LÉVY-DRIVEN AND FRACTIONALLY INTEGRATED ARMA PROCESSES WITH CONTINUOUS TIME PARAMETER

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Abstract

The definition and properties of Lévy-driven CARMA (continuous-time ARMA) processes are reviewed. Gaussian CARMA processes are special cases in which the driving Lévy process is Brownian motion. The use of more general Lévy processes permits the specification of CARMA processes with a wide variety of marginal distributions which may be asymmetric and heavier tailed than Gaussian. Non-negative CARMA processes are of special interest, partly because of the introduction by Barndorff-Nielsen and Shephard (2001) of non-negative Lévy-driven Ornstein-Uhlenbeck processes as models for stochastic volatility. Replacing the Ornstein-Uhlenbeck process by a Lévy-driven CARMA process with non-negative kernel permits the modelling of non-negative, heavy-tailed processes with a considerably larger range of autocovariance functions than is possible in the Ornstein-Uhlenbeck framework. We also define a class of zero-mean fractionally integrated Lévy-driven CARMA processes, obtained by convoluting the CARMA kernel with a kernel corresponding to Riemann-Liouville fractional integration, and derive explicit expressions for the kernel and autocovariance functions of these processes. They are long-memory in the sense that their kernel and autocovariance functions decay asymptotically at hyperbolic rates depending on the order of fractional integration. In order to introduce long-memory into *non-negative* Lévy-driven CARMA processes we replace the fractional integration kernel with a closely related absolutely integrable kernel. This gives a class of stationary non-negative continuous-time Lévy-driven processes whose autocovariance functions at lag h also converge to zero at asymptotically hyperbolic rates.

KEYWORDS: *continuous-time ARMA process, Lévy process, stochastic volatility, long memory, fractional integration.*

1. Introduction

Continuous-time models for time series which exhibit both heavy-tailed and long-memory behaviour are of considerable interest, especially for the modelling of financial time series where such behaviour is frequently observed empirically. A recent paper of Anh, Heyde and Leonenko (2002) develops such models via the Green-function solution of fractional differential equations driven by Lévy processes. A very general class of Gaussian fractionally integrated continuous time models with extensive financial applications has also been introduced by Comte and Renault

(1996, 1998). An alternative approach to generating slowly decaying autocorrelation functions by randomizing the time-scale of a CARMA process has been developed by Ma (2003). For financial applications of the models which we discuss in this paper, see the recent work of Todorov and Tauchen (2004).

We consider the class of second-order Lévy-driven continuous-time ARMA (CARMA) processes and the fractionally integrated (FICARMA) processes obtained by fractional integration of the kernel of the CARMA process. In Section 2 we review the definition and properties of Lévy-driven CARMA processes, deriving the kernel and autocovariance functions, specifying the joint characteristic functions and discussing the issue of causality. In Section 3 we indicate the relevance of CARMA processes with non-negative kernel to the stochastic volatility model of Barndorff-Nielsen and Shephard (2001), giving an example of a CARMA(3,2) process with non-negative kernel and non-monotone autocovariance function. In Section 4, following Brockwell (2003), zero-mean Lévy-driven FICARMA processes are defined and the asymptotic forms of the kernel and autocovariance functions determined. In Sections 5 and 6, explicit expressions are derived for the kernel and autocovariance functions in the case when the autoregressive zeroes are distinct. Using these results, a comparison is made in Section 7 of the autocorrelation functions at integer times of the fractionally integrated Ornstein-Uhlenbeck process and the fractionally integrated (in the discrete time sense) sampled Ornstein-Uhlenbeck process. In Section 8 we introduce a related class of *non-negative*, stationary, Lévy-driven processes whose kernel and autocorrelation functions at lag h also decay at hyperbolic rates for large h.

For a discussion of the relative merits of stochastic volatility and GARCH models in finance we refer the reader to the article by Shephard (1996), where it is pointed out that although the stochastic volatility models are more difficult to handle statistically, their properties are easier to understand and manipulate, and the multivariate and continuous-time generalizations are more straightforward. These properties and the discontinuous nature of high-frequency transaction data point naturally to the use of the Barndorff-Nielsen and Shephard Lévy-driven Ornstein-Uhlenbeck stochastic volatility model and its generalization to Lévy-driven CARMA volatility.

Before proceeding further we need a few essential facts regarding Lévy processes. For a detailed account of the pertinent properties of Lévy processes see Protter (2004). Suppose we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq\infty}, P)$, where $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and $(\mathcal{F}_t)$ is right-continuous.

Definition 1 (Lévy Process). An adapted process $\{L(t), t \geq 0\}$ with $L(0) = 0$ a.s. is said to be a Lévy process if

(i) $L(t) - L(s)$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$,
(ii) $L(t) - L(s)$ has the same distribution as L_{t-s} and
(iii) $L(t)$ is continuous in probability.

Every Lévy process has a unique modification which is càdlàg (right continuous with left limits) and which is also a Lévy process. We assume that our Lévy process has these properties. The characteristic function of $L(t)$, $\phi_t(\theta) := E(\exp(i\theta L(t)))$, has the form

$$\phi_t(\theta) = \exp(t\xi(\theta)),\ \theta \in \mathbb{R}, \tag{1.1}$$

where

$$\xi(\theta) = i\theta m - \frac{1}{2}\theta^2 s^2 + \int_{\mathbb{R}_0} (e^{i\theta x} - 1 - \frac{ix\theta}{1+x^2})\nu(dx), \tag{1.2}$$

for some $m \in \mathbb{R}$, $s \geq 0$, and measure ν on the Borel subsets of $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. The measure ν is called the Lévy measure of the process L and has the property,

$$\int_{\mathbb{R}_0} \frac{u^2}{1+u^2}\nu(du) < \infty.$$

If ν is the zero measure then $\{L(t)\}$ is Brownian motion with $E(L(t)) = mt$ and $\mathrm{Var}(L(t)) = s^2 t$. If $m = s^2 = 0$ and $\nu(\mathbb{R}_0) < \infty$, then $L(t) = at + P(t)$, where $\{P(t)\}$ is a compound Poisson process with jump-rate $\nu(\mathbb{R}_0)$, jump-size distribution $\nu/\nu(\mathbb{R}_0)$, and $a = -\int_{\mathbb{R}_0} \frac{u}{1+u^2}\nu(du)$. A wealth of distributions for $L(t)$ is attainable by suitable choice of the measure ν. See for example Barndorff-Nielsen and Shephard (2001). For the second-order Lévy processes (with which we are concerned in this paper), $E(L(1))^2 < \infty$ and there exist real constants μ and σ such that

$$EL(t) = \mu t,\ t \geq 0, \tag{1.3}$$

and

$$\mathrm{Var}(L(t)) = \sigma^2 t,\ t \geq 0. \tag{1.4}$$

2. Second-order Lévy-driven CARMA Processes

A second-order Lévy-driven continuous-time ARMA(p, q) process is defined (see Brockwell (2001)) in terms of the following state-space representation of the formal equation,

$$a(D)Y(t) = b(D)DL(t),\ t \geq 0, \tag{2.1}$$

in which D denotes differentiation with respect to t, $\{L(t)\}$ is a Lévy process with $EL(1)^2 < \infty$,

$$a(z) := z^p + a_1 z^{p-1} + \cdots + a_p,$$

$$b(z) := b_0 + b_1 z + \cdots + b_{p-1} z^{p-1},$$

and the coefficients b_j satisfy $b_q \neq 0$ and $b_j = 0$ for $q < j < p$. To avoid trivial and easily eliminated complications we assume that $a(z)$ and $b(z)$ have no common factors. The state-space representation consists of the *observation* and *state* equations,

$$Y(t) = \mathbf{b}'\mathbf{X}(t), \tag{2.2}$$

$$dX(t) - AX(t)dt = e\ dL(t), \tag{2.3}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}.$$

(If $p = 1$, A is defined to be $-a_1$.) In the special case when $\{L(t)\}$ is Brownian motion, (2.3) is an Ito equation with solution $\{\mathbf{X}(t), t \geq 0\}$ satisfying

$$\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-u)}\mathbf{e}\ dL(u), \tag{2.4}$$

(where the integral is defined as the L^2 limit of approximating Riemann-Stieltjes sums), and more generally,

$$\mathbf{X}(t) = e^{A(t-s)}\mathbf{X}(s) + \int_s^t e^{A(t-u)}\mathbf{e}\ dL(u), \quad \text{for all } t > s \geq 0. \tag{2.5}$$

Equations (2.2) and (2.5), with $\{L(t)\}$ a general second-order Lévy process (i.e. satisfying $E(L(1))^2 < \infty$) are the starting point for our definition of a second order Lévy-driven CARMA process (Definition 2 below). Equation (2.5) clearly shows (by the independence of increments of $\{L(t)\}$) that $\{\mathbf{X}(t)\}$ is Markov. The following propositions give necessary and sufficient conditions for stationarity of $\{\mathbf{X}(t)\}$.

Proposition 1 If $\mathbf{X}(0)$ is independent of $\{L(t), t \geq 0\}$ and $E(L(1)^2) < \infty$, then $\{\mathbf{X}(t)\}$ is weakly stationary if and only if the eigenvalues of the matrix A all have strictly negative real parts and $\mathbf{X}(0)$ has the mean and covariance matrix of $\int_0^\infty e^{Au}\mathbf{e}\ dL(u)$.

Proof. The eigenvalues of A must have negative real parts for the sum of the covariance matrices of the terms on the right of (2.4) to be bounded in t. If this condition is satisfied, then $\{\mathbf{X}(t)\}$ converges in distribution as $t \to \infty$ to a random variable with the distribution of $\int_0^\infty e^{Au}\mathbf{e}\ dL(u)$. Hence, for weak stationarity, $\mathbf{X}(0)$ must have the mean and covariance matrix of $\int_0^\infty e^{Au}\mathbf{e}\ dL(u)$. Conversely if the eigenvalues of A all have negative real parts and if $\mathbf{X}(0)$ has the mean and covariance matrix of $\int_0^\infty e^{Au}\mathbf{e}\ dL(u)$, then a simple calculation using (2.4) shows that $\{\mathbf{X}(t)\}$ is weakly stationary.

Proposition 2. If $\mathbf{X}(0)$ is independent of $\{L(t), t \geq 0\}$ and $E(L(1)^2) < \infty$, then $\{\mathbf{X}(t)\}$ is a strictly stationary second-order process if and only if the eigenvalues of the matrix A all have strictly negative real parts and $\mathbf{X}(0)$ has the distribution of $\int_0^\infty e^{Au}\mathbf{e}\ dL(u)$.

Proof. Necessity follows from Proposition 1. If the conditions are satisfied then strict stationarity follows from the fact that $\{\mathbf{X}(t)\}$ is a Markov process whose initial distribution is the same as its limit distribution.

Remark 1. It is convenient to extend the state process $\{\mathbf{X}(t), t \geq 0\}$ to a process with index set $(-\infty, \infty)$. To this end we introduce a second Lévy process $\{M(t), 0 \leq t < \infty\}$, independent of L and with the same distribution, and then define the following extension of L:

$$L^*(t) = L(t)I_{[0,\infty)}(t) - M(-t-)I_{(-\infty,0]}(t), \quad -\infty < t < \infty.$$

Then, provided the eigenvalues of A all have negative real parts, the process $\{\mathbf{X}(t)\}$ defined by

$$\mathbf{X}(t) = \int_{-\infty}^{t} e^{A(t-u)}\mathbf{e}\ dL^*(u), \tag{2.6}$$

is a strictly stationary process satisfying (2.5) (with L replaced by L^*) for all $t > s$ and $s \in (-\infty, \infty)$. Henceforth we shall refer to L^* as the **background driving Lévy process** (**BDLP**) and denote it for simplicity by L rather than L^*.

Remark 2. It is easy to check that the eigenvalues of the matrix A, which we shall denote by $\lambda_1, \ldots, \lambda_p$, are the same as the zeroes of the autoregressive polynomial $a(z)$. The corresponding right eigenvectors are $[1\ \lambda_j\ \lambda_j^2\ \ldots\ \lambda_j^{p-1}]'$. $j = 1, \ldots, p$. We are now in a position to define the CARMA process $\{Y(t), -\infty < t < \infty\}$ via (2.2) under the condition that

$$\mathcal{R}e(\lambda_j) < 0,\ j = 1, \ldots, p. \tag{2.7}$$

Definition 2 (Causal CARMA Process). If the zeroes $\lambda_1, \ldots, \lambda_p$ of the autoregressive polynomial $a(z)$ satisfy (2.7), then the CARMA(p, q) process with second-order BDLP $\{L(t), -\infty < t < \infty\}$ and coefficients $\{a_1, \ldots, a_p, b_0, \ldots, b_q\}$ is the strictly stationary process, $Y(t) = \mathbf{b}'\mathbf{X}(t)$, where $\mathbf{X}(t) = \int_{-\infty}^{t} e^{A(t-u)}\mathbf{e}\ dL(u)$, so

$$Y(t) = \int_{-\infty}^{t} \mathbf{b}' e^{A(t-u)}\mathbf{e}\ dL(u). \tag{2.8}$$

Remark 3 (Causality and Non-causality). Under Condition (2.7) we see from (2.8) that $\{Y(t)\}$ is a *causal* function of $\{L(t)\}$, since it has the form

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)\ dL(u), \tag{2.9}$$

where

$$g(t) = \begin{cases} \mathbf{b}' e^{At}\mathbf{e} & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

The function g is referred to as the **kernel of the CARMA process** $\{Y(t)\}$. Under (2.7), the function g defined by (2.10) can be written as

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \frac{b(i\lambda)}{a(i\lambda)}\ d\lambda. \tag{2.11}$$

(To establish (2.11) when the eigenvalues $\lambda_1, \ldots, \lambda_p$ are distinct, we use the explicit expressions for the eigenvectors of A to replace e^{At} in (2.10) by its spectral representation. The same expression is obtained when the right side of (2.11) is evaluated by contour integration. When there are multiple eigenvalues, the result is obtained by separating the eigenvalues slightly and taking the limit as the repeated eigenvalues converge to their common value.) It is of interest to observe that the representation (2.9) and (2.11) of $\{Y(t)\}$ defines a strictly stationary process even under conditions less restrictive than (2.7), namely

$$\mathcal{R}e(\lambda_j) \neq 0, \; j = 1, \ldots, p. \tag{2.12}$$

Thus (2.9), (2.11) and (2.12) provide a more general definition of CARMA process than Definition 2 above. However if any of the zeroes of $a(z)$ has real part greater than 0, the representation (2.9) of $\{Y(t)\}$ in terms of $\{L(t)\}$ will no longer be causal as is the case when (2.7) is satisfied. This distinction between causal and non-causal CARMA processes is analogous to the classification of discrete-time ARMA processes as causal or otherwise, depending on whether or not the zeroes of the autoregressive polynomial lie outside the unit circle (see e.g. Brockwell and Davis (1991)). From now on **we shall restrict attention to causal CARMA processes**, i.e. we assume(2.7), so that the general expression (2.11) for the kernel g can also be written in the form (2.10). However both forms of the kernel will prove to be useful.

Remark 4 (Second-order Properties). From the representation (2.8) of the causal CARMA process with BDLP $\{L(t)\}$ satisfying (1.3) and (1.4), we immediately find that $EY(t) = -\mathbf{b}'A^{-1}\mathbf{e}\mu$. From the representation (2.9) of $Y(t)$ we see that its autocovariance function can be expressed as

$$\gamma(h) = \text{cov}(Y(t+h), Y(t)) = \sigma^2 \int_{-\infty}^{\infty} \tilde{g}(h-u)g(u)du,$$

where $\tilde{g}(x) = g(-x)$ and g is defined in (2.11). Using the convolution theorem for Fourier transforms, we find that

$$\int_{-\infty}^{\infty} e^{-i\omega h}\gamma(h)dh = \sigma^2 \left|\frac{b(i\omega)}{a(i\omega)}\right|^2,$$

showing that the spectral density of the process is

$$f(\omega) = \frac{\sigma^2}{2\pi}\left|\frac{b(i\omega)}{a(i\omega)}\right|^2 \tag{2.13}$$

and the autocovariance function is

$$\gamma(h) = \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} e^{i\omega h}\left|\frac{b(i\omega)}{a(i\omega)}\right|^2 \, d\omega. \tag{2.14}$$

Remark 5 (Distinct Autoregressive Zeroes). When the zeroes $\lambda_1, \ldots, \lambda_p$ of $a(z)$ are distinct and satisfy (2.7), the expression for the kernel g takes an especially simple form. Expanding the integrand in (2.11) in partial fractions and integrating each term gives the expression

$$g(h) = \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r h} I_{[0,\infty)}(h). \tag{2.15}$$

Applying the same argument to (2.14) gives a corresponding expression for the autocovariance function:

$$\gamma(h) = \text{cov}(Y(t+h), Y(t)) \;=\; \sigma^2 \sum_{j=1}^{p} \frac{b(\lambda_j)b(-\lambda_j)}{a'(\lambda_j)a(-\lambda_j)} e^{\lambda_j |h|}. \tag{2.16}$$

For the stationary Ornstein-Uhlenbeck (or CAR(1)) process, $b(z) = 1$ and $a(z) = z + c$ for some $c > 0$. From (2.15) and (2.16) we immediately find that $g(h) = e^{-ch} I_{[0,\infty)}(h)$ and $\gamma(h) = \frac{\sigma^2}{2c} e^{-c|h|}$ where $\sigma^2 = \text{Var}(L(1))$.

Remark 6 (The Joint Distributions). Since the study of Lévy-driven CARMA processes is largely motivated by the need to model processes with non-Gaussian joint distributions, it is important to go beyond a second-order characterization of these processes. From Proposition 2 we already know that the marginal distribution of $Y(t)$ is that of $\int_0^\infty g(t-u)dL(u)$, where g is given by (2.11) or, under the conditions of Remark 5, by (2.15). Using the expression (1.1) for the characteristic function of $L(t)$, we find that the cumulant generating function of $Y(t)$ is

$$\log E(\exp(i\theta Y(t))) = \int_0^\infty \xi(\theta g(u))du. \tag{2.17}$$

More generally it can be shown (see Brockwell (2001)) that the cumulant generating function of $Y(t_1), Y(t_2), \ldots, Y(t_n)$, $(t_1 < t_2 < \cdots < t_n)$ is

$$\begin{aligned} &\log E[\exp(i\theta_1 Y(t_1) + \cdots + i\theta_n Y(t_n))] = \\ &\int_0^\infty \xi\left(\sum_{i=1}^{n} \theta_i g(t_i + u)\right) du + \int_0^{t_1} \xi\left(\sum_{i=1}^{n} \theta_i g(t_i - u)\right) du \;+ \\ &\int_{t_1}^{t_2} \xi\left(\sum_{i=2}^{n} \theta_i g(t_i - u)\right) du + \cdots + \int_{t_{n-1}}^{t_n} \xi\left(\theta_n g(t_n - u)\right) du. \end{aligned} \tag{2.18}$$

If $\{L(t)\}$ is a compound Poisson process with finite jump-rate λ and bilateral exponential jump-size distribution with probability density $f(x) = \frac{1}{2}\beta e^{-\beta|x|}$, then by (2.17), the corresponding CAR(1) process of Remark 4 has marginal cumulant generating function, $\kappa(\theta) = \int_0^\infty \xi(\theta e^{-cu})du$, where $\xi(\theta) = \lambda\theta^2/(\beta^2 + \theta^2)$. Straightforward evaluation of the integral gives

$$\kappa(\theta) = -\frac{\lambda}{2c} \log\left(1 + \frac{\theta^2}{\beta^2}\right),$$

showing that $Y(t)$ has a symmetrized gamma distribution, or more specifically that $Y(t)$ is distributed as the difference between two independent gamma distributed random variables with exponent $\lambda/(2c)$ and scale parameter β. In particular, if $\lambda = 2c$, the marginal distribution is bilateral exponential. For more examples see Barndorff-Nielsen and Shephard (2001).

3. An Application to Stochastic Volatility Modelling

Barndorff-Nielsen and Shephard (2001) introduced a model for asset-pricing in which the logarithm of an asset price is the solution of the stochastic differential equation

$$dX^*(t) = (\mu + \beta\sigma^2(t))dt + \sigma(t)dW(t),$$

where $\{\sigma^2(t)\}$, the instantaneous volatility, is a non-negative Lévy-driven Ornstein-Uhlenbeck process, $\{W(t)\}$ is standard Brownian motion and μ and β are constants. With this model they were able to derive explicit expressions for quantities of fundamental interest such as the integrated volatility. A crucial feature of volatility modelling is the requirement that the volatility must be non-negative, a property achieved by the Lévy-driven Ornstein-Uhlenbeck process since its kernel is non-negative and the driving Lévy process is chosen to be non-decreasing. A limitation of the use of the Ornstein-Uhlenbeck process (and of convex combinations of independent Ornstein-Uhlenbeck processes) is the constraint that the autocorrelations $\rho(h)$, $h \geq 0$, necessarily decrease as the lag h increases.

Much of the analysis of Barndorff-Nielsen and Shephard can however be carried out after replacing the Ornstein-Uhlenbeck process by a CARMA process with non-negative kernel driven by a non-decreasing Lévy process. This has the advantage of allowing the representation of volatility processes with a larger range of autocorrelation functions than is possible in the Ornstein-Uhlenbeck framework. For example, the CARMA(3,2) process with $a(z) = (z + 0.1)(z + 0.5 - i\pi/2)(z + 0.5 - i\pi/2)$ and $b(z) = 2.792 + 5z + z^2$ has non-negative kernel and autocovariance functions,

$$g(t) = 0.8762e^{-0.1t} + \left(0.1238\cos\frac{\pi t}{2} + 2.5780\sin\frac{\pi t}{2}\right)e^{-0.5t}, \quad t \geq 0,$$

and

$$\gamma(h) = 5.1161e^{-0.1h} + \left(4.3860\cos\frac{\pi h}{2} + 1.4066\sin\frac{\pi h}{2}\right)e^{-0.5h}, \quad h \geq 0,$$

respectively, both of which exhibit damped oscillatory behaviour.

Remark 7. In the next section, following Brockwell (2003), we define a fractionally integrated Lévy-driven CARMA process by fractional integration of the corresponding CARMA kernel.

4. Fractionally Integrated Lévy-driven CARMA Processes

The discrete-time process $\{X_t, t = 0, \pm1, \pm2, \ldots\}$ is said to be a fractionally integrated ARMA process of order (p, d, q), with $p, q \in \{0, 1, 2, \ldots\}$ and $0 < d < 0.5$ if $\{X_t\}$ is a stationary solution of the equations

$$(1 - B)^d\phi(B)X_t = \theta(B)Z_t, \tag{4.1}$$

where $\phi(B)$ and $\theta(B)$ are polynomials of degrees p and q in the backward shift operator B, $\{Z_t\}$ is a sequence of uncorrelated random variables with mean zero and variance σ^2, and $\phi(z) \neq 0$ for all complex z such that $|z| \leq 1$.

If $d = 0$ in (4.1), $\{X_t\}$ is an ARMA(p,q) process with the mean square and almost surely absolutely convergent representation

$$X_t = \sum_{j=0}^{\infty} \alpha_j Z_{t-j}, \tag{4.2}$$

where $\sum_{j=0}^{\infty} \alpha_j z^j = \theta(z)/\phi(z),\ |z| < 1$. If $d \in (0, 0.5)$, X_t has the mean-square convergent representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \tag{4.3}$$

where $\{\psi_j\}$ is the convolution of the sequences $\{\alpha_j\}$ and $\{\beta_j\}$ and $\sum_{j=0}^{\infty} \beta_j z^j = (1 - B)^{-d},\ |z| < 1$. The slow rate of decay of the sequence $\{\psi_j\}$ as compared with $\{\alpha_j\}$ and the resulting long-memory properties when $d > 0$ can be attributed directly to the convolving of $\{\alpha_j\}$ with the hyperbolically decaying sequence $\{\beta_j\}$. In fact from Stirling's formula it is easy to check that $\beta_j \sim j^{d-1}/\Gamma(d)$ as $j \to \infty$.

In order to incorporate long-memory into the class of causal Lévy-driven CARMA processes, this suggests convolving the kernel g as defined by (2.10) with the function $h(t) = t^{d-1} I_{(0,\infty)}(t)/\Gamma(d),\quad 0 < d < 0.5$. The resulting kernel,

$$g_d(t) = \int_0^t g(t-u) \frac{u^{d-1}}{\Gamma(d)}\, du, \tag{4.4}$$

is then the Riemann-Liouville fractional integral of the kernel g of the CARMA process defined by (2.8).

Definition 3 (FICARMA(p,d,q) Process). If $0 < d < 0.5$, the roots of $a(z) = 0$ all have negative real parts and L is a second-order Lévy process with mean zero, then the FICARMA(p,d,q) process with coefficients $a_1, \ldots, a_p, b_0, \ldots, b_q$ and driving process L is defined by (2.9) with $g(t)$ replaced by $g_d(t)$ as in (4.4), or equivalently,

$$g_d(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} (i\lambda)^{-d} \frac{b(i\lambda)}{a(i\lambda)}\, d\lambda, \quad 0 < d < 0.5. \tag{4.5}$$

(The equivalence of (4.4) and (4.5) follows from the fact that g_d is the convolution of the functions g and h, with Fourier transforms $\int_{-\infty}^{\infty} e^{-i\lambda t} g(t) dt = b(i\lambda)/a(i\lambda)$ and $\int_{-\infty}^{\infty} e^{-i\lambda t} h(t) dt = (i\lambda)^{-d}$ respectively.)

From (4.5) and the expression $\gamma_d(h) = \sigma^2 \int_0^{\infty} g_d(u + |h|) g_d(u) du$ for the autocovariance function of the CARMA(p,d,q) process, Brockwell (2003) derived the asymptotic expressions,

$$g_d(t) \sim \frac{t^{d-1}}{\Gamma(d)} . \frac{b(0)}{a(0)} \text{ as } t \to \infty, \tag{4.6}$$

$$(4.7) \qquad \gamma_d(h) \sim h^{2d-1}\frac{\sigma^2\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\left[\frac{b(0)}{a(0)}\right]^2 \text{ as } h\to\infty,$$

showing that the asymptotic behaviour of the kernel $g_d(t)$ and of the autocovariance function $\gamma_d(h)$ is analogous to that of the corresponding functions for the discrete time process $\{X_t\}$ defined by (4.1) (see e.g. Beran (1994)):

$$(4.8) \qquad \psi_j \sim \frac{j^{d-1}}{\Gamma(d)}.\frac{\theta(1)}{\phi(1)} \text{ as } j\to\infty,$$

$$(4.9) \qquad \gamma_X(h) \sim h^{2d-1}\frac{\sigma^2\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\left[\frac{\theta(1)}{\phi(1)}\right]^2 \text{ as } h\to\infty.$$

Our goal in the following two sections is to determine the kernel and autocovariance functions g_d and γ_d of the fractionally integrated process. The joint distributions of the FICARMA process have cumulant generating functions which are obtained from (2.18) on replacing the kernel g by g_d.

5. The Kernel of the Fractionally Integrated CARMA Process

Our starting point is the kernel of the CARMA(p,q) process with autoregressive polynomial $a(z)$ and moving average polynomial $b(z)$. We assume that the p roots $\lambda_1,\ldots,\lambda_p$, of $a(z)=0$ are distinct with real parts less than zero. The kernel g_d of the fractionally integrated process is then given by (4.4) with g given by (2.15).

The evaluation of the convolution in (4.4) is quite straightforward, leading to the expression

$$(5.1) \qquad g_d(t) = \sum_{j=1}^{p}\frac{b(\lambda_j)}{a'(\lambda_j)}u(d,\lambda_j,t),$$

where

$$(5.2) \qquad u(d,\lambda,t) = \lambda^{-d}e^{\lambda t}P(\lambda t,d)I_{[0,\infty)}(t).$$

In this expression, $\lambda^{-d}=r^{-d}e^{-id\theta}$, where (r,θ) is the polar representation of λ with $-\pi<\theta\le\pi$, and $P(z,d)$ is the incomplete gamma function with complex argument,

$$P(z,d) = \frac{1}{\Gamma(d)}\int_0^z e^{-x}x^{d-1}dx,$$

where integration is along the radial line in the complex plane from 0 to z. The function P can also be expressed as

$$P(z,d) = \frac{z^d}{\Gamma(d+1)}\ {}_1F_1(d;d+1;-z),$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind. This is a standard function, available for example in MATLAB. Thus

$$(5.3) \qquad g_d(t) = \sum_{j=1}^{p}\frac{b(\lambda_j)}{a'(\lambda_j)}\lambda_j^{-d}e^{\lambda_j t}P(\lambda_j t,d)I_{[0,\infty)}(t).$$

The asymptotic form of $g_d(t)$ as $t \to \infty$ is given by (4.6).

Example 1. (The fractionally integrated Ornstein-Uhlenbeck process) For the Ornstein-Uhlenbeck process, $a(z) = z + c$ for some $c > 0$ and $b(z) = 1$. From (2.15) we obtain the familiar expression for the kernel,

$$g(t) = e^{-ct} I_{[0,\infty)}(t), \tag{5.4}$$

and from (5.3) we obtain the fractionally integrated kernel,

$$g_d(t) = (-c)^{-d} e^{-ct} P(-ct, d) I_{[0,\infty)}(t). \tag{5.5}$$

From (4.6), the asymptotic form of $g_d(t)$ in this special case is

$$g_d(t) \sim \frac{t^{d-1}}{c\Gamma(d)} \text{ as } t \to \infty. \tag{5.6}$$

If $c = 1$ and $d = 0.2$, the exact and asymptotic expressions (5.5) and (5.6) agree to within 1 percent for $h \geq 100$. The exact and asymptotic expressions for $g_{0.2}(100)$ are $.005516$ and $.005472$ respectively, as compared with the much smaller value of the unintegrated kernel, $g(100) = 3.72 \times 10^{-44}$.

Example 2. (A fractionally integrated CAR(2) process) For the fractionally integrated CAR(2) process with distinct complex conjugate autoregressive roots λ and $\overline{\lambda}$, equation (5.3) gives $g_d(h) = 2\mathcal{R}e\left[(\lambda^{-d} e^{\lambda h} P(\lambda h, d))/(\lambda - \overline{\lambda})\right] I_{[0,\infty)}(h)$ and (4.6) gives $g_d(h) \sim h^{d-1}/(|\lambda|^2 \Gamma(d))$ as $h \to \infty$.

6. The Autocovariance Function of the Fractionally Integrated CARMA Process

The autocovariance function of the fractionally integrated CARMA process $Y_d(t) = \int_{-\infty}^{\infty} g_d(t-u) dL(u)$, with $g_d(u)$ defined as in (4.4), can be expressed as

$$\gamma_d(h) = \text{cov}(Y_d(t+h), Y_d(t)) = \sigma^2 \int_{-\infty}^{\infty} \tilde{g}_d(h-u) g_d(u) du,$$

where $\tilde{g}_d(x) = g_d(-x)$. Using the representation (4.5) of $g_d(x)$, and the convolution theorem for Fourier transforms, we find that

$$\int_{-\infty}^{\infty} e^{-i\omega h} \gamma_d(h) dh = \frac{\sigma^2}{|\omega|^{2d}} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2, \tag{6.1}$$

showing that the spectral density of the fractionally integrated process is

$$f_d(\omega) = \frac{\sigma^2}{2\pi |\omega|^{2d}} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2.$$

Applying the convolution theorem again to (6.1), we find that γ_d can be expressed as

$$\gamma_d(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \gamma(h-u) r(u) du, \tag{6.2}$$

where

$$r(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega u}}{|\omega|^{2d}} d\omega \;=\; \sqrt{\frac{2}{\pi}} \sin(\pi d)\Gamma(1-2d)|u|^{2d-1}. \tag{6.3}$$

Substituting from (2.16) and (6.3) into (6.2) we find, by an argument analogous to that used in Section 4, that

$$\gamma_d(h) = \sum_{j=1}^{p} \frac{b(\lambda_j)b(-\lambda_j)}{a'(\lambda_j)a(-\lambda_j)} v(d, \lambda_j, h), \tag{6.4}$$

where

$$v(d,\lambda,h) = \frac{\sigma^2}{2\cos(\pi d)} \left[2(-\lambda)^{-2d}\cosh(\lambda h) + \lambda^{-2d} e^{\lambda h} P(\lambda h, 2d) - (-\lambda)^{-2d} e^{-\lambda h} P(-\lambda h, 2d) \right], \tag{6.5}$$

and the complex-valued incomplete gamma function is defined as in Section 5. The asymptotic behaviour of $\gamma_d(h)$ as $h \to \infty$ was specified in (4.7).

Example 3. (The fractionally integrated Ornstein-Uhlenbeck process) For the Ornstein-Uhlenbeck process, $a(z) = z + c$ for some $c > 0$ and $b(z) = 1$. From (2.16) we obtain the familiar expression for the autocovariance function, $\gamma(h) = \sigma^2 e^{-c|h|}/(2c)$, and from (6.4) and (6.5) we find, for the fractionally integrated process, that the variance is

$$\gamma_d(0) = \frac{\sigma^2}{2c^{2d+1}\cos(\pi d)} \tag{6.6}$$

while the autocorrelation function, $\rho_d(h) = \gamma_d(h)/\gamma_d(0)$, is

$$\rho_d(h) = \cosh(ch) - \frac{e^{ch}}{2} P(ch, 2d) + \frac{e^{-ch}}{2}(-1)^{-2d} P(-ch, 2d), \;\; h \geq 0. \tag{6.7}$$

(In a related paper, Hϕg (2000) derived expressions for the autocovariance function of the *non-stationary* fractionally integrated Ornstein-Uhlenbeck process with initial value zero.)

The autocorrelation function (6.7), interestingly, depends on c and h only through the value of ch. The following table displays the autocorrelation function for $d = .01, .05, .1, .2, .3, .4, .45, .49$ and for $ch = 0, 5, 10, 15, 20, 25, 30$.

Table 1. *The autocorrelation function (6.9)*

$ch \backslash d$	.01	.05	.10	.20	.30	.40	.45	.49
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5	.01133	.03345	.07070	.18462	.36528	.63081	.80127	.95788
10	.00221	.01354	.03514	.11450	.26901	.54338	.74421	.94393
15	.00144	.00926	.02513	.08918	.22790	.50029	.71414	.93618
20	.00108	.00712	.01990	.07490	.20289	.47209	.69373	.93077
25	.00087	.00582	.01663	.06545	.18547	.45138	.67835	.92661
30	.00072	.00493	.01436	.05864	.17237	.43516	.66606	.92323

From (4.7) and (6.6) we obtain the asymptotic expression for the autocorrelation function,

$$\rho_d(h) \sim (ch)^{2d-1} \frac{2\Gamma(1-2d)\cos(\pi d)}{\Gamma(d)\Gamma(1-d)} \text{ as } h \to \infty. \tag{6.8}$$

The relative error of the asymptotic approximation when $ch = 30$ is less than 0.3% across the range of d-values tabulated. The variances $\gamma_d(0)$ are readily calculated from (6.6) and range from $0.50025\sigma^2$ when $d = .01$ to $15.91811\sigma^2$ when $d = .49$. $(\gamma_d(0) \to \infty$ as $d \to 0.5.)$

7. Comparison with a Discrete-time Fractionally Integrated ARMA Process

From (4.7) and (4.9) we see that the asymptotic behaviour of the autocovariance function of the fractionally integrated CARMA process is closely analogous to that of the discrete-time fractionally integrated ARMA process defined by (4.1). In this section we compare in more detail the autocorrelation structure of a fractionally integrated CARMA process with the process obtained by fractionally integrating (in the discrete time sense) the ARMA process obtained by sampling the CARMA process at integer times.

Starting from a continuous-time fractionally integrated process, the first step is to determine the parameters of the discrete-time ARMA process obtained by sampling the CARMA process at integer times. We restrict attention here to the simplest case, namely the Ornstein-Uhlenbeck process of Example 1. The sampled process in this case clearly satisfies the discrete-time AR(1) equations

$$X_t = e^{-c}X_{t-1} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2(1-e^{-2c})/(2c)). \tag{7.1}$$

The comparison to be made in this case is therefore between the autocovariance functions of the process obtained by (discrete-time) fractional integration with order d of the AR(1) process (7.1) and the continuous-time autocovariance function γ_d defined by (6.6) and (6.7).

Example 4. (The fractionally integrated Ornstein-Uhlenbeck process) For the Ornstein-Uhlenbeck process of Example 1 and for the sampled process (7.1), we find from (4.7) and (4.9) that the autocovariance functions of the corresponding fractionally integrated processes have the asymptotic forms

$$\gamma_d(h) \sim h^{2d-1} \frac{\sigma^2\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\ c^{-2}, \tag{7.2}$$

$$\delta_d(h) \sim h^{2d-1} \frac{\sigma^2\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\ \frac{\coth(c/2)}{2c}, \tag{7.3}$$

respectively. Of course the functions $\gamma_0(h)$ and $\delta_0(h)$ are identical, but for $d > 0$, (7.2) and (7.3) show that

$$\frac{\delta_d(h)}{\gamma_d(h)} \ \to\ \frac{c}{2}\coth\frac{c}{2} \text{ as } h \to \infty. \tag{7.4}$$

For $0 < c < 2.2$ the corresponding discrete-time AR(1) process has coefficient e^{-c} between 1 and .1108 and the asymptotic ratio in (7.4) is between 1 and 1.37.

To compare the autocovariance functions at finite lags we need to compare the expression

$$\gamma_d(h) = \frac{\sigma^2}{2c^{2d+1}\cos(\pi d)}\left[\cosh(ch) - \frac{e^{ch}}{2}P(ch, 2d) + \frac{e^{-ch}}{2}(-1)^{-2d}P(-ch, 2d)\right] \tag{7.5}$$

for the fractionally integrated CARMA process with the expression

$$\delta_d(h) = \frac{\sigma^2\Gamma(1-2d)\Gamma(d+1-h)}{2c\ \Gamma(1-d)\Gamma(2-d-h)\Gamma(d)} \tag{7.6}$$

$$\times\left[e^{-c}F\left(d+1-h, 1; 2-d-h; e^{-c}\right) + e^{c}F\left(d-1+h, 1; h-d; e^{-c}\right) - e^{-c}\right]$$

from Sowell's formula (see also Hosking (1981)) for the fractionally integrated discrete time AR(1) with coefficient e^{-c} and white noise variance $\sigma^2(1-e^{-2c})/(2c)$. The following tables show the corresponding autocorrelation functions when $c = 1$.

Continuous-time ACF, $\gamma_d(h)/\gamma_d(0)$.

$d\backslash h$	1	2	5	10	20	100	500
0	.36788	.13534	.00674	.00005	.00000	.00000	.00000
0.01	.38086	.14582	.01133	.00221	.00108	.00022	.00005
0.25	.70131	.49109	.26550	.17994	.12640	.05642	.02523
0.49	.98940	.97774	.95788	.94393	.93077	.90125	.87270

Discrete-time ACF, $\delta_d(h)/\delta_d(0)$.

$d\backslash h$	1	2	5	10	20	100	500
0	.36788	.13534	.00674	.00005	.00000	.00000	.00000
0.01	.37871	.14523	.01156	.00236	.00116	.00024	.00005
0.25	.66499	.46686	.25592	.17414	.12241	.05465	.02444
0.49	.98644	.97468	.95515	.94132	.92823	.89878	.87031

These autocorrelations illustrate the general qualitative similarity between the behaviour of the FICARMA and fractionally integrated ARMA autocorrelation functions. The Lévy-driven FICARMA processes constitute a very convenient parametric family of processes exhibiting long memory, a large variety of marginal distributions and a broad range of correlation structures.

8. Stationary Long-memory Non-negative CARMA Processes.

In Definition 3 we restricted the mean of the background driving Lévy process (BDLP) to be zero. For a *non-negative* Lévy-driven CARMA process the BDLP, $\{L(t)\}$, is necessarily non-decreasing and therefore $E(L(1)) > 0$. The kernel g_d as defined in (4.4) cannot be applied to $\{L(t)\}$ as in previous sections to generate a non-negative stationary long-memory process, since a kernel which is both integrable and square integrable on $[0,\infty)$ is required. However a convenient family of "moderately long memory" Lévy-driven processess with asymptotically hyperbolically decreasing kernel and autocovariance function can be generated from (2.9) on replacing the kernel g by

$$g_{a,d}(t) = \int_0^t g(t-u)h_{a,d}(u)du, \quad a>0,\ d<0, \tag{8.1}$$

where

$$h_{a,d}(t) = K_{a,d}\min(a^{d-1}, t^{d-1})I_{(0,\infty)}(t), \tag{8.2}$$

and $K_{a,d}$ is chosen (for convenience) so that $h_{a,d}$ is a probability density, i.e.,

$$K_{a,d} = a^{|d|}|d|/(1+|d|). \tag{8.3}$$

Other probability densities with slowly decreasing tails could be chosen instead of $h_{a,d}$, but this particular choice generates a convenient two-parameter family of convolution operations with the property that as $a \to 0$ and $d \to -\infty$, $g_{a,d}(t) \to g(t)$ for each $t > 0$. Moreover calculations analogous to those of Sections 5 and 6 can be carried out for the kernels $g_{a,d}$ and the corresponding autocovariance functions. The asymptotic rates of convergence to zero of the kernel and autocovariance functions are analogous to those of (4.6) and (4.7), but of course the rates are somewhat faster since d is restricted to be negative. We now sketch the details.

Repeating the argument of Section 5, with $h_{a,d}$ replacing the kernel h, and assuming as before that the p roots of $a(z) = 0$ are distinct with real parts less than zero, we can evaluate the kernel $g_{a,d}$ as

$$g_{a,d}(t) = K_{a,d}\sum_{j=1}^{p} \frac{b(\lambda_j)}{a'(\lambda_j)} w(a,d,\lambda_j,t), \tag{8.4}$$

where

$$w(a,d,\lambda,t) = \begin{cases} \frac{a^{d-1}}{\lambda}(e^{\lambda t}-1) & \text{if } t \le a, \\ e^{\lambda t}\left[\frac{a^{d-1}}{\lambda}(1-e^{-\lambda a}) + \frac{t^d}{d}\,{}_1F_1(d;d+1;-t\lambda) - \frac{a^d}{d}\,{}_1F_1(d;d+1;-a\lambda)\right] & \text{if } t > a. \end{cases}$$

Like the kernel g_d in (5.3), the kernel $g_{a,d}$ can easily be computed using MATLAB.

If λ has negative real part and $d < 0$, straightforward integration shows that, as $x \to \infty$,

$$x^{-d+1}\left[\int_0^a e^{\lambda(x-u)}a^{d-1}du + \int_a^x e^{\lambda(x-u)}u^{d-1}du\right] \to -\lambda^{-1}.$$

Hence, from (8.4), as $t \to \infty$,

$$t^{-d+1}g_{a,d}(t) \to -K_{a,d}\sum_{j=1}^{p}\frac{b(\lambda_j)}{a'(\lambda_j)}\frac{1}{\lambda_j} = K_{a,d}\frac{b(0)}{a(0)}. \tag{8.5}$$

This demonstrates the hyperbolic asymptotic rate of decay of the kernel.

The autocovariance function $\gamma_{a,d}$ of the process (2.9) with kernel g replaced by $g_{a,d}$ is the convolution,

$$\gamma_{a,d} = \gamma * h_{a,d} * \tilde{h}_{a,d}, \tag{8.5}$$

where $\tilde{h}_{a,d}(x) := h_{a,d}(-x)$. As $t \to \infty$,

$$h_{a,d} * \tilde{h}_{a,d}(t) \sim K_{a,d} t^{d-1},$$

and

$$t^{-d+1}\gamma_{a,d}(t) \to -2K_{a,d} \sum_{j=1}^{p} \frac{b(\lambda_j)b(-\lambda_j)}{a'(\lambda_j)a(-\lambda_j)} \frac{1}{\lambda_j} = K_{a,d} \left[\frac{b(0)}{a(0)}\right]^2. \tag{8.6}$$

From (8.5) and (8.6) we see that the rate of approach to zero of both the kernel and the autocovariance function is the same in this case, in contrast with the different rates in (4.6) and (4.7).

Acknowledgments We are indebted to two referees for their valuable comments and to the National Science Foundation for its support of PJB under NSF Grant DMS 0308109.

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