_{S \in K}(v_i(S) - p_i(S))\ \forall i$*. The same holds for CE with anonymous non-linear prices and* $CAP - II$*. The same holds for CE with personalized non-linear prices and* $CAP - III$*.*

The seminal proof utilizes basic duality theory such as complementary slackness conditions. Its most innovative part are the different ways of modeling

[10] No bidder ever wants to get two or more items.

$CAP$ that enable the interpretation of the dual variables as prices of different types and payoffs. $CAP - III$ and an extension of the proof to handle allocation constraints will be presented in Chapter 3.

The definition of the universal CE (UCE) concludes this section. While CE is needed as a certificate for an auction to realize an efficient allocation, the UCE is needed as certificate for the realization of the VCG outcome.

**Definition 2.28.** *(Mishra and Parkes, 2007)* $(X, p)$ *is a* ***universal competitive equilibrium*** *if:*
*i)*$(X, p)$ *is a CE*
*ii)*$\forall i$*, in the economy without bidder* $i$*,* $p_{-i}$ *support an efficient allocation in CE, where* $p_{-i} = (p_1, ..., p_{i-1}, p_{i+1}, ..., p_m)$ *and* $p_i$ *is the price vector for bidder* $i$*.*

Thus UCE prices are CE prices in the main economy and in every marginal economy, where one bidder each time is removed.

**Theorem 2.21.** *(Mishra and Parkes, 2007) UCE prices are necessary and sufficient for computing VCG payments.*

### 2.4.3 Bidder Submodularity: Expost Nash Equilibrium

The theory of the preceding subsection enabled the design of efficient iterative CAs which converge to a CE. As the reader may anticipate, they utilize non-linear personalized prices. All of them are efficient and admit an expost Nash equilibrium if the bidder submodularity condition is satisfied. A representative format of this auction family is the *iBundle* auction (Parkes and Ungar, 2000). The prices are initialized to zero and each bidder submits his bids using the XOR bidding language. Then the $CAP$ is solved to determine the winning bids. If every bidder is happy, i.e. wins a package, the auction ends. Else, price $p_i(S)$ increases if $i$ an unhpapy bidder and $S$ a non-empty package he bid on. The amount of increase $\epsilon$ is called bid increment and is predefined by the auctioneer. The larger the bid increment, the less rounds are needed for the auction to converge. This comes at the expense of efficiency loss since the valuations are extracted with less precision. The auction is summarized in Algorithm 1.

If bidders follow the straightforward bidding strategy, *iBundle* is efficient.

**Algorithm 1** *iBundle*

1:Initialize prices $p_i(S) = 0 \quad \forall i, S$
2:Receive XOR bids from bidders
3:Solve $CAP$ to determine provisionally winning bids
4:**If** every bidder receives a package (including the empty) **terminate**
5:**For each** bidder $i$ who wins no package (except the empty) and every package $S$ on which $i$ bid
6: Set $p_i(S) := p_i(S) + \epsilon$
7:**End for**
8:**Go to Line 2** (a new round begins)

**Definition 2.29.** *A bidder follows the* ***straightforward bidding strategy*** *if in every round he bids exactly on the packages that maximize his payoff (this can include the empty package), i.e. on every $S \in D_i(p)$.*

The empty package has always the prize of zero and a bidder bids on it only when all prices are greater or equal than his valuations. By bidding on the empty package he signalizes he is not interested in winning any items.

**Theorem 2.22.** *(Parkes and Ungar, 2000) If every bidder follows the straightforward bidding strategy, iBundle is efficient (within $3min(\mathcal{I}, \mathcal{K})$ of the optimal allocation) and reaches a competitive equilibrium.*

For the first part of the theorem it is shown that *iBundle* implements $CAP-III$. When *iBundle* terminates, all complementary slackness conditions are satisfied and the solution is optimal, hence efficient. The second part follows from the theory of the preceding section. As far as incentives are concerned, although BAS guarantees that the Vickrey outcome is in the core, a slightly stronger condition, BSM, is required for the iterative CAs to reach a true core outcome. Bidders have incentives to follow the straightforward bidding strategy when the bidder submodularity condition is satisfied.

**Definition 2.30.** *The condition* ***bidder submodularity*** *(BSM) is satisfied if*

$$V(J \cap i) - V(J) \geq V(J' \cap i) - V(J') \quad \forall J \subseteq J' \subseteq \mathcal{I}$$

In words, a bidder added to coalition $J$ contributes more than added to a larger coalition $J' \supseteq J$. The BSM condition relates to GAS and BAS.

**Theorem 2.23.** *(Ausubel and Milgrom, 2006a) GAS implies BSM.*

**Theorem 2.24.** *(Ausubel and Milgrom, 2006a) BAS implies BSM.*

**Theorem 2.25.** *(Ausubel and Milgrom, 2006a) If BSM holds, straightforward bidding in iBundle is an expost Nash equilibrium.*

Straightforward bidding is also an expost Nash equilibrium for the iterative auctions Ascending Proxy Auction (Ausubel and Milgrom, 2006a) and dVSV (de Vries et al., 2007). The only difference of the Ascending Proxy Auction to *iBundle* is that bidders input their valuation profiles to proxies before the start of the auction and the proxies bid then on their behalf straightforward. The dVSV auction differs in the price update rules. It identifies a set of bidders who are minimally undersupplied and increases their prices. Extensions of *iBundle* and dVSV have been designed that reach an UCE and compute VCG payments. Their outcome is only in the core if BAS holds but also straightforward bidding is always an expost Nash equilibrium.

### 2.4.4 No Bidder Submodularity: Perfect Nash Equilibrium

When BSM does not hold, the equilibrium of the iterative CAs is generally unknown. Sano provides a complete and an incomplete information analysis for restricted settings. When all valuations are public information, and the strategy space is restricted to semi-truthful strategies, the Vickrey-target strategy is a Perfect Nash equilibrium (Sano, 2011b). A semi-truthful strategy prescribes shading all valuations by a constant amount (thus every strategy in the single-minded setting is semi-truthful since there is only one positive valuation). After having subtracted this constant amount from all valuations, straightforward bidding is a semi-truthful strategy, whereby the demand set is computed and the new decreased valuations. The Vickrey-target strategy is a semi-truthful strategy with the constant which is subtracted equals the Vickrey payoff. In a complete information setting, every bidder can compute his Vickrey payoff since all valuations are publicly known.

When the valuations are not known, Sano (2012) derives a perfect Bayes Nash equilibrium of the setting with two items and many global and local bidders.

When a certain condition is satisfied, every local bidders wants to drop right at the beginning of the auction. On the contrary, global bidders are always truthful. Chapter 4 presents this work and examines the impact of risk aversion. While risk aversion has no impact on the dominant and expost Nash equilibria presented in this thesis, it affects the optimal bidding strategy in (perfect) Bayes Nash equilibria.

## 2.5 Limitations of Theory and Experiments

Despite the substantial progress theory in CAs has made the recent years, the bidder behavior and the performance of various auction formats cannot be always predicted. When BAS (BSM) does not hold, the optimal bidding strategy in sealed-bid (iterative) CAs is unknown for the general setting where bidders are multi-minded, i.e. they have positive valuations for more than one package. Even in cases where the theory provides solution concepts, the underlying assumptions may be violated: Bidders' utility functions may not be quasi linear due to risk aversion or budget constraints. Their valuations may have common value elements and be interdependent. A common prior about the bidders' valuation may not be available. Also externalities may be present. To design a CA or market generally that works well in practice, theory is the point of departure but not the end of the design process. The 2012 Nobel prize winner A. Roth, considers the economist as an engineer and mentions that "experimental and computational economics are natural complements to game theory in the work of design" (Roth, 2002). Theory tells what has to be done in order to achieve desiderata such as full efficiency but often the design has to be curtailed to address practical considerations.

The theory of iterative CAs calls for instance for non-linear personalized prices. Even when a dozen of items is auctioned, the amount of different prices that have to be communicated and the amount of bids that have to be submitted is prohibitive. While decision support tools or proxy agents may facilitate this process, there are cases where the usage of simple prices, like per item or linear prices, is preferable, and the goal of full efficiency has to be abandoned. This is also the case when the core property is favored over full efficiency. An auction format that is used widely for the sale of spectrum is a hybrid format, the Combinatorial Clock Auction. In its first stage ascending linear prices are used and the second stage is a sealed-bid core-selecting auction (Cramton, 2013). Further auction formats with only linear prices have been proposed by Porter et al. (2003), Kwasnica et al. (2005) and Bichler et al. (2009). The review of laboratory and experimental work is out of this thesis' scope. Results of simulations and experiments that among others compare the performance of linear versus non-linear CA formats as well as CAs versus non-CA auctions can be found in Bichler et al. (2009, 2013a,b); Brunner et al. (2010); Goeree and Holt (2010); Schneider et al. (2010).

# Chapter 3

# Combinatorial Auctions with Allocation Constraints

> Build me a future, splendid and
> graceful, make it better by design.
>
> VNV Nation

Combinatorial auctions are used in a variety of application domains such as transportation or industrial procurement using a variety of bidding languages and different allocation constraints. This flexibility in the bidding languages and the allocation constraints is essential in these domains, but has not been considered in the theoretical literature so far. In this work, we analyze two different pricing rules for ascending combinatorial auctions which allow for such flexibility: winning levels, and deadness levels. We determine the computational complexity of these pricing rules and show that deadness levels actually satisfy an ex-post equilibrium, while winning levels do not allow for a strong game-theoretical solution concept. We investigate the relationship of deadness levels and the simple price update rules used in efficient ascending combinatorial auction formats. We show that ascending combinatorial auctions with deadness level pricing rules maintain a strong game theoretical solution concept and reduce the number of bids and rounds required at the expense of higher computational effort. The calculation of exact deadness levels is a $\Pi_2^P$-complete problem. Nevertheless, numerical experiments show that for mid-sized auctions this is a feasible approach. This work provides a foundation for allocation constraints in combinatorial auctions and a theoretical framework for recent Information Systems contributions in this field.

## 3.1 Introduction

Combinatorial auctions allow selling or buying a set of heterogeneous items to or from multiple bidders. Bidders can specify package bids, i.e., a bid price is defined for a subset of the items for auction (Cramton et al., 2006). The price is only valid for the entire package and the package is indivisible. For example, in a CA a bidder might want to buy items $x$, $y$, and $z$ for a package price of $100, which might be more than the total of the prices for the items sold individually. CAs can be seen as generic mechanisms for multi-object markets as they allow expressing complex preferences such as items being substitutes or complements for bidders. They have found application in a variety of domains such as the auctioning of spectrum licenses (Cramton, 2009), truck load transportation (Caplice, 2007), bus routes (Cantillon and Pesendorfer, 2006), or industrial procurement (Bichler et al., 2006).

The design of efficient CAs has drawn considerable attention, as they raise fundamental questions on pricing and efficiency in multi-object markets. If bidders revealed their preferences truthfully, the auctioneer only had to solve an optimization problem to find the efficient allocation. Typically bidders have incentives to speculate and deviate from truthful bidding. The Vickrey-Clarke-Groves (VCG) mechanism has therefore been a significant contribution in auction theory. Green and Laffont (1977) proved that the sealed-bid VCG mechanism is the unique auction mechanism in which truthful bidding is a dominant strategy. In spite of this powerful result, this mechanism has a number of practical problems when used in multi-object markets (Ausubel and Milgrom, 2006b; Rothkopf, 2007). Also, many auctioneers want to have a transparent open-cry bidding process rather than a sealed-bid format. For example, almost all spectrum auctions organized throughout the world are iterative auctions with multiple rounds.

It is interesting to understand, whether there are also iterative CA formats, which also satisfy strong game-theoretical solution concepts, which limit incentives for speculation. Preference elicitation in iterative auctions can invalidate dominant strategy equilibria existing in a single-step version of a mechanism (Conitzer and Sandholm, 2002), but ex post equilibria can be achieved which also do not require agents to speculate about other bidders' valuations (Shoham and Leyton-Brown, 2009). Much progress has been made on this question in the theoretical literature in the recent years. *iBundle* (Parkes and Ungar, 2000), the Ascending Proxy Auction (APA) (Ausubel and Milgrom, 2002), and dVSV (de Vries et al., 2007) lead to an efficient allocation, if bidders bid straightforward [1], which is an ex post equilibrium as long as the buyer submodularity condition holds for all valuations. Buyer submodularity is a restriction on the valuations of bidders, which defines that bidders are more valuable when added to a smaller coalition. It is interesting that such a strong solution concept is possible in iterative CAs, even though the result does not hold for general valuations. This line of work is heavily based on duality theory in linear programming.

All these auction formats use non-linear and personalized ask prices and increase these ask price for losing bids by a minimum increment, which causes a large number of auction rounds (Schneider et al., 2010).[2] The APA uses proxy bidders in order to cope with the large number of auction rounds, and to make sure that bidders follow the straightforward strategy, which turns the mechanism into a sealed bid auction. Apart from these theoretical advances, a number of linear or item-price auction formats have been developed. These include versions of the combinatorial clock auction (Bichler et al., 2013b; Porter et al., 2003), where item-level ask prices rise, whenever there is overdemand on an item, but also the family of auction formats with pseudo-dual linear ask prices (Bichler et al., 2009; Kwasnica et al., 2005). As of yet, there is little theory on equilibrium bidding strategies in such auctions, although lab experiments yielded high levels of allocative efficiency.

Adomavicius and Gupta (2005) introduce different pricing rules for CAs. *Pricing rules* refer to functions by which the auctioneer determines ask prices based on bids submitted in the auction. Winning levels ($WLs$) describe the lowest possible bid which would win if no other bids are submitted, whereas deadness levels ($DLs$) are a lower bound to bids, which still can become winning in the course of the auction. $DLs$ and $WLs$ describe natural bounds and interesting feedback for a losing bidder. While there is no rationale to bid below the $DL$, bidding at the $WL$ could be rational in some situations, where a bidder is the only one able to outbid a winning coalition of bidders. $WLs$ are equivalent to the minimal winning bids described by Rothkopf et al. (1998). These pricing rules are independent of the allocation rules, and laboratory experiments with respective auction formats yielded high levels of efficiency (Adomavicius et al., 2012). They are very generic and can be considered a fundamental contri-

[1] Bidders follow the straightforward strategy if they bid only on the surplus maximizing package(s) in every round

[2] We will refer to *ask prices* as the prices set by the auctioneer during the auction, and *bid prices* as the prices specified by bidders, if it is necessary to distinguish prices. In this work, we assume that bidders always bid on the ask price.

bution in the emerging Information Systems literature on decision support in smart markets (Bapna et al., 2007; Bichler et al., 2010, 2009; Guo et al., 2007; Scheffel et al., 2011; Xia et al., 2004) and in the general literature on CAs.

The analysis in Adomavicius and Gupta (2005) is focused on a pure OR bidding language and assumes no substitutes valuations. OR bids can represent only bids that do not have any substitutabilities, i.e., purely additive and superadditive valuations (Nisan, 2006). Also, this initial work did not analyze equilibrium strategies in such auctions. However, $DLs$ and $WLs$ introduce a very generic concept, which can be applied to any bidding language. It is natural to ask, how these generic pricing rules can be defined for auctions with XOR bidding languages and other types of allocation constraints as well, and which game-theoretical solution concepts can be satisfied. Auctions with strong game-theoretical solution concepts are strategically easier for bidders because there is no need to speculate about other bidders' valuations. This makes bidding strategically easy and respective auction formats are more likely to lead to high efficiency.

In this work, we introduce $WLs$ and $DLs$ for general CAs, which allow for XOR bids and other constraints. Auctioneers use various types of allocation constraints to limit the number of winners or the number of items allocated to one bidder or to a group of bidders. Such constraints are actually the rule rather than the exception in application domains such as industrial procurement (Bichler et al., 2006; Sandholm and Suri, 2006) or transportation (Caplice, 2007). But they are typically not considered in the literature on iterative CAs. The beauty of $DLs$ and $WLs$ is that their definition is independent of the type of allocation constraints used in the winner determination. The resulting theory is applicable to a much broader set of real-world applications, where allocation constraints play a considerable role. We will refer to ascending CAs allowing for different types of allocation constraints as *flexible combinatorial auction (FCA)* in this text. Such constraints might be added by the auctioneer and the bidders, which allows for considerably more flexibility in the specification of the preferences of market participants.

While generic pricing rules for different CAs with allocation constraints would also have significant practical importance, we show that these pricing rules bare significant theoretical challenges. The main goal of this work is not to introduce a new auction format, but *to define and analyze computational and game-theoretical properties of* $WLs$ *and* $DLs$ for FCAs. In particular, we want to understand which pricing rules allow for ex post equilibria and how these pricing rules relate to the theoretical framework on efficient and ascending CAs (de Vries et al., 2007; Parkes and Ungar, 2000). This connects also recent IS contributions to the game-theoretical literature in this field.

The chapter is structured as follows: In Section 2, we briefly discuss allocation constraints in CAs. We define winning and deadness levels and describe respective algorithms in Section 3.3. In Section 3.4 we determine the computational complexity for these pricing rules. In Section 3.5 we analyze the impact of allocation constraints on efficient CAs and perform an equilibrium analysis of ascending CAs using either winning or deadness levels as ask prices in Section 3.6. In Section 3.7 we report on our computational experiments, before we provide conclusions in Section 3.8.

## 3.2 Allocation Constraints in Combinatorial Auctions

*Allocation constraints* specify limits on the allocation of the available items to the bidders without explicitly limiting bid prices or revenue of a bidder or a group of bidders. On the other hand *price constraints* set price limits on items, packages, a bidder's budget or auctioneer revenue. It has been shown that incentive-compatible auctions are impossible in general if there are private budget limits (Dobzinski et al., 2008), and also reserve prices by the auctioneer increase expected revenue at the expense of efficiency (Myerson, 1981). Table 3.1 provides an overview of allocation and price constraints in CAs, subsumed by the term *side constraints*. We focus on efficient auctions, and therefore will limit ourselves to allocation constraints. One can also divide side constraints into bidder specific ones (bidder level) and such constraints, which concern more than a single bidder (group level). The latter are typically specified by the auctioneer, while bidder specific constraints could be imposed by the bidder or the auctioneer, especially when the OR bidding language is used.

Allocation constraints are important in many domains. Spectrum auctions, which have been the driving application for much research in this area, regularly face spectrum caps (max # items/bidder) (Seifert and Ehrhart, 2005). In industrial procurement or the auction for transportation services allocation constraints are the rule rather than the exception (Bichler et al., 2006; Caplice, 2007; Sandholm and Suri, 2006; Sandholm, 2003). Buyers need to specify lower or upper bounds on the number of suppliers overall or per group (min/max # winning bidders): lower bounds in order to hedge the risk that some suppliers fail to deliver and upper bounds in order to avoid administrative expenses. Market share constraints are defined on a group of bidders. For example, due to corporate requirements at least one minority supplier must be included in the set of winners on a particular set of items. Auctioneers also need to impose lower or upper bounds on the number of items (min/max # items/bidder), which can be awarded to a particular bidder or a group of bidders. Exclusive disjunction describes constraints such that a bidder is allowed to win one set of items or another one, but not both. For example, an auctioneer allows a bidder to win one of two different regions A and B, but not both. Such constraints can also be specified on a group of bidders. The XOR language can be seen as a separate bidder specific allocation constraint.

While the XOR language already captures many further allocation constraints, experimental analysis has been shown that OR bidding languages are often more efficient than XOR languages due to the reduced number of package bids that need to be submitted (Brunner et al., 2010). OR bids can represent only bids that do not have any substitutabilities, i.e., purely additive and superadditive valuations (Nisan, 2006). If a bidder uses an OR bidding language, it might also be useful to specify constraints on the number of items (min/max # items/bidder) or budget awarded, in order to avoid the exposure problem, which can occur with substitutes valuations (e.g., a bidder is winning packages AB and CD, but only wants two lots at a maximum) or to express his capacity constraints in case he is a supplier in a procurement auction. Constraints of exclusive disjunction are relevant to the auctioneer in case of an OR bidding language, when a bidder is allowed to win one set of items or another one, but not both. Carriers in a transportation auction use such disjunctive constraints to communicate the message "give me this set of lanes, or this set of lanes, but not both" (Caplice, 2007).

| **side constraint** | **allocation constraint** | **price constraint** |
|---|---|---|
| **bidder level** | min/max # items/bidder | budget |
| | exclusive disjunction | |
| **group level** | min/max # winning bidders | reserve prices<br>budget |
| | market share | |
| | exclusive disjunction | |

Table 3.1: Side constraints

Having flexibility in the bidding language and the allocation constraints used by the auctioneer and the bidders allows for a much broader applicability of CAs, and this can be considered a prerequisite for most applications in transportation and industrial procurement.

## 3.3 Pricing Rules

Before we provide a formal definition, we will first provide an example to informally introduce the pricing rules $DL$ and $WL$. We will contrast these with pricing rules as they are used in *iBundle* or in auction formats with pseudo-dual linear ask prices, such as RAD (Kwasnica et al., 2005) to highlight the connections among these different pricing rules.

| packages | $AB$ | $BC$ | $AC$ | $B$ | $C$ |
|---|---|---|---|---|---|
| bids | $22^*_1$,$16_2$ | $24_3$ | $20_4$ | $7_5$ | $8^*_6$ |
| $RAD$ | 22 | 24 | 14 | 16 | 8 |
| $iBundle$ | $22_1$, $17_2$ | $25_3$ | $21_4$ | $8_5$ | $8_6$ |
| $DL$ | 22 | 24 | 20 | 7 | 8 |
| $WL$ | 22 | 30 | 23 | 10 | 8 |

TABLE 3.2: Example with six bids and different ask prices.

The upper part of Table 3.2 describes six bids from different bidders $B1$ to $B6$, submitted on subsets of three items $A$, $B$, and $C$, while the lower part shows the resulting prices in various auction formats. Subscripts indicate bidders, i.e., $22_1$ indicates a bid of \$22 from bidder $B1$. Prices have subscripts only if they are personalized. Asterisks denote the provisional winning bids.

The calculation of $RAD$ prices requires solving a number of linear programs. Note that these linear prices can be lower than a losing bid (see the bid on $AC$ of bidder $B4$). Another problem with pseudo-dual ask prices is the fact that they can also decrease if the competition shifts, and that the calculation does not take into account allocation constraints explicitly. This makes it difficult to define an equilibrium analysis and as of now there is little theory. Bichler et al. (2009) analyze the problems in defining pseudo-dual linear ask prices.

For the rest of the work, we will focus on the family of efficient and ascending auctions (*iBundle*, APA, and dVSV). Apart from the use of proxy agents, APA and *iBundle* are equivalent and follow a subgradient algorithm, whereas dVSV uses a primal-dual approach (de Vries et al., 2007). Line 4 in our example shows a new set of ask prices in *iBundle*. The auction format increases the bids of all losing bidders by a minimum bid increment (\$1 in this example) (Parkes and Ungar, 2000). $DLs$ and $WLs$ describe generic pricing rules, i.e., they are independent of the allocation constraints used by the auctioneer. $DL$s describe deadness level ask prices, which are simply the bids of all bidders in the last round in this example without any constraints. In this case, $DL$s do not need to be personalized. Losing bidders need to bid higher than this by a minimum bid increment. With a minimum bid increment of one, the $DL$ would be at the *iBundle* ask price for the losing bidders $B3$, $B4$, and $B5$. In the presence of allocation constraints, $DL$ ask prices can be much higher than the ones of *iBundle*, as we will see later.

The $WL$ describes the lowest bid prices, at which a single bid would become winning without a new complementary bid of another bidder. Bidders $B4$ and $B5$ could become winning at a lower price, if they would form a coalition. A known problem with package bidding is the threshold problem, in which bidders seeking larger packages may be favored, because small bidders do not have the incentive or capability to top the tentative winning bids of the large bidder. In such a problem, the $WL$ for a small bidder might be way too high to outbid a winning bidder unilaterally, and with only $WL$ ask prices, it will become difficult for bidders to coordinate. The spread between $DLs$ and $WLs$ can often be quite large in realistic value models.

Of course, one can also think of personalized and non-linear ask prices in between $DLs$ and $WLs$. The coalition of $B4$ and $B5$ in our example would need to increase their bids by a combined \$3 plus increment in order to become winning. Both bidders would become winning, if bidder $B4$ bid \$21.5 and bidder $B5$ bid \$8.5, for example We refer to such prices as coalitional winning levels ($CWL$). A $CWL$ could provide an alternative to linear-price auction formats, and help mitigate threshold problems and coordinate bidders. However, computing $CWLs$ turns out to be challenging. First, the number of losing coalitions can grow very fast with the number of items and bids in the auction. Second, and more importantly, there are a many ways how the costs to outbid the winning coalition can be shared among the bidders in a losing coalition. One can think about cost sharing which is proportional to the bids in the previous round or a cost sharing that satisfies certain fairness criteria. Independent of how $CWLs$ are set, there will be incentives to free-ride on other bidders in a losing coalition. Due to the ambiguities in the definition of $CWLs$, we will focus on $DLs$ and $WLs$ in this work, which describe natural upper and lower bounds for bids in an ascending auction, and leave the analysis of $CWLs$ for future research.

### 3.3.1 Winning Levels

We first introduce the necessary terminology. Let $\mathcal{K}$ denote the set of items and $\mathcal{I}$ the set of bidders. A subauction on itemset $S \subseteq \mathcal{K}$ refers to an auction where only items $l \in S$ are auctioned. Auction state $k \in \mathbb{N}$ refers to the auction after the first $k$ bids are submitted. A bid is a tuple $b = (S, v, k, i)$ where $S$ denotes the package the bid refers to, $v$ the amount bid, $k$ the auction state after the bid submission [3] and $i$ the bidder. Given bid $b$, we use the notation $S(b)$, $v(b)$, $k(b)$ and $i(b)$ to refer to its respective elements. A bid $b = (S, v, k, j)$ is foreign w.r.t. bidder $i$ if $j \neq i$. $B_{S,k} = \{(T, v, k', i)|T \subseteq S, k' \leq k\}$ is the set of all bids in $S$ and $B_{S,k,-i} = \{b \in B_{S,k}|i(b) \neq i\}$ is the set of all bids in $S$ which are foreign to bidder $i$. An allocation $X$ of packages to bidders is a set of non overlapping bids, i.e. for every $b, b' \in X, b \neq b' \Rightarrow S(b) \cap S'(b) = \emptyset$. An *allocation constraint* is a function $X \to \{0, 1\}$ where the value 1 indicates fulfillment of the constraint. Our analysis applies to every possible allocation constraint in this form which can be verified in polynomial time [4]. An allocation $X$ is feasible only if it satisfies every allocation constraint, in which case we write $feas(X) = 1$. The set of all feasible allocations is denoted by $\mathbb{C}_k = \{X \subseteq B_{\mathcal{K},k}|b, b' \in X, b \neq b' \Rightarrow S(b) \cap S'(b) = \emptyset, feas(X) = 1\}$. The combinatorial allocation problem ($CAP$), also known as winner determination problem (WDP), solves $max_{X \in \mathbb{C}_k} \sum_{b \in X} v(b)$. $CAP^k(\mathcal{K})$ denotes the maximum value of this optimization problem (henceforth we will refer to it simply as value) and $WIN_k(\mathcal{K}) \in \mathbb{C}_k$ the value-maximizing allocation at state $k$.[5] $CAP(S)$ is the value of subauction $S$.

The *winning level* of a package $S$ for bidder $i$ at auction state $k$, $WL_k(S, i)$, is the minimal price $i$ must bid to win $S$ at auction state $k + 1$.

**Definition 3.1.** $WL_k(S, i) = \underset{v}{argmin} : (S, v, k + 1, i) \in WIN_{k+1}(\mathcal{K})$.

Adomavicius and Gupta (2005) define the anonymous $WL$ of package $S$ at auction state $k$ by [6]

$$WL_k^{OR}(S) = CAP^k(\mathcal{K}) - CAP^k(\mathcal{K} \backslash S) \tag{3.1}$$

[3] For example, a bid with state $k = 3$ is the third bid submitted in chronological order.

[4] To the best our knowledge no allocation constraint has been proposed which does not fulfill this property.

[5] Depending on the context we may omit index $k$.

[6] The problem of finding this minimal price is introduced by Rothkopf et al. (1998) and referred to as the minimal winning bid problem.

Intuitively a bid on $S$ can only win, if the bid price together with the value of the complementary set of items $\mathcal{K}\backslash S$ exceeds the actual value of the whole auction. Implicitly, the following assumptions are made: (i) OR bidding language and (ii) absence of allocation constraints. When these assumptions are relaxed, the calculation of $WLs$ as in equation (3.1) is inappropriate. A first reason is that allocation constraints, which are not bidder specific, cannot be globally validated when solving subauction $CAP(\mathcal{K}\backslash S)$. In addition, $WLs$ must be personalized as the following example demonstrates.

**Example 3.1.** *Consider an auction with items A, B, C, bidders B1, B2 and B3 and the constraint that each bidder cannot win more than two items. Bidder* $B1$ *bids at* $k = 1$ *\$5 on AB, B2 at* $k = 2$ *\$1 on AB and at* $k = 3$ *\$2 on* $C$ *and* $B3$ *bids at* $k = 4$ *\$3 on* $AB$ *.* $WL(C)$ *is for B1 \$4 whereas for B2 it is only \$2.*

We introduce the following formula to calculate personalized $WLs$ that takes XOR bidding and allocation constraints into account:

**Proposition 3.1.**

$$WL_k(S, i) = CAP^k(\mathcal{K}) - CAP^k(\mathcal{K}, S_i) \tag{3.2}$$

$CAP^k(\mathcal{K}, S_i)$ denotes the value of the whole auction provided that bidder $i$ wins package $S$ for free. Thus the auction value $CAP^k(\mathcal{K}, S_i)$ is raised from the items in $\mathcal{K}\backslash S$, as it is the case in $CAP(\mathcal{K}\backslash S)$. The proof follows:

*Proof.* A self-contained proof is given. It is based on the proof of Theorem 2 of Adomavicius and Gupta (2005). We introduce the symbol $\mathbb{C}_k^E(S) = \{X \in C \cup E | C \in \mathbb{C}_k, E \in \bigcup_{i=1}^{I}(S, 0, 0, i) \cup \emptyset\}$[7] to denote the set of feasible allocations that can also include bids of zero value on S [8]. We define a binary relation $\prec$ on bid allocations to compare the values of two allocations: $C' \prec X'' \Rightarrow v(X') < v(X'')$ where $v(X) = \sum_{b \in X} v(b)$ is the value of the allocation $C$. $S(X) = \bigcup_{b \in X} S(b)$ denotes the items covered in allocation $X$. $WIN_k^E(\mathcal{K}, S, i) = max_{\prec}\{X \in \mathbb{C}_k^E(\mathcal{K}) | (S, 0, 0, i) \in X\}$ represents

[7] In the referenced proof $\mathbb{C}_k(S)$ denotes the set of feasible allocations for subauction $S$. In our setting, note that $B_1 \in \mathbb{C}_k(S)$ and $B_2 \in \mathbb{C}_k(\mathcal{K}\backslash S)$ does not imply that $B_1 \cup B_2 \in \mathbb{C}_k(\mathcal{K})$ due to allocation constraints. Thus, generally $\mathbb{C}_k(S)$ where $S \subset K$ is not defined.

[8] We need this extension since otherwise allocations where a bidder wins a package for free would not be feasible. These allocations are considered in $CAP_k(\mathcal{K}, S_i)$.

the winning allocation of the whole auction at state $k$ subject to the condition that bidder $i$ wins $S$ for free. We consider a new bid $b_{k+1}$ of bidder $i$ on package $S$ at state $k+1$. Let $X_1 = \{C \in \mathbb{C}_{k+1}(\mathcal{K}) | b_{k+1} \in X\}$ and $X_2 = \{C \in \mathbb{C}_{k+1}(\mathcal{K}) | b_{k+1} \notin C\}$ be the set of all allocations with and without $b_{k+1}$ respectively. It holds $X_1 \cap X_2 = \emptyset$ since they cannot share a common allocation (every allocation in $X_1$ contains $b_{k+1}$ and every allocation in $X_2$ does not) and $X_1 \cup X_2 = \mathbb{C}_{k+1}(\mathcal{K})$. Therefore:

$$\begin{aligned} WIN_{k+1}(\mathcal{K}) &= max_{\prec}\{X \in \mathbb{C}_{k+1}(\mathcal{K})\} = max_{\prec}\{X_1 \cup X_2\} \\ &= max_{\prec}\{max_{\prec}X_1, max_{\prec}X_2\} \end{aligned} \tag{3.3}$$

$$\begin{aligned} \text{and since } b_{k+1} \notin X \ \forall X \in X_2, \text{ it follows } X_2 \\ = \mathbb{C}_k(\mathcal{K}) \text{ and } max_{\prec}X_2 = max_{\prec}\mathbb{C}_k(K) = WIN_k(\mathcal{K}) \end{aligned} \tag{3.4}$$

Furthermore:

$$\begin{aligned} max_{\prec}X_1 &= max_{\prec}\{C \in \mathbb{C}_{k+1}(K) | b_{k+1} \in X\} \\ &= \{b_{k+1}\} \cup \max_{\prec}\{C \backslash \{b_{k+1}\} | X \in \mathbb{C}_{k+1}, b_{k+1} \in X\} \\ &= \{b_{k+1}\} \cup \max_{\prec}\{X | X \in \mathbb{C}_k(K), S(X) \cap S(b_{k+1}) = \emptyset\} \\ &= \{b_{k+1}\} \cup \max_{\prec}\{X \in \mathbb{C}_k^E(\mathcal{K}), (S(b_{k+1}), 0, 0, i(b_{k+1})) \in X\} \\ &= \{b_{k+1}\} \cup WIN_k^E(\mathcal{K}, S(b_{k+1}), i(b_{k+1})) \end{aligned}$$

The last equation together with (3.3) and (3.4) imply:

$$WIN_{k+1}(\mathcal{K}) = max_{\prec}\{WIN_k(\mathcal{K}), \{b_{k+1}\} \cup WIN_k^E(\mathcal{K}, S(b_{k+1}), i(b_{k+1}))\}$$

and $b_{k+1} \in WIN_{k+1} \iff v(WIN_k) < v(b_{k+1}) + v(WIN_k^E(\mathcal{K}, S(b_{k+1}), i(b_{k+1})))$

Thus, for a new bid $b_{k+1}$ to win, its value $v(b_{k+1})$ together with $v(WIN_k^E(\mathcal{K}, S(b_{k+1}), i(b_{k+1})))$ which is the value of $CAP$ subject to the constraint that the bidder $i(b_{k+1})$ wins $S(b_{k+1})$ for free (we denoted this $CAP$ as $CAP^k(\mathcal{K}, S_i)$), must exceed $v(WIN_k)$ which is the current value $CAP^k(\mathcal{K})$. This completes the proof. □

If the OR bidding language is used and no allocation constraints exist, the computation in (3.2) yields the same $WLs$ as (3.1).

### 3.3.2 Deadness Levels

The *deadness level* of package $S$ at auction state $k$, $DL_k(S,i)$, is the minimal price a bidder must bid to have a chance to win $S$ in any future auction state.

**Definition 3.2.** $DL_k(S,i) = \underset{v}{argmin} : \exists k' > k : (S, v, k+1, i) \in WIN_{k'}(\mathcal{K})$.

$DLs$ constitute lower bounds on bid prices and all future bids below are "dead", i.e., they cannot win any more no matter which bids are submitted in future auction states $k' > k$. This implies that $DL_k(S,i)$ is monotonically increasing through the progress of any iterative auction.[9]

**Proposition 3.2.**

$a)DL_k(S,i) \leq DL_{k+1}(S,i) \qquad b)DL_k(S,i) \leq WL_k(S,i) \qquad c)DL_k(\mathcal{K},i) = WL_k(\mathcal{K},i)$

*Proof.* [10] a)Assume $DL_{k+1}(S,i) < DL_k(S,i)$. Denote with $\varepsilon$ a very small positive number. Then $DL_k(S,i)$ implies that $\nexists k' > k : (S, DL_k(S,i) - \varepsilon, k', i) \in WIN_{k'}(\mathcal{K}) \Rightarrow \nexists k' > k : (S, DL_{k+1}(S,i), k', i) \in WIN_{k'}(\mathcal{K})$ (since we assumed $DL_{k+1}(S,i) < DL_k(S,i)$). But $DL_{k+1}(S,i)$ implies that $\exists k' > k+1 : (S, DL_{k+1}(S,i), k', i) \in WIN_{k'}(\mathcal{K})$. Contradiction.

In words, all bids below $DL_k(S,i)$ are destined to lose whatever happens in future auction states. But in the future state $k$+1 a bid amounting to $DL_{k+1}(S,i)$ and thus below $DL_k$ has a chance to win in a state greater than $k$+1. Therefore $DL_k(S,i)$ is not minimal and by definition not a $DL$.

b)Assume $DL_k(S,i) > WL_k(S,i)$. The $WL$ definition implies that the bid $(S, WL_k(S,i), k{+}1, i) \in WIN_{k+1}(\mathcal{K})$. The $DL$ definition together with the assumption $DL_k(S,i) > WL_k(S,i)$ implies that $\nexists k' > k : (S, WL_k(S,i), k', i) \in WIN_{k'}(\mathcal{K})$. Contradiction.

In words, the $DL$ is the minimal price to win the item in a possible future auction state. The $WL$ is the minimal price to win it at the next state, thus it cannot be lower.

c)$(\mathcal{K}, v, k+1, i) \in WIN_{k+1}(\mathcal{K}) \Rightarrow v \geq CAP_k(\mathcal{K})$. Thus the minimum $v$, i.e. the $WL$, is $WL_k(\mathcal{K}) = CAP_k(\mathcal{K})$. Furthermore if $v < CAP_k(\mathcal{K}) \Rightarrow \nexists k' > k$ :

[9] Through our analysis, we assume no bid revocability, otherwise $DLs$ would be always zero.

[10] We provide a definition-based proof without using a formula which calculates $DLs$. We have not derived such a formula yet.

$(S, v, k+1, i) \in WIN_{k'}(\mathcal{K})$ since by CAP definition $k' > k \Rightarrow CAP_{k'}(\mathcal{K}) \geq CAP_k(\mathcal{K})$. Thus $DL_k(\mathcal{K}, i) = CAP_k(\mathcal{K}) = WL_k(\mathcal{K}, i)$.

In words, to win in the next auction state all auctioned items, a bidder must bid at least the current auction value. Every lower bid will not be winning in any future auction state since the auction value will not sink. □

Adomavicius and Gupta (2005) define the anonymous $DL$ of a package $S$ by

$$DL_k^{OR}(S) = CAP^k(S) \tag{3.5}$$

In words, a bid on $S$ cannot be part of the winning allocation if it is below the value of subauction $S$ (i.e. $CAP(S)$). For example if $S = AB$ and there are already bids $A = \$10$ and $B = \$15$, then any bid on $AB$ below \$25 is dead.

The $DL$ in equation (3.5) is not valid if there are allocation constraints or any of the bidders uses an XOR bidding language. In these cases $DLs$ must be personalized. We also need to understand what influence the additional allocation constraints might have in future auction states. For instance, consider again Example 3.1. $B3$ loses $AB$ at current state $k = 4$. But if in the future state $k = 5$ $B1$ bids \$8 on $C$, then his previous winning bid on $AB$ loses due to the allocation constraint [11]. Thus $B3$ can win $AB$ for \$3 and his personal $DL$ at $k = 4$ is \$3. We say that the bid of $B1$ on $AB$ has been "blocked" at $k = 5$ due to the constraint.

A bid on package $S$ loses in an auction with allocation constraints if at least one of the following conditions is met: (i) There exists a higher bid or bid combination in subauction $S$ that wins in the whole auction (without violating allocation constraints). (ii) $S$ is not part of the value maximizing allocation. (iii) The bid in interaction with other bids that win in the whole auction violates an allocation constraint.

The first two conditions are common for every CA, with or without constraints. The third one leads us to the definition of *blocked bids*, which we use later on; a bid $b$ is blocked if it does not win due to allocation constraints. In this case, one or more winning bids in the complementary subauction, which we denote by $B'$, prevent the blocked bid to win. If the bids in $B'$ had not existed, $b$ would have won. The winning bids after the removal of a bid set $B'$ are denoted by $WIN_k^{-B'}(\mathcal{K})$. In the context of Example 3.1 extended by a bid of \$8 of bidder

[11] A bid of \$5 would produce the same effect. With \$8 we indicate here and in the sequel an arbitrarily high future bid, submitted due to increased competition.

$B1$ on item $C$, and of \$8 of $B1$ on additional item $D$, the bid set $B'$ comprises these two bids. The removal of both bids in $B'$ would turn the bid $b$ of $B1$ on $AB$ into winning. Since $b \in WIN_k^{-B'}(\mathcal{K})$, $b$ is blocked by $B'$ and does not win due to allocation constraints.

**Definition 3.3.** *Bid $b = (S, v, k^-, i)$ is blocked at state $\bar{k} \geq k^-$ if $b \notin WIN_{\bar{k}}(\mathcal{K})$ and $\exists B' \subseteq B_{\mathcal{K}\setminus S,\bar{k}}$ with $b \in WIN_{\bar{k}}^{-B'}(\mathcal{K})$.*

Computing $DL_k(S, i)$ requires knowing which currently winning bids in $S$ can be blocked in future auction states. We call these bids blockable.

**Definition 3.4.** *Bid $b = (S, v, k^-, i)$ is blockable at state $k$ if $b \in WIN_k(\mathcal{K})$ with $k \geq k^-$ and $\exists \bar{k} > k$ so that $b$ is blocked at state $\bar{k}$.*

Removing a blockable foreign to $i$ bid in $S$ causes another lower foreign bid (or bids) in $S$ to win. If a distinct future state $\bar{k}$ can be reached where both the second and the first bid are blocked, then the $DL_k(S, i)$ becomes even lower, since $i$ does not have to overbid any of them. We will say that the two bids are simultaneously blockable at current state $k$ and simultaneously blocked at $\bar{k} > k$. To see this happen, consider again Example 3.1 with an additional item $D$ and seek for $DL(AB, B2)$. We have already argued that the highest bid on $AB$ by $B1$ is blockable. Removing it causes the bid of $B3$ to win. This second bid is also blockable due to a similar reason as the first blockable bid ($B3$ may bid high on $D$). More importantly, these two bids are simultaneously blockable since there is a future state with both bids blocked (see Table 3.3). Had item $D$ not existed, this would not have been possible. Only one of the bids would have been blockable but the two of them would not have been simultaneously blockable.

<table>
<tr><th></th><th>A</th><th>B</th><th>C</th><th>D</th></tr>
<tr><td>$B1$</td><td colspan="2">$\not{5}^{4}$</td><td>$8^{5}$</td><td>0</td></tr>
<tr><td>$B2$</td><td colspan="2">$1^{2}$</td><td>$2^{3}$</td><td>0</td></tr>
<tr><td>$B3$</td><td colspan="2">$\not{3}^{4}$</td><td>0</td><td>$8^{6}$</td></tr>
</table>

TABLE 3.3: Bids in bold block crossed out bids and $DL(AB, B2) = 1$, $k$ is denoted as superscript

We just highlighted the central role of simultaneously blockable bid sets in computing $DL_k(S, i)$. These are bids by rivals of $i$, thus foreign to him, submitted in $S$. We denote a *simultaneously blockable bid set* in subauction $S$, at state $k$, comprising bids foreign to bidder $i$, as $B^{block}_{S,k,-i}$. Undoubtedly, foreign bids on packages which overlap with $S$ but are not entirely in $S$ are also preventing $i$ to win $S$ but these bids are not relevant at the most favorable future state for $i$ that $DL(S,i)$ represents. The reason is that in this most favorable state the value of subauction $\mathcal{K}\backslash S$ is high enough such that $S$ is part of the winning allocation. Before providing the formal definition for $B^{block}_{S,k,-i}$, it is meaningful to sketch the algorithm which utilizes these sets to compute $DL(S,i)$.

The algorithm identifies all simultaneously blockable bid sets. In our running example there was only one set when item $D$ was present, comprising the bids of $B1$ and $B3$ on $AB$ and two single-item sets when $D$ was not present, which were the two bids separately. The algorithm removes each identified set in turn and computes the value of $CAP(S)$. The lowest value is $DL(S,i)$, because $DL(S,i)$ represents the most favorable future state for $i$ to win $S$.

Having outlined the algorithm, we can state the conditions a set must fulfill in order to be declared as simultaneously blockable at current state $k$, with $\bar{k}$ being the future state where the set will be actually blocked. We begin with the indispensable conditions and continue with conditions which only accelerate the sketched algorithm. Their enumeration and motivation precedes the formal description given in 3.5. We have already discussed that going from $k$ to $\bar{k}$ in order to block bids in $S$, we add only bids in $\mathcal{K}\backslash S$ since if they would overlap with $S$, they would prevent $i$ from winning $S$. Their set is $B' = B_{\mathcal{K},\bar{k}} \backslash B_{\mathcal{K},k}$, i.e. all bids between $k$ and $\bar{k}$. Bids in $B'$ must be winning else they cannot block any bids, or generally impact the allocation or prices. The first condition states that after the submission of $B'$ it must be feasible that $i$ wins $S$ (condition i). Secondly, inherent to the definition of a blocked bid is the condition that every bid in $B^{block}_{S,k,-i}$ must be losing at $\bar{k}$ (condition ii).

Moving to conditions which only accelerate the sketched algorithm, we observe that if all bids in the simultaneously blockable set are losing at $k$, their removal will not lower the price $i$ has to pay for $S$, hence at least one of them must be winning at $k$ (condition iii). In Table 3.3 removing only bid 4 has no impact. Furthermore, it is not meaningful that a blockable bid set contains a bid that never wins, even after the removal of the rest bids of the set. As we begin to block some bids, previously losing bids turn to winning and these are the ones that lower the price of $S$ if blocked. Hence we demand $b \in B^{block}_{S,k,-i} \Rightarrow b \in WIN_k^{-(B^{block}_{S,k,-i} \backslash b)}(\mathcal{K})$ (condition iv). Furthermore, we observe that it is redundant to block a foreign bid on $T \subseteq S$ if it is lower than a bid already submitted by $i$ on $T' \subseteq T$, since $i$ can never pay less than his own bid. The set of all these bids, which we call non-$i$-dominated, is denoted by $B^{ndom}_{S,\bar{k},-i}$ and the corresponding condition states that all bids in $B^{block}_{S,\bar{k},-i}$ are non-$i$-dominated (condition v). The concluding condition specifies that simultaneously blockable bid sets must be maximal (condition vi) since blocking and removing an extra bid can only lower the value of $CAP(S)$. A set is defined as maximal if it satisfies a property (in our case the property is the satisfaction of conditions i to v) and there exist no strict superset of it that also satisfies the property. Hence adding an item in a maximal set causes violation of the property. In Table 3.3 bid 1 is not identified as simultaneously blockable only because it is not maximal.

**Definition 3.5.** *Bid set $B^{block}_{S,k,-i}$ is simultaneously blockable at state k if:*
$\exists B' = (B_{\mathcal{K},\bar{k}} \backslash B_{\mathcal{K},k}) \subseteq (B_{\mathcal{K}\backslash S,\bar{k}} \cap WIN_{\bar{k}}(\mathcal{K}))$ :

*i)* $feas(WIN_{\bar{k}}(\mathcal{K}\backslash S) \cup b) = 1$, *where* $S(b) = S, i(b) = i$

*ii)* $B^{block}_{S,k,-i} \cap WIN_{\bar{k}}(\mathcal{K}) = \emptyset$

*iii)* $B^{block}_{S,k,-i} \cap WIN_k(\mathcal{K}) \neq \emptyset$

*iv)* $b \in B^{block}_{S,k,i} \Rightarrow b \in WIN_k^{-(B^{block}_{S,k,-i}\backslash b)}(\mathcal{K})$

*v)* $B^{block}_{S,k,-i} \subseteq B^{ndom}_{S,k,-i}$

*vi)* $\nexists B'' \supset B^{block}_{S,k,-i}$ *with* $B''$ *satisfying conditions i) to v).*

With this definition we can introduce the general method to compute $DL_k(S,i)$ for arbitrary allocation constraints.

**General Method to Compute** $DL_k(S,i)$

In the first phase, the method takes as input all bids which are candidates to form simultaneously blockable bid sets. These are the bids in $B^{ndom}_{S,k,-i}$. The output of the first phase is the set of all simultaneously blockable bid sets in $S$. We can represent the first phase as a function $f$ with argument $B^{ndom}_{S,k,-i}$ and value the output of this phase, i.e. $f : B^{ndom}_{S,k,-i} \rightarrow \{B^{block}_{S,k,-i}\}$. We consider here $f$ as a black box since for the general description of the method only $f$'s argument and value matters. In the second phase each of the identified sets $B^{block}_{S,k,-i} \in f(B^{ndom}_{S,k,-i})$ is removed consecutively and the value of subauction $S$, denoted as $CAP^k(S, B_{S,k} \backslash B^{block}_{S,k,-i})$ is computed. The lowest of these values equals to $DL(S,i)$. It is not always necessary to compute all these values. If we encounter a value not greater than the value of the bids of $i$ in $S$, we do not need to continue since $i$ must overbid the own bids. Equation 3.6 summarizes the general two-phase method.

**Proposition 3.3.**

$$DL_k(S,i) = \min_{B^{block}_{S,k,-i}} \left\{CAP^k(S, B_{S,k} \backslash B^{block}_{S,k,-i}) : B^{block}_{S,k,-i} \in f(B^{ndom}_{S,k,-i})\right\} \quad (3.6)$$

*Proof.* The proof draws on the definition of $DL_k(S,i)$ and of simultaneously blockable bid sets. Let $c$ be the right side of equation 3.6 and $B^{block}_*$ the argument of the minimum. First we show that for $b = (S, c, k^-, i)$ $\exists k^* > k^- : b \in WIN_{k^*}(\mathcal{K})$. We define $k^*$ as the state at which: a) the bids in $B^{block}_*$ are actually blocked and do not win due to condition i, b) a bid set $B'$ in $\mathcal{K} \backslash S$ is submitted so that $B' \subseteq WIN_{k^*}(\mathcal{K})$ and c) $\forall b$ with $k(b) \in (k, k^*], i(b) \neq i, S(b) \subseteq \mathcal{K} \backslash S$ (i.e. no new foreign bids overlapping with $S$ are submitted). We can ensure that the bids in $B^{block}_*$ lose at $k^*$ without the need of submission of overlapping bids at states in $(k, k^*]$ due to condition ii. Thus the bid $b^*$ with value $c = CAP^k(S, B_{S,k} \backslash B^{block}_*)$ is part of the winning allocation (due to condition i and that the winning allocation maximizes $CAP$ over all feasible allocations). Similarly it can be argued that for $b' = (S, c - \epsilon, k, i)$ $\nexists k' > k : b \in WIN'_k(\mathcal{K})$: After removing each simultaneously blockable bid set $B^{block}_{S,k,-i}$, there is a feasible allocation which includes the bids which maximize $CAP^k(S, B_{S,k} \backslash B^{block}_{S,k,-i})$ and have a value at least $c$. Thus $b'$ cannot be part of the winning allocation. Collectively, we showed that $c = \underset{v}{argmin} : \exists k' > k : (S, v, k+1, i) \in WIN_{k'}(\mathcal{K})$. This is the definition of $DL_k(S,i)$. □

With an OR bidding language and without allocation constraints (3.6) reduces to (3.5), since without these constraints, there are no blockable bids and thus $DL_k(S,i) = CAP^k(S, B_{S,k} \backslash B^{block}_{S,k,-i}) = CAP^k(S, B_{S,k} \backslash \emptyset) = CAP^k(S, B_{S,k}) = CAP^k(S)$.

**Example 3.2.** *Consider an auction with items A, B, C, D, bidders $B1$ to $B5$ and the XOR bidding language. The bids in $AB$ are: $(AB, \$9, 1, B1)$, $(A, \$5, 2, B2)$, $(B, \$8, 3, B2)$, $(A, \$10, 4, B3)$, $(AB, \$15, 5, B4)$ and $(AB, \$19, 6, B5)$. We compute $DL_6(AB, B1)$ using the above formula. Function f 's argument $B^{ndom}_{AB,6,-B1}$ contains all bids except the bid of $B1$ and f returns all simultaneously blockable sets $B^{block}_{AB,6,-B1}$ and*

$$f(B^{ndom}_{AB,6,-B1}) = \{\{2,3,6\}, \{4,6\}, \{5,6\}\}$$

*Bids are referred by their state. Bids {2,3,6} are blockable due to XOR bidding if $B2$ and $B5$ win C and D respectively, bids {4,6} if $B3$ and $B5$ win*

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $B1$ | $9^1$ | | 0 | |
| $B2$ | $5^2$ | $8^3$ | 0 | |
| $B3$ | $10^4$ | 0 | | |
| $B4$ | $15^5$ | | 0 | |
| $B5$ | $19^{6*}$ | | 0 | |

TABLE 3.4: Bids of five bidders on items A,B,C and D. The superscript denotes $k$ and * currently winning. What is the $DL$ of the package AB for bidder B1?

*C and D respectively and bids {5,6} if $B4$ and $B5$ win C and D respectively. Note that {2,5,6} is not listed since even if bids 5,6 are removed, bid 2 does not win and violates condition iv of definition 3.5. Sets without bid 6, like {2,3,4}, {2,3,5}, {4,5} violate condition iii since none of their bids currently wins. The sets {3,6} and {2,6} are not maximal, since their superset {2,3,6} is simultaneously blockable. Subsequently we remove each simultaneously blockable set from subauction AB and compute $CAP$. After removing the first set $CAP = \$15$, after the second one $CAP = \$15$ and after the third one \$18. The minimum CAP is \$15, thus $DL_6(AB, B1) = \$15$. $B1$ could win AB for \$15 in a future state in which $B2$ wins C and $B5$ wins D.*

Function $f$ is specific to allocation constraints and the bidding language, which determine whether a bid set is simultaneously blockable or not. We provide now an algorithm to calculate $DLs$ for the XOR bidding language without additional allocation constraints. The XOR bidding language is fully expressive (Nisan and Segal, 2006), and it is used in high-stakes applications such as in spectrum auctions across Europe nowadays.

**Computation of $DLs$ for an XOR bid language**

We proceed according to the two-phase method and firstly seek for simultaneously blockable bid sets. We observe that a winning bid of bidder $j$ on a single item $l \in \mathcal{K} \backslash S$ suffices to simultaneously block all his bids in $S$. Consequently the number of foreign bidders, whose bids in $S$ are simultaneously blockable, amounts to $|\mathcal{K} \backslash S|$. If this number is greater or equal than the number of all foreign bidders in $S$ who have a least one non-$i$-dominated bid in $S$, denoted as $n_S$, we are done. All of them can be blocked and $DL$ is equal to the highest bid of $i$ in $S$, i.e he must overbid only his own bids. Otherwise we proceed to phase 2 and determine which bidder set leads, if removed from subauction $S$,

**Algorithm 2** *DL* XOR algorithm

**Input:** package $S$, bidder $i$
set of bids $B^{ndom}_{S,k,-i}$
**Output:** $DL(S,i)$

```
lowerBound ← max_b{v(b) : i(b) = i, S(b) ⊆ S}
DL(S,i) ← ∞
𝓕 ← getForeignBiddersOnPackage(S,i)
if |𝓕| ≤ |𝓚\S| then
  DL(S,i) ← lowerBound
else
  for all 𝓕_i ⊂ 𝓕 : |𝓕_i| = |𝓚\S|  do
    thisPrice ← CAP(S, B_{S,k}\{b'|b' ∈ B^{ndom}_{S,k,−i}, i(b') ∈ 𝓕_i})
    if thisPrice < DL(S,i)
      DL(S,i) ← thisPrice
    end if
    if DL(S,i) = lowerBound then
      break for
    end if
  end for
end if
return DL(S,i)
```

to the minimum $CAP$ value in $S$.[12] If $|S|=1$ or if all bids in $S$ are overlapping, it is easy to find which bidders should be removed; the ones with the highest bids. But the general case is obviously combinatorial and requires solving the CAP for each of the $\binom{n_S}{n_S-|\mathcal{K}\backslash S|} = \binom{n_S}{|\mathcal{K}\backslash S|}$ bidder sets remaining after removing the blockable ones. The pseudocode is given in 2.

The outlined calculation of XOR $DL$ can serve as a basis to calculate $DL$ for a number of other constrained cases, surprisingly even for cases with the OR bidding language. We provide an example, as this cannot be claimed for every conceivable case.

**Proposition 3.4.** *The $DL(S,i)$ for the OR bidding language in presence of the constraint "max a winners" can be calculated as XOR DL.*

*Proof.* The most opportune case for bidder $i$ to win $S$ is when he places a huge bid on a item of the complementary subauction $\mathcal{K}\backslash S$ so that he is surely among the $a$ winners [13] and additionally some bidders leading to the highest $CAP(S)$ are not among the $a$ winners because other "low" bidders in $S$ place huge bids

[12] Here it is more convenient to speak about removal of bidder sets instead of bid sets. The removal of a bidder set from subauction $S$ corresponds to the removal of all the bids in $S$ submitted from bidders in the bidder set.

[13] It is not known whether $i$ truly desires items in $\mathcal{K}\backslash S$. We take it for granted since we want to minimize his price on $S$. Knowing additional information such maximal willingness

in $\mathcal{K}\backslash S$. Together with bidder $i$, $min(a-1, |\mathcal{K}\backslash S|-1)$ foreign bidders can win in $\mathcal{K}\backslash S$ and all other $max(n_S - min(a-1, "|\mathcal{K}\backslash S|-1)), 0)$ foreign bidders in $S$ can be simultaneously blocked. Knowing this, we can proceed to calculate $DL(S, i)$ as in the XOR $DL$ case by solving one $CAP$ for each bidder sets of the derived cardinality. □

## 3.4 Computational Complexity

Computational complexity has turned out to be a practically relevant topic for the winner determination problem in CAs. We show that computational complexity is also a problem when computing pricing rules and that the computation of $DLs$ is even one of the rare examples of a $\Pi_2^P$-complete problem.

The $CAP$ is a well-known NP-complete problem in its decision version. The NP-completeness is proven for a variety of cases and bidding languages. Lehmann et al. (2006) prove the NP-completeness of $CAP$ for the OR and XOR bidding language by reducing from the independent set problem (Garey and Johnson, 1972) using intersection graphs. Sandholm and Suri (2001) prove the completeness for numerous cases dealing with side constraints such as bounds on the maximal number of winners. It is easy to see that the computation of $WLs$ must be at least as hard as $CAP$.

**Proposition 3.5.** *Deciding* $WL(S, i)$ *is* $NP - complete$.

*Proof.* Every instance of $CAP$ can be viewed as an instance of $WL$ for $S = \mathcal{K}$. For $S = \mathcal{K}$, equation (3.2) yields $WL(K, i) = CAP(\mathcal{K}) - CAP(\mathcal{K}, \mathcal{K}_i) = CAP(\mathcal{K})$. Thus $WL$ contains $CAP$ as a special case which is $NP$-complete. □

to pay in $\mathcal{K}\backslash S$ leads to additional constraints in the minimization problem of $DL$ and the optimal value increases. In our analysis we do not cope with this issue but with the standard definition of $DLs$ of Adomavicius and Gupta (2005) which is $DL(S) = CAP(S)$. Thereby winning subauction $S$ does not implies winning $S$. This especially will not happen if we incorporate the additional information that every bidder is interested in packages overlapping with $S$ and there is little interest in $\mathcal{K}\backslash S$ so that $S$ and $\mathcal{K}\backslash S$ will never be part of the efficient allocation. Although incorporating such information is beyond the scope of this work, we believe that our analysis and algorithms provide the necessary tools to guide the computation of $DLs$ even in such cases. What changes is how the simultaneously blocked bids are computed and this has to be engineered for every concrete case.

Interestingly, the $DL$ problem (equation (3.6)) is even harder to solve. It requires to solve one $CAP$ for every simultaneously blockable bid set. The number of bids in $S$ is $O(2^{|S|}|\mathcal{I}|)$. Hence the number of bid sets is $O(2^{2^{|S|}|\mathcal{I}|})$. Due to the set maximality property many of these double exponential bid sets need not to be evaluated. That means in the worst case only $\binom{2^{|S|}|\mathcal{I}|}{\lfloor (2^{|S|}|\mathcal{I}|)/2 \rfloor}$ $CAPs$ need to be solved, which is still super exponential to the number of items in $S$. The actual number of blockable bid sets depends strongly on the allocation constraints.

Obviously the problem belongs to a higher complexity class than $NP$. This class is $\Pi_2^P$ if the decision version is formulated as $DL(S,i) > c$ and the $\Sigma_2^P$ if $DL(S,i) \leq c$. Completeness in these classes informs of what can (or cannot) be done in polynomial time with access to an $NP$ oracle [14], i.e. an oracle which is able to decide the decision version of the winner determination problem in a single operation. Every CA employs a method to solve $CAP$. If we assume the presence of such an efficient method, completeness in these classes plays exactly the role of $NP$-completeness for "ordinary" optimization problems - it distinguishes the intractable from the efficiently solvable (Umans, 2000). $\Pi_2^P = coNP^{NP}$ and $\Sigma_2^P = NP^{NP}$.

We first show $DL(S,i) \in \Pi_2^P$ with the decision problem $DL(S,i) > c$.

**Lemma 3.1.** *$DL(S,i)$ is in $\Pi_2^P$.*

*Proof.* We show that there exists a polynomially balanced, polynomial-time decidable 3-ary relation $R$ such that $DL = \{x : \forall y_1 \exists y_2$ such that $(x, y_1, y_2 \in R)\}$. $x$ represents the graph containing all bids in $S$. Each node in $x$ represents a \$1 bid and nodes are connected if and only if the bids they represent are compatible (e.g. do not overlap). $y_1$ represents a subgraph of $x$ which is induced according to function $f$ (i.e. each subgraph represents a bid set after removing a simultaneously blockable bid set). $y_2$ is an independent set of $y_1$. Note that its cardinality equals the value of the auction. Relation $R$ decides in polynomial time that $y_1$ is a subgraph of $x$, $y_2$ an independent set of $y_1$ and that $y_2$ contains more than $c$ nodes. Thus $R$ is polynomially decidable. Furthermore $R$ is polynomially balanced (since the lengths of $y_1$ and $y_2$ are bounded by a polynomial in the length of $x$). □

[14] The exponent $NP$ denotes that the non-deterministic Turing machine accepting $\Sigma_2^P$ and the complement of $\Pi_2^P$ uses a $NP$ oracle. We refer to Papadimitriou (1993) for formal definitions.

Intuitively the $DL$ is greater than $c$ if all bid combinations prescribed by $f$ result in a subauction value greater than $c$. In graph terms, all of the induced subgraphs must have an independent set of cardinality greater than $c$. Remember that $DL$ is a minimization problem and if it exists one subgraph with value not greater than $c$ then the answer to the question becomes negative. That explains the necessity of the first quantifier $\forall$ and is central for the complexity of the problem, which is a min-max optimization problem (Ko and Lin, 1995).

We now prove the completeness of $DL(S,i)$ in $\Pi_2^P$. We make use of the structures of a fairly restrictive case of the problem, as it is common ground in such proofs (Lehmann et al., 2006), to reduce from the minmax-Clique problem defined in Ko and Lin (1995).

**Definition 3.6.** *minmax-Clique: Given is a graph* $G = (V, E)$ *with its vertices* $V$ *partitioned into subsets* $V_{i,j}$, $1 \leq i \leq I$, $1 \leq j \leq J$. *For any function* $t : \{1, ..., I\} \rightarrow \{1, ..., J\}$, $G_t$ *denotes the induced subgraph of* $G$ *on the vertex set* $V_t = \bigcup_{i=1}^{I} V_{i,t(i)}$. *Find* $f_{Clique}(G) = \min_t \max_Q\{|Q| \colon Q \subset V \text{ is a clique in } G_t\}$.

Intuitively the graph represents a network with $I$ components, with each component $V_i$ having $J$ subcomponents $V_{i,1}, ..., V_{i,J}$. At any time $t$ only one subcomponent $V_{i,t(i)}$ of each $V_i$ is active and the problem is to find the maximum clique size of all possible active subgraphs $G_t$. The minmax-Clique problem is $\Pi_2^P$-complete by reduction from $SAT_2$. The completeness is shown for $J = 2$ and subsets $V_{i,j}$ of same cardinality.

**Theorem 3.1.** *Deciding* $DL(S,i)$ *is* $\Pi_2^P$*-complete.*

*Proof.* We reduce from minmax-Clique. For each subcomponent $V_{i,j}$ we create a bidder who participates in subauction $S$. Since $J = 2$, each component corresponds to a pair of bidders. We introduce the allocation constraint that no two bidders belonging to the same pair are allowed to win together in $\mathcal{K} \backslash S$. For each node we create a \$1 bid on $T$ with $T \subseteq S$ submitted from the bidder associated to the subcomponent the node belongs to. Whenever two nodes in $G$ are connected, their associated bids are compatible.[15] It can be observed now that $f_{Clique}(G) = DL(S,i)$. There is a one-to-one correspondence between the $2^{|I|}$ possible active subgraphs $G_t$ and the bidder (bid) sets that remain after removing each of the $2^{|I|}$ simultaneously blockable bidder (bid)

[15] The graph we work on is complementary to the intersections graphs described by Lehmann et al. (2006).

sets of cardinality $|I|$, due to the constraint in $\mathcal{K}\backslash S$. Furthermore, the max-Clique problem on the complementary intersection graphs is equivalent to the maximum independent set problem on the actual intersection graph and thus equivalent to $CAP$. □

An interpretation of the constraint we introduced to prove completeness is that the auctioneer is not willing to deal with more than a single winner from a region, assuming that bidders are paired according to their region. We conjecture that the $DL$ problem without such constraints cannot be easier since the number of the simultaneously blockable bidders and thus the number of $CAPs$ which must be solved becomes much greater. To see this, consider an example with 8 foreign bidders in $S$ and $|\mathcal{K}\backslash S| = 4$. Without the constraint the bidder sets amount to $\binom{8}{4}$ whereas with the constraint only $2^4$ bidder sets need to be evaluated.

## 3.5 Allocation Constraints and their Impact on Equilibrium Strategies in Efficient Auction Designs

In what follows, we want to understand equilibrium strategies in CAs with allocation constraints. In order to analyze such CAs with respect to efficiency and incentive compatibility we first need to understand the impact of allocation constraints on those CA formats, which are known to be efficient with a strong game-theoretical solution concept. First, we analyze the VCG auction, which is known to be the unique CA format that is strategy proof, efficient and individually rational (Green and Laffont, 1977). Second, we focus on ascending CAs as *iBundle*, the APA, and dVSV in which straightforward bidding is an ex post equilibrium for buyer submodular valuations. The analysis integrates the auction mechanisms in Adomavicius and Gupta (2005) in this game-theoretical framework and provides conditions, when this auction format leads to an ex post Nash equilibrium.

**Definition 3.7.** *A strategy profile $s^* = (s_1^*, ..., s_n^*)$ is an ex-post Nash equilibrium iff the utility functions $u_i$ satisfy*

$$u_i(s^*, t) \geq u_i(s_i^{'}, s_{-i}^*, t) \quad \forall s_i^{'}, t, i$$

In other words, truthful bidding in every round of an auction is an ex-post (Nash) equilibrium, if for every bidder $i \in \mathcal{I}$, if all other bidders follow the truthful bidding strategy, then bidder $i$ maximizes his payoff in the auction by following the truthful bidding strategy independent of the type $t$ of other bidders (Mishra and Parkes, 2007). Ex post equilibria avoid speculation about other bidders' valuations or types and could therefore reduce the strategic complexity for bidders considerably, leading to higher efficiency, and also an increased adoption of ascending CAs. Note that this is weaker than a dominant strategy equilibrium, where bidders do not have to speculate about other bidders' valuations and strategies. In contrast to dominant strategy and ex post equilibria, Bayes-Nash equilibria do always exist, but they require bidders to speculate on both, the type and the strategy of others. We refer to dominant and ex post equilibria as *strong solution concepts*. For ascending auctions we focus on ex post equilibria, as preference elicitation in an indirect mechanism typically does not allow for dominant strategy equilibria (Conitzer and Sandholm, 2002).

Let us first introduce two definitions to describe bidder valuations, before we discuss individual auction formats.

**Definition 3.8.** *A coalitional value function $V$ maps a set of bidders $J$ to a real number $V(J)$, equal to the total value created from trade among these bidders and the auctioneer.*

$CAP$ implements a coalitional value function in the context of CAs. Bidder submodularity describes a property of the coalitional value function, which allows for strong solution concepts in ascending CAs and core outcomes in the VCG auction, as we will see below.

**Definition 3.9.** *(bidder submodular (BSM) condition) A coalitional value function $V$ is bidder submodular if bidders are more valuable when added to smaller coalitions: for all $i \in \mathcal{I}$ and all coalitions $J$ and $J'$ satisfying $J \subset J'$, $V(J \cup \{i\}) - V(J) \geq V(J' \cup \{i\}) - V(J')$.*

Since allocation constraints may alter (lower) $V(J)$, imposing them may turn a coalitional value function from not BSM to BSM or vice versa [16]. The simplest example to see this, is to impose the allocation constraint "max one winner". Any function will then turn into BSM. The constraint "min three

[16] Also the implication by Ausubel and Milgrom (2002) (Goods Are Substitutes) ⇒ BSM does not hold in the presence of allocation constraints.

winners" turns any function into not BSM. In the following, when we refer to $V(J)$ or BSM, these are computed by taking into account present allocation constraints.

### 3.5.1 VCG Mechanism

The VCG outcome serves as a baseline for all other efficient auction formats. For example, under BSM valuations APA, *iBundle*, and dVSV terminate with VCG prices which are in the core, eliminating incentives for speculation. Core prices have the property that no coalition of bidders can renegotiate the outcome with the auctioneer in order to increase everyones payoff in this coalition.

The VCG auction is a sealed bid auction allowing for package bids on all combinations of items. Bidders place sealed XOR bids on their desired packages without getting any feedback by the auctioneer or knowing bids of other bidders. The auctioneer calculates a feasible allocation $X^*$ that maximizes the sum of bid prices. Bidders payments are calculated in a second step. Winning bidders pay their bid prices $b_i(S)$ reduced by a discount which is equal to their marginal contribution to the whole economy. $p_i(S) = b_i(S) - (V(\mathcal{I}) - V(\mathcal{I} \backslash i)) \quad \forall S \in X^*$ and zero otherwise.

We concentrate on allocation constraints in this work. While Ausubel and Milgrom (2006b) show that budget constraints can lead to inefficiency in the VCG mechanism, allocation constraints do not affect its properties. However, the calculation of the VCG prices and in particular of the coalitional value from $V(J)$ with $J \subset \mathcal{I}$ has to consider the allocation constraints, as otherwise the auctioneer could suffer a negative payoff and participation would not be individually rational.

**Definition 3.10.** *(Shoham and Leyton-Brown (2009)) An environment exhibits the no-single-agent effect if* $\forall i, \forall v_{-i}, \forall X$ *there exists an allocation* $X'$ *that is feasible without* $i$ *and* $\sum_{j \neq i} v_j(X') \geq \sum_{j \neq i} v_j(X)$.

A mechanism is weakly budget balanced when it will not lose money, this means if the mechanism is not weakly budget balanced the auctioneer might be confronted with a negative payoff which would contradict individual rationality of the mechanism.

**Theorem 3.2.** *(Shoham and Leyton-Brown (2009)) The VCG mechanism is weakly budget balanced when the no single agent effect property holds.*

CAs without allocation constraints are always weakly budget balanced since the no single agent effect property always holds (Shoham and Leyton-Brown, 2009). The theorem extends to VCG auctions with allocation constraints but in this case the no single agent effect property may not hold. Hence these auctions are not always weakly budget balanced.

**Corollary 3.1.** *The VCG mechanism with allocation constraints is not always weakly budget balanced.*

*Proof.* It may happen that $V(\mathcal{I}\backslash i)$ in the VCG payment computation is zero because of allocation constraints. Consider an example where the auctioneer requires 2 winning bidders and only 2 bidders participate, such that the no-single-agent effect does not hold. Bidder $B1$ values item $A$ at \$10 and $B2$ values $B$ at \$10. The Vickrey payments are $p_1(A) = p_2(B) = 10-(20-0) = -10 \leq 0$ and the auctioneer loses money. □

If the no-single-agent effect does not hold, the auctioneer might want to consider bids by bidders $i \notin J$ to assure a feasible allocation, while maximizing $V(J)$.

### 3.5.2 Efficient Ascending CAs

The recent game-theoretical research has led to a coherent theoretical framework and a family of ascending CAs (*iBundle*, APA, dVSV) which satisfy an ex post equilibrium under BSM. These efficient ascending CAs use personalized and non-linear prices. They calculate a provisional value maximizing allocation at the end of every round and increase the prices for a certain group of bidders. The different approaches can be interpreted as implementations of primal-dual algorithms (dVSV) or subgradient algorithms (*iBundle*, APA) to solve an underlying linear programming problem (de Vries et al., 2007). This linear program ($CAP_3$) always yields integral solutions and the dual variables have a natural interpretation as non-linear and personalized ask prices (Bikhchandani and Ostroy, 2002).

We want to understand, whether additional allocation constraints have an impact on equilibrium strategies and efficiency in these auction formats. For this reason, we analyze the impact of allocation constraints on $CAP_3$. The original $CAP_3$ formulation changes with additional allocation constraints. An arbitrary allocation constraint can make certain allocations infeasible. Rather than modeling specific allocation constraints, we keep our analysis general and partition the set of all allocations in two subsets: the feasible allocations $\mathbb{C}$ and the infeasible ones $\mathbb{C}_u$, which turn infeasible due to the violation of certain allocation constraints (e.g. the maximum number of winners). This extends $CAP_3$ by constraint set (LP4):

$$\begin{array}{rcllll}
\max_{x_i(S)} \sum_{S\subseteq\mathcal{K}} \sum_{i\in\mathcal{I}} x_i(S)v_i(S) & & & & & \\
s.t. & & & & & \\
\sum_{S\subseteq\mathcal{K}} x_i(S) & \leq & 1 & \forall i & (\pi_i) & (LP1) \\
x_i(S) & \leq & \sum_{X\in\mathbb{C}\cup\mathbb{C}_u : S_i\in X} y(X) & \forall i, S & (p_i(S)) & (LP2) \\
\sum_{X\in\mathbb{C}\cup\mathbb{C}_u} y(X) & \leq & 1 & & (\pi_s) & (LP3) \\
y(X) & \leq & 0 & \forall X\in\mathbb{C}_u & (t(X)) & (LP4) \\
x_i(S), y(X) & \geq & 0 & \forall i, S, X\in\mathbb{C}\cup\mathbb{C}_u & &
\end{array} \tag{3.1}$$

The dual to the extended $CAP_3$ in (3.1) is:

$$\begin{array}{rcllll}
\min_{\pi_i,\pi_s} \sum_{i\in\mathcal{I}} \pi_i + \pi_s & & & & & \\
s.t. & & & & & \\
\pi_i + p_i(S) & \geq & v_i(S) & \forall i, S & (x_i(S)) & (DLP1) \\
\pi_s - \sum_{S_i\in X} p_i(S) & \geq & 0 & \forall X\in\mathbb{C} & (y(X)) & (DLP2a) \\
\pi_s + t(X) - \sum_{S_i\in X} p_i(S) & \geq & 0 & \forall X\in\mathbb{C}_u & (y(X)) & (DLP2b) \\
\pi_i, \pi_s, p_i(S) & \geq & 0 & \forall i, S & &
\end{array} \tag{3.2}$$

The decision variables of the primal are: $x_i(S)$ denoting whether bidder $i$ wins package $S$ and $y(X)$ denoting whether allocation $X$ is realized or not. In the dual, $\pi_s$ and $\pi_i$ denote the auctioneers' and bidder $i$'s profit respectively whereas $p_i(S)$ denotes the ask price for the package $S$ and bidder $i$. Prices are summarized by $p$. Theorem 3.1 in Bikhchandani and Ostroy (2002) shows that allocation $X$ and prices $p$ form a competitive equilibrium (CE), if $X$ is an optimal solution to the primal $CAP_3$ and $(p, \pi_i, \pi_S)$, where $\pi_i, \pi_S$ are payoffs resulting from $X$ and prices $p$ are an optimal solution to the corresponding dual linear program. Their proof is based on the resulting complementary slackness conditions. We show that additional allocation constraints causing additional infeasible solutions do not impact the theorem and the equivalence between competitive equilibrium and optimal solution to (3.1) is still given. Let us first enumerate the complementary slackness (CS) conditions:

$$\begin{aligned}
\left(\sum_S x_i(S) - 1\right)\pi_i &= 0 \quad \forall i & (CS1)\\
\left(x_i(S) - \sum_{S_i \in k} y(X)\right) p_i(S) &= 0 \quad \forall i, S & (CS2)\\
\left(\sum_X y(X) - 1\right)\pi_s &= 0 & (CS3)\\
y(X)t(X) &= 0 \quad \forall X \in \mathbb{C}_u & (CS4)
\end{aligned}$$

$$\begin{aligned}
(\pi_i + p_i(S) - v_i(S))\, x_i(S) &= 0 \quad \forall i, S & (CS5)\\
\left(\pi_s - \sum_{S_i \in X} p_i(S)\right) y(X) &= 0 \quad \forall X \in \mathbb{C} & (CS6)\\
\left(\pi_s + t(X) - \sum_{S_i \in X} p_i(S)\right) y(X) &= 0 \quad \forall X \in \mathbb{C}_u & (CS7)
\end{aligned}$$

The competitive equilibrium (CE) conditions are:

$$\pi_i = \max_S (v_i(S) - p_i(S)) \quad \forall i \quad (CE1) \qquad \pi_s = \max_{X \in \mathbb{C}} \sum_{S_i \in X} p_i(S) \quad (CE2)$$

**Lemma 3.2.** *$(X^*, p^*)$ is a CE if and only if the integral solution dictated by $X^*$ is an optimal solution to primal $CAP_3$ (3.1) and $(p^*, \pi_i^*, \pi_S^*)$ is an optimal solution to dual $CAP_3$ (3.2).*

*Proof.* We follow the proof of Theorem 3.1 by Bikhchandani and Ostroy (2002), and show that the additional infeasible allocations due to additional allocation constraints do not violate the equivalence of competitive equilibrium and optimality of the winner determination problem. The due to the allocation constraints additional (CS4) and (CS7) do always hold, as $y(X) = 0$ for the infeasible allocations (cf. (LP4) and $y(X) \geq 0$). Denote with $S_i^*$ the package bidder $i$ is assigned under allocation $X^*$.

Sufficiency: Suppose the LP (3.1) has an integral solution $X^*$ with $x_i(S) = 1$ iff $S = S_i^*$ and $y(X) = 1$ iff $X = X^*$. Let $(\pi_s^*, \pi_i^*, p^*, t(X^*))$ be an optimal solution of the DLP (3.2). $t(X^*) \geq \sum_{S_i \in X} p_i(S)$ because it does not appear anywhere else than in (DLP2b) and the program minimizes $\pi_s$. (CS5) and (DLP1) imply the first CE condition (CE1). (DLP2a) and (DLP2b) imply

$$\pi_s \geq \max \left\{ \max_{X \in \mathbb{C}} \sum_{S_i \in X} p_i(S), \max_{X \in \mathbb{C}_u} \sum_{S_i \in X} (p_i(S) - t(X)) \right\}.$$

Due to $t(X^*) \geq \sum_{S_i \in X} p_i(S)$ the last term is always smaller or equal to zero, while the first term is always greater or equal to zero. Due to (CS6) the above inequality implies the second CE condition (CE2). Hence $(X^*, p^*)$ is a CE.

Necessity: Let $(X^*, p^*)$ be a CE. Therefore by definition:

$$\begin{aligned}
\pi_i^* &\equiv v_i(S^*) - p_i(S_i^*) = \max_S (v_i(S) - p_i(S)) \quad \forall i & (CE1)\\
\pi_s^* &\equiv \sum_{S_i \in X^*} p_i(S) = \max_{X \in \mathbb{C}} \sum_{S_i \in X} p_i(S) & (CE2)
\end{aligned}$$

Let $x_i(S) = 1$ iff $S = S_i^*$ and $y(X) = 1$ iff $X = X^*$, else 0. $X^*$ is a feasible solution to LP (3.1) since the allocation is supported in a CE equilibrium. Similarly $(\pi_s^*, \pi_i^*, p^*, t(X^*))$ is feasible to DLP (3.2). The dual variable $t(X)$ does not impact this equivalence. The remaining proof showing that the integral solution we just constructed is optimal, is identical to the proof of Theorem 3.1 in Bikhchandani and Ostroy (2002).

□

**Definition 3.11.** *A straightforward bidder bids only for those packages that maximize his payoff given the current ask prices.*

**Corollary 3.2.** *The $CAP_3$ formulation with allocation constraints (3.1) yields integral solutions and thus the efficient ascending CAs (iBundle, APA, dVSV) terminate at a CE even if allocation constraints are present and bidders follow the straightforward bidding strategy.*

This follows directly from Lemma 3.2 and the original proofs of the efficiency of *iBundle* (Parkes and Ungar, 2000) and dVSV (de Vries et al., 2007). Parkes and Ungar (2000) show that all complementary slackness conditions except (CS1) are satisfied in each round of the *iBundle* auction. (CS1) states that every bidder with a positive utility for some packages at the current prices must receive a package in the allocation. Only in the last round this condition is satisfied for all bidders. The new complementary slackness conditions (CS4) and (CS7) due to allocation constraints are trivially satisfied, because $y(X)$ is null, and do not impact the proof. While the price updates in dVSV follow a primal-dual algorithm, *iBundle* and APA can be considered subgradient algorithms (de Vries et al., 2007).

Ausubel and Milgrom (2006a) show that the APA (and therefore *iBundle*) terminates with an efficient solution and straightforward bidding is an ex post Nash equilibrium strategy when the BSM condition holds. The proof is defined on some coalitional value function, which might be implemented by $CAP_3$ but also a $CAP_3$ with additional allocation constraints, and it is therefore not affected by allocation constraints. In summary, allocation constraints neither have an impact on the efficiency of the family of efficient ascending CAs, nor on the incentive properties. While *iBundle*, the APA, and dVSV allow for allocation constraints, they do not explicitly take them into account in the pricing rule, but implement simple price increments for subsets of bidders.

## 3.6 Efficiency and Equilibrium Analysis of FCAs

In what follows, we want to understand economical characteristics of FCAs and whether pricing rules such as $WL$, and $DL$ can also achieve 100% efficiency with a strong solution concept, and how they relate to other efficient ascending CAs such as *iBundle*, APA, and dVSV introduced earlier. Except from the ask price calculation (i.e., pricing rules) the following auctions ($\text{FCA}_{WL}$ and $\text{FCA}_{DL}$) are equivalent to APA and *iBundle*. As in all other efficient ascending CAs we will first assume a straightforward bidding strategy where the bidders only have to reveal their demand set in each round. We will show that while $\text{FCA}_{WL}$ does not even lead to an efficient solution with this bidding strategy, $\text{FCA}_{DL}$ leads to an efficient outcome and straightforward bidding is an ex-post equilibrium with buyer submodular valuations. Throughout we will assume an XOR bidding language.

### 3.6.1 $\text{FCA}_{WL}$

In the $\text{FCA}_{WL}$ auction losing bidders in a round get an ask price of $WL(S,i)+\epsilon$ for a package $S$. In each round $WLs$ for losing bids of losing bidders have to be calculated. This causes the $\text{FCA}_{WL}$ to be an ascending CA, although generally, $WLs$ are not monotonically increasing as $DLs$ are (cf. Section 3, Proposition 3.2).

**Proposition 3.6.** *$\text{FCA}_{WL}$ is an ascending CA.*

*Proof.* It is more convenient to speak of auction rounds $r$ than of states here. $WL_r(S,i)$ is only updated right after the submission of a bid $(S,v,r-1,i)$ and therefore $WL_r(S,i)$ can never be lower than the bid's value $v$. But $v$ is equal to the $WL_{r'}(S,i)$ presented to the bidder before the submission, hence for any $r' < r$ it holds $WL_{r'}(S,i) \leq WL_r(S,i)$ (the round $r'$ is either the previous round the bidder had bid on $S$ or the first round). □

The efficiency of a $\text{FCA}_{WL}$ can be as low as 0% if the bidders bid straightforward and valuations are *demand masking*.

**Definition 3.12.** *A demand masking set of bidder valuations is given if the following properties are fulfilled. For each item, there is one bidder. Each $l$-th* *bidder values the big package which contains all items with $V_b$ and the $l$-th single item with $V_s$. All other package valuations are zero. We set $mV_s > V_b > V_s$ so that at the efficient allocation every bidder wins a single item.*

| | $\mathcal{K}$ | item $l$ | item $l' \neq l$ |
|---|---|---|---|
| $l$-th bidder | $V_b$ | $V_s$ | 0 |

TABLE 3.5: Demand masking set of bidder valuations

Let $n = |\mathcal{K}|$ be the number of items. We will first provide an example with $n = 4$, $V_s = \$2$ and $V_b = \$5$, where $FCA_{WL}$ is inefficient.

**Example 3.3.** *There are four bidders, B1 to B4 and four items A to D. Table 3.6 indicates the auction progress. Prices are initialized to \$0. At the beginning, all bidders bid on the big package. When its price increases to \$3, then the losing bidders bid also on the single items, since their payoff is \$2, i.e. equals the payoff of the big package. Their bids on the single items are unsuccessful and the prices are updated to \$4. These updated prices exceed their valuations $V_s = \$2$, therefore they never bid again on the single items and the auction fails to reach the efficient solution.*

| | $\text{FCA}_{WL}$ | | | | | |
|---|---|---|---|---|---|---|
| packages | A | B | C | D | ABCD | $\emptyset$ |
| valuations | $2_1$ | $2_2$ | $2_3$ | $2_4$ | $5_1, 5_2, 5_3, 5_4$ | |
| round 1 | | | | | $0_1^*, 0_2, 0_3, 0_4$ | |
| round 2 | | | | | $0_1, 1_2^*, 1_3, 1_4$ | |
| round 3 | | | | | $2_1^*, 1_2, 2_3, 2_4$ | |
| round 4 | | $0_2$ | $0_3$ | $0_4$ | $2_1, 3_2^*, 3_3, 3_4$ | |
| round 5 | $0_1$ | | | | $3_2, 4_3^*, 4_4$ | |
| round 6 | | | | | $5_1^*, 5_2, 4_3, 5_4$ | $0_1, 0_2^*, 0_4^*$ |
| round 7 | | | | | $5_1^*, 5_2, 5_3, 5_4$ | $0_1, 0_2^*, 0_3^*, 0_4^*$ |
| | Termination | | | | | |

TABLE 3.6: $\text{FCA}_{WL}$ process

**Theorem 3.3.** *If bidder valuations are demand masking, the efficiency of $FCA_{WL}$ with straightforward bidding converges to $2/m$ in the worst case with $m > 1$.*

*Proof.* We distinguish two cases:

$$V_b \geq 2V_s \tag{3.1}$$

If inequality (3.1) holds, FCA$_{WL}$ is inefficient. At the beginning, all bidders bid on the big package $\mathcal{K}$. They do so until the round $r$ at which its winning level exceeds $V_b - V_s$ and thus its payoff falls below the payoff of a single item. Denote with $w$ the winner of the big package at this round. All other bidders lose. Their payoffs are higher for the single items than the big package and therefore they bid on them at the current price 0 (payoff(item $l$) $= V_s - 0 >$ payoff($\mathcal{K}$) $= V_b - (V_b - V_s + \delta) = V_s - \delta$, whereby $\delta$ denotes a small constant). These bids of zero value are unsuccessful and the single item prices (i.e. winning levels) for the next round amount to the winning bid of $w$ which was $V_b - V_s$. Their payoff(item $l$) $= V_s - V_b + V_s = 2V_s - V_b$ and payoff($\mathcal{K}$) $= V_b - V_b + V_s = V_s >$ payoff(item $l$) since $V_b > V_s$. Thus they bid again on the big package. Furthermore, due to (3.1) their payoffs for the single items (equal to $2V_s - V_b$) are negative and they will never bid on them again. The auction ends by assigning the big package to an arbitrary bidder. The efficiency is $V_b/(nV_s)$ and for $V_b = 2V_s + \delta$, it becomes $(2 + \delta')/n$. Thus for a large $m$, it converges to 0%. On the contrary, if (3.1) does not hold, the bidders will bid again on the small items at price $V_b - V_s$ when the payoff of the big package falls below payoff(item $l$) $= 2V_s - V_b$. They win the single items and the auction ends with the efficient outcome. Note also that for $m = 2$ the reverse inequality (3.1) cannot be satisfied due to requirement $nV_s > V_b$ and hence the auction is efficient for $n = 2$ and inefficient for $n > 2$. □

While there might also be other bidder valuations leading to low efficiency, it is sufficient for our purposes to show that the efficiency of FCA$_{WL}$ can actually be as low.

### 3.6.2 FCA$_{DL}$

Contrary to the negative results on FCA$_{WL}$, we show that FCA$_{DL}$ leads to full efficiency with straightforward bidding, but it requires less rounds and less bids than *iBundle* and the APA. The only difference between FCA$_{DL}$ and FCA$_{WL}$ is that in FCA$_{DL}$, ask prices are updated to $DL(S, i) + \epsilon$.

**Lemma 3.3.** *DL ask prices are always higher or equal to iBundle prices given the same bids.*

*Proof.* Let $p_i^k(S)$ be the *iBundle* price after $k$ bids are submitted. *iBundle* uses an XOR bidding language, and let $CAP_k(S,i)$ denote the highest bid of $i$ in $S$ (bids of one bidder cannot be combined due to XOR): $CAP_k(S,i) = max\{v|(T,v,k',i), T \subseteq S, k' \leq k\}$. Equation (3.6) implies $DL_k(S,i) \geq CAP_k(S,i)$. The price update rules in *iBundle* ensure that in each round $p_i^k(S) = max\{v|(T,v,k',i),\ T \subseteq S, k' \leq k\} + \epsilon$ (Parkes and Ungar, 2000) [17]. Thus $CAP_k(S,i) + \epsilon = p_i^k(S)$ and $DL_k(S,i) + \epsilon \geq p_i^k(S)$. □

To prove the efficiency of FCA$_{DL}$, we draw on the proof of optimality by Parkes and Ungar (2000) and show that optimality is not affected by the requirement to bid $DLs + \epsilon$ instead of only an $\epsilon$ above the last losing bid. Their proof works on a primal and a dual version of $CAP$, due to Bikhchandani and Ostroy (2002). This version is known as $CAP_2$ and is very similar to $CAP_3$. The main difference is that its prices are anonymous and it corresponds to the auction *iBundle*(2). To prove the efficiency of *iBundle*(2), it is assumed that no single bidder bids on two non-overlapping (safety condition. The efficiency of *iBundle*(3) or simply *iBundle* follows then directly from *iBundle*(2) (Parkes and Ungar, 2000) and it can be dispensed with the safety condition.

**Lemma 3.4.** *FCA$_{DL}$ terminates with the efficient solution and with CE prices if bidders bid straightforward.*

*Proof.* The only modification of FCA$_{DL}$, i.e. to quote $DLs$ instead of simple price updates, only affects the proof with respect to the complementary slackness condition CS-6. We only need to show that CS-6, which states that "the allocation must maximize the auctioneer's profit at prices $p(S)$, over all possible allocations and irrespective of bids received by agents", is satisfied by FCA$_{DL}$ too. Replace $p(S)$ by $DL(S)$. From the $DL$ computation follows that there is always a bidder or group of bidders willing to pay $DL(S)$ for every package in the value-maximizing allocation $X^*_{DL}$ that is computed based on the prices ($DLs$) and irrespective of the bids. For this, observe that the highest possible $DL(S)$, which is the case when no bids are blockable, is equal to $CAP(S)$ considering all submitted bids. Hence there is always a bidder or a group of bidders willing to pay $DL(S)$. Therefore, allocation $X^*_{DL}$ with auctioneer's profit $\sum_{S^* \in X^*_{DL}} DL(S^*)$ can be realized by assigning each $S^*$ to a subset of bidders $J(S^*)$, $J(S^*) \subseteq \mathcal{I}$ with $\bigcap J(S^*) = \emptyset$. Every bidder receives at most one package and hence the XOR constraint is not violated. The reason is

[17] Due to the free disposal assumption implying that packages are priced at least as high as the greatest price of any package they contain, i.e. $p_i^k(S) \geq p_i^k(T)$ for $S \supseteq T$.

that packages $S^*$ form a feasible allocation and are obviously non-overlapping and no single bidder bids on non-overlapping packages due to the bid safety condition. In summary, we showed that it is always possible for the auctioneer to realize the profit-maximizing allocation at prices $DL(S)$ irrespective of bids received, since the computation of $DLs$ ensures there are always bidders willing to take these prices [18]. □

**Theorem 3.4.** *$FCA_{DL}$ is efficient if bidders follow the straightforward bidding strategy. This strategy is an ex-post Nash equilibrium if the BSM condition holds.*

Theorem 3.4 follows directly from Lemma 3.4 and Ausubel and Milgrom (2006a).

FCA$_{DL}$ can reduce the number of auction rounds, which is a considerable problem of *iBundle* as shown by Scheffel et al. (2011) and Schneider et al. (2010). The reason is that dead bids in *iBundle*, which will never be part of the winning allocation, are skipped and prices increase faster. We provide a simple example that FCA$_{DL}$ can terminate with strictly less rounds than *iBundle*.

**Example 3.4.** *Consider items A,B,C are auctioned among bidders $B1$ to $B4$ in iBundle and $FCA_{DL}$ using an increment of $\epsilon = 1$. Bidders bid straightforward and are single minded which means they value only one package positively and all others with zero. The exact valuations of each bidder and the auction rounds are described in Table 3.7. Ties are broken in favor of more winners.*

The example illustrated in Table 3.7 shows that FCA$_{DL}$ reduces the number of auction rounds, the communication effort (since dead bids are not submitted) and also the computational effort. In general the reduction of auction rounds and communication effort comes at the price of higher computational effort as the $\Pi_2^P$-hard $DL$ determination problem has to be solved several times.

In what follows, we introduce two economically motivated value models to demonstrate the benefits of $FCA_{DL}$ concerning the number of auction rounds and the communication effort. Let $RRR = \frac{rounds\_iBundle - rounds\_FCA_{DL}}{rounds\_iBundle}$ denote the round reduction rate. Let $C$ denote the communication effort measured as the number of all bids and ask prices exchanged. $CRR = \frac{C\_iBundle - C\_FCA_{DL}}{C\_iBundle}$ is the corresponding reduction rate.

[18] To see why this is not the case by $WLs$, consider only one bid \$10 on $AB$. Then $WL(A) = \$10$ but no bidder is willing to pay the price. $DL(A)$ is instead \$0.

| | *iBundle* | | | | | FCA$_{DL}$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| packages | A | B | C | ABC | | A | B | C | ABC |
| valuations | $5_1$ | $5_2$ | $5_3$ | $8_4$ | | $5_1$ | $5_2$ | $5_3$ | $8_4$ |
| round 1 | $1_1^*$ | $1_2^*$ | $1_3^*$ | $1_4$ | | $1_1^*$ | $1_2^*$ | $1_3^*$ | $1_4$ |
| round 2 | $1_1^*$ | $1_2^*$ | $1_3^*$ | $2_4$ | | $1_1$ | $1_2$ | $1_3$ | $4_4^*$ |
| round 3 | $1_1^*$ | $1_2^*$ | $1_3^*$ | $3_4$ | | $2_1^*$ | $2_2^*$ | $2_3^*$ | $4_4$ |
| round 4 | $1_1$ | $1_2$ | $1_3$ | $4_4^*$ | | $2_1$ | $2_2$ | $2_3$ | $7_4^*$ |
| round 5 | $2_1^*$ | $2_2^*$ | $2_3^*$ | $4_4$ | | $3_1^*$ | $3_2^*$ | $3_3^*$ | $7_4$ |
| round 6 | $2_1^*$ | $2_2^*$ | $2_3^*$ | $5_4$ | | $3_1^*$ | $3_2^*$ | $3_3^*$ | $\emptyset_4^*$ |
| round 7 | $2_1^*$ | $2_2^*$ | $2_3^*$ | $6_4$ | | Termination | | | |
| round 8 | $2_1$ | $2_2$ | $2_3$ | $7_4^*$ | | | | | |
| round 9 | $3_1^*$ | $3_2^*$ | $3_3^*$ | $7_4$ | | | | | |
| round 10 | $3_1^*$ | $3_2^*$ | $3_3^*$ | $8_4, \emptyset_4^*$ | | | | | |
| | Termination | | | | | | | | |

TABLE 3.7: *iBundle* and FCA$_{DL}$ process

**Definition 3.13.** *Value model $VM1$ comprises $m$ single minded regional bidders who value pairwise non-overlapping packages to at least \$$\epsilon$ and one global bidder who values package $\mathcal{K}$ of all items to at least the sum of the valuations of the regional bidders.*

**Definition 3.14.** *Value model $VM2$ comprises $m$ single minded bidders with identical valuations for a specific package $S$, with $|S| > m - |\mathcal{K}| + 1$, i.e. the complementary to $S$ subauction is not too large and ensures the existence of blockable bids and thus high $DL$.*

**Theorem 3.5.** *In $VM1$ $RRR = \frac{m-1}{m+1}$ and $CRR = \frac{1}{2} - \frac{1}{2m}$. In $VM2$ $RRR = CRR = \frac{1}{m}$.*

*Proof.* $VM1$: In each round, either the coalition of the global bidder alone wins or the coalition of all regional bidders. Denote a round as $G$ if the global bidder wins, else as $S$. We consider the sequence of rounds which comprises of two consecutive winning rounds for the global bidder. In $FCA_{DL}$ the coalitions win alternately, thus the sequence is $GSG$. After a $G$, each regional bidder increases his bid by $\epsilon$ and after a $S$, the global bidder by $m\epsilon$ since his $DL$ increases by this amount. In $FCA_{DL}$ the sequence is $GSG$ while in *iBundle* $G\overbrace{S \cdots S}^{m}G$. $RRR$ and $CRR$ equal to the reductions rates of this cyclical sequence, excluding the last $G$. Thus $RRR = \frac{m+1-2}{m+1} = \frac{m-1}{m+1}$ and $CRR = \frac{2m-(m+1)}{2m} = \frac{1}{2} - \frac{1}{2m}$

$VM2$: Every bidder bids on the same package. Let $p_r$ denote its highest price over all bidders at round $r$. In $FCA_{DL}$ $p_r$ increases by $\epsilon$ after each round. In *iBundle* it can be easily seen that in every $m$ consecutive rounds, there is one round where $p_r$ remains unchanged since all losing bidders just level the price of the previously winning bid, thus on average the price increase is $(1-\frac{1}{m})\epsilon$. The number of rounds is equal to the final price $p^T$ divided by the average price increase, thus $RRR = \frac{(1-\frac{1}{m})\epsilon-\epsilon}{(1-\frac{1}{m})\epsilon} = \frac{m-1}{m+1}$. Regarding $CRR$, in *iBundle* each bidder submit $\frac{p^{final}}{\epsilon}+1$ bids. In $FCA_{DL}$, w.l.o.g. we assume that all but two bidders always lose (tie breaking). These two bidders alternately increase their bids by $2\epsilon$ and the computational effort for these two is equal to the effort for one of the always losing bidders. Thus $CRR = \frac{m(\frac{p^T}{\epsilon}+1)-(m-1)(\frac{p^T}{\epsilon}+1)}{m(\frac{p^T}{\epsilon}+1)} = \frac{1}{m}$. □

It follows that in $VM1$ $RRR \to 100\%$ and $CRR \to 50\%$ for $m \to \infty$. Surprisingly, in a realistic value model with regional and global bidders, which resembls the setting of FCC sprectum auctions, $FCA_{DL}$ can substantially decrease the number of rounds and communication effort. In more generic $VM2$, which is the case when many bidders have homogeneous valuations, $RRR \to 50\%$ and $CRR \to 50\%$ for $m = 2$.

## 3.7 Numerical Experiments

The computational complexity results for exact *DLs* suggest that the computational costs outweigh the benefits. We supplement the theoretical analysis with an experimental comparison of FCA$_{DL}$ and *iBundle*. Since there are hardly any real-world CA data sets available, we have adopted value models of the Combinatorial Auctions Test Suite (CATS) (Leyton-Brown et al., 2000). In addition to CATS value models, we have used an extended version of the Pairwise Synergy value model from An et al. (2005). A decscription of the value models follows:

The ***Real Estate*** value model is based on the *Proximity in Space* model from CATS (Leyton-Brown et al., 2000). Items sold in the auction are the real estate lots $l$, which have valuations $v_l$ drawn from the same normal distribution for each bidder. Adjacency relationships between two pieces of land $m$ and $n$ ($e_{mn}$) are created randomly for all bidders. There is a 90% probability of a vertical or horizontal edge, and an 80% probability of a diagonal edge. Edge weights $w_{mn}$ are then generated randomly for each bidder (mean 0.4, deviation 0.2), and they are used to determine package valuations of adjacent pieces of land: $v(S) = (1 + \sum_{e_{mn}:m,n \in S} w_{mn}) \sum_{l \in S} v_l$.

The ***Transportation*** value model uses the *Paths in Space* model from CATS (Leyton-Brown et al., 2000). It models a nearly planar transportation graph in Cartesian coordinates, where each bidder is interested in securing a path between two randomly selected vertices (cities). The items traded are edges (routes) of the graph. Parameters for the Transportation value model are the number of items (edges) $m$ and graph density $\rho$, which defines an average number of edges per city, and is used to calculate the number of vertices as $(m * 2)/\rho$. The bidder's valuation for a path is defined by the Euclidean distance between two nodes multiplied by a random number, drawn from a uniform distribution. Consequently only a limited number of packages, which represent paths between both selected cities, are valuable for the bidder. This allows us to consider even larger transportation networks in a reasonable time. In our simulations we set the mean of $\rho$ to 1.8 and 2.5 for the small and medium size auctions respectively. The standard deviation was set to 0.25.

The ***Pairwise Synergy*** value model in An et al. (2005) is defined by a set of valuations of individual items $\{v_l\}$ and a matrix of pairwise item synergies $\{syn_{k,l} : k, l \in \mathcal{K}, syn_{k,l} = syn_{l,k}, syn_{k,k} = 0\}$. The valuation of a package $S$ is then calculated as $v(S) = \sum_{k=1}^{|S|} v_k + \frac{1}{|S|-1} \sum_{k=1}^{|S|} \sum_{l=k+1}^{|S|} syn_{k,l}(v_k + v_l)$. A synergy value of 0 corresponds to completely independent items, and the synergy value of 1 means that the package valuation is twice as high as the sum of the individual item valuations. The relevant parameters for the Pairwise Synergy value model are the interval for the randomly generated item valuations, set to $[0.0, 30.0]$, and the interval for the randomly generated synergy values, set to $[0.0, 2.0]$. In ***Pairwise Synergy+*** we specified additionally two bidder segments. In small size auctions, six bidders were interested in packages of cardinality 1 and three bidders of cardinality 3. In medium size auctions, eight bidders were interested in packages of cardinality in the interval $[1, 3]$ and four in the interval $[7, 8]$.

For each value model we created 30 auction instances with different valuations, and ran each of them with both auction formats. All auctions used a bid increment of 1. Table 3.8 depicts the results for small and medium sized auctions. We used the Symphony MIP solver (http://www.coin-or.org/SYMPHONY/) and computers with an Intel Core 2 CPU with 2.67GHz and 4GB main memory. Efficiency and final pay prices of both auction formats were the same (in accordance with Section 3.6.2). Hence, we report only on the reduction in rounds $RRR$ and communication effort $CRR$ and the computation times.

In small size auctions (see Table 3.8) we observe a considerable reduction of the auction rounds across all value models and particularly in Pairwise Synergy+, where the maximum over the 30 auctions ($RRR_{max}$) is 47.945%. This is due to the existence of two bidder segments, regional and global, which leads to high $RRR$. The average $CRR$ is over 30% in all value models except the Transportation value model, where bidders are only interested in a very limited set of packages. $RF$ is the ratio of total run time of FCA$_{DL}$ to *iBundle*. Surprisingly, despite of the high computational complexity of $DLs$, the run times of FCA$_{DL}$ is sometimes even lower than that of *iBundle* (see 3.8 where $RF < 1$). The reason is the lower number of bids submitted, which shortens the winner determination in each round. The average computation time of a $DL$, denoted as $PCT$, ranges from 1.0 to 3.4 milliseconds only.

The mid-sized auctions with up to 9 items led to higher computation times than *iBundle* since many more $DLs$ had to be computed. Also in these experiments computing a $DL$ took between 1ms and 170ms only. The communication between auctioneer and bidders ($CRR$) was reduced substantially in all value models, but the total runtime $RF$ increased.

We conducted additional experiments with 10 items and 9 bidders in different value models, where bidders submit bids on every possible package. Also in these larger instances, the computation lasted only 1.48s on average. Obviously, the computation time depends on many parameters such as the package size, the size of the complementary subauction, the number of bidders and their bids. However, our results indicate that computing $DLs$ in ascending auctions might well be used in practical applications.

## 3.8 Conclusion

Designing efficient combinatorial auctions turned out to be a challenging task. A few recent papers have described efficient and ascending combinatorial auctions which satisfy strong game-theoretical solution concepts. In many applications the consideration of additional allocation constraints and flexibility in the choice of the bidding language are essential. These requirements have not been considered in the design of price feedback in the theoretical literature so far. It is important to extend the theory respectively. This could increase

| | iBundle(3) | DL | |
|---|---|---|---|
| **Real Estate** | | $RRR_{max}$ | 7.547% |
| | | $CRR_{max}$ | 49.222% |
| 2x2 items | ∅ R = 43.8 | ∅ $RRR$ | 3.395% |
| 10 bidders | ∅ C = 3943.1 | ∅ $CRR$ | 44.995% |
| | | ∅ $RF$ | 1.425 |
| | | ∅ $PCT$ | 3.402ms |
| **Transportation** | | $RRR_{max}$ | 17.021% |
| | | $CRR_{max}$ | 13.584% |
| 4 items | ∅ R = 108.4 | ∅ $RRR$ | 5.333% |
| 10 bidders | ∅ C = 842.2 | ∅ $CRR$ | 5.635% |
| | | ∅ $RF$ | 0.986 |
| | | ∅ $PCT$ | 1.252ms |
| **Pairwise Synergy+** | | $RRR_{max}$ | 47.945% |
| | | $CRR_{max}$ | 39.447% |
| 3 items | ∅ R = 110.9 | ∅ $RRR$ | 36.625% |
| 9 bidders | ∅ C = 1055.9 | ∅ $CRR$ | 30.028% |
| | | ∅ $RF$ | 0.714 |
| | | ∅ $PCT$ | 1.025ms |

| | iBundle(3) | DL | |
|---|---|---|---|
| **Real Estate** | | $RRR_{max}$ | 1.190% |
| | | $CRR_{max}$ | 49.029% |
| 3x3 items | ∅ R = 96.3 | ∅ $RRR$ | 0.513% |
| 12 bidders | ∅ C = 341261.6 | ∅ $CRR$ | 45.976% |
| | | ∅ $RF$ | 30.756 |
| | | ∅ $PCT$ | 169.377ms |
| **Transportation** | | $RRR_{max}$ | 0.000% |
| | | $CRR_{max}$ | 19.790% |
| 9 items | ∅ R = 104.7 | ∅ $RRR$ | 0.000% |
| 12 bidders | ∅ C = 1448.4 | ∅ $CRR$ | 4.519% |
| | | ∅ $RF$ | 1.094 |
| | | ∅ $PCT$ | 1.596ms |
| **Pairwise Synergy+** | | $RRR_{max}$ | 29.132% |
| | | $CRR_{max}$ | 24.314% |
| 8 items | ∅ R = 302.5 | ∅ $RRR$ | 5.001% |
| 12 bidders | ∅ C = 115947.0 | ∅ $CRR$ | 21.022% |
| | | ∅ $RF$ | 5.467 |
| | | ∅ $PCT$ | 8.862ms |



TABLE 3.8: Comparison of $FCA_{DL}$ to $iBundle$. Left part for small size and right part for medium size auctions.
$RRR_{max}$ = Max Round Reduction Rate, $CRR_{max}$ = Max Communication Reduction Rate,
∅ $RRR$ = Average Round Reduction Rate, ∅ $CRR$ = Average Communication Reduction Rate,
∅ $RF$ = Average Runtime Factor, ∅ $PCT$ = Average Price

the applicability of ascending combinatorial auctions in domains such as transportation or industrial procurement considerably and bare significant practical potential.

We consider ascending combinatorial auctions allowing for side constraints and OR as well as XOR bidding languages. We draw on the work by Adomavicius and Gupta (2005) and define winning and deadness levels ($WLs$ and $DLs$) as a general pricing rule for ascending combinatorial auctions, which allow for different bidding languages and allocation constraints. This extension leads to a number of theoretical challenges. We show that straightforward bidding is an ex post equilibrium in ascending combinatorial auctions with $DLs$, and how this pricing rule can be integrated in the theoretical framework of efficient and ascending combinatorial auctions.

While both, *iBundle* and the $\text{FCA}_{DL}$ allow for allocation constraints, $DLs$ take allocation constraints into account and actually lead to a lower number of auction rounds and bids that need to be submitted. The high number of auction rounds turned out to be one of the main obstacles for efficient ascending combinatorial auctions such as *iBundle*, the Ascending Proxy Auction, and dVSV. $DLs$ come at a computational cost, however. The computation is a $\Pi_2^P$-complete problem. We show, however, that such ask prices can be calculated for up to 10 items and 9 bidders with realistic value models in less than 1.5 seconds in experiments, which suggests that these approaches might well be used in applications. Approximations to the exact computation of $DLs$ could potentially be an area of future research.

These results provide a theoretical foundation for practical auction design. Such designs can leverage different pricing rules. Experimental research is required to gain insights on bidding behavior and efficiency in complex markets with allocation constraints.

# Chapter 4

# Ascending Core-Selecting Auctions with Risk Averse Bidders

When we passionately believe in
something non-existing, we create
it at the end.

N. Kazatzakis

Core-selecting combinatorial auctions are being used in an increasing number of spectrum sales worldwide. Only recently, such auctions have been analyzed as Bayesian games with risk-neutral bidders. It was shown that equilibrium strategies lead to outcomes that are further from the core than the VCG outcome in sealed-bid auctions. Ascending core-selecting auctions can even lead to inefficient non-bidding equilibria. Risk aversion is arguably a significant driver of bidding behavior in high-stakes auctions. We analyze the impact of risk aversion on equilibrium bidding strategies and efficiency of core-selecting auctions in a threshold problem with one global and several local bidders. Our first main result is a generalization of the condition for a non-bidding equilibrium in Goeree and Lien (2009) and Sano (2012), which allows for arbitrary concave utility functions. Second, we discuss this condition in the presence of asymmetries, and third, we provide comparative statics. We show that risk aversion and bidder asymmetries affect the equilibrium outcomes in ways that can be systematically analyzed. In particular, risk-aversion reduces the scope of non-bidding equilibrium in the sense that dropping at the reserve price ceases to be equilibrium as bidders become more risk averse. Different wealth levels and stochastic dominance orderings of the valuation distributions and their impact on non-bidding are analyzed as well.

## 4.1 Introduction

Combinatorial auctions have been suggested as an alternative to the simultaneous multi-round auction for the sale of spectrum auctions, and they have also been used in other areas such as transportation and industrial procurement. CAs solve the exposure problem of bidders with complementary valuations for bundles of items. In a simultaneous multi-round auction such a bidder might bid for a bundle, but end up winning only parts of the bundle and effectively making a loss.

The Vickrey-Clarke-Groves mechanism is the unique mechanism with a dominant strategy equilibrium (Green and Laffont, 1979; Holmstrom, 1979), but it is rarely used in practice. One of the problems of the VCG mechanism is that the outcomes might not be in the core (Ausubel and Milgrom, 2006b). This leads to very low revenue for bidders and possibilities for shill bidding.

Day and Milgrom (2008); Day and Raghavan (2007) and Day and Cramton (2012) introduce core-selecting payment rules, which impose core constraints first and then try to approximate incentive constraints. Day and Milgrom (2008) show that in a complete information game bidder-optimal core-selecting auctions minimize incentives to deviate from truthful bidding. Day and Cramton (2012) propose a quadratic core-selecting payment rule, which minimizes the Euclidean distance from the Vickrey payments and which is nowadays in use in spectrum auctions around the world. Since 2008, a two-stage core-selecting combinatorial clock auction (CCA) has been used in several countries world-wide to sell spectrum (Cramton, 2013). The CCA has an initial phase with an ascending combinatorial clock auction and a second stage with a sealed-bid core-selecting auction.

After initial analyses of the complete information game by Day and Milgrom (2008), more recently Goeree and Lien (2012), Sano (2011a) and Ausubel and Baranov (2010) analyzed the Bayesian Nash equilibria of sealed-bid core-selecting auctions. They study environments with only two items for sale. There are local bidders interested in a single item only and a global bidder who is interested in both items only (aka. local-local-global environment). It is easy to show that the VCG auction is not in the core, when the two local bidders have a higher valuation than the global bidder. Goeree and Lien (2012) derive an equilibrium of the nearest-Vickrey core-selecting auction and show that in a private values model with rational bidders, auctions with a core-selecting payment rule are on average further from the core than auctions with a VCG payment rule. They also show that no Bayesian incentive-compatible core-selecting auction exists, when the VCG outcome is not in the core. Ausubel and Baranov (2010) analyze different payment rules and different levels of correlation among bidders.

Goeree and Lien (2009) already provide a Bayesian analysis of ascending CAs in a threshold model as described above. They look at a specific version of the latter, where, if two or more local bidders drop out at the same time, one is selected to drop out, while the others are allowed to continue. While this additional tie breaking rule differs from some activity rules in the field, it can be used in practical designs and allows for a succinct pure perfect Bayesian strategy, one where all local bidders try to drop out at a price of null. Their experiments show that local bidders typically do not drop out immediately. With only two local bidders the level of overbidding was higher than with five local bidders. Sano (2012) then characterizes the results for ascending CAs with the same tie-breaking rule and provides a condition for a non-bidding equilibrium for general continuous distribution functions and non-zero reserve prices. In both papers, bidders can make each other bid truthfully by stopping early, such that each of the small bidders wants to stop first if a certain condition on the prior distributions is satisfied or the distributions are uniform.

Even if the environments are simple with two items only, the free-rider problem that arises is fundamental and can also be found in spectrum auctions in the field where regional bidders compete against national bidders. When a bidder submits a bid for a package of items, other bidders who are interested in different subsets of items in the package face a free-rider problem: even if the sum of their values exceeds the bid they need to overbid, each bidder has incentives to free-ride on the contribution of the other. Free-riding is a well-known phenomenon in public goods problems.

Goeree and Lien (2009, 2012); Sano (2011a, 2012) and Ausubel and Baranov (2010) all assume risk-neutral bidders. In high-stakes spectrum auctions it is unlikely that bidders are risk-neutral as the outcome of such an auction can impact the economic fate of a telecom substantially. So, smaller payoffs have a higher utility as long as bidders win, which can be modeled with a concave utility function. Risk aversion is an important phenomenon to be considered in auctions due to the uncertainties faced by bidders in auctions in general. It is a fundamental concept in expected utility theory (Arrow, 1965; Pratt, 1964) and widely used to explain overbidding behavior in first-price sealed-bid auctions (Bajari and Hortacsu, 2005; Cox et al., 1988). Risk aversion leads to different revenue rankings of single-item auctions and is likely to have an ample effect on equilibrium strategies in core-selecting CAs. In single-item first price sealed-bid auctions, Riley and Samuleson (1981) show that risk aversion leads to uniformly higher bids and thus higher revenue. Bidders increase their bids since they want insurance against the possibility of losing.

While risk aversion does not affect the equilibrium strategy under the single-item English auction, it might well have an impact on the condition for a non-bidding equilibrium in an *ascending* core-selecting CA. The higher one local bidder bids, the higher is the probability of the other local bidders to drop out. The consequences on the equilibrium strategy are not as obvious and quite different to single-item auctions. Already, the experiments in Goeree and Lien (2009) show, however, that local bidders do not drop out at a price of null in experiments, as theory would suggest, but they continue to bid further. Risk aversion can be used to explain such behavior in the lab as we will show.

**Contributions**

The contributions of this work are the following: We consider the environments that have been analyzed in the Bayesian literature on ascending core-selecting CAs, but want to understand the impact of risk aversion in these environments as they have been analyzed by Goeree and Lien (2009) and Sano (2012).

*First*, we characterize the necessary and sufficient conditions for the perfect Bayesian equilibrium of the ascending core-selecting auction mechanism to have the small bidders to drop at the reserve price for general environments with risk-averse bidders as well as arbitrary asymmetries across bidders with respect to initial wealth, value distributions, and risk attitudes. Our first main result is a generalization of the condition for a non-bidding equilibrium in Goeree and Lien (2009) and Sano (2012), which allows for arbitrary concave utility functions, reserve prices, and differences in initial wealth.

*Second*, we provide comparative statics and show that risk aversion and bidder asymmetries affect the equilibrium outcomes in ways that can be systematically analyzed. The impact of risk aversion on equilibrium bidding strategies is not obvious, since bidding higher can allow the other local bidder to drop out. Unlike super modular games or games fulfilling the single-crossing property

(Athey, 2001), such as the first price single-item sealed-bid auction, the impact of risk aversion is not obvious in the ascending auction game. The game does not fulfill these properties, as we will show, and bidders cannot buy insurance against the possibility of losing by increasing their bids. Increasing bids may lead to a lower probability of winning. Our results indicate that even in this case, risk-aversion reduces the scope of the non-bidding equilibrium in the sense that dropping at the reserve price ceases to be an equilibrium as the bidders become more risk averse. We also analyze different wealth levels, asymmetries across local bidders, and stochastic dominance orderings of the distributions of valuations and their impact on non-bidding. Similar to Goeree and Lien (2009) and Sano (2012) we do not analyze the continuation game, when the condition for a non-bidding equilibrium does not hold. So far, there are no Bayesian models of ascending CAs in general to our knowledge, and it is likely that no closed-form solutions of Bayesian equilibria exist. The analysis of the continuation game is a significant challenge beyond the focus of this work.

The remainder of the chapter is structured as follows. Section 4.2 contains a formulation of the model with descriptions of the environment and the auction procedure, as well as some preliminary lemmas on the model elements. In Section 4.3, we characterize the necessary and sufficient conditions for an ascending core-selecting auction to have a perfect Bayesian equilibrium in which small bidders stop bidding at the reserve price. Section 4.5 discusses parametric cases with specific distribution and utility functions. Finally, Section 4.6 provides conclusions. Throughout the chapter, proofs that are technical in nature are placed in the Appendix.

## 4.2 Model

We consider a stylized multiple-unit auction environment with two objects and three bidders. Two objects are to be sold to one or two of three potential buyers through a core-selecting ascending auction. The two objects are indistinguishable from the bidders' valuation perspective. As in Goeree and Lien (2009), Sano (2012) and others (Ausubel and Baranov (2010), Sano (2011a)) the environment has an a priori asymmetry in one of the bidders, referred to as the 'large bidder'. He demands two units of the object and has value zero for a single unit. The other bidders, referred to as 'small bidders,' each demand a single unit and have zero valuation for the second unit. We consider an independent private values (IPV) environment where each bidder has a private value for the object(s) which is unknown to the others. In addition to the asymmetry of small and large bidders with respect to the number of units demanded, we allow a full range of ex ante asymmetries across bidders with respect to the distribution of valuations, utility functions and initial wealth levels. We first describe the elements of the bidding environment followed by the details of auction mechanism.

### 4.2.1 Environment: Information, Valuations and Utility Functions

The auction environment with three bidders, $N = 1, 2, 3$, is represented by the collection $e = \{(F_1, F_2, G), (u_1, u_2, u_3)\}$, where the two tuples denote the valuation distributions and utility functions, respectively, of the three bidders. We provide the details of the two parts of an environment next.

#### 4.2.1.1 Information and Valuations

Each bidder has a private value for the object(s) which is unknown to the others. We denote the valuations by $V$ and index it by the bidders. Bidder $i$'s valuation is a random variable $V_i$ (its realization is denoted by $v_i$) with distribution function $F_i(\cdot)$ for $i = 1, 2$ or $G(\cdot)$ for $i = 3$ which has a strictly positive and continuously differentiable density function $f_i(\cdot)$ or $g(\cdot)$ on its support $[\underline{v_i}, \overline{v_i}]$. We set $\underline{v_1} = \underline{v_2} = \underline{v_3} = 0, \overline{v_1} = \overline{v_2} = 1, \overline{v_3} = 2$. Bidders $i = 1, 2$ are only interested in a single unit, while bidder $i = 3$ is only interested in the package of two units. Each bidder is interested in a single package only, and which package bidders are interested in is assumed to be public knowledge. All already referenced papers modeling core-selecting auctions as a Bayesian game are using the same environment, which can be considered the simplest scenario where the Vickrey auction is not in the core. In our work, we want to understand what risk aversion does to core-selecting auctions in this environment, where we still have closed form solutions for risk-neutral bidders.

#### 4.2.1.2 Utility Functions and Risk Aversion

Bidders are expected utility maximizers. Bidder $i$ has von-Neumann-Morgenstern utility function $u_i : \mathbb{R} \Rightarrow \mathbb{R}$. If a bidder with value $v$ and current wealth $\omega$ wins and pays a price $b$ his utility is $u_i(v - b + \omega)$; his utility is

$u_i(\omega)$ if he loses. We assume $u_i$ is twice continuously differentiable, with $u_i'(x) > 0$ and $u_i''(x) < 0 \quad \forall x$. Therefore, $u_i$ is concave and $i$ risk-averse. Since von-Neumann-Morgenstern utility functions are unique up to affine transformations, i.e., $u(x)$ and $a + cu(x)$ represent the same underlying preferences for any choice of real numbers $a$ and $c > 0$, we normalize the utility functions so that $u(\underline{\omega}) = 0$ and $u(\overline{\omega}) = 1$ for some wealth levels $\underline{\omega}$ and $\overline{\omega}$ with $0 < \underline{\omega} < \overline{\omega}$. We use the measure $A(x) = -u''(x)/u'(x) \in \mathbb{R}^+$ introduced by Pratt in his seminal work (Pratt, 1964), to compare the risk aversion of different bidders. Bidder $i$ is more risk-averse than bidder $j$ iff $A_i(x) > A_j(x) \quad \forall x$.

**Increasing absolute risk aversion (IARA) family:**
A bidder exhibits increasing absolute risk aversion if $A(x)$ is of the form:
$A(x) = \frac{1}{ax+b},\ a < 0$

**Decreasing absolute risk aversion (DARA) family:**
A bidder exhibits decreasing absolute risk aversion if $A(x)$ is of the form:
$A(x) = \frac{1}{ax+b},\ a > 0$. $Log\,(x + c)$ is such a function. By setting $b = 0$ it becomes evident that the CRRA functions discussed next belong to the family of DARA functions.

**Constant relative risk aversion (CRRA) family**:
A bidder exhibits constant relative (to his wealth) risk aversion if his utility function is of the form:
$u(x, \underline{\omega}, \overline{\omega}, \rho) = \frac{x^{1-\rho} - \underline{\omega}^{1-\rho}}{\overline{\omega}^{1-\rho} - \underline{\omega}^{1-\rho}}$ where $0 < \rho \neq 1,\ 0 \leq \underline{\omega} < \overline{\omega}$.[1]
$u'(x) = \frac{(1-\rho)x^{-\rho}}{\overline{\omega}^{1-\rho} - \underline{\omega}^{1-\rho}}$, $u''(x) = \frac{-\rho(1-\rho)x^{-\rho-1}}{\overline{\omega}^{1-\rho} - \underline{\omega}^{1-\rho}}$ and $A(x) = \frac{-u''(x)}{u'(x)} = \frac{\rho x^{-\rho-1}}{x^{-\rho}} = \frac{\rho}{x}$

For the case $\rho = 1$, taking the limit $\rho \to 1$ we obtain the logarithmic utility function [2]: $u(x, \underline{\omega}, \overline{\omega}, 1) = \frac{\ln(x/\overline{\omega})}{\ln(\underline{\omega}/\overline{\omega})}$
For the logarithmic utility function we have
$u'(x) = \frac{1}{x \ln(\underline{\omega}/\overline{\omega})}$, $u''(x) = \frac{-1}{x^2 \ln(\underline{\omega}/\overline{\omega})}$ and $A(x) = \frac{-u''(x)}{u'(x)} = \frac{1}{x}$.
Hence $A_i(x) > A_j(x)\, \forall x \Leftrightarrow \rho_i > \rho_j$

**Constant absolute risk aversion (CARA) family:**
A bidder exhibits constant absolute (independent to his wealth) risk aversion if his utility function is of the form:

[1] This representation follows from using the base utility function $\frac{x^{1-\rho}}{1-\rho}$ and selecting the parameters $a$ and $c$ in the representation $u(x) = c\left(\frac{x^{1-\rho}-a}{1-\rho}\right)$ so that $u(\underline{\omega}) = 0$ and $u(\overline{\omega}) = 1$. Specifically, $u(\underline{\omega}) = 0 \Rightarrow a = \underline{\omega}^{1-\rho}$, and $u(\overline{\omega}) = c\left(\frac{\overline{\omega}^{1-\rho} - \underline{\omega}^{1-\rho}}{1-\rho}\right) = 1 \Rightarrow c = \frac{1-\rho}{\overline{\omega}^{1-\rho} - \underline{\omega}^{1-\rho}}$.

[2] This is obtained by using positive affine transformations of the base utility function $\ln(x)$ , i.e., $u(x) = c \quad \ln(x) \ + a$ with the normalization $u(\underline{\omega}) = 0$ and $u(\overline{\omega}) = 1$.

$u\left(x,\underline{\omega},\overline{\omega},\lambda\right)=\frac{e^{-\lambda\underline{\omega}}-e^{-\lambda x}}{e^{-\lambda\underline{\omega}}-e^{-\lambda\overline{\omega}}}$ where $\lambda>0,\quad 0\leq\underline{\omega}<\overline{\omega}$. [3]
$u'\left(x\right)=\frac{\lambda\ e^{-\lambda x}}{e^{-\lambda\underline{\omega}}-e^{-\lambda\overline{\omega}}}, u''\left(x\right)=\frac{-\lambda^{2}e^{-\lambda x}}{e^{-\lambda\underline{\omega}}-e^{-\lambda\overline{\omega}}}$ and $A\left(x\right)=\frac{-u''(x)}{u'(x)}=\lambda$
Hence $A_i\left(x\right)>A_j\left(x\right)\forall x\Leftrightarrow\lambda_i>\lambda_j$

### 4.2.2 Ascending Core-selecting Auction

We study the same multi-unit clock auction with package bidding as in Sano (2012), and follow his description and notation. Let $r\in[0,\ 1)$ be the per-unit reserve price. Before the auction starts, bidders decide whether they participate or not. This decision is made public. In this ascending clock auction, a single clock visible to all bidders is used to indicate the per-unit price $p$ of the items. The clock starts at an initial price $p=r$ and increases continuously.

Each bidder responds with the demand at the current prices. Bidders are restricted by the activity rule: they can never increase their demands. For the specific environments where each bidder can demand 0 or $k$ units, bidder messages can take only two values and a bidder can indicate whether she is 'in' or 'out' (equivalently, 'continue' or 'stop') by pressing or releasing a button. In the case we consider, a bidder pressing the button at a price $p$ indicates her decision to stop bidding, that is, that her demand is zero at that price.

We assume that small bidders can demand either one unit or zero and large bidders can demand either two units or zero. Large bidders' submit 'package bids' on both items. The auctioneer raises the price until the allocation is determined on the basis of these reported values. In a core-selecting auction payments of winners are not necessarily the stopping prices. They are discounted so that the final outcome is in the bidder-optimal core. Let $p_i$ be bidder $i$'s stopping price.

We briefly discuss the outcomes, which can arise in this auction in the case that all three bidders participate (Sano 2011), as they will also be relevant to our analysis:

- Case 1: Bidder 3 first stops at $p_3$. Then, bidders 1 and 2 each win one unit. Both bidders 1 and 2 pay $p_3$.

[3]This representation is based on positive affine transformations of the base utility function $-\ e^{-\lambda x}$. Selecting the parameters $a$ and $c$ in the representation $u\left(x\right)=u\left(x\right)=c\left(a-\ e^{-\lambda x}\right)$ so that $u\left(\underline{\omega}\right)=0$ and $u\left(\overline{\omega}\right)=1$ yields the functional form we use. Specifically, $u\left(\underline{\omega}\right)=0\Rightarrow a=e^{-\lambda\underline{\omega}}$, and $u\left(\overline{\omega}\right)=c\left(e^{-\lambda\underline{\omega}}-e^{-\lambda\overline{\omega}}\right)=1\Rightarrow c=\frac{1}{e^{-\lambda\underline{\omega}}-e^{-\lambda\overline{\omega}}}$.

- Case 2: Bidder 1 first stops at $p_1$ and bidder 3 stops next at $p_3 > p_1$. Note that the efficient allocation has not yet been determined. The price continues to increase. If bidder 2 is active until $2p_3$ - $p_1$, then bidders 1 and 2 each win one unit, since the total value for small bidders is $p_1 + p_2 > 2p_3$. Bidder 1 pays $p_1$, and bidder 2 pays $2p_3 - p_1$.

- Case 3: Bidder 1 first stops at $p_1$ and bidder 3 stops next at $p_3 > p_1$. If bidder 2 stops at $p_2 < 2p_3$ - $p_1$, bidder 3 wins both units. Since the total value for small bidders is $p_1 + p_2 < 2p_3$, bidder 3 pays the amount $p_1 + p_2$ by the bidder-optimal core discounting. For types bidder 2 in the interval $v_2 < 2p_3 - p_1$, any action in the interval $[p_3,\ 2p_3 - p_1)$ yields the same maximal payoff (zero) hence is optimal. Sano (2012) adopts the assumption that a small bidder does not drop until the price level reaches his valuation, i.e. , $p_2(v_2; \{(1, p_1), (3, p_3)\} = \min\{v_2, 2p_3 - p_1\}$ .

- Case 4: Bidders 1 and 2 stop first and second at $p_1$ and $p_2$ respectively. Then, bidder 3 wins both units with price $p_1 + p_2$ by bidder-optimal core discounting.

Bidders 1 and 2 can be inverted in Cases 2 and 3. Bidder-optimal core discounting is adopted in order to promote truthful bidding. This discounting affects the bidders' incentives. However, as we will see later, a considerable part of the analysis is, in fact, independent of the discounting.

Sano (2012) shows that the large bidder has a weakly dominant strategy of truthful bidding independent of the history of other bidders' bids. He also shows that a small bidder has a weakly dominant strategy of truthful bidding after the other small bidder stops bidding. All bidders in his analysis are assumed to be risk neutral.

While payment rules matter in sealed-bid mechanisms (as noted by Ausubel and Baranov (2010)), an interesting implication of the analysis in Sano (2011) is that the payment rule does not matter for the ascending mechanism we study. A winning small bidder 2 in case 2 would always drop out at the price $p_2$ where $p_1 + p_2 > 2p_3$. Bidder 2 would not have an incentive to increase his bid in the supplementary sealed-bid phase, and the VCG payment would be in the core with respect to the bids submitted. The same is true for the winners in all other cases, who would always pay what they bid.

## 4.3 Equilibrium Analysis

In this section we derive the non-bidding equilibrium condition for general types of risk-averse utility functions and extend the analysis of risk-neutral bidders of Sano (2012). Subsequently we describe the non-bidding equilibrium with respect to asymmetry in the small bidders' utility functions, wealth levels, and value distributions. Our initial analysis concerns an economy with two small and one large bidder but we will generalize the results to many bidders.

### 4.3.1 Necessary and Sufficient Conditions for Non-bidding Equilibrium

Throughout the chapter, we use the term *'non-bidding equilibrium'* to refer to a perfect Bayesian equilibrium in which $\beta_i(v_i, \emptyset) = \min\{v_i, r\}$ for each $i = 1, 2$ and all $v_i \in V = [0, 1]$. Second, we will use the terms symmetry and asymmetry to refer to ex ante symmetry and asymmetry of small bidders. Thus, *'symmetric bidders'* is used instead of more cumbersome 'ex ante symmetric small bidders' to refer to an environment where small bidders are identical ex ante with respect to valuation distributions, risk attitudes and wealth levels, i.e., when $F_1 = F_2 = F$, $u_1 = u_2 = u$, and $\omega_1 = \omega_2 = \omega$.

#### 4.3.1.1 Symmetric Bidders

**Theorem 4.1.** *In environments with ex ante symmetric small bidders the necessary and sufficient condition for existence of a non-bidding equilibrium is*

$$\int_r^1 G(t+r)\left(\frac{f(t)}{1-F(r)} - L(u;1,t,r,\omega)\right)dt \geq 0 \tag{4.1}$$

For a utility function $u(z)$ the functional $L(u;x,y,z) = \frac{u'(x+z)}{u(y+z)-u(z)}$ , for $0 \leq x \leq y$, plays a central role in the statement and derivation of our main result. Some properties of this functional, such as its relation to Arrow-Pratt measure of absolute risk aversion, may be of independent interest. We collect some observations on this functional before we state our results.

$$L(u;v,t,r,\omega) = \frac{u'(v-t+\omega)}{u(v-r+\omega)-u(\omega)}$$

For a utility function in CRRA family this functional takes the form

$$L\left(u;v,t,r,\omega\right)=\begin{cases}\frac{(1-\rho)(v-t+\omega)^{-\rho}}{(v-r+\omega)^{1-\rho}-\omega^{1-\rho}}=\frac{(1-\rho)(\frac{v-t}{\omega}+1)^{-\rho}}{\omega\left\{(\frac{v-r}{\omega}+1)^{1-\rho}-1\right\}} & if\ \rho\neq 1\\ \frac{\ln(\underline{\omega}/\overline{\omega})}{(v-t+\omega)\ \ln((v-r+\omega)/\omega)} & if\ \rho=1\end{cases}$$

For the CARA family of utility functions

$$L\left(u;v,t,r,\omega\right)=\frac{\lambda e^{-\lambda(v-t+\omega)}}{e^{-\lambda\omega}-e^{-\lambda(v-r+\omega)}}=\frac{\lambda e^{-\lambda(v-t)}}{1-e^{-\lambda(v-r)}}$$

**Corollary 4.1.** *When small bidders are ex ante symmetric and risk-neutral, the necessary and sufficient condition for non-bidding equilibrium reduces to the condition in Sano Theorem 1:* [4]

$$\int_r^1 G\left(t+r\right)\left(\frac{f\left(t\right)}{1-F\left(r\right)}-\frac{1}{1-r}\right)dt\geq 0 \tag{4.2}$$

Suppose $r=0$. If the bidders are risk neutral 4.2 reduces to $\int_0^1 g\left(t\right)\left(F\left(t\right)-t\right)dt\leq 0$, i.e. $F$ first-order stochastically dominates the uniform distribution. In environments with risk-aversion this is not a sufficient condition: Suppose $F,G$ are uniform distributions, $r=0$ and $u$ is a CRRA with $\rho=1,\ \omega=1$. The non-bidding equilibrium condition is violated (the integral in 4.1 evaluates to -0.029 whereas under risk-neutrality to 0). This fact impacts positively on both efficiency and revenue of the package clock auction since the bidder who successfully stops at $r$, now reveals his valuation to a greater extent.

### 4.3.2 Asymmetric Bidders

So far, we have assumed complete symmetry among the small bidders. In the following, we deviate from this assumption and analyze the impact of asymmetry with respect to the small bidders' utility function, wealth levels, and value distributions.

[4] The condition in Sano is $\int_r^1\left(G\left(t+r\right)-G\left(2r\right)\right)\left(\frac{f(t)}{1-F(r)}-\frac{1}{1-r}\right)dt\geq 0$ but the term $-G\left(2r\right)$ is redundant since $\int_r^1 G\left(2r\right)\left(\frac{f(t)}{1-F(r)}-\frac{1}{1-r}\right)dt=G\left(2r\right)\left(1-1\right)=0$

**Theorem 4.2.** *In environments with asymmetric small bidders the necessary and sufficient condition for existence of a non-bidding equilibrium is*

$$\int_r^1 G(t+r)\left(\frac{f_2(t)}{1-F_2(r)} - L(u_1;1,t,r,\omega_1)\right)dt \geq 0 \tag{4.3}$$

$$\int_r^1 G(t+r)\left(\frac{f_1(t)}{1-F_1(r)} - L(u_2;1,t,r,\omega_2)\right)dt \geq 0 \tag{4.4}$$

In the following corollaries, we note various special cases where asymmetries take special forms.

**Corollary 4.2.** *When small bidders are symmetric with respect to risk attitudes and wealth levels, i.e., when* $u_1 = u_2 = u$, *and* $\omega_1 = \omega_2 = \omega$, *the necessary and sufficient condition for existence of a non-bidding equilibrium reduces to*

$$\int_r^1 G(t+r)\left(\frac{f_2(t)}{1-F_2(r)} - L(u_1;1,t,r,\omega)\right)dt \geq 0 \tag{4.5}$$

$$\int_r^1 G(t+r)\left(\frac{f_1(t)}{1-F_1(r)} - L(u_2;1,t,r,\omega)\right)dt \geq 0 \tag{4.6}$$

**Corollary 4.3.** *When small bidders share the same utility function but differ in wealth levels, i.e., when* $u_1 = u_2 = u$, *and* $\omega_1 \neq \omega_2$, *the necessary and sufficient condition for existence of a non-bidding equilibrium reduces to*

$$\int_r^1 G(t+r)\left(\frac{f_2(t)}{1-F_2(r)} - L(u;1,t,r,\omega_1)\right)dt \geq 0 \tag{4.7}$$

$$\int_r^1 G(t+r)\left(\frac{f_1(t)}{1-F_1(r)} - L(u;1,t,r,\omega_2)\right)dt \geq 0 \tag{4.8}$$

**Corollary 4.4.** *When small bidders are symmetric with respect to valuations, i.e., when* $F_1 = F_2 = F$, *the necessary and sufficient condition for existence of a non-bidding equilibrium reduces to*

$$\int_r^1 G(t+r)\left(\frac{f(t)}{1-F(r)} - L(u_1;1,t,r,\omega_1)\right)dt \geq 0 \tag{4.9}$$

$$\int_r^1 G(t+r)\left(\frac{f(t)}{1-F(r)} - L(u_2;1,t,r,\omega_2)\right)dt \geq 0 \tag{4.10}$$

### 4.3.3 Many Bidders Case

In this section, we generalize the case with three bidders and analyze cases with $m$ small and $l$ large bidders. As in Sano (2012), large bidders retain the dominant strategy property. Also, the free-rider problem does not arise when there are many small bidders, and there is competition among small bidders. However, the non bidding equilibrium condition in changes as follows:

**Theorem 4.3.** *In environments with $m$ ex ante symmetric small bidders and $l$ ex ante symmetric large bidders, if*

$$\int_s^1 \left(G\left(t+s\right) - G(2s)\right)^l \left(\frac{f\left(t\right)}{1-F\left(s\right)} - L(u;1,t,s,\omega)\right) dt \geq 0 \tag{4.11}$$

*for each $l = 1, \ldots, n$ and all $s \in [r,1]$, then the following strategies constitute a perfect Bayesian equilibrium:*

1. *Each large bidder follows a truthful strategy.*
2. *If more than two small bidders continue bidding, then each small bidder follows a truthful strategy.*
3. *If only two small bidders continue bidding (along with a large bidders), then each small bidder stops immediately.*
4. *If a bidder is the only active small bidder, then he follows a truthful strategy.*

## 4.4 Comparative Statics

We will now discuss the impact that different levels of risk aversion, different wealth levels, and stochastic orderings of valuation distributions have on the result.

### 4.4.1 Different Levels of Risk Aversion

We compare two environments $e, \tilde{e}$ which differ only in the risk-aversion of small bidders, with $\tilde{e}$ being the less risk-averse one. The risk-aversion of the large bidder is dispensable in our analysis since he always has the dominant strategy of truthful bidding. Our main question is whether the increase of the risk-aversion of small bidders makes their non-bidding strategy less attractive. The role of risk aversion has a particular importance in this setting since we cannot draw on the theory of sinlge-item auctions and the fact that risk aversion leads bidders buy insurance against the posibility of losing and to increase their bids. Such a behavior can only be expected in games where increasing the bid increases the probability of winning. Such games are supermodular games or games that fulfill the single crossing property (Athey, 2001). The ascending auction we analyze however is not such a game:

**Example 4.1.** *Suppose bidder 1 is a small bidder with a low valuation while bidder 2 a small bidder with a high valuation. If bidder 1 increases his bid, i.e. stays longer in the auction, the probability that bidder 2 becomes the first one to stop increases. On the other side, if bidder 1 drops earlier, the probability that he becomes the first one to stop and thus force bidder 2 to bid up his true valuation increases. Bidder 1 has obviously a higher probability of winning if he stops first and let bidder 2 with the higher valuation compete with the large bidder. Hence, increasing his bid does not lead in an increase in the probability of winning.*

Thus, risk averse bidders cannot buy insurance against losing by increasing their bids. Despite this, Theorem 4.4 answers affirmatively our main question. Thus we conclude that risk aversion has a positive impact on the efficiency and revenue of the auction. There are situations where risk-averse bidders continue bidding whereas risk neutral (or less risk-averse) bidders do not.

Firstly we provide intuition about the result. Let $\widetilde{e}$ be an environment with risk neutral small bidders and e with risk-averse small bidders, ceteris paribus. We examine how the utilities change for the two strategies 'drop at the reserve price $r$' and 'continue'. In the risk-neutral case the profit of bidder 1 when dropping at $r$ is $(v_1 - r)$ multiplied by the probability of winning when dropping at $r$. It is important to observe that this probability is independent of his valuation $v_1$ since bidder 1 bids the amount $r$ and whether he wins or not, depends solely on the other two bidders' bids, who behave truthfully, thus on their valuations. Additionally, this probability of winning is the same in both environments $e$, $\tilde{e}$. On the other hand, if bidder 1 decides to continue, his profit is $(v_1 + r - v_3)$ times the probability of winning which is equal to the probability of the large bidder $v_3$ being in the interval $[2r, v_1 + r]$ , since bidders 1 and 3 bid truthfully after bidder 2 has dropped at $r$. Also this probability is invariant in environments $e$, $\tilde{e}$. Since $v_3 \in [2r, v_1 + r]$ when bidder

1 wins, his profit of continuing given that he wins is in the interval $[0, v_1 - r]$ and is smaller than when dropping at $r$. The non-bidding equilibrium condition holds when the profits of dropping at $r$ are greater or equal than of continuing:$(v_1 - r)\ \ Prob(win|drop) \geq\ \ (v_1 + r - v_3)\, Prob(win|cont)$. We argued that changing the utility function from risk neutral to a strictly concave doesn't affect $Prob\,(win \mid drop)\,,\ Prob\,(win \mid cont)$ but only the utility of the quantities $(v_1 - r)$, and $(v_1 + r - v_3)$. Figure 4.1 shows how these change. For expositional purposes the two curves intersect at $(v_1 - r)$.

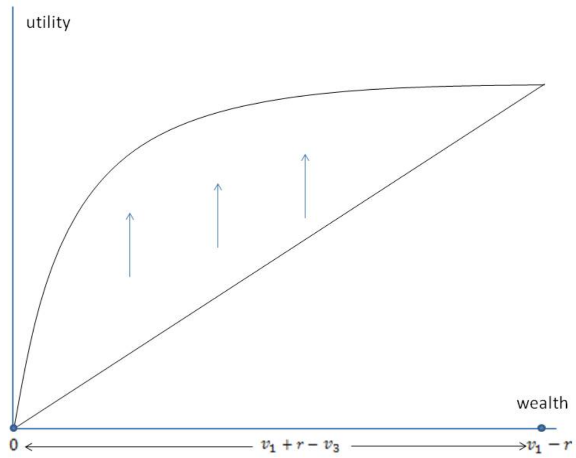


FIGURE 4.1: Utility of wealth gained when winning by dropping at $r$, $(v_1 - r)$ and continuing $(v_1 + r - v_3)$ for a risk-neutral and a concave utility function.

The shape of the utility functions provides rationale for our main result in this section. The utility when winning by dropping at $r$ is the same in both environments e, $\widetilde{\text{e}}$. On the contrary, the utility when winning by continuing at $r$ is higher in the risk-averse environment e for any money amount received. Therefore, the strategy of continuing becomes more attractive for risk-averse bidders and non-bidding ceases to be an equilibrium.

Theorem 4.4 formalizes and generalizes these thoughts to arbitrary utility functions. To compare risk-aversion the Arrow-Pratt measure is employed. Following Pratt (1964), bidder $i$ is considered as more risk-averse than $j$ if and only if the Arrow-Pratt measure of his utility function is greater or equal than

$j$'s for any wealth level.

**Theorem 4.4.** *If the environment* $e = \{F_1, F_2, G, u_1, u_2, u_3, \omega_1, \omega_2, \omega_3\}$ *admits a non-bidding equilibrium then so does the environment* $\tilde{e} = \{F_1, F_2, G, \tilde{u}_1, \tilde{u}_2, u_3, \omega_1, \omega_2, \omega_3\}$ *where* $\tilde{u}_i(x)$ *is such that* $\tilde{A}_i(x) < A_i(x)$, *for* $\forall x$ *and* $i = 1,\ 2$, *i.e., where small bidders are less risk-averse. Conversely, if the environment* $\tilde{e}$ *does not admit non-bidding in a perfect Bayesian equilibrium, neither does the environment* $e$.

**Corollary 4.5.** *Theorem 4.4 holds if* $\tilde{u}_1(x) = a + bx,\ \tilde{u}_2(x) = a' + b'x\ (a, b, a', b,\ \in \mathbb{R})$, *and* $u(x)$ *is strictly concave for at least one* $i = 1, 2$ *(otherwise* $u_i(x) = \tilde{u}_i(x)$*), i.e. in environment* $\tilde{e}$ *both local bidders are risk-neutral and in* $e$ *at least one of them is risk averse.*

**Corollary 4.6.** *Theorem 4.4 holds if* $u_1, u_2, \tilde{u}_1, \tilde{u}_2$ *belong to the family of CARA utility functions and* $\lambda_1 \leq \widetilde{\lambda}_1, \lambda_2 \leq \widetilde{\lambda}_2$.

**Corollary 4.7.** *Theorem 4.4 holds if* $u_1, u_2,\ \tilde{u}_1, \tilde{u}_2$ *belong to the family of CRRA utility functions and* $\rho_1 \leq \widetilde{\rho}_1, \rho_2 \leq \widetilde{\rho}_2$.

**Corollary 4.8.** *Theorem 4.4 The efficiency and revenue in environment* $e$ *are greater than or equal to the efficiency and revenue in environment* $\tilde{e}$.

### 4.4.2 Wealth Levels and Non-bidding Equilibrium

We now examine the case where the initial wealth of one bidder increases by a positive amount $\delta$. CARA utility functions are independent of wealth. If bidders exhibit a CRRA or DARA utility function, then their risk aversion decreases whereas if they exhibit an IARA utility function risk aversion increases.

We will show that the left hand side of the non-bidding equilibrium 4.1 decreases if we add a positive amount of wealth $\delta$. The proof will follow the one of Theorem 4.4. However, we cannot apply the results of this theorem directly, since the utility function now remains the same and we cannot leverage $\widetilde{A}_i(x) < A_i(x)\ \forall x$. Instead, we depart from the condition $A_i(\omega + \delta) < A_i(\omega),\ \delta > 0$ (this is the case for CRRA, DARA) or $A_i(\omega + \delta) > A_i(\omega),\ \delta > 0$ (this is the case for IARA)

**Theorem 4.5.** *Suppose* $u_1$ *is a CRRA or DARA utility function. If the environment* $e = \{F_1,\ F_2, G, u_1, u_2, u_3, \omega_1, \omega_2, \omega_3\}$ *admits a non-bidding equilibrium then so does the environment* $\tilde{e} = \{F_1,\ F_2, G, u_1, u_2, u_3, \widetilde{\omega}_1, \omega_2, \omega_3\}$

*where* $\widetilde{\omega}_1 > \omega_1$ *. Conversely, if the environment* $\tilde{e}$ *does not admit non-bidding in a perfect Bayesian equilibrium, neither does the environment* $e$*.*

**Corollary 4.9.** *Suppose* $u_1$ *is a IARA utility function. If the environment* $\tilde{e} = \{F_1,\ F_2, G, u_1, u_2, u_3, \widetilde{\omega}_1, \omega_2, \omega_3\}$ *admits a non-bidding equilibrium then so does the environment* $e = \{F_1,\ F_2, G, u_1, u_2, u_3, \omega_1, \omega_2, \omega_3\}$ *where* $\widetilde{\omega}_1 > \omega_1$ *. Conversely, if the environment* $e$ *does not admit non-bidding in a perfect Bayesian equilibrium, neither does the environment* $\tilde{e}$*.*

### 4.4.3 Stochastic Dominance Orderings and Non-bidding Equilibrium

Conditions of first order stochastic dominance (FSD) and second order stochastic dominance (SSD):

$$\tilde{G} \succcurlyeq_{FSD} G \Longleftrightarrow \tilde{G}(x) \leq G(x)\, \forall x \text{ (with strict inequality for some x)}$$

$$\tilde{G} \succcurlyeq_{SSD} G \Longleftrightarrow \int_0^x \left(G(t) - \tilde{G}(t)\right) dt \geq 0\ \forall x \text{ (with strict inequality for some x)}$$

FSD implies SSD but not vice versa, hence FSD is a stronger condition. Variable $x$ first order stochastically dominates $y$ iff the probability that $x$ is higher than an amount $z$ is higher than the probability that $y$ is higher than this amount, for any $z$. Hence $x$ is stochastically larger than $y$ (it has a higher expected value). SSD mirrors the riskiness of two variables. If $x$ second order stochastically dominates $y$, then $x$ is less risky. Additionally, if the distributions of the random variables satisfy the single crossing property, then $x$ is also stochastically larher than $y$.

We examine the impact of changing the distributions of the bidders' valuations to new ones that stochastically dominates the former ones. We show that changing the distribution of a valuation of a small bidder to a distribution which stochastically dominates the former leads to the non-bidding equilibrium being more probable. The reason is that the incentives to free ride increase in accordance with the probability that one small bidder outbids alone the large bidder.

**Theorem 4.6.** *Suppose* $r = 0$*. If the environment* $e = \{F_1,\ F_2, G, u_1, u_2, u_3, \omega_1, \omega_2, \omega_3\}$ *admits a non-bidding equilibrium then so does the environment* $\tilde{e} = \left\{\tilde{F}_1,\ F_2, G, u_1, u_2, u_3, \omega_1, \omega_2, \omega_3\right\}$ *where* $\tilde{F}_1 \succcurlyeq_{FSD} F_1$ *. Conversely, if the environment* $\tilde{e}$ *does not admit non-bidding in a perfect Bayesian equilibrium, neither does the environment* $e$*.*

**Theorem 4.7.** *Suppose additionally $g$ is nonincreasing. Then the weaker condition $\tilde{F}_1 \succcurlyeq_{SSD} F_1$ is sufficient for the theorem to hold.*

Theorem 4.6 holds for any reserve price $r$ if we impose first order stochastic dominance to the left-truncated versions of the cumulative distributions: $\tilde{F}_1(\cdot|r) \succcurlyeq_{FSD} F_1(\cdot|r)$ . Note that this condition is not implied by $\tilde{F}_1 \succcurlyeq_{FSD} F_1$ since the stochastic dominance is not necessarily preserved after truncating.

## 4.5 Analysis of Parametric Cases

We will illustrate selected parametric cases assuming different types of distribution functions. Let bidders be *risk-neutral* and all $v_i$ be drawn from a uniform distribution, then 4.2 evaluates to 0 for all $r \in [0..1]$ leading to a non-bidding equilibrium. If $F$ is a truncated Gaussian distribution $F \sim N(0.5, 0.25)$ on the interval [0..1] and G$\sim N(1, 0.5)$, then condition 4.2 is negative for $r \in [0..1]$, and risk-neutral bidders would continue bidding. Now if we increase the valuations of the small bidders and $F \sim N(0.8, 0.4)$ then condition (4.2) is positive for most values of $r$ and the small bidders would try to drop (Figure 4.2). The reason for free riding is that the probability for the competing small bidder to outbid the large bidder increases.

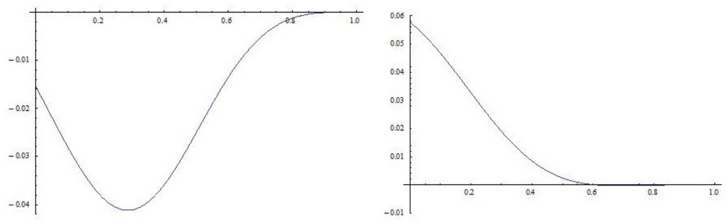


FIGURE 4.2: Condition 4.2 for different values of $r$ with $F \sim N(0.5, 0.25)$ on the left and $F \sim N(0.8, 0.4)$ on the right with risk-neutral bidders.

It is now interesting to understand, how *risk aversion* impacts this condition. If $v_i$ are uniformly distributed, the value of 4.1 is negative for both CARA and CRRA utility functions, and as opposed to the risk neutral case, the bidders would continue to bid (Figure 4.3).

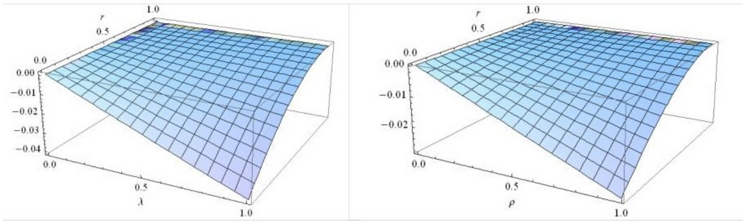

FIGURE 4.3: Condition 4.1 for different values of $r$, $F\sim U(0,1)$, $G\sim U(0,2)$. On the left for a CARA with $\lambda = 0..1$ and on the right CRRA utility function with $\rho = 0..1$, $\omega = 1$.

Figure 4.4 shows condition 4.1 for different values of $r$ when $F$ is a truncated Normal distribution $F \sim N(0.5, 0.25)$ or $F \sim N(0.8, 0.4)$ on the interval [0..1], $G \sim N(1, 0.5)$ and we have a CARA utility function with $\lambda$=0.9. In Figure 4.5 we change the utility function to a CRRA with $\rho$=0.9, $\omega$=1.

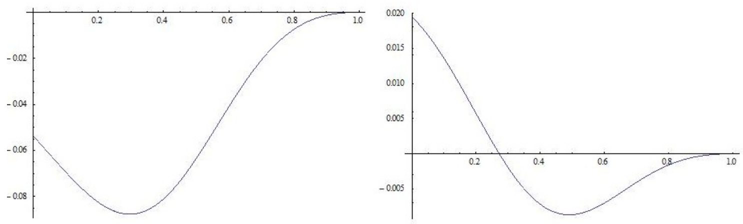

FIGURE 4.4: Condition 4.1 for different values of $r$ with $F\sim N(0.5, 0.25)$ on the left and $F\sim N(0.8, 0.4)$ on the right with CARA utility functions.

Last but not least, we vary the parameters $\lambda$ and $\rho$ and keep $r$=0, $F \sim N(0.8, 0.4)$, $G \sim N(1, 0.5)$ (Figure 4.6). Condition 4.1 holds and lead to a non-bidding equilibrium for all values of $\rho$ and for $\lambda < 1.4$. Higher values of $\lambda$ lead to such high risk aversion which eliminates the non-bidding equilibrium.

## 4.6 Conclusions

We have characterized the necessary and sufficient conditions for small bidders to drop at the reserve price for general environments allowing arbitrary

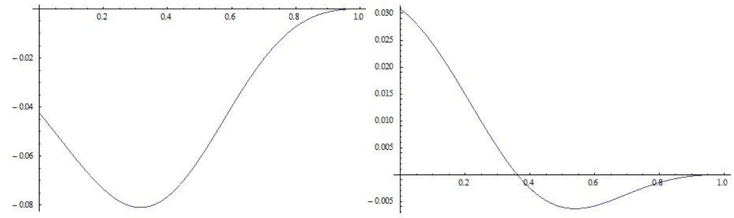

FIGURE 4.5: Condition 4.1 for different values of $r$ with $F \sim N(0.5, 0.25)$ on the left and $F \sim N(0.8, 0.4)$ on the right with CRRA utility functions and $\rho = 0.9$, $\omega = 1$.

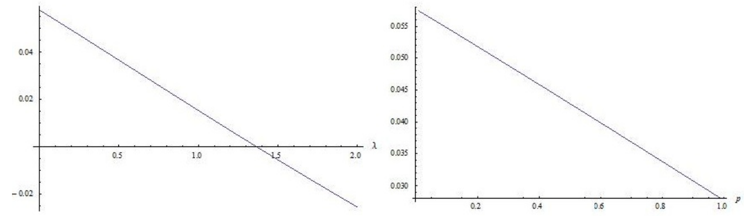

FIGURE 4.6: Condition 4.1 with $F \sim N(0.8, 0.4)$ for $r = 0$ and different parameters of a CARA and a CRRA utility function.

risk averse bidders as well arbitrary asymmetries across bidders with respect to both information each bidder has and risk attitudes. We have used this characterization to identify the set of environments that yield non-bidding under the ascending core-selecting auction mechanism. We have shown that risk aversion and bidder asymmetries affect the equilibrium outcomes in ways that can be systematically analyzed. In particular, risk-aversion reduces the scope of non-bidding equilibrium in the sense that dropping at the reserve price ceases to be equilibrium as the bidders become more risk averse.

Interesting questions remain open. The nature of equilibria when the non-bidding condition fails is perhaps the most important gap to be filled. In a companion work (Chapter 5) we explore the implications of risk aversion and bidder asymmetries in a sealed-bid core-selecting mechanism. We also explore how reserve price affects revenue and efficiency and compare ascending and sealed-bid mechanisms for general environments where the form of equilibria under both mechanisms can be characterized.

# Chapter 5

# Sealed Bid Core-Selecting Auctions with Risk Averse Bidders

Gedanken sind die Proben zu
Handlungen.

S. Freud

We analyze the impact of risk aversion on equilibrium bidding strategies and efficiency of core-selecting auctions in a threshold problem with one global and several local bidders. First, we derive equilibrium bidding strategies for sealed-bid core-selecting auctions. Second, we analyze the role of reserve prices and compare the sealed-bid auctions with the ascending auctions analyzed in Chapter 4 and with the VCG auction. Surprisingly, in spite of the non-bidding equilibrium in the ascending auction, setting optimal reserve prices in each format leads to higher efficiency and revenue in the ascending than in the sealed-bid auction.

## 5.1 Introduction

This chapter complements the analysis of Chapter 4 by examining the role of risk aversion, reserve prices and asymmetries in sealed-bid core-selecting auctions. The derivation of the equilibrium bidding strategies in both formats enables their comparison with respect to efficiency and revenue.

### Contributions

The contributions of this work are the following: We consider the environments that have been analyzed in the Bayesian literature on sealed-bid CAs, but want to understand the impact of risk aversion in these environments as they have been analyzed by Goeree and Lien (2012) and Ausubel and Baranov (2010).

*First*, we discuss the impact of risk aversion on the nearest-Vickrey core-selecting sealed-bid auction, to get an understanding how risk aversion impacts sealed-bid auctions. We show that, unlike in the ascending auction, the single crossing property holds and there exist pure strategy equilibria.[1] We derive the Bayes-Nash equilibrium strategy for CARA and CRRA utility functions and uniform distributions and prove the existence of pure-strategy Nash equilibria for general valuations and arbitrarily concave utility functions. Here we also establish interesting connections to the recent literature on public goods problems.

*Second*, we analyze the impact of reserve prices on the efficiency and revenue of the two formats for parametric cases, where the non-bidding equilibrium holds and compare them to the outcome of the VCG auction. Despite of the non-bidding equilibrium in the ascending auction, efficiency and revenue in the ascending core-selecting auction are surprisingly high, as one of the local bidders has a dominant strategy. Setting optimal reserve prices in each format leads to higher efficiency and revenue in the ascending core-selecting auction compared to the sealed-bid auction.

The remainder of the chapter is structured as follows. Section 5.2 contains a descriptions of the the sealed-bid core-selecting auction procedure. In Section 5.3, we derive the Bayes-Nash equilibrium strategy for risk-averse bidders. Section 5.6 discusses parametric cases with specific distribution and utility functions and provides the comparison of the sealed-bid auction with the ascending and the VCG auctin. Finally, Section 5.7 provides conclusions. Throughout the chapter, proofs that are technical in nature are placed in the Appendix.

[1] Schneider and Day (2011) presented the impact of loss and risk aversion with CRRA utility functions and uniform distributions in core-selecting sealed-bid auctions.

# 5.2 Model

## 5.2.1 Environment: Information, Valuations and Utility Functions

We consider the environment described in Section 4.2.

## 5.2.2 Bidder-optimal Core-selecting Vickrey-nearest (BCV) Auction

We will introduce the bidder-optimal, core-selecting, and Vickrey-nearest sealed-bid auction proposed by Day and Cramton (2012) and refer to it as BCV auction. Let $r \in [0, 1)$ be the per-unit reserve price. Before the auction starts, bidders decide whether they participate or not. This decision is made public. Bidders submit sealed-bids, the auctioneer selects the revenue maximizing allocation and computes payments which are in the core with respect to the bids submitted.

Different types of payment rules for core-selecting auctions have been discussed in the literature, including the one of the ascending proxy auction, the nearest-Vickrey rule, the nearest bid rule, and the proportional rule (Ausubel and Baranov, 2010). The nearest-Vickrey rule (Day and Cramton, 2012; Day and Raghavan, 2007) has been used in spectrum auctions in the field. The central idea of the nearest-Vickrey payment rule is to select the bidder-optimal core allocation that minimizes the distance to the VCG point.[2] For complete-information environments the BCV auction yields the Vickrey outcome when it is in the core and results in higher seller revenues when it is not Day and Milgrom (2008).

# 5.3 Bayes-Nash Equilibrium in BCV Auction

We will now analyze the sealed-bid BCV auction. We will first discuss equilibrium strategies for uniform distributions of $G$ and $F$, as they have been

[2]Note that Erdil and Klemperer (2010) defined 'reference rules', which reduce the marginal incentive to deviate as compared to other payment rules. They show that there always exists a reference rule, which dominates the nearest-Vickrey rule and has a lower sum-across-bidders of marginal deviation incentives for all possible valuation vectors.

analyzed in the risk-neutral case and they yield easy to characterize equilibrium strategies. But Theorem 5.1 shows that the single-crossing property Athey (2001) holds in the sealed-bid BCV auction, and therefore shows that such pure-strategy equilibrium strategies also exist for general valuations.

**Theorem 5.1.** *Suppose $f$ bounded and $G$ continuous. Then, there exists a pure-strategy equilibrium in increasing strategies for the (two) small bidders.*

This theorem also holds for many bidders and risk-averse bidders.

**Corollary 5.1.** *Suppose $f$ bounded and $G$ continuous. Then, there exists a pure-strategy equilibrium in increasing strategies for many small and global bidders and arbitrarily concave utility functions.*

## 5.4 Equilibrium Analysis

Based on the existence proof of Theorem 5.1, we can now derive equilibrium strategies for the BCV auction. To derive analytical results, we will consider parametric forms of utility functions and uniform distributions with $v_i \in [0,1]$ for $i = 1,2$ and $v_3 \in [0,2]$. In particular, we will analyze equilibrium bid functions for CARA and CRRA utility functions considering reserve prices. We can dispense with concrete forms in the subsequent section where we characterize the impact of increasing risk aversion to the bidding strategy.

**Theorem 5.2.** *The Bayes-Nash equilibrium of the BCV auction with CARA utility functions and uniform distributions of valuations is given by $B(V) = V$ and $b(v_1) \cong \max(r, v_1 - c + r)$ if $v \geq r$, else $b(v_1) = 0$ where $c = r + \frac{ln\left(1+\frac{1}{2}\lambda a\right)}{\lambda}$ and $a = \frac{1-c^2}{2} + c(c-1)$.*

**Corollary 5.2.** *The Bayes-Nash equilibrium of the BCV auction with risk neutral utility functions and uniform distributions of valuations is given by $B(V) = V$ and $b(v_1) \cong \max(r, v_1 - (3 - \sqrt{8+r^2}))$ if $v \geq r$, else $b(v_1) = 0$.*

The corollary is an application of the formula of Theorem 5.2 with $\lambda$=0.

**Theorem 5.3.** *The Bayes-Nash equilibrium of the BCV auction with CRRA utility functions and uniform distributions of valuations is given by $B(V) = V$ and $b(v_1) \cong \max(r, v_1 - c)$ if $v \geq r$ ,else $b(v_1) = 0$. where $c$ is the solution to $(c+\omega) - \omega^{1-\rho}(c+\omega)^{\rho} \cong \frac{1}{4}(1-\rho)\left((c-1)^2 - r^2\right)$*

## 5.5 Comparative Statics

Similarly to the ascending auction game, we examine the impact of risk aversion on the equilibrium bidding strategies. As in the first-price single-item auction case, risk aversion leads to increased bids and revenue.

**Theorem 5.4.** *Suppose there are two environments* $e = \{F_1, F_2, G, u, u, u_3\}$ *and* $\tilde{e} = \{F_1, F_2, G, \tilde{u}, \tilde{u}, u_3\}$ *with uniform distributions* $F, G$ *and a symmetric Bayes Nash equilibrium strategy* $b(v)$ *and* $\tilde{b}(v)$ *for the small bidders in the two environments, respectively. If* $u(x), \tilde{u}(x)$ *is such that* $\tilde{A}(x) < A(x)$*, for* $\forall x$ *and* $i = 1,\ 2$*, then* $\tilde{b}(v) \geq b(v)$*.*

**Corollary 5.3.** *Suppose local bidder 1 has a concave utility function, while bidder 2 is risk neutral, then the Bayes Nash equilibrium strategies satisfy the condition* $b_1(v) \geq b_2(v)\, \forall v$*.*

Next we examine the impact of changing the distributions of the bidders' valuations to ones that stochastically dominate the former. We show that changing the distribution of a valuation of small bidder 2 to a distribution which is first order stochastically dominated by the former, leads to a decrease in his bid and an increase in the bid of bidder 1.

**Theorem 5.5.** *Suppose two risk neutral environments* $e = \{F_1,\ F_2, G\}$ *and* $\tilde{e} = \left\{F_1, \tilde{F}_2, G\right\}$ *where* $F_2 \succcurlyeq_{FSD} \tilde{F}_2$ *and* $G$ *uniform, then* $\tilde{b}_1(v) \geq b_1(v)$ *and* $\tilde{b}_2(v) \leq b_2(v)$*.*

**Corollary 5.4.** *Theorem 5.5 holds also if* $\tilde{F}_2 \succcurlyeq_{SSD} F_2$*.*

*Proof.* (of corollary) It is a property of second order stochastic dominance that for every concave utility function $g$, $\int g(v)\tilde{f}_2(v)\,dv < \int g(v) f_2(v)\,dv$ □

### 5.5.1 Relationship to Public Goods Problems

The BCV auction exhibits surprising similarities to the private-information subscription game for the provision of a public good. The large bidder in the BCV auction can be interpreted as the cost threshold of the subscription game (Barbieri and Malueg, 2010). The bids of the small bidders can be interpreted as the contributions of the players of the subscription game. If the sum of these contributions surpasses the threshold, the public good is allocated, otherwise they are refunded. Similarly, if the sum of small bidder's bids surpass the global bidder's bid, small bidders win and otherwise they don't pay anything. Both the large bidder and the threshold are non-strategic (the large bidder has a truthful dominant strategy). A difference is that winning small bidders in the BCV auction receive discounts and pay less than they bid.

Due to this very fact, we expect that they bid higher amounts than the contributions of the players of the subscription game. . It turns out that the equilibrium bid in the BCV auction is exactly two times the contribution in the subscription game. A risk neutral small bidder in a BCV auction with no reserves bids $\max(0, v-k)$ whereas a player in the subscription game contributes $\max\left(0, \frac{v-k}{2}\right)$ . The amount k can be interpreted as bid shading in the BCV auction. In the following, we take an example from the paper by Barbieri and Malueg (2010) and compute the bid shading in the BCV auction to illustrate this connection. Bids and contributions are for $v = 1$.

**Example 5.1.** $F_1 = U[0,1]$ $F_2 = U[0,1]$ $F'_2(x) = 1-(1-x)^2$ $F''_2(x) = 3v^2 - 2v^3$. *We have* $F_2 >_{FOSD} F'_2$, $F_2 <_{SOSD} F''_2$

| **This work** | | | **Barbieri and Malueg (2010)** | |
|---|---|---|---|---|
| **Distribution** | **Bid** | | **Distribution** | **Contribution** |
| $(F_1, F_2)$ | (0.828 , 0.828) | | $(F_1, F_2)$ | (0.414 , 0.414) |
| $(F_1, F'_2)$ | (0.918 , 0.789) | | $(F_1, F'_2)$ | (0.459 , 0.395) |
| $(F_1, F''_2)$ | (0.835 , 0.826) | | $(F_1, F''_2)$ | (0.418 , 0.413) |

TABLE 5.1: Comparison of bid shading in the BCV auction and in the public goods subscription game.

# 5.6 Analysis of Parametric Cases

## 5.6.1 Impact of Reserve Prices

The bidding strategy of the local bidder depending on reserve price in the sealed-bid BCV auction is illustrated in the left diagram of Figure 5.1 assuming a uniform distribution. Increasing the reserve price only leads to increases in the bids of the local bidders with high values of the reserve price relative to their valuation. For instance, for $v = 0.6$ increasing $r$ from 0 to 0.47 increases the bid only from 0.43 to 0.47. Only in the interval $[0.47, 0.6]$ the slope is one and an increase in the reserve price has a higher impact on the bid.

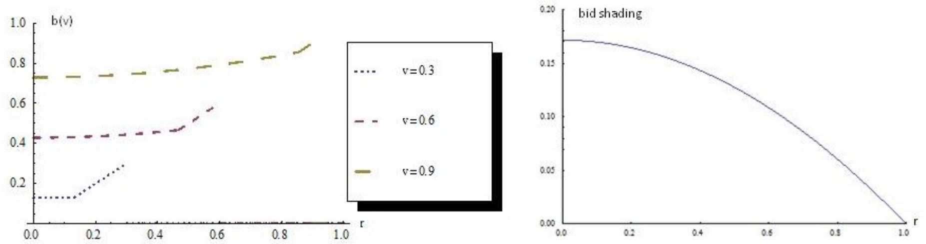


FIGURE 5.1: Equilibrium bids in the sealed-bid auctions for various valuations and bid shading as a function of reserve prices.

The bid shading in the right diagram of Figure 5.1 falls as $r$ increases and is independent on the valuation. Whenever the valuation minus the (maximum) bid shading is below the reserve price, then the bidder bids $r$, as explained in Section 5.5.

### 5.6.2 Impact of Risk Aversion

Figure 5.2 shows the amount of bid shading as risk aversion increases. The left diagram is for a CARA utility function and the right diagram for a CRRA ($\omega = 1$) with uniformly distributed bidder valuations and zero reserve prices. When $\lambda = 0$ or $\rho = 0$ resp., we have the risk neutral case while as the risk aversion parameter increases, bidders converge to the truth-telling behavior.

### 5.6.3 Comparison of Efficiency and Revenue

In this subsection we compare the revenue and efficiency of the ascending, the sealed-bid BCV, and the VCG auctions. We look at environments where the bidding functions can be derived analytically. First, we assume uniform distributions and risk-neutrality such that the non-bidding equilibrium holds in the ascending core-selecting auction. Figure 5.3 and Figure 5.4 plot revenue and efficiency for two environments based on different levels of the reserve price, which illustrates that reserve price is an important variable to consider in the analysis.

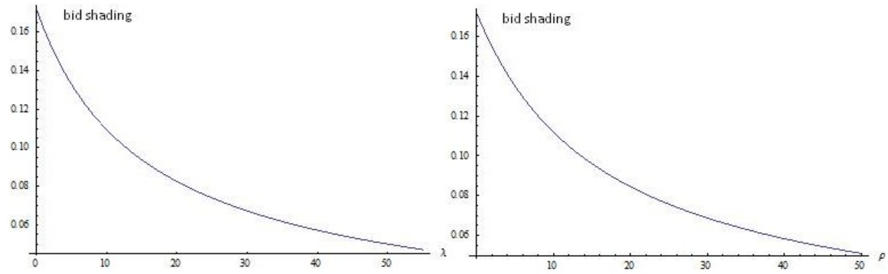


FIGURE 5.2: Bid shading in a sealed-bid BCV auction with CARA (left) and CRRA (right) utility functions and uniform distributions $F \sim U(0,1)$ and $G \sim U(0,2)$ of the bidder valuations.

The plots in Figure 5.3 describe revenue and efficiency in an environment 1, which has risk-neutral bidders and prior distributions of $F \sim U[0,1]$ and $G \sim U[0,2]$ such that the non-bidding equilibrium condition holds in the ascending auction in both environments.

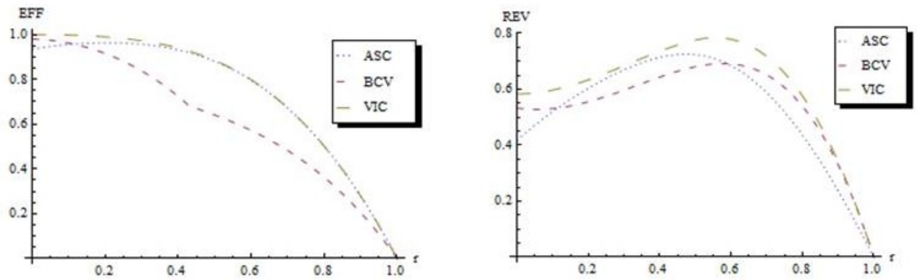


FIGURE 5.3: Efficiency and revenue of three auction formats for environment 1.

Interestingly, the ascending auction almost achieves the efficiency of the VCG auction in environment 1. Overall, higher reserve prices have a negative impact on efficiency in all three auction formats. If the reservation price is higher, the efficiency lost in an ascending auction is less than in the case of low reserve prices, because the bidder who drops out contributes at least the reserve price. In the BCV auction, both local bidders shade their bids, which can lead to lower efficiency levels with high reservation prices compared to the ascending auction. In terms of revenue, the higher reserve price in the auction can lead to higher revenue in an expected sense in all auction formats. The VCG auction dominates the BCV and the ascending core-selecting auction in environment 1.

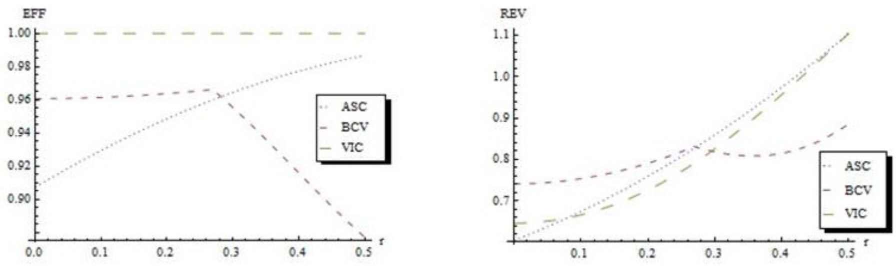


FIGURE 5.4: Efficiency and revenue of three auction formats for environment 2.

Figure 5.4 describe environment 2, which assumes bidders with a CARA utility function ($\lambda$=1), and value distributions of $F \sim U\,[0.5, 1]$ and $G \sim U\,[0, 2]$. Also in environment 2 the non-bidding equilibrium holds for the ascending core-selecting auction, but we add risk aversion. Reservation prices of $r < 0.5$ do not restrict the local bidders, so both bidders can always bid and both, efficiency and revenue increase in the ascending auction. We only plot the reservation price until 0.5 as the non-bidding equilibrium does not hold any more beyond. The VCG auction always achieves full efficiency up until a reserve price of 0.5, because until a reserve price of 0.5 the local bidders would always win.

The BCV auction exhibits a different curve. Due to the high valuation of the local bidders their bid shading is also high in spite of risk aversion (0.25 for zero reserve price as compared to 0.17 in environment 1). So, if both local bidders had a value of 0.5, they would only bid 0.25 each in the worst case. If the global bidder had a value larger than 0.5, then he could become winning just due to the bid shading of the local bidders. Also in environment 2, the revenue of the ascending core-selecting auction is surprisingly high compared to the outcome of the VCG auction, in spite of the non-bidding equilibrium.

For environment 2, we will now fix the reservation price at zero, but vary the level of risk aversion with parameter $\lambda$ (Figure 5.5). Risk aversion obviously has no impact on the efficiency and revenue of the Vickrey auction. It does also not have an impact on the ascending auction either, because the non-bidding equilibrium holds for all levels $\lambda < 2$. Efficiency and revenue of the BCV auction increase slightly. The impact can of course be higher with higher levels of risk aversion or different types of utility functions. However, levels of $\lambda > 2$ violate the condition for a non-bidding equilibrium in the ascending auction.

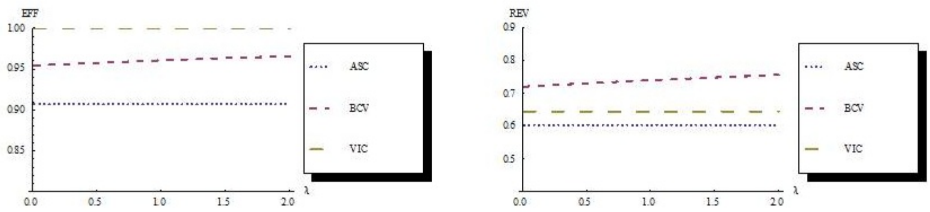


FIGURE 5.5: Efficiency and revenue of environment 2 with zero reserve price and different levels of risk aversion.

## 5.7 Conclusions

Core-selecting combinatorial auctions have been used in high-stakes spectrum auctions across the world. Only recently, such auctions have been analyzed as Bayesian games. The local-local-global model has been used throughout as a simple environment, where the VCG outcome is not in the core and truthful bidding is not a dominant strategy. The analysis of sealed-bid BCV auctions yielded that the outcomes of this auction might be further from the core than that of the VCG auction in the local-local-global environment (Goeree and Lien, 2012). The analysis of ascending core-selecting auctions with a specific tie-breaking rule can even results in non-bidding equilibria (Goeree and Lien, 2009; Sano, 2012).

Previous work assumes risk-neutral bidders. Risk aversion is an important phenomenon in high-stakes auctions and it is important to understand its impact on equilibrium strategies, revenue and efficiency of these auctions. In this work, we analyze the impact of risk aversion on bid shading in the BCV sealed-bid auction and thus complement the analysis of ascending auctions in Chapter 4. Parametric cases showed that even in environments where the non-bidding equilibrium of the ascending auction holds, efficiency and revenue of the ascending core-selecting auction are high. If the reserve price is set optimally, the efficiency of the ascending core-selecting auction was even higher than that of the sealed-bid BCV auction.

The extension of the analysis to environments with more items and multi-minded bidders is an important topic for future research.

# Chapter 6

# Conclusions and Future Work

> So far from shores I'd left behind,
> still far from shores I've yet to
> reach.
>
> VNV Nation

Combinatorial auctions gained increasing importance the recent years and it is being realized that they improve decisively the efficiency of complex markets. They find application in any market where more than a single indivisible item is offered for sale by a single seller and bidders do not value all items additively. Their most prominent application is currently the auctioning of spectrum licenses and typically induces revenues of billion dollars. In the B2B sector, CAs are conducted for the procurement of goods or services and for logistics operations (transportation, scheduling). Other applications involve the sale of tv ads, airport time slots and space in a new building. On a technical level, CAs mitigate the exposure problem a bidder faces who is wishing to buy more than one item but these items are auctioned sequentially or parallelely by means of separate single-item auctions. If the bidder wins the first item but fails to win more items in the other auctions he participates, he ends up paying for something he does not need. It is like paying to receive only.. a left shoe.

The main theoretical desideratum when designing an auction is usually to maximize the efficiency (social welfare). Hence, auction design should incentivize bidders to reveal their preferences (valuations) truthfully. To predict the bidding behavior and thus argue about the efficiency of a mechanism, is the main task of game theory and a large part of this thesis was devoted to game-theoretical analysis. A further important desideratum are core outcomes. Outcomes out of the core are perceived as unfair and evoke renegotiations after the end of the auction, which are strictly undesired. On the other hand, core outcomes ensure that there is no subset of bidders that can negotiate with the seller an alternative outcome after the end of the auction which improves (or does not change) the payoff of each participant in this negotiation. CAs that fulfill this property are called core-selecting auctions and are used widely nowadays. These auctions formats are treated in the thesis.

The first part of the thesis dealt with allocation constraints. These constraints arise naturally in most real-life applications of CAs and pose restrictions on the set of feasible allocations. The seller may not wish to allocate more than half of the items to a single bidder to eliminate oligopolies. A bidder may want to submit multiple bids whereby ensuring that he wins at most a certain amount of items. The main contribution of the thesis was to design price feedback that incorporates such constraints, something that the literature so far had not addressed. Thereby, the concepts of winning and deadness levels by Adomavicius and Gupta were utilized. Algorithms to compute them in the presence of allocation constraints were designed and implemented. While winning levels turned to be a NP-complete problem, it was shown that deadness levels are even harder to compute and the problem is $\Pi_2^P - complete$. Nevertheless, simulations showed that they can be computed in small and mid-sized auctions. Concerning incentive compatibility, the auction based on deadness levels admits an expost Nash equilibrium in the BSM world and produces core outcomes, whereas winning levels do not allow for a strong solution concept. A first question for future research is how to develop faster algorithms to compute deadness levels, despite their complexity class, and extend them to the multi-unit case. Further, since deadness levels explore the whole space of possible future auction states and pick the most opportune one, by systematically describing this space and leaving out of consideration cases that are highly unlikely, deadness levels can increase faster from round to round, their computational time can be reduced and hence their usage can become more practical. A related question is how they can be computed under the assumption that bidders bid straightforward. This reduces the space of future states that have to be examined, adds a restriction to the minimization problem and hence yields higher values of deadness levels. Experiments on auctions with allocation constraints can help to better understand bidding behavior in such complex environments.

In the second part of the thesis, core-selecting CAs and the impact of risk aversion were analyzed game-theoretically in the non-BSM world, since in the BSM world it is well known that truthful bidding is a dominant strategy (for sealed-bid formats) or expost equilibrium (for ascending formats) and this holds for any risk profile. The result of the analysis highlights the positive impact of risk aversion in both ascending and sealed-bid formats on efficiency but also revenue. In a sealed bid auction, the higher the risk aversion, the less bidders shade their valuations. Whereas this result can be expected since risk averse bidders need to buy insurance against the possibility of losing and hence increase their bids, the positive results concerning the ascending auction are more surprising. The reason is that the ascending auction is not a supermodular game or a game that fulfills the single-crossing property. This means that if a bidders increases his bid, his probability of winning does not necessarily increases. Therefore, it cannot be argued that risk averse bidders will increase their bids in order to increase their probability of winning (buy insurance against losing). Nevertheless, it was found that risk aversion has a positive impact on the non-bidding equilibrium derived by Sano. The higher the risk aversion, the more likely that non-bidding does not occur. Stated differently, risk aversion mitigates the incentives to free ride. A further surprising finding was the contradiction of the (false) intuition that sealed bid formats outperform ascending formats. This intuition is based on the fact that in ascending formats bidders want to stop right at the beginning of the auction. As shown, setting reserve prices optimally in each format can cause that the ascending format outperforms the sealed-bid one. Generally, theory in the non-BSM world is still in its infancy. The bidding behavior is only known in the restricted setting presented in the thesis with local and global single minded bidders.

Core-selecting auctions have been proposed only recently and require extensive analysis to better understand bidder behavior. Thereby relaxing assumptions of existing Bayesian Nash models that do not always hold in practice is essential. Such assumptions are risk neutrality, single-mindedness and no externatilities: The thesis at hand addressed risk neutrality. Single-mindedness implies that every bidder wants exactly one package. While this simplifies the analysis, typically bidders desire more than one package. Interestingly, there is no Bayes Nash equilibrium analysis at this point that dispenses with this assumption and traditional methods to derive equilibrium strategies become not applicable since bidders' types become multidimensional. Moving to externalities, a bidder often cares about who wins the lots that he does not get allocated. Some of the opponents are direct rivals whereas others operate in different regions or market segments. Such considerations play a crucial role in spectrum auctions for example. Further assumptions of existing models are:

no budget restrictions, no spiteful bidding, no interdependence in bidders' valuations and that all valuations of a bidder being known with certainty at no cost. The relaxation of each of these assumptions separately represents a big challenge.

Furthermore, the iterative formats analyzed until now are all ascending and admit strong free riding incentives. The negative results presented in this thesis highlight the necessity of analyzing alternative auction formats. A question worth pursuing is whether descending auction formats can mitigate this problem. Contrary to the single-item auction theory, in CAs the revenue equivalence theorem does not hold and ascending formats are not strategically equivalent to descending ones. My conjecture is that the dynamics of descending auction formats prevent bidders to free ride on each other. Free riding by dropping out of the auction early and thus forcing remaining bidders to bid up to their valuations is not possible in descending auctions. Early dropping has the opposite effect and allows remaining bidders to free ride on early (and thus high) bids. Different pricing, activity and visibility rules lead to different strategies in the descending auctions and various design alternatives remain to be explored.

To conclude, since complementarities and substitutabilities and economies of scale and scope arise naturally in a multitude of real-life environments, I conjecture that the current positive trend in using CAs will be sustained, with the advances in information systems technology being the enabling factor. CAs are complex mechanisms in various perspectives and significant research effort is still needed both of theoretical and of experimental nature.

# Appendix A

# Proofs

## A.1 Proofs of Chapter 4

### A.1.1 Proof Theorem 4.1

*Proof.* Suppose bidder 2 selects to drop at $r$. For bidder 1, if he also selects to drop at $r$, his expected payoff is determined by the randomization that determines who gets to continue. If bidder 1 drops at $r$, and bidder 2 is selected to continue bidding in the tie-breaking lottery, bidder 1's expected utility is:

$$\begin{aligned}\pi\left(drop;v,r\right) &= u\left(v-r+\omega\right) Prob\left\{1 \text{ wins with bid } r|\left(\left(r,v\right),\beta_{-1}\right)\right\} \\ &+ u\left(\omega\right)\; Prob\left\{1 \text{ loses with bid } r|\left(\left(r,v\right),\beta_{-1}\right)\right\} \\ &= u\left(\omega\right)+\left(u\left(v-r+\omega\right)-u\left(\omega\right)\right) Prob\left\{1 \text{ wins with bid } r|\left(\left(r,v\right),\beta_{-1}\right)\right\} \\ &= u\left(\omega\right)+\frac{u\left(v-r+\omega\right)-u\left(\omega\right)}{1-G\left(2r\right)}\left\{\int_r^1 G\left(t+r\right)\frac{f_2\left(t\right)}{1-F_2\left(r\right)}dt-G\left(2r\right)\right\}\end{aligned} \tag{A.1}$$

where we used the following derivations to evaluate the probability term:

$$
\begin{aligned}
&Prob\left\{1 \ wins \ with \ bid \ r|\left(\left(r,v\right),\beta_{-1}\right)\right\} = Prob\left\{v_2 + r > w_3|\overline{v}_2 \geq v_2 \geq r, w_3 \geq 2r\right\} \\
&= \frac{Prob\left\{v_2 + r > w_3, \overline{v}_2 \geq v_2 \geq r, w_3 \geq 2r\right\}}{Prob\left\{\overline{v}_2 \geq v_2 \geq r, w_3 \geq 2r\right\}} = \frac{Prob\left\{v_2 + r > w_3, \overline{v}_2 \geq v_2 \geq r, w_3 \geq 2r\right\}}{Prob\left\{\overline{v}_2 \geq v_2 \geq r\right\} Prob\left\{w_3 \geq 2r\right\}} \\
&= \frac{Prob\left\{v_2 + r > w_3 \geq 2r, \overline{v}_2 \geq v_2 \geq r\right\}}{\left(1 - F_2\left(r\right)\right)\left(1 - G\left(2r\right)\right)} = \frac{\int_r^{\overline{v}_2} \int_{2r}^{v_2 + r} dG\left(w_3\right) dF_2\left(v_2\right)}{\left(1 - F_2\left(r\right)\right)\left(1 - G\left(2r\right)\right)} \\
&= \frac{\int_r^{\overline{v}_2} \left(G\left(v_2 + r\right) - G(2r)\right) dF_2(v_2)}{\left(1 - F_2\left(r\right)\right)\left(1 - G\left(2r\right)\right)} = \frac{\int_r^{\overline{v}_2} G\left(t + r\right) f_2(t) dt - G\left(2r\right)\left(1 - F_2\left(r\right)\right)}{\left(1 - F_2\left(r\right)\right)\left(1 - G\left(2r\right)\right)} \\
&= \frac{\int_r^{\overline{v}_2} G\left(t + r\right) \frac{f_2(t)}{1 - F_2(r)} dt - G\left(2r\right)}{1 - G\left(2r\right)}
\end{aligned}
$$

If bidder 1 does not drop at $r$, or if he drops but he is selected to continue in the tie-breaking lottery, his expected utility is

$$
\pi\left(continue; v, r\right) = u\left(\omega\right) + \frac{u\left(v - r + \omega\right) - u\left(\omega\right)}{1 - G\left(2r\right)} \left\{\int_r^v \frac{u'\left(v - y + \omega\right)}{u\left(v - r + \omega\right) - u\left(\omega\right)} G\left(y + r\right) dy - G\left(2r\right)\right\} \tag{A.2}
$$

where we used the fact that bidder 1 wins in the continuation game against bidder 3 in the event that $\{v + r > w_3\}$ and pays $w_3 - r$ when he wins.

$$
\begin{aligned}
\pi\left(continue; v, r\right) &= \frac{\int_{2r}^{v+r} u\left(v + r - s + \omega\right) g\left(s\right) ds + \int_{v+r}^{\overline{w}_3} u\left(\omega\right) g\left(s\right) ds}{1 - G\left(2r\right)} \\
&= u\left(\omega\right) + \frac{\int_{2r}^{v+r} \left(u\left(v + r - s + \omega\right) - u\left(\omega\right)\right) g\left(s\right) ds}{1 - G\left(2r\right)}
\end{aligned}
$$

Integration by parts, setting $y := s - r$ and rearranging terms gives A.2.

If the ties are broken via a lottery that selects bidder 1 with probability $q$ and bidder 2 with probability 1-$q$, expected utility of dropping at r for bidder 1 is

$$
\begin{aligned}
EU\left(drop; v, r\right) &= q \ \pi\left(drop; v, r\right) + \left(1 - q\right) \pi\left(continue; v, r\right) \\
&= \pi\left(continue; v, r\right) + q\left(\pi\left(drop; v, r\right) - \pi\left(continue; v, r\right)\right)
\end{aligned}
$$

Therefore, the expected utility difference between the two actions for bidder 1 is

$$\begin{aligned}\Delta EU\left(v,r\right) &:= EU\left(drop;v,r\right)-EU\left(continue;v,r\right)=q\left(\pi\left(drop;v,r\right)-\pi\left(continue;v,r\right)\right)\\ &= q\frac{u\left(v-r+\omega\right)-u\left(\omega\right)}{1-G\left(2r\right)}\left\{\int_r^1 G\left(t+r\right)\frac{f_2(t)}{1-F_2\left(r\right)}dt-G\left(2r\right)\right\}\\ &- q\frac{u\left(v-r+\omega\right)-u\left(\omega\right)}{1-G\left(2r\right)}\left\{\int_r^v \frac{u''\ \left(v-y+\omega\right)}{u\left(v-r+\omega\right)-u\left(\omega\right)}G\left(y+r\right)dy-G\left(2r\right)\right\}\\ &= q\frac{u\left(v-r+\omega\right)-u\left(\omega\right)}{1-G\left(2r\right)}\left\{\int_r^1 G\left(t+r\right)\frac{f_2(t)}{1-F_2\left(r\right)}dt-\int_r^v \frac{u^{'}\ \left(v-y+\omega\right)}{u\left(v-r+\omega\right)-u\left(\omega\right)}G\left(y+r\right)dy\right\}\end{aligned}$$

**Remark:** Note that the sign of the expected utility difference is independent of the value of $q$.
sign $\Delta EU\left(v,r\right)=$ sign $\left\{\int_r^1 G\left(t+r\right)\frac{f_2(t)}{1-F_2(r)}dt-\int_r^v L\left(v,t,r,\omega\right)G\left(t+r\right)dt\right\}$
Thus the condition that bidder 1's expected utility from dropping at $r$ is at least as high as his expected utility from continuing becomes:

$$\begin{aligned}\Delta EU\left(v,r\right)>0\quad\forall v &\Leftrightarrow \int_r^1 G\left(t+r\right)\frac{f_2\left(t\right)}{1-F_2\left(r\right)}dt>\int_r^v L\left(v,t,r,\omega\right)G\left(t+r\right)dt\quad\forall v\\ &\Leftrightarrow \int_r^1 G\left(t+r\right)\frac{f_2\left(t\right)}{1-F_2\left(r\right)}dt>\max_v\int_r^v L\left(v,t,r,\omega\right)G\left(t+r\right)dt\end{aligned}$$

We show in Lemma A.1 that $T(v):=\int_r^v L\left(v,t,r,\omega\right)G\left(t+r\right)dt$ is monotone increasing in $v$ and thus it is maximized when $v=1$ with maximum value that is equal to $\int_r^1 L\left(1,t,r,\omega\right)G\left(t+r\right)dt$

Therefore:

$$\begin{aligned}\Delta EU\left(v,r\right)>0\ \ \forall v &\Leftrightarrow \int_r^1 G\left(t+r\right)\frac{f_2\left(t\right)}{1-F_2\left(r\right)}dt>\int_r^1 L\left(1,t,r,\omega\right)G\left(t+r\right)dt\\ &\Leftrightarrow \int_r^1 G\left(t+r\right)\left(\frac{f_2\left(t\right)}{1-F_2\left(r\right)}-L\left(1,t,r,\omega\right)\right)dt>0\end{aligned}$$

□

**Lemma A.1.** *$T\left(v\right):=\int_r^v L\left(v,t,r,\omega\right)G\left(t+r\right)dt$ is monotone increasing in $v$.*

*Proof.* It will be shown that $T'\left(v\right)>0$ by using Lemma A.2,Lemma A.3 and

Lemma A.4 listed after the proof of this central lemma.

$$\begin{aligned}
T'(v) &= L(v,v,r,\omega)\,G(v+r) + \int_r^v \frac{\partial L(v,t,r,\omega)}{\partial v} G(t+r)\,dt \\
&= L(v,v,r,\omega)\,G(v+r) - \int_r^v \frac{-u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} G(t+r)\,dt \\
&\quad - L(v,r,r,\omega)\int_r^v L(v,t,r,\omega)\,G(t+r)\,dt \qquad \text{(Lemma A.2a)} \\
&= L(v,v,r,\omega)\,G(v+r) + \int_r^v \frac{u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} G(t+r)\,dt \\
&\quad - L(v,r,r,\omega)\,T(v) \qquad (T(v)\,\text{definition}) \\
&> L(v,v,r,\omega)\,G(v+r) + G(v+r)\,\frac{u'(v-r+\omega)-u'(\omega)}{u(v-r+\omega)-u(\omega)} \\
&\quad - L(v,r,r,\omega)\,T(v) \qquad \text{(Lemma A.3))} \\
&= L(v,v,r,\omega)\,G(v+r) + G(v+r)\,(L(v,r,r,\omega)-L(v,v,r,\omega)) \\
&\quad - L(v,r,r,\omega)\,T(v) \qquad \text{(Lemma A.2b))} \\
&= L(v,r,r,\omega)\,(G(v+r)-T(v)) > 0 \qquad \text{(Lemma A.4 and } u(x) \text{ concave)}
\end{aligned}$$

□

**Lemma A.2.** *a)*$\frac{\partial L(v,t,r,\omega)}{\partial v} = -\frac{\partial L(v,t,r,\omega)}{\partial t} - L(v,t,r,\omega)\,L(v,r,r,\omega) = \frac{u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} - L(v,t,r,\omega)\,L(v,r,r,\omega)$
*b)*$L(v,r,r,\omega) - L(v,v,r,\omega) = \frac{u'(v-r+\omega)-u'(\omega)}{u(v-r+\omega)-u(\omega)}$

*Proof.* Directly from $L(v,t,r,\omega)$ definition. □

**Lemma A.3.** $\int_r^v \frac{u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} G(t+r)\,dt > G(v+r)\,\frac{u'(v-r+\omega)-u'(\omega)}{u(v-r+\omega)-u(\omega)}$

*Proof.*

$$\begin{aligned}
&G(t+r) < G(v+r) \Leftrightarrow \frac{u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} G(t+r) > \frac{u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} G(v+r) \\
&\Leftrightarrow \int_r^v \frac{u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} G(t+r)\,dt > G(v+r)\int_r^v \frac{u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)}\,dt \\
&\Leftrightarrow \int_r^v \frac{u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} G(t+r)\,dt > G(v+r)\,\frac{u'(v-r+\omega)-u'(\omega)}{u(v-r+\omega)-u(\omega)}
\end{aligned}$$

□

**Lemma A.4.** $T(v) < G(v+r)$

*Proof.* Using the fact that $G(x)$ is increasing, we get

$$T(v) = \int_r^v L(v,t,r,\omega)\, G(t+r)\, dt = \frac{\int_r^v u'(v-t+\omega)\, G(t+r)\, dt}{u(v-r+\omega) - u(\omega)}$$
$$< \frac{G(v+r)\int_r^v u'(v-t+\omega)\, dt}{u(v-r+\omega)-u(\omega)} = G(v+r)$$

□

## A.1.2 Proof Theorem 4.3

*Proof.* Each large bidder follows obviously a truthful strategy. If more than two small bidders continue bidding, each of them follows a truthful strategy (otherwise who stops loses immediately). Hence, it only need to be shown than if two small bidders continue bidding (along with a large bidder), then each small bidders stops immediately.

Let $s$ be the current price level.

$H_l(\cdot \mid \cdot)$ denotes the conditional CDF of $w^{(1)}$ among l valuations and $h_l(\cdot \mid \cdot)$ the corresponding pdf

e.g. $H_l\left(v_2 + s \mid w^{(l)} \geq 2s\right) = \frac{(G(v_2+s)-G(2s))^l}{(1-G(2s))^l}$ and $h_l\left(v_2 + s \mid w^{(l)} \geq 2s\right) = \frac{g(v_2+s)(G(v_2+s)-G(2s))^{l-1}}{(1-G(2s))^l}$

If bidder 1 stops at $s$ and if it is accepted, his ex interim payoff at s is:

$$\begin{aligned}\pi(drop; v, s) &= u(v-s+\omega)\, Prob\{1 \text{ wins with bid } s|((s,v),\beta_{-1})\} \\ &\quad + u(\omega)\ Prob\{1 \text{ loses with bid } s|((s,v),\beta_{-1})\} \\ &= u(\omega) + (u(v-s+\omega) - u(\omega))\, Prob\{1 \text{ wins with bid } s|((s,v),\beta_{-1})\} \\ &= u(\omega) + \frac{u(v-s+\omega)-u(\omega)}{(1-F_2(s))(1-G(2s))^l}\left\{\int_s^1 (G(v_2+s) - G(2s))^l f_2(v_2)\, dv_2\right\}\end{aligned} \tag{A.3}$$

where we used the following derivations to evaluate the probability term $Prob\{1\ wins\ with\ bid\ s|((s,v),\beta_{-1})\} = Prob\left\{v_2 + s > w^{(1)}|\overline{v}_2 \geq v_2 \geq s, w^{(l)} \geq 2s\right\} = \frac{\int_s^{\overline{v}_2} H_l\left(v_2+s\middle|w^{(l)}\geq 2s\right) f_2(v_2)dv_2}{1-F_2(s)} = \frac{\int_s^{\overline{v}_2} (G(v_2+s)-G(2s))^l f_2(v_2)dv_2}{(1-F_2(s))(1-G(2s))^l}$

If bidder 1 does not stop at $s$ he bids up truthfully since bidder 2 stops. Bidder 1 wins in the continuation game against bidder 3 in the event that $\left\{v+s>w^{(1)}\right\}$ and pays $w^{(1)}-s$ when he wins.

$$\begin{aligned}
\pi\left(continue; v,s\right) = & \int_{2s}^{v+s} u\left(v+s-w^{(1)}+\omega\right) h_l\left(w^{(1)} \mid w^{(l)} \geq 2s\right) dw^{(1)} \\
& + u\left(\omega\right)(1- \int_{2s}^{v+s} h_l\left(w^{(1)} \mid w^{(l)} \geq 2s\right) dw^{(1)}) \\
= & \, u\left(\omega\right) + \int_{2s}^{v+s} \left(u\left(v+s-w^{(1)}+\omega\right) - u\left(\omega\right)\right) h_l\left(w^{(1)} \mid w^{(l)} \geq 2s\right) dw^{(1)} \\
= & \, u\left(\omega\right) + \int_{2s}^{v+s} \left(u\left(v+s-t+\omega\right) - u\left(\omega\right)\right) h_l\left(t \mid t \geq 2s\right) dt \\
= & \, u\left(\omega\right) + \left(u\left(v-s+\omega\right) - u\left(\omega\right)\right)\left(-H_l\left(2s \mid 2s \geq 2s\right)\right) \\
& + \int_{2s}^{v+s} u'\left(v+s-t+\omega\right) H_l\left(t \mid t \geq 2s\right) dt \\
= & \, u\left(\omega\right) + \int_{2s}^{v+s} u'\left(v+s-t+\omega\right) H_l\left(t \mid t \geq 2s\right) dt \\
& \left(since\ H_l\left(2s \mid 2s \geq 2s\right) = 0\right) \qquad \text{(A.4)}
\end{aligned}$$

The difference (A.4)-(A.3) is:

$$\begin{aligned}
\Delta EU\left(v,s\right) &:= \pi\left(continue;v,s\right)-\pi\left(drop;v,s\right)\\
&= \frac{u\left(v-s+\omega\right)-u\left(\omega\right)}{(1-F_2\left(s\right))(1-G\left(2s\right))^l}\left\{\int_s^1 \left(G(v_2+s)-G\left(2s\right)\right)^l f_2\left(v_2\right)dv_2\right\}\\
&\quad -\int_{2s}^{v+s} u'\left(v+s-t+\omega\right)\frac{\left(G(t)-G\left(2s\right)\right)^l}{\left(1-G\left(2s\right)\right)^l}dt
\end{aligned}$$

$$\begin{aligned}
sign\Delta EU\left(v,r\right) &=\\
&= sign\left\{\frac{u\left(v-s+\omega\right)-u\left(\omega\right)}{(1-F_2\left(s\right))}\left(\int_s^1 \left(G(v_2+s)-G\left(2s\right)\right)^l f_2\left(v_2\right)dv_2\right)\right.\\
&\quad \left.-\int_s^v u'\left(v-y+\omega\right)\left(G(y+s)-G\left(2s\right)\right)^l\, dy\right\}\\
&\quad \text{(change variable } y=t-s)\\
&= sign\left\{\left(u\left(v-s+\omega\right)-u\left(\omega\right)\right)\left(\int_s^1 \left(G(v_2+s)-G\left(2s\right)\right)^l \frac{f_2\left(v_2\right)}{(1-F_2\left(s\right))}dv_2\right.\right.\\
&\quad \left.\left.-\int_s^v \frac{u'\left(v-y+\omega\right)}{u\left(v-s+\omega\right)-u\left(\omega\right)}\left(G(y+s)-G\left(2s\right)\right)^l\, dy\right)\right\}\\
&= sign\left\{\int_s^1 \left(G(t+s)-G\left(2s\right)\right)^l \frac{f_2\left(t\right)}{(1-F_2\left(s\right))}dt-\right.\\
&\quad \left.\int_s^v L(v,t,s,\omega)(G(t+s)-G\left(2s\right))^l\, dt\right\}
\end{aligned}$$

We will show in Lemma A.5 that $Q\left(v\right) := \int_s^v L(v,t,s,\omega)(G(t+s)-G\left(2s\right))^l\, dt$ is monotone increasing in $v$ and thus it is maximized at $v$=1.

Therefore we will show

$$\begin{aligned}
&\Delta \mathrm{EU}\left(\mathrm{v},\mathrm{s}\right) > 0 \forall v\\
&\Leftrightarrow \int_s^1 \left(G(t+s)-G\left(2s\right)\right)^l \frac{f_2\left(t\right)}{(1-F_2\left(s\right))}dt > \int_s^1 L(1,t,s,\omega)(G(t+s)-G\left(2s\right))^l\, dt\\
&\Leftrightarrow \int_s^1 \left(G\left(t+s\right)-G\left(2s\right)\right)^l \left(\frac{f_2\left(t\right)}{(1-F_2\left(s\right))}-L\left(1,t,s,\omega\right)\right)\, dt > 0
\end{aligned}$$

□

**Lemma A.5.** *$Q(v) := \int_s^v L(v,t,s,\omega)(G(t+s)-G\left(2s\right))^l\, dt$ is monotone increasing in $v$.*

*Proof.* It will be shown that $Q'(v) > 0$ by using Lemma A.6 and Lemma A.7 listed after the proof of this central lemma.

$$\begin{aligned}
Q'(v) = {} & L(v,v,s,\omega)(G(v+s)-G(2s))^l + \int_s^v \frac{\partial L(v,t,s,\omega)}{\partial v}(G(t+s)-G(2s))^l dt \\
= {} & L(v,v,s,\omega)(G(v+s)-G(2s))^l - \int_r^v \frac{\partial L(v,t,s,\omega)}{\partial t}(G(t+s)-G(2s))^l dt \\
& - L(v,s,s,\omega)\int_r^v L(v,t,s,\omega)(G(t+s)-G(2s))^l dt \qquad \text{(Lemma A.2a)} \\
= {} & L(v,v,s,\omega)(G(v+s)-G(2s))^l + \int_r^v \frac{u''(v-t+\omega)}{u(v-s+\omega)-u(\omega)}(G(t+s)-G(2s))^l dt \\
& - L(v,s,s,\omega)Q(v) \\
> {} & L(v,v,s,\omega)(G(v+s)-G(2s))^l + (G(v+s)-G(2s))^l \frac{u'(v-s+\omega)-u'(\omega)}{u(v-s+\omega)-u(\omega)} \\
& - L(v,s,s,\omega)Q(v) \qquad \text{(Lemma A.6)} \\
= {} & L(v,v,s,\omega)(G(v+s)-G(2s))^l + (G(v+s)-G(2s))^l (L(v,s,s,\omega)-L(v,v,s,\omega)) \\
& - L(v,s,s,\omega)Q(v) \qquad \text{(Lemma A.2b)} \\
= {} & L(v,s,s,\omega)\Big((G(v+s)-G(2s))^l - Q(v)\Big) > 0 \qquad \text{(Lemma A.7 and } u(x) \text{ concave)}
\end{aligned}$$

□

**Lemma A.6.**

$$\int_r^v \frac{u''(v-t+\omega)}{u(v-s+\omega)-u(\omega)}(G(t+s)-G(2s))^l dt > (G(v+s)-G(2s))^l \frac{u'(v-s+\omega)-u'(\omega)}{u(v-s+\omega)-u(\omega)}$$

*Proof.* Similar to Lemma A.3 □

**Lemma A.7.** $(G(v+s)-G(2s))^l > Q(v)$

*Proof.* Similar to Lemma A.4 □

### A.1.3 Proof Theorem 4.4

*Proof.* We will show that the left hand side of the non bidding equilibrium condition 4.1 decreases as $A(x)$ increases. First, we describe the left hand side as a function of the utility function, $NB(u) :=$

$\int_r^1 G(t+r)\left(\frac{f(t)}{1-F(r)} - L(u;;1,t,r,\omega)\right) dt$. Let $\tilde{u}(x)$, $u(x)$ be two utility functions and $\tilde{A}(x), A(x)$ the corresponding Arrow-Pratt measures such that $\tilde{A}(x) < A(x)$ for $\forall x$. We need to show

$$NB(\tilde{u}) > NB(u) \Leftrightarrow \int_r^1 G(t+r)\left(L(u;1,t,r,\omega) - L(\tilde{u};1,t,r,\omega)\right) dt > 0 \tag{A.5}$$

Define $\Delta(t) := L(u;1,t,r,\omega) - L(\tilde{u};1,t,r,\omega)$.
Since $\int_r^1 L(u;1,t,r,\omega)\, dt = 1\ \forall u,t,r,\omega$, it holds:

$$\int_r^1 \Delta(t)\, dt = 0 \tag{A.6}$$

We will show that $\Delta(t)$ has always the form in Figure A.1. Precisely we will show $\Delta(1) > 0$ and $\Delta(t) = 0$ has exactly one root. Due to A.6 the positive and the negative areas are equal. Since $G(x)$ is increasing, the positive area is multiplied by greater values than the negative area, hence inequality A.5 is true.

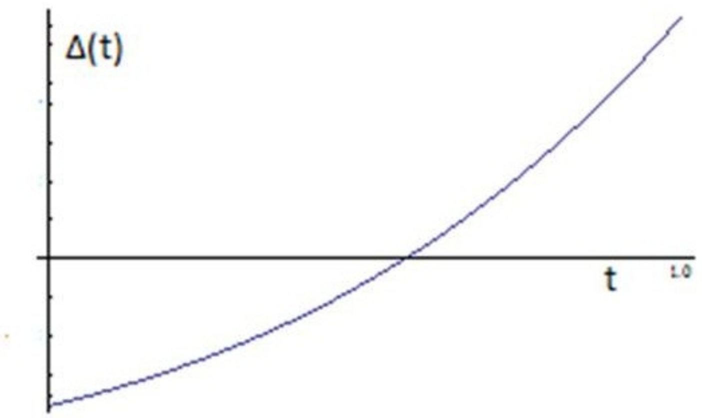


FIGURE A.1: Difference of functionals $L$ with different Arrow Pratt measures

To show $\Delta(1) > 0$ we make use of the following lemma, proven in Pratt (1964):

**Lemma A.8.** *(Pratt, 1964)*
$A_2(x) > A_1(x)\ \ \forall x \Leftrightarrow \frac{u_2(y)-u_2(x)}{u_2'(w)} < \frac{u_1(y)-u_1(x)}{u_1'(w)} \forall (w,x,y):\ w \le x \le y$

Let $w = x = \omega$, $y = \omega + 1 - r$, then we get:

$$\begin{aligned} A(x) > \tilde{A}(x)\,\forall x &\Leftrightarrow \frac{u(\omega+1-r)-u(\omega)}{u'(\omega)} < \frac{\tilde{u}(\omega+1-r)-\tilde{u}(\omega)}{\tilde{u}'(\omega)} \\ &\Leftrightarrow \frac{u'(\omega)}{u(\omega+1-r)-u(\omega)} > \frac{\tilde{u}'(\omega)}{\tilde{u}(\omega+1-r)-\tilde{u}(\omega)} \\ &\Leftrightarrow L(u;1,t,r,\omega) > L(\tilde{u};1,t,r,\omega) \\ &\Leftrightarrow \Delta(1) > 0 \end{aligned}$$

We proceed to show $\Delta(t) = 0$ has exactly one root: Since

$$\begin{aligned} \frac{\partial L(v,t,r,\omega)}{\partial t} &= \frac{-u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} = \frac{-u''(v-t+\omega)}{u'(v-t+\omega)} \frac{u'(v-t+\omega)}{u(v-r+\omega)-u(\omega)} \\ &= A(v-t+\omega)L(v,t,r,\omega) \end{aligned}$$

we have

$$\frac{\partial \Delta(t)}{\partial t} = A(1-t+\omega)\,L(u;1,t,r,\omega) - \tilde{A}(1-t+\omega)\,L(\tilde{u};1,t,r,\omega)$$

Suppose there are two values of t, $\mathrm{t}_2^*$ and $\mathrm{t}_1^*$ with $\mathrm{t}_2^* > \mathrm{t}_1^*$ such as $\Delta(\mathrm{t}_1^*) = \Delta(\mathrm{t}_2^*) = 0$.
Since $L(u;1,\mathrm{t}_1^*,r,\omega) = L(\tilde{u};;1,\mathrm{t}_1^*,r,\omega)$ and $A(1-\mathrm{t}_1^*+\omega) > \tilde{A}(1-\mathrm{t}_1^*+\omega)$ it follows $\frac{\partial\Delta(\mathrm{t}_1^*)}{\partial t} > 0$. Thus:

$$\forall t \in (\mathrm{t}_1^*, \mathrm{t}_2^*) \quad \Delta(\mathrm{t}) > 0 \tag{A.7}$$

Using the mean value theorem of differential calculus, we get that $\exists \mathrm{t}^c \in (\mathrm{t}_1^*, \mathrm{t}_2^*)$ with

$$\begin{aligned} &\frac{\partial \Delta(\mathrm{t}^c)}{\partial t} = \frac{\Delta(\mathrm{t}_2^*) - \Delta(\mathrm{t}_1^*)}{\mathrm{t}_2^* - \mathrm{t}_1^*} = 0 \\ &\Leftrightarrow A(1-\mathrm{t}^c+\omega)\,L(u;1,\mathrm{t}^c,r,\omega) - \tilde{A}(1-\mathrm{t}^c+\omega)\,L(\tilde{u};1,\mathrm{t}^c,r,\omega) = 0 \\ &\Leftrightarrow L(u;1,\mathrm{t}^c,r,\omega) < L(\tilde{u};1,\mathrm{t}^c,r,\omega) \;\; (\textit{since } A(1-\mathrm{t}^c+\omega) > \tilde{A}(1-\mathrm{t}^c+\omega)) \\ &\Leftrightarrow \Delta(\mathrm{t}^c) < 0 \end{aligned}$$

which is a **contradiction** to A.7. Thus there cannot exist two (or more) values of $t$, $\mathrm{t}_1^*$ and $\mathrm{t}_2^*$ with $\mathrm{t}_2^* > \mathrm{t}_1^*$ such as $\Delta(\mathrm{t}_1^*) = \Delta(\mathrm{t}_2^*) = 0$. Due to (A.6) and A.7 there is at least one root. Thus there is exactly one root. The last step of the proof is to formally show that the positive area is multiplied by greater values than the negative area, which implies that the integral in A.5 is positive.

Define $t^*$ as $\Delta(t^*) = 0$. Then,

$$-\int_r^{t^*} G(t+r)\,\Delta(t)dt < -G(t^*+r)\int_r^{t^*}\Delta(t)dt < \int_{t^*}^1 G(t+r)\,\Delta(t)dt$$

(since for any $t \in [r,\ t^*] G(t+r) <\ G(t^*+r)$ and $G$ increasing)

$$\Leftrightarrow \int_r^{t^*} G(t+r)\,\Delta(t)dt + \int_{t^*}^1 G(t+r)\,\Delta(t)dt > 0$$

$$\Leftrightarrow \int_r^1 G(t+r)\,\Delta(t)\,dt > 0$$

$$\Leftrightarrow \int_r^1 G(t+r)\,(L(u;1,t,r,\omega) - L(\tilde{u};1,t,r,\omega))\,dt > 0$$

□

### A.1.4 Proof Theorem 4.5

*Proof.* First, we express the left hand side as a function of $\omega$ as
$NB(\omega) := \int_r^1 G(t+r)\left(\frac{f(t)}{1-F(r)} - L(u;1,t,r,\omega)\right)dt$
We need to show
$NB(\omega+\delta) > NB(\omega) \iff \int_r^1 G(t+r)\,(L(u;1,t,r,\omega) - L(u;1,t,r,\omega+\delta))\,dt > 0$
Define $\Delta(t,\delta) := L(u;1,t,r,\omega) - L(u;1,t,r,\omega+\delta)$
We first show $\Delta(1,\delta) > 0\ \forall\delta > 0$ :
Integrating $A(\omega+\delta) < A(\omega)$ from $x$ to $y$ we get

$$\int_x^y \frac{u''(\omega+\delta)}{u'(\omega+\delta)}d\omega > \int_x^y \frac{u''(\omega)}{u'(\omega)}d\omega \Leftrightarrow \log\left(\frac{u'(y+\delta)}{u'(x+\delta)}\right) > \log\left(\frac{u'(y)}{u'(x)}\right)$$

$$\Leftrightarrow \frac{u'(y+\delta)}{u'(x+\delta)} > \frac{u'(y)}{u'(x)} for x < y, \delta > 0$$

$$q(y) := \frac{u(y) - u(x)}{u'(x)} - \frac{u(y+\delta) - u(x+\delta)}{u'(x+\delta)}$$

$$q'(y) = \frac{u'(y)}{u'(x)} - \frac{u'(y+\delta)}{u'(x+\delta)} < 0 \text{ due to the last inequality}$$

Apply the mean value theorem of differential calculus on $[x, y]$:
$\frac{q(y)-q(x)}{y-x} < 0 \Longleftrightarrow q(y) < 0$ since $q(x) = 0,\ y - x > 0$
Now replace in $q(y) < 0$ $y$ with $\omega + 1 - r$ and $x$ with $\omega$ :

$$\frac{u(\omega+1-r)-u(\omega)}{u'(\omega)} - \frac{u(\omega+1-r+\delta)-u(\omega+\delta)}{u'(\omega+\delta)} < 0$$
$$\Leftrightarrow\ L(1,1,r,\omega+\delta) < L(1,1,r,\omega) \Leftrightarrow\ \Delta(1,\delta) > 0$$

We proceed to show $\Delta(t) = 0$ has exactly one root:
Since

$$\frac{\partial L(v,t,r,\omega)}{\partial t} = \frac{-u''(v-t+\omega)}{u(v-r+\omega)-u(\omega)} = \frac{-u''(v-t+\omega)}{u'(v-t+\omega)}\frac{u'(v-t+\omega)}{u(v-r+\omega)-u(\omega)}$$
$$= A(v-t+\omega)L(v,t,r,\omega)$$

we have

$$\frac{\partial\Delta(t)}{\partial t} = \ A(1-t+\omega)L(u;1,t,r,\omega) - \tilde{A}(1-t+\omega+\delta)L(u;1,t,r,\omega+\delta)$$

Suppose there are two values of $t$, $\mathrm{t}_2^*$ and $\mathrm{t}_1^*$ with $\mathrm{t}_2^*>\mathrm{t}_1^*$ such as $\Delta(\mathrm{t}_1^*) = \Delta(\mathrm{t}_2^*) = 0$. Since

$$L(u;1,\mathrm{t}_1^*,r,\omega) = L(u;1,\mathrm{t}_1^*,r,\omega+\delta) \text{ and } A(1-\mathrm{t}_1^*+\omega) > A(1-\mathrm{t}_1^*+\omega+\delta)$$

it follows $\frac{\partial\Delta(\mathrm{t}_1^*)}{\partial t} > 0$. Thus:

$$\forall t\ \in(\mathrm{t}_1^*,\mathrm{t}_2^*)\Delta(\mathrm{t}) > 0 \tag{A.8}$$

Using the mean value theorem of differential calculus, we get that $\exists \mathrm{t}^c \in (\mathrm{t}_1^*, \mathrm{t}_2^*)$ with

$$\frac{\partial\Delta(\mathrm{t}^c)}{\partial t} = \frac{\Delta(\mathrm{t}_2^*)-\Delta(\mathrm{t}_1^*)}{\mathrm{t}_2^*-\mathrm{t}_1^*} = 0$$
$$\Leftrightarrow A(1-\mathrm{t}^c+\omega)L(u;1,\mathrm{t}^c,r,\omega) - A(1-\mathrm{t}^c+\omega+\delta)L(u;1,\mathrm{t}^c,r,\omega+\delta) = 0$$
$$\Leftrightarrow L(u;1,\mathrm{t}^c,r,\omega) < \ L(u;1,\mathrm{t}^c,r,\omega+\delta)\,(\text{since } A(1-\mathrm{t}^c+\omega) > \mathrm{A}(1-\mathrm{t}^c+\omega+\delta))$$
$\Leftrightarrow \Delta(\mathrm{t}^c) < 0$which is a **contradiction**to $A.8$.

□

### A.1.5 Proof Theorem 4.6

*Proof.*

$$NB\left(\tilde{F}_1\right) > NB\left(F_1\right) \Leftrightarrow \int_0^1 G\left(t\right)\left(\tilde{f}_1\left(t\right) - f_1\left(t\right)\right)dt \geq 0$$

$$\Leftrightarrow G\left(1\right)\left(\tilde{F}_1\left(1\right) - F_1\left(1\right)\right) - \int_0^1 g\left(t\right)\left(\tilde{F}_1\left(t\right) - F_1\left(t\right)\right)dt \geq 0$$

$$\Leftrightarrow \int_0^1 g\left(t\right)\left(\tilde{F}_1\left(t\right) - F_1\left(t\right)\right)dt \leq 0 \tag{A.9}$$

The last inequality holds since $g\left(t\right) > 0$ and $\tilde{F}_1 \succcurlyeq_{FSD} F_1$. □

### A.1.6 Proof Theorem 4.7

*Proof.* If $g\left(t\right)$ is non-increasing, then only the weaker second-order stochastic dominance is required: $\Delta\left(t\right) := F_1\left(t\right) - \tilde{F}_1\left(t\right)$.
Since $\Delta\left(0 + \varepsilon\right) \geq 0$ (for a small $\varepsilon$ – due to SSD), if $\Delta$ has up to one roots, A.9 holds immediately. Suppose now $\Delta$ has *n+1* roots $t_0 < ... < t_n$. $\tilde{F}_1 \succcurlyeq_{SSD} F_1$ $\Rightarrow \int_{t_0}^{t_i} \Delta\left(t\right)dt \geq 0\ \forall i$.

$\Delta\left(t\right) \geq 0\ \forall t \in \left[t_{i-1}, t_i\right]$, if $i$ odd and $\Delta\left(t\right) \leq 0\ \forall t \in \left[t_i, t_{i+1}\right]$ if $i$ even.

Let $I_i := \int_{t_{i-1}}^{t_i} g\left(t\right)\Delta\left(t\right)dt$ as in Figure A.2. Since $g$ nonincreasing, $I_{2k-1} \geq g\left(t_{2k-1}\right)\int_{t_{2k-2}}^{t_{2k-1}} \Delta\left(t\right)dt \geq 0$ and
$I_{2k} \geq g\left(t_{2k-1}\right)\int_{t_{2k-1}}^{t_{2k}} \Delta\left(t\right)dt \forall k \in \mathbb{N}$ It can be observed that the sign of $\Delta\left(\mathrm{t}\right)$ is positive in $\left[\mathrm{t_{i-1}}, \mathrm{t_i}\right]$, if i odd, else negative. The inequalities concerning the integrals $\mathrm{I_i}$ hold since $g$ nonincreasing.

Now let $S_n := \sum_{i=1}^{n} I_i$. We will show by induction that $S_n \geq 0\ \forall n$ (this immediately implies A.9). $S_1 \geq 0$ since $S_1 = I_1 \geq g\left(t_1\right)\int_{t_0}^{t_1} \Delta\left(t\right)dt \geq 0$ $S_2$=$I_1$+$I_2 \geq g\left(t_1\right)\int_{t_0}^{t_2} \Delta\left(t\right)dt \geq 0$ due to $\tilde{F}_1 \succcurlyeq_{FSD} F_1$.

We assume $S_{2k} \geq g\left(t_{2k-1}\right)\int_{t_0}^{t_{2k}} \Delta\left(t\right)dt \geq 0$ and will show

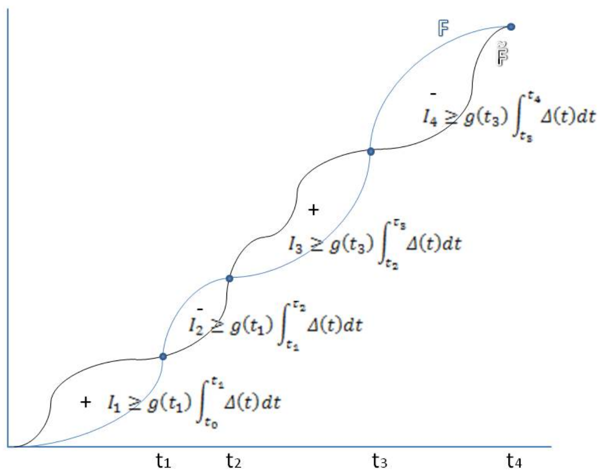


FIGURE A.2: Second order stochastic dominance $\widetilde{F} \succcurlyeq_{SSD} F$ but not $\widetilde{F} \succcurlyeq_{FSD} F$.

$S_{2k+2} \geq g\left(t_{2k+1}\right) \int_{t_0}^{t_{2k+2}} \Delta\left(t\right) dt \geq 0$ :

$$
\begin{aligned}
S_{2k+2} &= S_{2k} + I_{2k+1} + I_{2k+2} \\
&\geq S_{2k} + g\left(t_{2k+1}\right) \int_{t_{2k}}^{t_{2k+2}} \Delta\left(t\right) dt \\
&\geq g\left(t_{2k-1}\right) \int_{t_0}^{t_{2k}} \Delta\left(t\right) dt + g\left(t_{2k+1}\right) \int_{t_{2k}}^{t_{2k+2}} \Delta\left(t\right) dt \\
&\geq g\left(t_{2k+1}\right) \int_{t_0}^{t_{2k}} \Delta\left(t\right) dt + g\left(t_{2k+1}\right) \int_{t_{2k}}^{t_{2k+2}} \Delta\left(t\right) dt \\
&= g\left(t_{2k+1}\right) \int_{t_0}^{t_{2k+2}} \Delta\left(t\right) dt \\
&\geq 0
\end{aligned}
$$

In addition $S_{2k+1} = S_{2k} + I_{2k+1} \geq 0$ □

# A.2 Proofs of Chapter 5

## A.2.1 Proof Theorem 5.1

*Proof.* Athey's measurability assumption A1 is satisfied since we assumed bounded and continuous $f, G$. Hence, the players' types have bounded and atomless joint density. The payoff of bidder $i$ also exists and is finite for all nondecreasing $\beta_j(v_j)$ , $j \neq i$. Her condition that players' bids can be restricted in a compact interval is satisfied since bidding above the true valuation $\overline{w}$ , is dominated by bidding $\overline{w}$ and negative bids are not possible.

Her next condition states that the payoff

$$\Pi(b|v) = \int_0^1 \int_0^{b+\beta(v_{B1})} \Bigg(v - \Bigg(\max\{0, w_{AB} - \beta(v_{B1})\} + \frac{w_{AB} - \max\{0, w_{AB} - \beta(v_{B1})\} \; - \max\{0, w_{AB} - b\}}{2}\Bigg)\Bigg) f(v_{B1})\, g(w_{AB})\, dw_{AB} dv_{B1}$$

must be continuous. This holds since $f$ and $g$ are continuous and the payoff is differentiable.
We only consider the payoff $\Pi(b|v)$ of a local bidder with valuation $v$ for item $A$ and bid $b$ since the global bidder has a truthful strategy. The valuation of the second local bidder for item B is denoted by $v_B1$ and his bid by $\beta(v_{B1})$. The global bidder's bid is denoted by $w_{AB}$.
Next, the single crossing property has to be proved. We have to show that the following two inequalities hold for any bids $b_H > b_L$ and valuations $v_H > v_L$.

Weak inequality: $\Pi(b_H|v_L) - \Pi(b_L|v_L) \geq 0 \Rightarrow \Pi(b_H|v_H) - \Pi(b_L|v_H) \geq 0$

Strong inequality: $\Pi(b_H|v_L) - \Pi(b_L|v_L) > 0 \Rightarrow \Pi(b_H|v_H) - \Pi(b_L|v_H) > 0$

We first show the weak inequality by using the payoff formula $\Pi(b|v)$:

$$\Pi(b_H|v_H) - \Pi(b_H|v_L) = (v_H - v_L) \int_0^1 \int_0^{b_H+\beta(v_{B1})} f(v_{B1})\, g(w_{AB})\, dw_{AB} dv_{B1}$$

and

$$\Pi(b_L|v_H) - \Pi(b_L|v_L) = (v_H - v_L) \int_0^1 \int_0^{b_L+\beta(v_{B1})} f(v_{B1})\, g(w_{AB})\, dw_{AB} dv_{B1}$$

Since $b_H > b_L$ the last integral is smaller or equal than the second to last. Note that we cannot establish strict inequality since a higher bid does not imply higher probability of winning for every g (this would have been the case though if $g > 0$ everywhere).
Hence

$$\Pi(b_H|v_H) - \Pi(b_H|v_L) \geq \Pi(b_L|v_H) - \Pi(b_L|v_L) \quad \Leftrightarrow$$
$$\Pi(b_H|v_H) - \Pi(b_L|v_H) \geq \Pi(b_H|v_L) - \Pi(b_L|v_L)$$

Thus it holds:

$$\Pi(b_H|v_L) - \Pi(b_L|v_L) \geq 0 \Rightarrow \Pi(b_H|v_H) - \Pi(b_L|v_H) \geq 0$$

(if the smaller term is >0, then the greater is also $> 0$)

Next we show the strong inequality. We rewrite the payoff function so that we eliminate the $\max\{0, \mathrm{w_{AB}} - b\}$ term.

$$\Pi(b \mid v) = \int_0^1 \int_0^b \left( v - \left( \max\{0, w_{AB} - \beta(v_{B1})\} + \frac{w_{AB} - \max\{0, w_{AB} - \beta(v_{B1})\}}{2} \right)\right)$$
$$f(v_{B1})\, g(w_{AB})\, dw_{AB} dv_{B1} \, +$$
$$\int_0^1 \int_b^{b+\beta(v_{B1})} \left( v - \left( \max\{0, w_{AB} - \beta(v_{B1})\} + \frac{w_{AB} - \max\{0, w_{AB} - \beta(v_{B1})\} - w_{AB} + b}{2} \right)\right)$$
$$f(v_{B1})\, g(w_{AB})\, dw_{AB} dv_{B1}$$

For convenience we define the following abbreviations and rewrite the payoff function:

$$k(v_L) := v - \max\{0, w_{AB} - \beta(v_{B1})\} \ - \frac{w_{AB} - \max\{0, w_{AB} - \beta(v_{B1})\}}{2},$$
$$dX := f(v_{B1})\, g(w_{AB})\, dw_{AB} dv_{B1}$$
$$\varepsilon := b_H - b_L > 0$$
$$\Pi(b \mid v) = \int_0^1 \int_0^b k(v_L) dX + \int_0^1 \int_b^{b+\beta(v_{B1})} \left( k(v_L) - \frac{-w_{AB} + b}{2} \right) dX$$
$$\Delta := \Pi(b_H | v_L) - \Pi(b_L | v_L) =$$
$$\int_0^1 \int_{b_L}^{b_L+\varepsilon} k dX - \int_0^1 \int_{b_L}^{b_L+\varepsilon} \left( k(v_L) - \frac{-w_{AB} + b_L}{2} \right) dX +$$
$$\int_0^1 \int_{b_L+\beta(v_{B1})}^{b_L+\beta(v_{B1})+\varepsilon} \left( k(v_L) - \frac{-w_{AB} + b_L}{2} \right) dX + \int_0^1 \int_{b_L+\varepsilon}^{b_L+\beta(v_{B1})+\varepsilon} -\frac{\varepsilon}{2} dX$$
$$\Delta = \int_0^1 \int_{b_L}^{b_L+\varepsilon} \frac{-w_{AB} + b_L}{2} dX + \int_0^1 \int_{b_L+\beta(v_{B1})}^{b_L+\beta(v_{B1})+\varepsilon} \left( k(v_L) - \frac{-w_{AB} + b_L}{2} \right) dX +$$
$$\int_0^1 \int_{b_L+\varepsilon}^{b_L+\beta(v_{B1})+\varepsilon} -\frac{\varepsilon}{2} dX$$

In the second integral where $k(v_L)$ appears, it holds $w_{AB} > b_L$ and $k$ simplifies to:
$k(v_L) = v_L - w_{AB} + \beta(v_{B1}) - \frac{\beta(v_{B1})}{2} = v_L - w_{AB} + \frac{\beta(v_{B1})}{2}$
We rewrite $\Delta$:

$$\Delta = \int_0^1 \int_{b_L}^{b_L+\varepsilon} \frac{-w_{AB} + b_L}{2} dX + \int_0^1 \int_{b_L+\beta(v_{B1})}^{b_L+\beta(v_{B1})+\varepsilon} \left( v_L - \frac{w_{AB}}{2} + \frac{\beta(v_{B1})}{2} \ - \frac{b_L}{2} \right) dX +$$
$$\int_0^1 \int_{b_L+\varepsilon}^{b_L+\beta(v_{B1})+\varepsilon} -\frac{\varepsilon}{2} dX$$

Our point of departure is $\Delta > 0$

The first and the third integral are strictly negative. In the second integral it holds due to the integration limits (the first integration variable is $\mathrm{w_{AB}}$) $\mathrm{w_{AB}} > \beta(v_{\mathrm{B1}})$ and $b_L > 0$ thus the following quantity is strictly positive:

$$\Delta > 0 \Rightarrow \int_0^1 \int_{b_L+\beta(v_{B1})}^{b_L+\beta(v_{B1})+\varepsilon} v_L dX > 0$$

For strictly positive $v_L$ (see proof below that this always holds), it must hold:

$$\int_0^1 \int_{b_L+\beta(v_{B1})}^{b_L+\beta(v_{B1})+\varepsilon} dX > 0 \;\Leftrightarrow\; \int_0^1 \int_0^{b_L+\beta(v_{B1})+\varepsilon} dX > \int_0^1 \int_0^{b_L+\beta(v_{B1})} dX$$
$$\Leftrightarrow\; \Pi\left(b_H|v_H\right) - \Pi\left(b_H|v_L\right) > \Pi\left(b_L|v_H\right) - \Pi\left(b_L|v_L\right)$$

For the last equivalence we use the payoff formulas of the weak inequality proof.

$$\Leftrightarrow\; \Pi\left(b_H|v_H\right) - \Pi\left(b_L|v_H\right) > \Pi\left(b_H|v_L\right) - \Pi\left(b_L|v_L\right)$$

Thus we showed $\Pi\left(b_H|v_L\right) - \Pi\left(b_L|v_L\right) > 0 \Rightarrow \Pi\left(b_H|v_H\right) - \Pi\left(b_L|v_H\right) > 0$ which is the strong inequality of the single crossing property. It remains to show that $v_L$ is strictly positive:

Proof that $v_L > 0$

$\Delta > 0$ implies that the 2nd integral is strictly positive:

$$\int_0^1 \int_{b_L+\beta(v_{B1})}^{b_L+\beta(v_{B1})+\varepsilon} \left(v_L - w_{AB} + \frac{\beta\left(v_{B1}\right)}{2} \;-\; \frac{-w_{AB}+b_L}{2}\right) dX =$$
$$\int_0^1 \int_{b_L+\beta(v_{B1})}^{b_L+\beta(v_{B1})+\varepsilon} \left(v_L - \frac{w_{AB}}{2} + \frac{\beta\left(v_{B1}\right)}{2} \;-\; \frac{b_L}{2}\right) dX$$

Since with increasing $w_{AB}$, the value in the integral decreases, it must be strictly positive at the lower limit $w_{AB} = b_L + \beta\left(v_{B1}\right)$:

$$v_L - \frac{b_L}{2} - \frac{\beta\left(v_{B1}\right)}{2} + \frac{\beta\left(v_{B1}\right)}{2} \;-\; \frac{b_L}{2} > 0 => v_L - b_L > 0 \Rightarrow v_L > 0$$

□

### A.2.2 Proof Corollary 5.1

*Proof.* We only consider local bidders since global bidders have a truthful strategy. Assume that the highest bid on AB is $w_{AB}$. Without loss of generality we focus on a bidder A1 desiring item A and bidding $b$ for it. There are m bidders in total for A denoted by A1, A2.. Am and $n$ bidders for B denoted by B1, B2.. Bn

Let $\beta_{Amax}$ be the highest bid on A by a bidder other than A1, $\beta_{Bmax}$ the highest bid on B and $\beta_{Bmax2}$ the second highest bid on B. With a slight abuse of notation let h( · ) be the joint pdf of the variables in the argument.

$$\Pi(b \mid v) =$$

$$\mathbb{I}_{b>\beta_{Amax}} \int_0^1 .. \int_0^1 \int_0^1 \int_0^{b+\beta_{Bmax}} \left( v - \left( \max\{\beta_{Amax}, w_{AB} - \beta_{Bmax}\} + \frac{w_{AB} - \max\{\beta_{Amax}, w_{AB} - \beta_{Bmax}\} \ - \max\{\beta_{Bmax2}, w_{AB} - b\}}{2} \right)\right)$$

$$h(v_{A1}, ..v_{An}, , v_{B1}, ..v_{Bn}, w_{AB})\, dw_{AB} d_{A1}..d_{An} dv_{B1}..dv_{Bn}$$ Let $k(v) := \left( v - \left( \max\{\beta_{Amax}, w_{AB} - \beta_{Bmax}\} \ + \frac{w}{}\right.\right.$

$\varepsilon := b_H - b_L > 0$

We rewrite payoff (we suppress the dots between integral symbols):

$$\Pi(b \mid v) = \mathbb{I}_{b>\beta_{Amax}} \left( \int_0^1 \int_0^{b+\beta_{Bmax2}} (k(v) + \frac{\beta_{Bmax2}}{2}) dX + \int_0^1 \int_{b+\beta_{Bmax2}}^{b+\beta_{Bmax}} \left( k(v) - \frac{-w_{AB} + b}{2} \right) dX \right)$$

$$\begin{aligned}
\Delta :=& \Pi(b_H|v_L) - \Pi(b_L|v_L) \\
=& \mathbb{I}_{\beta_{Amax}<b_L} \left\{ \int_0^1 \int_{b_L}^{b_L+\varepsilon} (k(v) + \frac{\beta_{Bmax2}}{2}) dX - \int_0^1 \int_{b_L}^{b_L+\varepsilon} \left( k(v_L) - \frac{-w_{AB} + b_L}{2} \right) dX + \right. \\
& \left. \int_0^1 \int_{b_L+\beta_{Bmax}}^{b_L+\beta_{Bmax}+\varepsilon} \left( k(v_L) - \frac{-w_{AB} + b_L}{2} \right) dX + \int_0^1 \int_{b_L+\varepsilon}^{b_L+\beta_{Bmax}+\varepsilon} -\frac{\varepsilon}{2} dX \right\} + \\
& \mathbb{I}_{b_H>\beta_{Amax}>b_L} \left\{ \int_0^1 \int_0^{b_L+\varepsilon} (k(v_L) + \frac{\beta_{Bmax2}}{2}) dX + \int_0^1 \int_{b_L+\varepsilon}^{b_L+\varepsilon+\beta_{Bmax}} \left( k(v_L) - \frac{-w_{AB} + b_L + \varepsilon}{2} \right) dX \right\}
\end{aligned}$$

If the first case applies, i.e. $\mathbb{I}_{\beta_{Amax}<b_L} = 1$ then the proof reduces to the proof of the previous setting (only the distribution of the highest bid on AB changes and the highest bids on B but the proofs are distribution free)

Now consider the case $\mathbb{I}_{\beta_{Amax}<b_L} = 0$

Since with bidding $b_L$ the bidder loses, it holds $\Pi(b_L|v_L) = \Pi(b_L|v_H) = 0$

Thus the 2nd Athey condition reduces to $0 > 0 \Rightarrow \Pi(b_H|v_H) - \Pi(b_H|v_L) > 0$

and holds trivially.

***Extension to risk aversion***

We assumed throughout risk neutrality. The proof holds for every utility function since i) we didn't make any distributional assumptions to compute expected payoffs and ii)utility functions are strictly increasing functions and it holds for every $u_1$, $u_2$ utility functions and any $b, v_H, v_L$ that

$$\Pi\left(u_1; b|v_H\right) - \Pi\left(u_1; b|v_L\right) > 0 \ \Leftrightarrow \Pi\left(u_2; b|v_H\right) - \Pi\left(u_2; b|v_L\right) > 0$$

□

### A.2.3 Proof Theorem 5.2

*Proof.* The first part of the proof follows Goeree and Lien (2012) for risk-neutral bidders. We assume the local bidders' values $v_1$ and $v_2$ to be uniformly distributed on $[0, 1]$, and the global bidders' value $v_3$ to be distributed between $[0, 2]$, and a Vickrey-closest bidder-optimal core-selecting auction. The optimal bid of local bidder 1 is where the cost and benefit of a marginal deviation cancel. Let local bidder 1 with value $v_1$ act as if he would have value $v_1 + \varepsilon$, so that he increases his bid from $b\left(v_1\right)$ to $b\left(v_1\right) + \varepsilon b'\left(v_1\right)$.

The local bidder becomes a winner, if the value of the global bidder $v_3$ who follows the truthful strategy $B(v_3) = v_3$ lies between $b\left(v_1\right) + b\left(v_2\right)$ and $b\left(v_1\right) + b\left(v_2\right) + \varepsilon b'\left(v_1\right)$. Assuming a uniform distribution of $v_3$ and integrating yields $\frac{1}{2}\varepsilon b'(v_1)$. In this case of turning from losing to winning, the global bidder's bid and the sum of the local bidders' bids are equal, and the local bidders' Vickrey payment is their bid $b\left(v_1\right)$. If we assume a concave utility function with $u_i'\left(x\right) > 0$ and $u_i''\left(x\right) < 0 \ \forall x$, then the *expected gain* in utility of the deviation is $\frac{1}{2}\varepsilon b'\left(v_1\right)\left(u\left(v_1 - b\left(v_1\right) + \omega\right) - u\left(\omega\right)\right)$ In a Vickrey-closest bidder-optimal core-selecting auction, the payment of bidder 1 is given by

$$p_1 = \max\left\{0, B(v_3) - b(v_2)\right\} + \frac{1}{2}(B(v_3) - \max\left\{0, B(v_3) - b\left(v_1\right)\right\} - \max\left\{0, B(v_3) - b(v_2)\right\})$$

Therefore, the payment of bidder 1 only goes up $\frac{1}{2}\varepsilon b'\left(v_1\right)$ if the global bidder's value is between $b\left(v_1\right)$ and $b\left(v_1\right) + b\left(v_2\right)$. The *expected cost* of increasing the bid by $\varepsilon b'\left(v_1\right)$ is:

$$\left(u\left(v_1 - b\left(v_1\right) + \omega\right) - \ u\left(v_1 - b\left(v_1\right) - \frac{1}{2}\varepsilon b'\left(v_1\right) + \omega\right)\right) \int_0^1 \int_{b(v_1)}^{b(v_1)+b(v_2)} \frac{1}{2} dv_3 dv_2$$

$$= \left(u\left(v_1 - b\left(v_1\right) + \omega\right) - \ u\left(v_1 - b\left(v_1\right) - \frac{1}{2}\varepsilon b'\left(v_1\right) + \omega\right)\right) \frac{1}{2} \int_0^1 b\left(v\right) dv$$

The requirement that expected gain and expected cost cancel yields the following condition:

$$\varepsilon b'(v_1)(u(v_1 - b(v_1) + \omega) - u(\omega))$$
$$= \left(u(v_1 - b(v_1) + \omega) - u\left(v_1 - b(v_1) - \frac{1}{2}\varepsilon b'(v_1) + \omega\right)\right)\int_0^1 b(v)\,dv$$

Let $k := v_1 - b(v_1) + \omega$, then

$$\varepsilon b'(v_1)(u(k) - u(\omega)) = \left(u(k) - u\left(k - \frac{1}{2}\varepsilon b'(v_1)\right)\right)\int_0^1 b(v)\,dv$$

Taylor expansion:

$$u\left(k - \frac{1}{2}\varepsilon b'(v_1)\right) = u(k) - \frac{1}{2}\varepsilon b'(v_1)\,u'(k) + o\varepsilon$$

We can drop the $o\varepsilon$ terms as inconsequential since $\varepsilon$ is 'small'. Replacing the Taylor expansion in previous equation:

$$\varepsilon b'(v_1)(u(k) - u(\omega)) \cong \frac{1}{2}\varepsilon b'(v_1)\,u'(k)\int_0^1 b(v)\,dv$$

$$\Leftrightarrow b'(v_1)\left(u(k) - u(\omega) - \frac{1}{2}u'(k)\int_0^1 b(v)\,dv\right) \cong 0 \qquad \text{(A.1)}$$

Replacing $u(\cdot)$ with the CARA utility function:

$$b'(v_1)\left(\frac{e^{-\lambda\underline{\omega}} - e^{-\lambda(v_1 - b(v_1) + \omega)}}{e^{-\lambda\underline{\omega}} - e^{-\lambda\overline{\omega}}} - \frac{e^{-\lambda\underline{\omega}} - e^{-\lambda\omega}}{e^{-\lambda\underline{\omega}} - e^{-\lambda\overline{\omega}}} - \frac{1}{2}\frac{\lambda\, e^{-\lambda(v_1 - b(v_1) + \omega)}}{e^{-\lambda\underline{\omega}} - e^{-\lambda\overline{\omega}}}\int_r^1 b(v)\,dv\right) \cong 0$$

If $b'(v_1) \neq 0$ :

$$\Leftrightarrow e^{-\lambda\underline{\omega}} - e^{-\lambda(v_1 - b(v_1) + \omega)} - e^{-\lambda\underline{\omega}} + e^{-\lambda\omega} - \frac{1}{2}\,\lambda\, e^{-\lambda(v_1 - b(v_1) + \omega)}\int_r^1 b(v)\,dv \cong 0$$

$$\Leftrightarrow -e^{-\lambda(v_1 - b(v_1) + \omega)} + e^{-\lambda\omega} - \frac{1}{2}\,\lambda\, e^{-\lambda(v_1 - b(v_1) + \omega)}\int_r^1 b(v)\,dv \cong 0$$

$$\Leftrightarrow -e^{-\lambda(v_1 - b(v_1))} + 1 - \frac{1}{2}\,\lambda\, e^{-\lambda(v_1 - b(v_1))}\int_r^1 b(v)\,dv \cong 0$$

$$\Leftrightarrow e^{-\lambda(v_1 - b(v_1))}\left(1 + \frac{1}{2}\lambda\int_r^1 b(v)\,dv\right) \cong 1$$

Taking the natural logarithm of both sides:

$$-\lambda\left(v_1-b\left(v_1\right)\right)+ln(1+\frac{1}{2}\lambda\int_r^1 b\left(v\right)dv)\cong 0 \Leftrightarrow\ b\left(v_1\right)=v_1-\ \frac{ln(1+\frac{1}{2}\lambda\int_r^1 b\left(v\right)dv)}{\lambda}$$

We will solve the integral equation, which is similar to a Fredholm integral equation of $2^{nd}$ order, using the direct computation method.

$a\equiv\displaystyle\int_r^1 b\left(v\right)dv$ and $b\left(v\right)\cong v-\dfrac{ln\left(1+\frac{1}{2}\lambda a\right)}{\lambda}$

The right equation into the left gives:

$a=\displaystyle\int_c^1 v-\frac{ln\left(1+\frac{1}{2}\lambda a\right)}{\lambda}dv$

The lower integration limit $c$ is set to $c=r+\dfrac{ln\left(1+\frac{1}{2}\lambda a\right)}{\lambda}$ due to $b\left(v\right)\geq r\ \forall v\geq r$

Integrating:

$a=\ \dfrac{v^2}{2}-cv\ \Big|_c^1=\dfrac{1}{2}-c-\dfrac{c^2}{2}+cc\Leftrightarrow a=\dfrac{1-c^2}{2}+c(c-1)$

Overall, this leads to

$b\left(v_1\right)\cong\max\left\{r,v_1-c+r\right\}$

where $c=r+\dfrac{ln\left(1+\frac{1}{2}\lambda a\right)}{\lambda}$ and $a=\dfrac{1-c^2}{2}+c(c-1)$

This equation has no closed-form solution. For given $\lambda$, the amounts $\alpha$ and $c$ can be easily calculated. Then $b\left(v_1\right)\cong\max\left\{r,v_1-c+r\right\}$ □

### A.2.4 Proof Theorem 5.3

*Proof.* Equation (A.1) for CRRA utility functions becomes:

$$b'\left(v_1\right)\left(\frac{\left(v_1-b\left(v_1\right)+\omega\right)^{1-\rho}-\ \underline{\omega}^{1-\rho}}{\overline{\omega}^{1-\rho}-\underline{\omega}^{1-\rho}}-\frac{\omega^{1-\rho}-\ \underline{\omega}^{1-\rho}}{\overline{\omega}^{1-\rho}-\underline{\omega}^{1-\rho}}-\frac{1}{2}\frac{\left(1-\rho\right)\left(v_1-b\left(v_1\right)+\omega\right)^{-\rho}}{\overline{\omega}^{1-\rho}-\underline{\omega}^{1-\rho}}\int_r^1 b\left(v\right)dv\right)\cong 0$$

If $b'\left(v_1\right)\neq 0:$

$$\Leftrightarrow\left(\left(v_1-b\left(v_1\right)+\omega\right)^{1-\rho}-\omega^{1-\rho}-\frac{1}{2}\left(1-\rho\right)\left(v_1-b\left(v_1\right)+\omega\right)^{-\rho}\int_r^1 b\left(v\right)dv\right)\cong 0$$

$$\Leftrightarrow\left(v_1-b\left(v_1\right)+\omega\right)-\omega^{1-\rho}(v_1-b\left(v_1\right)+\omega)^{\rho}\cong\frac{1}{2}\left(1-\rho\right)\int_r^1 b\left(v\right)dv$$

We can express $b(v_1)$ as $b(v_1) = v_1 - c$ since we show in the proof of Theorem 5.4 that $b'(v_1) = 1$.

The lower integration limit is set to $c + r$ due to $b(v) \geq r \ \forall v \geq r$

$$(c+\omega) - \omega^{1-\rho}(c+\omega)^{\rho} \cong \frac{1}{2}(1-\rho)\int_{c+r}^{1}(v-c)\,dv$$
$$\Rightarrow (c+\omega) - \omega^{1-\rho}(c+\omega)^{\rho} \cong \frac{1}{4}(1-\rho)\left((c-1)^2 - r^2\right)$$

For given $\rho$ and $\omega$, we can compute $c$. Then $b(v_1) \cong \max\{r, v_1 - c\}$ □

### A.2.5 Proof Theorem 5.4

*Proof.* Departing from the differential equation (A.1)

$$b'(v_1)\left(u(v_1 - b(v_1) + \omega) - u(\omega) - \frac{1}{2}u'(v_1 - b(v_1) + \omega)\int_0^1 b(v)\,dv\right) \cong 0$$

we will show that risk aversion leads to more aggressive bidding (for every u). Taking $b'(v_1) \neq 0$ and rearranging terms:

$$\int_0^1 b(v)\,dv = \frac{u(v_1 - b(v_1) + \omega) - u(\omega)}{\frac{1}{2}u'(v_1 - b(v_1) + \omega)}$$

Differentiating both sides w.r.t $v_1$:

$$0 = 2(1 - b'(v_1)) - \frac{2(-u(\omega) + u(\omega - b(v_1) + v_1))(1 - b'(v_1))u''(\omega - b(v_1) + v_1)}{u'(\omega - b(v_1) + v_1)^2}$$
$$\Leftrightarrow 0 = 2(1 - b'(v_1))\left\{1 - \frac{(-u(\omega) + u(\omega - b(v_1) + v_1))u''(\omega - b(v_1) + v_1)}{u'(\omega - b(v_1) + v_1)^2}\right\}$$

Due to concavity of $u$, the fraction is negative thus the expression in {} is strictly positive. Thus for every concave utility function:

$$b'(v_1) = 1, \quad \forall v_1 \geq c, \text{ where c a threshold below which } b'(v_1) = b(v_1) = 0$$

Next, we compare the equilibrium bidding functions $b_1(v), b_2(v)$ of two utility functions $u_1, u_2$ with Arrow-Pratt measures $A_1, A_2$, where $u_1$ exhibits a higher degree of risk aversion, i.e. $A_1(x) > A_2(x) \ \forall x$. We will show that:

$$\int_0^1 b_1(v)\,dv \geq \int_0^1 b_2(v)\,dv \text{ by establishing that}$$
$$\varphi(x) \equiv \frac{u_1(x) - u_1(\omega)}{\frac{1}{2}u_1'(x)} - \frac{u_2(x) - u_2(\omega)}{\frac{1}{2}u_2'(x)} \geq 0 \quad \forall x \geq \omega$$

Since $\varphi(\omega) = 0$, it remains to be shown that $\varphi(x)$ is strictly increasing for all $x$ such that $\varphi(x) = 0$. Differentiating $\varphi(x)$ gives:

$$\begin{aligned}
\varphi(x) =& -\frac{(u_1(x) - u_1(\omega))\, u_1{}''(x)}{\frac{1}{2}u_1{}'(x)^2} + \frac{(u_2(x) - u_2(\omega))\, u_2{}''(x)}{\frac{1}{2}u_2{}'(x)^2} \\
=& \frac{(u_1(x) - u_1(\omega))\, A_1(x)}{\frac{1}{2}u_1{}'(x)} - \frac{(u_2(x) - u_2(\omega))\, A_2(x)}{\frac{1}{2}u_2{}'(x)} \\
\text{Using } \varphi(x) =& 0 \Leftrightarrow \frac{u_1(x) - u_1(\omega)}{\frac{1}{2}u_1{}'(x)} = \frac{u_2(x) - u_2(\omega)}{\frac{1}{2}u_2{}'(x)} \\
\varphi'(x) =& \frac{u_1(x) - u_1(\omega)}{\frac{1}{2}u_1{}'(x)}(A_1(x) - A_2(x))
\end{aligned}$$

Since $u_1$ is concave and $A_1(x) > A_2(x)$, we conclude that $\varphi'(x) > 0$, thus $\int_0^1 b_1(v)\, dv \geq \int_0^1 b_2(v)\, dv$ . Together with the fact, that for every concave utility function $b'(v) = 1, \quad \forall v \geq c$, it follows that

$$b_1(v) \geq b_2(v) \qquad \forall v \geq r$$

□

### A.2.6 Proof Corollary 5.3

*Proof.* Bidder 1 is risk averse, bidder 2 risk neutral.
For bidder 1 we have:

$$\int_0^1 b_2(v)\,dv = \frac{u(v_1 - b_1(v_1) + \omega) - u(\omega)}{\frac{1}{2}u'(v_1 - b_1(v_1) + \omega)}$$

Differentiating both sides w.r.t $v_1$: (here $b(v)$ is used interchangeably with $b_1(v)$)

$$0 = 2(1 - b'(v_1)) - \frac{2(-u(\omega) + u(\omega - b(v_1) + v_1))(1 - b'(v_1))u''(\omega - b(v_1) + v_1)}{u'(\omega - b(v_1) + v_1)^2}$$

$$\Leftrightarrow 0 = 2(1 - b'(v_1))\left\{1 - \frac{(-u(\omega) + u(\omega - b(v_1) + v_1))u''(\omega - b(v_1) + v_1)}{u'(\omega - b(v_1) + v_1)^2}\right\}$$

Due to concavity of $u$, the fraction is negative thus the expression in $\{\}$ is strictly positive. Thus for every concave utility function:

$b_1'(v_1) = 1, \quad \forall v_1 :\geq b_1(v_1) > r$

For bidder 2 we have:

$$b_2'(v_1)\left(v_1 - b_2(v_1) - \frac{1}{2}\int_0^1 b_1(v)\,dv\right) \cong 0$$

$$b_2(v_1) = v_1 - \frac{1}{2}\int_0^1 b_1(v)\,dv$$

Thus it also holds (integrate $v_1$)

$$2\int_0^1 b_2(v)\,dv + \int_0^1 b_1(v)\,dv = 1$$

$$\begin{aligned} \text{Define } \varphi(x) &\equiv \int_0^1 b_1(v)\,dv - \int_0^1 b_2(v)\,dv = 1 - 3\int_0^1 b_2(v)\,dv \\ &= 1 - 6\frac{u_2(x) - u_2(\omega)}{u_2'(x)},\ x \geq \omega \text{ ( since } v \geq b(v)) \\ \varphi'(x) &= -\frac{(u_1(x) - u_1(\omega))\,u_1''(x)}{u_1'(x)^2} > 0 \end{aligned}$$

since $u_1$ increasing and concave($u_1'' < 0$).

Together with $\varphi(\omega) = 1 > 0$ we establish

$$0\int_0^1 b_1(v)\,dv > \int_0^1 b_2(v)\,dv$$

The last fact and that $b'_1(v) = b'_2(v) = 1$ imply $b_1(v) \geq b_2(v)$ $\forall v$ (strict inequality for $b_1(v) \neq 0$) □

### A.2.7 Proof Theorem 5.5

*Proof.* We have an asymmetric equilibrium which is described by:

$$b_1(v) = \max\left(0, v - \frac{1}{2}a_2\right), b_2(v) = \max\left(0, v - \frac{1}{2}a_1\right)$$

$$a_1 = \int_{r+a_2/2}^{1}\left(v - \frac{1}{2}a_2\right) f_1(v)\, dv, a_2 = \int_{r+a_1/2}^{1}\left(v - \frac{1}{2}a_1\right) f_2(v)\, dv$$

*Let* $\tilde{b}_1,\ \tilde{b}_2, \tilde{a}_1, \tilde{a}_2$ describe the equilibrium in $\tilde{e}$.

We proceed similarly to Barbieri and Malueg (2010) and write $a_2$ as a function of $a_1$.

$$\tilde{a}_2(a_1) = \int_{\frac{a_1}{2}+r}^{1}\left(v - \frac{1}{2}a_1\right)\tilde{f}_2(v)\, dv < \int_{\frac{a_1}{2}+r}^{1}\left(v - \frac{1}{2}a_1\right) f_2(v)\, dv = a_2(a_1)$$

Here $a_1$ is fixed. The inequality holds since $F_2 >_{FSD} \widetilde{F}_2 \Rightarrow \int g(v)\tilde{f}_2(v)\, dv < \int g(v) f_2(v)\, dv$ for every concave $g(v)$. The best response function of bidder 1, denoted as $R_1(\alpha_2)$, is a decreasing function of $a_2$ and since $\tilde{a}_2 < a_2 \forall a_1$, $R_1(\tilde{a}_2) > R_1(\alpha_2)$, the curve moves to the right. The best response function of bidder 2 $R_2(\alpha_1)$ is a decreasing function of $a_1$. Both functions are depicted in Figure A.3. The equilibrium bids are at the intersection of the two curves. Hence, moving from $e$ to $\widetilde{e}$, the intersection point moves to the southeast, i.e. $\widetilde{b}_1(v) \geq b_1(v)$ and $\widetilde{b}_2(v) \leq b_2(v)$

□

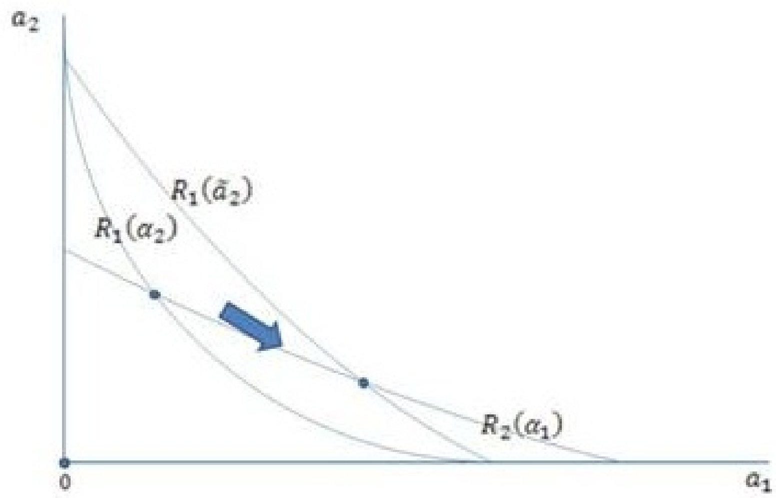


FIGURE A.3: Best reply functions and stochastic dominance.

# Appendix B

# List of Symbols

Symbols are listed according to the chapter they appear for the first time, excluding the introductory chapter. For the ease of exposition certain symbols denote different things in different contexts.

**Chapter 2 - Section 2.1**

| | |
|---|---|
| $\mathcal{I}$ | set of bidders (players, agents) |
| $i, j \in \mathcal{I}$ | bidder |
| $m = \|\mathcal{I}\|$ | number of bidders |
| $A$ | action sets of all players |
| $A_i$ | action set available to player $i$ |
| $a_i$ | action (pure strategy) of player $i$ |
| $u$ | utility functions of all players |
| $u_i$ | utility function of player $i$ |
| $S_i$ | set of (mixed) strategies available to player $i$ |
| $s_i \in S_i$ | a (mixed strategy) of player $i$ |
| $s_{-i}$ | strategies of all players except $i$ |
| $s$ | strategy profile of all players |
| $\Theta_i$ | type space of player $i$ |
| $\theta_i$ | type of player $i$ |
| $\theta_{-i}$ | types of all players except $i$ |
| $p$: | prior over types |
| $\beta(v)$ | bidding strategy (amount bid dependend on valuation $v$) |

**Chapter 2 - Section 2.2 to end**

| | |
|---|---|
| $\mathcal{K}$ | set of items |
| $k, l \in \mathcal{K}$ | item |
| $S, T \subseteq \mathcal{K}$ | package of items |
| $n = \lvert\mathcal{K}\rvert$ | number of items |
| $v_i(S)$ | valuation of bidder $i$ for package $S$ |
| $b_i(S)$ | bid of bidder $i$ on package $S$ |
| $X, Y$ | allocation |
| $X_i$ | package bidder $i$ gets in allocation $X$ |
| $Y_i$ | package bidder $i$ gets in allocation $Y$ |
| $\mathbb{C}$ | set of feasible allocations |
| $p$ | vector with payements of bidders to seller |
| $p$ | price vector |
| $p_i$ | price vector for bidder $i$ |
| $p_{-i}$ | price vector for all bidders except $i$ |
| $p_i(S)$ | (personalized non-linear) price of package $S$ for bidder $i$ |
| $p(S)$ | (anonymous non- linear) price of package $S$ |
| $p(l)$ | (anonymous linear) price of item $l$ |
| $p_i(l)$ | (personalized(linear) price of item $l$ for bidder $i$ |
| $p_i^{VCG}$ | VCG price (payment) |
| $J, J' \subseteq \mathcal{I}$ | coalition of bidders |
| $V(J)$ | (true) coalitional value function for coalition $J$ |
| $\widehat{V}(J)$ | coalitional value function for coalition $J$ based on the reported (not necessarily true) valuations |
| 0 | the seller |
| $\pi$ | payoff vector of all bidders |
| $\pi_i$ | payoff of bidder $i$ |
| $\pi_s$ | seller's payoff |
| $R_i$ | rivals of bidder $i$ |
| $b_i$ | bid of bidder $i$ |
| $\Sigma$ | strategy space of all players |
| $\mathcal{X}$ | set of alternatives |
| $x$ | outcome function of a mechanism |
| $t$ | payment function of a mechanism |
| $D_i(p)$ | demand set of bidder $i$ at prices $p$ |
| $\epsilon$ | bid increment |

**Chapter 3**

| | |
|---|---|
| $k$ | auction state |
| $b$ | bid |
| $(S, v, k, i)$ | bid on package $S$ with value $v$ at auction state $k$ by bidder $i$ |
| $S(b)$ | package the bid $b$ is submitted on |
| $v(b)$ | value of bid $b$ |
| $k(b)$ | auction state of bid $b$ |
| $i(b)$ | bidder who submitted bid $b$ |
| $B, B'$ | a set of bids |
| $B_{S,k}$ | set of all bids in subauction $S$ until auction state $k$ |
| $B_{S,k,-i}$ | set of all bids in subauction $S$ until auction state $k$ excepct bids of bidder $i$ |
| $B^{block}_{S,k,-i}$ | simultaneously blockable bid set in subauction $S$, state $k$ comprising bids foreign to bidder $i$ |
| $B^{ndom}_{S,k,-i}$ | non $i$-dominated bids in $B_{S,k,-i}$ |
| $\mathbb{C}_k$ | set of feasible allocations involving bids submitted until auction state $k$ (index $k$ may be omitted) |
| $\mathbb{C}_u$ | set of infeasible allocations |
| $CAP^k(\mathcal{K})$ | value of CAP at state $k$ when the whole itemset $\mathcal{K}$ is auctioned (value of the whole auction) |
| $WIN_k(\mathcal{K})$ | value maximizing allocation at state $k$ for the whole auction (when the whole itemset $\mathcal{K}$ is auctioned) |
| $WIN_k^{-B'}(\mathcal{K})$ | value maximizing allocation at state $k$ for the whole auction after removal of bid set $B'$ |
| $WL_r(S, i)$ | winning level for package $S$ and bidder $i$ at round $r$ |
| $CAP(S)$ | value of subaction $S$ (value of CAP when only items in $S$ are auctioned) |
| $WL_k(S, i)$ | winning level at state $k$ for package $S$ and bidder $i$ |
| $WL_k^{OR}(S, i)$ | winning level at state $k$ for package $S$ when the OR language is used |
| $CAP^k(\mathcal{K}, S_i)$ | value of CAP with the additional constraint that bidder $i$ gets package $S_i$ for free |
| $DL_k(S, i)$ | deadness level at state $k$ for package $S$ and bidder $i$ |
| $DL_k^{OR}(S)$ | deadness level at state $k$ for package $S$ when the OR language is used |
| $x_i(S)$ | binary decision variable indicated whether bidder $i$ gets pacakge $S$ |
| $t(X)$ | a dual variable corresponding to constraint LP4 concerning infeasible allocations |
| $p_i^k(S)$ | iBundle price $p_i(S)$ after submition of $k$ bids |
| $C_k^E(S)$ | set of feasible allocations that can also include bids of zero value on $S$ |
| $WIN_k^E(\mathcal{K}, S, i)$ | winning allocation of the whole auction at state $k$ subject to the condition that bidder $i$ wins $S$ for free. |

**Chapter 4**

| | |
|---|---|
| $e, \tilde{e}$ | environment |
| $F_i$ | cumulative distribtuion function of small bidder's $i$ valuation which is common knowledge |
| $G$ | cumulative distribtuion function of large bidder's valuation which is common knowledge |
| $f_i$ | first derivative of $F_i$ (probability density function) |
| $g$ | first derivative of $G$ (probability density function) |
| $\omega_i$ | initial wealth of bidder $i$ |
| $A_i(x)$ | Arrow-Pratt measure of risk aversion of bidder $i$ |
| $\lambda$ | risk aversion parameter of CARA utility functions |
| $\rho$ | risk aversion parameter of CRRA utility functions |
| $r$ | reserve price |
| $p_i$ | dropping price of bidder $i$ |
| $L(\cdot)$ | functional defined in Section 4.3 appearing in the condition for the D of non-bidding equilibrium |

**Chapter 5**

| | |
|---|---|
| $a, a_1, a_2, c, k$ | constants used in the descriptions of equilibrium bidding strategies |
| $\Pi(b\|v)$ | (expected) payoff of a local bidder with valuation $v$ and bid $b$ |
| $\Pi(u; b\|v)$ | (expected) payoff of a local bidder with valuation $v$, bid $b$ and utility function $u$ |
| $v_{B1}$ | valuation of the local bidder who desires $B$ |
| $\beta(v_{B1})$ | bid of the local bidder who desires $B$ |
| $w_{AB}$ | global bidder's bid on $AB$ |
| $b_H, b_L$ | bids such that $b_H > b_L$ |
| $v_H, v_L$ | bids such that $v_H > v_L$ |
| $k(\cdot)$ | abbreviation used in proof of Theorem 5.1 |
| $dX$ | abbreviation used in proof of Theorem 5.1 |
| $\epsilon$ | abbreviations used in proof of Theorem 5.1 |
| $dX$ | abbreviation used in proof of Theorem 5.1 |
| $A_i$ | $i - th$ bidder desiring item $A$ |
| $B_i$ | $i - th$ bidder desiring item $B$ |
| $\beta(A_{max})$ | highest bid on item $A$ by a bidder other than $A1$ |
| $\beta(B_{max})$ | highest bid on item $B$ |
| $\beta(B_{max2})$ | second- highest bid on item $B$ |
| $h(\cdot)$ | joint probability density function of the variables in the argument |
| $\mathbb{I}$ | indicator function |
| $B(v_3)$ | (truthful) bidding strategy of global bidder with valuation $v_3$ |
| $\varphi(x)$ | function defined in the proofs of Theorem 5.4 and Corollary 5.3 |
| $R_i(\cdot)$ | best response function of bidder $i$ |

# Appendix C

# List of Abbreviations

| | |
|---|---|
| **APA** | **A**scending **P**roxy **A**uction |
| **BAS** | **B**idders **A**re **S**ubstitutes condition |
| **BCV** | **B**idder-optimal **C**ore-selecting **V**ickrey-nearest |
| **BSM** | **B**idder **S**ub**m**odularity condition |
| **CA** | **C**ombinatorial **A**uction |
| **CAP** | **C**ombinatorial **A**llocation **P**roblem |
| **CARA** | **C**onstant **A**bsolute **R**isk **A**version |
| **CATS** | **C**ombinatorial **A**uction **T**est **S**uite |
| **CCA** | **C**ombinatorial **C**lock **A**uction |
| **CE** | **C**ompetitive **E**quilibrium |
| **CRR** | **C**ommunication effort **R**eduction **R**ate |
| **CRRA** | **C**onstant **R**elative **R**isk **A**version |
| **CWL** | **C**oalitional **W**inning **L**evel |
| **DARA** | **D**ecreasing **A**bsolute **R**isk **A**version |
| **DL** | **D**eadness **L**evel |
| **DLP** | **D**ual **L**inear **P**rogram |
| **dVSV** | **de V**ries **S**chummer **V**ohra auction |
| **EFF** | **EFF**iciency |
| **FCA** | **F**lexible **C**ombinatorial **A**uction |
| **FCC** | **F**ederal **C**ommunication **C**ommission |
| **FSD** | **F**irst-order **S**tochastic **D**ominance |
| **GAS** | **G**oods **A**re **S**ubstitutes condition |
| **DSS** | Chair of **D**ecision **S**ciences and **S**ystems |
| **IARA** | **I**ncreasing **A**bsolute **R**isk **A**version |
| **IPV** | **I**ndependent **P**rivate **V**aluations |

| | |
|---|---|
| **IS** | **I**nformation **S**ystems |
| **ISR** | **I**nformation **S**ystem **R**esearch Journal |
| **LP** | **L**inear **P**rogram |
| **NP** | **N**ash **E**quilibrium |
| **NP** | **N**on/deterministic **P**olynomial time |
| **OR** | additive-**OR** (bidding language) |
| **PCT** | **P**rice **C**alculation **T**ime |
| **RAD** | **R**esource **A**llocation **D**esign |
| **RO** | **R**esearch **O**bjective |
| **RRR** | **R**ound **R**eduction **R**ate |
| **RF** | **R**untime **F**actor |
| **SMRA** | **S**imultaneous **M**ulti-**R**ound Design **A**uction |
| **SSD** | **S**econd-order **S**tochastic **D**ominance |
| **TUM** | **T**echnische **U**niversiät **M**ünchen |
| **VCG** | **V**ickrey-**C**larke **G**roves mechanism |
| **UCE** | **U**niversal **C**ompetitive **E**quilibrium |
| **VM** | **V**alue **M**odel |
| **WL** | **W**inning **L**evel |
| **XOR** | e**X**clusive-**OR** (bidding language) |

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