Ruin estimation in multivariate models with Clayton dependence structure

Dedicated to Hans Bühlmann on the Occasion of His Seventy-Fifth Birthday

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Abstract

We consider the estimation of the ruin probability in a linear portfolio of insurance risk processes. We model the total claim amount of different business activities by compound Poisson processes. We allow for dependence of the components, which we model by a Lévy copula. We study in detail a Clayton-Pareto model as representative for a large claims model and a Clayton-exponential model as a small claims model. We compare the ruin probability in the Clayton dependence model with the corresponding independent and completely dependent models.

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1 Introduction

Risk modelling in insurance by means of estimating the ruin probability for a single compound Poisson processes has a long tradition. In this paper we consider not only one single risk process but a portfolio of risk processes, whose sum describes the risk of an insurance company. In most realistic situations these risk processes would be dependent, and it is to be expected that the dependence structure influences the ruin probability, i.e. the risk of the portfolio.

The core problem here is the dynamic multivariate modeling. We model the net risk portfolio of an insurance company as a multivariate Lévy process $R = (R_t^1, R_t^2, \ldots, R_t^d)_{t\geq 0}$. This means that R has stationary and independent increments and we assume that its sample paths are cadlag (right continuous with left limits). The corresponding net risk reserve of the insurance company is given by the stochastic process $R^+ = (R_t^+)_{t\geq 0}$, where

$$R_t^+ := R_t^1 + R_t^2 + \cdots + R_t^d \,, \quad t \geq 0 \,.$$

We can think of each component as a risk process $R_t^i = x_i + c_i t - C_t^i$, $t \geq 0$, for initial risk reserves $x_i \geq 0$ and premium rates $c_i > 0$ for all $i = 1, \ldots, d$. Then

$$R_t^+ = x + ct - C_t^+ \,, \quad t \geq 0 \,, \tag{1.1}$$

for $x = \sum_{i=1}^d x_i$, $c = \sum_{i=1}^d c_i$ and $C_t^+ = \sum_{i=1}^d C_t^i$ describes the total risk reserve of the company at time $t \geq 0$. Note that R^+ is again a Lévy process as summation of the components over disjoint increments remain independent and also stationarity of increments of the sum is inherited from the components. As summation is continuous, also the cadlag property prevails.

In this model we consider the *ruin probability* as risk measure. For the initial risk reserve $x \geq 0$ the ruin probability is defined as

$$\Psi(x) := P(R_t^+ < 0 \quad \text{for some } t \geq 0).$$

It is to be expected that the dependence structure of the components will have an effect on the ruin probability. The question is how to model the dependence structure between the risk processes. A unifying approach for dependence modelling is the notion of a copula, which has become popular; see e.g. Joe [7]. Since the law of a Lévy process is completely determined by its distribution at time t, the dependence structure of the components of a Lévy process could in principle be parameterized by a distributional copula C_t of the d.f. of its components at some fixed time t. However, this parametrization is not convenient as C_t depends on t. Moreover, given the copula C_t at time t it is not clear, how the copula C_s at some other time s looks like. Infinite divisibility is a property of the law of any Lévy process at any time. It is not clear, what copulae correspond to infinite divisible laws. Moreover, copulae are invariant under strictly increasing transformations, however, infinite divisibility is not; see Tankov [9], Example 3.1. From these considerations one can conclude that distributional copulae may not be an appropriate concept to model the dependence structure in a multivariate Lévy process.

The properties of the Lévy process suggest another related approach. From independence and stationarity of the increments follows that the law of the Lévy process $X = (X_t)_{t\geq 0}$ is for any fixed t infinitely divisible and can be represented by the *Lévy-Khintchine formula*

$$E(e^{i\,(z,X_t)}) = \exp\left\{t\left(i\,(a,z) - \frac{1}{2}(z,Qz) - \int_{\mathbb{R}^d}(e^{i(z,x)} - 1 - i(z,x)I_{\{|x|\geq 1\}})\Pi(dx)\right)\right\}$$

for all $z \in \mathbb{R}^d$, where $a \in \mathbb{R}^d$, Q is a positive definite $(d\times d)-$matrix and Π is a positive measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(dx) < \infty$, called the *Lévy measure* of X. The triplet (a, Q, Π) is called the *characteristic triplet* of the Lévy process X. The dependence structure between the components of X is given by the matrix Q for the Gaussian part and by the Lévy measure Π for the jump part.

Whereas the dependence structure in the Gaussian part is well-understood, the dependence in the Lévy measure is much less obvious. However, as the Lévy measure is independent of t it suggests itself for modelling the dependence in the jump part. In this paper we shall introduce parametric models for the Lévy measure and study their impact on the ruin probability of the risk process R.

As suggested in Tankov [9] we use Lévy copulae for the modelling of dependence for a multivariate risk process. Since we are going to use Lévy copulae for modelling of subordinator dependency, we prefer the notion "subordinator copulae" (S-copulae) in order to emphasize that we consider only subordinators.

Our paper is organised as follows. In Section 2 we define a subordinator copula and present some properties and examples. We define the ruin probability and summarize asymptotic results in the realm of heavy- and light-tailed claim size distributions. As the tail integral is the main ingredient for estimating the ruin probability we present its represention for bivariate subordinators. Section 3 is dedicated to the Clayton risk process, the sum of two dependent compound Poisson processes, whose Lévy measures are dependent with a Clayton dependence structure. We identify the sum of two such dependent compound Poisson processes as a compound Poisson process with new Poisson intensity and claim size distribution. This allows us to estimate the ruin probability for certain Pareto and exponential models.

2 Preliminaries

2.1 Subordinator copulae

A *distributional copula* in $\mathbb{R}^d$ is a d-dimensional distribution function (d.f.) with uniform marginals. For an arbitrary d-dimensional d.f. Sklar's theorem provides the theoretical basis for modelling.

Theorem 2.1. [Sklar's Theorem] *Let F be a d.f. on $\mathbb{R}^d$ with marginals $F_1, \dots, F_d$. Then there exists a copula $C : [0,1]^d \to [0,1]$ such that for all $x_1, \dots, x_d \in \overline{\mathbb{R}} = [-\infty, \infty]$*

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \tag{2.1}$$

If the marginals are continuous, then C is unique. Otherwise it is unique on $\mathrm{Ran}F_1 \times \dots \times \mathrm{Ran}F_d$. *Conversely, if C is a copula and $F_1, \dots, F_d$ are d.f.s, then* (2.1) *defines a joint d.f. with marginals $F_1, \dots, F_d$.*

Lévy measures are in general unbounded on $\mathbb{R}^d$ and may have a non-integrable singularity at 0. This causes problems for the copula idea. Insurance risk processes, however, are usually modelled by spectrally positive Lévy processes having only positive jumps. This corresponds to a Lévy measure concentrated on the positive quadrant $[0, \infty)^d$. Subordinators are prominent examples of such processes as they have only increasing sample paths, implying also that they are a.s. of bounded variation on every compact time interval. For instance, compound Poisson processes are subordinators with finite Lévy measures.

The Lévy-Khintchine formula for subordinators simplifies to

$$E(\exp\{i\,(z, X_t)\}) = \exp\{t \int_{(0,\infty)^d} (e^{i\,(z,x)} - 1)\Pi(dx)\}, \quad z \in \mathbb{R}^d.$$

For subordinators Lévy measures play essentially the same role as probability measures for random variables. Still the problem with the singularity at 0 prevails. To circumvent this problem we follow Tankov [9] and define a copula for the tail integral.

Definition 2.2. [Tail integral] *Let X be a spectrally positive Lévy process in $\mathbb{R}^d$ with Lévy measure Π. Its* tail integral *is the function $\overline{\Pi} : [0,\infty]^d \to [0,\infty]$ satisfying for $x = (x_1, \dots, x_d)$,*

(1) $$\overline{\Pi}(x) = \begin{cases} \Pi([x_1, \infty) \times \dots \times [x_d, \infty)) & \text{for } x \in [0,\infty)^d \setminus \{0\}, \\ \infty & \text{for } x = 0; \end{cases}$$

(2) $\overline{\Pi}$ is equal to 0, if one of its arguments is ∞;

(3) $\overline{\Pi}(0,\ldots,x_i,0,\ldots,0) = \overline{\Pi}_i(x_i)$ *for* $(x_1,\ldots,x_d) \in \mathbb{R}_+^d$*, where* $\overline{\Pi}_i(x_i) = \Pi([x_i,\infty))$ *is the tail integral of component* i*.*

Definition 2.3. [Subordinator copula/S-copula] *A d-dimensional S-copula is a measure defining function* $S : [0,\infty]^d \to [0,\infty]$ *with marginals, which are the identity functions on* $[0,\infty]$*.*

Proposition 2.4. *Let* $S : [0,\infty]^d \to [0,\infty]$ *be a S-copula with domain* $D_1 \times \cdots \times D_d$ *and denote* $\ell_i = \min D_i$*,* $u_i = \max D_i$ *for* $i = 1,\ldots,d$*. Then*

(1) $S(y_1,\ldots,y_d)$ *is increasing in each component.*

(2) $S(y_1,\ldots,y_{i-1},\ell_i,y_{i+1}\ldots,y_d) = 0$ *for all* $i \in \{1,\ldots,d\}$*,* $y_i \in D_i$*.*

(3) $S(u_1,\ldots,u_{i-1},y_i,u_{i+1}\ldots,u_d) = y_i$ *for all* $i \in \{1,\ldots,d\}$*,* $y_i \in D_i$*.*

(4) For all $(a_1,\ldots,a_d),(b_1,\ldots,b_d) \in D_1 \times \cdots \times D_d$ *with* $a_i \le b_i$ *we have*

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} S(y_{1i_1},\ldots,y_{di_d}) \ge 0$$

where $y_{j1} = a_j$ *and* $y_{j2} = b_j$ *for all* $j \in \{1,\ldots,d\}$*.*

The following is Sklar's theorem for S-copulae. For a proof we refer to Tankov [9].

Theorem 2.5. [Tankov [9], Theorem 3.1] *Let* $\overline{\Pi}$ *denote the tail integral of a d-dimensional subordinator whose components have Lévy measures* $\Pi_1,\ldots,\Pi_d$*. Then there exists a S-copula* $S : [0,\infty]^d \to [0,\infty]$ *such that for all* $x_1,\ldots,x_d \in [0,\infty]$

$$\overline{\Pi}(x_1,\ldots,x_d) = S(\overline{\Pi}_1(x_1),\ldots,\overline{\Pi}_d(x_d)). \tag{2.2}$$

If the marginal tail integrals $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$ *are continuous then this S-Copula is unique. Otherwise, it is unique on* $\mathrm{Ran}\overline{\Pi}_1 \times \cdots \times \mathrm{Ran}\overline{\Pi}_d$*.*
Conversely, if S *is a S-Copula and* $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$ *are marginal tail integrals of subordinators, then* (2.2) *defines the tail integral of a d-dimensional subordinator and* $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$ *are tail integrals of its components.*

Example 2.6. [Complete dependence S-copula]
Let $X = (X_1,\ldots,X_d)$ be a subordinator with equal components. Its Lévy measure is given by $\nu(x_1,\ldots,x_d) = \nu_1(x_1)I_{\{x_1=x_2=\cdots=x_d\}}$. For the tail integral this means for $x = (x_1,\ldots,x_d)$

$$\overline{\Pi}(x) = \int_{\max(x_1,\ldots,x_d)}^{\infty} \Pi_1(du) = \min(\overline{\Pi}_1(x_1),\ldots,\overline{\Pi}_d(x_d)).$$

This implies the S-copula

$$S_{\parallel}(x_1,\ldots,x_d) = \min(x_1,\ldots,x_d). \qquad \square$$

Example 2.7. [Independence S-copula]
Let $X = (X_1, \ldots, X_d)$ be a subordinator with characteristic triplet $(c, 0, \nu)$. Its componenents are independent if and only if

$$\nu(A) = \sum_{i=1}^{d} \nu_i(A_i), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $A_i = \{x \in \mathbb{R} : (0, \ldots, 0, x, 0, \ldots, 0) \in A\}$, where x stands at the ith component. For the tail integral this means for $x = (x_1, \ldots, x_d)$

$$\overline{\Pi}(x) = \overline{\Pi}_1(x_1) I_{\{x_2 = \ldots = x_d = 0\}} + \cdots + \overline{\Pi}_d(x_d) I_{\{x_1 = \ldots = x_{d-1} = 0\}}.$$

This implies the S-copula

$$S_\perp(x_1, \ldots, x_d) = x_1 I_{\{x_2 = \ldots = x_d = \infty\}} + \cdots + x_d I_{\{x_1 = \ldots = x_{d-1} = \infty\}}. \qquad \square$$

Example 2.8. [Archimedian S-copula]
Let $\phi : [0, \infty] \mapsto [0, \infty]$ be a strictly decreasing function with $\phi(0) = \infty$ and $\phi(\infty) = 0$. Assume furthermore that ϕ^{-1} is completely monotone. Then the following is an S-copula

$$S(x_1, \ldots, x_d) = \phi^{-1}(\phi(x_1) + \cdots + \phi(x_d)). \qquad \square$$

Example 2.9. [Clayton S-copula]
The Archimedian S-copula with $\phi(t) = t^{-\theta}$ for $\theta \in (0, \infty)$ yields

$$S_\theta(x_1, \ldots, x_d) = (x_1^{-\theta} + \cdots + x_d^{-\theta})^{-1/\theta}.$$

The family $(S_\theta, \theta > 0)$ is called the *Clayton family of S-copulae*. This family includes as limits for $\theta \to \infty$ the complete dependence and for $\theta \to 0$ the independence S-copulae:

$$\begin{aligned} \lim_{\theta \to \infty} S_\theta(x_1, \ldots, x_d) &= S_{||}(x_1, \ldots, x_d). \\ \lim_{\theta \to 0} S_\theta(x_1, \ldots, x_d) &= S_\perp(x_1, \ldots, x_d). \end{aligned}$$

This means that the parameter θ allows us to adjust the dependence of d subordinators from complete dependence to independence. These copulae correspond to subordinators with time constant S-copulae. As shown in Proposition 4.4 of Barndorff-Nielsen and Lindner [1], S_θ must have mass on the axes. $\square$

2.2 Ruin theory

The tail integral is the most interesting object, when studying ruin probabilities. We shall investigate two regimes: a heavy-tailed regime, represented by subexponential claim sizes, and a light-tailed regime, represented by exponential claim sizes. Although ruin theory has been developed in the general class of spectrally positive Lévy processes (see Klüppelberg, Kyprianou and Maller [8] for the subexponential theory and Bertoin and Doney [4] for the light-tailed theory), we shall restrict ourselves in this paper to compound Poisson models. The reason for this is that the Lévy measure of a compound Poisson process is finite, which allows for an explicit representation of the sum of dependent processes. Moreover, the tail integral simplifies considerable. It also allows for an immediate comparison of the dependent model with the extreme models given by the sum of independent and complete dependent processes. We start with some definitions.

Definition 2.10. *(a) A Lebesgue-measurable function* $h : [0,\infty) \to (0,\infty)$ *is* regularly varying with index $\gamma \in \mathbb{R}$ $(h \in \mathcal{R}_\gamma)$, *if*

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^\gamma, \quad t > 0.$$

If $\gamma = 0$ *then* h *is called* slowly varying.

(b) Let F *be the d.f. of a positive rv whose tail* $\overline{F}(x) = 1 - F(x) > 0$ *for all* $x > 0$. *Denote by* $F^{2*} = F * F$ *the convolution of* F *and by* $\overline{F^{2*}} = 1 - F^{2*}$ *its tail.* F *or* Y *is called* subexponential, *if*

$$\overline{F^{2*}}(x) \sim 2\overline{F}(x)\,, \quad x \to \infty\,,$$

where $\sim$ *means that the quotient of lhs and rhs tends to 1 as* $x \to \infty$.

All d.f.s with regularly varying tail are subexponential, but the class is much richer.

The following Theorem is due to Embrechts and Veraverbeke [6]; see e.g. Embrechts, Klüppelberg and Mikosch [5], Theorem 1.3.6.

Theorem 2.11. *Let* C *be a compound Poisson process with rate* $\lambda > 0$, *i.e.*

$$R_t = x + ct - C_t = x + ct - \sum_{i=1}^{N_t} Y_i\,, \quad t \geq 0\,,$$

where the tail integral of the Lévy measure is $\overline{\Pi}(z) = \lambda\overline{F}(z)$ *for* $z \geq 0$. *If* F *is subexponential, then under the net profit condition* $c - \lambda EY > 0$ *we obtain*

$$\Psi(x) \sim \frac{\lambda}{c - \lambda EY} \int_x^\infty \overline{F}(y)dy, \quad x \to \infty.$$

Remark 2.12. By Karamata's theorem (see Bingham, Goldie and Teugels [3], p. 28), if $\overline{F} \in \mathcal{R}_{-b}$ and $b > 1$, then

$$\Psi(x) \sim \frac{\lambda}{c - \lambda EY}\frac{x}{b-1}\overline{F}(x), \quad x \to \infty. \qquad \square$$

The following classical Cramér-Lundberg theorem can be found e.g. in Embrechts et al. [5], Theorem 1.2.2.

Theorem 2.13. *Let C be a compound Poisson process with rate $\lambda > 0$, i.e.*

$$R_t = x + ct - C_t = x + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0.$$

Assume that the net profit condition $\lambda EY - c > 0$ holds and that there exists a constant $\kappa > 0$ such that

$$\widehat{f}_I(\kappa) = \int_0^\infty e^{\kappa z} \overline{F}(z) dz = \frac{c}{\lambda}. \tag{2.3}$$

Then

$$\Psi(x) \sim K e^{-\kappa x}, \quad x \to \infty,$$

for some $K \in [0, \infty)$ given by

$$K = \left(\frac{\kappa \lambda}{c - \lambda EY} \int_0^\infty z e^{\kappa z} \overline{F}(z) dz \right)^{-1},$$

where the right hand side is taken as 0, if the integral is infinite.

2.3 The tail integral

Theorem 2.5 allows for a representation of the Lévy measure of a sum of dependent compound Poisson processes by means of the S-copula. Our main object to study will be the following.

Definition 2.14. *Let $C = (C^1, C^2)$ denote a bivariate subordinator and define $C^+ := C^1 + C^2$. Then C^+ has tail integral*

$$\overline{\Pi}^+(z) = \Pi^+([z, \infty)) = \Pi(\{(x, y) \in [0, \infty)^2 : x + y \geq z\}, \quad z \geq 0.$$

As is common in one-dimensional Lévy process theory we identify $\overline{\Pi}^+(0) = \overline{\Pi}^+(0+)$, though this is in contrast to Definition 2.2(1).

Before we start with Clayton dependent risk processes, we explain the situation for two independent and completely dependent processes respectively.

Example 2.15. Let C^1, C^2 be compound Poisson processes and define $C^+ := C^1 + C^2$. Then the risk process is given as in (1.1) by

$$R_t^+ = x + ct - C_t^+ = x + ct - (C_t^1 + C_t^2), \quad t \geq 0.$$

We shall always assume the net profit condition to hold for two independent total claim amount processes C^1, C^2, i.e. $c > EC_1^+ = \lambda_1 EY_1 + \lambda_2 EY_2$, where C^i is characterized by the Poisson rate λ_i and the claim size Y_i for $i = 1, 2$. As the mean of C^+ is not affected by the dependence, the net profit condition holds for any dependent model if and only if it holds for the independent processes.

(i) Assume that C^1, C^2 are independent with the same claim size distribution, i.e. $Y_1 \stackrel{d}{=} Y_2$ with d.f. F. Then C^+ is compound Poisson with rate $\lambda_1 + \lambda_2$ and claim size d.f. F. This implies that $\overline{\Pi}^+(z) = (\lambda_1 + \lambda_2)\overline{F}(z)$, $z \geq 0$.

(ii) Assume that C^1, C^2 are completely dependent, i.e. $C^+ = 2C^1$. Then C^+ is compound Poisson with rate λ and claim size distribution $2Y$. This implies that $\overline{\Pi}^+(z) = \lambda\overline{F}(z/2)$, $z \geq 0$. □

We formulate our next result for the general class of subordinators.

Proposition 2.16. [Sum of two subordinators] *Let $C = (C^1, C^2)$ denote a bivariate subordinator and define $C^+ := C^1 + C^2$. If the tail integrals $\overline{\Pi}_i$ for $i = 1, 2$ are absolutely continuous on $(0, \infty)$ and the dependence between C^1, C^2 is given by a two times continuously differentiable S-copula S, then the tail integral of the Lévy measure of C^+ can be calculated by*

$$\begin{aligned}\overline{\Pi}^+(z) &= \int_{(0,\infty)} \frac{\partial S(u,v)}{\partial u}\Big|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2((z-x)\vee 0)} \Pi_1(dx) \\ &\quad + \Pi(\{0\} \times [z,\infty)) + \Pi([z,\infty) \times \{0\}), \quad z > 0. \end{aligned} \tag{2.4}$$

Proof. Let Π denote the Lévy measure of C. As indicated in the introduction C^+ is a Lévy process. The Lévy-Khintchine formula gives for any Borel set $B \subseteq [0, \infty]^2$

$$\Pi^+(B) = \Pi(\{(x,y) \in [0,\infty]^2 : x + y \in B\}).$$

From Theorem 2.5 we know

$$\overline{\Pi}(x,y) = S(\overline{\Pi}_1(x), \overline{\Pi}_2(y)) \quad \text{for} \quad (x,y) \in [0,\infty]^2.$$

From this follows that

$$\begin{aligned}\overline{\Pi}^+(z) &= \Pi^+([z,\infty)) \\ &= \Pi(\{(x,y) \in (0,\infty)^2 : x + y \geq z\}) + \Pi(\{0\} \times [z,\infty)) + \Pi([z,\infty) \times \{0\}) \\ &=: \Pi^+_{ac}(z) + \Pi(\{0\} \times [z,\infty)) + \Pi([z,\infty) \times \{0\}). \end{aligned} \tag{2.5}$$

Since S is a two times continuously differentiable positive function on $(0,\infty)^2$ and the tail integrals $\overline{\Pi}_1, \overline{\Pi}_2$ are absolutely continuous on $(0,\infty)$, there exists a bivariate Lévy density $\Pi(dx,dy)$ on $(0,\infty)^2$ given by

$$\Pi(dx,dy) = \frac{\partial^2 S(u,v)}{\partial u \partial v}|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2(y)} \Pi_1(dx)\Pi_2(dy), \quad x,y \in (0,\infty).$$

For the absolutely continuous part of (2.5) we calculate

$$\begin{aligned}
\Pi_{ac}^+(z) &= \int_{(0,\infty)} \int_{(0,\infty)} I_{\{x+y\geq z\}} \Pi(dx,dy) \\
&= \int_{(0,\infty)} \int_{((z-x)\vee 0,\infty)} \Pi(dx,dy) \\
&= \int_{(0,\infty)} \int_{((z-x)\vee 0,\infty)} \frac{\partial^2 S(u,v)}{\partial u \partial v}|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2(y)} \Pi_2(dy)\Pi_1(dx) \\
&= \int_{(0,\infty)} \int_{(0,\overline{\Pi}_2((z-x)\vee 0))} \frac{\partial^2 S(u,v)}{\partial u \partial v}|_{u=\overline{\Pi}_1(x)} dv \Pi_1(dx) \\
&= \int_{(0,\infty)} \left[\frac{\partial S(u,v)}{\partial u}|_{u=\overline{\Pi}_1(x)} \right]_0^{\overline{\Pi}_2((z-x)\vee 0)} \Pi_1(dx) \\
&= \int_{(0,\infty)} \left[\frac{\partial S(u,v)}{\partial u}|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2((z-x)\vee 0)} - \frac{\partial S(u,v)}{\partial u}|_{u=\overline{\Pi}_1(x), v=0} \right] \Pi_1(dx). \\
&= \int_{(0,\infty)} \frac{\partial S(u,v)}{\partial u}|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2((z-x)\vee 0)} \Pi_1(dx),
\end{aligned}$$

since $S(u,0)=0$ for all $u \geq 0$. □

Example 2.17. [Continuation of Example 2.15]
(i) For the independent case the integral in equation (2.4) vanishes and

$$\Pi(\{0\}\times[z,\infty)) + \Pi([z,\infty)\times\{0\}) = \overline{\Pi}_1(z) + \overline{\Pi}_2(z), \quad z \geq 0,$$

as all mass is concentrated on the axes.

(ii) In contrast to that, for the complete dependence case the above sum vanishes and all the mass is concentrated on the 45 degree line in $(0,\infty)^2$.

(iii) In all other cases, for a homogeneous S-copula, there is mass on the axes as well as in $(0,\infty)^2$; see Proposition 4.4. of Barndorff-Nielsen and Lindner [1]. □

3 The Clayton risk process

Let C^1, C^2 denote *compound Poisson processes* with rates $\lambda_1, \lambda_2 > 0$ and claim size d.f.s F_1, F_2. Let the dependence between C^1, C^2 be given by the Clayton S-copula S_θ with parameter $\theta > 0$ as in Example 2.9.

Proposition 3.1. *Consider the situation above.*

(a) The process C^+ is a compound Poisson process with tail integral given by

$$\overline{\Pi}^+(z) = I_1(z) + I_2(z) + I_3(z), \tag{3.1}$$

where for $z > 0$

$$\begin{aligned} I_1(z) &= \lambda_1 \lambda_2^{\theta+1} \int_{(0,z)} \Big(\frac{\overline{F}_2^\theta(z-x)}{(\lambda_1^\theta \overline{F}_1^\theta(x) + \lambda_2^\theta \overline{F}_2^\theta(z-x))} \Big)^{\frac{\theta+1}{\theta}} F_1(dx), \\ I_2(z) &= \lambda_1 \lambda_2 \overline{F}_1(z) \left(\lambda_1^\theta \overline{F}_1^\theta(z) + \lambda_2^\theta \right)^{-1/\theta}, \\ I_3(z) &= \lambda_1 \overline{F}_1(z) + \lambda_2 \overline{F}_2(z) - (\lambda_1^{-\theta} \overline{F}_1^{-\theta}(z) + \lambda_2^{-\theta})^{-1/\theta} - (\lambda_1^{-\theta} + \lambda_2^{-\theta} \overline{F}_2^{-\theta}(z))^{-1/\theta}. \end{aligned}$$

Moreover,

$$I_2(z) \ \sim \ \lambda_1 \overline{F}_1(z), \quad z \to \infty.$$

(b) Assume now that $F = F_1 = F_2$ and $\lambda = \lambda_1 = \lambda_2$. Then

$$\overline{\Pi}^+(z) = \lambda(I_1(z) + I_2(z) + 2I_3(z)),$$

where

$$I_1(z) = \int_{(0,z)} \left(\frac{\overline{F}^\theta(z-x)}{\overline{F}^\theta(z-x) + \overline{F}^\theta(x)} \right)^{\frac{\theta+1}{\theta}} F(dx), \tag{3.2}$$

$$I_2(z) = \overline{F}(z) \left(\overline{F}^\theta(z) + 1 \right)^{-1/\theta} \sim \overline{F}(z), \quad z \to \infty, \tag{3.3}$$

$$I_3(z) = \overline{F}(z)(1 - (1 + \overline{F}^\theta(z))^{-1/\theta}) \ = \ o(\overline{F}(z)), \quad z \to \infty. \tag{3.4}$$

Proof. (a) By (2.4) we get for the tail integral of the Lévy measure Π^+ of the process $C^+ = C^1 + C^2$ for $z > 0$

$$\begin{aligned} \overline{\Pi}^+_{ac}(z) &= \int_{(0,z)} \lambda_1^{-(\theta+1)} \overline{F}_1^{-(\theta+1)}(x) (\lambda_1^{-\theta} \overline{F}_1^{-\theta}(x) + \lambda_2^{-\theta} \overline{F}_2^{-\theta}(z-x))^{-\frac{\theta+1}{\theta}} \lambda_1 F_1(dx) \\ &\quad + \int_z^\infty \lambda_1^{-(\theta+1)} \overline{F}_1^{-(\theta+1)}(x) (\lambda_1^{-\theta} \overline{F}_1^{-\theta}(x) + \lambda_2^{-\theta})^{-\frac{\theta+1}{\theta}} \lambda_1 F_1(dx) \\ &= \lambda_1 \lambda_2^{\theta+1} \int_{(0,z)} \overline{F}_2^{\theta+1}(z-x) (\lambda_1^\theta \overline{F}_1^\theta(x) + \lambda_2^\theta \overline{F}_2^\theta(z-x))^{-\frac{\theta+1}{\theta}} F_1(dx) \\ &\quad + \lambda_1 \lambda_2^{\theta+1} \int_z^\infty (\lambda_1^\theta \overline{F}_1^\theta(x) + \lambda_2^\theta)^{-\frac{\theta+1}{\theta}} F_1(dx) \\ &=: I_1(z) + I_2(z). \end{aligned}$$

To calculate $I_3(z)$ we use Definition 2.2(3) and Theorem 2.5.

$$\begin{aligned}\Pi(\{0\}\times[z,\infty)) &= \lim_{x\downarrow 0}\Pi(([0,\infty)\times[z,\infty))\setminus([x,\infty)\times[z,\infty)))\\ &= \overline{\Pi}(0,z)-\lim_{x\downarrow 0}\Pi([x,\infty)\times[z,\infty))\\ &= \overline{\Pi}_2(z)-\lim_{x\downarrow 0}S(\overline{\Pi}_1(x),\overline{\Pi}_2(z)).\end{aligned}$$

As a compound Poisson process has finite Lévy measure and S is for $\theta\in(0,\infty)$ continuous in $[0,\infty)^2$, we obtain

$$\Pi(\{0\}\times[z,\infty))=\overline{\Pi}_2(z)-S(\overline{\Pi}_1(0),\overline{\Pi}_2(z))=\lambda_2\overline{F}_2(z)-(\lambda_1^{-\theta}+\lambda_2^{-\theta}\overline{F}_2^{-\theta}(z))^{-1/\theta},$$

and similarly for $\Pi([z,\infty)\times\{0\})$. This gives the expression for $I_3(\cdot)$.

Setting $z=0$ we obtain $I_1(0)=0$ and

$$I_3(0) \;=\; \lambda_1\big(1-(1+((\lambda_1/\lambda_2))^\theta)^{-1/\theta}\big)+\lambda_2\big(1-(1+(\lambda_2/\lambda_1)^\theta)^{-1/\theta}\big)\in[0,\lambda_1+\lambda_2].$$

This implies

$$\begin{aligned}\overline{\Pi}^+(0) &\le I_3(0)+\lambda_1\lambda_2^{\theta+1}(\lambda_1^\theta+\lambda_2^\theta)^{-\frac{\theta+1}{\theta}}\\ &= \lambda_1+\lambda_2-2\lambda_1\lambda_2(\lambda_1^\theta+\lambda_2^\theta)^{-1/\theta}+\lambda_1\lambda_2^{\theta+1}(\lambda_1^\theta+\lambda_2^\theta)^{-\frac{\theta+1}{\theta}}\\ &= \lambda_1+\lambda_2-\lambda_1\lambda_2(\lambda_1^\theta+\lambda_2^\theta)^{-1/\theta}\left(2-\frac{\lambda_2^\theta}{\lambda_1^\theta+\lambda_2^\theta}\right)\\ &\le \lambda_1+\lambda_2\;<\;\infty.\end{aligned}$$

Finiteness of the Lévy measure of C^+ implies that C^+ is a compound Poisson process; see e.g. Bertoin [2], Chapter 1, Proposition 2 and Corollary 3.

To estimate $I_2(z)$ note that $\lambda_2^\theta(\lambda_1^\theta v^\theta+\lambda_2^\theta)^{-\frac{\theta+1}{\theta}}$ has antiderivative $v(\lambda_1^\theta v^\theta+\lambda_2^\theta)^{-1/\theta}$ giving

$$I_2(z)=\lambda_1\lambda_2\left(\lambda_1^\theta\overline{F}_1^\theta(z)+\lambda_2^\theta\right)^{-1/\theta}\overline{F}_1(z);$$

hence

$$I_2(z)\sim\lambda_1\overline{F}_1(z),\quad z\to\infty\,.$$

(b) Now (3.1) reduces to

$$\begin{aligned}\overline{\Pi}^+(z) &= \lambda\int_0^z\left(\frac{\overline{F}^\theta(z-x)}{\overline{F}^\theta(z-x)+\overline{F}^\theta(x)}\right)^{\frac{\theta+1}{\theta}}F(dx)+\lambda\overline{F}(z)(\overline{F}^\theta(z)+1)^{-1/\theta}\\ &\quad +2\lambda\overline{F}(z)-2\lambda(\overline{F}^{-\theta}(z)+1)^{-1/\theta}\\ &=: \lambda(I_1(z)+I_2(z)+2I_3(z)).\end{aligned}$$

The limit relation (3.3) follow immediately from (a) and (3.4) is obvious. □

Remark 3.2. As shown in Proposition 3.1(a), the asymptotic behaviour of $I_3(z)$ for $z \to \infty$ depends on the tail behaviour of F_1 and F_2. As in the proof of Proposition 3.1(b), note that for $z \to \infty$

$$\begin{aligned} A_1(z) &:= (\lambda_1^{-\theta}\overline{F}_1^{-\theta}(z) + \lambda_2^{-\theta})^{-1/\theta} \sim \lambda_1 \overline{F}_1(z), \\ A_2(z) &:= (\lambda_1^{-\theta} + \lambda_2^{-\theta}\overline{F}_2^{-\theta}(z))^{-1/\theta} \sim \lambda_2 \overline{F}_2(z). \end{aligned}$$

Now assume that $\lim_{z\to\infty} \overline{F}_2(z)/\overline{F}_1(z) = c \in [0, \infty)$. (Obviously, the roles of F_1 and F_2 can be exchanged.) Then

$$A_1(z) + A_2(z) = \lambda_1 \overline{F}_1(z)\left(1 + \frac{\lambda_2}{\lambda_1}\frac{\overline{F}_2(z)}{\overline{F}_1(z)} + o(1)\right) = \lambda_1 \overline{F}_1(z)\left(1 + \frac{\lambda_2}{\lambda_1}c + o(1)\right).$$

This implies that $A_1(z)+A_2(z) \sim \lambda_1\overline{F}_1(z)$ if $c = 0$, and $A_1(z)+A_2(z) \sim (\lambda_1+c\lambda_2)\overline{F}_1(z)$ if $c > 0$. □

The proof of Proposition 3.1 suggests the following construction, which will be used later.

Proposition 3.3. *Consider the representation of $\overline{\Pi}^+(\cdot)$ as in Proposition 3.1(a). Define*

$$\widetilde{\lambda} = \lambda_1 + \lambda_2 - (\lambda_1^{-\theta} + \lambda_2^{-\theta})^{-1/\theta}. \tag{3.5}$$

Then $C^+ = C_1 + C_2$ can be identified with a compound Poisson process with rate $\widetilde{\lambda}$ and claim size distribution with tail $\overline{G}(z) = (1/\widetilde{\lambda})(I_1(z) + I_2(z) + I_3(z))$, $z \geq 0$, satisfying $\overline{G}(0) = 1$.

Proof. First note that $I_1(0) = 0$. Then, by change of variables, setting $v = \overline{F}_1(z)$, we obtain

$$I_2(0) = \lambda_1\lambda_2(\lambda_1^{\theta} + \lambda_2^{\theta})^{-1/\theta}.$$

Setting $\widetilde{\lambda} = I_2(0) + I_3(0)$ and $\overline{G}(z) := (1/\widetilde{\lambda})(I_1(z) + I_2(z) + I_3(z))$ for $z \geq 0$, this model defines a compound Poisson process with the same Lévy measure as C^+. □

Corollar 3.4. *If $\lambda_1 = \lambda_2 = \lambda$, then the expression for $\widetilde{\lambda}$ in* (3.5) *reduces to*

$$\widetilde{\lambda} = 2\lambda(1 - 2^{-\frac{\theta+1}{\theta}}).$$

3.1 Clayton-subexponential models

The next result concerns the heavy-tailed regime. We start with Pareto claim sizes.

Theorem 3.5. *Let C^1, C^2 be compound Poisson processes, both with rate $\lambda > 0$ and Pareto claim sizes with d.f.*

$$\overline{F}(x) = \left(\frac{a}{a+x}\right)^b, \quad x > 0,$$

for $a > 0$ and $b > 1$. Assume that the dependence between C^1, C^2 is given by the Clayton S-copula S_θ for $\theta \in (0,\infty)$ as in Example 2.9. Then

$$\overline{\Pi}^+(x) = \lambda(Kx^{-b} + \overline{F}(x) + o(\overline{F}(x))), \quad x \to \infty, \tag{3.6}$$

for some constant $K = a^b \tilde{K}(b,\theta) > 0$. In particular, $\overline{\Pi}^+(\cdot) \in \mathcal{R}_{-b}$.

Proof. From (3.2) we obtain for any $\theta \in (0,\infty)$ and $z > 0$

$$\begin{aligned} I_1(z) &= ba^b \int_0^z \frac{(a+x)^{b\theta-1}}{\left((a+z-x)^{b\theta} + (a+x)^{b\theta}\right)^{\frac{\theta+1}{\theta}}} dx \\ &= ba^b \int_a^{z+a} \frac{y^{b\theta-1}}{((2a+z-y)^{b\theta} + y^{b\theta})^{\frac{\theta+1}{\theta}}} dy \qquad (3.7) \\ &= b \int_1^{1+z/a} \frac{x^{b\theta-1}}{((2+z/a-x)^{b\theta} + x^{b\theta})^{\frac{\theta+1}{\theta}}} dx. \qquad (3.8) \end{aligned}$$

Define for $w > 2$ the function $L(w) := I_1((w-2)a)$; i.e.

$$L(w) := b \int_1^{w-1} \frac{x^{b\theta-1}}{((w-x)^{b\theta} + x^{b\theta})^{\frac{\theta+1}{\theta}}} dx.$$

For any $t > 0$ we obtain

$$\begin{aligned} L(tw) &= b \int_1^{tw-1} \frac{x^{b\theta-1}}{((tw-x)^{b\theta} + x^{b\theta})^{\frac{\theta+1}{\theta}}} dx \\ &= \frac{b}{t^b} \left\{ \int_{1/t}^1 + \int_1^{w-1} + \int_{w-1}^{w-1/t} \right\} \frac{x^{b\theta-1}}{((w-x)^{b\theta} + x^{b\theta})^{\frac{\theta+1}{\theta}}} dx \\ &=: L_1(w) + L_2(w) + L_3(w). \end{aligned}$$

We see immediately that

$$L_2(tw) = t^{-b} L_2(w).$$

This implies that $L_2(w) = \tilde{K}\, w^{-b}$ for all $w > 2$ and for some constant $\tilde{K} = \tilde{K}(b,\theta) > 0$.

To estimate L_1 and L_2 observe that by the mean value theorem we obtain for some $\xi \in [1/t, 1]$,

$$L_1(w) = \frac{b}{t^b} \frac{\xi^{b\theta-1}(1-\frac{1}{t})}{((w-\xi)^{b\theta}+\xi^{b\theta})^{\frac{\theta+1}{\theta}}} \sim \frac{b\xi^{b\theta-1}(1-\frac{1}{t})}{t^b w^{b(\theta+1)}}, \quad w \to \infty.$$

Moreover, for some $\xi(w) \in [w-1, w-1/t]$

$$L_3(w) = \frac{b(1-\frac{1}{t})\xi^{b\theta-1}}{t^b\left((w-\xi)^{b\theta}+\xi^{b\theta}\right)^{\frac{\theta+1}{\theta}}} \sim \frac{b(1-\frac{1}{t})}{t^b w^{b+1}}, \quad w \to \infty.$$

This implies that $L_1(w) = o(L_2(w))$ and $L_3(w) = o(L_2(w))$ as $w \to \infty$ hold.

Finally,

$$I_1(z) = L(2+z/a) = (2+z/a)^{-b}\tilde{K}(b,\theta) \sim \tilde{K}(b,\theta)a^b z^{-b}, \quad z \to \infty.$$

The result follows then from Proposition 3.1(b). □

Corollar 3.6. *Suppose that the conditions of Theorem 3.5 hold. Assume also that the net profit condition $c - 2\lambda EY > 0$ holds. Then*

$$\Psi(x) \sim \frac{\lambda}{c-2\lambda EY}\frac{a^b}{b-1}(\tilde{K}(b,\theta)+1)\, x^{-(b-1)}, \quad x \to \infty.$$

In particular, $\Psi \in \mathcal{R}_{-(b-1)}$.

Proof. From (3.6) we know that

$$\overline{\Pi}^+(z) \sim \lambda(\tilde{K}(b,\theta)+1)\overline{F}(z).$$

Then by Remark 2.12 we obtain the ruin estimate

$$\Psi(x) \sim \frac{\lambda}{c-2\lambda EY}\frac{x}{b-1}(\tilde{K}(b,\theta)+1)\overline{F}(x), \quad x \to \infty,$$

by Karamata's theorem. □

For some special parameters we can calculate K explicitly. We start with the independent and complete dependent models.

Example 3.7. [Pareto model, continuation of Example 2.17]
(i) In the independent case we get

$$\overline{\Pi}^+(z) = \overline{\Pi}_1(z) + \overline{\Pi}_2(z) = 2\lambda\overline{F}(z), \quad z \geq 0.$$

This implies for the ruin probability

$$\Psi_\perp(x) = \frac{2\lambda}{c-2\lambda EY}\int_x^\infty \overline{F}(z)dz \sim \frac{2\lambda}{c-2\lambda EY}\frac{x}{b-1}\overline{F}(x), \quad x\to\infty.$$

(ii) For the complete dependent model we have

$$\overline{\Pi}^+(z) = \overline{\Pi}_1(z/2) = \lambda\overline{F}(z/2), \quad z \ge 0.$$

This implies for the ruin probability as $x\to\infty$,

$$\begin{aligned}\Psi_\parallel(x) &= \frac{2\lambda}{c-2\lambda EY}\int_{x/2}^\infty \overline{F}(z)dz \sim \frac{2\lambda}{c-2\lambda EY}\frac{x/2}{b-1}\overline{F}(x/2)\\ &\sim \frac{2\lambda}{c-2\lambda EY}\frac{x}{b-1}2^{b-1}\overline{F}(x)\\ &\sim 2^{b-1}\Psi_\perp(x) > \Psi_\perp(x).\end{aligned}$$

□

Example 3.8. [Clayton-Pareto model, $b\theta = 1$, $b > 1$]
Suppose that the conditions of Theorem 3.5 hold. Assume also that the net profit condition $c - 2\lambda EY > 0$ holds. Take $b\theta = 1$. Then $(\theta+1)/\theta = b+1$ and by (3.7),

$$I_1(z) = \frac{ba^b z}{(2a+z)^{b+1}} \sim \frac{ba^b}{z^b} \sim b\overline{F}(z), \quad z\to\infty$$

Proposition 3.1(b) implies

$$\overline{\Pi}^+(z) \sim \lambda(b+1)\overline{F}(z), \quad z\to\infty.$$

Then since $b > 1$,

$$\Psi(x) \sim \frac{\lambda}{c-2\lambda EY}\frac{b+1}{b-1}x\overline{F}(x) \gtrsim \Psi_\perp(x), \quad x\to\infty,$$

where $\gtrsim$ means that the quotient of lhs and rhs remains bounded away from 0. Since $b+1 < 2^{b-1}$ for $b > b_0$ and $b+1 > 2^{b-1}$ for $b < b_0$ the ruin probability of the dependent model with Clayton S-copula $S_{1/b}$ can be smaller or larger than the completely dependent model. □

Example 3.9. [Clayton-Pareto model, $b = 2$, $\theta = 1$]
Consider the Pareto claim size distribution as in Theorem 3.5 and assume that $b = 2$ and $\theta = 1$. Then by (3.7), setting $u := a + z/2$

$$\begin{aligned}
I_1(z) &= 2a^2 \int_a^{a+z} \frac{y}{((2a+z-y)^2+y^2)^2} dy \\
&= \frac{a^2}{2} \int_a^{a+z} \frac{y}{((y-u)^2+u^2)^2} dy \\
&= \frac{a^2}{2} \left(\int_{-\frac{z}{2}}^{\frac{z}{2}} \frac{t}{(t^2+u^2)^2} dt + \int_{-\frac{z}{2}}^{\frac{z}{2}} \frac{u}{(t^2+u^2)^2} dt \right) \\
&= \frac{a^2}{2} \int_{-\frac{z}{2}}^{\frac{z}{2}} \frac{u}{(t^2+u^2)^2} dt \\
&= \frac{a^2}{2} \left(\frac{1}{u} \frac{2z}{z^2+4u^2} + \frac{1}{u^2} \arctan \frac{z}{2u} \right) \\
&= \frac{a^2}{2} \left(\frac{2}{2a+z} \frac{2z}{z^2+(2a+z)^2} + \frac{4}{(2a+z)^2} \arctan \frac{z}{2a+z} \right) \\
&\sim a^2 \left(\frac{z}{z^3} + \frac{2}{z^2} \frac{\pi}{4} \right) = \frac{a^2}{z^2}(1 + \frac{\pi}{2}).
\end{aligned}$$

Proposition 3.1(b) implies

$$\overline{\Pi}^+(z) \sim \lambda(2 + \frac{\pi}{2}) \frac{a^2}{z^2}, \quad z \to \infty.$$

Using again Karamata's theorem we obtain for the ruin probability

$$\Psi_1(x) \sim \frac{\lambda(2+\pi/2)}{c - 2\lambda EY} \frac{a^2}{x} \sim \left(1 + \frac{\pi}{4}\right) \Psi_\perp(x) > \Psi_\perp(x).$$

As $\Psi_\parallel(x) \sim 2\Psi_\perp(x)$ as $x \to \infty$ the ruin probability Ψ_1 is asymptotically smaller than $\Psi_\parallel$. □

We conclude this section with an example to show that there exist subexponential d.f.s such that the tail integral $\overline{\Pi}_+$ is heavier than the tail $\overline{F}$ itself. This means that we can certainly not hope to extend our results to the full subexponential class.

Example 3.10. [Clayton-Weibull model]
Let $\overline{F}(z) = \exp(-\sqrt{z})$, $z \geq 0$, and take the Clayton S-copula S_1 as in Example 2.9.

For the integral I_1 as given in (3.2) we calculate

$$\begin{aligned}
I_1(z) &= \int_0^z \left(\frac{\exp(-\sqrt{z-x})}{\exp(-\sqrt{z-x})+\exp(-\sqrt{x})}\right)^2 \frac{\exp(-\sqrt{x})}{2\sqrt{x}}\,dx \\
&= \int_0^z \frac{\exp(-\sqrt{x})}{(\exp(\sqrt{z-x}-\sqrt{x})+1)^2}\frac{1}{2\sqrt{x}}\,dx \\
&= \int_0^{\sqrt{z}} \frac{\exp(-x)}{(\exp(\sqrt{z-x^2}-x)+1)^2}\,dx \\
&= \int_0^{\sqrt{z/2}} \frac{\exp(-x)}{(\exp(\sqrt{z-x^2}-x)+1)^2}\,dx + \int_{\sqrt{z/2}}^{\sqrt{z}} \frac{\exp(-x)}{(\exp(\sqrt{z-x^2}-x)+1)^2}\,dx \\
&=: A(z)+B(z).
\end{aligned}$$

Next we calculate

$$\begin{aligned}
\frac{B(z)}{\overline{F}(z)} &\geq \int_{\sqrt{z/2}}^{\sqrt{z}} \frac{\exp(\sqrt{z}-x)}{(\exp(\sqrt{z/2}-x)+1)^2}\,dx \\
&= e^{\sqrt{z}} \int_{\sqrt{z/2}}^{\sqrt{z}} \frac{e^x}{(e^{\sqrt{z/2}}+e^x)^2}\,dx \\
&= e^{\sqrt{z}} \int_{e^{\sqrt{z/2}}}^{e^{\sqrt{z}}} \frac{dx}{(e^{\sqrt{z/2}}+x)^2} = -\left.\frac{e^{\sqrt{z}}}{e^{\sqrt{z/2}}+x}\right|_{e^{\sqrt{z/2}}}^{e^{\sqrt{z}}} \\
&= \frac{e^{\sqrt{z}}}{e^{\sqrt{z/2}}}(\frac{1}{2}+o(1)) \;\to\; \infty, \quad z\to\infty.
\end{aligned}$$

This implies that

$$\frac{\overline{\Pi}^+(z)}{\overline{F}(z)} \geq \frac{B(z)}{\overline{F}(z)} \;\to\; \infty, \quad z\to\infty. \qquad \square$$

3.2 Clayton-exponential models

Next we investigate the Cramér case. Here the role of $\overline{\Pi}^+$ is not as immediate as in the heavy-tailed case. We can, however, apply Cramér's theorem to C^+.

We concentrate on exponential claim sizes and some special cases of θ, where we can calculate I_2 and estimate I_1.

Example 3.11. [Exponential distribution, $\theta = 1$]
Let $\overline{F}(x) = e^{-ax}$, $x > 0$, for some $a > 0$. Then, setting $t := e^{a\theta z}$, we obtain

$$\begin{aligned} I_1(z) &= \int_0^z \left(\frac{e^{-a\theta(z-x)}}{e^{-a\theta(z-x)} + e^{-\theta ax}} \right)^{\frac{\theta+1}{\theta}} ae^{-ax}dx \\ &= \int_0^z \left(1 + te^{-2a\theta x}\right)^{-\frac{\theta+1}{\theta}} ae^{-ax}dx \\ &= \int_{e^{-az}}^1 \frac{dy}{(1+ty^{2\theta})^{\frac{\theta+1}{\theta}}}. \end{aligned}$$

If $\theta = 1$ we can calculate this integral with sufficient precision:

$$\begin{aligned} I_1(z) &= \int_{e^{-az}}^1 \frac{dy}{(1+ty^2)^2} = \frac{1}{\sqrt{t}} \int_{e^{-\frac{1}{2}az}}^{e^{\frac{1}{2}az}} \frac{dy}{(1+y^2)^2} \\ &= \frac{1}{2\sqrt{t}} \left(\frac{e^{\frac{1}{2}az}}{e^{az}+1} - \frac{e^{-\frac{1}{2}az}}{e^{-az}+1} + \arctan(e^{\frac{1}{2}az}) - \arctan(e^{-\frac{1}{2}az}) \right) \\ &= \frac{1}{2} \left(\frac{1}{e^{az}+1} - \frac{e^{-az}}{e^{-az}+1} + e^{-\frac{1}{2}az} \left(\arctan e^{\frac{1}{2}az} - \arctan e^{-\frac{1}{2}az} \right) \right) \\ &= \frac{1}{2} e^{-\frac{1}{2}az} \left(\arctan e^{\frac{1}{2}az} - \arctan e^{-\frac{1}{2}az} \right) \\ &\le \frac{\pi}{4} e^{-\frac{1}{2}az}, \quad z \ge 0, \end{aligned}$$

and

$$I_1(z) \sim \frac{\pi}{4} e^{-\frac{1}{2}az}, \quad z \to \infty.$$

Moreover,

$$I_2(0) = \frac{1}{2} \quad \text{and} \quad I_2(z) \le e^{-az}, \quad z \ge 0. \tag{3.9}$$

Finally, $I_3(0) = 1/2$. This implies that C^+ is a compound Poisson process with $\widetilde{\lambda} = \overline{\Pi}^+(0) = \lambda(I_1(0) + I_2(0) + 2I_3(0)) = (3/2)\lambda$ and claim size distribution G with tail

$$\overline{G}(z) \le \frac{\pi}{6} e^{-\frac{1}{2}az} + 2e^{-az}, \quad z > 0. \tag{3.10}$$

□

Theorem 3.12. *Let R^+ be the risk process as in (1.1) with C^1, C^2 compound Poisson processes, both having rate $\lambda > 0$ and exponential claim sizes with $\overline{F}(x) = e^{-ax}$, $x \ge 0$, for $a > 0$. Assume that the dependence between C^1, C^2 is given by the Clayton S-copula S_1 as in Example 2.9. Assume also that the net profit condition $c - 2\lambda EY > 0$ holds. Then the independent model satisfies the Cramér-Lundberg condition (2.3); i.e. $\kappa_\perp = a - 2\lambda/c > 0$ satisfies*

$$\widehat{f_I}(\kappa_\perp) = \int_0^\infty e^{\kappa_\perp x} \overline{F}(x)dx = \frac{c}{2\lambda}.$$

Then the Clayton-exponential model allows for a Cramér-Lundberg ruin estimate

$$\Psi(x) \sim K_1 e^{-\kappa_1 x}, \quad x \to \infty, \tag{3.11}$$

for $K_1, \kappa_1 > 0$. *Moreover,* $0 < \kappa_1 < \kappa_\perp < a$.

Denoting by $\kappa_\parallel$ *the Lundberg coefficient of the completely dependent model, then* $\kappa_\parallel = a/2 - \lambda/c = \frac{1}{2}\kappa_\perp$ *and both,* $\kappa_\parallel$ *and* κ_1 *are less than* $\frac{a}{2}$. *The relation between* $\kappa_\parallel$ *and* κ_1 *depends on the choice of the parameter* a.

Proof. As calculated in Example 3.11 C^+ can be identified with a compound Poisson process with rate $\widetilde{\lambda} = (3/2)\lambda$ and claim size distribution G satisfying (3.10). Since the tail of G is exponentially decreasing at least with exponent $a/2$, the function

$$\widehat{g}_I(s) := \int_0^\infty e^{sx}\overline{G}(x)dx,$$

is finite at least for $s < a/2$. Recall that the net profit condition holds if and only if it holds for the independent processes giving

$$c - 2\lambda EY = c - 2\lambda/a > 0 \quad \Longleftrightarrow \quad a > 2\lambda/c. \tag{3.12}$$

This allows a comparison of the two models and we obtain first

$$2\widehat{f}_I(0) = \frac{3}{2}g_I(0)\,,$$

giving

$$\widehat{g}_I(0) = \int_0^\infty \overline{G}(y)dy = \frac{4}{3}EY = \frac{4}{3}\widehat{f}_I(0).$$

Since by (3.9) we have $I_1(z) \sim \frac{\pi}{4}e^{-\frac{1}{2}az}$ as $z \to \infty$, we must also have $\lim_{s\to a/2}\widehat{g}_I(s) = \infty$. Consequently, $\widehat{g}_I$ is a convex continuous function on $[0, \frac{a}{2})$ satisfying by the net profit condition

$$\widehat{g}_I(0) < \frac{2c}{3\lambda} \quad \text{and} \quad \lim_{s\to a/2}\widehat{g}_I(s) = \infty.$$

So there exists a positive solution κ_1 of the equation

$$\widehat{g}_I(s) = \frac{2c}{3\lambda},$$

i.e. C^+ satisfies the Cramér condition with Lundberg coefficient κ_1. This implies (3.11).

For a comparison of the different Lundberg coefficients recall that we can calculate it in the independent exponential model explicitly, getting $\kappa_\perp = a - 2\lambda/c$. As by the net profit condition (3.12) $\widehat{f}_I(a/2) = 2/a < c/\lambda$ this implies that $\kappa_\perp > a/2$ and hence $\kappa_\perp > \kappa_1$. The remainder of the proof is obvious. □

Example 3.13. [Exponential distribution, $\theta = 1/2$]
We have restricted ourselves above to the Clayton-exponential model for $\theta = 1$. There is one further case, which we can analyse in a similar way. If $\theta = \frac{1}{2}$, then

$$\begin{aligned} I_1(z) &= \int_{e^{-az}}^1 \frac{dy}{(1+ty)^3} = -\frac{1}{2t(1+ty)^2}\Big|_{e^{-az}}^1 \\ &= \frac{1}{2} e^{-\frac{1}{2}az}((1+e^{-az/2})^{-2} - (1+e^{az/2})^{-2}) \\ &=: \frac{1}{2} e^{-\frac{1}{2}az}(h_1(z) - h_2(z)) \le \frac{1}{2} e^{-\frac{1}{2}az}, \end{aligned}$$

since $h_1(0) = h_2(0) = 1/4$ and $h_1(z) \uparrow 1$ and $h_2(z) \downarrow 0$, hence $h_1(z) - h_2(z) \le 1$ for all $z \ge 0$. From this follows also that

$$I_1(z) \sim \frac{1}{2} e^{-\frac{1}{2}az}, \quad z \to \infty.$$

Since $I_2(z),\ I_3(z) \le e^{-az}$, $\overline{\Pi}^+$ is exponentially decreasing at least with exponent $a/2$. Now one can proceed as in Example 3.11. □

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