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***Abstract*—Parallel multiple-input multiple-output (MIMO) broadcast channels are separable from an information theoretic point of view, i.e., the whole capacity region can be achieved with strategies that perform independent encoding and decoding on each carrier subject to a power allocation across carriers. However, recent research has revealed that such a separate transmission can become suboptimal if the transmit strategy is restricted to linear transceivers. The answer to the question whether parallel broadcast channels with linear transceivers are separable or not is a property of the system parameters and the particular channel realization, and judging whether a certain scenario is separable or inseparable is still an open problem for most scenarios. In this paper, we consider some special cases of parallel MIMO broadcast channels and classify them as separable or inseparable, respectively. The insights and the applied proof techniques might be helpful to derive more general classification methods in future research.**

I. Introduction

In a communication system with parallel orthogonal resources (e.g., carriers), data transmission can be performed separately on each resource or jointly across the resources (also known as carrier-noncooperative and carrier-cooperative transmission, respectively [1]). A mathematical description of these two types of transmit strategies is presented along with the system model in Section II. Following the nomenclature of [2], [3], we say that a scenario is *inseparable* whenever a carrier-cooperative (joint) strategy can outperform the best carrier-noncooperative (separate) strategy. Conversely, a scenario is separable if this is not possible, i.e., if separate transmission is optimal. Throughout the paper, we use the Shannon rates achievable with circularly symmetric complex Gaussian input signals as performance measure.

Even though the capacity achieving strategy in multiple-input multiple-output (MIMO) broadcast channels, namely dirty paper coding (DPC, e.g., [4]), can be applied separately on each carrier [5], the best transmit strategy with linear transceivers might require joint transmission [2]. This fact is of particular interest since even approximate DPC as in [6] has prohibitive complexity for online implementation. Linear transceivers are an attractive alternative for application in a practical system due to their simplicity and low complexity. On the other hand, the optimization of linear strategies becomes more complicated than the optimization procedure in the DPC case, and new effects such as inseparability can occur.

The fact that MIMO broadcast channels can be inseparable under the assumption of linear transceivers was proven in [2]. The proof was based on the construction of an appropriate example scenario in which there exist tuples of per-user rates that are achievable by a joint strategy with a given sum transmit power, but not by the globally optimal linear separate strategy with the same sum transmit power. As the optimization of joint strategies is more involved and has a higher computational complexity than the optimization of separate strategies (cf. [2]), it would be desirable to have a simple test to tell whether a setting is separable or not before performing the actual optimization. However, such a classification is a nontrivial problem in general.

As a first step towards developing such a separability test, a reasonable approach is to find more examples of scenarios for which the question of separability can be answered. In a later step, it could then be investigated which properties are typical for separable and for inseparable scenarios, respectively, in order to obtain a characterization.

In this paper, we study the separability of the following three scenarios. First, we reconsider the system configuration of the minimal example that was used to prove the inseparability theorem in [2] and study the setting for more general channel realizations (cf. Section III). Second, we extend this example to a setting with a higher number of carriers in order to show that inseparability can occur with nontrivial channel realizations in systems with a number of carriers in a realistic order of magnitude (cf. Section IV). Finally, we study a two-user multicarrier MIMO broadcast channel with full multiplexing, which can be proven to be separable in the high SNR domain (cf. Section V).

In each of the considered cases, the paper answers the question of inseparability for some particular scenarios (i.e., for some particular choices of the system dimensions, the channel realizations, the rate requirements, and/or the SNR regime) for which it had not yet been answered. However, at the same time, the three settings are used as examples to reveal the limits of our current knowledge about inseparability and to show which inseparability-related questions could be studied in the near future.

Note that all results presented in this paper—especially the numerical ones—have to be understood in this light. We do not present any Monte Carlo simulations showing the average performance of particular separate or joint transmit strategies for channel realizations drawn from a certain distribution. The reason for this is that averaging the gains would destroy the information regarding which particular channels realizations contribute to these gains and which do not. Therefore, we systematically construct a variety of qualitatively different channel realizations and present simulation results for the individual channel realizations instead of average results. Only such an individual analysis enables us to draw conclusions about the circumstances under which inseparability can occur.

Notation: We write vectors as boldface lowercase letters and matrices as boldface uppercase letters, and we use $\bullet^{\mathrm{T}}$ and $\bullet^{\mathrm{H}}$ for the transpose and conjugate-transpose operation, respectively. The matrix $\mathbf{I}_L$ is the identity matrix of size L, and $\mathbf{1}$ is the all-ones vector.

II. System Model and Problem Formulation

In a set of C parallel broadcast channels,[1] each subchannel $c \in \{1, \dots, C\}$ is characterized by channel matrices $\boldsymbol{H}_k^{(c),\mathrm{H}} \in \mathbb{C}^{N_k \times M}$ and noise covariance matrices $\boldsymbol{C}_{\boldsymbol{\eta}_k}^{(c)} \in \mathbb{C}^{N_k \times N_k}$ for all users $k \in \{1, \dots, K\}$. The number of transmit antennas is denoted by M, and N_k is the number of antennas at receiver k. The additive noise is assumed to be circularly symmetric complex Gaussian. For the simulations in this paper, we assume that $\boldsymbol{C}_{\boldsymbol{\eta}_k}^{(c)} = \mathbf{I}_{N_k}$ for all k and c, but all considerations could also be extended to other noise covariance matrices.

Data transmission with linear transceivers can be described in the overall system by

$$\hat{\boldsymbol{x}}_k = \boldsymbol{V}_k^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \sum_{k'=1}^{K} \boldsymbol{B}_{k'} \boldsymbol{x}_{k'} + \boldsymbol{V}_k^{\mathrm{H}} \boldsymbol{\eta}_k \tag{1}$$

with the circularly symmetric complex Gaussian symbol vectors[2] $\boldsymbol{x}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{S_k})$ and the corresponding estimates $\hat{\boldsymbol{x}}_k \in \mathbb{C}^{S_k}$, each containing $S_k \leq C \min\{N_k, M\}$ independent data streams for user k, and

$$\boldsymbol{H}_k^{\mathrm{H}} = \operatorname{blockdiag}\left(\boldsymbol{H}_k^{(1),\mathrm{H}}, \dots, \boldsymbol{H}_k^{(C),\mathrm{H}}\right) \in \mathbb{C}^{N_k C \times MC} \tag{2}$$

$$\boldsymbol{C}_{\boldsymbol{\eta}_k} = \operatorname{blockdiag}\left(\boldsymbol{C}_{\boldsymbol{\eta}_k}^{(1)}, \dots, \boldsymbol{C}_{\boldsymbol{\eta}_k}^{(C)}\right) \in \mathbb{C}^{N_k C \times N_k C}. \tag{3}$$

The matrices $\boldsymbol{B}_k \in \mathbb{C}^{MC \times S_k}$ are the beamforming matrices, and $\boldsymbol{V}_k^{\mathrm{H}} \in \mathbb{C}^{S_k \times N_k C}$ are the receive filters.

If we do not impose any constraints on the structure of the filter matrices, we allow that a transmit symbol is spread over several subchannels, which is the case of joint (carrier-cooperative) transmission. By introducing the constraint that the filter matrices have to match the block-diagonal structure of the channel matrices and the noise covariance matrices, we obtain separate (carrier-noncooperative) transmission strategies, which can also be expressed as

$$\hat{\boldsymbol{x}}_k^{(c)} = \boldsymbol{V}_k^{(c),\mathrm{H}} \boldsymbol{H}_k^{(c),\mathrm{H}} \sum_{k'=1}^{K} \boldsymbol{B}_{k'}^{(c)} \boldsymbol{x}_{k'}^{(c)} + \boldsymbol{V}_k^{(c),\mathrm{H}} \boldsymbol{\eta}_k^{(c)} \tag{4}$$

[1]The concept of parallel broadcast channels shall not be confused with the case of interfering broadcast channels (e.g., [7]). The latter describes several interfering base stations and is an interference channel scenario, while the former includes only one base station, but parallel carriers, and is a classical broadcast scenario.

[2]Note that $\boldsymbol{x}_k$ is a concatenation of all symbols intended for user k no matter across which subchannel(s) they are transmitted.

on all subchannels c. In this case, the only coupling between the subchannels is that the per-user rate $r_k = \sum_{c=1}^{C} r_k^{(c)}$ and the sum power $P = \sum_{c=1}^{C} P^{(c)}$ are obtained by summing up the respective per-carrier quantities. Despite its potential suboptimality in the linear precoding case, separate transmission is widely adopted in the multicarrier literature (e.g., [8]–[11]) due to the advantage that far less variables need to be optimized (only the diagonal blocks) and that decomposition techniques can be applied to optimize the various carriers individually.

Note that we consider the case where time-sharing between various strategies is allowed, which does not contradict the assumption of linear transceivers with circularly symmetric Gaussian codebooks. This means that different sets of transmit and receive filters can be applied subsequently, each during a time interval that corresponds to an arbitrary fraction of the total transmission time. The resulting data rates and transmit powers are the time averages of the respective quantities.

Our aim is to judge whether a given setting is inseparable with linear transceivers, i.e., whether the best joint strategy has a better performance than the best separate strategy. Inseparability can be proven by finding an arbitrary joint strategy that can outperform the globally optimal separate strategy, whereas separability can be proven by showing that the globally optimal joint strategy cannot outperform separate transmission. The problem is, however, that finding such globally optimal solutions in the general case is an open problem in MIMO broadcast channels with linear transceivers. To gain more insights about the question under which circumstances parallel MIMO broadcast channels are inseparable, we consider some special cases for which inseparability (or separability) can be proven since the globally optimal separate (or joint) solution is known.

In the numerical simulations performed in this paper, we compare a suboptimal joint transmission strategy with the optimal separate strategy. The results of such simulations have to be interpreted as follows. Whenever the joint strategy outperforms the globally optimal separate strategy, inseparability of the respective setting is shown. On the other hand, if the separate strategy performs better, we cannot draw conclusions about separability or inseparability. Since the algorithm used to find the joint strategy is not globally optimal, there might still exist a different joint strategy that outperforms the separate solution.

III. Small MISO Example with Three Users

In our previous work [2], a small example with $K = 3$ single-antenna receivers, $M = 2$ transmit antennas, and $C = 2$ carriers was considered. In such a setting, the channel matrices can be written as

$$\boldsymbol{H}_k = \begin{bmatrix} \boldsymbol{h}_k^{(1),\mathrm{H}} & \\ & \boldsymbol{h}_k^{(2),\mathrm{H}} \end{bmatrix} \tag{5}$$

where the set of channel vectors $\boldsymbol{h}_k^{(c)}$ can be parametrized by the norms $\ell_k^{(c)} = \|\boldsymbol{h}_k^{(c)}\|_2$ and the Hermitian angles [12] between the channel vectors:

$$\theta_{k,j}^{(c)} = \arccos \frac{\left| \boldsymbol{h}_k^{(c),\mathrm{H}} \boldsymbol{h}_j^{(c)} \right|}{\|\boldsymbol{h}_k^{(c)}\|_2 \|\boldsymbol{h}_j^{(c)}\|_2} \in [0, \tfrac{\pi}{2}]. \quad (6)$$

For notational brevity, let $\alpha_1^{(c)} = \theta_{1,2}^{(c)}$, $\alpha_2^{(c)} = \theta_{2,3}^{(c)}$, and $\alpha_3^{(c)} = \theta_{3,1}^{(c)}$. Given a set of norms and Hermitian angles, there are only two possible sets of channels in the considered three-user MISO system if negligible rotations are ignored [2]. These two channel realizations differ only in the sign of the phase $\psi^{(c)}$ of one of the complex channel coefficients, which can be chosen to be positive on one carrier without loss of generality, but has to be considered on the other carrier(s) [2].

For proving that inseparability can occur and for further analysis, completely symmetric channel realizations with unit norm $\ell_k^{(c)} = 1$ and symmetric angular separations $\alpha_i^{(c)} = \alpha \; \forall i, \forall c$ were considered in [2]. Inseparability could be shown for some intermediate values of α while $\alpha \to 0$ led to provable separability. When choosing the arbitrary sign of $\psi^{(c)}$ differently on the two carriers, inseparability appeared to be more pronounced. Therefore, we stick to this choice in all scenarios considered in this section.

According to [2], the fact that inseparability can occur in MIMO broadcast channels with linear transceivers can be proven by showing that a certain power P_{sep} is needed to achieve the rates $r_k = 1 \; \forall k$ with separate transmission in the channel realization with $\alpha = \frac{\pi}{4}$, but a strictly lower power $P_{\text{joint}} < P_{\text{sep}}$ suffices to achieve the same rates with joint transmission.

A. Results for High and Low SNR

To obtain inseparability results for a larger SNR range, we now repeat the sufficient separability test of [2] with different rate values: we choose per-user rates $r_k = r \; \forall k$, compute the optimal power P_{sep} necessary to achieve these per-user rates with separate transmission on each resource, and compare it with the (possibly suboptimal) power P_{joint} needed with joint transmission. The results are given in Fig. 1.

It can be seen that the considered channel realization is not only inseparable for $r = 1$, but for a much larger range of rate requirements. Only in the the high rate regime (equivalent to the high SNR regime), the algorithm applied for the case of joint transmission found solutions that perform worse than the optimal separate solution. For this case, the simulation results do not allow us to draw conclusion about (in)separability: as every separate transmit strategy is also a feasible joint strategy (cf., e.g., [13]), optimal joint transmission can always perform at least as good as the optimal separate solution, but the remaining question is if a joint strategy that strictly outperforms separate transmission can also be found in the high SNR regime. We will come back to this question later in this paper (cf. Section IV-B).

B. Asymmetry in the Channel Directions

As a second extension to the investigations of [2], we study the same minimal example for asymmetric unit-norm channel realizations, where the Hermitian angles $\alpha_i^{(c)}$ take different values for different i. In [2], it was shown that only combinations of Hermitian angles fulfilling

$$\sum_{i=1}^{3} \alpha_i \leq \pi \quad \text{and} \quad 2\alpha_k - \sum_{i=1}^{3} \alpha_i \leq 0 \; \forall k \quad (7)$$

are possible in the considered system, where we have omitted the superscript index c for notational brevity. It can be seen from the visualization in Fig. 3 that the four linear inequalities in (7) describe a regular tetrahedron.

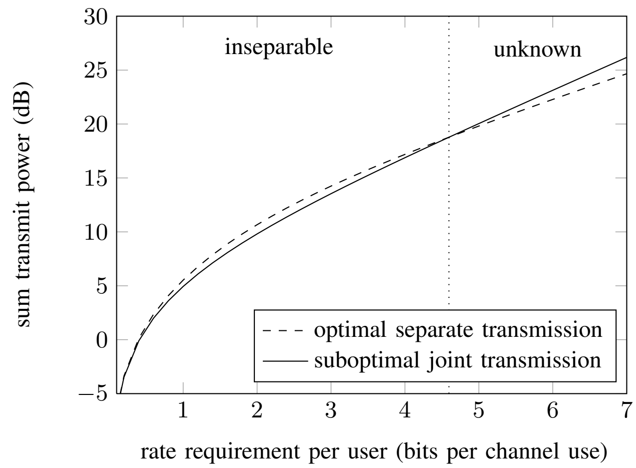

Fig. 1. Sum power for the small MISO example with symmetric unit-norm channels ($\alpha = 45°$).

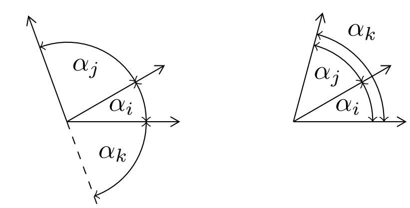

Fig. 2. Angles between three vectors in $\mathbb{R}^2$.

To get some intuition about the meaning of these inequalities, let us first consider the real-valued case by studying the possible angles between three vectors in $\mathbb{R}^2$. In Fig. 2, two qualitatively different scenarios can be seen. We observe that in the real-valued case, the angles either have to sum up to 180°, i.e., $\sum_{i=1}^{3} \alpha_i = \pi$, or one angle must be the sum of the two others, i.e., $2\alpha_k = \sum_{i=1}^{3} \alpha_i$ for some k. These equations correspond to the faces of the tetrahedron of the complex-valued case [cf. (7)]. In $\mathbb{C}^2$, we have more freedom in arranging the vectors so that combinations of angles corresponding to points in the interior of the tetrahedron become possible. The possible real-valued scenarios are then limiting special cases of the set of possible complex-valued scenarios.

Let us consider the case where we have the same unit-norm channels on both carriers apart from the differing sign of the phases $\psi^{(2)} = -\psi^{(1)}$. In this example, we have the three Hermitian angles $\alpha_i^{(c)} = \alpha_i, \; i = 1, 2, 3$ as free parameters, and

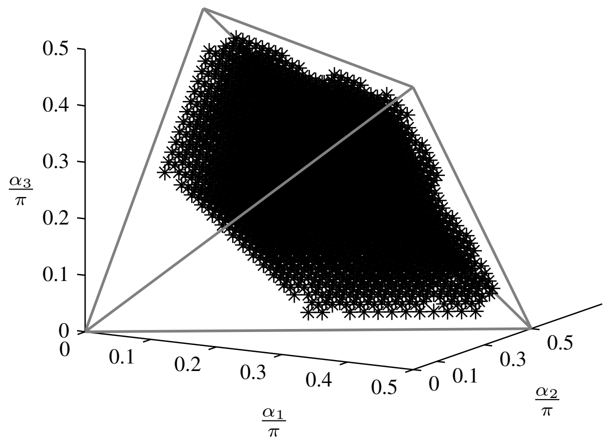

Fig. 3. Possible combinations of angles between three vectors in $\mathbb{C}^2$ (gray tetrahedron) and unit-norm channel realizations for which inseparability could be shown at $r_k = 1\ \forall k$ in the small MISO example (black markers).

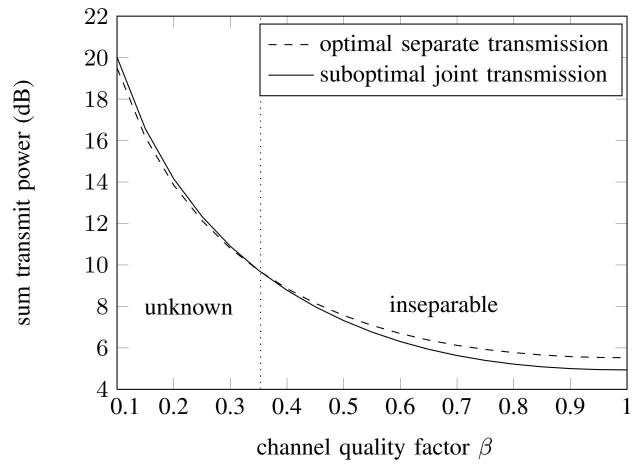

Fig. 4. Sum power for the small MISO example with symmetric channel directions ($\alpha = 45°$) and user-dependent channel quality.

each possible combination of these parameters corresponds to a point in the tetrahedron in Fig. 3.

For any such point, a sufficient inseparability test as in [2] can be performed: using the algorithm from [11], the globally minimal power needed to achieve the rate requirements $r_k = 1\ \forall k$ can be computed. A sufficient condition for inseparability is that a (possibly suboptimal) algorithm for power minimization with joint transmission (e.g., [14]) can find a solution that needs less power than the optimum for separate transmission.

The solution of our study can be seen in Fig. 3. Every black marker corresponds to a channel realization for which a joint strategy that performs better than the optimal separate solution could be found by the suboptimal algorithm [14], i.e., for which inseparability could be shown. For the regions in the tetrahedron where no markers are present, we cannot tell whether the setting is separable or not since the test is only a sufficient condition and not a necessary one.

The most pronounced one out of these regions is the lower left part of the plot, where all three angles are small. This is in compliance with the study of the symmetric scenario in [2], where inseparability was not observed for angles $\alpha_k = \alpha \leq \frac{\pi}{6}$. A special point in this region is $\alpha_k = 0\ \forall k$, for which the scenario is provably separable [2].

As in the case of small angular separation, we cannot observe inseparability for the faces of the tetrahedron and their neighborhood, either. Noting that the faces are the extremal cases, which correspond to channel vectors with real-valued coefficients, this leaves room for several possibilities: either joint strategies outperforming the separate solution exist also for these regions, but have not yet been found, or real-valued channel realizations are separable—either as a general rule or at least in this scenario.

This reveals another task for future research, namely to study whether and in which cases inseparability can happen with real-valued channel coefficients. An example for a situation where a channel model with real-valued coefficients plays an important role is the case where so-called widely linear processing is applied to treat the real and imaginary part of the transmit signals separately (cf., e.g., [15], [16]). Consequently, the (in)separability of real-valued channels needs to be studied in order to judge possible benefits of carrier-cooperative transmission in combination with widely linear transceivers.

C. Asymmetries in the Channel Quality

In the last subsection, we studied how far we can deviate from the symmetry in terms of channel directions and still obtain an inseparable channel realization. We now go back to the symmetric case with $\alpha_1^{(c)} = \alpha_2^{(c)} = \alpha_3^{(c)} = \alpha = \frac{\pi}{4}$ for all c, but we introduce asymmetry in terms of channel strengths, i.e., we modify the norms $l_k^{(c)}$.

Let us first consider the case where the channel conditions are the same on both carriers, but different for the three users. To this end, we choose

$$\ell_1^{(c)} = \beta\ \ \forall c, \qquad \ell_2^{(c)} = 1\ \ \forall c, \qquad \ell_3^{(c)} = \frac{1}{\beta}\ \ \forall c \tag{8}$$

and plot the simulation results in Fig. 4 over the channel quality factor β. In particular, we again consider the sum power needed to achieve the rates $r_k = 1\ \forall k$ with optimal separate and suboptimal joint transmission.

For $\beta = 1$, we have the original symmetric scenario, which is known to be inseparable. As β gets smaller, the asymmetry of the scenario increases, which leads to higher sum power since the best situation to transmit symmetric rate requirements is to have constant channel quality across users. This observation holds for both separate and joint transmission. Up to a certain degree of asymmetry, the joint solution is still the better one, i.e., we still observe inseparability, but for very unequal channel quality, the applied suboptimal joint algorithm can no longer outperform the optimal separate strategy. For this range of β, we cannot tell whether or not the scenario is separable.

As a second example, we study the influence of an imbalance between the channel quality on the two carriers while

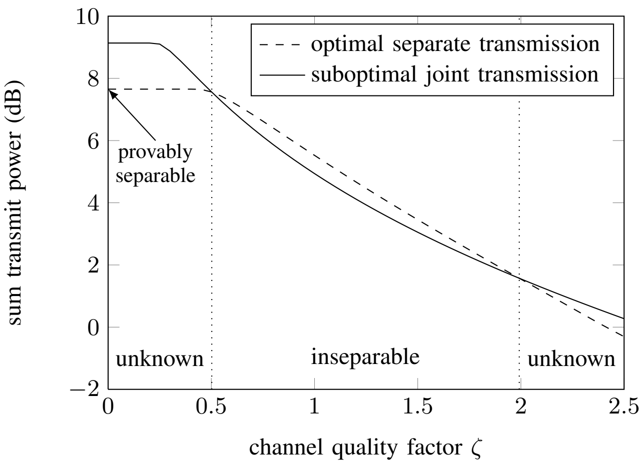

Fig. 5. Sum power for the small MISO example with symmetric channel directions ($\alpha = 45°$) and carrier-dependent channel quality.

keeping a symmetric situation with respect to the users. For the simulation, we again choose $\alpha = \frac{\pi}{4}$ and $r_k = 1\ \forall k$, but the channel norms are now given by

$$\ell_k^{(1)} = \zeta \ \ \forall k, \qquad\qquad \ell_k^{(2)} = 1 \ \ \forall k. \tag{9}$$

The results can be seen in Fig. 5.

For small values of ζ, one of the carriers has very bad channel quality and is used by neither of the transmit schemes. In this range, the difference between the transmit power of the separate and the joint strategy is a consequence of the suboptimality of the joint solution (recall that the optimal joint solution is at least as good as the optimal separate solution). For the special case $\zeta = 0$, the first carrier becomes totally useless, and the scenario is provably separable since transmitting only on the second carrier is a separate strategy (cf. [2]).

With increasing ζ, the channel quality on the first carrier increases so that the same rates can be achieved using a lower sum transmit power. It turns out that the setting is inseparable not only for the well known symmetric situation with $\zeta = 1$, but also for small and moderate imbalances of the channel quality on the two carriers.

For a very strong first carrier, i.e., for high ζ, we again come to a range where we are not able to find an advantageous joint solution. However, this does not imply separability since comparison with a globally optimal joint method would be necessary to show separability. Only for $\zeta \to \infty$, we can say that the scenario is separable, since transmitting anything on the second carrier no longer makes sense and the optimal strategy—transmitting only on the first carrier—is a separate strategy.

IV. Three-User MISO with many Carriers

As practical communication systems such as OFDM systems usually have large numbers of carriers (e.g., 1024), the question arises whether or not inseparability is a phenomenon specific to the small number of carriers considered up to this point. Moreover, in larger systems, we have to ask if the gains of joint transmission require all carriers to be treated jointly or if it suffices to treat only small groups of carriers jointly while treating each group separately from the others. The latter would be desirable for implementation in a real system since the required linear filters would be much less complex than in the case where all carriers form one large group. Answers to both questions are given in this section.

A trivial inseparable example could be created based on the small system from Section III by adding many carriers on which all channel coefficients are zero. Instead, we are looking for a nontrivial example with nonzero channels on all carriers. To this end, we consider above example as part of a larger system with more carriers and use similar nontrivial channels on the other carriers.

The simplest method is to copy the two carriers such that the resulting system has $\frac{C}{2}$ carriers that are a copy of carrier 1 and $\frac{C}{2}$ carriers that are a copy of carrier 2. However, since the last section has revealed that there is a wide range of channel realizations that lead to inseparability in the small scenario, we can also take various such groups of channels on two carriers and combine the different groups to get a large system.

To show inseparability, we consider a sum rate maximization with fixed sum power. For the separate case, this can be written as

$$\max_{(P^{(c)}\geq 0,\ \boldsymbol{r}^{(c)})_{c=1,\ldots,C}} \quad \sum_{c=1}^{C} \mathbf{1}^{\mathrm{T}} \boldsymbol{r}^{(c)}$$
$$\text{s.t.} \quad \boldsymbol{r}^{(c)} \in \mathcal{R}^{(c)}(P^{(c)}) \ \text{ and } \ \sum_{c=1}^{C} P^{(c)} = P. \tag{10}$$

Here, the rate vector $\boldsymbol{r}^{(c)}$ contains the rates $r_k^{(c)}$ of all users on carrier c, and $\mathcal{R}^{(c)}(P^{(c)})$ is the set of rate vectors achievable on carrier c with transmit power $P^{(c)}$. If any joint strategy with sum transmit power P can achieve a sum rate strictly higher than the solution of (10), inseparability is proven for a given setting.

Note that this is again only a sufficient, but not a necessary criterion for inseparability. First of all, we again would have to know the globally optimal joint strategy as a reliable reference value in order to judge whether or not the optimal sum rate is achievable with separate transmission. But then, even if we could judge this, we would still not know whether the rate regions of joint and separate transmission coincide since the sum rate optimal point is only one out of many Pareto optimal points of a rate region.

By allowing time-sharing as stated in Section II, the duality gap of the problem vanishes [17], and the optimization can be solved by means of a dual decomposition approach. Therefore, we introduce the Lagrangian multiplier λ and write down the dual problem

$$\min_{\lambda} \quad \lambda P \ + \ \sum_{c=1}^{C} \max_{\substack{P^{(c)}\geq 0 \\ \boldsymbol{r}^{(c)}\in\mathcal{R}^{(c)}(P^{(c)})}} \left(\mathbf{1}^{\mathrm{T}}\boldsymbol{r}^{(c)} - \lambda P^{(c)}\right). \tag{11}$$

A. Symmetric Setting with 1024 Carriers

We first consider the aforementioned simple extension of the small system to the case of $C = 1024$ carriers, i.e., we have a system with three users and completely symmetric unit-norm channels with $\alpha = \frac{\pi}{2}$ on all carriers (cf. Section III). Let the sign of the phase $\psi^{(c)}$ be positive on half of the carriers and negative on the others.

As the sign of $\psi^{(c)}$ is irrelevant when considering the carriers separately [2], the function mapping the transmit power to a rate region is the same for all carriers, i.e., $\mathcal{R}^{(c)}(P^{(c)}) = \mathcal{R}^{(1)}(P^{(c)})\ \forall c$. As a consequence, for a given Lagrangian multiplier λ, we have exactly the same subproblem on each carrier c so that we obtain the same optimal rate vector and sum power on each carrier, i.e., $P^{(c)}(\lambda) = P^{(1)}(\lambda)\ \forall c$. In the outer optimization of (11), the optimizer $\lambda^\star$ has to be such that

$$P = \sum_{c=1}^{C} P^{(c)}(\lambda^\star) = C P^{(1)}(\lambda^\star) \tag{12}$$

which yields $P^{(c)} = \frac{P}{C}$ in the optimum.

Plugging this into the original problem (10), the optimization decomposes into C identical per-carrier sum rate maximizations

$$\max_{\boldsymbol{r}^{(c)}} \ \mathbf{1}^{\mathrm{T}} \boldsymbol{r}^{(c)} \quad \text{s.t.} \quad \boldsymbol{r}^{(c)} \in \mathcal{R}^{(1)}\left(\tfrac{P}{C}\right). \tag{13}$$

For given sum transmit power P, we have to solve only one sum rate maximization, e.g., by means of the method proposed in [18] or by a branch and bound method (cf. [19]).[3] The overall sum rate is then given by $R = C\mathbf{1}^{\mathrm{T}}\boldsymbol{r}^{(1)}$. Note that both numerical methods not only deliver an approximately optimal achievable value, but also a rigorous upper bound to the actual optimum. Considering the value of the upper bound as the optimal sum rate for the separate case, we can be sure to not mistakenly detect inseparability due to numerical inaccuracies.

As reference algorithm with joint transmission, we apply the successive-stream allocation method combined with gradient-projection steps proposed in [21] to the block-diagonal channel matrices (2) and the block-diagonal covariance matrices (3), but we choose the initial filters randomly as proposed in [13]. The latter is necessary to ensure that the algorithm can converge to solutions with joint transmission [13]. As the algorithm delivers an achievable value, i.e., a lower bound to the optimal joint solution, we also avoid mistakes caused by numerical inaccuracies from this side.

In Fig. 6, we can see the sum rate achieved for various values of the transmit power. Instead of the sum rate R and the sum power P, we have plotted the per-carrier values $\frac{R}{C}$ and $\frac{P}{C}$. Similar as in the small example in Fig. 1, we observe a significant gain of joint transmission especially for moderate, but also for high per-carrier transmit powers $\frac{P}{C}$. Thus, the large scenario is inseparable as well, which shows that inseparability can occur in nontrivial parallel broadcast channels with practically relevant numbers of carriers. Note, however, that numerical results for high, but finite values of the SNR cannot provide evidence about the asymptotic behavior, which still remains an open question.

[3]Note that the optimal solution of (13) does not even require time-sharing since time-sharing corresponds to taking the convex hull of $\mathcal{R}^{(1)}$, which cannot increase the sum rate [20, Corollary A5.9].

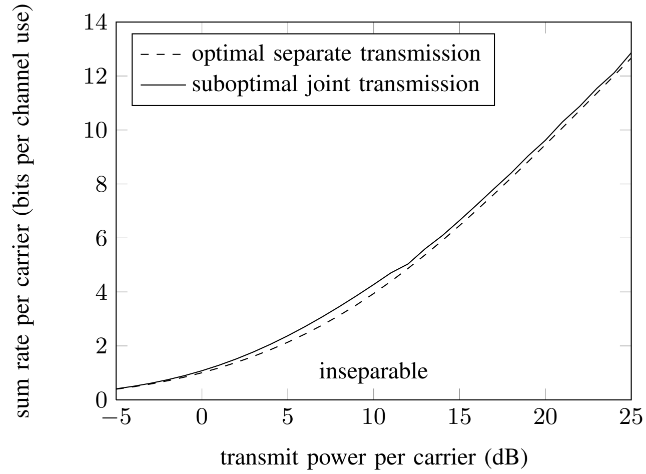

Fig. 6. Sum rate for the three-user MISO example with $C = 1024$ carriers and symmetric unit-norm channels ($\alpha = 45°$).

B. Partly Joint Transmission

An interesting fact is that we can obtain the same sum rate achieved by the joint solution using the following partly joint strategy. We apply the joint sum rate maximization described above to groups of two carriers, where the sign of $\psi^{(c)}$ differs for the two carriers within a group. To each group, we allocate the transmit power $\frac{2P}{C}$. Due to the special choice of the channel realizations on all carriers, the same per-group rate R_{group} is achieved in all groups. Performing the data transmission for each group of carriers separately from the transmission on carriers belonging to other groups, the sum rate is given by the sum of all per-group rates, i.e., $R = \frac{C}{2} R_{\text{group}}$.

Even though it is clear that the construction of a partly joint strategy with good performance is more involved in general, this simple example reveals an important property: if parallel broadcast channels are inseparable under linear transceivers, it can happen that joint transmission across small groups (e.g., two out of 1024 carriers) can already lead to significant gains so that the more complex method of performing precoding across all carriers can be avoided. The question how (and to which group size) the carriers should be combined for a given channel realization is left open for future research.

From the way the partly joint strategy and the optimal separate strategy are constructed, it is easy to see that above results also apply for any other even number of carriers C. In particular, they also hold for a sum rate maximization in the small scenario considered in Section III-A. This shows that the symmetric small MISO example is indeed inseparable at high SNR as well. A possible explanation why this could not be shown by the simulation presented in Fig. 1 is as follows: the suboptimal algorithms used in this paper do

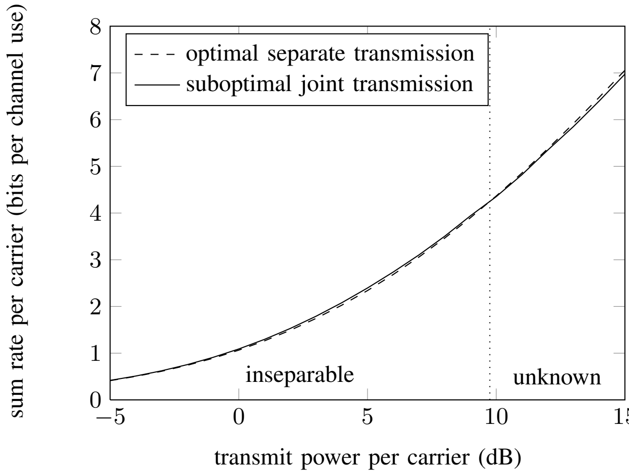

Fig. 7. Sum rate for the three-user MISO example with $C = 16$ carriers and different channels on the various carriers.

not apply time-sharing, which can lead to a penalty for the power minimization in Section III-A, but not for the sum rate maximization considered here (cf. Footnote 3, which analogously applies to the case of joint transmission).

C. Asymmetric Setting with 16 Carriers

In order to show that inseparability also occurs with more diversified channels on the various carriers, we include Fig. 7. We have taken eight different points with black markers from Fig. 3 together with eight random channel quality factors $\zeta \in [0.9, 1.1]$ to create eight groups with two carriers per group. We again choose two different signs of $\psi^{(c)}$ within each group.

For the resulting channel realization, we again find the optimal separate solution by solving (11) and compare the result to the suboptimal method for joint transmission (successive stream allocation and gradient-projection method from [21] combined with random initialization from [13]). Note that the simplifications described before (same solution on all carriers or in each group of two carriers, respectively) are no longer possible. Therefore, (11) has to be solved with a nested optimization (similar to the one in [11]), and the suboptimal joint method has to be applied to the complete system at once.

In this case, we observe inseparability only for moderate SNR. For the high SNR regime, we cannot draw any conclusions since the applied test is only sufficient, but not necessary for inseparability.

V. Two-User MIMO with Full Multiplexing at High SNR

In this section, we consider the case of two multiantenna receivers, i.e., a MIMO system with two users. We restrict ourselves to the case of full multiplexing, i.e., the number of transmit antennas at the base station is at least as high as the total number of receive antennas. Our aim is to show that the regions of achievable rates for the case of joint transmission and the case of separate transmission coincide. This implies that no joint strategy can outperform separate transmission, i.e., inseparability cannot occur.

We use the so-called rate-profile method [5] to characterize the rate region for joint transmission, i.e., we make use of the fact that every Pareto optimal point of the rate region is the solution of the rate balancing problem

$$\max\ R_1 + R_2 \quad \text{s.t.} \quad \frac{R_1}{\gamma_1} = \frac{R_2}{\gamma_2} \quad \text{and} \quad \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \in \mathcal{R}(P) \quad (14)$$

for some choice of the ratio $\frac{\gamma_1}{\gamma_2}$. The optimization in (14) has to be carried out over all possible joint transmit strategies. As explained in [13], such an optimization can be performed by applying results derived for single-carrier MIMO broadcast channels to the block-diagonal channel matrices (2) and the block-diagonal covariance matrices (3).

For our analysis, we make use of [22], where the optimal rate balancing solution at high SNR, i.e., $P \to \infty$, was derived for a single-carrier MIMO broadcast channel with full multiplexing. According to this derivation, the optimal strategy in the dual uplink [23] must involve an uplink transmit filter $\boldsymbol{V}_k$ whose columns are the principal generalized eigenvectors of the matrix pair $\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{H}_k$ and $\boldsymbol{H}_k^{\mathrm{H}}\left(\mathbf{I}_{MC} - \boldsymbol{H}_\ell\left(\boldsymbol{H}_\ell^{\mathrm{H}}\boldsymbol{H}_\ell\right)^{-1}\boldsymbol{H}_\ell^{\mathrm{H}}\right)\boldsymbol{H}_k$ for either $(k,\ell) = (1,2)$ or $(k,\ell) = (2,1)$ depending on the particular values of γ_1, γ_2, N_1, and N_2. Since $\boldsymbol{H}_k$ and $\boldsymbol{H}_\ell$ have the same block structure (2), each of these filter vectors has non-zero elements only within a block that corresponds to one of the carriers, i.e., the transmission of user k in the dual uplink is performed separately on each carrier. Moreover, according to [22], user ℓ has to apply a unitary matrix as uplink transmit filter in the optimal solution, resulting in the identity matrix as uplink transmit covariance matrix for user ℓ. This covariance matrix also corresponds to a separate strategy as it is block-diagonal. Since both users apply a separate transmit strategy in the dual uplink, all useful signals, interfering signals, and noise contributions have block-diagonal covariance matrices. This implies that separate receive processing on each carrier is optimal in the dual uplink (cf., e.g., [13]), which is equivalent to the statement that separate transmit processing is optimal in the original broadcast channel, as can be seen from the uplink-to-downlink transformation rule from [23] (cf. the discussion in [13]).

This proves that all Pareto optimal points of the high SNR rate region can be achieved by separate strategies, i.e., the scenario is separable at high SNR. Note that for above proof to be applicable, the following three assumptions are necessary: two-user system, number of transmit antennas at least as high as total number of receive antennas, and operation in the high SNR regime.

It is not clear which of the assumptions are the decisive factors for separability and which are only necessary due to the chosen line of argument. However, based on the observations for high, but finite SNR in the previous section, we conjecture that it is not (alone) the infinite SNR which renders the scenario separable. Rather, we think that the reason for separability is the limitation to two users, the assumption of full multiplexing, or the combination of both. We come to this conjecture since among the scenarios for which inseparability could be proven (cf. [2] and Sections III and IV), there is neither one with two users nor one with full multiplexing.

Future research should investigate which of the assumptions made in this section is/are actually responsible for separability.

VI. Discussion

We have studied special cases of parallel MIMO broadcast channels with linear transceivers, and some of them turned out to be separable while others turned out to be inseparable. This reveals that it would be advantageous to have simple criteria to judge whether a certain scenario is separable or not since the high complexity of optimizing a joint strategy should only be invested in cases where it is known that a gain can be achieved.

In this paper, we have identified symmetry among users and among carriers as potentially conducive to inseparability. On the other hand, real-valued channel coefficients, two-user systems, and full multiplexing might be factors that promote separability. Finally, the SNR and the number of carriers do not necessarily contribute to the question whether a scenario is separable or not (cf. the symmetric example in Section IV-A), but for the general general case, their influence is not yet clear (cf. the asymmetric example in Section IV-C).

These first hints should be investigated rigorously for more general scenarios in future research in order to identify criteria for inseparability. For instance, a next step could be a high SNR analysis of systems with more than two users or an analysis of the two-user case for finite SNR. Another line of research could be to further study the real-valued case, which is relevant for the extension of the results to widely linear transceivers.

Our results also give some indications on the relevance of joint (carrier-cooperative) transmit strategies for practical implementations. We have not only shown that such strategies can be advantageous in systems whose number of carriers lies in a realistic order of magnitude, but we have also obtained a hint on how potential gains can be exploited at a reasonable computational complexity: even if the number of carriers is high, it can be sufficient to perform carrier-cooperation only within small groups of carriers. This makes the concept much more attractive than it would be if coding across all carriers was necessary.

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