REPRESENTATIONS OF CONTINUOUS-TIME ARMA PROCESSES

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Abstract

Using the kernel representation of a continuous-time Lévy-driven ARMA (autoregressive-moving average) process, we extend the class of non-negative Lévy-driven Ornstein-Uhlenbeck processes employed by Barndorff-Nielsen and Shephard (2001) to allow for non-monotone autocovariance functions. We also consider a class of fractionally integrated Lévy-driven continuous-time ARMA processes obtained by a simple modification of the kernel of the continuous-time ARMA process. Asymptotic properties of the kernel and of the autocovariance function are derived.

KEYWORDS: *continuous-time ARMA process, Lévy process, stochastic volatility, long memory, fractional integration.*

1. Introduction

A recent paper of Anh, Heyde and Leonenko (2002) introduced a class of continuous-time long-memory processes via the Green function solutions of fractional differential equations driven by Lévy processes. Such processes play a potentially important role in the modeling of financial time series, since such series frequently exhibit signs of both long memory and heavy tails. In this paper we adopt a different point of view, constructing a continuous-time analogue of a discrete-time fractionally integrated ARMA process by making a natural modification to the kernel of a continuous-time ARMA (henceforth referred to as CARMA) process. We begin by deriving this kernel representation and pointing out its relevance to the stochastic volatility model of Barndorff-Nielsen and Shephard (2001). In the latter paper an Ornstein-Uhlenbeck process driven by a non-decreasing Lévy process was used to model volatility in a stochastic volatility model for asset prices. The stationary Ornstein-Uhlenbeck process,

$$X(t) = \int_{-\infty}^{t} e^{-c(t-y)} dL(y), \quad c > 0,$$

was chosen because it has a non-negative kernel $\left(g(t) = \exp(-ct)I_{[0,\infty)}(t)\right)$ and consequently, if the driving Lévy process L is non-decreasing, the process X will be non-negative as is necessary if it is to represent volatility. However the use of the Ornstein-Uhlenbeck process restricts the class of volatility autocorrelation functions to functions of the form $\rho(h) = \exp(-ch)$ for some $c > 0$. Barndorff-Nielsen and Shephard suggested extending this class by using linear combinations of independent

Ornstein-Uhlenbeck processes with positive coefficients, however the autocorrelation functions are still restricted to be monotone decreasing. If the Ornstein-Uhlenbeck process is replaced by a non-negative Lévy-driven CARMA process, a much larger class of autocorrelations can be modeled, and in particular the monotonicity constraint can be removed. This is illustrated by means of an example. Causal and non-causal CARMA processes are briefly discussed and in Section 4 we define fractionally integrated Lévy-driven CARMA processes and examine the asymptotic behaviour of their kernel and autocorrelation functions. From a second-order point of view the processes considered here constitute a subclass of the Gaussian continuous-time fractionally integrated processes introduced by Comte and Renault (1996).

2. CARMA Processes

A second-order Lévy-driven continuous-time ARMA(p,q) process is defined (see Brockwell (2000, 2001) in terms of the following state-space representation of the formal equation,

$$a(D)Y(t) = b(D)DL(t), \ t \geq 0, \tag{2.1}$$

in which D denotes differentiation with respect to t, $\{L(t)\}$ is a Lévy process with $EL(1)^2 < \infty$,

$$a(z) := z^p + a_1 z^{p-1} + \cdots + a_p,$$
$$b(z) := b_0 + b_1 z + \cdots + b_{p-1} z^{p-1},$$

and the coefficients b_j satisfy $b_q \neq 0$ and $b_j = 0$ for $q < j < p$. The state-space representation consists of the *observation* and *state* equations,

$$Y(t) = \mathbf{b}'\mathbf{X}(t), \tag{2.2}$$

and

$$d\mathbf{X}(t) - A\mathbf{X}(t)dt = \mathbf{e} \ dL(t), \tag{2.3}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}.$$

(If $p = 1$, A is defined to be $-a_1$.) The solution $\{\mathbf{X}(t), t \geq 0\}$ of equation (2.3) satisfies

$$\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-u)}\mathbf{e} \ dL(u), \tag{2.4}$$

where the integral is defined as the L^2 limit of approximating Riemann-Stieltjes sums. The process $\{\mathbf{X}(u), u \geq 0\}$ also satisfies the relations,

$$\mathbf{X}(t) = e^{A(t-s)}\mathbf{X}(s) + \int_s^t e^{A(t-u)}\mathbf{e} \ dL(u), \quad \text{for all } t > s \geq 0, \tag{2.5}$$

which clearly show (by the independence of increments of $\{W(t)\}$) that $\{\mathbf{X}(t)\}$ is Markov.

We shall assume henceforth that $\mathbf{X}(0)$ is independent of $\{L(t), t \geq 0\}$. Then $\{\mathbf{X}(t), t \geq 0\}$ is strictly stationary if and only if $\mathbf{X}(0)$ has the same distribution as $\int_0^\infty e^{A(t-u)}\mathbf{e}\, dL(u)$ and the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A (i.e. the zeroes of the autoregressive polynomial $z^p + a_1 z^{p-1} + \cdots + a_p$) have negative real parts, i.e

$$\Re(\lambda_i) < 0, \;\; i = 1, \ldots, p. \tag{2.6.}$$

Definition 1: If the eigenvalues $\lambda_1, \ldots, \lambda_p$ have negative real parts, the CARMA(p, q) process $\{Y(t), t \geq 0\}$ with coefficients $a_1, \ldots, a_p, b_1, \ldots, b_q$ and driving Lévy process L is defined in terms of the strictly stationary solution $\mathbf{X}(t)$ by equation (2.2). It is clearly both strictly and weakly stationary.

In order to define a CARMA process indexed by $(-\infty, \infty)$, we introduce a second Lévy process $\{M(t), 0 \leq t < \infty\}$, independent of L and with the same distribution, and then define the following extension of L:

$$L^*(t) = L(t)I_{[0,\infty)}(t) - M(-t-)I_{(-\infty,0]}(t), \;\; -\infty < t < \infty.$$

Then, provided the eigenvalues of A all have negative real parts, the process $\{\mathbf{X}(t)\}$ defined by

$$\mathbf{X}(t) = \int_{-\infty}^{t} e^{A(t-u)}\mathbf{e}\; dL^*(u),$$

is a strictly stationary solution of (2.3) for $t \in (-\infty, \infty)$ with corresponding CARMA process,

$$Y(t) = \int_{-\infty}^{t} \mathbf{b}' e^{A(t-u)}\mathbf{e}\; dL^*(u), \;\; -\infty < t < \infty. \tag{2.7}$$

Definition 2: If the eigenvalues $\lambda_1, \ldots, \lambda_p$ have negative real parts, the CARMA(p, q) process $\{Y(t), -\infty < t < \infty\}$ with coefficients $a_1, \ldots, a_p, b_1, \ldots, b_q$ and driven by the (extended) Lévy process L^* is defined by (2.7). It is clearly both strictly and weakly stationary and the distribution of $\{Y(t), t \geq 0\}$ is the same as that of the CARMA process in Definition 1.

The representation (2.7) generalizes the corresponding representation of the Lévy-driven Ornstein-Uhlenbeck process,

$$Y(t) = \int_{-\infty}^{t} e^{-c(t-u)} dL^*(u), \;\; c > 0,$$

used by Barndorff-Nielsen and Shephard (2001) in their model for stochastic volatility.

For the remainder of this paper we shall be concerned with the representation (2.7) and its extensions. We first observe that (2.7) can be greatly simplified when the eigenvalues $\lambda_1, \dots, \lambda_p$ are distinct. In this case the corresponding right eigenvectors of the matrix A are

$$[1 \ \lambda_j \ \lambda_j^2 \ \dots \ \lambda_j^{p-1}]', \ j = 1, \dots, p,$$

from which the spectral expansion of A is readily computed, yielding the expression,

$$\mathbf{b}' e^{Au} \mathbf{e} = \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r u}. \tag{2.8}$$

If we now define

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu\lambda} \frac{b(i\lambda)}{a(i\lambda)} \, d\lambda, \tag{2.9}$$

the change of variable $z = i\lambda$ and a simple contour integration with respect to z shows that

$$g(u) = \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r u} I_{(-\infty,0)}(\Re(\lambda_r u)),$$

so that the representation (2.7) of the CARMA process can be rewritten as

$$Y(t) = \int_{-\infty}^{\infty} g(t-u) dL^*(u), \quad -\infty < t < \infty. \tag{2.10}$$

Thus, in the case of distinct eigenvalues with negative real parts, we can define the CARMA processs by equations (2.9) and (2.10).

The representation (2.9) and (2.10) of $Y(t)$ remains valid for repeated eigenvalues with negative real parts (as seen by allowing a group of distinct eigenvalues to approach a common limit and using the continuity of $a(\cdot)$ as a function of its zeroes). Moreover the process defined by (2.9) and (2.10) is a strictly stationary solution of the CARMA equations for $t \in (-\infty, \infty)$, *even when one or more of the eigenvalues has (strictly) positive real part.* If all the eigenvalues of A have negative real part the process is **causal**, i.e. $g(u) = 0$ for $u < 0$. If all the eigenvalues have positive real part then $g(u) = 0$ for $u > 0$. If some have positive and some have negative real parts, then $g(u)$ is non-zero for all u. This classification is analogous to the classification of discrete-time ARMA processes as causal or otherwise, depending on whether or not the zeroes of the autoregressive polynomial lie outside the unit circle (see e.g. Brockwell and Davis (1991). These considerations lead to the following definition (which reduces to Definition 2 in the case when all the eigenvalues have negative real parts).

Definition 3: If the eigenvalues $\lambda_1, \dots, \lambda_p$ have non-zero real parts, the CARMA(p, q) process $\{Y(t), -\infty < t < \infty\}$ with coefficients $a_1, \dots, a_p, b_1, \dots, b_q$ and driven by the (extended) Lévy process L^* is defined by (2.9) and (2.10).

In the case when the eigenvalues, $\lambda_1, \dots, \lambda_p$, are distinct and have negative real parts,

$$g(u) = \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r u} I_{(0,\infty)}(u), \tag{2.11}$$

and the autocovariance function is

$$\gamma(h) = \mathrm{cov}(Y_{t+h}, Y_t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ih\lambda}\left|\frac{b(i\lambda)}{a(i\lambda)}\right|^2 d\lambda \;=\; \sum_{r=1}^{p} \frac{b(\lambda_r)b(-\lambda_r)}{a'(\lambda_r)a(-\lambda_r)} e^{\lambda_r|h|}. \tag{2.12}$$

In particular, for the (causal) Lévy-driven Ornstein-Uhlenbeck process, $b(z) = 1$, $a(z) = z + c$ with $c > 0$,

$$g(u) = e^{-cu} I_{(0,\infty)}(u) \quad \text{and} \quad \gamma(h) = \frac{1}{2c} e^{-c|h|}.$$

3. Application to Stochastic Volatility Modelling

Barndorff-Nielsen and Shephard (2001) introduced a model for asset-pricing in which the logarithm of an asset price is the solution of the stochastic differential equation

$$dX^*(t) = (\mu + \beta\sigma^2(t))dt + \sigma(t)dW(t),$$

where $\{\sigma^2(t)\}$, the instantaneous volatility, is a non-negative Lévy-driven Ornstein-Uhlenbeck process, $\{W(t)\}$ is standard Brownian motion and μ and β are constants. With this model they were able to derive explicit expressions for quantities of fundamental interest such as the integrated volatility. A limitation of the use of the Ornstein-Uhlenbeck process (and of convex combinations of independent Ornstein-Uhlenbeck processes) is the constraint that the autocorrelations $\rho(h)$, $h \geq 0$, necessarily decrease as the lag h increases.

Much of the analysis of Barndorff-Nielsen and Shephard can however be carried out after replacing the Ornstein-Uhlenbeck process by a CARMA process with non-negative kernel driven by a non-decreasing Lévy process. This has the advantage of allowing the representation of volatility processes with a larger range of autocorrelation functions than is possible in the Ornstein-Uhlenbeck framework. The following example illustrates the larger class of autocovariance functions attainable by a Lévy-driven CARMA process with non-negative kernel.

Example 1. From equation (2.11) we find that the kernel function, g, of a CARMA$(3,2)$ process $\{Y(t)\}$ with

$$a(z) = (z + 0.1)(z + 0.5 - i\pi/2)(z + 0.5 - i\pi/2) \quad \text{and} \quad b(z) = 2.792 + 5z + z^2$$

is

$$g(t) = 0.8762e^{-0.1x} + \left(0.1238\cos\frac{\pi x}{2} + 2.5780\sin\frac{\pi x}{2}\right)e^{-0.5x}.$$

This function is plotted in Figure 1. For any driving Lévy process with $\mathrm{Var}(L(1)) = 1$, the autocovariance function of the CARMA process $\{Y(t)\}$ is, from (2.12),

$$\gamma(h) = 5.1161e^{-0.1x} + \left(4.3860\cos\frac{\pi x}{2} + 1.4066\sin\frac{\pi x}{2}\right)e^{-0.5x}.$$

This function is shown in Figure 2. If the driving Lévy process is non-decreasing, the corresponding CARMA process will have non-negative sample paths. This is an example of a non-negative Lévy-driven CARMA process with non-monotone autocorrelation function.

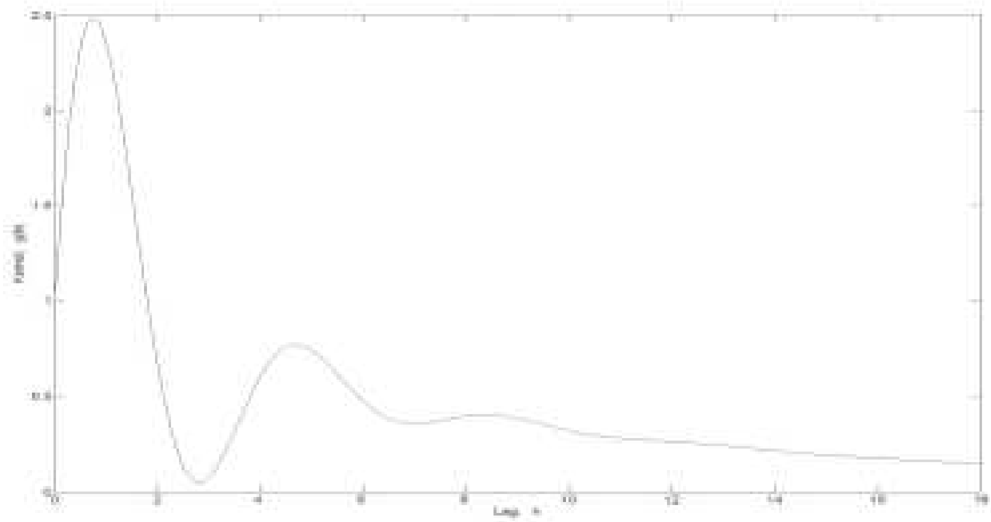

Fig. 1. *The kernel of the process in Example 1.*

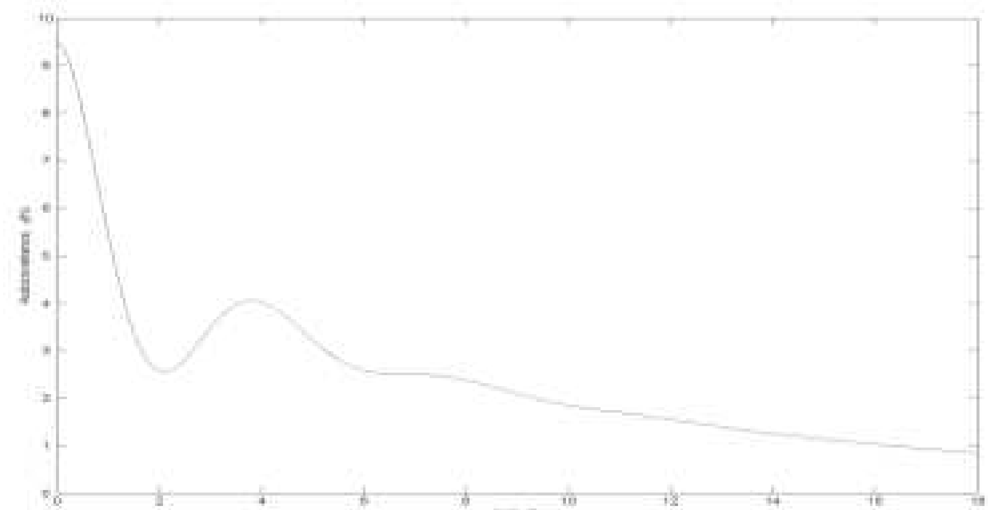

Fig. 2. *The autocovariance function of the process in Example 1.*

4. Fractionally Integrated Lévy-driven CARMA Processes

The discrete-time process $\{X_t, t = 0, \pm 1, \pm 2, \ldots\}$ is said to be a fractionally integrated ARMA process of order (p, d, q), with $p, q \in \{0, 1, 2, \ldots\}$ and $0 < d < 0.5$ if $\{X_t\}$ is a stationary solution of the equations

$$(1 - B)^d \phi(B) X_t = \theta(B) Z_t, \tag{4.1}$$

where $\phi(B)$ and $\theta(B)$ are polynomials of degrees p and q in the backward shift operator B, $\{Z_t\}$ is a sequence of uncorrelated random variables with mean zero and variance σ^2 and $\phi(z) \neq 0$ for all complex z such that $|z| \leq 1$.

If $d = 0$ in (4.1), $\{X_t\}$ is an ARMA(p,q) process with the almost surely and mean-square convergent representation

$$X_t = \sum_{j=0}^{\infty} \alpha_j Z_{t-j}, \tag{4.2}$$

where $\sum_{j=0}^{\infty} \alpha_j z^j = \theta(z)/\phi(z),\ |z| < 1$. If $d \in (0, 0.5)$, X_t has the mean-square convergent representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \tag{4.3}$$

where $\{\psi_j\}$ is the convolution of the sequences $\{\alpha_j\}$ and $\{\beta_j\}$ and $\sum_{j=0}^{\infty} \beta_j z^j = (1-B)^{-d}$, $|z| < 1$. The essential distinction between the series (4.2) and (4.3) is in the rates of convergence of the sequences $\{\alpha_j\}$ and $\{\psi_j\}$ to zero as $j \to \infty$. Thus there exist $K, M > 0$ and $r \in (0,1)$ such that $|\alpha_j| < Kr^j$, while $|\psi_j| \sim Mj^{d-1}$ as $j \to \infty$. Correspondingly the autocovariance function $\gamma_0(h)$ of the process (4.2) is geometrically bounded, while the autocovariance function of the process (4.3) satisfies $\gamma_d(h) \sim Ch^{2d-1}$ as $h \to \infty$. As a result of the slow rate of decay of ψ_j and $\gamma_d(h)$ as $j \to \infty$ and $h \to \infty$, the process (4.3) is said to have long memory. For more details see e.g. Brockwell and Davis (1991).

In order to incorporate long-memory into the class of causal Lévy-driven CARMA processes we make a simple modification to the kernel g as defined by (2.9) which amounts to convolving g with the kernel,

$$h(t) = \frac{t^{d-1}}{\Gamma(d)} I_{(0,\infty)}(t), \quad 0 < d < 0.5.$$

The resulting kernel,

$$g_d(t) = \int_0^t g(t-u) \frac{u^{d-1}}{\Gamma(d)}\ du, \tag{4.4}$$

is then the Riemann-Liouville fractional integral of the kernel g of the CARMA process defined by (2.10). (Notice also that the tail behaviour of the function h is analogous to that of the coefficients β_j in the power series expansion of $(1-B)^{-d}$. Application of Stirling's formula shows that $\beta_j \sim j^{d-1}/\Gamma(d)$ as $j \to \infty$.)

Definition: If $0 < d < 0.5$ and the roots of $a(z) = 0$ all have negative real parts, then the long-memory CARMA(p,d,q) process with coefficients $a_1, \ldots, a_p, b_1, \ldots, b_q$ and driven by the second-order (extended) Lévy process L^* is defined by (2.10) with $g(t)$ replaced by (4.4), or equivalently,

$$g_d(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} (i\lambda)^{-d} \frac{b(i\lambda)}{a(i\lambda)}\ d\lambda, \quad 0 < d < 0.5. \tag{4.5}$$

[The equivalence of (4.4) and (4.5) follows from the fact that g_d is the convolution of the functions g and h, with Fourier transforms $\int_{-\infty}^{\infty} e^{-i\lambda t} g(t)dt = b(i\lambda)/a(i\lambda)$ and $\int_{-\infty}^{\infty} e^{-i\lambda t} h(t)dt = (i\lambda)^{-d}$ respectively.]

Using the equivalent representations (4.4) and (4.5) we now derive asymptotic expressions for the kernel $g_d(t)$ and the corresponding autocovariance function $\gamma_d(h)$, showing that they converge to zero at hyperbolic rates as $t \to \infty$ and $h \to \infty$ respectively.

A simple change of variable allows us to rewrite the integral in (4.5) as

$$g_d(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{tz} z^{-d} \frac{b(z)}{a(z)}\ dz.$$

The asymptotic behaviour of $g_d(t)$ as $t \to \infty$ then follows at once from a Tauberian theorem of Doetsch (1974), p.254. Thus

$$g_d(t) \sim \frac{t^{d-1}}{\Gamma(d)} . \frac{b(0)}{a(0)}. \tag{4.6}$$

The autocovariance function of a CARMA(p, d, q) process, $\{Y(t)\}$, with kernel g_d is given by

$$\gamma_d(h) = \mathrm{Var}(L(1)) \int_0^{\infty} g_d(u + |h|) g_d(u) du,$$

since for $h \geq 0$ we can write

$$\begin{aligned} \mathrm{cov}(Y(t+h), Y(t)) &= \mathrm{cov}\left(\int_0^{\infty} g_d(u) dL^*(t+h-u), \int_0^{\infty} g_d(u) dL^*(t-u) \right) \\ &= \mathrm{cov}\left(\int_0^{\infty} g_d(u+h) dL^*(t-u), \int_0^{\infty} g_d(u) dL^*(t-u) \right) \\ &= \mathrm{Var}(L^*(1)) \int_0^{\infty} g_d(u+h) g_d(u) du. \end{aligned}$$

Assuming that the zeroes of the autoregressive polynomial are distinct, substituting for g_d from (4.3) and (2.11) and letting $h \to \infty$ gives the result,

$$\gamma_d(h) h^{1-2d} \to \frac{\mathrm{Var}(L^*(1))\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{b(\lambda_i) b(\lambda_j)}{\lambda_i \lambda_j a'(\lambda_i) a'(\lambda_j)}.$$

From the partial fraction expansion of $b(\lambda)/a(\lambda)$, we see that

$$\sum_{i=1}^{p} \frac{b(\lambda_i)}{\lambda_i a'(\lambda_i)} = -\frac{b(0)}{a(0)}$$

and hence, as $h \to \infty$,

$$\gamma_d(h) \sim h^{2d-1} \frac{\mathrm{Var}(L^*(1))\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \left[\frac{b(0)}{a(0)} \right]^2, \tag{4.7}$$

the latter result being valid also in the case when the autoregressive polynomial has multiple zeroes.

Example 2. The CARMA$(1, d, 0)$ process driven by a Lévy process with $\mathrm{Var}(L^*(1)) = 1$ has $b(z) = 1$ and $a(z) = z + c$ for some $c > 0$. Substituting in (4.6) and (4.7) we find that the asymptotic behaviour of the kernel and autocovariance functions are given by

$$g_d(t) \sim \frac{t^{d-1}}{c\Gamma(d)} \text{ as } t \to \infty$$

and

$$\gamma_d(h) \sim h^{2d-1} \frac{\mathrm{Var}(L^*(1))\Gamma(1-2d)}{c^2\Gamma(d)\Gamma(1-d)} \text{ as } h \to \infty.$$

Thus, in this example, and more generally, as indicated by the relations (4.6) and (4.7), the asymptotic behaviour as $t \to \infty$ and $h \to \infty$ of the kernel $g_d(t)$ and of the autocovariance function $\gamma_d(h)$ mimic the corresponding behaviour of the discrete time analogues given at the beginning of this section.

We note finally that fractional integration of a non-negative kernel preserves the non-negativity. Consequently fractional integration can be used to incorporate long memory into the stochastic volatility models discussed in Section 2. Figure 3 shows the kernel g_d obtained by fractional integration with $d = 0.25$ of the non-negative kernel shown in Figure 1.

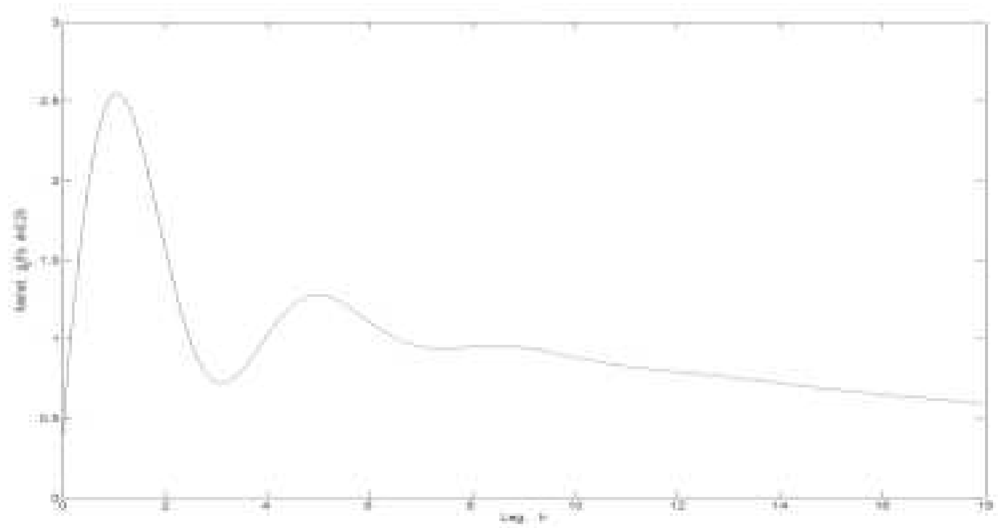

Fig. 3. *The kernel of the fractionally integrated ($d = .25$) process of Example 1.*

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