$\mathcal{H}_\infty$ bound for linear port-Hamiltonian systems

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In this article, a new bound on the $\mathcal{H}_\infty$ norm for linear port-Hamiltonian systems with or without the collocated output is proposed. It is shown that its computation is efficient, compared to the general algorithms used in numerical analysis software. The requirements for the applicability of the bound are little restrictive, and proven to be met by a large class of technically relevant systems. Additionally, for purely damped port-Hamiltonian systems without an interconnection matrix, the bound is shown to be exact. The performance of the newly introduced bound is illustrated by two numerical examples.

1 Introduction

Port-Hamiltonian systems are a geometrically defined class of systems developed in recent years. The modeling approach unifies concepts from theoretical mechanics and electrical engineering. Using energy as a link, the port-Hamiltonian formalism provides a unified framework for the interdisciplinary modeling of complex dynamical systems by interconnecting subsystems from different physical domains. Due to its conceptual simplicity and modularity the port-Hamiltonian approach is suitable for automated model generation, processing and simulation. Furthermore, the resulting state-space representation reveals physical insight into the system.

Two widely-used measurements used for characterizing linear dynamical systems are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm. For a single-input single-output system these can be interpreted as the integral of the transfer function over the imaginary axis and the peak in the associated *Bode* plot of the system, respectively. The norms are employed in several tasks of modern control and systems theory, such as optimal or robust control or model order reduction.

In order to compute the norms of linear dynamical systems different algorithms are implemented successfully in numerical analysis software, see e. g. [2]. However, dense matrix computations are required for its implementation. In this article, a new bound on the $\mathcal{H}_\infty$ norm for linear port-Hamiltonian systems is proposed, offering faster processing with the drawback of possibly introducing some overestimation. In addition, the results are generalized to systems with a port-Hamiltonian state-equation and an arbitrary output vector.

The article is organized as follows. In section 2 some preliminaries are reviewed, such as the port-Hamiltonian system representation, and a problem formulation is given. The new bound on the $\mathcal{H}_\infty$ norm is presented for port-Hamiltonian systems in section 3 and for arbitrary outputs in section 4. Purely damped and second-order systems exhibit special properties regarding the bound, which is discussed in section 5. Two numerical examples demonstrate the performance of the bound in section 6.

2 Preliminaries and problem statement

In this section some preliminaries are given, together with the problem formulation.

2.1 Port-Hamiltonian systems

Consider the linear time invariant (LTI) port-Hamiltonian system of the form

$$\begin{aligned}\dot{\mathbf{x}}(t) =& \left(\mathbf{J}-\mathbf{R}\right)\mathbf{Q}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t),\\ \mathbf{y}(t) =& \mathbf{B}^T\mathbf{Q}\mathbf{x}(t),\end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^p$ and $\mathbf{y}(t) \in \mathbb{R}^p$ denote the states, inputs and outputs of the system, respectively. The skew-symmetric matrix $\mathbf{J} = -\mathbf{J}^T \in \mathbb{R}^{n\times n}$ defines the interconnection of the energy-storing states, while the positive semi-definite matrix $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0} \in \mathbb{R}^{n\times n}$ describes the dissipation in the system. The Hamiltonian $H(\mathbf{x}) = {}^1\!/_2\,\mathbf{x}^T\mathbf{Q}\mathbf{x}$ with the positive-definite matrix $\mathbf{Q} = \mathbf{Q}^T > \mathbf{0} \in \mathbb{R}^{n\times n}$ gives the total stored energy in the system. The collocated input- and output-matrix, $\mathbf{B} \in \mathbb{R}^{n\times p}$ and $\mathbf{B}^T\mathbf{Q}$, respectively, lead to

$$\dot{H}(\mathbf{x}) \leq \mathbf{y}^T\mathbf{u}, \tag{2}$$

verifying the passivity of port-Hamiltonian systems. Stability in the sense of Lyapunov follows from the positive-definiteness of the Hamiltonian $H > 0$. In case the dissipation matrix has full rank ($\mathbf{R} > \mathbf{0}$), asymptotic stability is given, whereas in case $\mathbf{R}$ is singular, asymptotic stability can be checked by the invariance principle of Krassowskij-LaSalle. A detailed overview on port-Hamiltonian systems can be found in [6]. The transfer-matrix $G(s)$ from input to output of system (1) is readily given by

$$G(s) = \mathbf{B}^T\mathbf{Q}\left(s\mathbf{I}-\left(\mathbf{J}-\mathbf{R}\right)\mathbf{Q}\right)^{-1}\mathbf{B}, \tag{3}$$

where s denotes the Laplace operator. With the usual abuse of notation, both the dynamical system (1) and the transfer function (3) will be denoted by $G(s)$. For the ease of presentation, the following abbreviations are introduced:

$$\mathbf{A} := \left(\mathbf{J}-\mathbf{R}\right)\mathbf{Q}, \tag{4}$$

$$\mathbf{C}_{PH} := \mathbf{B}^T\mathbf{Q}. \tag{5}$$

2.2 Port-Hamiltonian and general LTI-systems

When considering port-Hamiltonian systems, the output is restricted to the collocated output $\mathbf{C}_{PH}$, defined by the input vector and the energy matrix. However, the output of interest might not coincide with this collocated output. Nevertheless, the results on the $\mathcal{H}_\infty$ norm of port-Hamiltonian systems will be later generalized to port-Hamiltonian state-equations equipped with any output equation. The dynamics of such systems read as

$$\begin{aligned}\dot{\mathbf{x}}(t) =& \left(\mathbf{J}-\mathbf{R}\right)\mathbf{Q}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t),\\ \mathbf{y}(t) =& \mathbf{C}\mathbf{x}(t),\end{aligned} \tag{6}$$

where $\mathbf{C} \neq \mathbf{B}^T\mathbf{Q} \in \mathbb{R}^{q\times n}$ is an arbitrary output vector. Thus, the dimension of the output q needs not necessarily to be same as the dimension of the input p. Note that in case of non-collocated outputs the passivity property might be lost. However, the conclusions about stability are equal to real port-Hamiltonian systems (1). This can be readily verified, since the state equation remains unchanged.

The only difference of system (6) to a general LTI-system is the decomposition of the matrix $\mathbf{A}$. The following lemma shows the relation between these systems.

Lemma 1. *The state-equation of a given stable LTI-system*

$$\begin{aligned}\dot{\mathbf{x}}(t) =& \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ \mathbf{y}(t) =& \mathbf{C}\mathbf{x}(t),\end{aligned} \tag{7}$$

can be written in port-Hamiltonian form (6).

Proof. For a stable LTI-system there exist many solutions $\mathbf{Q} = \mathbf{Q}^T > \mathbf{0}$ to the Lyapunov-equation:

$$\mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A} \leq \mathbf{0}. \tag{8}$$

In [5] it is verified that with the choice

$$\mathbf{J} = \frac{1}{2}\left(\mathbf{A}\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}^T\right), \tag{9}$$

$$\mathbf{R} = -\frac{1}{2}\left(\mathbf{A}\mathbf{Q}^{-1} + \mathbf{Q}^{-1}\mathbf{A}^T\right), \tag{10}$$

a port-Hamiltonian decomposition of the dynamic matrix $\mathbf{A} = (\mathbf{J} - \mathbf{R})\mathbf{Q}$ can be obtained by starting from any Lyapunov solution $\mathbf{Q}$. □

Remark. The results of this paper are presented for port-Hamiltonian state equations. But keeping Lemma 1 in mind, the results are valid for any stable LTI-system, once a solution $\mathbf{Q}$ to the Lyapunov equation (8) is known. However, since the Lyapunov solution introduces solely a virtual energy, physical insight might be lost.

2.3 $\mathcal{H}_\infty$ norm

The $\mathcal{H}_\infty$ norm of a dynamical system $G(s)$ is defined in the frequency-domain as

$$\|G(s)\|_\infty := \sup_\omega \sigma_{\max}\{G(j\omega)\}, \tag{11}$$

where $\sigma_{\max}\{\cdot\}$ denotes the maximum singular value. For a single-input single-output system the $\mathcal{H}_\infty$ norm of a system can be identified with the peak in its associated bode-plot. Information on standard algorithms to compute the $\mathcal{H}_\infty$ norm of a system can be found in [2].

2.4 The problem

The $\mathcal{H}_\infty$ norm can be used to characterize dynamical systems. However, when dealing with large systems the standard algorithms to compute the norm become numerically demanding. In this paper, a numerical efficient bound on the norm is presented, that possibly introduces some overestimation. Based on the results it will be shown, when the new bound is reasonably applied.

3 Bound on the $\mathcal{H}_\infty$ norm of linear port-Hamiltonian systems

In this section a bound on the $\mathcal{H}_\infty$ norm of linear port-Hamiltonian systems is proposed. Before presenting the bound as the main contribution of this paper, the following auxiliary lemma is derived, based on the so-called *Bounded Real Lemma*.

Lemma 2. *For a linear port-Hamiltonian systems (1), let* $\gamma > 0 \in \mathbb{R}$ *be a solution to the matrix inequality* $\gamma\mathbf{R} - \mathbf{B}\mathbf{B}^T \geq \mathbf{0}$*, then* $\|G(s)\|_\infty \leq \gamma$.

Proof. The proof is based on a rather unusual formulation of the Bounded Real Lemma, to be found in [8]. It states that $\|G(s)\|_\infty < \gamma$ for a $\gamma > 0$ if and only if there exists a solution $\mathbf{X} = \mathbf{X}^T$ to the linear matrix inequality

$$\mathbf{M} := \begin{pmatrix} \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} & \mathbf{X}\mathbf{B} & \mathbf{C}_{PH}^T \\ \mathbf{B}^T\mathbf{X} & -\gamma\mathbf{I}_p & \mathbf{0} \\ \mathbf{C}_{PH} & \mathbf{0} & -\gamma\mathbf{I}_p \end{pmatrix} < \mathbf{0}, \tag{12}$$

where $\mathbf{I}_p \in \mathbb{R}^{p\times p}$ denotes the identity matrix. For the purpose of this paper, the non-strict version of the Bounded Real Lemma is preferable. In [7] it is shown, that if there is a solution $\mathbf{X} = \mathbf{X}^T$ to the non-strict inequality $\mathbf{M} \leq \mathbf{0}$, then $\|G(s)\|_\infty \leq \gamma$. Thus, by application of the *Schur*-Lemma, $\mathbf{M} \leq \mathbf{0}$ is equivalent to:

$$\begin{aligned} -\gamma\, \mathbf{I}_{2p} &\leq \mathbf{0}, &(13)\\ \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \frac{1}{\gamma}\left(\mathbf{X}\mathbf{B}\mathbf{B}^T\mathbf{X} + \mathbf{C}_{PH}^T\mathbf{C}_{PH}\right) &\leq \mathbf{0}. &(14)\end{aligned}$$

Obviously, equation (13) holds true. A natural choice for port-Hamiltonian systems is $\mathbf{X} := \mathbf{Q}$. Together with (4) and (5), equation (14) becomes:

$$\mathbf{Q}\left(-\mathbf{J} - \mathbf{R}\right)\mathbf{Q} + \mathbf{Q}\left(\mathbf{J} - \mathbf{R}\right)\mathbf{Q} + \frac{2}{\gamma}\mathbf{Q}\mathbf{B}\mathbf{B}^T\mathbf{Q} \leq \mathbf{0}. \tag{15}$$

Multiplication from left and right with $\mathbf{Q}^{-1}$ leads to

$$-\mathbf{R} + \frac{1}{\gamma}\mathbf{B}\mathbf{B}^T \leq \mathbf{0}, \tag{16}$$

and completes the proof. □

With the help of Lemma 2 the main contribution of this paper can be stated, after giving the following definition.

Definition 1. Let $\mathbf{Y} \in \mathbb{R}^{n\times k}$ and $\mathbf{Z} \in \mathbb{R}^{n\times(n-k)}$ be bases of the image and kernel of $\mathbf{R}$, respectively, where $k = \operatorname{rank}\{\mathbf{R}\}$. Then define:

$$\begin{aligned} \widehat{\mathbf{R}} &:= \mathbf{Y}^T\mathbf{R}\mathbf{Y}, &(17)\\ \widehat{\mathbf{B}} &:= \mathbf{Y}^T\mathbf{B}. &(18)\end{aligned}$$

Theorem 1. *Given a linear port-Hamiltonian system (1), assume that* $\mathbf{B}$ *lies in the range of* $\mathbf{R}$, *i. e.* $\mathbf{B} \subset \operatorname{span}\{\mathbf{Y}\}$. *Then a bound on the* $\mathcal{H}_\infty$ *norm is given by* $\|G(s)\|_\infty \leq \lambda_{\max}\left\{\widehat{\mathbf{B}}^T\widehat{\mathbf{R}}^{-1}\widehat{\mathbf{B}}\right\}$, *where* $\lambda_{\max}\{\cdot\}$ *denotes the largest eigenvalue of a matrix.*

Proof. The proof starts with Lemma 2. Multiplying equation (16) from the right with the square and full-rank matrix $[\mathbf{Y}\ \mathbf{Z}]$ and from the left with its transpose, yields:

$$\gamma \begin{bmatrix} \mathbf{Y}^T \\ \mathbf{Z}^T \end{bmatrix} \mathbf{R} \begin{bmatrix} \mathbf{Y} & \mathbf{Z} \end{bmatrix} - \begin{bmatrix} \mathbf{Y}^T \\ \mathbf{Z}^T \end{bmatrix} \mathbf{B}\mathbf{B}^T \begin{bmatrix} \mathbf{Y} & \mathbf{Z} \end{bmatrix} \geq \mathbf{0}. \tag{19}$$

Due to $\mathbf{Z}$ being a basis of the kernel of $\mathbf{R}$, $\mathbf{R}\mathbf{Z} = \mathbf{Z}^T\mathbf{R} = \mathbf{0}$. Further note that the assumption $\mathbf{B} \subset \operatorname{span}\{\mathbf{Y}\}$ implies $\mathbf{B}^T\mathbf{Z} = \mathbf{0}$, since $\mathbf{R}$ is symmetric and thus its range and kernel are orthogonal to each other. For this reason, equation (19) becomes:

$$\gamma \begin{bmatrix} \widehat{\mathbf{R}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \widehat{\mathbf{B}}\widehat{\mathbf{B}}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \geq \mathbf{0}. \tag{20}$$

Thus, what remains is

$$\gamma\, \mathbf{v}^T\widehat{\mathbf{R}}\mathbf{v} - \mathbf{v}^T\widehat{\mathbf{B}}\widehat{\mathbf{B}}^T\mathbf{v} \geq \mathbf{0}, \qquad \forall\, \mathbf{v} \in \mathbb{R}^k, \tag{21}$$

being equivalent to

$$\gamma \geq \frac{\mathbf{v}^T\widehat{\mathbf{B}}\widehat{\mathbf{B}}^T\mathbf{v}}{\mathbf{v}^T\widehat{\mathbf{R}}\mathbf{v}}, \qquad \forall\, \mathbf{v} \in \mathbb{R}^k, \tag{22}$$

due to the positive definiteness of $\widehat{\mathbf{R}}$. Introducing the *Cholesky*-factorization of $\widehat{\mathbf{R}} = \mathbf{L}\mathbf{L}^T$, and performing a coordinate change $\mathbf{w} = \mathbf{L}^T\mathbf{v}$ leads to:

$$\gamma \geq \frac{\mathbf{w}^T\mathbf{L}^{-1}\widehat{\mathbf{B}}\widehat{\mathbf{B}}^T\mathbf{L}^{-T}\mathbf{w}}{\mathbf{w}^T\mathbf{w}}, \qquad \forall\, \mathbf{w} \in \mathbb{R}^k. \tag{23}$$

Equation (23) denotes a *Rayleigh quotient*, whose maximum value is the largest eigenvalue of the matrix $\mathbf{L}^{-1}\widehat{\mathbf{B}}\widehat{\mathbf{B}}^T\mathbf{L}^{-T}$. For arbitrary matrices $\mathbf{G}$ and $\mathbf{H}$ of appropriate size $\lambda_{\max}\{\mathbf{GH}\} = \lambda_{\max}\{\mathbf{HG}\}$, and thus:

$$\gamma \geq \lambda_{\max}\left\{\mathbf{L}^{-1}\widehat{\mathbf{B}}\widehat{\mathbf{B}}^T\mathbf{L}^{-T}\right\} = \lambda_{\max}\left\{\widehat{\mathbf{B}}^T\mathbf{L}^{-T}\mathbf{L}^{-1}\widehat{\mathbf{B}}\right\} = \lambda_{\max}\left\{\widehat{\mathbf{B}}^T\widehat{\mathbf{R}}^{-1}\widehat{\mathbf{B}}\right\}, \tag{24}$$

which completes the proof. □

Remark. The assumptions made for Theorem 1 are that the image of $\mathbf{R}$ has to be known or easily computable and additionally, that $\mathbf{B}$ has to lie in the range of $\mathbf{R}$. As will be shown in section 5 this does not constitute a restriction to a large class of technically relevant systems.

Remark. Note that, for the calculation of the bound the inverse $\widehat{\mathbf{R}}^{-1}$ does not have to be known explicitly; it is sufficient to solve the linear system of equations $\widehat{\mathbf{R}}\mathbf{N} = \widehat{\mathbf{B}}$ for $\mathbf{N} \in \mathbb{R}^{k\times p}$. Furthermore, it is stressed, that $\widehat{\mathbf{B}}^T\widehat{\mathbf{R}}^{-1}\widehat{\mathbf{B}}$ is of small dimensions $p \times p$, such that the eigenvalues can be easily computed.

Lemma 3. *The H_∞ bound for linear port-Hamiltonian systems proposed in Theorem 1 is invariant to state-space transformations of the form $\mathbf{z} = \mathbf{T}\mathbf{x}$.*

Proof. Let $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{R}}$, $\widetilde{\mathbf{B}}$ denote the respective matrices of the state-space representation in transformed coordinates $\mathbf{z}$. To proof the Lemma, it is sufficient to show the equivalence of

(i) $\gamma\mathbf{R} - \mathbf{B}\mathbf{B}^T \geq \mathbf{0}$, and

(ii) $\gamma\widetilde{\mathbf{R}} - \widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^T \geq \mathbf{0}$.

Applying the state-transformation $\mathbf{x} = \mathbf{T}^{-1}\mathbf{z}$ to the original formulation (1) leads to:

$$\widetilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \mathbf{T}\left(\mathbf{J} - \mathbf{R}\right)\mathbf{T}^T\mathbf{T}^{-T}\mathbf{Q}\mathbf{T}^{-1}. \tag{25}$$

Thus, the dissipation matrix in transformed coordinates $\mathbf{z}$ reads as $\widetilde{\mathbf{R}} := \mathbf{T}\mathbf{R}\mathbf{T}^T$. It is easy to verify that $\widetilde{\mathbf{B}} = \mathbf{T}\mathbf{B}$. Therefore, (ii) can be rewritten as

$$\gamma\mathbf{T}\mathbf{R}\mathbf{T}^T - \mathbf{T}\mathbf{B}\mathbf{B}^T\mathbf{T}^T \geq \mathbf{0}, \tag{26}$$

which is equivalent to (i). □

4 Bound on the $\mathcal{H}_\infty$ norm of stable LTI-systems

In this section the results of the bound on the $\mathcal{H}_\infty$ norm is generalized to systems of the form (6), having a port-Hamiltonian state-equation and an arbitrary output. Note that with Lemma 1 this is in fact a generalization to stable LTI-systems where any Lyapunov solution is known. For the ease of presentation, the matrix $\mathbf{G}$ is introduced, collecting the input and output vectors into one matrix:

$$\mathbf{G} := \left[\begin{array}{cc} \mathbf{B} & \mathbf{Q}^{-1}\mathbf{C}^T \end{array} \right]. \tag{27}$$

Note that for the calculation of $\mathbf{G}$ the inverse $\mathbf{Q}^{-1}$ does not have to be known explicitly. Instead, it is sufficient to solve the linear set of equations $\mathbf{QN} = \mathbf{C}^T$ for $\mathbf{N} \in \mathbb{R}^{n\times q}$.

Lemma 4. *For a linear systems of the form (6), let $\gamma > 0 \in \mathbb{R}$ be a solution to the matrix inequality $\gamma\, 2\mathbf{R} - \mathbf{G}\mathbf{G}^T \geq \mathbf{0}$, then $\|G(s)\|_\infty \leq \gamma$.*

Proof. Note that $\mathbf{B}\mathbf{B}^T + \mathbf{Q}^{-1}\mathbf{C}^T\mathbf{C}\mathbf{Q}^{-1} = \mathbf{G}\mathbf{G}^T$. The rest of the proof is similar to the one of Lemma 2, and hence omitted. □

Accordingly to Definition 1, the following abbreviation is introduced:

$$\widehat{\mathbf{G}} := \mathbf{Y}^T\mathbf{G} = \left[\begin{array}{cc} \mathbf{Y}^T\mathbf{B} & \mathbf{Y}^T\mathbf{Q}^{-1}\mathbf{C}^T \end{array} \right]. \tag{28}$$

Theorem 2. *Given a linear system of the form (6), assume that $\mathbf{G}$ lies in the range of $\mathbf{R}$, i.e. $\mathbf{G} \subset \operatorname{span}\{\mathbf{Y}\}$. Then a bound on the $\mathcal{H}_\infty$ norm is given by $\|G(s)\|_\infty \leq \frac{1}{2}\lambda_{\max}\left\{\widehat{\mathbf{G}}^T\widehat{\mathbf{R}}^{-1}\widehat{\mathbf{G}}\right\}$.*

Proof. The proof is conducted similarly to the one of Theorem 1 by substituting $2\,\widehat{\mathbf{R}}$ for $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{G}}$ for $\widehat{\mathbf{B}}$. □

Remark. Theorem 2 shows that the results for port-Hamiltonian systems can be generalized to arbitrary but stable LTI-systems, with the restriction that a Lyapunov solution $\mathbf{Q}$ and the directions $\mathbf{Q}^{-1}\mathbf{C}^T$ have to be known. Then the largest eigenvalue of a matrix having the dimension $(p+q)\times(p+q)$ has to be computed.

Lemma 5. *The H_∞ bound for linear systems of the form (6) proposed in Theorem 2 is invariant to state-space transformations of the form $\mathbf{z} = \mathbf{T}\mathbf{x}$.*

Proof. Let $\widetilde{\mathbf{G}}$ denote the matrix $\mathbf{G}$ in transformed coordinates. Then,

$$\widetilde{\mathbf{G}} = \left[\begin{array}{cc} \widetilde{\mathbf{B}} & \widetilde{\mathbf{Q}}^{-1}\widetilde{\mathbf{C}}^T \end{array} \right] = \left[\begin{array}{cc} \mathbf{TB} & \mathbf{T}\mathbf{Q}^{-1}\mathbf{T}^T\mathbf{T}^{-T}\mathbf{C}^T \end{array} \right] = \mathbf{T}\left[\begin{array}{cc} \mathbf{B} & \mathbf{Q}^{-1}\mathbf{C}^T \end{array} \right] = \mathbf{TG}. \tag{29}$$

The rest of the proof is similar to the one of Lemma 3, and hence omitted. □

5 Relevant system classes

In this section two particular system classes are highlighted, together with their attributes according to the proposed H_∞ bound in the previous sections.

5.1 Systems with $\mathbf{J} = \mathbf{0}$

This paper proposes a *bound* on the H_∞ norm, implicating an overestimation of the real H_∞ norm. However, the following Lemma shows that for the case of $\mathbf{J} = \mathbf{0}$ the bound is exact.

Lemma 6. *Let $G(s)$ be a linear port-Hamiltonian system (1) with $\mathbf{J} = \mathbf{0}$, then $\|G(s)\|_\infty = \lambda_{\max}\left\{\mathbf{B}^T\mathbf{R}^{-1}\mathbf{B}\right\}$.*

Proof. The upper bound introduced in Theorem 1 holds true also for $\mathbf{J} = \mathbf{0}$, i. e. $\|G(s)\|_\infty \leq \lambda_{\max}\left\{\widehat{\mathbf{B}}^T\widehat{\mathbf{R}}^{-1}\widehat{\mathbf{B}}\right\}$. The proof will be completed by showing that this bound is actually achieved at some frequency for the matrices $\mathbf{R}$ and $\mathbf{B}$ instead of $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{B}}$. Choose $\omega = 0$, then:

$$G(0) = \mathbf{B}^T\mathbf{Q}\left(0\cdot\mathbf{I} - \mathbf{R}\mathbf{Q}\right)^{-1}\mathbf{B} = \mathbf{B}^T\mathbf{R}^{-1}\mathbf{B}. \tag{30}$$

Due to the symmetry of $\mathbf{R}$:

$$\sigma_{\max}\left\{\mathbf{B}^T\mathbf{R}^{-1}\mathbf{B}\right\} = \lambda_{\max}\left\{\mathbf{B}^T\mathbf{R}^{-1}\mathbf{B}\right\}, \tag{31}$$

which yields:

(i) $\sigma_{\max}\left\{G(j0)\right\} = \lambda_{\max}\left\{\mathbf{B}^T\mathbf{R}^{-1}\mathbf{B}\right\}$,

(ii) $\|G(s)\|_\infty = \sup_\omega \sigma_{\max}\left\{G(j\omega)\right\} \leq \lambda_{\max}\left\{\widehat{\mathbf{B}}^T\widehat{\mathbf{R}}^{-1}\widehat{\mathbf{B}}\right\}$.

In the case $\mathbf{J} = \mathbf{0}$ the system has to be fully damped – i.e. $\operatorname{rank}\left\{\mathbf{R}\right\} = n$ – for asymptotic stability. Therefore $\mathbf{R} = \widehat{\mathbf{R}}$ and $\mathbf{B} = \widehat{\mathbf{B}}$ which completes the proof for asymptotically stable systems. If $\mathbf{R}$ was singular, the system would have a pole in $\lambda = 0$ leading to $\|G(s)\|_\infty \to \infty$. Accordingly, $\lambda_{\max}\left\{\mathbf{B}^T\mathbf{R}^{-1}\mathbf{B}\right\} \to \infty$, which completes the proof. □

5.2 Second-order systems

A technically important class of linear models are second-order systems. These arise for example from the modeling of a mechanical system by the *finite-element method* (FEM). Likewise, *nodal analysis* (NA) of RLC circuits leads to second-order systems, as well. The general formulation reads as follows:

$$\mathbf{M}\,\ddot{\mathbf{q}}(t) + \mathbf{D}\,\dot{\mathbf{q}}(t) + \mathbf{K}\,\mathbf{q}(t) = \mathbf{F}\,\mathbf{u}(t), \tag{32}$$

where $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K} > \mathbf{0} \in \mathbb{R}^{m\times m}$ are symmetric positive definite, denoting for mechanical systems the mass, damping and stiffness, respectively. Introducing the state-vector $\mathbf{x}(t) = \begin{bmatrix} \mathbf{q}^T(t) & \mathbf{p}^T(t) \end{bmatrix}^T \in \mathbb{R}^n$, where $\mathbf{p}(t) = \mathbf{M}\,\dot{\mathbf{q}}(t)$ and $n = 2\,m$, leads to a port-Hamiltonian state-space representation (1), with:

$$\begin{aligned} \mathbf{R} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}, & \mathbf{Q} &= \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1} \end{bmatrix}, \\ \mathbf{J} &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}, & \mathbf{B} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix}. \end{aligned} \tag{33}$$

Note that the collocated output in this case is $\mathbf{y}(t) = \mathbf{F}^T\dot{\mathbf{q}}(t)$. Clearly,

$$\mathbf{Y} := \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \tag{34}$$

defines a basis for $\mathrm{span}\left\{\mathbf{R}\right\}$, which proofs that the assumptions made in Theorem 1, i. e. $\mathbf{B} \subset \mathrm{span}(\mathbf{R})$, always apply to second-order systems. Therefore, the proposed bound on the H_∞ norm in Theorem 1 can always be used for second-order systems in a straight-forward manner, as $\widehat{\mathbf{R}} = \mathbf{Y}^T\mathbf{R}\mathbf{Y} = \mathbf{D}$ and $\widehat{\mathbf{B}} = \mathbf{Y}^T\mathbf{B} = \mathbf{F}$, which leads to:

$$\|G(s)\|_\infty \leq \lambda_{\max}\left\{\mathbf{F}^T\mathbf{D}^{-1}\mathbf{F}\right\}. \tag{35}$$

6 Technical example

In this section, the performance and suitability of the proposed H_∞ bound is illustrated by two numerical exmples.

6.1 Thermal Model of an H-Bridge

As a first example, the thermal fitting model for the STMicroelectronics H-bridge motor driver VNH2SP30-E is considered. The integrated power circuit consists of four power MOSFETs arranged on three separate chip areas (dice) – the high potential MOSFETs mounted on a common die. The thermal behavior is described by an equivalent network composed of capacitances and resistances. Due to the irreversible transformation of thermal energy, the interconnection matrix $\mathbf{J} = \mathbf{0}$ is absent in the resulting port-Hamiltonian model of order $n = 20$. The thermal power introduced at each MOSFET junction and the resulting temperatures constitute the four inputs and collocated outputs of the system, respectively. Detailed information on the modeling can be found in [3], whereas the numerical values are given in [1].

Recalling section 5.1, the bound given by Theorem 1 should be the exact quantity. Indeed, $\lambda_{\max}\left\{\mathbf{B}^T\mathbf{R}^{-1}\mathbf{B}\right\} = 82.31$ gives the same as executing the command „`norm`“ in *Matlab*. However, the command „`norm`“ requires $3.3 \cdot 10^{-2}$ seconds for computation, whereas the simple matrix-vector products along with an eigenvalue computation of small order – necessary for the calculation of the newly introduced bound – needs merely $6.7 \cdot 10^{-4}$ seconds. This results in an almost 50 times faster calculation of the H_∞ norm of the system at hand. Certainly, one might argue that the time for computation of the command „`norm`“ is still fast enough; however, this might become crucial for large-scale systems, as discussed in the following.

6.2 FEM-model of a Timoshenko beam

The second example deals with a 3D cantilever Timoshenko beam [4], modeled by the Finite Element Method. The resulting second-order system (32) is transformed into a first-order state-space system as shown in section 5.2. Note that the mass matrix $\mathbf{M}$ is diagonal, due to lumping of mass [9], and thus can easily be inverted. The model input is the vertical force applied at the beam's free end, leading to the velocity at this point as the collocated output. The beam is modeled with 200 nodes, resulting in a port-Hamiltonian system of order $n = 2400$.

Computing the command „`norm`“ requires 65.3 seconds and results in $\|G\left(s\right)\|_\infty = 9.87$ for the system. In contrast, the calculation of $\lambda_{\max}\left\{\widehat{\mathbf{B}}^T\widehat{\mathbf{R}}^{-1}\widehat{\mathbf{B}}\right\}$ takes only 0.077 seconds, providing a bound on the H_∞ norm of 25.5. This shows the enormous advantage of the newly introduced H_∞ bound over the command „`norm`“ in *Matlab*, when it comes to simulation time. However, an overestimation of approximately 2.5 is observed for the example at hand. Nevertheless, when it comes to large-scale systems – such as $n > 10^5$ – the proposed bound might be the only possibility to gain information about the H_∞ norm of a system at all.

7 Conclusion

A bound on the $\mathcal{H}_\infty$ norm of linear port-Hamiltonian systems is proposed in this article. To sum up its properties and performance, the bound constitutes a trade-off between tightness and computational speed. This is due to the fact that it saves computational costs compared to the general algorithms used in numerical analysis software, but then introduces an overestimation on the real $\mathcal{H}_\infty$ norm.

Considering these characteristics of the new $\mathcal{H}_\infty$ bound, the author suggests its employment in three different settings. First of all, for purely damped port-Hamiltonian systems ($\mathbf{J} = \mathbf{0}$) the bound is preferable since it is shown to be exact with less numerical effort. Secondly, for large-scale systems the bound might be the sole chance for investigating the norm at all. In the end, in all other circumstances the decision is more involved. The applicability of the bound is determined by the order of the model, by the system class, if the assumptions of Theorem 1 are met (though being little restrictive), etc. and last but not least by the eye of the beholder, whether tightness or computational speed is more crucial. However, it is still an open question, how the interconnection matrix introduces the overestimation into the bound.

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