On the Equivalence of Two Nonlinear Control Approaches: Immersion and Invariance and IDA-PBC

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Abstract

In this paper we compare the two well known nonlinear control design techniques *Interconnection and Damping Assignment Passivity Based Control* (IDA-PBC) and *Immersion and Invariance* (I&I) at the example of the so-called Acrobot underactuated mechanical system. The immersion and matching equations in both approaches have a similar structure which is exploited to derive equivalent control laws, each of them providing a different perspective on the stabilization problem. In particular, the coordinate change which renders the potential energy matching PDE in IDA-PBC an ordinary differential equation is used to define the immersion map in I&I. It is shown that the energy shaping part of the IDA-PBC controller makes the closed-loop system an interconnection of two lower-dimensional port-Hamiltonian (pH) systems in the on- and off-manifold coordinates that appear in the I&I framework. The effect of damping injection output feedback can be identified with dissipation in the off-manifold part of the interconnected system. Dissipation is propagated to the on-manifold part which results in asymptotic stability of the system's equilibrium. The particular choice of the I&I design parameters in the present example, including the unconventional definition of coordinates on the invariant manifold, provides an interesting re-interpretation of the IDA-PBC control law from the I&I perspective. Finally, a discussion on the equivalence of the two approaches is presented by examining the cases of linear mechanical systems with one unactuated pivot as well as of general linear mechanical systems.

Keywords: Underactuated mechanical systems, port-Hamiltonian systems, passivity based control, Immersion and Invariance.

1 Introduction

The I&I methodology was originally introduced for the stabilization of general nonlinear systems in [3]. This work was further developed in a series of publications that have been summarized in [2]. In the I&I approach the desired behavior of the system to be controlled is captured by the choice of a target dynamical system of lower dimension than the original system. The control objective is to find a controller which guarantees that the closed-loop system asymptotically behaves like the target system achieving asymptotic model matching. This should be contrasted with the more restrictive exact matching techniques such as the IDA-PBC methodology. The success of I&I is witnessed by the wide range of applications such as electrical, mechanical and power systems. Recently, there have been several developments, for example in the approach for stabilization [16], [17], [15] and speed observation [4] of mechanical systems as well as for the adaptive control of nonlinearly parameterized systems [11].

The IDA-PBC controller design technique on the other hand, cf. [12] for an overview, aims at transforming a nonlinear dynamical system by state feedback into a (full order) port-Hamiltonian (pH) system. The target pH structure, cf. the book [20] on the pH state representation, allows for an easy passivity/Lyapunov based stability proof of the closed-loop equilibrium. By the shape of the energy function which is constructed in the course of the design process analytically, an estimate of the domain of attraction is given. A wide range of application examples are solved using IDA-PBC. Recently, some effort has been made in finding sets of reasonable design parameters to enhance transparency with respect to dynamics assignment [8].

Especially for mechanical systems it is convenient in the I&I approach to represent the dynamics on the target manifold as a pH system in order to keep its physical structure and which dynamics, in contrast to IDA-PBC, are lower dimensional. Both design procedures of IDA-PBC and I&I require solutions of partial differential equations, which have certain structural similarities. Note however, that I&I can be seen as a relaxation of the IDA-PBC approach, in the sense that less structure is prespecified in advance for the closed-loop system.

The goal of the present note is to identify some of these similar structures, in the design procedures, such as coordinate changes and part of the immersion mapping, and based thereon to elaborate on equivalences of both methods. *Equivalence* is basically meant in the sense that identical controllers result from the application of the different methodologies, which adopt different perspectives on the design problem.

Some first relationships on the structurally similar control laws were shown for the inertia wheel pendulum example in [15], [17]. In this note, the focus is on another underactuated mechanical system, the so-called Acrobot, see e. g. [19] for a description of the system and the model. It is shown that structural knowledge from the solution of the IDA-PBC energy shaping problem can be exploited to find an immersion map for I&I. Furthermore, the IDA-PBC control law has an interesting interpretation in the context of I&I. In contrast to the usual I&I procedure where the target system of the reduced system is selected to have an asymptotically stable desired equilibrium, here only Lyapunov stable target dynamics are chosen in an initial step. This corresponds to the *energy shaping* step of IDA-PBC where a presumably lossless mechanical system is endowed with a new virtual energy, while the system remains lossless. In the subsequent *damping injection* step asymptotical stabilization is provided by passive output feedback in IDA-PBC. This measure, from the I&I point of view, makes the immersion manifold attractive, and at the same time, when the damping is pervasive, the equilibrium of the total system is asymptotically stabilized.

From a *practical* point of view it might appear unconventional to parameterize a controller (I&I) knowing the solution of another design problem (IDA-PBC). From an *analytical* perspective, however, the study presented in this note can be a first step to identify the additional degrees of freedom in the I&I procedure, compared to IDA-PBC.

The present paper is an extended version of the conference paper [10]. Besides some more detailed explanations and illustrations we appended a section on some results for linear mechanical systems. It is organized as follows: Section 2 provides a brief review of the considered methods IDA-PBC and I&I. In Section 3 the equivalence of both approaches is studied at the Acrobot example. In Section 4 the case of linear mechanical systems is discussed from the two perspectives. The paper closes with a discussion and an outlook in Section 5.

Notation: The Jacobian (row) and gradient (column) vector of a mapping $H : \mathbb{R}^n \to \mathbb{R}$ are denoted as follows:

$$\partial_x H(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial H(\boldsymbol{x})}{\partial x_1} & \dots & \frac{\partial H(\boldsymbol{x})}{\partial x_n} \end{bmatrix}, \qquad \partial_x^T H(\boldsymbol{x}) = \nabla H(\boldsymbol{x}).$$

2 IDA-PBC and I&I for underactuated mechanical systems

Both design methodologies are briefly introduced, and their structural similarities are highlighted. We consider lossless mechanical systems in Hamiltonian representation

$$(\Sigma): \quad \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \partial_q^T H(\boldsymbol{q},\boldsymbol{p}) \\ \partial_p^T H(\boldsymbol{q},\boldsymbol{p}) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{G}(\boldsymbol{q}) \end{bmatrix} \boldsymbol{u}. \tag{1}$$

$\boldsymbol{q} \in \mathbb{R}^n$ and $\boldsymbol{p} \in \mathbb{R}^n$ denote the vectors of generalized coordinates and momenta, $\boldsymbol{u} \in \mathbb{R}^m$, $m < n$, the vector of input forces/torques and $\boldsymbol{G}(\boldsymbol{q}) \in \mathbb{R}^{n\times m}$ the input matrix which has full column rank m in the considered operation range of the system. $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ is the corresponding identity matrix. The Hamiltonian or total energy of the system is assumed to have a simple mechanical structure, i. e. it is composed of a kinetic energy $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ plus potential energy $V : \mathbb{R}^n \to \mathbb{R}$:

$$H(\boldsymbol{q},\boldsymbol{p}) = \underbrace{\frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{p}}_{T(\boldsymbol{q},\boldsymbol{p})} + V(\boldsymbol{q}), \qquad \boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^T(\boldsymbol{q}) > 0, \tag{2}$$

where $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ denotes the mass matrix.

Remark 1 *Physical damping, especially in unactuated coordinates, can produce additional difficulties in the design process, and is therefore excluded in this contribution. In IDA-PBC, it is not possible to generate a closed-loop system of the above simple mechanical form if a dissipation condition [6] is violated. By relaxing the structure of the target system, the problem can be obviated and stabilizing IDA-PBC controllers can be derived [9], though in a more complicated procedure. Interestingly, the resulting controllers contain structures which also appear when sign-indefinite damping in mechanical systems is considered [18], a relation which gives rise to future investigations.*

2.1 IDA-PBC

The IDA-PBC state feedback control law

$$\boldsymbol{u}_{IDA} = \boldsymbol{u}_{es} + \boldsymbol{u}_{di}$$

is composed of two parts corresponding to the design steps *energy shaping* and *damping injection.*

Energy shaping. The goal of energy shaping is to find a nonlinear state feedback law $\boldsymbol{u}_{es}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{v})$ to transform (1) into another lossless pH system

$$(\Sigma_d): \quad \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{J}_1(\boldsymbol{q}) \\ -\boldsymbol{J}_1^T(\boldsymbol{q}) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \partial_q^T H_d(\boldsymbol{q},\boldsymbol{p}) \\ \partial_p^T H_d(\boldsymbol{q},\boldsymbol{p}) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{G}(\boldsymbol{q}) \end{bmatrix} \boldsymbol{v} \tag{3}$$

with new input $\boldsymbol{v}$ and a closed-loop (virtual) energy

$$H_d(\boldsymbol{q},\boldsymbol{p}) = \underbrace{\frac{1}{2}\boldsymbol{p}^T\boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p}}_{T_d(\boldsymbol{q},\boldsymbol{p})} + V_d(\boldsymbol{q}), \qquad \boldsymbol{M}_d(\boldsymbol{q}) = \boldsymbol{M}_d^T(\boldsymbol{q}) > 0, \tag{4}$$

which possesses again a simple mechanical structure. The virtual potential energy $V_d(\boldsymbol{q})$ must admit a minimum at the new desired equilibrium $\boldsymbol{q}^*$ such that $H_d(\boldsymbol{q},\boldsymbol{p})$ serves as a storage function for (3) with the collocated output

$$\boldsymbol{y} = \boldsymbol{G}^T(\boldsymbol{q})\partial_p^T H_d(\boldsymbol{q},\boldsymbol{p}). \tag{5}$$

Accounting for the fact that generalized velocities and momenta also in the closed-loop are related by $\dot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{p}$, it follows that

$$\boldsymbol{J}_1(\boldsymbol{q}) = \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{M}_d(\boldsymbol{q}).$$

Furthermore, it must be ensured that the dynamics, which is not affected by the control input, remains unchanged. This is expressed by the matching PDE

$$\boldsymbol{G}^{\perp}(\boldsymbol{q})\partial_q^T H(\boldsymbol{q},\boldsymbol{p}) = \boldsymbol{G}^{\perp}(\boldsymbol{q})\boldsymbol{J}_1^T(\boldsymbol{q})\partial_q^T H_d(\boldsymbol{q},\boldsymbol{p}). \tag{6}$$

The full rank left hand annihilator $\boldsymbol{G}^{\perp}(\boldsymbol{q}) \in \mathbb{R}^{(n-m)\times n}$ with the property $\boldsymbol{G}^{\perp}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{0}$ eliminates the input functions from the matching equation, which results from the comparison of the second rows of (1) and (3). The energy shaping control finally follows from solving the matching equation for $\boldsymbol{u}$, using the pseudo-inverse $\boldsymbol{G}^+(\boldsymbol{q}) = (\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q}))^{-1}\boldsymbol{G}^T(\boldsymbol{q})$:

$$\boldsymbol{u} = \boldsymbol{u}_{es} + \boldsymbol{v} = \boldsymbol{G}^+(\boldsymbol{q})(-\boldsymbol{J}_1^T(\boldsymbol{q})\partial_q^T H_d(\boldsymbol{q},\boldsymbol{p}) + \partial_q^T H(\boldsymbol{q},\boldsymbol{p})) + \boldsymbol{v}. \tag{7}$$

Damping injection. With a feedback of the collocated output, e. g. by a constant $m \times m$ matrix $\boldsymbol{K}_{di} > 0$,

$$\boldsymbol{v} = \boldsymbol{u}_{di} = -\boldsymbol{K}_{di}\boldsymbol{G}^T(\boldsymbol{q})\partial_p^T H_d(\boldsymbol{q},\boldsymbol{p})$$

(damping injection), the new equilibrium $(\boldsymbol{q}^*, \boldsymbol{0})$ of the passive system (3), (5), is stabilized asymptotically under the condition of zero state detectability [5]. As a result, the interconnection and damping matrix in (3) becomes

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{J}_1(\boldsymbol{q}) \\ -\boldsymbol{J}_1^T(\boldsymbol{q}) & -\boldsymbol{R}_2(\boldsymbol{q}) \end{bmatrix}, \qquad \boldsymbol{R}_2(\boldsymbol{q}) = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{K}_{di}\boldsymbol{G}^T(\boldsymbol{q}) \geq 0.$$

2.2 I&I

On the other hand, in the I&I approach for pH systems the desired behavior of the system to be controlled is captured by the choice of a target dynamical pH system, of lower dimension than the original system $2s < 2n$,

$$(\Sigma_t): \begin{bmatrix} \dot{\boldsymbol{\xi}}_q \\ \dot{\boldsymbol{\xi}}_p \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\Lambda}(\boldsymbol{\xi}_q) \\ -\boldsymbol{\Lambda}^T(\boldsymbol{\xi}_q) & -\boldsymbol{R}_t(\boldsymbol{\xi}) \end{bmatrix} \begin{bmatrix} \partial^T_{\xi_q} H_t(\boldsymbol{\xi}) \\ \partial^T_{\xi_p} H_t(\boldsymbol{\xi}) \end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\xi}_q \\ \boldsymbol{\xi}_p \end{bmatrix} \tag{8}$$

with the Hamiltonian function $H_t : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$

$$H_t(\boldsymbol{\xi}_q, \boldsymbol{\xi}_p) = \underbrace{\frac{1}{2}{\boldsymbol{\xi}_p}^T \boldsymbol{M}_t^{-1}(\boldsymbol{\xi}_q)\boldsymbol{\xi}_p}_{T_t(\boldsymbol{\xi}_q,\boldsymbol{\xi}_p)} + V_t(\boldsymbol{\xi}_q),$$

and states $\boldsymbol{\xi}_q, \boldsymbol{\xi}_p \in \mathbb{R}^s$. Moreover, $\boldsymbol{M}_t = \boldsymbol{M}_t^T > 0$ is the target inertia matrix, $\boldsymbol{R}_t = \boldsymbol{R}_t^T \geq 0$ is the target damping matrix, both of dimension $s \times s$, and T_t, V_t define the target kinetic and potential energy functions, respectively. (Σ_t) has an asymptotically stable equilibrium at $(\boldsymbol{\xi}_q^*, \mathbf{0})$ if $V_t(\boldsymbol{\xi}_q)$ has a strict minimum at $\boldsymbol{\xi}_q^*$.

The objective is to find a controller which guarantees that the closed-loop system asymptotically behaves like the target system (Σ_t) achieving *asymptotic* model matching, as opposed to the more restrictive *exact* matching of IDA-PBC (or correspondingly the Controlled Lagrangians method). This is formalized by finding a manifold in state space that can be rendered invariant and attractive, with internal dynamics a copy of the desired closed-loop dynamics, and designing a control law that steers the system state towards the manifold. The following steps summarize the I&I approach [2]:

Immersion map. Choosing the immersion mapping $\boldsymbol{\pi}(\boldsymbol{\xi})$ as

$$\boldsymbol{\pi}(\boldsymbol{\xi}) = \begin{bmatrix} \boldsymbol{\pi}_1(\boldsymbol{\xi}) \\ \boldsymbol{\pi}_2(\boldsymbol{\xi}) \end{bmatrix},$$

with $\boldsymbol{\pi}_i : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}^n$, the matching (or immersion) conditions can be written, with $\boldsymbol{x} = [\boldsymbol{q}^T\ \boldsymbol{p}^T]^T \in \mathbb{R}^{2n}$ the vector of coordinates and momenta of the original system:

$$\partial_p^T H|_{\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})} = \partial_\xi \boldsymbol{\pi}_1 \dot{\boldsymbol{\xi}}$$
$$-\partial_q^T H|_{\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})} + \boldsymbol{G}(\boldsymbol{\pi}_1(\boldsymbol{\xi}))\boldsymbol{c}(\boldsymbol{\pi}(\boldsymbol{\xi})) = \partial_\xi \boldsymbol{\pi}_2 \dot{\boldsymbol{\xi}}.$$

Both immersion conditions must be satisfied by all available design parameters in the target system (8) as well as the immersion mapping $\boldsymbol{\pi}(\boldsymbol{\xi})$ and the virtual feedback control law $\boldsymbol{c}(\boldsymbol{\pi}(\boldsymbol{\xi}))$ that renders the manifold invariant. The first equation is related to the equation $\partial_p^T H = \boldsymbol{J}_1 \partial_p^T H_d$ in the IDA-PBC approach, which defines the matrix $\boldsymbol{J}_1(\boldsymbol{q})$. Multiplying the second equation with the left hand annihilator $\boldsymbol{G}^\perp(\boldsymbol{\pi}(\boldsymbol{\xi}))$ and replacing the target dynamics on the right hand side, a PDE is obtained which has a structure similar to the IDA-PBC matching PDE (6):

$$-\boldsymbol{G}^{\perp}(\boldsymbol{\pi}(\boldsymbol{\xi}))\partial_q^T H|_{\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})} = \boldsymbol{G}^{\perp}(\boldsymbol{\pi}(\boldsymbol{\xi}))\partial_\xi \boldsymbol{\pi}_2 \dot{\boldsymbol{\xi}}. \tag{9}$$

[Figure 1 about here]

Implicit manifold. The manifold which is to be rendered attractive and invariant is defined as

$$\mathcal{M} \;=\; \{\boldsymbol{x} \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{0}\} \;=\; \{\boldsymbol{x} \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{\pi}(\boldsymbol{\xi}) \text{ for some } \boldsymbol{\xi} \in \mathbb{R}^s \times \mathbb{R}^s\}.$$

The mapping $\boldsymbol{\phi}(\boldsymbol{x})$ describes the distance of the state $\boldsymbol{x}$ from the target manifold and therefore defines the so-called off-manifold coordinates $\boldsymbol{\zeta} = \boldsymbol{\phi}(\boldsymbol{x})$. Together with the on-manifold coordinates $\boldsymbol{\xi}$, a regular coordinate change can be defined, which will be exploited later.

Manifold attractivity and trajectory boundedness. The I&I control law $\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{\zeta})$ has to ensure that the trajectories of the closed-loop system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \partial_q^T H(\boldsymbol{q}, \boldsymbol{p}) \\ \partial_p^T H(\boldsymbol{q}, \boldsymbol{p}) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{G}(\boldsymbol{q}) \end{bmatrix} \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{\zeta})$$

remain bounded, while the target manifold is approached asymptotically, i. e. the off-manifold coordinates which satisfy

$$\dot{\boldsymbol{\zeta}} = \partial_x \boldsymbol{\phi}(\boldsymbol{x}) \left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \partial_q^T H(\boldsymbol{q}, \boldsymbol{p}) \\ \partial_p^T H(\boldsymbol{q}, \boldsymbol{p}) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{G}(\boldsymbol{q}) \end{bmatrix} \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{\zeta}) \right\},$$

tend to zero: $\lim_{t\to\infty} \boldsymbol{\zeta}(t) = \boldsymbol{0}$.

Immersion vs. matching PDE. At this point, the IDA-PBC matching PDE (6) is reminded and compared to the I&I immersion PDE (9). Both PDEs can be split into a kinetic and a potential energy part. Below, the equations are set one above the other in order to underscore the structural similarity (arguments are omitted for brevity):

Kinetic energy:

$$\begin{aligned} \boldsymbol{G}^{\perp}\big|_{\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})} \left(\partial_q^T T\big|_{\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})} - \partial_{\xi_p}\boldsymbol{\pi}_2 \boldsymbol{\Lambda}^T \partial_{\xi_q}^T T_t + \left(\partial_{\xi_q}\boldsymbol{\pi}_2\boldsymbol{\Lambda} - \partial_{\xi_p}\boldsymbol{\pi}_2 \boldsymbol{R}_t\right) \partial_{\xi_p}^T T_t \right) &= \boldsymbol{0} \\ \boldsymbol{G}^{\perp}\left(\partial_q^T T - \boldsymbol{M}_d \boldsymbol{M}^{-1} \partial_q^T T_d - \boldsymbol{R}_2 \partial_p^T T_d\right) &= \boldsymbol{0} \end{aligned}$$

Potential energy:

$$\begin{aligned} \boldsymbol{G}^{\perp}\big|_{\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})} \left(\partial_q^T V\big|_{\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})} - \partial_{\xi_p}\boldsymbol{\pi}_2 \partial_{\xi_q}^T V_t \right) &= \boldsymbol{0} \\ \boldsymbol{G}^{\perp}\left(\partial_q^T V - \boldsymbol{M}_d \boldsymbol{M}^{-1} \partial_q^T V_d\right) &= \boldsymbol{0} \end{aligned}$$

This similarity will be exploited in the following section at the example of the Acrobot in order to derive corresponding control laws from both perspectives.

3 A comparative study with the Acrobot

The considered system is the so-called Acrobot, a 2 DOF underactuated mechanical system with two rigid links, see Fig. 2. The pivot rotates freely, the intermediate joint is actuated. The control task is to stabilize the unstable upright stretched-out configuration of the mechanism $q_1^* = q_2^* = 0$.

3.1 Model

The Hamiltonian description of the system dynamics with generalized coordinates and momenta $\boldsymbol{q} = [q_1\ q_2]^T$, $\boldsymbol{p} = [p_1\ p_2]^T$, is given by (1) and (2), with the inertia matrix

$$\boldsymbol{M}(q_2) = \begin{bmatrix} c_1 + c_2 + 2c_3 \cos q_2 & c_2 + c_3 \cos q_2 \\ c_2 + c_3 \cos q_2 & c_2 \end{bmatrix},$$

which only depends on q_2, and the potential energy

$$V(\boldsymbol{q}) = c_4 \cos q_1 + c_5 \cos(q_1 + q_2).$$

The constants $c_1, \ldots, c_5$ depend on masses, lengths and inertia. Physical dissipation is not considered, as it has been already discussed above. Only the intermediate joint is actuated, consequently the input vector is $\boldsymbol{G} = \boldsymbol{e}_2$, and the corresponding left hand annihilator is $\boldsymbol{G}^\perp = \boldsymbol{e}_1^T$, where $\boldsymbol{e}_i$ denotes the i-th unit column vector.

[Figure 2 about here]

3.2 IDA-PBC

The IDA-PBC matching PDE according to (6) is

$$\partial_{q_1}(\frac{1}{2}\boldsymbol{p}^T\boldsymbol{M}^{-1}(q_2)\boldsymbol{p} + V(\boldsymbol{q})) = \boldsymbol{e}_1^T\boldsymbol{M}_d(\boldsymbol{q})\boldsymbol{M}^{-1}(q_2)\partial_q^T(\frac{1}{2}\boldsymbol{p}^T\boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p} + V_d(\boldsymbol{q})). \tag{10}$$

The design parameters in $\boldsymbol{M}_d(\boldsymbol{q})$ and $V_d(\boldsymbol{q})$ must be chosen such that (10) is satisfied. It turns out that the kinetic part of this PDE is trivially true for a closed-loop inertia matrix $\boldsymbol{M}_d = const.$, and only the potential energy matching PDE

$$\boldsymbol{m}_{d,1}^T\boldsymbol{M}^{-1}(q_2)\partial^T V_d(\boldsymbol{q}) = \partial_{q_1}V(\boldsymbol{q}) \tag{11}$$

has to be considered in the sequel. Notice that $\boldsymbol{M}_d = const.$ is *one possible* parameterization to solve the IDA-PBC matching problem, which in the course of the paper is used as a reference for the I&I approach.

In the following, the inverse original mass matrix will be denoted as $\boldsymbol{M}^{-1}(q_2) = \overline{\boldsymbol{M}}(q_2)$, and column/elementwise:

$$\overline{\boldsymbol{M}}(q_2) = [\overline{\boldsymbol{m}}_1(q_2)\ \overline{\boldsymbol{m}}_2(q_2)] = \begin{bmatrix} \overline{m}_{11}(q_2) & \overline{m}_{12}(q_2) \\ \overline{m}_{12}(q_2) & \overline{m}_{22}(q_2) \end{bmatrix}.$$

The new mass matrix is written row/elementwise

$$\boldsymbol{M}_d = \begin{bmatrix} \boldsymbol{m}_{d,1}^T \\ \boldsymbol{m}_{d,2}^T \end{bmatrix} = \begin{bmatrix} m_{d,11} & m_{d,12} \\ m_{d,12} & m_{d,22} \end{bmatrix}.$$

The linear matching PDE (11) then has the structure

$$a_1(q_2)\partial_{q_1} V_d(q) + a_2(q_2)\partial_{q_2} V_d(q) = \partial_{q_1} V(q) \tag{12}$$

with the coefficients

$$a_i(q_2) = \boldsymbol{m}_{d,1}^T \overline{\boldsymbol{m}}_i(q_2), \quad i = 1, 2,$$

which are summarized in the vector $\boldsymbol{a}(q_2) = [a_1(q_2)\ a_2(q_2)]^T$.

Solution of the matching PDE. With the coordinate change $\boldsymbol{z} = \boldsymbol{t}(\boldsymbol{q})$,

$$\begin{aligned} z_1 = \ & t_1(q_1, q_2) & = q_1 - \int_{q_2^*}^{q_2} \frac{a_1(\theta)}{a_2(\theta)} d\theta \\ z_2 = \ & t_2(q_1, q_2) & = q_2 \end{aligned}$$

and its inverse $\boldsymbol{q} = \boldsymbol{t}^{-1}(\boldsymbol{z})$,

$$\begin{aligned} q_1 = \ & t_1^{-1}(z_1, z_2) & = z_1 + \int_{z_2^*}^{z_2} \frac{a_1(\theta)}{a_2(\theta)} d\theta \\ q_2 = \ & t_2^{-1}(z_1, z_2) & = z_2 \end{aligned} \tag{13}$$

the potential energy matching PDE (12) is transformed into the ODE

$$\partial_{z_2} \tilde{V}_d(\boldsymbol{z}) = \frac{\partial_{q_1} V \circ \boldsymbol{t}^{-1}(\boldsymbol{z})}{a_2(z_2)}. \tag{14}$$

The right hand side can be simply integrated over z_2 and the solution

$$\tilde{V}_d(\boldsymbol{z}) = \tilde{V}_d^p(\boldsymbol{z}) + \tilde{V}_d^h(z_1)$$

consists of a particular part $\tilde{V}_d^p(\boldsymbol{z})$ and a homogeneous part $\tilde{V}_d^h(z_1)$, which is an *arbitrary* function of z_1. The function $t_1(\boldsymbol{q})$ solves the corresponding homogeneous PDE

$$\boldsymbol{a}^T(q_2)\partial_q^T t_1(\boldsymbol{q}) = 0,$$

and therefore arbitrary functions of $z_1 = t_1(\boldsymbol{q})$ can be added to the particular solution in order to shape the resulting potential energy. From previous work it is known that controller parameterizations can be easily derived for the Acrobot such that prespecified closed-loop local linear dynamics is achieved, while all IDA-PBC design parameters meet the definiteness requirements $\boldsymbol{M}_d > 0$, $\boldsymbol{q}^* = \arg\min V_d(\boldsymbol{q})$, and $\boldsymbol{R}_2 \geq 0$, see [7].

Energy shaping. With the appropriately tuned potential energy, the energy shaping control law (7) for the Acrobot is

$$u_{es} = -\boldsymbol{m}_{d,2}^T \overline{\boldsymbol{M}}(q_2)\partial^T V_d(\boldsymbol{q}) + \partial_{q_2} H(\boldsymbol{q}, \boldsymbol{p}). \tag{15}$$

The closed-loop system under this controller is a lossless pH system (3) with new positive definite energy $H_d(\boldsymbol{q}, \boldsymbol{p})$.

Damping injection. To asymptotically stabilize the new equilibrium $(\boldsymbol{q}^*, \boldsymbol{0})$, damping is injected via output feedback ($k_{di} > 0$)

$$u_{di} = -k_{di}\boldsymbol{G}^T\partial_p^T H_d(\boldsymbol{q}, \boldsymbol{p}) = -k_{di}\overline{\boldsymbol{m}}_{d,2}^T\boldsymbol{p}. \tag{16}$$

To prove that the equilibrium is indeed asymptotically stable, it can be shown that $(\boldsymbol{q}^*, \boldsymbol{0})$ is the only possible state which remains in the set where $\dot{H}_d = 0$ holds (LaSalle).

3.3 Immersion and Invariance

In contrast to IDA-PBC, in I&I lower dimensional target dynamics is specified. For the Acrobot a target pH system with only one degree of freedom is assumed:

$$\begin{bmatrix} \dot{\xi}_q \\ \dot{\xi}_p \end{bmatrix} = \begin{bmatrix} 0 & \lambda(\xi_q) \\ -\lambda(\xi_q) & 0 \end{bmatrix} \begin{bmatrix} \partial_{\xi_q} H_t(\xi_q, \xi_p) \\ \partial_{\xi_p} H_t(\xi_q, \xi_p) \end{bmatrix} \tag{17}$$

with

$$H_t(\xi_q, \xi_p) = \frac{1}{2}\frac{\xi_p^2}{m_t(\xi_q)} + V_t(\xi_q). \tag{18}$$

Observe that the target dynamics, when $H_t(\xi_q, \xi_p)$ is positive definite around its equilibrium $(\xi_q^*, 0)$ is *only Lyapunov stable*, whereas usually asymptotically stable target dynamics ($R_t > 0$) is specified in I&I.

The target dynamics must be immersed into an invariant manifold, i. e. a mapping $\boldsymbol{\pi}(\boldsymbol{\xi})$ must exist such that $\boldsymbol{x}^* = \boldsymbol{\pi}(\boldsymbol{\xi}^*)$ and, in addition, a virtual control law $c(\boldsymbol{\pi}(\boldsymbol{\xi}))$ such that the trajectories of the target system (on the manifold) are trajectories of the original system. When the structure of the mapping $\boldsymbol{\pi}(\boldsymbol{\xi})$ according to

$$\begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\pi}_1(\boldsymbol{\xi}) \\ \boldsymbol{\pi}_2(\boldsymbol{\xi}) \end{bmatrix}, \qquad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \pi_{11}(\boldsymbol{\xi}) \\ \pi_{12}(\boldsymbol{\xi}) \end{bmatrix}, \qquad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \pi_{21}(\boldsymbol{\xi}) \\ \pi_{22}(\boldsymbol{\xi}) \end{bmatrix},$$

is assumed, the immersion condition is

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \partial_q^T H \\ \partial_p^T H \end{bmatrix} \Bigg|_{\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{G} \end{bmatrix} c(\boldsymbol{\pi}(\boldsymbol{\xi})) = \begin{bmatrix} \partial_\xi \boldsymbol{\pi}_1 \\ \partial_\xi \boldsymbol{\pi}_2 \end{bmatrix} \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} \begin{bmatrix} \partial_{\xi_q} H_t \\ \partial_{\xi_p} H_t \end{bmatrix}. \tag{19}$$

This condition for the 2 DOF Acrobot system represents 3 scalar equations which must be met by all free design parameters and one equation which can be solved for the virtual control law $c(\boldsymbol{\pi}(\boldsymbol{\xi}))$.

Choice of design parameters. Multiplying the second row of (19) with the left hand annihilator $\boldsymbol{G}^\perp = \boldsymbol{e}_1^T$, where $m_t = const.$ is chosen (inspired by $\boldsymbol{M}_d = const.$ in IDA-PBC), yields

$$-\partial_{q_1} V|_{\boldsymbol{q}=\boldsymbol{\pi}_1(\boldsymbol{\xi})} = \begin{bmatrix} \partial_{\xi_q}\pi_{21} & \partial_{\xi_p}\pi_{21} \end{bmatrix} \begin{bmatrix} \lambda m_t^{-1}\xi_p \\ -\lambda \partial_{\xi_q} V_t \end{bmatrix}.$$

Defining

$$p_1 = \pi_{21}(\boldsymbol{\xi}) = \xi_p \qquad \text{and} \qquad \lambda(\xi_q) = a_2(\xi_q),$$

the equation becomes

$$\partial_{\xi_q} V_t(\xi_q) = \frac{\partial_{q_1} V \circ \pi_1(\boldsymbol{\xi})}{a_2(\xi_q)}.$$

The equation is structurally similar to (14), with the difference that $\tilde{V}_d$ has 2 arguments z_1 and z_2, while V_t is only a function in ξ_q.

Solution of the immersion PDE. Define the generalized coordinate part of the immersion map $\boldsymbol{q} = \boldsymbol{\pi}_1(\boldsymbol{\xi})$ in analogy to $\boldsymbol{q} = \boldsymbol{t}^{-1}(\boldsymbol{z})$ in (13), with z_1 restricted to the equilibrium value $z_1 = z_1^*$:

$$\begin{aligned} q_1 = \;& \pi_{11}(\xi_q) &= z_1^* + \int_{\xi_q^*}^{\xi_q} \frac{a_1(\theta)}{a_2(\theta)} d\theta \\ q_2 = \;& \pi_{12}(\xi_q) &= \xi_q. \end{aligned} \tag{20}$$

Then a solution of this ODE is identical with the (known) solution of the IDA-PBC potential energy PDE, restricted to $z_1 = z_1^*$:

$$V_t(\xi_q) = \tilde{V}_d(z_1^*, \xi_q).$$

The remaining design quantities $\pi_{22}(\xi_q, \xi_p)$ and m_t are fixed such that the first vector-valued row of

(19) is true:

$$\boldsymbol{M}^{-1}(\xi_q)\begin{bmatrix}\xi_p\\ \pi_{22}(\xi_q,\xi_p)\end{bmatrix}=\begin{bmatrix}\frac{a_1(\xi_q)}{a_2(\xi_q)} & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}a_2(\xi_q)m_t^{-1}\xi_p\\ -a_2(\xi_q)\partial_{\xi_q}V_t(\xi_q)\end{bmatrix}.$$

The two scalar equations can be solved (after multiplication with $\boldsymbol{M}(\xi_q)$) for the inertia of the target system

$$m_t = m_1^T(\xi_q)\overline{\boldsymbol{M}}(\xi_q)m_{d1} = m_{d,11},$$

which is constant, as has been assumed, as well as the remaining component of the immersion mapping

$$p_2 = \pi_{22}(\xi_p) = \frac{m_{d,12}}{m_{d,11}}\xi_p.$$

Intermediate result: target dynamics. Given a solution of the IDA-PBC energy shaping problem for the Acrobot as described in the previous section with $V_d(q_1,q_2)$ or $\tilde{V}_d(z_1,z_2)$, the positive definite closed-loop potential energy function and the constant positive definite mass matrix $\boldsymbol{M}_d = [m_{d,ij}]_{i,j=1,2}$.

The 1 DOF system (17) with

$$\begin{aligned}m_t &= m_{d,11}\\ V_t(\xi_q) &= \tilde{V}_d(z_1^*,\xi_q)\\ \lambda(\xi_q) &= \boldsymbol{m}_{d,1}^T\overline{\boldsymbol{m}}_2(\xi_q)\end{aligned}$$

is a feasible target system for I&I and

$$\begin{bmatrix}q_1\\ q_2\\ p_1\\ p_2\end{bmatrix}=\begin{bmatrix}z_1^*+\int_{\xi_q^*}^{\xi_q}\frac{\boldsymbol{m}_{d,1}^T\overline{\boldsymbol{m}}_1(\theta)}{\boldsymbol{m}_{d,1}^T\overline{\boldsymbol{m}}_2(\theta)}d\theta\\ \xi_q\\ \xi_p\\ \frac{m_{d,12}}{m_{d,11}}\xi_p\end{bmatrix} \tag{21}$$

is the corresponding immersion mapping $\boldsymbol{x}=\boldsymbol{\pi}(\boldsymbol{\xi})$. The restriction of the closed-loop potential energy $\tilde{V}_d$ from IDA-PBC to the equilibrium value of the characteristic coordinate $z_1 = z_1^*$ provides a potential energy function for the I&I target system.

What remains, is to express the invariant manifold implicitely (define off-manifold coordinates), choose a control law such that the off-manifold coordinates tend to zero and the overall system trajectories remain bounded. In the I&I approach the way how to derive this stabilizing control law is not explicitely specified. In contrast, the IDA-PBC procedure directly leads to the energy shaping and damping injection control laws (15) and (16).

3.4 I&I stabilization and relation to IDA-PBC

In this subsection, the uncontrolled system is expressed in the on- and off-manifold coordinates, corresponding to the above defined immersion mapping. An obvious choice for the I&I controller to keep the trajectories bounded turns out to be identically the IDA-PBC energy shaping control law. Moreover, dissipation introduced via IDA-PBC damping injection can be allocated in the off-manifold subsystem.

Implicit manifold and manifold attractivity. A mapping $\boldsymbol{\phi}(\boldsymbol{x})$, $\boldsymbol{x} = [\boldsymbol{q}^T\ \boldsymbol{p}^T]^T$ which satisfies the set identity $\{\boldsymbol{x} = \boldsymbol{\pi}(\boldsymbol{\xi})\} = \{\boldsymbol{\phi}(x) = \boldsymbol{0}\}$ is given by (for $q_1^* = 0$)

$$\boldsymbol{\phi}(\boldsymbol{x}) = \begin{bmatrix} q_1 - \int_{q_2^*}^{q_2} \frac{a_1(\theta)}{a_2(\theta)} d\theta \\ p_2 - \frac{m_{d,12}}{m_{d,11}} p_1 \end{bmatrix} = \begin{bmatrix} \zeta_q \\ \zeta_p \end{bmatrix}. \tag{22}$$

A control law $\psi(\boldsymbol{x}, \boldsymbol{\phi}(\boldsymbol{x}))$ has to be determined such that the trajectories of the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\psi(\boldsymbol{x}, \boldsymbol{\phi}(\boldsymbol{x}))$$

remain bounded and $\lim_{t\to\infty} \boldsymbol{\zeta}(t) = \boldsymbol{0}$, where

$$\dot{\boldsymbol{\zeta}} = \partial_x \boldsymbol{\phi}(x)[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\psi(\boldsymbol{x}, \boldsymbol{\phi}(\boldsymbol{x}))].$$

Then the invariant manifold is attractive and the equilibrium $\boldsymbol{x}^*$ is stable.

As the target dynamics (17) was defined only Lyapunov stable, asymptotic stability of the desired equilibrium can not be deduced from making the manifold attractive in the studied example. Therefore, having in mind the IDA-PBC control laws derived above, the solution of the stabilization problem is considered from another perspective:

System in on/off manifold coordinates. The immersion mapping (21) and the off-manifold coordinates (22) define the regular coordinate change $(\boldsymbol{\zeta}, \boldsymbol{\xi}) = \boldsymbol{\tau}(\boldsymbol{q}, \boldsymbol{p})$ with

$$\begin{bmatrix} \zeta_q \\ \zeta_p \\ \xi_q \\ \xi_p \end{bmatrix} = \begin{bmatrix} \phi_1(\boldsymbol{q}, \boldsymbol{p}) \\ \phi_2(\boldsymbol{q}, \boldsymbol{p}) \\ q_2 \\ p_1 \end{bmatrix}.$$

In the following, the original *open-loop* model of the Acrobot will be expressed in a pH form in the new coordinates $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$. However, the energy gradient on the right hand side will *not* be taken from the physical energy $H(\boldsymbol{q}, \boldsymbol{p})$, but the shaped *closed-loop* energy $H_d(\boldsymbol{q}, \boldsymbol{p})$ from the IDA-PBC design according to (4). Evaluated in the new coordinates , it can be written as

$$\hat{H}_d(\zeta,\xi) = \hat{V}_d(\zeta_q,\xi_q) + \hat{T}_d(\zeta_p,\xi_p)$$

with the expressions for the potential and kinetic energy part

$$\hat{V}_d(\zeta_q,\xi_q) = \tilde{V}_d(z_1 = \zeta_q, z_2 = \xi_q)$$

and

$$\hat{T}_d(\zeta_p,\xi_p) = \frac{1}{2}\frac{m_{d,11}}{|\boldsymbol{M}_d|}\zeta_p^2 + \frac{1}{2m_{d,11}}\xi_p^2.$$

The corresponding mass matrix for this kinetic energy term is diagonal:

$$\hat{\boldsymbol{M}}_d = \operatorname{diag}\left\{\frac{|\boldsymbol{M}_d|}{m_{d,11}}, m_{d,11}\right\}.$$

Using these conventions, the open-loop system can be written as

$$\begin{bmatrix}\dot{\zeta}_q\\ \dot{\zeta}_p\\ \dot{\xi}_q\\ \dot{\xi}_p\end{bmatrix} = \begin{bmatrix}0 & -j_{12} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & j_{32} & 0 & j_{34}\\ 0 & 0 & -j_{34} & 0\end{bmatrix}\begin{bmatrix}\partial_{\zeta_q}\hat{H}_d\\ \partial_{\zeta_p}\hat{H}_d\\ \partial_{\xi_q}\hat{H}_d\\ \partial_{\xi_p}\hat{H}_d\end{bmatrix} + \begin{bmatrix}0\\ f_{\zeta_p}+u\\ 0\\ 0\end{bmatrix}$$

with the elements

$$\begin{aligned}
j_{12}(\xi_q) &= \frac{|\boldsymbol{M}_d|}{|\boldsymbol{M}(\xi_q)|}\frac{1}{\boldsymbol{m}_{d,1}^T\overline{\boldsymbol{m}}_2(\xi_q)},\\
j_{32}(\xi_q) &= \frac{|\boldsymbol{M}_d|}{|\boldsymbol{M}(\xi_q)|}\frac{m_{11}(\xi_q)}{m_{d,11}},\\
j_{34}(\xi_q) &= \boldsymbol{m}_{d,1}^T\overline{\boldsymbol{m}}_2(\xi_q),
\end{aligned}$$

of the interconnection matrix. The expression

$$f_{\zeta_p}(\boldsymbol{\zeta},\boldsymbol{\xi}) = (-\partial_{q_2}H + \frac{m_{d,12}}{m_{d,11}}\partial_{q_1}H)\circ\boldsymbol{\tau}^{-1}(\boldsymbol{\zeta},\boldsymbol{\xi})$$

in the second ODE is nothing else than the drift term on the right hand side of $\dot{\zeta}_p = \dot{p}_2 - \frac{m_{d,12}}{m_{d,11}}\dot{p}_1$, expressed in transformed coordinates. It ensures the equivalence of the above state differential equations with the original open-loop system representation.

Trajectory boundedness. A control law which provides Lyapunov stability of the equilibrium $(\boldsymbol{\zeta}^*,\boldsymbol{\xi}^*) = \boldsymbol{\tau}(\boldsymbol{q}^*,\boldsymbol{0})$ is certainly given by

$$u_{es} = -f_{\zeta_p}(\boldsymbol{\zeta},\boldsymbol{\xi}) + j_{12}(\xi_q)\partial_{\zeta_q}\hat{H}_d(\boldsymbol{\xi},\boldsymbol{\zeta}) - j_{32}(\xi_q)\partial_{\xi_q}\hat{H}_d(\boldsymbol{\xi},\boldsymbol{\zeta}). \tag{23}$$

Thus, the closed-loop system becomes

$$\begin{bmatrix} \dot{\zeta}_q \\ \dot{\zeta}_p \\ \dot{\xi}_q \\ \dot{\xi}_p \end{bmatrix} = \begin{bmatrix} 0 & -j_{12}(\xi_q) & 0 & 0 \\ j_{12}(\xi_q) & 0 & -j_{32}(\xi_q) & 0 \\ 0 & j_{32}(\xi_q) & 0 & j_{34}(\xi_q) \\ 0 & 0 & -j_{34}(\xi_q) & 0 \end{bmatrix} \begin{bmatrix} \partial_{\zeta_q} \hat{H}_d \\ \partial_{\zeta_p} \hat{H}_d \\ \partial_{\xi_q} \hat{H}_d \\ \partial_{\xi_p} \hat{H}_d \end{bmatrix}.$$

Both the $\boldsymbol{\zeta}$- and the $\boldsymbol{\xi}$-subsystem now are coupled via (a) the mixed terms in $\hat{V}_d(\zeta_q, \xi_q)$ and (b) the element $j_{32}(\xi_q)$ of the interconnection matrix. Observe that

$$\hat{H}_d(\zeta_q = z_1^*, \zeta_p = 0) = H_t(\xi_q, \xi_p)$$

holds and indeed, with $\boldsymbol{\zeta}$ as output vector, the target system (17), (18) represents the corresponding zero dynamics.

The system at this stage is lossless and lacks damping for asymptotic stability of its equilibrium. A closer look at the control law (23) and its transformation into the original states $(\boldsymbol{q}, \boldsymbol{p})$ reveals that it *exactly matches* the IDA-PBC energy shaping state feedback (15). Thus it seems appropriate to apply in addition the damping injection feedback (16) derived above to achieve asymptotic stability.

Damping injection. The damping injection law

$$u_{di} = -k_{di} \partial_{p_2} H_d(\boldsymbol{q}, \boldsymbol{p}) = -k_{di} \partial_{\zeta_2} \hat{H}_d(\boldsymbol{\zeta}, \boldsymbol{\xi})$$

leads to the closed-loop state representation

$$\begin{bmatrix} \dot{\zeta}_q \\ \dot{\zeta}_p \\ \dot{\xi}_q \\ \dot{\xi}_p \end{bmatrix} = \begin{bmatrix} 0 & -j_{12}(\xi_q) & 0 & 0 \\ j_{12}(\xi_q) & -k_{di} & -j_{32}(\xi_q) & 0 \\ 0 & j_{32}(\xi_q) & 0 & j_{34}(\xi_q) \\ 0 & 0 & -j_{34}(\xi_q) & 0 \end{bmatrix} \begin{bmatrix} \partial_{\zeta_q} \hat{H}_d \\ \partial_{\zeta_p} \hat{H}_d \\ \partial_{\xi_q} \hat{H}_d \\ \partial_{\xi_p} \hat{H}_d \end{bmatrix},$$

which, as already has been mentioned in the IDA-PBC part, has the desired asymptotically stable equilibrium.

4 Linear mechanical systems

The equivalence result derived for the Acrobot system relies on the solution of the homogeneous matching PDE in IDA-PBC which defines the characteristic coordinate $z_1 = t_1(\boldsymbol{q})$. The corresponding (partial) transformation can be used at the same time to define a part of the immersion mapping in I&I. For a system with n degrees of freedom and only one passive joint, the homogeneous version of the potential energy matching PDE has $n-1$ locally independent solutions which define $n-1$

characteristic coordinates to shape the resulting closed-loop energy. However, the transformation can in general not be expressed in a simple analytic way as in the Acrobot 2 DOF example[1]. Determining expressions for the characteristic coordinate functions, which identically can be used in the I&I design process, is a problem itself.

This section briefly sketches some ideas on how to approach the equivalence question for linear mechanical systems. First, from the perspective of the mentioned class of systems with an unactuated pivot, then from a more general point of view.

4.1 Kinematic chains with unactuated pivot

In the linear (or linearized) case, considered in this section, the mass matrices $\boldsymbol{M}$ and $\boldsymbol{M}_d$ are both constant, and the expressions for the potential energies V and V_d are quadratic forms in $\boldsymbol{q}$. According to Section 3, the I&I target system is supposed to have 1 DOF.

IDA-PBC matching PDE. The potential energy matching PDE in IDA-PBC can be written, as a generalization of (12) to the n DOF case,

$$\partial_q V_d(\boldsymbol{q})\,\boldsymbol{a} = \partial_{q_1} V(\boldsymbol{q}) \quad \Leftrightarrow \quad a_1 \partial_{q_1} V_d(\boldsymbol{q}) + \ldots + a_n \partial_{q_n} V_d(\boldsymbol{q}) = \partial_{q_1} V(\boldsymbol{q}),$$

with the constants

$$a_i = \boldsymbol{m}_{d,1}^T \overline{\boldsymbol{m}}_i, \quad i = 1, \ldots, n,$$

and reminding that $\overline{\boldsymbol{m}}_i$, $\boldsymbol{m}_{d,i}$ denote the i-th columns of the matrices $\overline{\boldsymbol{M}} = \boldsymbol{M}^{-1}$ and $\boldsymbol{M}_d$, respectively. For $\boldsymbol{a} \in \mathbb{R}^n$, there exists a square matrix $\boldsymbol{T} \in \mathbb{R}^{n \times n}$ such that the first $n-1$ elements of the vector $\boldsymbol{T}\boldsymbol{a}$ are zero, e. g.

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_1^T \\ \vdots \\ \boldsymbol{t}_n^T \end{bmatrix} = \boldsymbol{I}_n - \begin{bmatrix} \boldsymbol{0}_{n \times n-1} & \dfrac{\boldsymbol{a}}{a_n} \end{bmatrix}.$$

The coordinate change $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{q}$ with

$$\begin{aligned} z_i &= q_i - \frac{a_i}{a_n} q_n, \qquad i = 1, \ldots, n-1 \\ z_n &= q_n \end{aligned}$$

and its inverse $\boldsymbol{q} = \boldsymbol{T}^{-1}\boldsymbol{z}$ with

$$\begin{aligned} q_i &= z_i + \frac{a_i}{a_n} z_n, \qquad i = 1, \ldots, n-1 \\ q_n &= z_n \end{aligned}$$

[1] At least, if the approach as sketched above is followed, with $\boldsymbol{M}_d = const.$ and a simple choice of $\boldsymbol{G}^{\perp}$.

renders the potential energy matching PDE an ODE

$$\partial_{z_n} \tilde{V}_d(\boldsymbol{z}) = \frac{\partial_{q_1} V \circ \boldsymbol{T}^{-1}\boldsymbol{z}}{a_n}.$$

The structure coincides with (14), and consequently, the solution can be written as

$$\tilde{V}_d(\boldsymbol{z}) = \tilde{V}_d^p(\boldsymbol{z}) + \tilde{V}_d^h(z_1, \ldots, z_{n-1}).$$

In this case, $n-1$ characteristic coordinates are available for energy shaping.

I&I immersion mapping. Inspired by the 2 DOF case, the following immersion mapping is proposed:

$$\begin{bmatrix} q_1 \\ \vdots \\ q_{n-1} \\ q_n \end{bmatrix} = \begin{bmatrix} \pi_{1,1}(\xi_q) \\ \vdots \\ \pi_{1,n-1}(\xi_q) \\ \pi_{1,n}(\xi_q) \end{bmatrix} = \begin{bmatrix} z_1^* + \frac{a_1}{a_n}\xi_q \\ \vdots \\ z_{n-1}^* + \frac{a_{n-1}}{a_n}\xi_q \\ \xi_q \end{bmatrix}$$

and

$$\boldsymbol{p} = \boldsymbol{\pi}_2(\xi_p) = \frac{\boldsymbol{m}_{d,1}}{m_{d,11}}\xi_p,$$

where $\boldsymbol{m}_{d,1}$ is the first column of $\boldsymbol{M}_d$. Moreover, defining

$$\lambda = \boldsymbol{m}_{d,1}^T \overline{\boldsymbol{m}}_n = a_n, \qquad m_t = m_{d,11},$$

it turns out that the unactuated parts of the immersion condition (19) are satisfied. Particularly, the immersion PDE

$$\partial_{\xi_q} V_t(\xi_q) = \frac{\partial_{q_1} V \circ \boldsymbol{\pi}_1(\xi_q)}{a_n}$$

has again the same structure as the transformed matching PDE from IDA-PBC, restricted to the equilibrium values of $z_1^*, \ldots, z_{n-1}^*$, and therefore the corresponding solution is

$$V_t(\xi_q) = \tilde{V}_d(z_1^*, \ldots, z_{n-1}^*, \xi_q).$$

Consequently, the 1 DOF target system (17), (18), with the above choices of λ, the target mass m_t and the potential energy $V_t(\xi_q)$ is a feasible target system for the n DOF linearized kinematic chain with unactuated pivot. The statement can be proven in a straightforward way by replacing the corresponding terms in the immersion condition (19).

The discussion on deriving feedback laws to keep the closed-loop trajectories bounded and the target manifold attractive, in accordance with the energy shaping and damping injection steps of IDA-PBC, as carried out in Subsection 3.3, is omitted here.

4.2 General linear mechanical systems

In the preceding subsection the target system was chosen to have only 1 DOF in order to capture the desired behavior of the unactuated pivot. In this part, our focus is turned to the case of linear mechanical systems with an even number of n DOF and a choice of a higher dimensional target system.

To this end, consider the linear version of the pH system (1) and express it as a partition of two pH subsystems with the unactuated state $\boldsymbol{x}_1 = [\boldsymbol{q}_1^T, \boldsymbol{p}_1^T]^T$ and the actuated state $\boldsymbol{x}_2 = [\boldsymbol{q}_2^T, \boldsymbol{p}_2^T]^T$, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, as

$$(\Sigma): \quad \begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_1\boldsymbol{x}_1 \\ \boldsymbol{P}_2\boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{G}_2 \end{bmatrix} \boldsymbol{u}, \tag{24}$$

with $\boldsymbol{u} \in \mathbb{R}^{n/2}$ the vector of inputs[2]. The corresponding Hamiltonian is given as

$$H(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2}\boldsymbol{x}_1^T\boldsymbol{P}_1\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{x}_2^T\boldsymbol{P}_2\boldsymbol{x}_2$$

with positive definite diagonal matrices $\boldsymbol{P}_1,\ \boldsymbol{P}_2 \in \mathbb{R}^{n\times n}$.

Remark 2 *Note that the dynamics (1) can always be transformed into the dynamics* (24)*, that have decoupled the x_1 from the x_2 dynamics, by an appropriate coordinate change, see for example [13].*

IDA-PBC. Consider now the coordinate change defined by

$$\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} \mapsto \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 - \boldsymbol{\Pi}\boldsymbol{x}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{\zeta} \end{bmatrix}$$

with $\boldsymbol{\Pi} \in \mathbb{R}^{n\times n}$, $\mathrm{rank}(\boldsymbol{\Pi}) = n$. The resulting pH dynamics take the form

$$(\tilde{\Sigma}): \quad \begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{\zeta}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_1 & -\boldsymbol{J}_1\boldsymbol{\Pi}^T \\ -\boldsymbol{\Pi}\boldsymbol{J}_1 & \boldsymbol{J}_2 + \boldsymbol{\Pi}\boldsymbol{J}_1\boldsymbol{\Pi}^T \end{bmatrix} \begin{bmatrix} (\boldsymbol{P}_1 + \boldsymbol{\Pi}^T\boldsymbol{P}_2\boldsymbol{\Pi})\boldsymbol{x}_1 + \boldsymbol{\Pi}^T\boldsymbol{P}_2\boldsymbol{\zeta} \\ \boldsymbol{P}_2\boldsymbol{\Pi}\boldsymbol{x}_1 + \boldsymbol{P}_2\boldsymbol{\zeta} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{G}_2 \end{bmatrix} \boldsymbol{u}, \tag{25}$$

with the transformed Hamiltonian function given by

$$\tilde{H}(\boldsymbol{x}_1, \boldsymbol{\zeta}) = \frac{1}{2}\boldsymbol{x}_1^T(\boldsymbol{P}_1 + \boldsymbol{\Pi}^T\boldsymbol{P}_2\boldsymbol{\Pi})\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{\zeta}^T\boldsymbol{P}_2\boldsymbol{\zeta} + \boldsymbol{\zeta}^T\boldsymbol{P}_2\boldsymbol{\Pi}\boldsymbol{x}_1.$$

Following the IDA-PBC methodology as presented in Subsection 2.1, the desired (transformed) pH dynamics are chosen similarly to (3) as [3]

[2]The choice of the dimensions $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^{n/2}$, where n is an even number, is made for simplicity of presentation.

[3]In the expressions of the desired closed-loop system presented here, the dissipation terms resulting from the damping injection are included in order to obtain an insight on how this would affect the matching conditions and the proof of asymptotic stability. Accordingly for the choice of the target system in I&I defined below.

$$(\tilde{\Sigma}_d): \quad \begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{\zeta}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_1^d - \boldsymbol{R}_1^d & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_2^d - \boldsymbol{R}_2^d \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_1^d \boldsymbol{x}_1 \\ \boldsymbol{P}_2^d \boldsymbol{\zeta} \end{bmatrix}, \tag{26}$$

with the desired energy function

$$\tilde{H}_d(\boldsymbol{x}_1, \boldsymbol{\zeta}) = \frac{1}{2}\boldsymbol{x}_1^T \boldsymbol{P}_1^d \boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{\zeta}^T \boldsymbol{P}_2^d \boldsymbol{\zeta},$$

and positive definite diagonal matrices $\boldsymbol{P}_1^d$, $\boldsymbol{P}_2^d \in \mathbb{R}^{n\times n}$.

Matching the transformed dynamics (25) with the desired dynamics (26) yields

Matching conditions:

$$\begin{aligned} \boldsymbol{J}_1\boldsymbol{P}_1\boldsymbol{x}_1 &= (\boldsymbol{J}_1^d - \boldsymbol{R}_1^d)\boldsymbol{P}_1^d\boldsymbol{x}_1 \\ (\boldsymbol{J}_2\boldsymbol{P}_2\boldsymbol{\Pi} - \boldsymbol{\Pi}\boldsymbol{J}_1\boldsymbol{P}_1)\boldsymbol{x}_1 + \boldsymbol{J}_2\boldsymbol{P}_2\boldsymbol{\zeta} + \boldsymbol{G}_2\boldsymbol{u} &= (\boldsymbol{J}_2^d - \boldsymbol{R}_2^d)\boldsymbol{P}_2^d\boldsymbol{\zeta}. \end{aligned}$$

The first of the matching conditions fixes the dynamics of the unactuated subsystem while multiplying the second matching equation by the left hand annihilator $\boldsymbol{G}_2^\perp$ of $\boldsymbol{G}_2$ yields, after comparing the terms in $\boldsymbol{x}_1$ and $\boldsymbol{\zeta}$,

$$\boldsymbol{G}_2^\perp(\boldsymbol{J}_2\boldsymbol{P}_2\boldsymbol{\Pi} - \boldsymbol{\Pi}\boldsymbol{J}_1\boldsymbol{P}_1) = \boldsymbol{0} \tag{27}$$

$$\boldsymbol{G}_2^\perp(\boldsymbol{J}_2\boldsymbol{P}_2 - (\boldsymbol{J}_2^d - \boldsymbol{R}_2^d)\boldsymbol{P}_2^d) = \boldsymbol{0}. \tag{28}$$

I&I. On the other hand, following the I&I approach presented in Subsection 2.2 the linear target system dynamics are chosen consistently to (8) as

$$\dot{\boldsymbol{\xi}} = (\boldsymbol{J}_t - \boldsymbol{R}_t)\boldsymbol{P}_t\boldsymbol{\xi}$$

with $\boldsymbol{\xi} \in \mathbb{R}^n$, i.e. $s = n/2$, and the target Hamiltonian function

$$H_t(\boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{P}_t\boldsymbol{\xi},$$

where $\boldsymbol{P}_t \in \mathbb{R}^{n\times n}$ a positive definite matrix. Choosing the target manifold to be such that the unactuated dynamics are matched to the target dynamics gives the expression

$$\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\pi}_1(\boldsymbol{\xi}) \\ \boldsymbol{\pi}_2(\boldsymbol{\xi}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\Pi}\boldsymbol{\xi} \end{bmatrix}.$$

The corresponding matching conditions for the I&I approach are in this case given by

Immersion condition:

$$\boldsymbol{J}_1\boldsymbol{P}_1\boldsymbol{\xi} = (\boldsymbol{J}_t - \boldsymbol{R}_t)\boldsymbol{P}_t\boldsymbol{\xi}$$
$$\boldsymbol{J}_2\boldsymbol{P}_2\boldsymbol{\Pi}\boldsymbol{\xi} + \boldsymbol{G}_2\boldsymbol{c}(\boldsymbol{\pi}(\boldsymbol{\xi})) = \boldsymbol{\Pi}(\boldsymbol{J}_t - \boldsymbol{R}_t)\boldsymbol{P}_t\boldsymbol{\xi}.$$

Similarly to the IDA-PBC part, the first matching condition fixes the target dynamics while multiplying the second matching equation by the left annihilator $\boldsymbol{G}_2^{\perp}$, and using the first matching equation, yields

$$\boldsymbol{G}_2^{\perp}(\boldsymbol{J}_2\boldsymbol{P}_2\boldsymbol{\Pi} - \boldsymbol{\Pi}\boldsymbol{J}_1\boldsymbol{P}_1) = \boldsymbol{0}. \tag{29}$$

Comparing the expressions in (27) and (29) shows that these equations are identical. The second matching condition of the IDA-PBC approach given in (28) is absent in the I&I side which is consistent to the fact that the I&I approach is aiming at asymptotic matching thus, imposing less constraints as those imposed by the exact matching of IDA-PBC.

Finally, choosing the I&I control law $\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{\zeta})$ to be the IDA-PBC controller with $\boldsymbol{J}_t = \boldsymbol{J}_d^1$, $\boldsymbol{R}_t = \boldsymbol{R}_d^1$ will result in the closed-loop system defined by (26). Note however that any other controller $\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{\zeta})$ ensuring that $\lim_{t\to\infty}\boldsymbol{\zeta}(t) = \boldsymbol{0}$ would result in a closed-loop system of the form

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{\zeta}} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{J}_t - \boldsymbol{R}_t) & \boldsymbol{L} \\ \boldsymbol{0} & (\boldsymbol{J}_\zeta - \boldsymbol{R}_\zeta) \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_t\boldsymbol{x}_1 \\ \boldsymbol{P}_\zeta\boldsymbol{\zeta} \end{bmatrix},$$

with a certain matrix $\boldsymbol{L}$, since it is known by [14] that any (asymptotically) stable linear system can always be written in the pH form with a skew-symmetric $\boldsymbol{J}_\zeta$, a positive semi-definite (positive definite) $\boldsymbol{R}_\zeta$ and a positive definite $\boldsymbol{P}_\zeta$, of corresponding dimensions.

5 Discussion and outlook

In the comparative study which formed the main part of this contribution, a solution of the IDA-PBC energy shaping plus damping injection problem is assumed to be given for the Acrobot example. In particular, the coordinate change, which transforms the potential energy matching PDE into an ODE is used to define one part of the immersion mapping in the corresponding I&I problem. The potential energy of the lower dimensional target system corresponds to the closed-loop potential energy in IDA-PBC, restricted to the equilibrium value of the characteristic coordinate. Accordingly, the inertia of the I&I target system equals the first element of the IDA-PBC closed-loop mass matrix.

The on-manifold coordinates $\boldsymbol{\xi}$ can be naturally complemented by a set of off-manifold coordinates $\boldsymbol{\zeta}$, which together constitute a regular coordinate change. Writing down the transformed state differential equations in pH form – again taking advantage of the known solution of the IDA-PBC energy shaping problem – provides a clear structure of the state representation. Based on this formulation an energy shaping control law can be simply derived such that the system becomes stable with an equilibrium at the minimum of the previously shaped energy. The system at this stage has the structure of two interconnected subsystems, where it has to be mentioned that both subsystems are not only coupled via the obvious coupling element j_{32} of the interconnection matrix, but already via the mixed terms in the potential energy. With straightforward computations it can be checked that the feedback law which provides this convenient structure is exactly the energy shaping controller from IDA-PBC.

In the final damping injection step, the passive IDA-PBC output feedback is applied, which provides damping in the $\boldsymbol{\zeta}$-subsystem. The system gets the structure of two coupled oscillators, one of them damped. The damping is propagated between both system parts and finally the system comes to rest at the minimum of its virtual energy.

As a result of the above study it can be claimed that the structural similarity (in parts) of the IDA-PBC and the I&I approaches can be fruitfully exploited. In particular, given a solution of the controller design problem using the IDA-PBC methodology, this solution has an equivalent representation in the context of I&I, which has an interesting structure.

At the end of this contribution the attention is directed to an instrumental question in I&I, namely how to choose the states of the target system. In [3] the unactuated part of a mechanism is mentioned as a sensible choice. However, it is also noted that the choice is not unique and can be driven by the solvability of the immersion conditions. This is what can be observed with the solution to the Acrobot problem presented here. While $\xi_q = q_2$ and $\xi_p = p_1$ is probably not the most intuitive choice for the target system states, it allows for a solution of the I&I problem, which is inspired from and equivalent to the (known) solution of the IDA-PBC problem.

Another motivation to study the relationship between the two approaches is based on the recent results in [17] which show that the I&I approach is able to stabilize a desired equilibrium for the Cart–Pendulum system having a *constant* desired inertia matrix which is known not to be possible using the IDA-PBC approach.

In the last section, some equivalences of IDA-PBC and I&I have been revealed for linear mechanical systems. The results in 4.1 can be a useful local starting point when the corresponding nonlinear problem for the considered class of systems with one degree of unacutation is attacked. In Subsection 4.2, direct equivalences of the IDA-PBC and I&I equations for a more general class of linear mechanical systems have become obvious.

Future research questions in the presented context are certainly (a) the solution of the nonlinear controller design problem for the class of systems with n DOF and unactuated pivot. To this end, at least an approximate expression of the coordinate change (13) must be derived, corresponding to Eqs. (17) and (18) in [7], using computer algebra. Then Eq. (20) can be correspondingly generalized for the $(n-1)$ DOF immersion mapping. (b) A further investigation of systems with different structure, like the Cart-Pendulum system would be a topic of interest. (c) Recently in [1] a methodology was proposed to replace the PDEs of IDA-PBC with algebraic inequalities by constructing an approximate integral and a dynamic extension as an alternative to solving the PDEs. Although this idea was first introduced in the observer design using the I&I framework, see for example the recent article on mechanical systems [4] and references therein, it has not been exploited in the I&I stabilization setting and is certainly to be investigated. And finally, (d) for the linear case, it would be desirable to obtain necessary and/or sufficient conditions for I&I stabilization similarly to the works [13], [14] on linear IDA-PBC.

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Figures

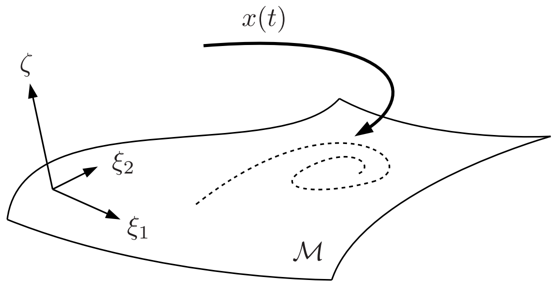

Figure 1: Illustration of asymptotic matching in I&I.

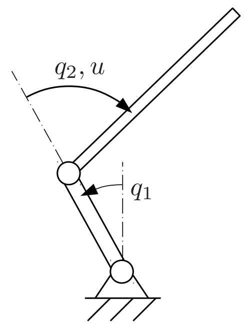

Figure 2: Sketch of the Acrobot, u denotes the torque applied to the intermediate joint.