Generalized fractional Lévy processes with fractional Brownian motion limit

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Abstract

Fractional Lévy processes generalize fractional Brownian motion in a natural way. We go a step further and extend the usual fractional Riemann-Liouville kernel to regularly varying functions with the same fractional integration parameter. We call the resulting stochastic processes generalized fractional Lévy processes (GFLP) and show that they may have short or long memory increments and that their sample paths may have jumps or not. Moreover, we define stochastic integrals with respect to a GFLP and investigate their second order structure and sample path properties. A specific example is the Ornstein-Uhlenbeck process driven by a time scaled GFLP. We prove a functional central limit theorem (FCLT) for such scaled processes to a fractional Ornstein-Uhlenbeck process. This approximation applies to a wide class of stochastic volatility models, which include models, where possibly neither the data nor the latent volatility process are semimartingales.

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1 Introduction

This paper contributes to current discussions in various areas of applications, where high-frequency and unequally spaced data lead to continuous-time modelling. This applies in particular to financial and computer network traffic data, but also to environmental and climate data, where remote sensing, satellite and/or radar data have become available.

Practitioners, engineers and scientists observe different characteristics in such data. In particular, we have to distinguish Gaussian and non-Gaussian distributions (specifically heavy tails), no jumps or jumps, which are triggered by market forces or discontinuities in physical processes, short and long memory of various origin, as well as stochastic variability (volatility) observed in high-frequency measurements. We shall define a new class of models, which allows for flexible modelling of the three essential properties: distributions, memory and jump behavior.

All stochastic objects are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, P)$, which satisfies the usual conditions of completeness and right continuity of the filtration. Recall from Marquardt (2006) that a fractional Lévy process (FLP) has the representation

$$S(t) = \int_{\mathbb{R}} \{(t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}\} dL(x)\,, \quad t \in \mathbb{R}\,, \tag{1.1}$$

where $u_+ = \max(u, 0)$, $H \in (0,1)$ and L is a two-sided Lévy process. For L being Brownian motion (BM) this process defines fractional Brownian motion (FBM) denoted by B^H and has been studied extensively. We extend the class of processes (1.1) to

$$S(t) = \int \{g(t-x) - g(-x)\} dL(x), \quad t \in \mathbb{R},$$

for appropriate functions g and call S a *generalized fractional Lévy process* (*GFLP*). The class of of functions g is determined such that $S(t)$ exists for all $t \in \mathbb{R}$.

As mentioned above, our motivation is based on three characteristics of a stochastic process, the distribution or tail probabilities, the dependence structure or memory, and the sample path properties. By introducing GFLPs, we provide a common platform for modeling these properties in a flexible way. Heavy-tailed distributions as well as long memory are often observed in environmental data, computer network traffic, and in volatility processes of financial data, see e.g. Doukhan et al. (2003). Sample path properties can be investigated based on high-frequency data from various applications, indicating that data and volatilities may exhibit jumps (statistical methods are presented in Jacod and Protter (2011)).

As a classic approach short range dependence models are integrated over a fractional kernel, thus obtaining long memory versions of such processes. This applies in particular for processes driven by BM. In the context of volatility modelling prominent papers using this method are Comte and Renault (1996), Comte and Renault (1998), and Comte et al. (2003).

A different approach modifies the driving BM to a FBM, thus obtaining stochastic differential equations driven by FBM; cf. Buchmann and Klüppelberg (2006) and Zähle (1998). It is then a natural step to extend FBM to FLPs providing more flexible distributions and tail behavior than Gaussian processes, retaining the long memory increments. This implies immediately that OU processes driven by FLP constitute a rich distributional class with long memory (cf. Marquardt (2006)). They have been extended to general SDEs driven by FLP by e.g. Fink and Klüppelberg (2011). However, all these processes can model only long memory and they have all continuous sample path.

On the other hand, OU processes driven by a Lévy process provide besides flexible distributions both continuous sample paths (when driven by BM) and sample paths with jumps (when driven by a Lévy process with jumps). In recent years substantial research focused on Lévy-driven models with mostly short memory, exemplified in Barndorff-Nielsen and Shephard (2001) or in Klüppelberg et al. (2004). However, all these processes have exponential autocovariances, hence short memory.

Certain models, which give more flexibility for distributions and memory have been considered; for instance, CARMA models (cf. Brockwell and Lindner (2009) and references therein) extend the class of Lévy-driven OU processes. Although they allow for more flexible autocovariance functions than simple exponentials, they are restricted to short range dependence modelling. Long range dependent models like the FICARMA (Brockwell and Marquardt (2005)) or the infinite factor supOU process by Barndorff-Nielsen (2001) have been suggested. However, FICARMA processes have again continuous sample paths, and the supOU process is a rather complex model.

As a result, we notice a lack of stochastic processes model, which have flexible memory, interpolating between long and exponential decay. Besides, most of the presently applied long memory processes do not allow for jumps. In the light of these facts, the proposed GFLP can contribute a more flexible model to the discussion.

Our paper is organized as follows. In Section 2 we define the GFLP S. We show that S can exhibit both short memory increments (with exponentially or fast polynomially decreasing autocovariances) and long memory increments (with slow polynomially decreasing autocovariances). We investigate the sample path behavior, where we show that S has a càdlàg version and can have continuous paths or jumps. It is an interesting feature that S can have jumps and long memory. In Section 3 we extend the classic Riemann-Liouville fractional integrals by allowing for more general kernel functions. For fixed kernel function we determine the class $\mathcal{H}$ of integrands such that the integral with respect to S exists. We present some analytic results for this integral. If the kernel function is positive (or negative) on $\mathbb{R}_+$ the isometry between the two inner product spaces $L^2(\Omega)$ and $\mathcal{H}$ is presented, giving the second moment structure of S. As a prominent example we consider the Ornstein-Uhlenbeck (OU) process driven by a GFLP and prove functional convergence of scaled versions to a fractional (Gaussian) OU process. In Section 5 we apply our results to stochastic volatility models, proving joint weak convergence of the data process (driven by BM or FBM) and the volatility process in the Skorokhod space $D(\mathbb{R}_+^2)$.

2 Generalized fractional Lévy processes

Throughout this paper we work with a two-sided Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ constructed by taking two independent copies $L_1 = \{L_1(t)\}_{t\geq 0}$ and $L_2 = \{L_2(t)\}_{t\geq 0}$ of a Lévy process and setting $L(t) := L_1(t)\mathbf{1}_{[0,\infty)}(t) - L_2((-t)-)\mathbf{1}_{(-\infty,0)}(t)$. Moreover, L is centered without Gaussian component and the Lévy measure ν satisfies $\int_{|x|>1} x^2\nu(dx) < \infty$, i.e. $E[(L(t))^2] = tE[(L(1))^2] = t\int_{\mathbb{R}} x^2\nu(dx) < \infty$ for all $t \in \mathbb{R}$. The distribution of L is uniquely defined by the characteristic function (ch.f.) $E[\exp\{i\theta L(t)\}] = \exp\{t\psi(\theta)\}$ for $t \geq 0$, where

$$\psi(\theta) = \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x)\nu(dx)\,, \quad \theta \in \mathbb{R}\,. \tag{2.1}$$

For more details on Lévy processes we refer to the excellent monograph of Sato (1999).

The following result is known and we recall it for later reference. It can be found in Proposition 2.1 and Theorem 3.5 of Marquardt (2006) or, in a more general version, in Rajput and Rosinski (1989).

Proposition 2.1. *Let L be a Lévy process. Assume that $E[L(1)] = 0$ and $E[(L(1))^2] < \infty$. For $t \in \mathbb{R}$ let $f_t \in L^2(\mathbb{R})$. Then the integral $S(t) := \int_{\mathbb{R}} f_t(u)dL(u)$ exists in the $L^2(\Omega)$ sense. Furthermore, for $s,t \in \mathbb{R}$ we obtain $E[S(t)] = 0$, the isometry*

$$E[(S(t))^2] = E[(L(1))^2]\|f_t(\cdot)\|^2_{L^2(\mathbb{R})} \tag{2.2}$$

holds, and

$$\widetilde{\Gamma}(s,t) = \mathrm{Cov}(S(s), S(t)) = E[(L(1))^2]\int_{\mathbb{R}} f_s(u)f_t(u)du. \tag{2.3}$$

Moreover, the ch.f. of $S(t_1),\ldots,S(t_m)$ for $t_1 < \cdots < t_m$ is given by

$$E\Big[\exp\Big\{\sum_{j=1}^m i\theta_j S(t_j)\Big\}\Big] = \exp\left\{\int_{\mathbb{R}} \psi\Big(\sum_{j=1}^m \theta_j f_{t_j}(s)\Big)ds\right\}\,, \tag{2.4}$$

for $\theta_j \in \mathbb{R}$, $j = 1,\ldots,m$, where ψ is given in (2.1).

We define now a generalized fractional Lévy process.

Definition 2.2. *Let L be a Lévy process with $E[L(1)] = 0$ and $E[(L(1))^2] < \infty$. Let $g : \mathbb{R} \to \mathbb{R}$ with $g(t) = 0$ for $t < 0$ and such that $\int_{\mathbb{R}}(g(t-s) - g(-s))^2ds < \infty$ for all $t \in \mathbb{R}$. The stochastic process $S = \{S(t)\}_{t\in\mathbb{R}}$ defined by*

$$S(t) = \int_{\mathbb{R}} \{g(t-u) - g(-u)\}dL(u)\,, \quad t \in \mathbb{R}\,, \tag{2.5}$$

is called generalized fractional Lévy process (GFLP).

The process S has stationary increments and is symmetric with $S(0) = 0$, i.e. $S(-t) \stackrel{d}{=} -S(t),\ t \geq 0$. By taking $g(u) = u_+^{H-\frac{1}{2}}$ we obtain a FLP.

The integral (2.5) obviously exists in the $L^2(\Omega)$ sense. In what follows we formulate assumptions on g needed for the existence of a stochastic integral with respect to S considered in Section 3 or for the existence of a functional limit of a scaled family of such processes in Section 4.

Assumption 2.3. *The function* $g : \mathbb{R} \to \mathbb{R}$ *satisfies* $g(t) = 0$ *for* $t < 0$ *and is continuously twice differentiable on* $(0, \infty)$*, the limit* $\lim_{u\downarrow 0} |g'(u)|$ *exists and is finite, and* $g''(u) = O(u^{-3/2-\varepsilon})$ *as* $u \to \infty$ *for sufficiently small* $\varepsilon > 0$.

We assume that Assumption 2.3 holds throughout the paper. We start with some sample path properties of a GFLP.

Lemma 2.4. *Let* L *be a Lévy process with* $E[L(1)] = 0$ *and* $E[(L(1))^2] < \infty$*. Under Assumption 2.3 on* g*, the GFLP* S *has a càdlàg version. Moreover,* S *has jumps if and only if* $g(0) \neq 0$.

Proof. We let $t > 0$ (w.l.o.g.) since the proof is analogous for $t \leq 0$. Write

$$S(t) = \int_0^t g(t-u)dL(u) + \int_{-\infty}^0 \{g(t-u) - g(-u)\}dL(u) =: S_1(t) + S_2(t).$$

Our assumption on the Lévy process implies the LILs (Sato, 1999, Propositions 47.11 and 48.9),

$$\limsup_{t\downarrow 0} \frac{|L(t)|}{(2t\log\log(1/t))^{1/2}} = 0 \quad a.s. \quad \text{and} \quad \limsup_{t\to\infty} \frac{|L(t)|}{(2t\log\log t)^{1/2}} = \left(E[(L(1))^2]\right)^{1/2} \quad a.s.$$

We use the fact that, if f is continuously differentiable,

$$\int_a^b f(s)dL(s) = f(b)L(b) - f(a)L(a) - \int_a^b L(s)df(s)$$

holds (see Lemma 2.1 of Eberlein and Raible (1999): for fixed ω the sets of jumps of L are an at most countable Lebesgue null set). This together with the LIL at the origin and the assumptions on g yields

$$\begin{aligned} S_1(t) &= \int_0^t g(t-u)dL(u) = g(0)L(t) - \lim_{s\downarrow 0} g(t-s)L(s) + \int_0^t L(u)g'(t-u)du \\ &= g(0)L(t) + \int_0^t L(u)g'(t-u)du, \end{aligned}$$

whereas this together with the LIL at infinity yields

$$\begin{aligned} S_2(t) &= \lim_{s\downarrow -\infty} \{g(t-s) - g(-s)\}L(s) + \lim_{s\downarrow -\infty} \int_s^0 \{g'(t-u) - g'(-u)\}L(u)du \\ &= \int_{-\infty}^0 \{g'(t-u) - g'(-u)\}L(u)du. \end{aligned}$$

As for the expression of S_2, we apply the dominated convergence theorem to

$$S_2(t) - S_2(s) = \int_{-\infty}^0 \{g'(t-u) - g'(s-u)\}L(u)du$$

to observe $\lim_{t\to s} |S_2(t) - S_2(s)| = 0$. Hence S_2 is a.s. continuous. Similarly, the integral term of S_1 is continuous. Since L is càdlàg without drift and Gaussian components, S_1 and hence S has jumps if and only if $g(0) \neq 0$. $\square$

GFLPs can exhibit both short and long memory increments. By Proposition 2.1, when w.l.o.g. $E[(L(1))^2] = 1$, the covariance function of the increments has for $t, s, h > 0$ the form

$$\begin{aligned}\gamma(t,h) &= E\big[\{S(t+s+h) - S(t+s)\}\{S(s+h) - S(s)\}\big] \\ &= \int_{-\infty}^{h} \{g(t+h-u) - g(t-u)\}\{g(h-u) - g(-u)\}du. \end{aligned} \tag{2.6}$$

Definition 2.5. *The GFLP S is said to have* long memory increments *if $\gamma(t,h) \sim Ct^{-\beta}$ as $t \to \infty$ with $C > 0$ and $\beta \leq 1$ for all $h > 0$. (In this case $\gamma(n,1)$ is non-summable.) If γ decreases faster we say S has* short memory increments.

Whether S has long or short memory increments depends on the asymptotic behavior of g.

Lemma 2.6. *Let $0 < \alpha < \frac{1}{2}$ and $c_1 > 0$. Assume that $g(x) = c_1 x^\alpha$ for $x \geq M > 0$. Then S has long memory increments.*

Proof. Set w.l.o.g. $c_1 = 1$. Write $\gamma(t,h) = \gamma_1(t,h) + \gamma_2(t,h)$ with

$$\begin{aligned}\gamma_1(t,h) &:= \int_{-M}^{h} \{g(t+h-u) - g(t-u)\}\{g(h-u) - g(-u)\}du \\ \gamma_2(t,h) &:= \int_{-\infty}^{-M/t} \{g(t+h-tv) - g(t-tv)\}\{g(h-tv) - g(-tv)\}tdv.\end{aligned}$$

By the mean value theorem, for $x \geq M$ and $y > 0$ we have

$$g(y+x) - g(x) = (y+x)^\alpha - x^\alpha = \alpha(x+\theta y)^{\alpha-1}y,$$

where the parameter $0 < \theta < 1$ depends on both x and y. We apply this mean value theorem to both γ_1 and γ_2 and observe for $\theta = \theta(t,h,u) \in (0,1)$

$$\begin{aligned}\gamma_1(t,h) &= \alpha t^{\alpha-1} h \int_{-M}^{h} (1 - u/t + \theta h/t)^{\alpha-1}\{g(h-u) - g(-u)\}du \\ &\sim \alpha t^{\alpha-1} h \int_{-M}^{h} \{g(h-u) - g(-u)\}du\end{aligned}$$

and, similarly,

$$\gamma_2(t,h) \sim \alpha^2 h^2 t^{2\alpha-1} \int_{-\infty}^{0} (1-v)^{\alpha-1}(-v)^{\alpha-1}dv = \alpha^2 h^2 t^{2\alpha-1} B(\alpha, 1-2\alpha),$$

where we have used the dominated convergence theorem. Hence, $\gamma(t,h) \sim Ct^{2\alpha-1}$ as $t \to \infty$ with $0 < \alpha < \frac{1}{2}$. $\square$

Example 2.7. (a) Let $g(x) = e^{-\lambda x}1_{\{x \geq 0\}}$. Then S is a Lévy OU process, whose properties are well-known. It has short memory increments, since (2.6) gives $\gamma(t,h) = e^{-\lambda t}\int_{-\infty}^{h}(e^{-\lambda(h-u)} - e^{-\lambda(-u)})^2 du$ for $t, h > 0$. Moreover, the sample paths of S exhibit jumps, since $g(0) \neq 0$.
(b) Let $g(x) = x^\alpha$ with $0 < \alpha < \frac{1}{2}$ for $x \geq M$ and some $M > 0$. Then the sample paths of S can

have jumps or not depending on the behavior of g in 0, while S has long memory increments by Lemma 2.6.
(c) Consider $g(x) = 1/(\alpha + \lambda x)^{\beta} 1_{\{x \geq 0\}}$ with parameters $\alpha, \lambda \geq 0$ and $\beta > -\frac{1}{2}$, $\beta \neq 0$ as on p. 635 of Gander and Stephens (2007), where they use this function g for stochastic volatility models driven by Lévy processes. Then the sample paths of S have jumps and S can exhibit short or long memory increments depending on β.

Remark 2.8. (a) Like a fractional Lévy process, the GFLP S has stationary increments and $S(0) = 0$ holds. Moreover, it inherits the symmetry from the driving Lévy process; i.e. $S(-t) \stackrel{d}{=} -S(t)$ for $t \geq 0$. A novelty of the GFLP is that the process class combines processes, which can have jumps without having independent increments, and without losing symmetry or its stationary increments. Moreover, while fractional Lévy processes exhibit always long memory behavior, the class of GFLPs can model both short and long memory.
(b) Continuity of $\int_0^t L(u)g'(t-u)du$ in S_1 and S_2 as proved in Lemma 2.4 also follows from the Kolmogorov-Čhentsov theorem.

3 Stochastic integrals with respect to a GFLP

Recall that the Riemann-Liouville fractional integrals $I^{\alpha}_{\pm}$ are defined for $\alpha \in (0,1)$ by

$$(I^{\alpha}_{-}h)(u) = \frac{1}{\Gamma(\alpha)} \int_u^{\infty} h(t)(t-u)^{\alpha-1} dt \quad \text{and} \quad (I^{\alpha}_{+}h)(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u} h(t)(u-t)^{\alpha-1} dt$$

for functions $h : \mathbb{R} \to \mathbb{R}$, provided that the integrals exist for almost all $u \in \mathbb{R}$. For details see e.g. Samko et al. (1993).

As a motivation for what follows note that for $t > 0$

$$g(t-u) - g(-u) = \int_{\mathbb{R}} \mathbf{1}_{(0,t]}(v) g'(v-u) dv \,.$$

We use now g' for an extension of the classical Riemann-Liouville kernel function and define for appropriate functions h

$$(I^{g}_{-}h)(u) := \int_u^{\infty} h(v)g'(v-u)dv = \int_{\mathbb{R}} h(v)g'(v-u)dv. \tag{3.1}$$

In what follows we assume that S is a GFLP driven by a Lévy process L with $E[L(1)] = 0$ and (w.l.o.g.) $E[(L(1))^2] = 1$. Starting from the fact that

$$S(t) = \int_{\mathbb{R}} (I^{g}_{-}\mathbf{1}_{(0,t]})(x) L(dx) \,, \quad t \in \mathbb{R} \,, \tag{3.2}$$

we shall define a stochastic integral for a function h in a similar way as in Marquardt (2006), Section 5. Since g' is continuous on $(0,\infty)$ by Assumption 2.3, the integral (3.2) is well defined as

$$(I^{g}_{-}\mathbf{1}_{(0,t]})(x) = -\int_t^0 g'(v-x)dv = g(t-x) - g(-x).$$

For a fixed function g as above define

$$\widetilde{\mathcal{H}} := \left\{ h : \mathbb{R} \to \mathbb{R}_+ : \int_{\mathbb{R}} (I_-^g h)^2(u) du < \infty \right\},$$

where $I_-^g h$ is as in (3.1). The proof of the following result is analogous to that of Proposition 5.1 of Marquardt (2006).

Proposition 3.1. *Suppose that g satisfies Assumption 2.3 and for its derivative g' $\int_0^1 |g'(s)|ds + \int_1^\infty (g'(s))^2 ds < \infty$ holds. If $h : \mathbb{R} \to \mathbb{R}_+$ satisfies $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $h \in \widetilde{\mathcal{H}}$.*

Proof. Starting from the fact that $I_-^g h \in L^2(\mathbb{R})$ if and only if $|\int_{\mathbb{R}} \varphi(u)(I_-^g h)(u)du| \le C\|\varphi\|_{L^2(\mathbb{R})}$ for all $\varphi \in L^2(\mathbb{R})$ for some $C > 0$, it suffices to show that for all $\varphi \in L^2(\mathbb{R})$

$$\int_{\mathbb{R}} \int_0^\infty |\varphi(u) g'(s) h(s+u)| ds du \le C \|\varphi\|_{L^2(\mathbb{R})} .$$

This holds, if $I_1 = \int_{\mathbb{R}} \int_0^1 |\varphi(u)g'(s)h(s+u)|dsdu \le C\|\varphi\|_{L^2(\mathbb{R})}$ and $I_2 = \int_{\mathbb{R}} \int_1^\infty |\varphi(u)g'(s)h(s+u)|dsdu \le C\|\varphi\|_{L^2(\mathbb{R})}$. Applying Fubini's theorem and the Hölder inequality we obtain

$$I_1 = \int_0^1 |g'(s)| \int_{\mathbb{R}} |\varphi(u)h(s+u)| du ds \le \|\varphi\|_{L^2(\mathbb{R})} \|h\|_{L^2(\mathbb{R})} \int_0^1 |g'(s)| ds < \infty .$$

Furthermore, setting $t = s + u$ and using Fubini's theorem and the Hölder inequality,

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} |h(t)| \int_1^\infty |\varphi(t-s) g'(s)| ds dt \le \int_{\mathbb{R}} \|\varphi\|_{L^2(\mathbb{R})} \Big(\int_1^\infty (g'(s))^2 ds \Big)^{1/2} |h(t)| dt \\ &\le \|\varphi\|_{L^2(\mathbb{R})} \|h\|_{L^1(\mathbb{R})} \Big(\int_1^\infty (g'(s))^2 ds \Big)^{1/2} < \infty . \end{aligned}$$

□

We define the space $\mathcal{H}$ as the completion of all functions $h : \mathbb{R} \to \mathbb{R}_+$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm

$$\|h\|_{\mathcal{H}} := \Big(E[(L(1))^2] \int_{\mathbb{R}} (I_-^g h)^2(u) du \Big)^{1/2} .$$

We shall need an additional condition on g:

Assumption 3.2. *In addition to Assumption 2.3, assume that g is monotone on $(0, \infty)$; i.e. $g' > 0$ or $g' < 0$ on $(0, \infty)$. We call g' a* kernel function.

Assumption 3.2 implies that the sign of $g' \cdot h$ is fixed on the whole of $\mathbb{R}$ and, thus, $\|\cdot\|_{\mathcal{H}}$ defines in fact a norm. For more details on such spaces for the classical Riemann-Liouville kernel we refer to Pipiras and Taqqu (2000).

From the proof of Proposition 3.1 follows immediately that for $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

$$\|h\|_{\mathcal{H}} \le C(\|h\|_{L^1(\mathbb{R})} + \|h\|_{L^2(\mathbb{R})}) .$$

Next we define the stochastic integral with integrator S, which gives the correspondence between the space $\mathcal{H}$ and that of stochastic integrals in $L^2(\Omega)$. Note that we should distinguish the functional space $\mathcal{H}$ of h from the space $\widetilde{\mathcal{H}}$ of $I_-^g h$.

Theorem 3.3. *Suppose that g satisfies Assumption 3.2. Let S be a GFLP and $h \in \mathcal{H}$. Then the left-hand side integral is defined in the $L^2(\Omega)$ sense and it holds that*

$$\int_{\mathbb{R}} h(u)dS(u) = \int_{\mathbb{R}} (I^g_- h)(u)dL(u)\,. \tag{3.3}$$

Moreover, the following isometry holds:

$$\| \int_{\mathbb{R}} h(u)dS(u)\|^2_{L^2(\Omega)} = \|h\|^2_{\mathcal{H}}\,.$$

Proof. To construct the integral $\int_{\mathbb{R}} h(t)dS(t)$ for $h \in \mathcal{H}$ we proceed as usual. For the indicator function $\varphi(\cdot) = \mathbf{1}_{(0,t]}(\cdot)$ for $t > 0$ we calculate

$$\int_{\mathbb{R}} \varphi(u)dS(u) = \int_{\mathbb{R}} \mathbf{1}_{(0,t]}(u)dS(u) = S(t)\,,$$

and for the right-hand side of (3.3) we obtain

$$\int_{\mathbb{R}} (I^g_- \varphi)(u)dL(u) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{(0,t]}(s)g'(s-u)dsdL(u) = \int_{\mathbb{R}} (g(t-u) - g(-u))dL(u) = S(t)$$

Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a step function; i.e. $\varphi(t) = \sum_{i=1}^{n-1} a_i \mathbf{1}_{(t_i,t_{i+1}]}(t)$, where $a_i \in \mathbb{R}_+$ for $i = 1,\dots,n-1$ and $-\infty < t_1 < \cdots < t_n < \infty$. Notice that $\varphi \in \mathcal{H}$. Define

$$\int_{\mathbb{R}} \varphi(t)dS(t) = \sum_{i=1}^{n-1} a_i(S(t_{i+1}) - S(t_i))\,,$$

then the right-hand side of (3.3) is

$$\begin{aligned}\int_{\mathbb{R}} (I^g_- \varphi)(u)dL(u) &= \int\int \sum_{j=1}^{n-1} a_j \mathbf{1}_{(t_j,t_{j+1}]}(s)g'(s-u)dsdL(u)\\ &= \int \sum_{j=1}^{n-1} a_j \int_{t_j}^{t_{j+1}} g'(s-u)dsdL(u)\\ &= \sum_{j=1}^{n-1} a_j(S(t_{j+1}) - S(t_j))\,.\end{aligned}$$

Moreover, for all step functions φ it follows from (2.2)

$$\| \int_{\mathbb{R}} \varphi(u)dS(u)\|^2_{L^2(\Omega)} = E\Big[\Big(\int_{\mathbb{R}} (I^g_-\varphi)(u)dL(u)\Big)^2\Big] = E[(L(1))^2]\int_{\mathbb{R}} (I^g_-\varphi)^2(u)du = \|\varphi\|^2_{\mathcal{H}}. \tag{3.4}$$

Since the non-negative step functions are dense in $\mathcal{H}$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of such functions such that $\|\varphi_k - h\|_{\mathcal{H}} \to 0$ as $k \to \infty$. It follows from the isometry property (3.4) that the integrals converge in $L^2(\Omega)$ and the isometry property is preserved in this procedure. Finally, (3.4) implies that the integral $\int_{\mathbb{R}} h(t)dS(t)$ is the same for all sequences of step functions converging to h. $\square$

The second order properties of integrals, which are driven by GFLPs follow by direct calculation. It is useful to observe that $L^2(\Omega)$ and $\mathcal{H}$ are inner product spaces with the inner products given for $h_1, h_2 \in \mathcal{H}$ by

$$\Big\langle \int_{\mathbb{R}} h_1(u)dS(u), \int_{\mathbb{R}} h_2(u)dS(u) \Big\rangle_{L^2(\Omega)} = \Big\langle h_1, h_2 \Big\rangle_{\mathcal{H}} .$$

The inner product in $L^2(\Omega)$ is the covariance, whereas an interpretation of the inner product in $\mathcal{H}$ can be found in the next Proposition.

Proposition 3.4. *Let S be a GFLP with kernel function g' satisfying Assumption 3.2 and let $h_1, h_2 \in \mathcal{H}$. Then*

$$\mathrm{Cov}\Big[\int_{\mathbb{R}} h_1(v)dS(v) \, , \int_{\mathbb{R}} h_2(u)dS(u) \Big] \;\; = \;\; \int_{\mathbb{R}} \int_{\mathbb{R}} h_1(u) h_2(v) \Gamma(u,v) du dv \, ,$$

where

$$\Gamma(u,v) = \frac{\partial^2 \mathrm{Cov}[S(u), S(v)]}{\partial u \partial v} = E[(L(1))^2] \int_{\mathbb{R}} g'(u-w) g'(v-w) dw \, . \tag{3.5}$$

In particular,

$$\Big\langle h_1, h_2 \Big\rangle_{\mathcal{H}} = E[(L(1))^2] \int_{\mathbb{R}} \int_{\mathbb{R}} h_1(u) h_2(v) \int_{\mathbb{R}} g'(u-w) g'(v-w) dw du dv \, .$$

Proof. Set w.l.o.g. $E[(L(1))^2] = 1$. It suffices to prove the identities for the indicator functions $h_1 = \mathbf{1}_{(0,s]}$ and $h_2 = \mathbf{1}_{(0,t]}$ for $0 < s < t$. For $s < 0$ or $t < 0$ we use the stationarity of the increments and the symmetry of S.

$$\begin{aligned}
\mathrm{Var}[S(t)] &= \|S(t)\|^2_{L^2(\Omega)} = \int (g(t-u) - g(-u))^2 du \\
&= \int_{\mathbb{R}} \Big(\int_u^\infty \mathbf{1}_{(0,t]}(v) g'(v-u) dv \Big)^2 du = \|\mathbf{1}_{(0,t]}\|^2_{\mathcal{H}} \, , \\
\mathrm{Cov}[S(s), S(t)] &= \langle S(s), S(t) \rangle_{L^2(\Omega)} \\
&= \int_{\mathbb{R}} \{g(s-w) - g(-w)\}\{g(t-w) - g(-w)\} dw \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{(0,s]}(u) \mathbf{1}_{(0,t]}(v) \int_{\mathbb{R}} g'(v-w) g'(u-w) dw du dv \\
&= \big\langle \mathbf{1}_{(0,s]}, \mathbf{1}_{(0,t]} \big\rangle_{\mathcal{H}} \, ,
\end{aligned}$$

where we have used Fubini's theorem for the second last identity, which is justified by the definition of $\mathcal{H}$. $\square$

Remark 3.5. Assumption 3.2 is needed to formulate the isometry between the space of stochastic integrals with respect to S, i.e. $L^2(\Omega)$, and the functional space of integrands $\mathcal{H}$, which depends on g. However, for defining the stochastic integral with integrator S, the weaker space $\widetilde{\mathcal{H}}$ suffices.

Next we define the OU process driven by a GFLP.

Definition 3.6. *Let S be a GFLP such that it satisfies Assumption 3.2, and let $\lambda, \gamma > 0$.*
(i) *For an initial finite random variable $V(0)$ we define an* OU process driven by a GFLP *as*

$$V(t) := e^{-\lambda t}\Big(V(0) + \gamma \int_0^t e^{\lambda u} dS(u)\Big), \quad t \in \mathbb{R}\,. \tag{3.6}$$

(ii) *If the initial random variable is given by $V(0) = \gamma \int_{-\infty}^0 e^{\lambda u} dS(u)$, the OU process driven by a GFLP is stationary and we denote its stationary version by*

$$\overline{V}(t) = \gamma \int_{-\infty}^t e^{-\lambda(t-u)} dS(u)\,, \quad t \in \mathbb{R}\,. \tag{3.7}$$

(iii) *Recall that, when S is replaced by FBM B^H for $H \in (1/2, 1)$ in* (3.6) *and* (3.7)*, we obtain the* fractional (Gaussian) Ornstein Uhlenbeck (FOU) process*; cf. (Cheridito et al., 2003, Lemma 2.1) or Pipiras and Taqqu (2000). We denote the stationary FOU by $\overline{Y} = \{\overline{Y}(t)\}_{t\in\mathbb{R}}$. It will appear as limit process in Section 4.*

We show the existence of $\overline{V}$ and formulate some properties.

Proposition 3.7. *Let S be a GFLP such that it satisfies Assumption 3.2 and let $\lambda > 0$ and set w.l.o.g. $\gamma = 1$. For all $t \in \mathbb{R}$ the stochastic integral*

$$\overline{V}(t) := \int_{-\infty}^t e^{-\lambda(t-u)} dS(u) = \int_{-\infty}^t (I_-^g e^{-\lambda(t-\cdot)})(u) dL(u)$$

exists in the $L^2(\Omega)$ sense. Furthermore, for all $s, t \in \mathbb{R}$ we have $E[\overline{V}(t)] = 0$ and

$$\mathrm{Cov}[\overline{V}(s), \overline{V}(t)] = \int_{-\infty}^t \int_{-\infty}^s e^{-\lambda(t-u)} e^{-\lambda(s-v)} \Gamma(u,v) du dv\,,$$

where Γ is given in (3.5)*. Moreover, the ch.f. of $\overline{V}(t_1), \ldots, \overline{V}(t_m)$ for $t_1 < \cdots < t_m$ is given by*

$$E\left[\exp\left\{\sum_{j=1}^m i\theta_j \overline{V}(t_j)\right\}\right] = \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^m \theta_j \int_{-\infty}^{t_j} e^{-\lambda(t_j - v)} g'(v-s) dv\right) ds\right\}\,,$$

where $\theta_j \in \mathbb{R}$, $j = 1, \ldots, m$, and ψ is given in (2.1)*.*

Proof. By Theorem 3.3 and Proposition 3.4 the existence of the integral and the autocovariance function is a consequence of the fact that $e^{-\lambda(t-\cdot)} 1_{\{t \ge \cdot\}} \in \mathcal{H}$. The ch.f. follows for $t \in \mathbb{R}$ from Proposition 2.1 by observing that $f_t(s)$ is replaced by

$$h_t(s) = \int_{-\infty}^t e^{-\lambda(t-v)} g'(v-s) dv\,, \quad s \in \mathbb{R}\,.$$

□

4 Limit theory for OU processes driven by time scaled GFLPs

Throughout this section we assume that $E[(L(1))^2] = 1$. Moreover, we work under Assumption 3.2 so that Theorem 3.3 and Proposition 3.4 apply. For $x > 0$ we denote $\sigma^2(x) := \mathrm{Var}[S(x)]$ and define the *time scaled GFLP* $S_x = \{S_x(t)\}_{t\in\mathbb{R}}$ by

$$S_x(t) := \frac{S(xt)}{\sigma(x)}, \quad t \in \mathbb{R}. \tag{4.1}$$

Recall the definition of Γ from (3.5) and of $\widetilde{\Gamma}$ from (2.3). Note that the expression (3.2) carries over to the time scaled GFLP as follows. For $x > 0$ we have

$$S(xt) = \int \mathbf{1}_{(0,tx]}(v)dS(v) = \int_{\mathbb{R}} (I^g_- \mathbf{1}_{(0,tx]})(u)dL(u)\,, \quad t \geq 0\,.$$

Consequently, we can formulate the following Lemma.

Lemma 4.1. *For $x > 0$ let S_x be the time scaled GFLP* (4.1) *and assume that Assumption 3.2 holds. Then for $s, t \in \mathbb{R}$ we have*

$$\begin{aligned}
\widetilde{\Gamma}_x(s,t) &:= \mathrm{Cov}[S_x(s), S_x(t)] = \frac{\mathrm{Cov}[S(xs), S(xt)]}{\mathrm{Var}[S(x)]} = \frac{\widetilde{\Gamma}(xs,xt)}{\sigma^2(x)} = \frac{\langle \mathbf{1}_{(0,xs]}, \mathbf{1}_{(0,xt]}\rangle_{\mathcal{H}}}{\|\mathbf{1}_{(0,x]}\|^2_{\mathcal{H}}}, \\
\widetilde{\Gamma}_x(t,t) &:= \mathrm{Var}[S_x(t)] = \frac{\|\mathbf{1}_{(0,xt]}\|^2_{\mathcal{H}}}{\|\mathbf{1}_{(0,x]}\|^2_{\mathcal{H}}}, \\
\Gamma_x(s,t) &= \frac{\partial^2}{\partial s \partial t}\mathrm{Cov}[S_x(s), S_x(t)] = \frac{1}{\sigma^2(x)}\frac{\partial^2}{\partial s\partial t}\mathrm{Cov}[S(xs), S(xt)] = \frac{x^2\Gamma(xs,xt)}{\sigma^2(x)}\,.
\end{aligned} \tag{4.2}$$

Proof. We prove the variance formula (4.2) for $t > 0$, the other formulas are proved analogously. For $s, t > 0$ we have (for $s < 0$ or $t < 0$ we use the symmetry of S_x)

$$\|\int_{\mathbb{R}} \mathbf{1}_{(0,t]}(u)dS_x(u)\|^2_{L^2(\Omega)} = \|S_x(t)\|^2_{L^2(\Omega)} = \frac{\|S(xt)\|^2_{L^2(\Omega)}}{\|S(x)\|^2_{L^2(\Omega)}} = \frac{\|\mathbf{1}_{(0,tx]}\|^2_{\mathcal{H}}}{\|\mathbf{1}_{(0,x]}\|^2_{\mathcal{H}}}\,.$$

□

Lemma 4.1 provides a general principle by using the same construction of the integral as in Theorem 3.3.

Theorem 4.2. *For $x > 0$ let S_x be the time scaled GFLP* (4.1) *and suppose that Assumption 3.2 holds.*
(i) *Then for $h \in \mathcal{H}$,*

$$\int_{\mathbb{R}} h(u)dS_x(u) = \int_{\mathbb{R}} h^x(u)dL(u)\,, \tag{4.3}$$

in the $L^2(\Omega)$ sense, where

$$h^x(u) = \frac{x}{\sigma(x)}\int_{\mathbb{R}} h(v)g'((xv-u)_+)dv\,. \tag{4.4}$$

(ii) *Assume that* $h_s, h_t \in \mathcal{H}$ *for* $s, t \in \mathbb{R}$. *Then*

$$\operatorname{Cov}\left(\int h_s(u)dS_x(u), \int h_t(u)dS_x(u)\right) = \int\int h_t(u)h_s(v)\Gamma_x(u,v)dudv\,,$$

where

$$\Gamma_x(u,v) = \frac{x^2\Gamma(xs,xt)}{\sigma^2(x)} = \frac{x^2}{\sigma^2(x)}\int g'((ux-w)_+)g'((vx-w)_+)dw\,.$$

(iii) *Defining* h_t^x *as in* (4.4) *with* h *replaced by* h_t, *the ch.f. of* $\int h_{t_1}(u)dS_x(u), \dots, \int h_{t_m}(u)dS_x(u)$ *for* $t_1 < \cdots < t_m$ *is given by*

$$E\left[\exp\left\{i\sum_{j=1}^m \theta_j \int h_{t_j}(u)dS_x(u)\right\}\right] = \exp\left\{\int \psi\left(\sum_{j=1}^m \theta_j h_{t_j}^x(u)\right)du\right\},$$

where $\theta_j \in \mathbb{R}$, $j = 1, \dots, m$, *and* ψ *is as in* (2.1).

Proof. To prove (4.3) it suffices to take an (interval)-indicator function as in the proof of Theorem 3.3 and we omit it. Part (ii) follows from Proposition 2.1. Finally, (iii) follows from the fact that

$$\left(\int h_{t_1}(u)dS_x(u), \dots, \int h_{t_m}(u)dS_x(u)\right) \stackrel{d}{=} \left(\int h_{t_1}^x(u)dL(u), \dots, \int h_{t_m}^x(u)dL(u)\right),$$

where $h_t^x(u) = \frac{x}{\sigma(x)}\int h_t(v)g'(xv-u)dv$ and $\stackrel{d}{=}$ denotes equality in distribution. $\square$

An important step in the proof of convergence of an OU process driven by a time scaled GFLP is the convergence of the covariance function and its second derivative. This requires that g' is *regularly varying*; i.e. for all $u > 0$

$$\lim_{x\to\infty}\frac{g'(xu)}{g'(x)} = u^{\rho-1} \tag{4.5}$$

for $\rho \in (0, \frac{1}{2})$, and we write $g' \in \mathrm{RV}_{\rho-1}$. Such properties have also been used in Klüppelberg and Mikosch (1995) and Klüppelberg and Kühn (2004) to prove convergence of scaled shot-noise processes to self-similar Gaussian processes, in particular, to FBM. Condition (4.5) on g implies in particular that $\operatorname{Cov}[S(s), S(t)]$ is bivariate regularly varying with index $1 + 2\rho$ and, hence, that $\sigma^2 \in \mathrm{RV}_{1+2\rho}$. For more details on regular variation we refer to Bingham et al. (1987). The following result exploits these properties.

Theorem 4.3. *Let* $\rho \in (0, \frac{1}{2})$ *and* $g' \in RV_{\rho-1}$. *Define* $H := \rho + \frac{1}{2}$. *Then for each* $s, t \in \mathbb{R}$,

$$\begin{aligned}\lim_{x\to\infty}\Gamma_x(s,t) &= \lim_{x\to\infty}\frac{x^2\int_{-\infty}^{s\wedge t} g'(x(s-w)_+)g'(x(t-w)_+)dw}{\sigma^2(x)} \\ &= \frac{\partial^2}{\partial t\partial s}\operatorname{Cov}(B^{\rho+1/2}(s), B^{\rho+1/2}(t)) = H(2H-1)|t-s|^{2H-2}. \end{aligned} \tag{4.6}$$

$$\lim_{x\to\infty}\widetilde{\Gamma}_x(s,t) = \operatorname{Cov}(B^{\rho+1/2}(s), B^{\rho+1/2}(t)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{4.7}$$

Proof. We use the second moment expressions from Theorem 3.4. To prove (4.6) write

$$\Gamma_x(s,t) = \left(\frac{xg'(x)}{g(x)}\right)^2 \frac{\int_{-\infty}^{s\wedge t} g'(x(s-w)_+)g'(x(t-w)_+)/(g'(x))^2 dw}{\int_{-\infty}^{1}\{g(x(1-v)_+) - g(x(-v)_+)\}^2/g^2(x)dv}. \tag{4.8}$$

Then by Karamata's theorem (cf. Theorem 1.5.11 of Bingham et al. (1987)),

$$\lim_{x\to\infty} \frac{xg'(x)}{g(x)} = \rho$$

and $g \in \mathrm{RV}_\rho$. We first show convergence of the numerator of the factor in (4.8) by deriving bounds in the spirit of Potter (cf. Bingham et al. (1987), Theorem 1.5.6). For $0 < \varepsilon < (1/2-\rho)\wedge\rho$ we have $x^{1-\varepsilon}g'(x) \in RV_{\rho-\varepsilon}$ and $\rho-\varepsilon \in (0,1/2)$. Hence, for every $\delta > 0$ there exists some x_0 such that for all $x \geq x_0$ and $|s-w| \leq M$ for some $M > 0$,

$$\left|\frac{g'(x(s-w))}{g'(x)}\right| = \frac{(x(s-w)_+)^{1-\varepsilon}g'(x(s-w)_+)}{(s-w)_+^{1-\varepsilon}x^{1-\varepsilon}g'(x)} \leq \frac{\delta + (s-w)_+^{\rho-\varepsilon}}{(s-w)_+^{1-\varepsilon}} \leq c_M(s-w)_+^{\varepsilon-1},$$

where $c_M > 0$ is some constant, depending on M. On the other hand, for $|s-w| > M$,

$$\left|\frac{g'(x(s-w)_+)}{g'(x)}\right| \leq (1+\varepsilon)(s-w)_+^{\rho-1+\varepsilon}$$

for sufficiently large x (cf. Propositions 0.5 & 0.8 of Resnick (1987)).
If we choose M appropriately, it follows that

$$\begin{aligned}
&\int_{-\infty}^{s\wedge t} \frac{g'(x(s-w)_+)g'(x(t-w)_+)}{(g'(x))^2}dw \\
\leq\ & (1+\varepsilon)^2\int_{-\infty}^{s-M}(s-w)_+^{\rho-1+\varepsilon}(t-w)_+^{\rho-1+\varepsilon}dw + c_M^2\int_{s-M}^{s\wedge t}(s-w)_+^{-1+\varepsilon}(t-w)_+^{-1+\varepsilon}dw.
\end{aligned}$$

Now we apply Lebesgue's dominated convergence theorem to the numerator of (4.8) and obtain convergence of this numerator to that of (4.6). As for the denominator of (4.6), its convergence follows (as also the convergence of $\widetilde{\Gamma}_x$ in (4.7)) by a dominated convergence argument as in the proof of Theorem 3.2 of Klüppelberg and Kühn (2004). □

Since S_x is a time changed version of S, $E[S_x(t)] = 0$ and $\mathrm{Var}[S_x(t)] = \sigma^2(xt)/\sigma^2(x)$ hold for all $t \in \mathbb{R}$. Hence, we can define the following time scaled version of V.

Definition 4.4. *For $x > 0$ let S_x be the time scaled GFLP* (4.1) *and suppose that Assumption 3.2 holds.*
(i) *For $\lambda, \gamma > 0$ we define the* OU process $V_x = \{V_x(t)\}_{t\in\mathbb{R}}$ driven by the time scaled GFLP S_x *by*

$$V_x(t) := e^{-\lambda t}\left(V_x(0) + \gamma\int_0^t e^{\lambda u}dS_x(u)\right), \quad t \geq 0.$$

(ii) *If the initial random variable is given by $V_x(0) = \gamma\int_{-\infty}^0 e^{\lambda u}dS_x(u)$, then V_x is stationary and we denote the stationary process by*

$$\overline{V}_x(t) := \gamma\int_{-\infty}^t e^{-\lambda(t-u)}dS_x(u), \quad t \in \mathbb{R}. \tag{4.9}$$

The following is a consequence of Theorem 4.2 and Proposition 2.1. We have set again $\gamma = 1$ for simplicity.

Proposition 4.5. *For $x > 0$ let S_x be the time scaled GFLP* (4.1) *and suppose that Assumption 3.2 holds.*
(i) *For $t \in \mathbb{R}$*

$$\overline{V}_x(t) = \int_{-\infty}^{t} e^{-\lambda(t-u)} dS_x(u) = \frac{x}{\sigma(x)} \int_{\mathbb{R}} \int_{-\infty}^{t} e^{-\lambda(t-v)} g'(xv-u) dv dL(u).$$

(ii) *For $s, t \in \mathbb{R}$, we have $E[\overline{V}_x(t)] = 0$ and*

$$\mathrm{Cov}[\overline{V}_x(s), \overline{V}_x(t)] = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\lambda(t-u)} \mathbf{1}_{(-\infty,t]}(u) e^{-\lambda(s-v)} \mathbf{1}_{(-\infty,s]}(v) \Gamma_x(u,v) du dv, \tag{4.10}$$

where

$$\Gamma_x(u,v) = \frac{x^2}{\sigma^2(x)} \int_{\mathbb{R}} g'(xu-w) g'(xv-w) dw.$$

(iii) *The ch.f. of $\overline{V}_x(t_1), \overline{V}_x(t_2), \ldots, \overline{V}_x(t_m)$ for $t_1 < t_2 < \cdots < t_m$ is given by*

$$E\left[\exp\left\{ i \sum_{j=1}^{m} \theta_j \overline{V}_x(t_j) \right\}\right] = \exp\left\{ \int_{\mathbb{R}} \psi\left(\sum_{j=1}^{m} \theta_j \frac{x}{\sigma(x)} \int_{-\infty}^{t_j} e^{-\lambda(t_j-v)} g'(xv-u) dv \right) du \right\},$$

where $\theta_j \in \mathbb{R}$ for $j = 1, \ldots, m$ and ψ is given in (2.1).

By extending earlier work of Lane (1984), who proved a CLT for the Poisson shot noise process, it was shown in Theorem 3.2 of Klüppelberg and Kühn (2004) that, if the driving Lévy process is compound Poisson, then the GFLP S_x converges weakly to B^H in the Skorokhod space $D(\mathbb{R}_+)$ equipped with the metric of uniform convergence on compacts. Since the limit process has continuous sample path, by Theorem 6.6 of Billingsley (1999) we can equivalently consider weak convergence with respect to the Skorohod d_∞^0-metric on $D(\mathbb{R}_+)$, which induces the J_1 topology. For a definition of d_∞^0 see e.g. (16.4) in Billingsley (1999). According to his Theorems 16.7 and 13.1 we have to show weak convergence of the finite dimensional distributions and tightness of $(\overline{V}_\cdot(t)|_{[0,T]})_{t\in\mathbb{R}}$ for every $T > 0$.

We extend this result two-fold. Firstly, we generalize the driving compound Poisson process to a Lévy process and, secondly, we consider the convergence of stochastic volatility models driven by a GFLP in Section 5.

Theorem 4.6. *For $x > 0$ let $\overline{V}_x$ be the stationary OU process driven by a time scaled GFLP as defined in (4.9). Let $\overline{Y}$ be the stationary FOU process from Definition 3.6(iii) with $H \in (1/2, 1)$. Then*

$$\overline{V}_x \xrightarrow{d} \overline{Y} \quad \text{as } x \to \infty,$$

where convergence holds in the Skorokhod space $D(\mathbb{R}_+)$ equipped with the metric which induces the Skorokhod J_1 topology.

Proof. We start proving convergence of the finite dimensional distributions. Let $0 = t_1 < t_2 < \cdots < t_m < T$ and $\theta_j \in \mathbb{R}$ for $j = 1, \ldots, m$. Recall from Proposition 4.5 (iii) the ch.f. of $\overline{V}_x(t)$:

$$\begin{aligned} E\Big[\exp\Big\{i\sum_{j=1}^m \theta_j \overline{V}_x(t_j)\Big\}\Big] &= \exp\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}} \phi\big(y\sum_{j=1}^m \theta_j h^x_{t_j}(u)\big)\nu(dy)du\Big\} \\ &= \exp\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}} x\phi\big(y\sum_{j=1}^m \theta_j h^x_{t_j}(xu)\big)\nu(dy)du\Big\}, \end{aligned} \tag{4.11}$$

where $\phi(x) = e^{ix} - 1 - ix$, and we set

$$h^x_t(s) := \frac{x}{\sigma(x)}\int_{-\infty}^t e^{-\lambda(t-v)} g'(xv - s)dv.$$

The outline of our prove is that we apply a Taylor expansion (Lemma 3.2 of Petrov (1995)) to $x\phi(\cdot)$ in (4.11) and observe that

$$x\phi\big(y\sum_{j=1}^m \theta_j h^x_{t_j}(xw)\big) \;\sim\; -\frac{y^2}{2}x\big(\sum_{j=1}^m \theta_j h^x_{t_j}(xw)\big)^2 \quad \text{as } x \to \infty. \tag{4.12}$$

Then, since $\int_{\mathbb{R}} y^2\nu(dy) = E[(L(1))^2] = 1$, we will show that

$$\int_{\mathbb{R}} x\big(\sum_{j=1}^m \theta_j h^x_{t_j}(xw)\big)^2 dw \;\to\; \sum_{j,k}^m \theta_j\theta_k \mathrm{Cov}(\overline{Y}(t_j)\overline{Y}(t_k)) \quad \text{as } x \to \infty, \tag{4.13}$$

which implies that the finite dimensional distributions convergence to the corresponding Gaussian process.

Firstly, in view of (4.10) and Theorem 4.3 we prove that for $s, t \geq 0$,

$$\lim_{x\to\infty}\int x h^x_s(xw)h^x_t(xw)dw = \lim_{x\to\infty}\mathrm{Cov}(\overline{V}_x(s), \overline{V}_x(t)) = \mathrm{Cov}(\overline{Y}(s), \overline{Y}(t)), \tag{4.14}$$

which is the key of the proof. In view of (4.10), since $\Gamma_x(u, v)$ should converge to the unbounded function $|u - v|^{2H-2}$ for $H = \rho + \frac{1}{2}$, there is some difficulty to apply the dominated convergence theorem directly; i.e., to find a dominant function. Alternatively, we work with the following representation, which is obtained from integration by parts:

$$\begin{aligned} h^x_t(xw) &= \frac{g(x(t-w))}{\sigma(x)} - \lambda\int_{-\infty}^t e^{-\lambda(t-v)}\frac{g(x(v-w))}{\sigma(x)}dv \\ &= \frac{f_{tx}(xw)}{\sigma(x)} - \lambda\int_{-\infty}^t e^{-\lambda(t-v)}\frac{f_{xv}(xw)}{\sigma(x)}dv, \end{aligned}$$

where we have set $f_t(w) = g(t - w) - g(-w)$. Now we apply dominated convergence to each term of the following representation

$$\begin{aligned} \int x h^x_s(xw)h^x_t(xw)dw = \widetilde{\Gamma}_x(s,t) &- \lambda\int_{-\infty}^s e^{-\lambda(s-u)}\int_{\mathbb{R}}\frac{x f_{xu}(xw)f_{xt}(xw)}{\sigma^2(x)}dwdu \\ &- \lambda\int_{-\infty}^t e^{-\lambda(t-u)}\int_{\mathbb{R}}\frac{x f_{xu}(xw)f_{xs}(xw)}{\sigma^2(x)}dwdu \end{aligned}$$

$$+\lambda^2\int_{-\infty}^{s}e^{-\lambda(s-u)}\int_{-\infty}^{t}e^{-\lambda(t-v)}\int_{\mathbb{R}}\frac{xf_{xu}(xw)f_{xv}(xw)}{\sigma^2(x)}dwdudv. \quad (4.15)$$

From Theorem 4.3 we find

$$\lim_{x\to\infty}\widetilde{\Gamma}_x(s,t)=\mathrm{Cov}(B^{\rho+1/2}(s),B^{\rho+1/2}(t)).$$

For the remaining terms, we only consider the third integral, since convergence of other integrals can be proved similarly. By the Cauchy-Schwarz inequality the integrand in the third integral is dominated by

$$e^{-\lambda(s-u)-\lambda(t-v)}\sqrt{\frac{\sigma^2(xu)}{\sigma^2(x)}\frac{\sigma^2(xv)}{\sigma^2(x)}}.$$

We give a uniform upper bound for $\sigma^2(ux)/\sigma^2(x)$. Since $\sigma^2\in\mathrm{RV}_{1+2\rho}$, for sufficiently small $\delta>0$, the function $\gamma(x):=\sigma^2(x)|x|^{-1-2\rho-\delta}$ for $x>0$ is regularly varying with index $-\delta$ and $\gamma(ux)/\gamma(x)$ converges to $|u|^{-\delta}$ uniformly in $|u|\in[1,\infty)$ as $x\to\infty$ (cf. Theorem 1.5.2 of Bingham et al. (1987)). Hence, we have

$$\frac{\sigma^2(ux)}{\sigma^2(x)}=|u|^{1+2\rho+\delta}\frac{\gamma(ux)}{\gamma(x)}\le|u|^{1+2\rho+\delta}(1+|u|^{-\delta})\le 2|u|^{1+2\rho+\delta},\quad |u|\in[1,\infty) \quad (4.16)$$

for sufficiently large x. Furthermore by Karamata's theorem $\sigma^2(ux)/\sigma^2(x)$ converges to $|u|^{1+2\rho}$ uniformly in $|u|\in(0,1]$, and this together with (4.16) implies

$$\frac{\sigma^2(ux)}{\sigma^2(x)}\le(c+c'|u|^{1+2\rho})1_{\{|u|\le 1\}}+2|u|^{1+2\rho+\delta}1_{\{|u|>1\}}, \quad (4.17)$$

for some $c,c'>0$. Thus the dominating function is uniformly integrable and the integral converges; i.e.,

$$\lim_{x\to\infty}\int_{-\infty}^{s}\int_{-\infty}^{t}\lambda^2e^{-\lambda(s-u)-\lambda(t-v)}\sqrt{\frac{\sigma^2(xu)}{\sigma^2(x)}\frac{\sigma^2(xv)}{\sigma^2(x)}}dudv<\infty.$$

Now we apply a generalized dominated convergence theorem (e.g. Theorem 1.21 of Kallenberg (1997)) to (4.15) and obtain for the third integral of (4.15) in the limit

$$\lambda^2\int_{-\infty}^{s}\int_{-\infty}^{t}e^{-\lambda(s-u)-\lambda(t-v)}\frac{1}{2}\Big(|u|^{2\rho+1}+|v|^{2\rho+1}-|u-v|^{2\rho+1}\Big)dudv.$$

Hence with (4.15) we conclude

$$\begin{aligned}\lim_{x\to\infty}\mathrm{Cov}(\overline{V}_x(s),\overline{V}_x(t)) &= \frac{1}{2}(|t|^{2\rho+1}+|s|^{2\rho+1}-|t-s|^{2\rho+1})\\ &\quad -\lambda\int_{-\infty}^{s}e^{-\lambda(s-u)}\frac{1}{2}(|t|^{2\rho+1}+|u|^{2\rho+1}-|t-u|^{2\rho+1})du\\ &\quad -\lambda\int_{-\infty}^{t}e^{-\lambda(t-u)}\frac{1}{2}(|s|^{2\rho+1}+|u|^{2\rho+1}-|s-u|^{2\rho+1})du\\ &\quad +\lambda^2\int_{-\infty}^{s}\int_{-\infty}^{t}e^{-\lambda(s-u)-\lambda(t-v)}\frac{1}{2}(|u|^{2\rho+1}+|v|^{2\rho+1}-|u-v|^{2\rho+1})dudv\end{aligned}$$

$$
\begin{aligned}
&= \rho(2\rho+1)\int_{-\infty}^{s} e^{-\lambda(s-u)}\int_{-\infty}^{t} e^{-\lambda(t-v)}|u-v|^{2\rho-1}dudv \\
&= \mathrm{Cov}(\overline{Y}(s),\overline{Y}(t)),
\end{aligned}
$$

which proves (4.14).

We turn to the proof of (4.12) and (4.13). From the representation

$$
\frac{f_{xt}(xw)}{\sigma(x)} = x^{-1/2}\frac{\{g(x(t-w))-g(x(-u))\}/g(x)}{(\int_{\mathbb{R}}(f_x(xu))^2du/g^2(x))^{1/2}}
$$

we observe that $f_{xt}(xw)/\sigma(x) = O(x^{-1/2})$ and hence $h_t^x(xw) = O(x^{-1/2})$. Then (4.12) is implied by a Taylor expansion: for sufficiently large x we have

$$
x\left|\phi\Big(y\sum_{j=1}^{m}\theta_j h_{t_j}^x(xw)\Big)+\frac{y^2}{2}\Big(\sum_{j=1}^{m}\theta_j(h_{t_j}^x(xw))\Big)^2\right| = \frac{y^3}{6}\Big(x^{1/3}\sum_{j=1}^{m}\theta_j h_{t_j}^x(xw)\Big)^3 = O(x^{-1/2}),
$$

and the right-hand side tends to 0 as $x\to\infty$. In the light of (4.14) and the same generalized dominated convergence theorem as above (e.g. Theorem 1.21 of Kallenberg (1997)), it suffices for the proof of (4.13) to show that the integral of a dominating function for $x\phi(y\sum_{j=1}^m\theta_j h_{t_j}^x(xw))$ converges as $x\to\infty$. We choose the dominating function by

$$
x\Big|\phi\Big(y\sum_{j=1}^{m}\theta_j h_{t_j}^x(xu)\Big)\Big| \le \frac{y^2}{2}x\Big|\sum_{j=1}^{m}\theta_j h_{t_j}^x(xu)\Big|^2 = a_x(u,y), \quad x\in\mathbb{R}.
$$

such that its integral can be estimated as

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} a_x(u,y)\nu(dy)du &\le \frac{x}{2}\int_{\mathbb{R}} y^2\nu(dy)\left\|\sum_{j=1}^{m}\theta_j h_{t_j}^x(x\cdot)\right\|_{L^2(\mathbb{R})}^2 \\
&\le \frac{x}{2}2^{m-1}\sum_{j=1}^{m}\left\|\theta_j h_{t_j}^x(x\cdot)\right\|_{L^2(\mathbb{R})}^2 \\
&= 2^{m-2}\sum_{j=1}^{m}\theta_j^2\int_{\mathbb{R}} x(h_t^x(xu))^2du, \qquad (4.18)
\end{aligned}
$$

where we use Minkowski's inequality and the fact $\int y^2\nu(dy) = 1$. Since the right-hand side converges as $x\to\infty$ by (4.14), we apply the generalized dominated convergence theorem to (4.11) and obtain

$$
\lim_{x\to\infty} E\left[\exp\left\{i\sum_{j=1}^{m}\theta_j\overline{V}_x(t_j)\right\}\right] = E\left[\exp\left\{i\sum_{j=1}^{m}\theta_j\overline{Y}(t_j)\right\}\right],
$$

which implies convergence of the finite dimensional distributions.

Next we prove tightness. For $0\le s<t<\infty$ choose $T>0$ such that $s,t\in[0,T]$. By equation (13.14) of Billingsley (1999) it suffices to show $E[(\overline{V}_x(t)-\overline{V}_x(s))^2]\le c_T(t-s)^{1+\rho}$ for some constant $c_T>0$, By stationarity of $\overline{V}_x$ we have

$$
\overline{V}_x(t)-\overline{V}_x(s) \stackrel{d}{=} \overline{V}_x(t-s)-\overline{V}_x(0) = \Big(e^{-\lambda(t-s)}-1\Big)\overline{V}_x(0)+\int_0^{t-s} e^{-\lambda(t-s-u)}dS_x(u).
$$

Applying Young's inequality gives

$$E\Big[\big(\overline{V}_x(t)-\overline{V}_x(s)\big)^2\Big] \le 2\left(e^{-\lambda(t-s)}-1\right)^2 E\left[(\overline{V}_x(0))^2\right] + 2E\Big[\Big(\int_0^{t-s} e^{-\lambda(t-s-u)}dS_x(u)\Big)^2\Big].$$

Since $|e^{-\lambda(t-s)}-1| \le c'_T(t-s)^{(1+\rho)/2}$ for $t>s$ and some constant $c'_T>0$, it suffices to show that

$$E\Big[\Big(\int_0^{t-s} e^{-\lambda(t-s-u)}dS_x(u)\Big)^2\Big] \le c''_T(t-s)^{1+\rho}$$

for some constant $c''_T>0$. Observe that by integration by parts as in (4.15),

$$\begin{aligned}
&E\Big[\Big(\int_0^{t-s} e^{-\lambda(t-s-u)}dS_x(u)\Big)^2\Big]\\
&= \int_{\mathbb{R}}\Big(\frac{x}{\sigma(x)}\int_0^{t-s} e^{-\lambda(t-s-w)}g'((xu-w)_+)du\Big)^2 dw\\
&\le 2\frac{\sigma^2((t-s)x)}{\sigma^2(x)} + 2\lambda^2 e^{-2\lambda(t-s)}\int_0^{t-s}\int_0^{t-s} e^{\lambda(u+v)}\sqrt{\frac{\sigma^2(xu)}{\sigma^2(x)}\frac{\sigma^2(xv)}{\sigma^2(x)}}dudv\\
&\le 2\frac{\sigma^2((t-s)x)}{\sigma^2(x)} + 2\lambda^2 c'''_T e^{-2\lambda(t-s)}\Big(\int_0^{t-s} e^{\lambda u}du\Big)^2\\
&= 2\frac{\sigma^2((t-s)x)}{\sigma^2(x)} + 2c'''_T(e^{-\lambda(t-s)}-1)^2
\end{aligned}$$

for some constant $c'''_T>0$. Since $\sigma^2\in RV_{1+2\rho}$, similarly as in the proof of Theorem 3.2 (p. 349) of Klüppelberg and Kühn (2004), the bounded function $\eta(x):=\sigma^2(x)/x^{1+\rho}$ is regularly varying with index ρ; i.e., $\eta(x(t-s))/\eta(x)$ converges to $(t-s)^\rho$ as $x\to\infty$ uniformly in $t>s$ on compact subsets of $\mathbb{R}_+$. This implies that for each $M>0$ and $x\ge x_M$ for some x_M

$$\frac{\sigma^2(x(t-s))}{\sigma^2(x)} \quad\le\quad (T^\rho+1)(t-s)^{1+\rho}.$$

This (together with the Cauchy-Schwarz inequality) implies the tightness condition (13.14) of Billingsley (1999), which gives our result. □

Remark 4.7. The proof of convergence of the finite dimensional distributions resembles that of Theorem 1 in Pipiras and Taqqu (2008). However, the ch.f. (4.11) of our stochastic integrals is more complicated than the one on p. 303, line 3 of that paper, as we have to deal with an additional integral with respect to Lebesgue measure.

5 Limits of stochastic volatility models

We propose a flexible model class for stochastic volatility (SV) models: the data are driven by BM or FBM, and the volatility process is an OU process driven by a time scaled GFLP. This allows for different distributions by varying the driving Lévy process, it gives flexible dependence structures, ranging from exponential short memory to polynomial, including long memory, and it also allows for jumps in the volatility by the behavior of g in 0.

Moreover, we allow for time scaled versions of the SV model, which gives, when we apply Theorem 4.6, in the limit a function of a FOU process with $H \in (1/2, 1)$. Consequently, we can adjust the model for the roughness of its sample paths, from those with jumps to continuous ones.

For $\overline{H} \in [1/2, 1)$ let $W^{\overline{H}}$ be FBM (BM corresponding to $\overline{H} = 1/2$). For $x > 0$ and a continuous function $f : \mathbb{R} \to \mathbb{R}_+$ we define the SV model

$$\begin{aligned} z_x(t) &:= \mu t + \beta \int_0^t v_x(s)ds + \int_0^t \sqrt{v_x(s-)}dW^{\overline{H}}(s), \\ v_x(t) &:= f(\overline{V}_x(t)). \end{aligned} \tag{5.1}$$

The integral in the data equation is for $\overline{H} > 1/2$ a path integral as defined in Young (1936) or Mikosch and Norvaiša (2000), and requires p-variation of the sample path v_x for appropriate p. For $\overline{H} = 1/2$ we take the usual Itô-integral.

We shall show that for $x \to \infty$ the bivariate process $\{(z_x(t), v_x(t))\}_{t\geq 0}$ converges in the Skorokhod space $D(\mathbb{R}_+^2)$ to

$$\begin{aligned} z(t) &:= \mu t + \beta \int_0^t v(s)ds + \int_0^t \sqrt{v(s-)}dW^{\overline{H}}(s), \\ v(t) &:= f(\overline{Y}(t)). \end{aligned}$$

Recall first that $v_x = f(\overline{V}_x)$, so that by the continuous mapping theorem weak convergence of v_x follows from that of $\overline{V}_x$.

Theorem 5.1. *For $x > 0$ let (z_x, v_x) be as in (5.1). Assume that $z_x = \{z_x(t)\}_{t\geq 0}$ is driven by FBM (or BM) with $\overline{H} \in [1/2, 1)$. Assume furthermore that $v_x = \{v_x(t)\}_{t\geq 0}$ is positive, has a.s. càdlàg sample paths, and that it is independent of $W^{\overline{H}}$. Suppose that for every $T > 0$ and $t \in [0, T]$ for all x sufficiently large*

$$E[(v_x(t))^2] \leq M, \quad t \in [0, T], \tag{5.2}$$

for some constant $M > 0$, which may depend on T. For $H > 1/2$ we additionally assume that $\sqrt{v_x}$ is of finite p-variation for $p < 1/(1 - \overline{H})$. If

$$v_x \xrightarrow{d} v \quad \text{as } x \to \infty, \tag{5.3}$$

in the Skorkohod space $D(\mathbb{R}_+)$ with the metric which induces the Skorokhod J_1 topology, and if $\sqrt{v}$ is again of finite p-variation with $p < 1/(1 - \overline{H})$, then also

$$(z_x, v_x) \xrightarrow{d} (z, v) \quad \text{as } x \to \infty,$$

in the Skorokhod space $D(\mathbb{R}_+^2)$ with the metric which induces the Skorokhod J_1 topology.

Proof. In order to prove weak convergence we show convergence of the finite dimensional distributions and tightness. We shall often condition z_x on the σ-field

$$\mathcal{G} := \sigma\{v_x(s), s \in [0, T], 0 < x < \infty\}$$

so that, given $\mathcal{G}$, the process z_x is a Gaussian process. Now we take $0 = t_1 < t_2 < \cdots < t_m \le T$ and $0 = t'_1 < t'_2 < \cdots < t'_n \le T$ for $m, n \in \mathbb{N}$ and prove

$$\begin{aligned}&(z_x(t_1), z_x(t_2), \dots, z_x(t_m), v_x(t'_1), v_x(t'_2), \dots, v_x(t'_n))\\ &\xrightarrow{d} (z(t_1), z(t_2), \dots, z(t_m), v(t'_1), v(t'_2), \dots, v(t'_n))\end{aligned}$$

by the Cramér-Wold device. For $(\gamma_{11}, \gamma_{12}, \dots, \gamma_{1m}, \gamma_{21}, \dots, \gamma_{2n}) \in \mathbb{R}^{m+n}$ we shall show that

$$\sum_{j=1}^m \gamma_{1j} z_x(t_j) + \sum_{k=1}^n \gamma_{2k} v_x(t'_k) \xrightarrow{d} \sum_{j=1}^m \gamma_{1j} z(t_j) + \sum_{k=1}^n \gamma_{2k} v(t'_k).$$

Observe that

$$\sum_{j=1}^m \gamma_{1j} z_x(t_j) + \sum_{k=1}^n \gamma_{2k} v_x(t'_k) = \sum_{j=2}^m \left(\sum_{h=j}^m \gamma_{1j} \right) (z_x(t_j) - z_x(t_{j-1})) + \sum_{k=1}^n \gamma_{2k} v_x(t'_k)$$

with $z_x(t_1) = 0$ and, hence, it suffices to show that

$$\sum_{j=2}^m \gamma_{1j}(z_x(t_j) - z_x(t_{j-1})) + \sum_{k=1}^n \gamma_{2k} v_x(t'_k) \xrightarrow{d} \sum_{j=2}^m \gamma_{1j}(z(t_j) - z(t_{j-1})) + \sum_{k=1}^n \gamma_{2k} v(t'_k).$$

We use the independence of v_x and $W^{\overline{H}}$ and the conditional Gaussianity of both z_x and z given the σ-field $\mathcal{G}$ to obtain the ch.f.

$$\begin{aligned}
&E\left[e^{i\lambda\{\sum_{j=2}^m \gamma_{1j}(z_x(t_j)-z_x(t_{j-1}))+\sum_{k=1}^n \gamma_{2k} v_x(t'_k)\}}\right]\\
&= E\left[E\left[e^{i\lambda\{\sum_{j=2}^m \gamma_{1j}(z_x(t_j)-z_x(t_{j-1}))+\sum_{k=1}^n \gamma_{2k} v_x(t'_k)\}} \,\middle|\, \mathcal{G}\right]\right]\\
&= E\left[e^{i\lambda\sum_{k=1}^n \gamma_{2k} v_x(t'_k)} E\left[e^{i\lambda\sum_{j=2}^m \gamma_{1j}(z_x(t_j)-z_x(t_{j-1}))} \,\middle|\, \mathcal{G}\right]\right]\\
&= E\Bigg[e^{i\lambda\sum_{k=1}^n \gamma_{2k} v_x(t'_k)+i\lambda\sum_{j=2}^m \gamma_{1j}\left(\mu(t_j-t_{j-1})+\beta\int_{t_{j-1}}^{t_j} v_x(u)du\right)}\\
&\qquad e^{-\frac{\lambda^2}{2}H(2H-1)\sum_{j,k}^m \gamma_{1j}\gamma_{1k}\int_{t_{j-1}}^{t_j}\int_{t_{k-1}}^{t_k}\sqrt{v_x(u)}\sqrt{v_x(w)}|u-w|^{2H-2}dudw}\Bigg]\\
&=: E[h(v_x)].
\end{aligned}$$

Since $h(\cdot)$ is continuous, the continuous mapping theorem yields $h(v_x) \xrightarrow{d} h(v)$. Furthermore, the fact that $|h| \le 1$ together with Lemma 3.11 of Kallenberg (1997) implies that $E[h(v_x)] \to E[h(v)]$ as $x \to \infty$ (see also 3.8 in Ch. VI of Jacod and Shiryaev (2003)). Again by conditional independence, reversing the argument which led to $E[h(v_x)]$ yields

$$E[h(v)] = E\left[e^{i\lambda\{\sum_{j=2}^m \gamma_{1j}(z(t_j)-z(t_{j-1}))+\sum_{k=1}^n \gamma_{2k} v(t'_k)\}}\right],$$

This concludes the first part of the proof.

Secondly, we prove tightness. For the process z_x we apply the tightness condition of (13.14) in Billingsley (1999). Since $W^{\overline{H}}$ has zero mean, it suffices to prove tightness of

$$I_x^{(1)}(t) := \int_0^t v_x(s)ds \quad \text{and} \quad I_x^{(2)}(t) := \int_0^t \sqrt{v_x(s)}dW^{\overline{H}}(s), \quad t \ge 0.$$

For $0 \le s < t$ we have

$$
\begin{aligned}
E\left[\left(I_x^{(1)}(t) - I_x^{(1)}(s)\right)^2\right] &= E\left[\int_s^t \int_s^t v_x(u)v_x(w)dudw\right] \\
&= \int_s^t \int_s^t E[v_x(u)v_x(w)]dudw \\
&\le \int_s^t \int_s^t \sqrt{E[(v_x(u))^2]E[(v_x(w))^2]}dudw \\
&\le M(t-s)^2,
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality and (5.2). This ensures the tightness condition for the Lebesgue integral $I_x^{(1)}$. As for tightness of the (fractional) Brownian integral $I_x^{(2)}$ recall that, given the σ-field $\mathcal{G}$, $I_x^{(2)}$ is Gaussian. We distinguish two cases.
For $\overline{H} > 1/2$ we calculate

$$
\begin{aligned}
E\left[\left(I_x^{(2)}(t) - I_x^{(2)}(s)\right)^2\right] &\le E\left[E\left[\left(I_x^{(2)}(t) - I_x^{(2)}(s)\right)^2 \Big| \mathcal{G}\right]\right] \\
&\le cE\left[\int_s^t \int_s^t \sqrt{v_x(u)}\sqrt{v_x(w)}|u-w|^{2\overline{H}-2}dudw\right] \\
&\le cM(t-s)^{2\overline{H}},
\end{aligned}
$$

where c is a finite positive constant, and apply (13.14) of Billingsley (1999) with $\beta = 1/2$.
For $\overline{H} = 1/2$ we apply the same condition with $\beta = 1$ giving

$$
E\left[\left(I_x^{(2)}(t) - I_x^{(2)}(s)\right)^4\right] = E\left[E\left[\left(I_x^{(2)}(t) - I_x^{(2)}(s)\right)^4 \Big| \mathcal{G}\right]\right] \le cE\left[\left(I_x^{(1)}(t) - I_x^{(1)}(s)\right)^2\right],
$$

using properties of the quadratic variation of BM, and c is again a finite positive constant. Now since the limit process z is continuous, the bivariate tightness of $\{(z_x(t), v_x(t))\}_{t\ge 0}$ follows from Corollary 3.33 of Ch. VI of Jacod and Shiryaev (2003). $\square$

Remark 5.2. (a) The same remark as made before Theorem 4.6 holds for the bivariate model. Since the bivariate limit process has continuous sample paths, weak convergence also holds in the Skorokhod space $D(\mathbb{R}_+^2)$ equipped with the metric of uniform convergence on compacts (Jacod and Shiryaev, 2003, 1.17 (b), VI).
(b) Assume that $v_x = \overline{V}_x$ is the stationary OU process driven by a time scaled GFLP as defined in (4.6). If $\overline{V}_x$ satisfies the moment condition (5.2) and the p-variation condition on the sample path, then Theorem 5.1 applies.

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