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Optimal Control of Energy Storages in a Mixed Power Market with Conventional Power Technologies and Renewable Energy Sources.

Diplomarbeit

von

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Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

Garching, den 25. Mai 2011

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Contents

List of Figures

List of Tables

List of Notation for Chapter 2

List of Notation for Chapter 3 and Chapter 4

List of Notation for Chapter 5

$\mathcal{U}^{\triangle}(t)$	set of all advisable control strategies in discrete time	41
$S^{\triangle}$	set of all $\mathbb{F}$ stopping times at discrete time grids	41
$S_v^{\triangle}$	set of all $S^{\triangle}$ stopping times greater than the stopping time v	41
$\psi_i^{\triangle}(t,r,d,l)$	reward function in regime i in discrete time	41
$\ell_{\rightarrow}(m,l,i)$	charging level at $(m+1)\triangle t$ in regime i and charging level l at $m\triangle t$	41
$\ell_{\leftarrow}(m+1,l,i)$	charging level at $m\triangle t$ in regime i and charging level l at $(m+1)\triangle t$	42
$\mathcal{B}$	set of functions	46
$N_{\mathcal{B}}$	number of functions	46
$B_j(x)$	function $j=1,\dots,N_{\mathcal{B}}$	46
$\Phi_m^{i,l}(r)$	mapping r onto $\mathbb{E}[J(m+1,R_{m+1},l,i)\mid R_m=r]$	46
$\hat{\Phi}_m^{i,l}(r)$	approximation of the mapping $\Phi_m^{i,l}(r)$	46
$\hat{v}(m,r,c_g,j)$	estimated reward until T	48
$\hat{J}(m,r,c_g,i)$	approximated maximal reward until T for starting at time $m\triangle t$	48
$\hat{\xi}(m,r,g,i)$	optimal regime at time $m\triangle t$	48
$\overline{\mathcal{B}}$	set of bivariate functions	52
$N_{\overline{\mathcal{B}}}$	number of bivariate functions	52
$\overline{B}_j(x,y)$	bivariate function $j=1,\dots,N_{\mathcal{B}}$	52
$\Phi_m^i(r,l)$	mapping r and $L_{m+1}=l$ onto $\mathbb{E}[J(m+1,R_{m+1},l,i)\mid R_m=r]$	52
$\overline{\Phi}_{t_1}^i(r,l)$	approximation of the bivariate mapping	52
$\overline{v}(m,r,c_g,j)$	estimated reward until T	53
$\overline{J}(m,r,l_m^n(i),i)$	approximated maximal reward until T	54
$\overline{\xi}(m,r,g,i)$	optimal regime at time $m\triangle t$	53
$'\vee'$	path wise maximum operator	42

Introduction

Electricity is one of the fundamental goods of modern civilization. In recent times many countries focused on nuclear energy to cover the growing energy appetite of their economy. But after the nuclear disaster following the earthquake on March 11 in Japan, the stance on nuclear energy has changed. Not only is Japan thinking about phasing-out nuclear energy and replacing it with renewable sources, but also Germany. The German "Sachverständigenrat für Umwelt" (a research organization specializing in environmental issues) has already pointed out the main challenges on Germany's way to 100% renewable feed-in, see Faulstich et al. [2011]. Beside a reconstruction of the German grid, a main challenge is the expansion of energy storage facilities. These facilities are necessary to ensure an uninterrupted power supply on a pure renewable energy market due to the intermittency of the energy production by those sources. The need to store energy in storage facilities raises the question of how to control such storages, taking into account the fluctuations of energy in the market. In addition to that it is reasonable to obtain a value for the investment in such a storage facility. We focus on the transition period between a pure conventional energy market and a pure renewable one, called a mixed energy market. In this market we are going to develop a response to both questions in this thesis.

The incorporation of renewable energy in the electricity market forces changes in this market, due to the differences between the renewable end conventional energy production. So are the marginal costs for renewable energy significant lower than the costs for conventional power technologies. Since the driving fuels such as sun and wind, that we focus on, are free of charge. The lower marginal costs imply that the producers of renewable energy offer their good already for a lower price than conventional energy producers. Thereby from a macro economic perspective the electricity price on the market is reduced in comparison to a pure conventional power market. This change in price is denoted as *merit order effect*. A summary of various papers concerning this effect is given in Morthost et al. [2010]. Moreover renewable energy production is not to an equal amount controllable as the conventional energy production. Environmental circumstances have a major influence on the amount renewable energy producing entities produce and can not be influenced by humans. Since energy needs to be consumed in a close timely relation to the production, this leads to the challenge to ensure an uninterrupted power supply. The common idea to respond to this challenge is to store the energy in times of overproduction and feed the power back to the gird at times of peak demands. Thus in a mixed power market energy storages gain importance. These changes caused by renewable energy sources are briefly summarized in Chapter 1.

In Chapter 2 we motivate the necessity of a large cumulated capacity of energy storages integrated in the grid. For this purpose we evaluate, based on historical time series, different development scenarios in the renewable energy sector and available storage capacities the minimal required backup from conventional plants to ensure an uninterrupted power supply, using a linear optimization approach. This means we minimize the required backup from conventional plants given the knowledge of all past and future demand for energy and renewable produced energy in the considered time interval by controlling the modes – charging, keeping the charging level constant and discharging – of the energy storages. In this setting we neglect the uncertainty inherited in the amount of energy contributed by renewable sources and demand on the electricity market.

To consider this uncertainty we develop a more elaborate problem in Chapter 3. This means we include the stochasticity of the renewable energy production and the demand by a dynamic stochastic model. In this framework we assume that an agent is controlling an energy storage focused on maximizing the expected reward. The agent is in this setting just able to control the respective modes – charging, keeping the charging level constant and discharging – of the storage, denoted by regimes. To specify this reward the generic features of an energy storage are pointed out and the market price for electricity is derived. The derivation of the market price is based on a *modified merit order curve*. This modified curve states, given the current energy contribution from renewable sources and the demand, the energy price on the market. The approach is based on the theory behind the merit order effect discussed in Chapter 1, where the merit order curve is used for the derivation. In this case the curve is a step function given by the marginal cost progression of conventional power technologies and the production capacities of the respective technologies. A problem including this way for the price derivation has not been considered in the literature before. The energy storage characteristics are mirrored by the change rates in the charging level depending on the regime, the maximal storage capacity and the minimal charging level. Respective costs are summarized in three categories, including management and operating costs, storage costs of one energy unit over one time period and switching costs between the regimes. Bearing these storage features in mind the agent requires the optimal dynamic strategy to maximize the reward by switching between the regimes.

Problems of finding an optimal strategy between multiple regimes, as derived in Chapter 3, are known as *multiple switching problems*. We show in Chapter 4 that a solution for the problem stated in Chapter 3 exists. The basic theory to show the existence is the Snell envelope theory. This theory is aimed at finding the optimal stopping time to maximize the reward represented by a stochastic reward process. The next step towards the multiple switching problems is the two regime switching problem. Pioneer papers in the area of the two regimes problem are Brennan and Schwartz [1985] and Dixit [1989]. The mathematical theory of the two regimes case was firstly studied by Brekke and Øksendal [1994] and has been extended by others including Hamadene and Jeanblanc [2007] using backwards stochastic differential equations (BSDE) as well as Ly Vath and Pham [2007] with viscosity solutions. The theory for the multiple switching case is based on the research on the two regimes case and has recently been published by Pham et al. [2009] and Djehiche et al. [2009]. To derive the existence of a solution we focus on the framework discussed in Djehiche et al. [2009] and extent this theory to cover also the additional charging level dependence in our setting. Especially, due to complexity included by the dependence on the charging level an analytical derivation of the solution is not viable, therefore we focus in the following on a numerical approach to obtain a solution.

In Chapter 5 we describe two algorithms constituted in Carmona and Ludkovski [2010], namely the Bivariate Least Squares Monte Carlo Method (BLSM) and the Mixed Interpolation Tsitsiklis-van Roy (MITvR), to compute a numerical solution. The key idea of both algorithms is a discretization of the switching times and an empirical regression to compute the conditional expectation. This idea was firstly pointed out by Longstaff and Schwartz [2001] to price an American option by an approximation with Bermuda options. The algorithm of Longstaff and Schwartz uses, besides transfer to discrete exercise times, the concept of backwards equations relaying on empirical regression as the base for the exercise decision. A quite similar approach was also proposed by Tsitsiklis and van Roy [2000]. The regression idea with an set of bases function has been extensively studied until now, among others Egloff [2005] extended the theory to general stochastic learning problems. An other application of this idea is the dimension reduction as used for example introduced in Hepperger [2010]. To employ the approach for our problem we show explicitly the steps to obtain the backward equation. Therefore the information needed for the switching decision is reduced to the conditional expectations of the future rewards in the different regimes. But by this formulation the nested dependence of the optimal dynamic strategy on the charging level and the charging level on the strategy is not considered. For the practical implementation this dependence needs to be incorporated. To include the dependence on the charging level of the optimal strategy we follow two different approaches. On the one hand, in the MiTvR scheme the optimal regime is evaluated at each switching time point and at each charging level grid point. On the other hand, in the BLSM scheme an expectation conditioned on the charging level and the renewable energy production is computed for the optimal regime decision.

Finally, in Chapter 6, we give a concrete storage example and show the influence of the different parameters on the maximal expected reward. For this example we focus on the German electricity market for the *merit order curve* and the energy demand. In this market environment we consider different storage characteristics, including the storage size, charging and discharging rates, and compute the maximal expected reward. The same computations are done for the costs occurring while running the storage, including switching costs between the regimes and costs to store one energy unit for one time period. Thereby the influences of the different parameters on the reward become transparent. Besides developments on the conventional power market can be examined, since the merit order curve is very flexible. So can consequences of recent developments in the marginal costs and capacities of the conventional technologies be easily mirrored by a modification of the merit order curve. For example the value of a storage can also be examined after a nuclear phase-out. Furthermore, we take a closer look at the resulting optimal strategy.

Chapter 1

Energy markets with renewable energy

The structure of energy markets changes with the contribution of energy produced by renewable sources. Furthermore, changes on the energy supply side are necessary to achieve way to 100% renewable energies. In this chapter the main changes required by renewable energies in the electricity market and the respective required additional technologies are introduced.

The technologies which are generally counted as renewable are biomass, wave power, tidal power, hydro power, solar power and wind power plants, but in this thesis we focus on wind and solar energy. One difference between these technologies and conventional ones is that the driving fuels, wind and sun, are free of charge, which implies low marginal costs. Thereby, from a macroeconomic perspective, the contribution of renewable energies reduces the price. This effect is known as merit order effect and is described in Section 1.1. But the dependence on those natural sources leads to intermittency effects in the course of the year and day as well as to short term fluctuations depending on environmental conditions, catalogued in Section 1.2. In Section 1.3 the currently available storage facilities are introduced, which can be used to ensure an uninterrupted energy service on a market with renewable sources.

1.1 Merit Order Effect

The merit order effect describes the price effect resulting from adding energy produced by renewable sources to the conventional power mix. To specify this effect in more detail we have to go back to the basic microeconomic model for price determination in a market, called the supply-and-demand model.

The supply-and-demand model assumes that buyers and sellers of a certain product interact on a market such that in the long run it reaches an economic equilibrium. This is the point, where the demand from the buyers' side for a certain price is equal to the supply by the sellers' side for the same price. This means that neither of the two sides has the incentive to offer higher prices or accept lower prices. To obtain the equilibrium the concepts of demand and supply curves are applied. The supply curve represents the

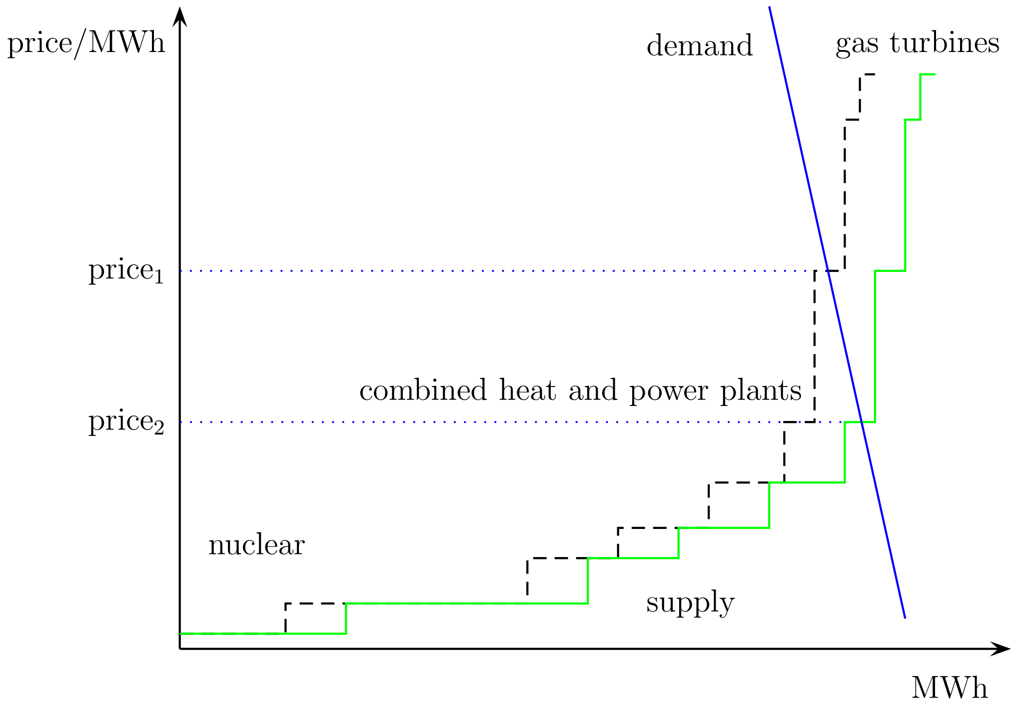

Figure 1.1: Resulting merit order curves with and without renewable energy in the power market for the supply-and-demand model. The dashed staircase represents the supply curve without contributions by the renewable energy sector and the green lined staircase represents the supply curve with contributions by the renewable energy sector. The blue line is the demand in MWh for the price per MWh in Euro. This results in the market clearing price, price$_1$, in the pure conventional power market and in the price$_2$ in the market with contribution by the renewable sector.

quantity producers' offer at a given price. This curve slopes upwards, since for a higher price it is advantageous to expand the production or to invest in plants to increase the supply. The demand curve represents the quantity buyers will purchase at a given price. This curve has the property that it slopes downwards due to the fact that consumers normally are willing to buy more at lower prices. In this setting the market-clearing equilibrium is the intersection of these two curves. The supply-and-demand model can be applied if the market is roughly competitive, see [Pindyck, 2009, p. 6].

Now coming back to the energy market, we need to find a supply and a demand curve. The merit order curve can be seen as the supply curve in power markets. It is an increasing step function with one step for each conventional power technology with equal marginal costs. It orders the marginal costs of the energy portfolio from the least to the most expensive. The step height represents the marginal costs and the length of the steps represents the amount of energy the respective power technology is able to produce. Since we focus on the marginal costs, it is worth noting that the differences in the prices are mainly driven by the costs of the fuel needed by the technology to produce electricity. The marginal costs represent the cost for one further quantity of the same good as already produced and hence disregard fixed costs. In this setting the technology order of conventional plants is nuclear as the cheapest, followed by combined heat and power plants (CHP) and ends with condensing plants Morthost et al. [2010].

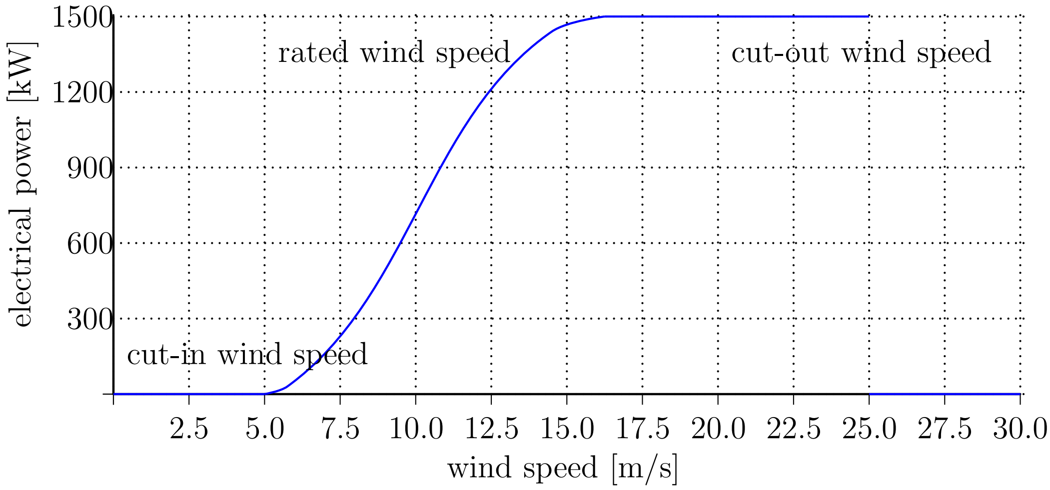

Figure 1.2: Example wind power curve.

For the demand curve in power markets, since electricity is one of the fundamental goods for modern civilization, which currently has no substitute, it is reasonable to assume that the demand curve is inelastic. This means that a change in the price of a good has almost no influence on the quantity demanded. Price inelasticity is normal for goods which are necessary and have almost no substitutes.

The dashed curve in Figure 1.1 shows the merit order curve without renewable energy contribution to the market. Thus, without a contribution of renewable energy to the electricity market, the equilibrium is achieved at the intersection of the supply curve, dashed, and the blue demand curve. Since natural resources like wind and sun are free of charge, renewable energy has almost no marginal costs. In the supply-and-demand model this means that, even for receiving almost no gains, the producers of renewable energy would still offer their good to the market. Hence, by adding the renewable energy stream to the power mix in our market, the merit order curve is shifted to the right. The respective shift is equal to the amount of renewable energy produced at this time. Consequently in the demand-and-supply model with renewable energy in the portfolio the price, price_2, is clearly below the price, price_1, in the purely conventional power mix. The difference between the equilibrium price with and without the renewable energy contribution is called merit-order-effect. In Figure 1.1 the merit order effect is price_1 - price_2.

1.2 Variability of renewable energy production

The fundamental difference between the renewable energy power production and the power production by conventional plants is that the amount produced can only be steered to certain degree. Most of the renewable energy sources exhibit seasonal and short term fluctuations, mainly driven by the weather, especially photovoltaic (PV) and wind we focus on in this thesis.

The energy production of a wind turbine depends mainly on the wind speed and the technical specifications of the turbine. For a given wind mill the dependence of the produced electricity and the wind speed can be read from the respective power curve, a function giving the power output for a given wind speed. Each mill needs a certain wind

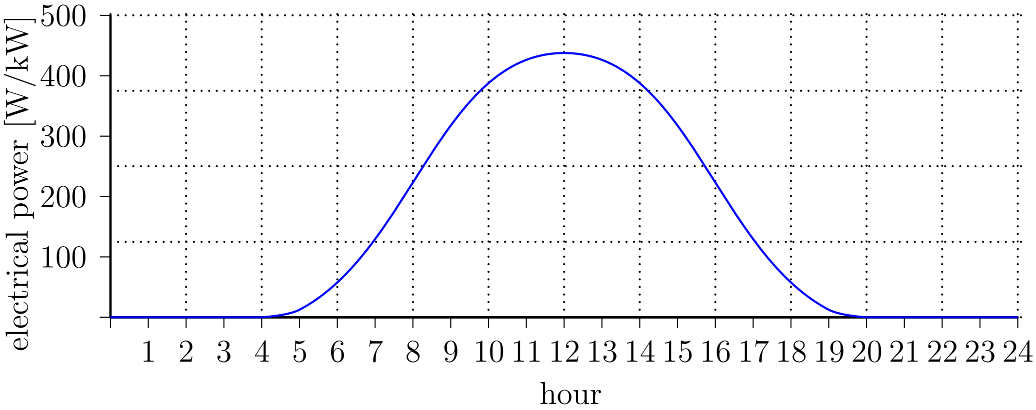

Figure 1.3: Average daily run of energy production by PV panels in summertime.

speed to start the energy production, called cut-in wind speed, and stops the production at a given wind speed, called cut-out wind speed, to prevent system damage. From the cut-in wind speed the produced energy rises by the power of three up to the maximal power production, the respective wind speed is called rated wind speed, and remains constant from this point on until the cut-out wind speed is reached.

For the electricity production with PV panels the produced amount depends on the radiation, the air temperature, the collector's angle of slope in relation to the sun and the PV module characteristics. With a rising radiation the current rises almost proportionally, but the voltages stay almost the same. A rise in air temperature reduces the voltage although it leaves the current unchanged. This implies that the power output of PV plants would be maximal in cool regions with long daylight hours. Apart from local weather driven fluctuations the energy production by PV panels follows the run of the sun. The average production of a PV plant in summer during the day is shown in Figure 1.3. To achieve additional efficiency so called solar trackers, which angel the panels in an optimal slope to the sun, can be applied. Thereby between 20 % and 50% more output is achieved.

1.3 Energy Storages

The storage of energy, in an electricity market with the contribution of renewable energy is essential due to the variability discussed in Section 1.2 and the fact that electricity needs to be consumed almost at the same time as it is produced. This section is mainly based on the Chapter 4.5.2. of the study by Faulstich et al. [2011].

The available storages can be characterized in four groups contingent on the storage form, potential, mechanic, chemical or electrochemical. Each technology has different advantages depending on the storage capacity, the storage performance, the efficiency, the economic life-time and the power density. Hence they have different favorable fields of application in concern to ensure an uninterrupted power supply.

Compressed air energy storage (CAES) is a mechanic energy storage. The mode of operation is to compress ambient air with electricity driven turbines in times of energy over production into aquifers or caverns. In times of high load the air is released through turbines to generate electricity. The compression generates as a by-product heat, which at the moment is not exploited in most of the plants. Subsequently, the compressed air cools down and therefore heat is required for decompression. This procedure reduces the efficiency to around 50%. In the future so called adiabatic CAES are projected, which store the heat and return it when the air is released to produce energy.

The group of chemical energy storages are based on hydrogen and methane production. These technologies are interesting since they can also be used in other sectors, such as transport or heating. The idea behind the hydrogen approach is to produce hydrogen by electrolysis in times of high supply of energy and low demand and store it. At peak demands electricity is produced by running gas fired plants with the stored hydrogen. For methane the idea is similar to the hydrogen storage, but has advantages in the high energy density of methane compared to hydrogen.

Electro chemical storages are based on a principle similar to the one of a classic battery. They are important as systemic reserve. Major exponents are lithium-ion, nickel cadmium and lead-acid batteries.

The main field of application of the pumped storage power stations is the intra day balancing of energy streams. The principle behind this power production is the potential energy between a higher water reservoir to a lower one. In periods of low demand and high production energy is consumed to pump the water from the lower storage basin into the higher one. During periods of high demand and low production the stored water is released through turbines into the lower reservoir to produce energy. The potential energy depends on the body of water and the height difference between the two basins and thereby the efficiency is ranging between 70-80%. Thereby the potential to run a storage of this type depends on topographical conditions. According to Faulstich et al. [2011] the available options in Germany are exhausted, but in Norway and in Sweden the potential is estimated above 118 TWh storage capacity. The main advantage of pumped storage power stations is that they can react to load demand changes within seconds.

Chapter 2

A Clairvoyant Approach

In this chapter we solve the optimal charging and discharging strategy of energy storages under the assumption that we know the energy contribution from renewable sources to the electricity market and the load, demanded energy, over a certain time horizon. The optimal dispatching strategy in this case is equal to minimizing the needed energy from conventional plants to ensure a load and supply equilibrium.

We derive the contribution of energy from renewable sources, wind and solar, as products of the installed capacity, β for wind and α for sun, and utilization factors, i.e. the energy produced under certain environmental conditions as a fraction of the installed capacity. The time series for the wind energy utilization is denoted by $\{wind_m\}_{m\in\{0,\ldots,M\}}$ and the solar energy utilization by $\{sol_m\}_{m\in\{0,\ldots,M\}}$, where $m \in \{0, \ldots, M\}$ denotes the measurement time points. The respective load time series is $\{load_m\}_{m\in\{0,\ldots,M\}}$ and quoted in GWh.

For given installed capacities and a given storage capacity CS we formulate a linear optimization problem to minimize the backup needed by conventional technologies $\{backup_m\}_{m\in\{0,\ldots,M\}}$ in Section 2.1. A detailed description of how the example data is obtained is given in Section 2.2. In Section 2.3 the implemented graphical user interface as a tool to evaluate different renewable energy development scenarios is introduced. Finally, we give a detailed influence study of different scenarios in the scope of the needed backup, storage and wasted energy in Section 2.4.

2.1 Linear Optimization for given Data

We denote the fixed maximal storage capacity by CS [GWh], the installed capacity of wind parks by β [GW] and the installed capacity of photovoltaic (PV) production entities by α [GW] integrated in the grid. The storage facilities' charging level is denoted by ST_m [GWh] at time $m \in \{0, \ldots, M\}$ and is dependent on the history up to time m. The storage facilities can be charged or discharged by st_m, $-ST_{m-1} \leq st_m \leq CS - ST_{m-1}$, to balance the energy supply and demand in the time interval from $m-1$ to m. At times of higher production from renewable energy sources than demanded and full storage capacity the excess energy is wasted, denoted by $\{w_m\}_{m\in\{0,\ldots,M\}}$. On the other side, at times, where the load cannot be met by the stored and renewable energy production, the energy demand

and supply equilibrium is ensured by energy produced by conventional power plants, denoted by $\{backup_m\}_{m\in\{0,\dots,M\}}$.

The respective optimization problem is given by:

$$\begin{aligned}
\min_{w_m,st_m,ST_m} \quad & \textstyle\sum_{m=1}^{M} backup_m && && (2.1)\\
\text{s.t. } \alpha\, sol_m + \beta\, wind_m + b_m - w_m - st_m = \; & load_m && m = 0,\dots,M && (2.2)\\
ST_m = \; & ST_{m-1} + st_m && m = 1,\dots,M && (2.3)\\
ST_0 = \; & ST_M + st_0 && && (2.4)\\
backup_m, w_m \geq \; & 0 && m = 0,\dots,M && (2.5)\\
0 \leq \; ST_m \leq \; & CS && m = 0,\dots,M && (2.6)
\end{aligned}$$

Equation (2.2) ensures that the load and the supply in the electricity market coincides. The equality in (2.3) updates the charging level of the storages based on the level ST_{m-1} and the balancing term st_m. At this point we assume for simplicity that the amount charged to the storage facilities is equal to the amount taken from the energy market for charging as well as that the charging (discharging) is restricted by the capacity CS (by the storage level ST_{m-1}). Furthermore, we assume that we start with the same joint level in inventories as we finish with, see (2.4). Within (2.5) we ensure that the waste $\{w_m\}_{m\in\{0,\dots,M\}}$ and the backup $\{backup_m\}_{m\in\{0,\dots,M\}}$ are non-negative, and in (2.6) we guarantee that the stored amount of energy $\{ST_m\}_{m\in\{0,\dots,M\}}$ fulfills the requirements.

To obtain the standard formulation of an linear optimization problem we introduce slack variables $\{RS_m\}_{m\in\{0,\dots,M\}}$ to achieve an equality for the constraints $ST_m \leq CS,\ m \in \{0,\dots,M\}$, i.e. $ST_m + RS_m = CS$ and $RS_m > 0$. Hence we define the vector in the following way:

$$x = \begin{pmatrix} b_0 & w_0 & st_0 + CS & ST_0 & RS_0 & backup_1 & \cdots & st_M + CS & ST_M & RS_M \end{pmatrix}^T .$$

Accordingly to this definition set $f := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \end{pmatrix}^T$ to meet (2.1). To transfer the equality constraints and (2.6) define the matrix

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \text{ and the vector } b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} .$$

The matrix A_1 and the respective b_1 represent Constraint (2.2) and are defined as follows:

$$A_1 = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & \cdots & & & & \\ & & & & & \ddots & & & & \\ & \cdots & & & 0 & 1 & -1 & -1 & 0 \end{pmatrix} \text{ and } b_1 = \begin{pmatrix} load_1 - \alpha sol_1 - \beta wind_1 - CS \\ \vdots \\ load_M - \alpha sol_M - \beta wind_M - CS \end{pmatrix} .$$

The matrix A_2 and b_2 are given by

$$A_2 = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 & & \cdots & & 0 & -1 \\ & \cdots & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & \cdots \\ & & & & & & & & & & \ddots \end{pmatrix} \text{ and } b_2 = \begin{pmatrix} -CS \\ \vdots \\ -CS \end{pmatrix}$$

and ensure Constraints (2.3) and (2.4). Finally, we define A_3 and b_3 to cover Constraint (2.5)

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & \cdots & \\ & & & & & \ddots & & \\ & & \cdots & & 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } b_3 = \begin{pmatrix} CS \\ \vdots \\ CS \end{pmatrix}.$$

Hence, the linear optimization problem in standard form is given by:

$$\begin{array}{rcl} \min & & f'x \\ Ax & = & b \\ 0 & \leq & x. \end{array} \tag{2.7}$$

This problem can now be solved using standard linear optimization methods, as the Simplex Algorithm [Vanderbei, 2008, Chapter 2] or Primal-Dual Interior points methods [Vanderbei, 2008, Part 3]. Interior point methods are superior for large scale linear programming and the Simplex algorithm for medium scale problems, see [Vanderbei, 2008, Simplex Methods vs Interior point methods Chapter 22] .

2.2 Description of the time series

The data we use are time series of the utilization for PV plants $\{sol_m\}_{m\in\{0,\dots,M\}}$, the utilization for wind farms $\{wind_m\}_{m\in\{0,\dots,M\}}$ and the load $\{load_m\}_{m\in\{0,\dots,M\}}$ aggregated over Europe and owned by Siemens. The time points $m \in \{0,\dots,M\}$ represent hourly measurements in the time horizon from 1st January 2000 until 31st December 2007, hence 70128 data points. The countries in this aggregation are all countries on the European continent except Estonia, Latvia, Lithuania, Belarus, Ukraine, Swiss, Moldova and Russia.

The load series are based on publicly available data of country or network transmission providers' load history. Before 2006, the demand data was in most cases not available in a sufficient small temporal resolution. For this reason the load streams have been replicated based on the known load performances and the larger scale available data. Especially weekend effects are reflected in this replication. From 2006 onwards the country specific load data in hourly resolution comes from the load database (https://www.entsoe.eu/resources /data-portal/consumption/), provided by the European Network of Transmission System Operators for Electricity (ENTSO-E). To obtain the Europe wide view the data is summed to one load history.

For the solar and wind time series the local weather conditions are used. Therefore, the countries are separated into a 50×50 km^2 spatial mesh on the mainland and off-shore regions on the waterside for the calculation of the utilization series. The specific surface roughness and mainland or seaside indices are accomplished for each grid-cell. The weather data for these grid-cells was provided by the private weather service provider Weather & Wind Energy Prognosis (WEPROG). This data consists of the wind speed and direction 100 m above the ground, the cloud cover, air temperature 2m above the ground, net short wave radiation at the surface and the cloud albedo. Moreover the surface albedo is given in monthly resolution.

The wind utilization time series are obtained by using the wind power curves and the impacting wind at hub hight. The wind power curve gives the power output for a given wind mill as a function of the wind speed, as for example Figure 1.2. The impacting wind at hub hight is obtained by the respective surface roughness given by the grid indices and the measured wind speed. For a more detailed description of the different influences of the parameters see Section 1.2. The utilization factors per grid cell are aggregated over Europe to obtain the European factor.

The PV plants utilization factor is achieved by the usage of the global radiations, temperatures, geographical coordinates and the assumptions towards the photovoltaic panels characteristic. The global radiation is obtained from the known cloud cover, cloud albedo and net short wave radiation. The influence of the different factors on the power output of PV panels are pointed out in Section 1.2. For each grid cell different PV panels with varying tilt angles and orientations as well as with or without solar trackers are considered. The resulting small spatial utilizations factors are aggregated to the European wide view.

2.3 Graphical User Interface

To give a simple tool to view the influences of different developments scenarios of the renewable energy sector we implemented a graphical user interface (GUI). This interface offers different variables to adjust the scenarios and gives a graphical overview as well as concrete figures of the respective result.

First of all we introduce the adjustment parameters. The quota of renewable energy on the load $\gamma \in \mathbb{R}_+$, defined by

$$\gamma := \frac{\sum_{m=1}^{M} \alpha sol_m + \beta wind_m}{\sum_{m=1}^{M} load_m},$$

adjusts the development of the renewable energy sector as part of the overall energy supply. The fraction of PV energy on the renewable energy $\mu \in (0, 1)$, defined by

$$\mu := \frac{\sum_{m=1}^{M} \alpha sol_m}{\sum_{m=1}^{M} \alpha sol_m + \beta wind_m},$$

is the adjustment tool for the development of the PV energy in the renewable energy sector. Given γ and μ we can derive the respective installed capacities of PV plants by

$$\alpha = \frac{\mu\, \gamma\, \sum_{m=1}^{M} load_m}{\sum_{m=1}^{M} sol_m}$$

and the capacities of wind plants by

$$\beta = \frac{(1-\mu)\, \gamma\, \sum_{m=1}^{M} load_m}{\sum_{m=1}^{M} wind_m},$$

needed to get the problem stated in (2.7). The parameters γ and μ, which are together with the storage capacity CS are the variable inputs and are collected in the parameter caption in the GUI, see Figure 2.1. This caption along with the data setting caption forms the input

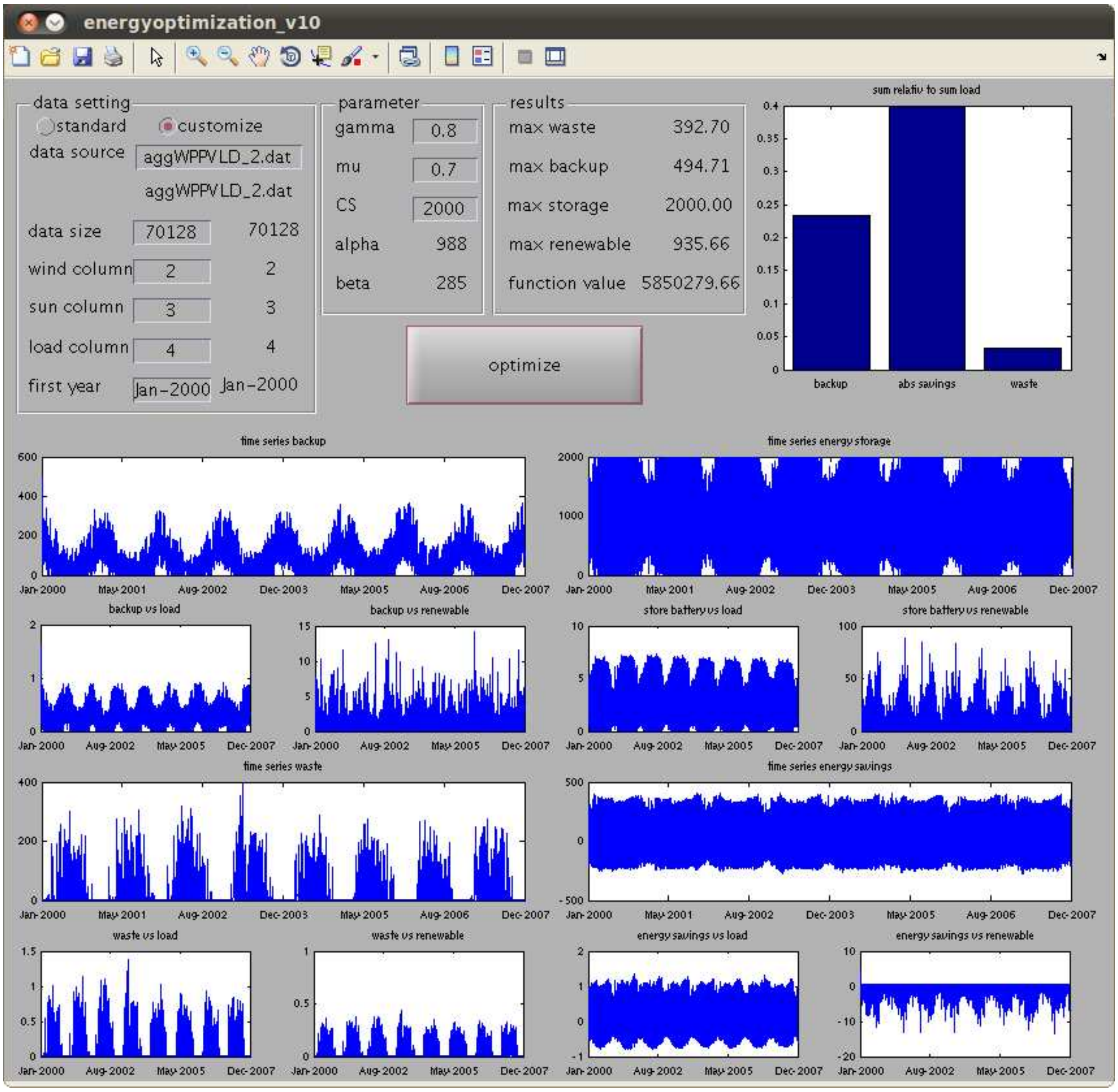

Figure 2.1: GUI to obtain the influence of different development scenarios of renewable energy production, given by the quota of renewable energy on load γ and fraction PV energy on renewable energy μ.

section of the interface. The data setting caption offers the features to upload data files and specify the columns corresponding to $\{sol_m\}_{m\in\{0,\dots,M\}}$,$\{wind_m\}_{m\in\{0,\dots,M\}}$,$\{load_m\}_{m\in\{0,\dots,M\}}$ and the considered time horizon.

Now by solving the optimization problem (2.7) with the respective α, β and CS we obtain four time series, including the needed backup $\{backup_m\}_{m\in\{0,\dots,M\}}$, the waste $\{w_m\}_{m\in\{0,\dots,M\}}$, the balancing energy $\{st_m\}_{m\in\{0,\dots,M\}}$ and the charging level of the storage facilities $\{ST_m\}_{m\in\{0,\dots,M\}}$, for further analysis. For each of the results the GUI offers three plots, including the pure time series, the time series relative to $\{load_m\}_{m\in\{0,\dots,M\}}$ and relative to the contribution by renewable energies $\{\alpha\ sol_m + \beta\ wind_m\}_{m\in\{0,\dots,M\}}$ for detailed examination. Moreover, the sum of the backup, the absolute values of the storage stream and the waste relative to the cumulative load are shown in a bar-plot. Additional to the visual examination part some resulting numbers are given in the result caption, namely the maximum of $\{w_m\}_{m\in\{0,\dots,M\}}$, $\{backup_m\}_{m\in\{0,\dots,M\}}$, $\{ST_m\}_{m\in\{0,\dots,M\}}$,$\{\alpha\ sol_m + \beta\ wind_m\}_{m\in\{0,\dots,M\}}$ and the resulting minimal problem value.

2.4 Results

We examine the influence of the different parameters, including the quota of the total renewable energy on the load $\gamma \in [0, 1]$, the fraction PV energy on renewable energy $\mu \in [0, 1]$ and the storage capacity $CS \in [0, 50000]$. The large interval for the storage capacity is reasonable since the potential storage capacity for pumped storages in Sweden and Norway are estimated to 118 TWh (see Faulstich et al. [2011]). Since for the given time series, as described in 2.2, the sparse matrix A of the standard problem (2.7) is of a large size, 210384×350640, we use a version of the interior point solver LIPSOL, implemented in MATLAB. LIPSOL is a variant of the Mehrotras's Algorithm as described in [Wright, 1997, Chapter 10].

For a fixed fraction of sun energy on the renewable energies $\mu = 0.6$ the results are shown in Figure 2.2. On the left hand side we see, that needed backup linearly demises for a rising quota of renewable energy on the load and for a large storage capacity $CS \in [1000, 50000]$. For a smaller storage capacity $CS \in [0, 1000]$ the needed backup for a high fraction of renewable energy on the load, i.e. $\gamma > 0.5$, descent much slower, which can be seen on the left plot in Figure 2.2. This means that, for a minor storage capacity the production of renewable energy is wasted, since peak productions are not storable. On the right hand side of Figure 2.2, we see that we start wasting a significant amount of energy at a renewable energy contribution over 40% and no storage available. This wasted energy raises up to 0.3 of the overall load for no storage capacity available and $\gamma = 1$. For a small storage capacity $CS \in [0, 5000]$ the lost energy steps up with the contribution of renewable energies and the reduction of the storage capacity. For a high storage capacity $CS \in [5000, 50000]$ the wasted energy is slowly, almost linear, falling with the quota of renewable energy contribution on the load and the storage size. It shows that the storage capacity has a major influence on the needed backup for conventional plants as well as on the energy wasted. This emphasize the importance of an adequate amount of storage capacity integrated in the grid.

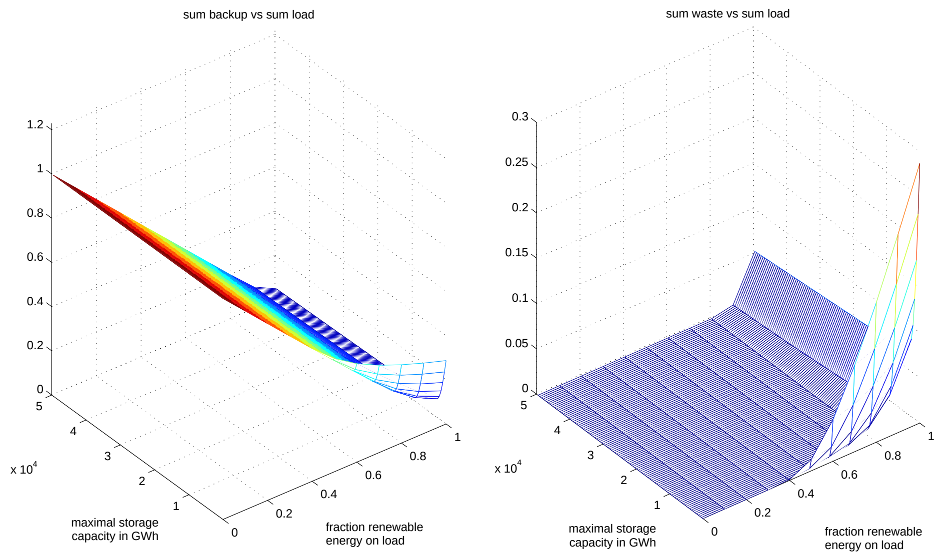

Figure 2.2: Shows the sum of the backup, to the left hand side, and waste, on the right hand side, versus the sum of the load for 60% sun energy contribution on the renewable energy contribution.

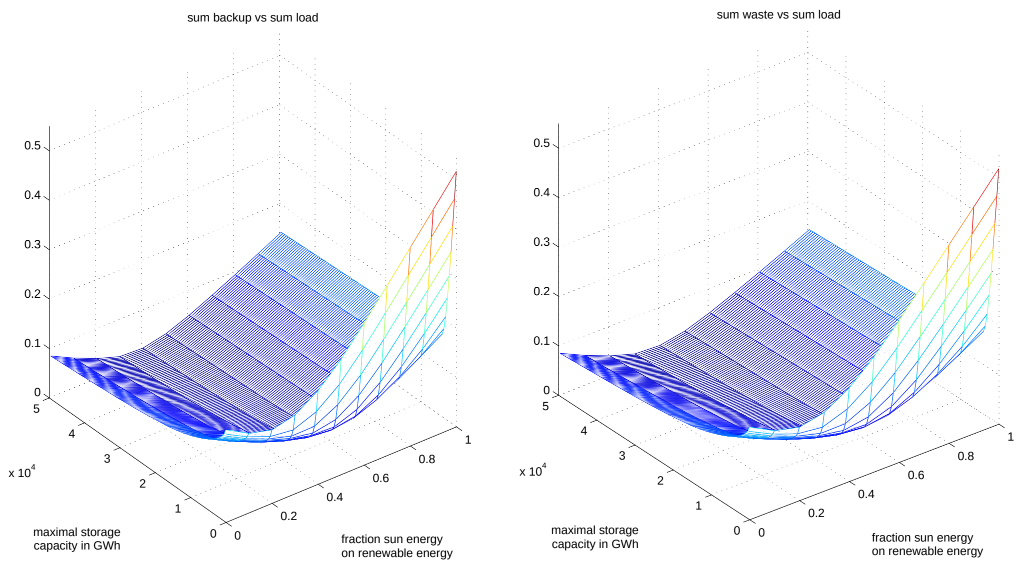

Figure 2.3: Shows the sum of the backup, to the left hand side, and waste, on the right hand side, versus the sum of the load for renewable energy quota $\gamma = 1$ on the load.

In the $\gamma = 1$ scenario, as shown in Figure 2.3, the influence of the fraction PV plants on the renewable energy contribution μ and the storage capacity CS is almost equal for the backup on the left hand side and the waste on the right hand side. We see that the fraction of energy generated by PV plants has a significant influence on the required backup and wasted energy, with a minimum wasted energy and needed backup of around $\mu = 0.4$. This result is confirmed by the results in [Heide et al., 2010, 4. Conclusion], where the optimal mix of PV and wind generated electricity in the 100% renewable scenario is 55%

wind energy and 45% PV energy.

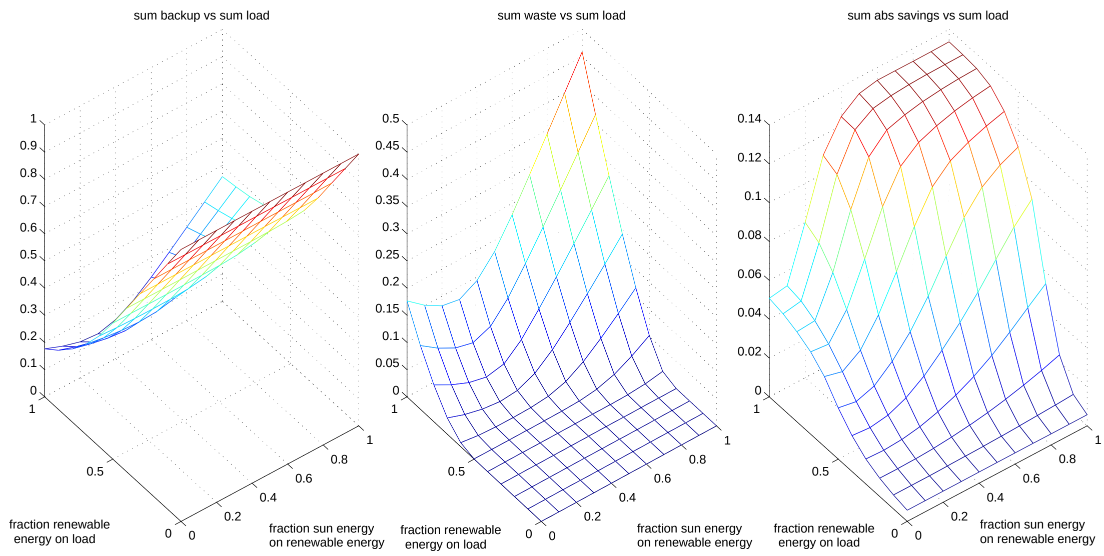

Figure 2.4: For 540 GWh storage capacity the sum of the backup, waste and absolute balancing streams versus the sum of the load are shown.

In Figure 2.4 we consider a constant storage capacity of 540 GWh. The left plot shows that the needed backup falls with the renewable energy quota on the load and depicts a minimum for a fraction around 30% contribution from solar energy at $\gamma = 1$. For an increase in the fraction of energy produced by PV plants the needed backup energy rises in the interval $\mu \in [0.4, 1]$ and falls in the interval $\mu \in [0, 0.2]$. Moreover, this effect flattens out in the size of fraction of renewable energy on the load γ. The middle plot in Figure 2.4 shows the dependence between wasted energy and the respective fractions is shown. The minimum wasted energy is produced for a combination of approximately 30% solar energy production on renewable energy contribution for a given quota of renewable energy contribution $\gamma \in [0.5, 1]$. With a rise of the fraction of PV energy on the renewable energy contribution in $\gamma \in [0.3, 1]$ and given quota renewable energy on the load $\gamma > 0.5$ the wasted energy rises as well. The storage streams soar with the fraction of PV energy due to the day-night fluctuations, which is shown in Figure 2.4 to the right. As well the storage streams rise with the quota of renewable sources, since for conventional power technologies a storage is meaning less.

In a nutshell we see that storages become more and more crucial with a rising quota of renewable energy on the electricity market, with an additional increase for high contributions from PV plants.

Chapter 3

Problem Setup

In Chapter 2 we developed an optimal storage strategy for a given time interval for all electricity storages connected to the grid and given the complete knowledge about the demanded and produced energy in this interval. This approach neglects certain aspects in reality, especially the uncertainty of the energy production by renewable sources. Thus, from now on we are going to focus on a more elaborate problem setting. This setting considers the uncertain production by the stochastic renewable energy process $\{R_t\}_{t\in[0,T]}$ and the variability of the demanded energy by the demand process $\{D_t\}_{t\in[0,T]}$. Moreover, we focus on one electricity storage facility, including all storages introduced in Section 1.3, leased by the agent over the finite time horizon $[t, \hat{T}]$. Since every finite time horizon of the form $[t, \hat{T}]$ can be transformed to $[0, T]$ without loss of generality, we do so from now on. Moreover, we assume that the agent is a price taker and that the electricity market is infinitely liquid. This chapter introduces our problem and the notations used in the following paragraphs, in which we derive an optimal dispatching strategy for an energy storage given variable energy prices.

In the following let $\{R_t\}_{t\in[0,T]}$ denote the renewable energy process, where R_t is the amount of energy in MW produced by renewable technologies at time $t \in [0, T]$. The demand process is denoted by $\{D_t\}_{t\in[0,T]}$ and D_t is the amount of energy in MW demanded at time $t \in [0, T]$. In this setting we assume that all stochastic quantities live on the filtrated probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions of right continuity and completeness.

The price on the electricity market in our setting results from the price function $\rho : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ of the renewable energy process $\{R_t\}_{t\in[0,T]}$ and the demand process $\{D_t\}_{t\in[0,T]}$. The price function ρ reflects the results for price derivation in the demand-and-supply model and the merit-order curve discussed in Section 1.1. We establish this function in Section 3.1. Given the price on the electricity market the agent wants to optimize his return in the time horizon $[0, T]$ by changing the possible regimes of the battery, namely charging, discharging and keeping the level constant. The essential reflections of the batterie's operational characteristics in gain and losses of the respective regimes are summarized in Section 3.2. In Section 3.3, the segments introduced in the previous sections are assembled into the control problem, which we solve in Chapter 4.

3.1 Price fixing

As already pointed out the market price is obtained from the renewable energy process $\{R_t\}_{t\in[0,T]}$, the demand $\{D_t\}_{t\in[0,T]}$ and a function similar to the merit order curve presented in Figure 1.1. Therefore we are going to formulate the results from Section 1.1 to obtain an energy price in a mathematical way.

Let us assume that we have N conventional power technologies. These could include fossil fuel driven plants as well as nuclear power and waste-to-energy plants. All power plants of a given technology $n \in \{1, ..., \mathsf{N}\}$ have the same non-negative marginal costs p_n for producing one MWh and all power plants of technology n have a capacity of c_n quoted in MW. The technologies are labelled such that for all $n_1, n_2 \in \{1, ..., \mathsf{N}\}$ with $n_1 < n_2$ it holds that $\mathsf{p}_{n_1} < \mathsf{p}_{n_2}$. Furthermore let $A_n = \{1, \ldots, n\}$ for $n \in \{1, ..., \mathsf{N}\}$, be the n energy portfolio. The capacity of the energy portfolio A_n, is given then given by $\sum_{k=1}^{n} \mathsf{c}_k$.

Given $\{\mathsf{p}_n\}_{n=1}^{\mathsf{N}}$ and $\{\mathsf{c}_n\}_{n=1}^{\mathsf{N}}$ we define a step function $\hat{\rho} : [0, T] \times \mathbb{R}_+ \to \{\mathsf{p}_1, \ldots, \mathsf{p}_\mathsf{N}\}$ by:

$$\hat{\rho}(t, D_t) = \sum_{n=1}^{\mathsf{N}} \mathsf{p}_n 1_{[\sum_{i=1}^{n-1} \mathsf{c}_i, \sum_{i=1}^{n} \mathsf{c}_i)}(D_t). \tag{3.1}$$

This function gives the marginal costs p_n of technology n such that the capacity of the energy portfolio A_n covers the demand D_t at every $t \in [0, T]$. Figure 3.1 gives a visualization of the function in (3.1).

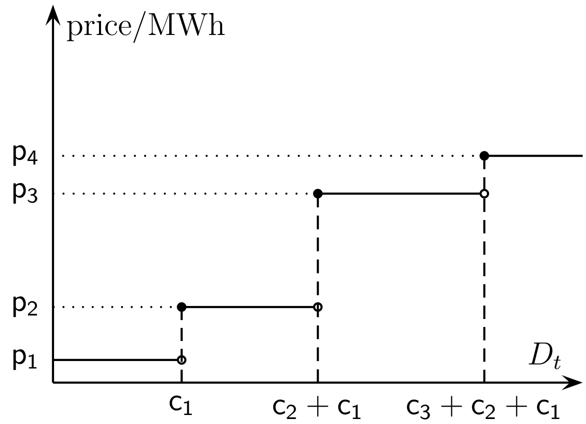

Figure 3.1: Classical merit order curve $\hat{\rho}$, (3.1).

The transfer of this definition from the pure conventional power market into a mix of conventional and renewable energy is straightforward. It is worth to notice, that the energy needed to be provided at time $t \in [0, T]$ by conventional technologies to meet the demand is $\max(0, D_t - R_t)$. Since we are not going to assume that $D_t \geq R_t$ almost surely for all $t \in [0, T]$, we introduce p_0 as the marginal costs for one MWh electricity from renewable technologies. Hence, together with the derivation of $\hat{\rho}$, in (3.1), we achieve a price function $\rho : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to [\mathsf{p}_0, \mathsf{p}_\mathsf{N}]$ for the energy market with contribution from renewable energies by:

$$\rho(t, R_t, D_t) = \begin{cases} \mathsf{p}_0 & \text{if } D_t - R_t < 0, \\ \hat{\rho}(t, D_t - R_t) & \text{if } D_t - R_t \geq 0. \end{cases} \tag{3.2}$$

3.2 Storage Characteristics

In this section we define the specific constraints, costs and the rewards the agent deals with.

First of all we state the constraints on the size of the storage facility. Assume that the facility has the maximal charging level c_{max} and the minimal charging level c_{min} quoted in MWh and $0 \leq \mathsf{c}_{min} < \mathsf{c}_{max} < \infty$. At each time point $t \in [0,T]$ the storage facility has a certain charging level $L_t \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, which depends on the strategy up to time $t \in [0,T]$, leading to the charging level process $\{L_t\}_{t\in[0,T]}$. Further details on $\{L_t\}_{t\in[0,T]}$ are given in Section 3.3.

The storage facility has three possible regimes $i \in \{-1,0,1\}$, corresponding to

$$i = \begin{cases} -1 & \text{charging the storage,} \\ 0 & \text{keeping the storage level constant,} \\ +1 & \text{discharging the storage.} \end{cases} \tag{3.3}$$

We postulate that given by the storage characteristics only a limited amount of energy per time unit can be transferred from the storage into the electricity network, denoted by a_{out}[MWh], and charged from the network into the storage, denoted by a_{in}[MWh]. Since the charging rate depends, besides a_{in} and a_{out}, also on the current charging level $L_t, t \in [0,T]$, we define the change function $a_i : \{-1,0,1\} \times [0,T] \times [\mathsf{c}_{min}, \mathsf{c}_{max}] \to \mathbb{R}_+$ for regime $i \in \{-1,0,1\}$ in the following way:

$$a_i(t, L_t) = \begin{cases} a_{in}(t, L_t) & \text{if } i = -1, \\ 0 & \text{if } i = 0, \\ a_{out}(t, L_t) & \text{if } i = +1, \end{cases} \tag{3.4}$$

where a_{in} and a_{out} are appropriate functions of the charging level. This function gives the changes in the storage level in MWh for a charging level L_t at time $t \in [0,T]$ and in regime $i \in \{-1,0,1\}$. Therefore it holds that $\frac{\partial a_{-1}(t,l)}{\partial t} \leq \mathsf{a}_{in}$ and $\frac{\partial a_1(t,l)}{\partial t} \leq \mathsf{a}_{out}$ for all $t \in [0,T]$ and $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, which means that the restriction given by the storage characteristics holds.

In the following, we introduce notations for the costs resulting from running the electricity storage. Let b denote the monetary costs of storing one MWh per time unit. These costs summarize various costs; for example the cost due to not influenceable relative discharging, mainly dependent on the charging level. As well we have operating and managing costs K_i depending on the regime $i \in \{-1,0,1\}$. Finally, changing the regime incurs switching costs. We denote the corresponding costs for switching from regime i to regime j by $C_{i,j}$, $i,j \in \{-1,0,1\}$, and make the following assumptions on these costs.

Assumption 3.1. *The costs $C_{i,j}$ are greater than $\varepsilon > 0$ and furthermore satisfy the triangle property, $C_{i,k} \leq C_{i,j} + C_{j,k}$ for all $i,j,k \in \{-1,0,1\}$ and $i \neq j$.*

Both assumptions on the switching costs are made to make a switching disadvantageous, the first one to switch in general away from the current regime and the second one to switch twice at the same time. This is needed “to role out chattering, where the owner would repeatedly change the regime back-and-forth"[Carmona and Ludkovski, 2010, Section 2.1]. Furthermore, Assumption 3.1 ensures the existence of a finite variation optimal strategy.

To get the reward of the storage in regimes $i \in \{-1, 0, 1\}$ and at time $t \in [0, T]$, given the contribution from the renewable energy sector R_t, the demand D_t and the charging level L_t we have to sum the costs and rewards established before. The reward function $\psi_i : \mathbb{R}_+ \times \mathbb{R}_+ \times [\mathsf{c}_{min}, \mathsf{c}_{max}] \to \mathbb{R}$ in regime $i \in \{-1, 0, 1\}$ is given by:

$$\psi_i(t, R_t, D_t, L_t) = i \; \rho(t, R_t, D_t) \; a_i(t, L_t) - \mathsf{b} \; L_t - \mathsf{K}_i, \tag{3.5}$$

where $\rho(t, R_t, D_t)$, defined in (3.2), is the price of one MWh, $a_i(t, L_t)$, given by (3.4), is the change rate in MWh, b are the costs of storing one MWh and K_i are the management and operating cost in regime $i \in \{-1, 0, 1\}$. Besides, under the assumption that selling and buying has an immediate influence on the charging level we can specify the dynamics of the charging level process by

$$dL_t = -i \; a_i(t, L_t)dt, \tag{3.6}$$

where $i \in \{-1, 0, 1\}$ is the current regime and a_i is the respective change function (3.4).

3.3 Control Problem

Finally, we are able to derive the control problem in our setting. This means, that given the time horizon $[0, T]$ and the uncertain cash flows driven by the renewable energy process $\{R_t\}_{t \in [0,T]}$ and the demand process $\{D_t\}_{t \in [0,T]}$, we attain an optimal strategy by steering the regimes $i \in \{-1, 0, 1\}$ such that the reward is maximized. Subsequently we going to obtain a strategy consisting of optimal regimes and the corresponding switching times between the regimes, which maximize the reward.

A strategy consists of a non-decreasing sequence of $\mathbb{F}$-stopping times $\{\tau_k\}_{k \geq 0}$, with $0 \leq \tau_k \leq \tau_{k+1}$ for all $k \geq 0$, and a sequence of regimes $\{\xi_k\}_{k \geq 0}$ with $\xi_k \neq \xi_{k+1}$ and $\xi_k \in \{-1, 0, 1\}$ for all $k \geq 0$. We denote such a strategy by $u = (\xi_k, \tau_k)_{k \geq 0}$ and by $\{u_t\}_{t \in [0,T]}$ the corresponding indicator of the current regime, given by:

$$u_t = \sum_{k \geq 0} \xi_k 1_{[\tau_k, \tau_{k+1})}(t) \, , t \in [0, T]. \tag{3.7}$$

Note that $\{u_t\}_{t \in [0,T]}$ is uniquely determined by $\{\tau_k\}_{k \geq 0}$ and $\{\xi_k\}_{k \geq 0}$ and vice versa, the càdlàg process $\{u_t\}_{t \in [0,T]}$ determines $\{\tau_k\}_{k \geq 0}$ and $\{\xi_k\}_{k \geq 0}$.

Definition 3.1 (Admissible Strategy [Djehiche et al., 2009, p. 2754]). *A strategy $u = (\xi_k, \tau_k)_{k \geq 0}$ is called admissible if it satisfies*

$$\lim_{k \to \infty} \tau_k = T \quad \mathbb{P} - a.s. \tag{3.8}$$

The set of admissible strategies with $\tau_0 \geq t$ is denoted by $\mathcal{U}(t)$ for all $t \in [0, T]$.

Moreover, for a given strategy $u \in \mathcal{U}(t)$ and a fixed initial charging level $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, the respective charging levels up to time T are uniquely determined by the following ordinary differential equation [Carmona and Ludkovski, 2010, Eq. (4)]:

$$dL_s^u = -u_s \, a_{u_s}(L_s^u)ds \,, \qquad L_0^u = l \;, \tag{3.9}$$

which is the strategy equivalent to the dynamics given in (3.6). Hence, every charging level L_t^u with $t \in [0,T]$, given a strategy u, is determined by $L_t^u = L_0^u + \int_0^t (-u_s) a_{u_s}(L_s^u) ds$. Since we are always dealing with the charging level in combination with a given strategy we suppress from now on u for simplicity, i.e. $L_t^u =: L_t, t \in [0,T]$.

For any strategy $u \in \mathcal{U}(t)$ being at time $t \in [0,T]$ in regime i, i.e. $u_t = i$, given the renewable energy contribution $R_t = r$, the demand $D_t = d$ and the charging level $L_t = l$ the expected gain or loss until time T is given by:

$$\begin{aligned} J(t,r,d,l,i;u) = \mathbb{E}\Bigg[\int_t^T & e^{\mathsf{r}_m(t-s)} \psi_{u_s}(s, R_s, D_s, L_s)\, ds - \\ & \sum_{\tau_k < T} e^{\mathsf{r}_m(t-\tau_k)} C_{u_{\tau_k -}, u_{\tau_k}} \,\Bigg|\, \mathcal{F}_t, L_t = l, u_t = i \Bigg] . \end{aligned} \tag{3.10}$$

The first term in (3.10) measures the rewards discounted to the present value at time t with the market interest rate r_m and the second term sums the discounted switching costs of the strategy u up to time T. The function $\psi_i(t, R_t, D_t, L_t)$ is the reward function defined in (3.5).

These preliminaries lead to the formal control problem

$$J(t,r,d,l,i) = \sup_{u \in \mathcal{U}(t)} J(t,r,d,l,i;u), \tag{3.11}$$

Until now constraints on conditions at time T have not been specified. Since such constraints are common in lending contracts, for example to return the storage facility with the same inventory level as the initial one, we include a penalty function at time T to cover that. The penalty function is the mathematical formulation of the respective 'buy back provisions'(Carmona and Ludkovski [2010]). A common example for a penalty function is $V : \mathbb{R}_+ \times \mathbb{R}_+ \times [\mathsf{c}_{min}, \mathsf{c}_{max}] \to \mathbb{R}$, given by:

$$V(R_T, D_T, L_T) = 2\, \rho(T, R_T, D_T)\; \max(0, L_0 - L_T), \tag{3.12}$$

where L_0 is the initial charging level at the beginning of the contract. Hence, the agent is charged twice the market price, if he returns the storage at a lower level than the initial level. This constraint is reasonable, since otherwise we would allow a risk free profit by simply discharging the storage facility.

Chapter 4

Recursive Optimal Stopping

This chapter is mainly based on Chapter 3 in the Ph.D. thesis of Michael Ludkovski [2005]. The difference between his assumptions and ours is that we do not obtain the price directly by a price process. Instead, we achieve the price by the modified merit order curve $\rho(t, R_t, D_t)$, (3.2), with renewable energy process $\{R_t\}_{t\in[0,T]}$ representing the amount of energy in MWh produced by renewable technologies at time $t \in [0, T]$ and the load process $\{D_t\}_{t\in[0,T]}$, as implemented in Section 3. Furthermore in [Ludkovski, 2005, Chapter 3] deals with the optimal switching problem without considering the charging level and its dependence to the reward and strategy. Thus, we extend the results to include the charging level dependence, as well.

From now on we assume that the driving process is just $\{R_t\}_{t\in[0,T]}$ and keep the load process $\{D_t\}_{t\in[0,T]}$ constant over time, i.e. $D_t = D > 0$ for $t \in [0, T]$. Based on the assumption of a fixed demand D the functions introduced in Chapter 3 simplify to:

$$\begin{aligned} \rho(t, r) &:= \rho(t, r, D), \\ \psi_i(t, r, l) &:= \psi_i(t, r, D, l), \end{aligned} \tag{4.1}$$

where $\rho : [0, T] \times \mathbb{R}_+ \to [\mathsf{p}_0, \mathsf{p}_\mathsf{N}]$ is the respective price function derived from (3.2) and $\psi_i : [0, T] \times O \times [\mathsf{c}_{min}, \mathsf{c}_{max}] \to \mathbb{R}$ is the reward function in regime $i \in \{-1, 0, 1\}$ derived from (3.5).

In Section 4.1 technical assumptions on the reward function ψ_i in regime i and the driving process $\{R_t\}_{t\in[0,T]}$ are stated and further notations are introduced. We recall general results for Snell envelope theory, which are important for solving the optimal switching problem (3.11) in Section 4.3, in Section 4.2.

4.1 Setup and Assumptions

We start with stating the assumptions on the renewable energy process $\{R_t\}_{t\in[0,T]}$. We assume that $\{R_t\}_{t\in[0,T]}$ is an Itô process on the open subset $O \subseteq \mathbb{R}_+$. The respective Brownian motion $\{B_t\}_{t\in[0,T]}$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the completed filtration generated by $\{B_t\}_{t\in[0,T]}$ and satisfies the usual condition of right continuity, i.e. $\cap_{\epsilon>0}\mathcal{F}_{t+\epsilon} = \mathcal{F}_t$ for all $t \in [0, T]$, and completeness, i.e. all set of $\mathcal{F}$ with probability zero are contained in $\mathcal{F}_t, \forall t \in [0, T]$. The corresponding filtrated probability space is $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

For ease of notation we introduce important sets. Define:

$$S := S_0 = \{\tau \ : \ \mathbb{F}\text{-stopping time } 0 \le \tau \le T\} \tag{4.2}$$

and the set

$$S_v := \{\tau \in S : v \le \tau\}, \tag{4.3}$$

for $v \in S$. Moreover, for $p \ge 1$ we define:

$$\mathcal{S}_T^p := \left\{ \{X_t\}_{t\in[0,T]} : X_t \in \mathcal{F}_t, \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p \right] < \infty \right\}, \tag{4.4}$$

and

$$\mathcal{M}_T^p := \left\{ \{X_t\}_{t\in[0,T]} : X_t \in \mathcal{F}_t, \mathbb{E}\left[\int_0^T |X_s|^p \, ds \right] < \infty \right\}. \tag{4.5}$$

Assumption 4.1 ([Ludkovski, 2005, Assumption 1])**.** *The reward function ψ_i is Borel measurable and $\psi_i(\cdot, R_\cdot, L_\cdot) \in \mathcal{M}_T^2$ for all initial charging levels $L_0 \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ and $i \in \{-1, 0, 1\}$.*

The first assumption is necessary that $\{\psi(t, R_t, l)\}_{t\in[0,T]}$ is a random process for all $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$. The second assumption is needed to make the Snell envelope uniformly integrable. The reward function ψ_i stated in (4.1) satisfies these assumptions for every regime $i \in \{-1, 0, 1\}$. Measurability is ensured, since the reward function ψ_i is a sum of constants and a measurable function ρ. Besides, $\psi_i(\cdot, R_\cdot, l) \in \mathcal{M}_T^2$, since:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T |\psi_i(s, R_s, L_s)|^2 ds \right] &= \mathbb{E}\left[\int_0^T |i \ \rho(t, R_s) \ a_i(t, L_s) - \mathsf{b} \ L_t - \mathsf{K}_i|^2 ds \right] \\
&\le \mathbb{E}\left[\int_0^T (|\rho(t, R_s) \ a_i(t, L_s)| + |\mathsf{b} \ L_t| + |\mathsf{K}_i|)^2 ds \right] \\
&\le \int_0^T (\mathsf{p_N} \ \mathsf{a}_i + \mathsf{b} \ \mathsf{c}_{max} + \mathsf{K}_i)^2 ds < \infty.
\end{aligned} \tag{4.6}$$

The first inequality is simply the triangle property. The second inequality follows from the operational characteristics, namely

$$\begin{aligned}
a_i(t, L_t) \le \ & \mathsf{a}_i \quad < \infty \quad \text{by the definition of the change function } a_i \text{ (3.4) and} \\
L_t \le \ & \mathsf{c}_{max} < \infty \quad \text{due to the charging level restriction,}
\end{aligned}$$

and the derivation of the electricity price, namely

$$\rho(t, R_t) \le \ \mathsf{p_N} < \infty \quad \text{by the definition of the price function (3.2).}$$

Moreover, if we define $\bar{\mathsf{a}} := \max_{i\in\{-1,0,1\}} \mathsf{a}_i$ and $\mathsf{K} := \max_{i\in\{-1,0,1\}} \mathsf{K}_i$ we obtain:

$$\mathbb{E}\left[\int_0^T \max_{i\in\{-1,0,1\}} |\psi_i(s, R_s, L_s)|^2 ds \right] \le \int_0^T (\mathsf{p_N} \ \bar{\mathsf{a}} + \mathsf{b} \ \mathsf{c}_{max} + \mathsf{K})^2 ds < \infty, \tag{4.7}$$

and thus $\{\sup_{i\in\{-1,0,1\}} \psi_i(t, R_t, L_t)\}_{t\in[0,T]} \in \mathcal{M}_T^2$.

4.2 Snell envelope in continuous time

This section is mainly based on Appendix D in Karatzas and Shreve [2001]. We also include results published by Fakeev [1970] in his introduction to stopping rules in continuous time, Hamadène and Lepeltier [2000] paper on the mixed game problem and Cvitanic and Karatzas [1996] Appendix A, as a more general summary. In addition the convergence proof by Djehiche et al. [2009] given in the theoretical introduction to optimal multiple switching problems is included.

Throughout this section let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be a filtration satisfying the usual conditions of right continuity and completeness. We define the set S, S_v and $\mathcal{S}_T^p$ also for the general filtration $\mathbb{F}$. Let $\{Z_t\}_{t\in[0,T]}$ be a càdlàg process adapted to $\mathbb{F}$ and $\mathbb{E}[\sup_{t\in[0,T]} |Z_t|^2] < \infty$. We call the process $\{Z_t\}_{t\in[0,T]}$, according to the optimal stopping context, reward process, as defined in Thompson [1971].

Definition 4.1 (Essential Supremum [Karatzas and Shreve, 2001, Def. A.1])**.** *Let $\{Z_l\}_{l\in\mathbb{L}}$ be an arbitrary family of random variables with the index set $\mathbb{L} \subseteq \mathbb{R}$ (or $\mathbb{Z}$). The essential supremum of $\{Z_l\}_{l\in\mathbb{L}}$ is the unique random variable Y satisfying that $Y \geq Z_l$ a.s. for all $l \in \mathbb{L}$ and $Y \leq X$ a.s for all random variables X such that $X \geq Z_l$ a.s. for all $l \in \mathbb{L}$. This random variable is denoted by $\operatorname{esssup}_{l\in\mathbb{L}} Z_l$.*

Definition 4.2 (Dominates [Björk, 2009, Def. 21.9])**.** *We say a process $\{Y_l\}_{l\in\mathbb{L}}$ dominates the process $\{Z_l\}_{l\in\mathbb{L}}$, if $Y_l \geq Z_l$ a.s. for all $l \in \mathbb{L} \subseteq \mathbb{R}$ (or $\mathbb{Z}$).*

Definition 4.3 (Snell Envelope [Karatzas and Shreve, 2001, D.7])**.** *The Snell envelope of the reward process $\{Z_t\}_{t\in[0,T]}$ is the smallest càdlàg supermartingale dominating $\{Z_t\}_{t\in[0,T]}$.*

Theorem 4.4 ([Fakeev, 1970, Theorem 1 and Theorem 2])**.** *The càdlàg process $\{Y_t\}_{t\in[0,T]}$, given by*

$$Y_t := \operatorname*{esssup}_{\tau\in S_t} \mathbb{E}[Z_\tau|\mathcal{F}_t], \quad t \in [0,T], \tag{4.8}$$

is the Snell envelope of the reward process $\{Z_t\}_{t\in[0,T]}$.

Proof. This proof is a summary of the respective proof of theorems [D.1]-[D.7] given in Karatzas and Shreve [2001], using the same notation and extend it to the uniformly integrable $\{Z_t\}_{t\in[0,T]}$ process, instead of a non-negative process.

For all $t \in [0,T]$ the process Y_t is adapted to $\mathcal{F}_t$ by definition, see (4.8). Moreover, we prove the existence of a sequence of stopping times $\{\tau_n\}_{n=0}^\infty$ in S_v such that $\{\mathbb{E}[Z_{\tau_n}|\mathcal{F}_v]\}_{n=1}^\infty$ is non-decreasing and $\lim_{n\to\infty} \mathbb{E}[Z_{\tau_n}|\mathcal{F}_v] = \operatorname{esssup}_{\tau\in S_v} \mathbb{E}[Z_\tau|\mathcal{F}_v]$ almost surely. To see this, we have to recognize that the family of random variables $\{\mathbb{E}[Z_\tau|\mathcal{F}_v]\}_{\tau\in S_v}$ is closed under pairwise maximization. This means that for every stopping time $\tau_1, \tau_2 \in S_v$ there exists a stopping time $\tau_3 := 1_B\tau_1 + 1_{B^c}\tau_2$, where $B = \{\omega \in \Omega : \ \mathbb{E}[Z_{\tau_1}|\mathcal{F}_v](\omega) \geq \mathbb{E}[Z_{\tau_2}|\mathcal{F}_v](\omega)\} \in \mathcal{F}_v$, such that the following holds:

$$\begin{aligned}\mathbb{E}[Z_{\tau_3}|\mathcal{F}_v] &= \mathbb{E}[1_B Z_{\tau_1}|\mathcal{F}_v] + \mathbb{E}[1_{B^c} Z_{\tau_2}|\mathcal{F}_v] = 1_B\mathbb{E}[Z_{\tau_1}|\mathcal{F}_v] + 1_{B^c}\mathbb{E}[Z_{\tau_2}|\mathcal{F}_v] \\ &= \mathbb{E}[Z_{\tau_1}|\mathcal{F}_v] \vee \mathbb{E}[Z_{\tau_2}|\mathcal{F}_v] \geq \mathbb{E}[Z_{\tau_i}|\mathcal{F}_v] \quad i = 1,2.\end{aligned}$$

The operator '$\vee$' is the pathwise maximum operator. For a given $A \in \mathcal{F}_v$ and $K \in \mathbb{N}$ choose disjoint sets $A_1, \dots, A_K$ in $\mathcal{F}_v$, such that $\cup_{k=1}^K A_k = A$, and $\tau_1, \dots, \tau_K$ are stopping times in S_v. We define the corresponding set by $\pi = \{K; A_1, \ldots, A_K; \tau_1, \ldots, \tau_K\}$ and denote it by $\{\mathbb{E}[Z_\tau|\mathcal{F}_v]\}_{\tau \in S_v}$-partition of A given K. In addition we define

$$\mu_\pi^\alpha(A) := \mathbb{E}\left[\sum_{k=1}^{K} 1_{A_k} \mathbb{E}[Z_{\tau_k}|\mathcal{F}_v] \wedge \alpha\right]$$

and

$$\mu^\alpha(A) := \sup\{\mu_\pi^\alpha | \ \pi \text{ a } \{\mathbb{E}[Z_\tau|\mathcal{F}_v]\}_{\tau \in S_v}\text{-partition of A }\}$$

For a given $n \in \mathbb{N}$ there exists a $\{\mathbb{E}[Z_\tau|\mathcal{F}_v]\}_{\tau \in S_v}$-partition of Ω denoted by

$$\pi^n = \{K^{(n)}; A_1^{(n)}, \ldots, A_K^{(n)}; \tau_1^{(n)}, \ldots, \tau_K^{(n)}\}$$

such that

$$\mu^n(\Omega) \leq \mathbb{E}\left[\sum_{k=1}^{K^{(n)}} 1_{A_k^{(n)}} \mathbb{E}[Z_{\tau_k^{(n)}}|\mathcal{F}_v] \wedge n\right] + \frac{1}{n},$$

since $\mu^n(\Omega) \leq n < \infty$. Then, due to $\{\mathbb{E}[Z_\tau|\mathcal{F}_v]\}_{\tau \in S_v}$ is closed under pairwise maximization, we get $F_n = \mathbb{E}[Z_{\tau_1^{(n)}}|\mathcal{F}_v] \vee \cdots \vee \mathbb{E}[Z_{\tau_K^{(n)}}|\mathcal{F}_v]$ and $E_n = F_1 \vee \cdots \vee F_n$ are in $\{\mathbb{E}[Z_\tau|\mathcal{F}_v]\}_{\tau \in S_v}$. Hence, we have that:

$$\mathbb{E}\left[\operatorname*{esssup}_{\tau \in S_v} \mathbb{E}[Z_\tau|\mathcal{F}]\right] = \mu^\infty(\Omega) \leq \lim_{n\to\infty} \mathbb{E}[E_n] = \mathbb{E}\left[\lim_{n\to\infty} E_n\right]$$

and since $E_n \leq \operatorname{esssup}_{\tau \in S_v} \mathbb{E}[Z_\tau|\mathcal{F}_v]$ for all $n \in \mathbb{N}$ a.s., we have equality. Thus, there exists a sequence of stopping times $\{\tau_n\}_{n \geq 0}$ such that:

$$\lim_{n\to\infty} \mathbb{E}[Z_{\tau_n}|\mathcal{F}_v] = \operatorname*{esssup}_{\tau \in S_v} \mathbb{E}[Z_\tau|\mathcal{F}_v]$$

Then we see that the first equality in

$$\mathbb{E}[|Y_t|] = \mathbb{E}\left[|\lim_{n\to\infty} \mathbb{E}[Z_{\tau_n}|\mathcal{F}_t]|\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} |Z_t|\right] < \infty \ \ \forall t \in [0,T] \tag{4.9}$$

holds. The inequality follows from $\mathbb{E}[Z_{\tau_n}|\mathcal{F}_t] \leq \mathbb{E}[\sup_{s\in[0,T]} Z_s|\mathcal{F}_t]$.

Since for all $v > \mu$, $v, \mu \in S$ the following inequality holds,

$$\mathbb{E}[Y_v|\mathcal{F}_\mu] = \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[Z_{\tau_n}|\mathcal{F}_v]|\mathcal{F}_\mu] = \lim_{n\to\infty} \mathbb{E}[Z_{\tau_n}|\mathcal{F}_\mu] = \operatorname*{esssup}_{\tau \in S_v} \mathbb{E}[Z_\tau|\mathcal{F}_\mu] \overset{S_v \subseteq S_\mu}{\leq} Y_\mu, \tag{4.10}$$

we see that $\{Y_t\}_{t\in[0,T]}$ is a supermartingale. The first equality in (4.10) is obtained as in (4.9). The second equality follows from the law of iterated conditional expectation. The third is a result of the following two inequalities. On the one hand from the definition of the essential supremum it holds that:

$$\lim_{n\to\infty} \mathbb{E}[Z_{\tau_n}|\mathcal{F}_\mu] \leq \operatorname*{esssup}_{\tau \in S_v} \mathbb{E}[Z_\tau|\mathcal{F}_\mu].$$

On the other hand $\mathbb{E}[Z_\rho|\mathcal{F}_v] \leq Y_v = \operatorname{esssup}_{\tau\in S_v} \mathbb{E}[Z_\tau|\mathcal{F}_v]$ for all $\rho \in S_v$ and $v \in S_\mu$ and, by taking the conditional expectation it follows that for all $\rho \in S_v$, $\mathbb{E}[Z_\rho|\mathcal{F}_\mu] \leq \mathbb{E}[Y_v|\mathcal{F}_\mu]$. This implies that

$$\operatorname*{esssup}_{\tau\in S_v} \mathbb{E}[Z_\tau|\mathcal{F}_\mu] \leq \mathbb{E}[Y_v|\mathcal{F}_\mu]. \tag{4.11}$$

Thus we see by comparing (4.11) with (4.10) that the third equality holds. The last inequality follows due to the fact that $S_v \subseteq S_\mu$. Hence, we see that $\{Y_t\}_{t\in[0,T]}$ is a supermartingale.

Let us define

$$S_v^* := \{\tau \in S_v : \tau > v \text{ a.s. on } \{\tau < T\}\}$$

and introduce $Y_v^* = \operatorname{esssup}_{\tau\in S_v^*} \mathbb{E}[Z_\tau|\mathcal{F}_v]$. Since $Y_v \geq Y_v^* \vee Z_v$ by the definition of the essential supremum, Theorem 4.1, and by taking the essential supremum above the following inequality

$$\mathbb{E}[Z_\rho|\mathcal{F}_v] = \mathbb{E}[Z_v 1_{\rho=v} + Z_\rho 1_{\rho>v}|\mathcal{F}_v] = Z_v 1_{\rho=v} + \mathbb{E}[Z_\rho 1_{\rho>v}|\mathcal{F}_v] \leq Z_v 1_{\rho=v} + Y_v^* 1_{\rho>v} \leq Y_v^* \vee Z_v,$$

it follows that

$$Y_v = Y_v^* \vee Z_v. \tag{4.12}$$

Define for every decreasing sequence $\{v_n\}_{n\geq 0}$ in S_v^* the respective stopping times $\{\tau_n\}_{n\geq 0}$ for a $\tau \in S_v$ by

$$\tau_n = \begin{cases} \tau \text{ if } \tau > v_n, \\ T \text{ if } \tau \leq v_n. \end{cases}$$

Moreover, since $\{Z_t\}_{t\in[0,T]}$ is in $\mathcal{S}_T^2$ there exists a sufficient large constant C such that $\{Z_t + C\}_{t\in[0,T]}$ is non negative. Denote $Z_t + C =: Z_t^C$ for $t \in [0,T]$. For the respective essential supremum $\{Y_t^{C^*}\}_{t\in[0,T]}$ and some $A \in \mathcal{F}_v$ the following holds:

$$\begin{aligned} \int_{A\cap\{v_n<T\}} Z_{\tau_n}^C dP - \int_{A\cap\{\tau\leq v_n<T\}} Z_T^C dP &= \int_{A\cap\{v_n<\tau<T\}} Z_\tau^C dP \\ &= \int_{A\cap\{v_n<\tau<T\}} \mathbb{E}[Z_\tau^C|\mathcal{F}_v] dP \leq \int_{A\cap\{v_n<\tau<T\}} Y_{v_n}^{C^*} dP \\ &\leq \int_{A\cap\{v<T\}} Y_{v_n}^{C^*} dP. \end{aligned}$$

Then for letting $n \to \infty$ it follows that the set $\{v_n < T\} \uparrow \{v < T\}$ and $\{\tau \leq v_n < T\} \downarrow \emptyset$. Hence, we have by the right continuity of $\{Z_t\}_{t\in[0,T]}$:

$$\int_{A\cap\{v<T\}} Z_\tau^C \, dP \leq \lim_{n\to\infty} \int_{A\cap\{v<T\}} Y_{v_n}^{C^*} dP,$$

and after subtracting $\int_{A\cap\{v<T\}} C \, dP$ from both sides we obtain:

$$\int_{A\cap\{v<T\}} Z_\tau \, dP \leq \lim_{n\to\infty} \int_{A\cap\{v<T\}} Y_{v_n}^* dP. \tag{4.13}$$

Furthermore for $\{v = T\}$ we see that,

$$\int_{A\cap\{v=T\}} Z_T \, dP \le \int_{A\cap\{v=T\}} Y^*_{v_n} dP. \tag{4.14}$$

By (4.14) and (4.13) we have $\int_A Z_\tau \, dP \le \lim_{n\to\infty} \int_A Y^*_{v_n} \, dP$ and thus

$$\int_A Y^*_v \, dP \le \lim_{n\to\infty} \int_A Y^*_{v_n} \, dP.$$

On the other hand from supermartingale property it follows that

$$\int_A Y^*_{v_n} \, dP = \int_A \mathbb{E}[Y^*_{v_n}|\mathcal{F}_v] \, dP \le \int_A Y^*_v \, dP. \tag{4.15}$$

As conclusion we obtain

$$\lim_{n\to\infty} \int_A Y^*_{v_n} \, dP = \int_A Y^*_v \, dP. \tag{4.16}$$

By the extension of Fatou's lemma for integrable lower bounds and the right continuity of $\{Z_t\}_{t\in[0,T]}$ we get

$$Y^*_v \ge \liminf_{n\to\infty} \mathbb{E}[Z_{v_n}|\mathcal{F}_v] \ge \mathbb{E}\left[\lim_{n\to\infty} Z_{v_n}\middle|\mathcal{F}_v\right] = Z_v. \tag{4.17}$$

It follows by (4.12) that

$$Y^*_v = Y_v \text{ a.s..}$$

Hence there exists a càdlàg version Y^0_v of Y_v, from now on denoted by Y_v.

Let $\{X_t\}_{t\in[0,T]}$ be a regular càdlàg supermartingale dominating $\{Z_t\}_{t\in[0,T]}$. Then for all $t \in [0,T]$ the following inequality holds

$$X_t \ge Z_t \quad \forall t \in [0,T].$$

Thus for every $\tau \in S_t$ get

$$X_t \ge \mathbb{E}[X_\tau|\mathcal{F}_t] \ge \mathbb{E}[Z_\tau|\mathcal{F}_t]$$

and by the definition of the essential supremum: $X_t \ge Y_t$, for all $t \in [0,T]$. Hence we have proofed that $\{Y_t\}_{t\in[0,T]}$ is the Snell envelope of $\{Z_t\}_{t\in[0,T]}$. □

Theorem 4.5 (Doob-Meyer decomposition supermartingale [Karatzas and Shreve, 2001, Theorem D.13])**.** *Let $\{Y_t\}_{t\in[0,T]}$ be a supermartingale of class $[D]$ (i.e. the set of random variables $\{Y_\tau,\ \tau \in S\}$ is uniformly integrable) the Doob Meyer decomposition implies that there exists a finite martingale $\{M_t\}_{t\in[0,T]}$ and an increasing process $\{A_t\}_{t\in[0,T]}$ such that*

$$Y_t = M_t - A_t, \quad t \in [0,T],$$

with $Y_0 = M_0$ and $A_0 = 0$.

Theorem 4.6 ([Cvitanic and Karatzas, 1996, A.10 and A. 11])**.** *Suppose that $\{Z_t\}_{t\in[0,T]}$ is upper semicontinuous (i.e. $\limsup_{n\to\infty} Z_{\tau_n} \leq Z_\tau$ for all $\tau \in S$ and $\tau_n \uparrow \tau$). Then, the Snell envelope $\{Y_t\}_{t\in[0,T]}$ of $\{Z_t\}_{t\in[0,T]}$ is a continuous and uniformly integrable process. Furthermore, for a given stopping time $v \in S$ the optimal strategy is given by:*

$$\tau_v = \inf\{s \geq v : Z_s = Y_s\} \wedge T \tag{4.18}$$

and we denote this strategy as Snell strategy.

Proof. We follow the proof as given in [Hamadène and Lepeltier, 2000, Lemma 1]. Since $|Y_t| \leq \mathbb{E}\left[\sup_{s\in[0,T]}|Z_s||\mathcal{F}_t\right]$, where $\mathbb{E}\left[\sup_{s\in[0,T]}|Z_s||\mathcal{F}_t\right]$ is a martingale, it follows by Doob's inequality that

$$\mathbb{E}\left[\sup_{s\in[t,T]}|Y_s|^2\right] \leq 4\mathbb{E}\left[\sup_{s\in[0,T]}|Z_s|^2\right] < \infty.$$

Moreover we see that $\{Y_\tau\}_{\tau\in S}$ is uniformly integrable. According to the Doob-Meyer decomposition, Theorem 4.5, have a martingale $\{M_t\}_{t\in[0,T]}$ and a nondecreasing process $\{A_t\}_{t\in[0,T]}$ such that $Y_t = M_t - A_t$ for all $t \in [0,T]$. Let $\{Y_t^-\}_{t\in[0,T]}$ be the left continuous version of $\{Y_t\}_{t\in[0,T]}$ and $\{\overline{Z_t}\}_{t\in[0,T]}$ defined by $\overline{Z}_\tau := \limsup_{n\to\infty} Z_{\tau_n}$ for all $\tau \in S$ and $\tau_n \uparrow \tau$. Hence all jumps of $\{A_t\}_{t\in[0,T]}$ are included in the set $\{Y^- = \overline{Z}\}$. For a stopping time $\tau \in S$, where the process $\{A_t\}_{t\in[0,T]}$ jumps, the following holds:

$$\mathbb{E}\left[Y_\tau^- 1_{\triangle A_\tau>0}\right] = \mathbb{E}\left[\overline{Z}_\tau 1_{\triangle A_\tau>0}\right] \leq \mathbb{E}\left[\mathbb{E}\left[Z_\tau|\mathcal{F}_{\tau-}\right]1_{\triangle A_\tau>0}\right] = \mathbb{E}\left[Z_\tau 1_{\triangle A_\tau>0}\right] \leq \mathbb{E}\left[Y_\tau 1_{\triangle A_\tau>0}\right],$$

where the first inequality is given by the upper semicontinuity and the second by the definition of the Snell envelope $\{Y_t\}_{t\in[0,T]}$. It follows that $\{A_t\}_{t\in[0,T]}$ and thus $\{Y_t\}_{t\in[0,T]}$ is continuous.

We now mimic the proofs of Proposition D.10 and Theorem D.12 in Karatzas and Shreve [2001] with our assumptions on $\{Z_t\}_{t\in[0,T]}$, i.e. $\{Z_t\}_{t\in[0,T]}$ is not non-negative and not continuous. To prove the existence of a snell strategy we define for $v \in S$ and $\alpha \in (0,1)$ the set

$$D^\alpha(v) := \inf\{t \in (v,T] : \alpha Y_t \geq Z_t\} \in S_v \tag{4.19}$$

and the set

$$D^*(v) := \lim_{\alpha\uparrow 1} D^\alpha(v).$$

Set (4.19) is right continuous by the right continuity of $\{Y_t\}_{t\in[0,T]}$ and $\{Z_t\}_{t\in[0,T]}$. Moreover it holds that:

$$\alpha Y_{D^\alpha(t)} \leq Z_{D^\alpha(t)}, \text{ for all } t \in [0,T]. \tag{4.20}$$

Furthermore since $t \leq D^\alpha(t)$, it follows that $Y_t \geq \mathbb{E}[Y_{D^\alpha(t)}|\mathcal{F}_t]$. Now define a non negative version of $\{Z_t\}_{t\in[0,T]}$ by $Z_t^C = Z_t + C$, where $C > 0$ is a sufficient large constant. Note that all inequalities given above are also valid for $\{Z_t^C\}_{t\in[0,T]}$ and the respective $\{Y_t^C\}_{t\in[0,T]}$. On the set $\{D^\alpha(t) = t\}$ we obtain that

$$\alpha Y_t^C + (1-\alpha)\mathbb{E}[Y_{D^\alpha(t)}^C|\mathcal{F}_t] = Y_t^C \geq Z_t^C$$

On the other hand on the set $\{D^\alpha(t) > t\}$ we get, due to the positivity of Z_t^C and hence the positivity of Y_t^C, that:

$$\alpha Y_t^C + (1-\alpha)\mathbb{E}[Y^C_{D^\alpha(t)}|\mathcal{F}_t] \geq \alpha Y_t^C \geq Z_t^C$$

Thus we get that $\mathbb{E}[Y^C_{D^\alpha(t)}|\mathcal{F}_t] \geq Z_t^C$ and together with (4.20) we have $\mathbb{E}[Y^C_{D^\alpha(t)}|\mathcal{F}_t] = Y_t^C$. By subtracting the constant C from both sides it follows that:

$$\mathbb{E}[Y_{D^\alpha(t)}|\mathcal{F}_t] = Y_t, \text{ for all } t \in [0,T]. \tag{4.21}$$

Combing (4.21) and (4.20) we obtain that.

$$Y_v = \mathbb{E}[Y_{D^\alpha(v)}|\mathcal{F}_t] \leq \frac{1}{\alpha}\mathbb{E}[Z_{D^\alpha(v)}|\mathcal{F}_t] \text{ a.s.}$$

Since $Z_{D^\alpha(v)} \leq \sup_{t\in[0,T]} Z_t$ a.s. , hence bounded and by the upper semicontinuity, we get

$$Y_v \leq \lim_{\alpha\uparrow 1}\mathbb{E}[Y_{D^\alpha(v)}|\mathcal{F}_t] \leq \mathbb{E}[Z_{D^*(v)}|\mathcal{F}_t] \text{ a.s.}$$

The reverse inequality is obtained by the supermartingale property of $\{Y_t\}_{t\in[0,T]}$, i.e.

$$\mathbb{E}[Z_{D^*(v)}|\mathcal{F}_v] = \mathbb{E}[Y_{D^*(v)}|\mathcal{F}_v] \leq Y_v.$$

As conclusion we obtain

$$\mathbb{E}[Z_{D^*(v)}|\mathcal{F}_v] = Y_v. \tag{4.22}$$

Thus $\mathbb{E}[Z_{D^*(v)}] = Y_v \geq \mathbb{E}[Y_{D^*(v)}]$, but since $\{Y_t\}_{t\in[0,T]}$ dominates $\{Z_t\}_{t\in[0,T]}$ it must be true that $Z_{D^*(v)} = Y_{D^*(v)}$. This implies:

$$D^*(v) \geq \inf\{t \in [v,T] : Y_t = Z_t\}.$$

On the other hand it follows from the definition (4.19), that

$$D^*(v) = \lim_{\alpha\uparrow 1} D^\alpha(v) \leq \inf\{t \in (v,T] : Y_t \geq Z_t\}.$$

Besides, on the set $\{Z_v = Y_v\}$ the right continuity of $\{Y_t\}_{t\in[0,T]}$ and $\{Z_t\}_{t\in[0,T]}$ implies that $D^\alpha(v) = v$ a.s. for all $\alpha \in (0,1)$ and thereby $D^*(v) = v$. Hence we conclude that:

$$D^*(v) = \inf\{t \in [v,T] : Y_t = Z_t\}$$

and by (4.22) we see that $D^*(v)$ is the optimal stopping time after $v \in S$. $\square$

Theorem 4.7 ([Djehiche et al., 2009, Proposition 2. iv]). *Let $\{Z_t^n\}_{t\in[0,T]}$ be a càdlàg and uniformly integrable for every stopping time $\tau \in S$ sequence of processes $n \geq 0$, such that $\{Z_t^n\}_{t\in[0,T]}$ converges increasingly and pointwisely to $\{Z_t\}_{t\in[0,T]}$. Then the Snell envelope $\{Y_t^n\}_{t\in[0,T]}$ of $\{Z_t^n\}_{t\in[0,T]}$ converges increasingly and point wise to $\{Y_t\}_{t\in[0,T]}$ the Snell envelope of $\{Z_t\}_{t\in[0,T]}$.*

Proof. Since $\{Z_t^n\}_{t\in[0,T]}$ is increasing in $n \geq 0$ and converges pointwise to $\{Z_t\}_{t\in[0,T]}$, it follows that $Z_t^n \leq Z_t$ a.s. for all $t \in [0,T]$ and $n \geq 0$. Hence $Y_t^n \leq Y_t$ for all $t \in [0,T]$ and every $n \geq 0$ and we have that

$$\lim_{n\to\infty} Y_t^n \leq Y_t, \quad t \in [0,T]. \tag{4.23}$$

The process $\{\lim_{n\to\infty} Y_t^n\}_{t\in[0,T]}$, as a limit of a non-decreasing càdlàg supermartingale sequence of class [D], also is a càdlàg supermartingale of class [D]. On the other hand since $Z_t^n \leq Y_t^n$ a.s. for all $t \in [0,T]$ and every $n \geq 0$, it follows that $\lim_{n\to\infty} Z_t^n = Z_t \leq \lim_{n\to\infty} Y_t^n$. Thus we see that:

$$Y_t \leq \lim_{n\to\infty} Y_t^n, \quad t \in [0,T]. \tag{4.24}$$

With (4.23) and (4.24) we conclude that $Y_t = \lim_{n\to\infty} Y_t^n$ for $t \in [0,T]$. □

4.3 Optimal switching via recursive optimal stopping

For the control problem in (3.11) we will now explain how we can find a solution. It is mainly based on two mathematical concepts, namely stochastic optimization and dynamic programming. We stay in the framework described in Section 3.3 and use the definitions and variables from that section. This includes the set of admissible strategies $\mathcal{U}(t)$ defined in Definition 3.1 as well as the use of an Itô process on $O \subseteq R_+$ as renewable energy processes $\{R_t\}_{t\in[0,T]}$, introduced in Section 4.1. Besides, we proved in (4.6) that the reward functions ψ_i (4.1) for $i \in \{-1,0,1\}$ and in (4.7) that $\sup_{i\in\{-1,0,1\}} \psi_i$ belongs to the set $\mathcal{M}_T^2$, where $\mathcal{M}_T^2$ was defined in (4.5). Furthermore we omit from now on for ease of notation the discounting terms, since they inherit no additional challenge in the solution finding.

For every strategy $u \in \mathcal{U}(t)$ being at time $t \in [0,T]$ in regime i, i.e. $u_t = i$, the renewable energy process $R_t = r$ and the charging level $L_t = l$ the expected gain or loss until time T, as given in (3.10), is in this setting given by:

$$J(t,R_t,l,i;u) = \mathbb{E}\left[\int_t^T \psi_{u_s}(s,R_s,L_s)\,ds - \sum_{\tau_k<T} C_{u_{\tau_k-},u_{\tau_k}} \,\middle|\, R_t = r, L_t = l, u_t = i\right]. \tag{4.25}$$

The function $\psi_i(t,R_t,L_t)$ is the reward function defined in (4.1) for the fixed demand D. In contrast to (3.10) the information up to time $t \in [0,T]$ can be reduced to $R_t = r$ since $\{R_t\}_{t\in[0,T]}$ is a Markov process.

A strategy $u \in \mathcal{U}(t)$ is called finite, if the agent makes only a finite number of decisions within $[t,T]$. This means for $u \in \mathcal{U}(t)$ the respective τ_k satisfies the condition $\mathbb{P}(\tau_k < T \text{ for all } k \geq 0) = 0$. Moreover, the optimal strategy lies in the set of finite strategies, denoted by $\mathcal{U}^f(t)$, as stated in Proposition 4.8.

Proposition 4.8 ([Djehiche et al., 2009, Proposition 1])**.** *The suprema over admissible strategies and finite admissible strategies coincide:*

$$\sup_{u\in\mathcal{U}(t)} J(t,r,l,i;u) = \sup_{u\in\mathcal{U}^f(t)} J(t,r,l,i;u), \tag{4.26}$$

It is worth to notice, that multiple switching at the same time, $\tau_k = \tau_{k+1}$ for $k \geq 0$, would in principle be possible, however by the subadditivity of $C_{i,j}$ as stated in Assumption 3.1 this would be suboptimal. Furthermore, it is worth to notice that by the construction of $\{u_s\}_{s\in[t,T]}$ via stopping times $\{\tau_k\}_{k\geq 0}$, it follows that $\{u_s\}_{s\in[t,T]}$ is adapted to $\mathbb{F}$. This implies, that the decisions of the agent only rely on the informations given by the renewable energy process $\{R_t\}_{t\in[0,T]}$.

The first idea leading to the solution is formulated by R. Bellman in the 1950'ies in the framework of Dynamic Programming and is known as the Bellman optimality principle.

Theorem 4.9 ([Bellman, 1954, Principle of optimality]). *Every optimal strategy to maximize* (4.25) *has the property that, whatever the initial regime, charging level and initial decisions are, the remaining decisions must constitute an optimal strategy with regard to the regime and the charging level resulting from the initial decisions.*

To apply this principle we restrict the total number of allowed switches to $K \in \mathbb{N}$. Accordingly, we define the set

$$\mathcal{U}^k(t) := \{u \in \mathcal{U}(t) : \tau_l = T \text{ for all } l \geq k\}$$

of all admissible strategies, as defined in Definition 3.1, with maximal $k \in \{1, \dots, K\}$ switches allowed and starting at time t. Now, we have on the one hand the classical control problem with at most K regime switches allowed in the setting (4.25), given by

$$\overline{J^K}(t,r,l,i) = \sup_{u\in U^K(t)} \mathbb{E}\left[\int_t^T \psi_{u_s}(s,R_s,L_s)\,ds - \sum_{0\leq k\leq K} C_{u_{\tau_k-},u_{\tau_k}} \,\middle|\, R_t = r, L_t = l, u_t = i\right], \tag{4.27}$$

which is the analogue to (3.11) with restricted number of switches and under the assumptions introduced in Section 4.1.

On the other hand, having the Bellman's optimality principle in mind we can formulate a recursive definition. By applying the principle we are left with finding the optimal solution for the next $K-1$ switches from the time the first switch occurred onwards and so forth. Hence, the problem reduces to finding the first stopping time under the condition that only one switch is allowed and then going bottom up in the number of allowed switches.

Since we stay in one regime until the next switch occurs, we can simplify the actual charging level opposed the dependence on the optimal strategy as given in (3.9) by a definition for a fixed regime $i \in \{-1,0,1\}$. This means that we just need to obtain the charging level at time s for staying in regime i from time t to time s and having charging level l at time t. Thus, we define the charging level function $\ell : [0,T]\times[0,T]\times[\mathsf{c}_{min},\mathsf{c}_{max}]\times\{-1,0,1\} \to [\mathsf{c}_{min},\mathsf{c}_{max}]$ by:

$$\ell(t,s,l,i) = l + \int_t^s (-i)\; a_i(m,\ell(t,m,l,i))dm, \tag{4.28}$$

where a_i is given in (3.4).

We define simpler recursive optimal stopping problems as stated in [Carmona and Ludkovski, 2010, Eq. 3.7]. Define $J^k(t,r,l,i)$ for $r \in \mathbb{R}_+$,$l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, $0 \le t \le T$ and $i \in \{-1,0,1\}$, by:

$$\begin{aligned} J^0(t,r,l,i) &:= \mathbb{E}\left[\int_t^T \psi_i(s,R_s,\ell(t,s,l,i))\,ds \;\middle|\; R_t = r \right], \\ J^k(t,r,l,i) &:= \sup_{\tau \in S_t} \mathbb{E}\left[\int_t^\tau \psi_i(s,R_s,\ell(t,s,l,i))\,ds + M^{k,i}(\tau,R_\tau,\ell(t,\tau,l,i)) \;\middle|\; R_t = r \right], \end{aligned} \tag{4.29}$$

for $k = 1,\dots,K$, where

$$M^{k,i}(t,r,l) := \max_{j \neq i} \left\{ -C_{i,j} + J^{k-1}(t,r,l,j) \right\} \tag{4.30}$$

is the optimal strategy, given the decision to switch away from regime $i \in \{-1,0,1\}$ at time $t \in [0,T]$ and in charging level $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$. Thus, $M^{k,i}(t,r,l)$ contains the recursive part of the definition and is the application of Bellman's principle in (4.29).

Now, to invoke stochastic control methods we have to show that we can rewrite (4.29) as:

$$J^k(t,r,l,i) = \operatorname*{esssup}_{\tau \in S_t} \mathbb{E}\left[\int_t^\tau \psi_i(s,R_s,\ell(t,s,l,i))\,ds + M^{k,i}(\tau,R_\tau,\ell(t,\tau,l,i)) \;\middle|\; R_t = r \right].$$

In order to do so we have to show that the regularity assumptions of Section 4.2 hold. For ease of notation we write

$$J_t^{k,i}(l) := J^k(t,R_t,l,i), \tag{4.31}$$

where the right hand side was introduced in (4.29). We are follow an idea given by Djehiche et al. [2009] and extend it in a way that it covers also the additional dependence on the charging level in our setting.

Theorem 4.10. *The processes $\{J_t^{k,i}(l)\}_{t\in[0,T]}$ are $\mathbb{F}$-adapted, continuous and in $\mathcal{S}_T^2$ for all $k \in \{0,\dots,K\}, l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ and $i \in \{-1,0,1\}$, .*

Proof. By induction we prove that for any regime $i \in \{-1,0,1\}$ and charging level $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ the theorem assumption holds. First, we shall prove that it holds for $k = 0$. We can rewrite the definition given in (4.31) as

$$J_t^{0,i}(l) = \mathbb{E}\left[\int_0^T \psi_i(s,R_s,\ell(t,s,l,i))\,ds \;\middle|\; \mathcal{F}_t \right] - \int_0^t \psi_i(s,R_s,\ell(t,s,l,i))\,ds, \tag{4.32}$$

where the first summand is a martingale with respect to a Brownian motion and the second is a continuous process. Hence, $J_t^{0,i}(l)$ is continuous. For all $t \in [0,T]$ the following inequality holds:

$$\begin{aligned} |J_t^{0,i}(l)| &\le \mathbb{E}\left[\int_t^T |\psi_i(s,R_s,\ell(t,s,l,i))|\,ds \;\middle|\; \mathcal{F}_t \right] \\ &\le \mathbb{E}\left[\int_0^T |\psi_i(s,R_s,\ell(t,s,l,i))|\,ds \;\middle|\; \mathcal{F}_t \right] < \infty \end{aligned}$$

since $\mathbb{E}[\int_0^T |\psi_i(s, R_s, \ell(t, s, l, i))|^2 \, ds] < \infty$. Together with Doob's inequality we obtain that $\{J_t^{0,i}(l)\}_{t\in[0,T]}$ is in $\mathcal{S}_T^2$. Now, suppose that the statement holds for k and show that it is also holds for $k+1$. The process given by

$$\left\{ \int_0^t \psi_i\left(s, R_s, \ell(t, s, l, i)\right) \, ds + M^{k+1,i}(t, R_t, l) \right\}_{t\in[0,T]}, \tag{4.33}$$

where $M^{k+1,i}(t, R_t, l)$ is defined as in (4.30) is right continuous and in $\mathcal{S}_T^2$ by (4.7). Furthermore, by assumption 3.1 on the regime switching costs $C_{i,j}$, it holds that

$$\max_{j\neq i, j\in\{-1,0,1\}} \left(-C_{i,j} + J_t^{k,i}(l) \right) |_{t\uparrow T} \leq -\varepsilon \leq 0.$$

Thus, since $J_T^{k,i}(l) = 0$, (4.33) is upper semi continuous. Hence, by Theorem 4.6 the respective Snell envelope, given by

$$\left\{ J_t^{k+1,i}(l) + \int_0^t \psi_i(s, R_s, \ell(t, s, l, i)) \, ds \right\}_{t\in[0,T]}, \tag{4.34}$$

is continuous and in $\mathcal{S}_T^2$. By subtracting the continuous term $\int_0^t \psi_i(s, R_s, \ell(t, s, l, i)) \, ds$ we obtain that $\{J_t^{k+1,i}(l)\}_{t\in[0,T]}$ is continuous and in $\mathcal{S}_T^2$. □

Since we are now in the setting as introduced in Theorem 4.6, we can also apply the Snell strategy derived in this theorem. The following Theorem 4.11 capitalizes the Snell strategy. The theorem is based on the Verification Theorem [Djehiche et al., 2009, Theorem 1] and transfers its results to the restricted switching case and the charging level dependences.

Theorem 4.11 (Optimal Strategy). *For a given $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ and $i \in \{-1, 0, 1\}$ the optimal strategy is given by the sequence of $\mathbb{F}$-stopping times $\{\tau_k\}_{k\in\{0,\dots,K\}}$ and indicators $\{\xi_k\}_{k\in\{0,\dots,K\}}$,defined by:*

$$\begin{aligned} \tau_0 &= 0, \\ \tau_{(K+1)-k} &= \inf \Big\{ s \geq \tau_{K-k} : \ J_t^{k,\xi_{K-k}}(\ell(\tau_{K-k}, s, l_{K-k}, \xi_{K-k})) \\ &\qquad = M^{k,\xi_{K-k}}(s, R_s, \ell(\tau_{K-k}, s, l_{K-k}, \xi_{K-k})) \Big\} \wedge T, \end{aligned} \tag{4.35}$$

where the charging levels $\{l_k\}_{k\in\{0,\dots,K\}}$ are determined by:

$$\begin{aligned} l_0 &= l, \\ l_{(K+1)-k} &= \ell(\tau_{K-k}, \tau_{(K+1)-k}, l_{K-k}, \xi_{K-k}), \end{aligned} \tag{4.36}$$

and the corresponding regimes are given by:

$$\begin{aligned} \xi_0 &= i, \\ \xi_{(K+1)-k} &= \operatorname*{argmax}_{j\neq \xi_{K-k}, j\in\{-1,0,1\}} \left(-C_{\xi_{K-k},j} + J^{k-1,j}_{\tau_{(K+1)-k}}(l_{(K+1)-k}) \right). \end{aligned} \tag{4.37}$$

Proof. By Theorem 4.10 we know that the processes $\{J_t^{k,i}(l)\}_{t\in[0,T]}$ are $\mathbb{F}$-adapted continuous processes in $\mathcal{S}_T^2$ for all $k = 0,\dots,K$, $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ and $i \in \{-1,0,1\}$. Moreover, we can apply the definition of the Snell strategy given in this theorem. Begin with K switches allowed and in regime $i \in \{-1,0,1\}$, i.e. $\xi_0 = i$, at time 0, i.e. $\tau_0 = 0$, and with charging level $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, i.e. $l_0 = l$. Going back to the results (4.33) and (4.34), the Snell envelope of the process

$$\left\{\int_0^t \psi_i\left(s, R_s, \ell(t,s,\ell(0,t,l,i),i)\right) ds + M^{K,i}(t, R_t, \ell(0,t,l,i))\right\}_{t\in[0,T]}$$

is given by

$$\left\{J_t^{K,i}(\ell(0,t,l,i)) + \int_0^t \psi_i(s, R_s, \ell(t,s,\ell(0,t,l,i),i))\, ds\right\}_{t\in[0,T]}.$$

The corresponding Snell strategy as given in (4.18) is:

$$\begin{aligned}\tau_1 = \inf\Bigg\{t \geq 0 : &\int_0^t \psi_i\left(s, R_s, \ell(t,s,\ell(0,t,l,i),i)\right) ds + M^{K,i}(t, R_t, \ell(0,t,l,i)) \\ &= J_t^{K,i}(\ell(0,t,l,i)) + \int_0^t \psi_i(s, R_s, \ell(t,s,\ell(0,t,l,i),i))\, ds\Bigg\} \wedge T \qquad (4.38)\\ = \inf\Big\{t \geq \tau_0 : &J_t^{K,i}(\ell(0,t,l,i)) = M^{K,i}(t, R_t, \ell(0,t,l,i))\Big\} \wedge T.\end{aligned}$$

Accordingly, by (4.28) the charging level at τ_1 is given by

$$l_1 = \ell(\tau_0, \tau_1, l_0, \xi_0),$$

where $\tau_0 = 0$, $l_0 = l$ and $\xi_0 = i$ are given and the stopping time τ_1 is determined by (4.38). The respective regime indicator ξ_1 is obtained by:

$$\xi_1 = \underset{j\neq\xi_0, j\in\{-1,0,1\}}{\operatorname{argmax}} \left(-C_{\xi_0,j} + J_{\tau_1}^{K-1,j}(l_1)\right),$$

Analogously the remaining stopping times $\{\tau_k\}_{k\in\{2,\dots,K\}}$, regime indicators $\{\xi_k\}_{k\in\{2,\dots,K\}}$ and charging levels $\{l_k\}_{k\in\{2,\dots,K\}}$ can be obtained. □

To apply the recursive definition given in (4.29) to solve the control problem in (4.27) we have to verify the equality of both formulations.

Theorem 4.12 ([Ludkovski, 2005, Theorem 1]). *For $r \in O$, $i \in \{-1,0,1\}$ and $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ the maximal expected reward $J^k(t,r,l,i)$ (4.29) is equal to the value function for the optimal switching problem $\overline{J^k}(t,r,l,i)$ (4.27) with at most $k = 0,\dots,K$ switches allowed.*

Proof. We show this by induction for starting at time $t \in [0,T]$ in any regime $i \in \{-1,0,1\}$ and charging level $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$. Since for $K = 0$ no decision is necessary we start with

$K = 1$ and obtain:

$$\begin{aligned}
\overline{J^1}(t,r,l,i) &= \sup_{u \in U^1(t)} \mathbb{E}\left[\int_t^T \psi_{u_s}(s, R_s, \ell(t,s,l,i))\,ds - C_{i,u_{\tau_1}} \,\middle|\, R_t = r\right] \\
&= \operatorname*{esssup}_{\tau \in S_t, j \neq i} \mathbb{E}\left[\int_t^\tau \psi_i(s, R_s, \ell(t,s,l,i))ds - C_{i,j}\right. \\
&\qquad\qquad \left. + \int_\tau^T \psi_j(s, R_s, \ell(t,s,l,i))ds \,\middle|\, R_t = r\right] \\
&= J^1(t,r,l,i).
\end{aligned} \tag{4.39}$$

Thus, with the results of Theorem 4.11 we get that the Snell strategy

$$\tau_1 = \inf\{s \geq t : J^1(s, R_s, \ell(t,s,l,i), i) = M^{i,1}(s, R_s, \ell(t,s,l,i))\} \wedge T \tag{4.40}$$

together with

$$\xi_1 = \operatorname*{argmax}_{j \neq i, j \in \{-1,0,1\}} \{-C_{i,j} + J^0(\tau, R_\tau, \ell(t,\tau,l,i), j)\}$$

forms the optimal strategy in $\mathcal{U}^1(t)$.

Let the assumption hold for $K-1$, hence

$$\mathbb{E}\left[\overline{J^{K-1}}(t,r,l,i)\middle|\mathcal{F}_t\right]$$

is equal to

$$\mathbb{E}\left[J^{K-1}(t,r,l,i)\middle|\mathcal{F}_t\right]$$

for all $r \in O$,$l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, $t \in [0,T]$ and $i \in \{-1,0,1\}$. Furthermore,

$$\begin{aligned}
J^K(t,r,l,i) &\geq \mathbb{E}\left[\int_t^\tau \psi_i(s, R_s, \ell(t,s,l,i))\,ds - M^{i,K}(\tau, R_\tau, \ell(t,\tau,l,i)) \,\middle|\, R_t = r\right] \\
&\geq \mathbb{E}\left[\int_t^\tau \psi_i(s, R_s, \ell(t,s,l,i))\,ds - C_{i,\xi} + J^{K-1}(\xi, R_\tau, \ell(t,\tau,l,i)) \,\middle|\, R_t = r\right] \\
&= \mathbb{E}\left[\int_t^\tau \psi_i(s, R_s, \ell(t,s,l,i))\,ds - C_{i,\xi} + \overline{J^{K-1}}(\xi, R_\tau, \ell(t,\tau,l,i)) \,\middle|\, R_t = r\right]
\end{aligned} \tag{4.41}$$

for all extensions $\tau \in S_t$ and $\xi \in \{-1,0,1\}$ of the optimal strategy $u \in \mathcal{U}^{K-1}(\tau)$ with $u_\tau = \xi$. Since $\tau \in S_t$ and $\xi \in \{-1,0,1\}$ is arbitrary it holds that $J^k(t,r,l,i) \geq \overline{J^K}(t,r,l,i)$.

By Theorem 4.11 we know, that by setting

$$\tau_1 = \inf\{s \geq t : J^K(s, R_s, \ell(t,s,l,i), i) = M^{i,K}(s, R_s, \ell(t,s,l,i))\} \wedge T$$

and

$$\xi_1 = \operatorname*{argmax}_{j \neq i, j \in \{-1,0,1\}} \{-C_{i,j} + J^{K-1}(\tau_1, R_{\tau_1}, \ell(t,\tau_1,l,i), j)\}$$

we get the optimal regime switching time and the respective regime after t. Together with the induction assumption we see that:

$$\begin{aligned} J^K(t,r,l,i) &= \mathbb{E}\left[\int_t^{\tau_1} \psi_i(s,R_s,L_s)\,ds - C_{i,\xi_1} + J^{k-1}(\xi,R_\tau,L_{\tau_1}) \,\middle|\, R_t = r\right] \\ &= \mathbb{E}\left[\int_t^{\tau_1} \psi_i(s,R_s,L_s)\,ds - C_{i,\xi_1} + \overline{J^{k-1}}(\xi_1,R_{\tau_1},L_{\tau_1}) \,\middle|\, R_t = r\right]. \end{aligned} \tag{4.42}$$

Hence, we have by (4.41) and (4.42) that $J^k = \overline{J^k}$. $\square$

Additionally, we obtain the optimal strategy $u \in \mathcal{U}^k(t)$ by extension of the optimal strategy $u^* \in \mathcal{U}^{k-1}(\tau)$ with initial regime ξ_1 in the following manner:

$$\begin{aligned} \tau_0 = t \ \text{ and } \quad & \tau_k = \tau^*_{k-1} \text{ for } k \in \{1,\dots,K\}, \\ \xi_0 = i \ \text{ and } \quad & \xi_k = \xi^*_{k-1} \text{ for } k \in \{1,\dots,K\}. \end{aligned}$$

Hence, with the induction starting at (4.40) and we see that the Snell strategy and the optimal strategy $u \in \mathcal{U}^k(t)$ coincides. Besides, under the assumption that $\{R_t\}_{t\in[0,T]}$ is Markovian it follows that $\tau^*_k = \inf\{s \geq \tau^*_{k-1} : J^{k-1}(s,R_s,L_s) = M^{i,k-1}(s,R_s,L_s)\} \wedge T$ is Markovian as well.

Now we defined everything on the basis of $k \in \{0,\dots,K\}$ switches allowed but we would like to come back to the general definition of $J(t,r,l,i)$ with an arbitrary number of switches. To come back to the general setting, we need the following theorem.

Theorem 4.13 ([Ludkovski, 2005, Theorem 2]). *Let $J(t,r,l,i)$ be defined as in (3.10) and $J^k(t,r,l,i)$ as in (4.29). Then $J^K(t,r,l,i)$ converges for $K \to \infty$ a.s. pointwise to $J(t,r,l,i)$, for all other parameters fixed.*

The proof can be obtained by mimicking the proof of Theorem 4.7 and noting the fact, that more allowed switches are advantageous. This is obvious since $\mathcal{U}^K(t) \subset \mathcal{U}^{K+1}(t)$, thus $J^K(t,r,l,i)$ is monotonously rising in $K \geq 0$, if we keep all other parameters fixed.

Chapter 5

Numerical Solution

In this chapter we transfer the findings from the previous chapters to a computational solution. To this end we switch from continuous time in discrete time.

Let M denote the number of time intervals with lag $\triangle t = \frac{T}{M}$, then the respective set of possible switching times is given by $\{m\triangle t\}_{m=0,\dots,M}$. Accordingly, define the discrete equivalent to (4.2) by

$$S^{\triangle} := \{\tau \in S : \tau = m\triangle t \text{ a } m = 0, \dots, M\} \tag{5.1}$$

and for $v \in S^{\triangle}$ equivalent to (4.3) by

$$S_v^{\triangle} := \{\tau \in S^{\triangle} : \ \tau \geq v\}. \tag{5.2}$$

The set of admissible strategies (3.1) in discrete time is

$$\mathcal{U}^{\triangle}(t) := \{u \in \mathcal{U}(t) : \ \tau_k = m\triangle t \text{ for } m \in \{0, \dots, M\}\}. \tag{5.3}$$

Furthermore, is the renewable energy process $\{R_m\}_{m\in\{0,\dots,M\}}$ in discrete time given by the continuous time renewable energy process $\{R_t\}_{t\in[0,T]}$ at the respective discrete time grid points, i.e. $R_m = R_{m\triangle t}$.

Denote by $\psi_i^{\triangle} : \{0, \dots, M\} \times O \times [\mathsf{c}_{min}, \mathsf{c}_{max}] \to \mathbb{R}$ the reward function in discrete time of regime $i \in \{-1, 0, 1\}$ from $m\triangle t$ to time $(m+1)\triangle t$. Defined by:

$$\psi_i^{\triangle}(m, r, l) = \psi_i(m\triangle t, r, l) \ \triangle t, \tag{5.4}$$

where ψ_i is the reward function as in (4.1) from the continuous time setting.

Since in the algorithms we need the charging level one time step ahead being in regime $i \in \{-1, 0, 1\}$ for having the current charging level $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, we define the function $\ell_{\rightarrow} : \{0, \dots, M\} \times \{-1, 0, 1\} \times [\mathsf{c}_{min}, \mathsf{c}_{max}] \to [\mathsf{c}_{min}, \mathsf{c}_{max}]$ by:

$$\ell_{\rightarrow}(m, l, i) = \begin{cases} l - i\ a_i(m\triangle t, l)\triangle t & \text{if } \mathsf{c}_{min} \leq l - i\ a_i(m\triangle t, l)\triangle t \leq \mathsf{c}_{max}, \\ \mathsf{c}_{max} & \text{if } l - i\ a_i(m\triangle t, l)\triangle t > \mathsf{c}_{max}, \\ \mathsf{c}_{min} & \text{if } l - i\ a_i(m\triangle t, l)\triangle t < \mathsf{c}_{min}, \end{cases} \tag{5.5}$$

where $a_i(t, l)$ is the level change function given by (3.4). As well we need the current charging level knowing the charging level $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ one time step ahead and the regime $i \in \{-1, 0, 1\}$ for the interval $[m\triangle t, (m+1)\triangle t)$. Therefore, we define the function $\ell_{\leftarrow} : \{0, \ldots, M\} \times \{-1, 0, 1\} \times [\mathsf{c}_{min}, \mathsf{c}_{max}] \to [\mathsf{c}_{min}, \mathsf{c}_{max}]$ to go backwards in time by:

$$\ell_{\leftarrow}(m+1, l, i) = \begin{cases} l + i\, a_i((m+1)\triangle t, l)\triangle t & \text{if } \mathsf{c}_{min} \le l + i\, a_i((m+1)\triangle t, l)\triangle t \le \mathsf{c}_{max}, \\ \mathsf{c}_{max} & \text{if } l + i\, a_i((m+1)\triangle t, l)\triangle t > \mathsf{c}_{max}, \\ \mathsf{c}_{min} & \text{if } l + i\, a_i((m+1)\triangle t, l)\triangle t < \mathsf{c}_{min}. \end{cases} \tag{5.6}$$

These are the discrete time formulations of the charging level function given in (4.28). By applying (5.5) we obtain the respective charging level process $\{L_m\}_{m\in\{0,\ldots,M\}}$ for a strategy $u \in \mathcal{U}^{\triangle}(0)$ with initial charging level $L_0 = l,\ l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, by:

$$L_m = \ell_{\rightarrow}(m-1, L_{m-1}, u_{m-1}), \quad m \in \{1, \ldots, M\}, \tag{5.7}$$

analogously to (3.9).

The respective control problem to (3.10) in discrete time is given by:

$$J(m, r, l, i) = \sup_{u\in\mathcal{U}^{\triangle}(m)} \mathbb{E}\left[\sum_{n=m}^{M} \psi^{\triangle}_{u_m}(n, R_n, L_n) - \sum_{\tau_k<T} C_{u_{\tau_{k-1}}, u_{\tau_k}} \,\middle|\, R_m = r, L_m = l, u_m = i \right]. \tag{5.8}$$

5.1 Snell envelope in discrete time

This section is based on Section VI in Neveu [1975]. We simplify the setting by assuming that $M < \infty$ and that the reward process $\{Z_m\}_{m\in\{0,\ldots,M\}}$ satisfies

$$\mathbb{E}[\sup_{m\in\{0,\ldots,M\}} |Z_m|] < \infty.$$

This assumption reflects our problem setting and simplifies the definitions. Throughout this section let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtrated probability space, where $\mathbb{F} := \{\mathcal{F}_m, m \in \{0, \ldots, M\}\}$ is the completed filtration generated by $\{Z_m\}_{m\in\{0,\ldots,M\}}$.

Definition 5.1 (Snell Envelope in discrete time [Neveu, 1975, Prop. VI-1-2])**.** *The Snell envelope $\{Y_m\}_{m\in\{0,\ldots,M\}}$ of $\{Z_m\}_{m\in\{0,\ldots,M\}}$ is the smallest supermartingale dominating $\{Z_m\}_{m\in\{0,\ldots,M\}}$.*

Since $M < \infty$ the solution of the $\{Z_m\}_{m\in\{0,\ldots,M\}}$-backward equations given by:

$$\Gamma_m = \begin{cases} Z_M & \text{if } m = M, \\ Z_m \vee \mathbb{E}[\Gamma_{m+1}|\mathcal{F}_m] & \text{else }, \end{cases} \tag{5.9}$$

is unique and takes for $m < M$ the value of Z_m on the set $\mathcal{G} := \{\omega \in \Omega : Z_m(\omega) \ge \mathbb{E}[\Gamma_{m+1}|\mathcal{F}_m](\omega)\}$ and $\mathbb{E}[\Gamma_{m+1}|\mathcal{F}_m]$ else. This sets give also the definition of the pathwise maximum '$\vee$' operator. Furthermore $\{\Gamma_m\}_{m\in\{0,\ldots,M\}}$ is the Snell envelope of $\{Z_m\}_{m\in\{0,\ldots,M\}}$. The solution of the $\{Z_m\}_{m\in\{0,\ldots,M\}}$-backward equation is adapted to $\mathbb{F}$,

$$\mathbb{E}[|\Gamma_m|] \le \mathbb{E}[\sup_{m\in\{0,\ldots,M\}} |Z_m|] < \infty$$

and $\mathbb{E}[\Gamma_{m+1}|\mathcal{F}_m] \leq \Gamma_m$. Hence, it is a supermartingale and by definition smaller than all other supermartingales dominating $\{Z_m\}_{m\in\{0,\dots,M\}}$. All in all we have that the solution to the $\{Z_m\}_{m\in\{0,\dots,M\}}$-backward equations is the Snell envelope of $\{Z_m\}_{m\in\{0,\dots,M\}}$, i.e. $\Gamma_m = Y_m$ for $m \in \{0,\dots,M\}$. A more detailed proof can be found in [Neveu, 1975, Proof of Prop. VI-1-2] and follows the same steps as the proof of Theorem 4.4 for the continuous Snell envelope.

Definition 5.2 (Snell strategy [Neveu, 1975, Prop. VI-1-3])**.** *Given the Snell envelope $\{Y_m\}_{m\in\{0,\dots,M\}}$ of $\{Z_m\}_{m\in\{0,\dots,M\}}$, the associated Snell strategy for all $m \in \{0,\dots,M\}$ is given by:*

$$\tau_m := \inf\{s \in \{m,\dots,M\} : Z_s = Y_s\} \wedge M. \tag{5.10}$$

This is the discrete time equivalent of the Snell strategy as stated in Theorem 4.6 for the continuous time case. Likewise, as for the Snell strategy there exists as well a discrete time equivalent of the Doob-Meyer decomposition Theorem 4.5.

Theorem 5.3 (Doob-Meyer decomposition [Neveu, 1975, Prop. VIII-1-1])**.** *Every finite supermartingale $\{Y_m\}_{m\in\{0,\dots,M\}}$ can be written as the difference between a finite positive martingale $\{M_m\}_{n\in\{0,\dots,M\}}$ and an increasing process $\{A_m\}_{m\in\{0,\dots,M\}}$,*

$$Y_m = M_m - A_m\,,\ m \in \{0,\dots,M\}, \tag{5.11}$$

where $M_0 = Y_0$ and $A_0 = 0$. Moreover the decomposition is unique.

5.2 Basic concept for the numerical solution

Within this section we derive the basic concept to solve the optimal control problem (5.8) with numerical methods. This solution is based on three results:

(i) The maximal reward in regime $i \in \{-1,0,1\}$ at time $m\triangle t$ with charging level $l \in [\mathsf{c}_{min},\mathsf{c}_{max}]$ and contribution from renewable sources $r \in O$ is given by:

$$\max_{j\in\{-1,0,1\}} \Big(-C_{i,j} + \psi_j^{\triangle}(m,r,l) + \mathbb{E}[J(m+1,R_{m+1},\ell_{\leftarrow}(m,l,j),j)|R_m = r]\Big)\,, \tag{5.12}$$

where $C_{i,i} := 0$ and $C_{i,j}$ for $j \in \{-1,0,1\}\setminus\{i\}$ the switching costs as defined in Assumption 3.1. The respective optimal regime is given by:

$$\operatorname*{argmax}_{j\in\{-1,0,1\}} \Big(-C_{i,j} + \psi_j^{\triangle}(m,r,l) + \mathbb{E}[J(m+1,R_{m+1},\ell_{\leftarrow}(m,l,j),j)|R_m = r]\Big)\,. \tag{5.13}$$

If (5.13) is i we stay and else we switch to the regime given by (5.13).

(ii) The reward $J(M,r,l,i)$ is known for all $r \in O$,$l \in [\mathsf{c}_{min},\mathsf{c}_{max}]$ and $i \in \{-1,0,1\}$ by the penalty function given in (3.12).

(iii) The conditional expectation for $j \in \{-1,0,1\}$

$$\mathbb{E}[J(m+1,R_{m+1},\ell_{\leftarrow}(m,l,j),j)|R_m = r] \tag{5.14}$$

can be approximated by regression of a set function of r against the known values $J(m+1,R_{m+1},l_1,i)$, for $l_1 \in [\mathsf{c}_{min},\mathsf{c}_{max}]$.

This forms a solution of (5.8), since (5.12) and (5.13) in (i) can be obtained if we know the conditional expectation (5.14). Due to the fact that the value (5.14) is not viable in most of the problem, we need the approximation introduced in (iii). But the approximation of (5.14) is based on the knowledge of $J(m+1, R_{m+1}, l, i)$, thus we need (ii) at time T and the value given in (5.12) for $(m+1) \in \{1, \dots, M-1\}$. Hence, we start with $J(M, R_M, l, i)$ as stated in (ii), then we approximate $\mathbb{E}[J(M, R_M, \ell_{\leftarrow}(M, l, j), j) | R_{M-1} = r]$ and consequently obtain by (5.12) $J(M-1, R_{M-1}, l, i)$, and so forth. A visualization of the consecutive steps is shown in Figure 5.1.

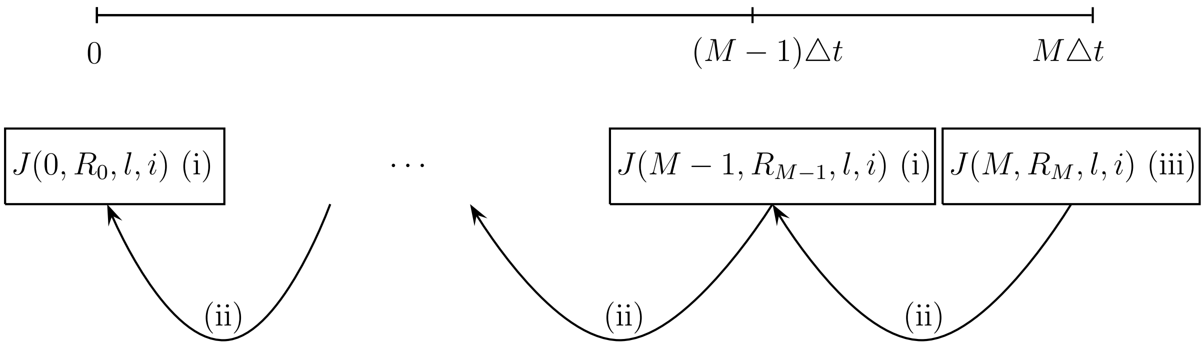

Figure 5.1: Visualization of the basic idea.

First of all we show that (5.12) holds. Therefore we have to bear in mind, that we are now dealing with discrete Snell envelopes rather than with continuous ones, so we can calculated them in standard manner via backward equations (5.9). Together with Assumption 3.1 for the switching costs the control problem in discrete time simplifies as follows.

Theorem 5.4 ([Carmona and Ludkovski, 2010, Eq. (12)])**.** *The control problem* (5.8) *is equivalent to*

$$\begin{aligned} J(m, r, l, i) = \max_{j \neq i, j \in \{-1,0,1\}} & \Big(-C_{i,j} + \psi_j^{\triangle}(m, r, l) + \mathbb{E}[J(m+1, R_{m+1}, \ell_{\rightarrow}(m, l, j), j) | R_m = r] \Big) \\ & \vee \Big(\psi_i^{\triangle}(m, r, l) + \mathbb{E}\left[J(m+1, R_{m+1}, \ell_{\rightarrow}(l, i), i) \mid R_m = r \right] \Big), \end{aligned} \tag{5.15}$$

where '$\vee$' is the pathwise maximum operator as introduced in (5.9)*.*

Proof. Due to the fact that the proof can not be found in Carmona and Ludkovski [2010] and the importance for the construction, we give a detailed proof of this equation. Analogously to the derivation of the continuous Snell envelope, see proof of Theorem 4.10, we find that

$$\left\{ J(m, r, l, i) + \sum_{n=0}^{m-1} \psi_i^{\triangle}(n, R_n, L_n^i) \right\}_{m \in \{0, \dots, M\}},$$

where the charging level $\{L_n^i\}_{n=0,\dots,m-1}$ is given by $L_n^i = \ell_{\leftarrow}(n+1, L_{n+1}^i, i)$ for and $L_m^i = l$, is the Snell envelope of

$$\left\{ M^i(m,r,l) + \sum_{n=0}^{m-1} \psi_i^{\triangle}(n, R_n, L_n^i) \right\}_{m\in\{0,\dots,M\}},$$

where $M^i(m,r,l) = \max_{j\neq i, j\in\{-1,0,1\}}(-C_{i,j} + J(m,r,l,j))$. Besides we define that

$$\sum_{n=0}^{-1} \psi_i^{\triangle}(n, R_n, L_n^i) := 0.$$

From the backward equation (5.9) it follows that:

$$\begin{aligned} J(m,r,l,i) + \sum_{n=0}^{m-1} \psi_i^{\triangle}(n, R_n, L_n^i) = &\left(M^i(m,r,l) + \sum_{n=0}^{m-1} \psi_i^{\triangle}(n, R_n, L_n^i) \right) \vee \\ &\mathbb{E}\left[J(m+1, R_{m+1}, l, i) + \sum_{n=0}^{m} \psi_i^{\triangle}(n, R_n, L_n^i) \,\middle|\, \mathcal{F}_m \right] \end{aligned}$$

and by subtracting $\sum_{n=0}^{m-1} \psi_i^{\triangle}(n, R_n, L_n^i)$ we obtain:

$$J(m,r,l,i) = M^i(m,r,l) \vee \mathbb{E}\left[\left(\psi_i^{\triangle}(m,r,l) + J(m+1, R_{m+1}, \ell_{\rightarrow}(m,l,i), i) \right) \,\middle|\, \mathcal{F}_m \right]. \tag{5.16}$$

Assume now, that for some $\omega \in \Omega$ the following holds:

$$J(m,r,l,i)(\omega) = -C_{i,j} + J(m,r,l,j)(\omega)\ , \quad j \neq i, j \in \{-1,0,1\},$$

and

$$J(m,r,l,j)(\omega) = -C_{j,k} + J(m,r,l,k)(\omega)\ , \quad k \neq j, k \in \{-1,0,1\}.$$

Then, we obtain by the definition of the switching costs that:

$$J(m,r,l,i)(\omega) = -C_{i,j} - C_{j,k} + J(m,r,l,k)(\omega) < \begin{cases} J(m,r,l,i)(\omega) & \text{for } k = i, \\ -C_{i,k} + J(m,r,l,k)(\omega) & \text{else .} \end{cases}$$

This is a contradiction to (5.16). Hence on the set $\{J(m,r,l,i) = -C_{i,j} + J(m,r,l,j)\}$ it holds that:

$$\begin{aligned} J(m,r,l,j) &= \mathbb{E}[\psi_j^{\triangle}(m,r,l) + J(m+1, R_{m+1}, \ell_{\rightarrow}(m,l,j), j) | R_m = r] \\ &= \psi_j^{\triangle}(m,r,l) + \mathbb{E}[J(m+1, R_{m+1}, \ell_{\rightarrow}(m,l,j), j) | R_m = r], \end{aligned}$$

for $j \in \{-1,0,1\} \setminus \{i\}$. Thus, on the set $\{J(m,r,l,i) = M^i(m,r,l)\}$ we have that:

$$\begin{aligned} M^i(m,r,l) &= \max_{j\in\{-1,0,1\}\setminus\{i\}} \mathbb{E}[-C_{i,j} + \psi_j^{\triangle}(m,r,l) + J(m+1, R_{m+1}, \ell_{\rightarrow}(m,l,j), j) | R_m = r] \\ &= \max_{j\in\{-1,0,1\}\setminus\{i\}} \left(-C_{i,j} + \psi_j^{\triangle}(m,r,l) + \mathbb{E}[J(m+1, R_{m+1}, \ell_{\rightarrow}(m,l,j), j) | R_m = r] \right). \end{aligned}$$

With the measurability of $\psi_i^{\triangle}(m,r,l)$, we see that (5.15) holds. □

Consequently to find a solution to the problem given in (5.15) it is sufficient to evaluate the conditional expectations. It is worth to notice, that the conditional expectations

$$\mathbb{E}\left[J(m+1, R_{m+1}, \ell_{\rightarrow}(l,i), i) \mid R_m = r\right] \tag{5.17}$$

exist for every $m \in \{0, \dots, M\}$, $i \in \{-1, 0, 1\}$ and $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$, since

$$|J(m+1, R_{m+1}, \ell_{\rightarrow}(l,i), i)| \leq \sum_{m+1}^{M} \max_{i \in \{-1,0,1\}} |\mathsf{a_i} \bigtriangleup t \ \mathsf{p_N}| < \infty,$$

by the same arguments as in (4.7). Furthermore the conditional expectation lies in the Hilbert space $L^2(\mathbb{P})$. A conditional expectation in a Hilbert space can be viewed as a projection

$$y \mapsto \Phi(y) = \mathbb{E}[X|Y = y].$$

Therefore we achieve a feasible projection representing the conditional expectation (5.17) denoted by:

$$\Phi_m^{i,l} : r \rightarrow \mathbb{E}\left[J(m+1, R_{m+1}, \ell_{\rightarrow}(l,i), i) \mid R_m = r\right]. \tag{5.18}$$

In most of the models it will not be possible to calculate this quantity explicitly. Consequently, the problem could be solved by finding an adequate approximation for $\Phi_m^{i,l}$, as claimed in (iii). The idea we focus on for this approximation was given by Longstaff and Schwartz [2001]. It is based on an approximation of $\Phi_m^{i,l}$ via a truncated Hilbert basis of $L^2(\mathbb{P})$. Let $N_{\mathcal{B}}$ denote the number of functions in the truncated Hilbert basis and

$$\mathcal{B} = \{B_j \text{ function } j, j = 1, \dots, N_{\mathcal{B}}\} \tag{5.19}$$

the set of basis functions. How to select those functions and the number of functions $N_{\mathcal{B}}$, which are sufficient to calculate the conditional expectation, is discussed in Section 5.5.

We obtain an approximation of the mapping $\Phi_m^{i,l}$ by a linear combination of functions in $\mathcal{B}$, i.e.

$$\Phi_m^{i,l}(r) \approx \sum_{n=1}^{N_{\mathcal{B}}} \alpha_n B_n(r), \tag{5.20}$$

where α_n are $\mathbb{R}$-valued coefficients and $B_n(r)$ is the function value of the n-th function in $\mathcal{B}$. The coefficients $\{\alpha_n\}_{n \in \{1,\dots,N_{\mathcal{B}}\}}$ for a given set of functions $\mathcal{B}$ can by derived by:

$$\alpha = \underset{\alpha \in \mathbb{R}^{N_{\mathcal{B}}}}{\operatorname{argmin}} \left\| \Phi_m^{i,l}(\cdot) - \sum_{n=1}^{N_{\mathcal{B}}} \alpha_n B_n(\cdot) \right\|_{\mathbb{P}}, \tag{5.21}$$

where $\|\Phi_m^{i,l}(\cdot)\|_{\mathbb{P}} = \left(\int_{r \in O} (\Phi_m^{i,l}(r))^2 \mathbb{P}(dr)\right)^{1/2}$.

Since the exact computation of the projection (5.21) is in general not viable, we introduce a sample based projection [Tsitsiklis and van Roy, 2000, Chapter C]. Therefore, we generate a Monte Carlo sample of the driving renewable energy process $\{R_m\}_{m \in \{0,\dots,M\}}$ starting at $R_0 = r_0 \in O$. Let N be the number of the Monte Carlo sample paths and denote by

$$\{r_m^n\}_{m=0}^{M} \tag{5.22}$$

the realization of the n-th path, $n \in \{0, \dots, N\}$.

Let us assume that we know $\{J(m+1, r^n_{m+1}, l, i)\}_{n\in\{0,\dots,N\}}$ for all $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$. The assumption is reasonable since we know $\{J(M, r^n_M, l, i)\}_{n\in\{0,\dots,N\}}$, see (ii), for all $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ and iteratively by going backwards we know the reward one time step ahead. The expectation at time m of the rewards $\{J(m+1, r^n_{m+1}, l, i)\}_{n\in\{0,\dots,N\}}$ at time $m+1$ can obtained by finding the linear combination of basis functions $\{B(r^n_m)\}_{n\in\{0,\dots,N\}}$ of the realization of the driving processes at time m which minimize the distance in the Euclidean norm. Estimates $\{\hat{\alpha}_n\}_{n\in\{1,\dots,N_\mathcal{B}\}}$ for the coefficients $\{\alpha_n\}_{n\in\{1,\dots,N_\mathcal{B}\}}$ can now be derived by:

$$\hat{\alpha} = \inf_{\hat{\alpha}\in\mathbb{R}^{N_\mathcal{B}}} \left\| \begin{pmatrix} J(m+1, r^1_{m+1}, l, i) \\ \vdots \\ J(m+1, r^N_{m+1}, l, i) \end{pmatrix} - \begin{pmatrix} \sum_{n=1}^{N_\mathcal{B}} \hat{\alpha}_n B_n(r^1_m) \\ \vdots \\ \sum_{n=1}^{N_\mathcal{B}} \hat{\alpha}_n B_n(r^N_m) \end{pmatrix} \right\|_2 , \tag{5.23}$$

where $\|\cdot\|_2$ is the Euclidean norm. This approach is numerically efficient, since we use the Monte Carlo sample paths for deriving the conditional expectation as well as for generating the pathwise reward values. For large samples the obtained coefficients $\{\hat{\alpha}_n\}_{n\in\{1,\dots,N_\mathcal{B}\}}$ become close to the exact coefficients in (5.21)[Tsitsiklis and van Roy, 2000, Chapter C].

The assumption that we know $\{J(m+1, r^n_{m+1}, l, i)\}_{n\in\{0,\dots,N\}}$ for all l in the subset of the real numbers $[\mathsf{c}_{min}, \mathsf{c}_{max}]$ neglects that this knowledge is numerical not feasible.
An intuitive idea is to find values for $\{J(m+1, r^n_{m+1}, l, i)\}_{n\in\{0,\dots,N\}}$ for all $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ by interpolation. Therefore construct a grid within the feasible charging levels, $\mathcal{L} = \{c_0 = \mathsf{c}_{min}, c_1, \dots, c_G = \mathsf{c}_{max}\}$, and interpolate between those grid points. This is the idea behind the 'Mixed Interpolation Tsitsiklis-van Roy ' (MITvR) scheme introduced in Section 5.3, formulated in the work of Ludkovski [2005].
But the MITvR scheme resembles a slow lattice scheme in the charging level variable $l \in [\mathsf{c}_{min}, \mathsf{c}_{max}]$ and is hence not numerically efficient (Carmona and Ludkovski [2010]). Accordingly we try to circumvent this lattice scheme in the charging level, by using the 'Bivariate Least Squares Monte Carlo' (BLSM) scheme invented by Ludkovski [2005], which is described Section 5.4.

5.3 Mixed Interpolation Tsitsiklis-van Roy (MITvR)

The MITvR scheme is based on the intuitive idea to pass the continuous charging level interval $[\mathsf{c}_{min}, \mathsf{c}_{max}]$ into the discrete set $\mathcal{L} = \{c_0 = \mathsf{c}_{min}, c_1, \dots, c_G = \mathsf{c}_{max}\}$. It is called Tsitsiklis-van Roy scheme, since we use the approximated values for the next step as established in Tsitsiklis and van Roy [2000] in comparison to the realized values as done in Longstaff and Schwartz [2001].

Let G be the number of equidistant intervals within $[\mathsf{c}_{min}, \mathsf{c}_{max}]$, hence the grid points are given by:

$$c_g = \mathsf{c}_{min} + g \triangle g \ , g \in \{0, \dots, G\},$$

where $\triangle g = (\mathsf{c}_{max} - \mathsf{c}_{min})/G$.

To make the regime decision at time $m\triangle t$ and charging levels $\{c_g\}_{g\in\{0,\dots,G\}}$ we need an approximation of the following conditional expectation:

$$\mathbb{E}\left[J(m+1, R_{m+1}, \ell_{\rightarrow}(m, c_g, i), i) \mid R_m = r\right]$$

for every $i \in \{-1,0,1\}$ and the respective charging level $\ell_{\rightarrow}(m,c_g,i)$ at $(m+1)\triangle t$ given by (5.5). Since we do not know the value $J(m+1,r^n_{m+1},\ell_{\rightarrow}(m,c_g,i),i)$ for the charging level $\ell_{\rightarrow}(m,c_g,i)$ we interpolate them by the known values $\{J(m+1,r^n_{m+1},c_g,i)\}_{g\in\{0,\dots,G\}}$ for each path $n \in \{0,\dots,N\}$. Then apply the method pointed out in Section 5.2. Obtain the coefficients $\{\hat{\alpha}_n\}_{n\in\{1,\dots,N_{\mathcal{B}}\}}$ as in (5.23) by replacing

$$\{J(m+1,r^n_{m+1},\ell_{\rightarrow}(m,c_g,i),i)\}_{n\in\{0,\dots,N\}}$$

with the interpolated values

$$\{\hat{J}(m+1,r^n_{m+1},\ell_{\rightarrow}(m,c_g,i),i)\}_{n\in\{0,\dots,N\}}.$$

For each $g \in \{0,\dots,G\}$ we get the approximation of the function $\Phi_m^{i,\ell_{\rightarrow}(c_g,i)}$ by:

$$\hat{\Phi}_m^{i,\ell_{\rightarrow}(c_g,i)}(r) = \sum_{n=1}^{N_{\mathcal{B}}} \hat{\alpha}_n B_n(r), \tag{5.24}$$

as given in (5.20). A visualization of this consecutive steps is shown in Figure 5.2.

Let us now assume at time $m\triangle t$ and path $n \in \{0,\dots,N\}$ that we are in regime $i \in \{-1,0,1\}$. Define the function $\hat{v} : \{0,\dots,M\} \times O \times \mathcal{L} \times \{-1,0,1\} \rightarrow \mathbb{R}$ by:

$$\hat{v}(m,r^n_m,c_g,j) = \begin{cases} \hat{\Phi}_m^{j,\ell_{\rightarrow}(m,c_g,j)}(r^n_m) + \psi_j^{\triangle}(m,r,c_g) - C_{i,j} & \text{if } j \neq i, \\ \hat{\Phi}_m^{i,\ell_{\rightarrow}(m,c_g,i)}(r^n_m) + \psi_i^{\triangle}(m,r,c_g) & \text{if } i = j, \end{cases} \tag{5.25}$$

where $\hat{\Phi}_m^{j,\ell_{\rightarrow}(m,c_g,j)}$ is given by (5.24), $\psi_i^{\triangle}$ is given by (5.4) and $C_{i,j}$ are the switching costs from regime i to regime j. The respective reward value is then given by:

$$\hat{J}(m,r^n_m,c_g,i) = \max_{j\in\{-1,0,1\}} \hat{v}(m,r^n_m,c_g,j).$$

Moreover, the optimal regime can be evaluated using the function $\hat{\xi} : \{0,\dots,M\} \times O \times [\mathsf{c}_{min},\mathsf{c}_{max}] \times \{-1,0,1\} \rightarrow \{-1,0,1\}$, defined by:

$$\hat{\xi}(m,r^n_m,g,i) = \operatorname*{argmax}_{j\in\{-1,0,1\}} \hat{v}(m,r^n_m,c_g,j). \tag{5.26}$$

The function $\hat{\xi}$ gives the optimal regime in the time interval from $m\triangle t$ until $(m+1)\triangle t$ for the renewable energy contribution r^n_m, coming from regime i and have the charging level c_g at time $m\triangle t$. The resulting optimal regimes are the computational equivalents to the optimal regime $(\xi_k)_{k\geq 0}$ in (3.7). The respective switching times $(\tau_k)_{k\geq 0}$ in (3.7) are those $m \in \{0,\dots,M\}$ for which $\hat{\xi}(m,r,g,i) \neq i$, but this knowledge is not required in the computational solution. Based on the fact, that the switching regions for each time $m\triangle t$, $m \in \{0,\dots,M\}$ and for all grid point $g \in \{0,\dots,G\}$ given by $(r^n_m,\hat{\xi}(m,r^n_m,g,i))_{n\in\{0,\dots,N\}}$ determines the optimal strategy uniquely.

Starting at time $T = M\triangle t$ in paths $n \in \{0,\dots,N\}$ with the value:

$$\hat{J}(M,r^n_M,c_g,i) = V(r^n_M,c_g,i) \quad \text{for } g \in \{0,\dots,G\},$$

where V is the penalty function, for example (3.12), and successively applying the steps, described above, we obtain the optimal expected value at time 0, $\hat{J}(0,r_0,L_0,i)$.

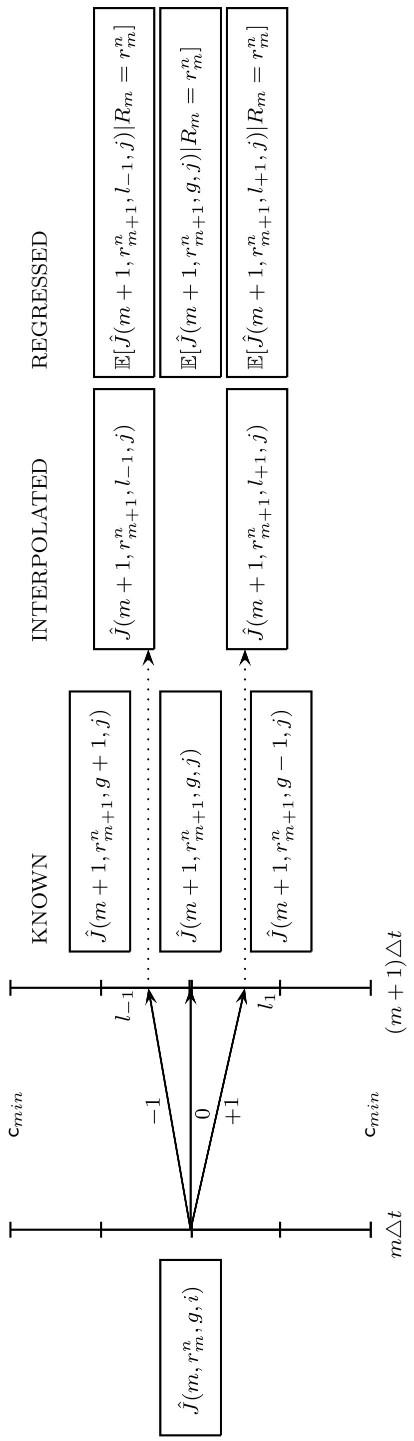

Figure 5.2: Visualization of one time step $m\triangle t$ in the MiTvR scheme. Starting at a given charging level c_g, $g \in \{0, \dots, G\}$, and regime $i \in \{-1, 0, 1\}$. Then go one time step ahead for every regime $j \in \{-1, 0, 1\}$. For the resulting charging levels l_{-1} and l_{+1} interpolate the reward given the reward at the grid points. Finally, regress the functions of $\mathcal{B}$ against this interpolated values to obtain the conditional expectation.

5.3.1 Algorithm

We stepwise present the algorithm introduced in Section 5.3:

(i) Select the algorithm parameters number of time grids M, N and G.

(ii) Generate sample paths $n \in \{0, \dots, N\}$ of the renewable energy process, $\{r_m^n\}_{m=0}^M$, with $r_0^n = r_0$.

(iii) Initialize the value function $\hat{J}(M, r_M^n, c_g, i)$ with a given penalty function, for example (3.12), for all $g \in \{0, \dots, G\}$, $n \in \{0, \dots, N\}$ and $i \in \{-1, 0, 1\}$.

(iv) Moving backwards in time $m = (M-1), \dots, 1$, repeat:

(a) For each $g \in \{0, \dots, G\}$:

i. Set $\ell_g^i = \ell_{\rightarrow}(m, c_g, i)$ for all $i \in \{-1, 0, 1\}$, where $\ell_{\rightarrow}$ is given by (5.5).

ii. Compute $\hat{J}(m+1, r_m^n, \ell_g^i, i)$ by interpolation and calculate the approximated conditional expectations $\hat{\Phi}_m^{i,\ell_g^i}(r)$, as given in (5.24) for every $i \in \{-1, 0, 1\}$.

iii. For every regime $i \in \{-1, 0, 1\}$ do:

A. Evaluate the expected rewards for being in regime $j \in \{-1, 0, 1\}$ after $m\triangle t$ $\hat{v}(m, r_m^n, c_g, j)$, as given in (5.25) for every $j \in \{-1, 0, 1\}$ and $n \in \{0, \dots, N\}$.

B. Set the maximal reward with

$$\hat{J}(m, r_m^n, g, i) = \max_{j \in \{-1,0,1\}} (\hat{v}(m, r_m^n, c_g, j))$$

for every $n \in \{0, \dots, N\}$.

C. Obtain the optimal regime by

$$\hat{\xi}(m, r_m^n, g, i) = \operatorname*{argmax}_{j \in \{-1,0,1\}} \hat{v}(m, r_m^n, c_g, j)$$

for all $n \in \{0, \dots, N\}$.

iv. End the regime loop

(b) End the loop for g

(v) End the loop for m

(vi) Interpolate $\hat{J}(1, r_m^n, \ell_{\rightarrow}(0, L_0, i), i)$ and set for $i \in \{-1, 0, 1\}$ and $n \in \{0, \dots, N\}$ $\overline{\hat{J}}(1, \ell_{\rightarrow}(0, L_0, i), i) = \frac{1}{N} \sum_{n=1}^N \hat{J}(1, r_m^n, \ell_{\rightarrow}(0, L_0, i), i)$. The maximal reward is then given by:

$$\begin{aligned}\hat{J}(0, r, L_0, 0) = \max \Big(& \overline{\hat{J}}(1, \ell_{\rightarrow}(0, L_0, 0), 0) + \psi_0^{\triangle}(0, r, L_0), \\ & \overline{\hat{J}}(1, \ell_{\rightarrow}(0, L_0, -1), -1) + \psi_{-1}^{\triangle}(0, r, L_0) - C_{0,-1}, \\ & \overline{\hat{J}}(1, \ell_{\rightarrow}(0, L_0, 1), 1) + \psi_1^{\triangle}(0, r, L_0) - C_{0,1} \Big).\end{aligned}$$

5.4 Bivariate Least Squares Monte Carlo Method

The Bivariate Least Squares Monte Carlo Method (BLSM) circumvent the discretization in the space of feasible charging levels. Since the BLSM scheme is described in two different ways, we split this Section into two parts. In the first subsection we describe the BLSM scheme with random mesh. This means that we draw a random mesh of a random mesh of inventory levels and renewable energy contributions, i.e. $\{r_m^n, l_m^n\}_{m\in\{0,\dots,M\}}$ for each path $n \in \{0,\dots,N\}$, is drawn, as pointed out in Carmona and Ludkovski [2007]. The last subsection describes the path dependent BLSM, as formulated in Carmona and Ludkovski [2010]. In this scheme the charging levels are propagated backward from in time from a random terminal charging level. This scheme is the more intuitive then the first on, but not stable under different guessing schemes of optimal regime to propagate the charging level backward in time. Therefore we concentrate on the BLSM scheme with random mesh.

5.4.1 BLSM with random mesh

The BLSM with random mesh works as follows. We draw, additional to the renewable energy paths $\{r_m^n\}_{m\in\{0,\dots,M\}}$, a charging level l_m^n uniformly distributed in $[\mathsf{c}_{min}, \mathsf{c}_{max}]$ for all $m \in \{0,\dots,M\}$ and $n \in \{0,\dots,N\}$. With these charging levels we evaluate the pathwise values of (5.15) by:

$$\begin{aligned} J(m, r_m^n, l_m^n, i) =& M^i(m, r_m^n, l_m^n) \vee \\ & \Big(\psi_i^{\triangle}(r_m^n, l_m^n) + \mathbb{E}\left[J(m+1, R_{m+1}, \ell_{\rightarrow}(l_m^n, i), i) \mid R_m = r_m^n\right]\Big), \end{aligned} \tag{5.27}$$

where $\ell_{\rightarrow}$ is given by (5.5) and

$$M^i(m,r,l) = \max_{j\neq i, j\in\{-1,0,1\}} (-C_{i,j} + \psi_j^{\triangle}(m,r,l) + \mathbb{E}[J(m+1, R_{m+1}, \ell_{\rightarrow}(m,l,j), j) | R_m = r]).$$

In this pathwise formulation '$\vee$' in just the standard maximum operator.

Using the function $\ell_{\rightarrow}(m, l_m^n, j)$ in (5.5) we obtain the charging level at time $(m+1)\triangle t$ resulting from l_m^n at time $m\triangle t$ and being in regime $j \in \{-1,0,1\}$. Hence, it ensures that the comparison in (5.27) is done on the same basis. It is necessary to have an equivalent basis for the inventory level, since the switching decision is not only depending on the current renewable energy process value r_m^n but also on the charging level $\ell_{\rightarrow}(m, l_m^n, j)$. The future return depends on the energy already charged to the battery, based on the fact that the charged energy can be materialized. This implies that the charging level l_m^n at time $m\triangle t$ has an significant influence on the respective reward $J(m+1, r_{m+1}^n, \ell_{\rightarrow}(m, l_m^n, j), j)$, $j \in \{-1,0,1\}$, up to time T. The value $J(m+1, r_{m+1}^n, \ell_{\rightarrow}(m, l_m^n, j), j)$ is unknown, since it is drawn randomly.

We are left with finding a solution towards the conditional expectation, but this time also the values $J(m+1, r_{m+1}^n, \ell_{\rightarrow}(m, l_m^n, j), j)$, $j \in \{-1,0,1\}$, are unknown. Recall, the idea of approximating the conditional expectation by a truncated Hilbert basis introduced in Section 5.2 and enhance this idea to work also with the unknown values

$$J(m+1, r_{m+1}^n, \ell_{\rightarrow}(m, l_m^n, j), j),\ j \in \{-1,0,1\},$$

for the respective charging level $\ell_{\rightarrow}(m, l_m^n, j)$ in the paths $n \in \{0, \dots, N\}$. Therefore, we follow an idea given by Carmona and Ludkovski [2010] and do a bivariate regression. A similar approach in the more general multidimensional case was also established in Tsitsiklis and van Roy [2000]. To characterize the bivariate regression in this context we carry out the same procedure as in Section 5.2.

The conditional expectation is again well defined as argued in Section 5.2. We choose $N_{\overline{\mathcal{B}}}$ bivariate linear independent functions $\overline{B}_n(x, y)$, which can be tensor products of univariate basis functions (see Carmona and Ludkovski [2007]) or other bivariate linear independent functions. Denote the set of this functions by

$$\overline{\mathcal{B}} = \{\overline{B}_j \text{ bivariate function } j, j = 1, \dots, N_{\overline{\mathcal{B}}}\}.$$

The function selection in the bivariate case is also discussed in Section 5.5. We approximate $\Phi_m^i : (r, l) \rightarrow \mathbb{E}[J(t_2, R_{m+1}, l, i) | R_m = r]$ analogously to (5.20) by:

$$\Phi_m^i(r, l) \approx \sum_{n=1}^{N_{\overline{\mathcal{B}}}} \alpha_n \overline{B}_n(r, l), \tag{5.28}$$

where α_n are $\mathbb{R}$-valued coefficients and $\overline{B}_n(r, l)$ is the function value for r and l of the n-th basis function in $\overline{\mathcal{B}}$. This function $\hat{\Phi}_m^i(r, l)$ (5.28) is the bivariate mirror of the function $\hat{\Phi}_m^{i,l}(r)$ (5.20). The estimates for the coefficients $\{\alpha_n\}_{n \in \{1, \dots, N_{\mathcal{B}}\}}$ of the projection on $\overline{\mathcal{B}}$ can be obtained by empirical regression against the known values $J(m+1, r_{m+1}^n, l_{m+1}^n, i)$,

$$\overline{\alpha} = \underset{\overline{\alpha} \in \mathbb{R}^{N_{\overline{\mathcal{B}}}}}{\operatorname{argmin}} \left\| \begin{pmatrix} J(m+1, r_{m+1}^1, l_{m+1}^1, i) \\ \vdots \\ J(m+1, r_{m+1}^N, l_{m+1}^N, i) \end{pmatrix} - \begin{pmatrix} \sum_{n=1}^{N_{\overline{\mathcal{B}}}} \overline{\alpha}_n \overline{B}_n(r_m^1, l_{m+1}^1) \\ \vdots \\ \sum_{n=1}^{N_{\overline{\mathcal{B}}}} \overline{\alpha}_n \overline{B}_n(r_m^N, l_{m+1}^N) \end{pmatrix} \right\|_2,$$

similar to (5.23). It is worth to notice, that we regress against a set of functions of the current renewable energy contribution $\{r_m^n\}_{n \in \{0, \dots, N\}}$ and the future charging level $\{l_{m+1}^n\}_{n \in \{0, \dots, N\}}$. We define $\overline{\Phi}_m^i(r, l)$ as the empirical version of (5.28) by:

$$\overline{\Phi}_m^i(r, l) = \sum_{n=1}^{N_{\overline{\mathcal{B}}}} \overline{\alpha}_n \overline{B}_n(r, l). \tag{5.29}$$

To turn the assumption, that we know $J(m+1, r_{m+1}^n, l_{m+1}^n, i)$ at time $m\triangle t$, to the truth, we have to initialize the starting point for the backward fashioned calculation. For every regime $i \in \{-1, 0, 1\}$ and path $n \in \{0, \dots, N\}$ set:

$$\overline{J}(M, r_M^n, l_M^n, i) = V(r_M^n, l_M^n, i)$$

at expire time $T = M\triangle t$, where $V(r, l, i)$ is the penalty function (3.12) reflecting the lending contract specifications.

Now going back in time and set $m = (M-1), \dots, 0$. For given m regress for all $i \in \{-1, 0, 1\}$ the known values $\overline{J}(m+1, r_{m+1}^n, l_{m+1}^n, i)$ for all paths $n \in \{0, \dots, N\}$ one time step ahead, against $\{\overline{B}_j(r_m^n, l_{m+1}^n)\}_{j=1}^{N_{\overline{\mathcal{B}}}}$.

We achieve the respective charging levels at time $(m+1)\triangle t$ given the level l_m^n at time $m\triangle t$ for every regime $j \in \{-1,0,1\}$ by $\ell_{\rightarrow}(m, l_m^n, j)$, see (5.5). By the step $\ell_{\rightarrow}(m, l_m^n, j)$, $j \in \{-1,0,1\}$, we obtain the charging level one time step ahead. Thereby we are in the situation for which obtained the approximated conditional expectation in (5.29). Thus, we can solve the optimal regime and the maximal reward for the charging level l_m^n and the fixed current regime $i \in \{-1,0,1\}$ at time $m\triangle t$. We define the function:

$$\overline{v}(m, r_m^n, l_m^n, j) = \begin{cases} \overline{\Phi}_m^j(r_m^n, \ell_{\rightarrow}(m, l_m^n, j)) + \psi_j^{\triangle}(m, r_m^n, l_m^n) - C_{i,j} & \text{if } j \neq i, \\ \overline{\Phi}_m^i(r_m^n, \ell^{\rightarrow}(m, l_m^n, i)) + \psi_i^{\triangle}(m, r_m^n, l_m^n) & \text{if } i = j, \end{cases} \tag{5.30}$$

where $j \in \{-1,0,1\}$. The function $\overline{\xi} : \{0,\dots,M\} \times O \times [\mathsf{c}_{min}, \mathsf{c}_{max}] \times \{-1,0,1\} \rightarrow \{-1,0,1\}$ defined by:

$$\overline{\xi}(m, r_m^n, l_m^n, i) = \underset{j\in\{-1,0,1\}}{\operatorname{argmax}} \left(\overline{v}(m, r_m^n, l_m^n, j)\right), \tag{5.31}$$

derives the optimal regime for the time period $[m\triangle t, (m+1)\triangle t)$ given the charging level l_m^n and renewable energy contribution r_m^n at time $m\triangle t$ and regime i before $m\triangle t$. This $\overline{\xi}(m, r_m^n, l_m^n, i)$ is the discrete time pathwise and charging level dependent equivalent to $(\xi_k)_{k\geq 0}$ in (3.7). Besides, this yields also the approximate reward value

$$\overline{J}(m, r_m^n, l_m^n, i) = \max_{j\in\{-1,0,1\}} (\overline{v}(m, r_m^n, l_m^n, j)). \tag{5.32}$$

The schemes described above to obtain the value of $\overline{J}(0, r_0, L_0, 1)$ is computational efficient since in comparison to the MiTvR scheme, Section 5.3, for each time $m \in \{0,\dots,M\}$ just one regression is performed instead of $G+1$.

5.4.2 Algorithm

In this section we give a scripted summary of the algorithm introduced in Section 5.4:

(i) Select the algorithm parameters M and N and set $\triangle t = \frac{T}{M}$.

(ii) Generate N paths of the driving renewable energy process $\{r_m^n\}_{m=0}^M$ as well as random charging levels $\{l_m^n\}_{m=0}^M$ for each path $n \in \{0,\dots,N\}$, sampled independently and uniformly distributed in $[\mathsf{c}_{min}, \mathsf{c}_{max}]$.

(iii) Moving backwards in time for $m = (M-1),\dots,0$ repeat:

   (a) Compute the approximated conditional expectations $\overline{\Phi}_m^j(r,l)$ as described in (5.28) and apply this mapping for $(r_m^n, \ell_{\rightarrow}(m, l_m^n, j))$, where the charging level $\ell_{\rightarrow}(m, l_m^n, j)$, (5.5), to obtain (5.30) for every $j \in \{-1,0,1\}$.

   (b) For every regime $i \in \{-1,0,1\}$ do

      i. Obtain $\overline{v}(m, r_m^n, l_m^n, j)$ in (5.30)

      ii. Evaluate the optimal regime $\overline{\xi}(m, r_m^n, l_m^n, i)$ in (5.31) and the maximal reward $\overline{J}(m, r_m^n, l_m^n, i)$ in (5.32), where $\overline{v}$ is given by (iii a) .

(c) End regime loop i

(iv) End the loop for m

(v) Interpolate $\overline{J}(0, r_0^n, l_0^n, i)$ for the initial charging level L_0 for all $i \in \{-1, 0, 1\}$ and denoted the interpolation by $\hat{\overline{J}}(1, r, L_0, i)$. Get the maximal reward for starting in regime 0 by:

$$\overline{J}(0, r, L_0, 0) = \max\left(\hat{\overline{J}}(1, r, L_0, 0), \hat{\overline{J}}(1, r, L_0, -1) - C_{0,-1}, \hat{\overline{J}}(1, r, L_0, i) - C_{0,1}\right)$$

5.4.3 Path dependent BLSM

In this subsection we give a brief stepwise overview of the pathwise scheme. For this scheme the charging level at time $m\triangle t$ is dependent on the regime i and on the path n, hence denoted by $l_m^n(i)$.

(i) Draw a terminal charging level $l_M^n(i)$ for each path $n \in \{0, \dots, N\}$ and regime $i \in \{-1, 0, 1\}$, sampled independently and uniformly distributed in $[\mathsf{c}_{min}, \mathsf{c}_{max}]$.

(ii) Guess on basis of r_m^n and the current regime i the optimal regime $\hat{\imath}$ at time $m\triangle t$ and set the charging level $l_m^n(i) = \ell_{\leftarrow}(m+1, l_m^n(\hat{\imath}), \hat{\imath})$, where $\ell_{\leftarrow}$ is given in (5.6).

(iii) Perform the same steps for the regression as in 5.4.1 to obtain (5.30).

(iv) In the cases that the guessed regime $\hat{\imath} = \overline{\xi}(m, r_m^n, l_m^n, i)$ set

$$\overline{J}(m, r_m^n, l_m^n(i), i) = \begin{cases} \overline{J}(m+1, r_{m+1}^n, l_{m+1}^n(j), j) + \psi_j^{\triangle}(m, r_m^n, l_m^n(i)) - C_{i,j} & \text{if } \hat{\imath} \neq i \\ \overline{J}(m+1, r_{m+1}^n, l_{m+1}^n(i), i) + \psi_i^{\triangle}(m, r_m^n, l_m^n(i)) & \text{if } i = \hat{\imath} \end{cases} \tag{5.33}$$

else obtain the value as in (5.32).

The reward value obtained in (iv) gives, beside the intuitive design, a second argument for this scheme, since the value $\max_{j\in\{-1,0,1\}}(\overline{v}(m, r_m^n, l_m^n(i), j))$ as the approximation of $\overline{J}(m, r_m^n, l_m^n(i), i)$ is potentially biased [Longstaff and Schwartz, 2001, p. 124 f]. Hence, it is advantageous to calculate $\overline{J}(m, r_m^n, l_m^n(i), i)$ looking ahead properly, given the decision made by the first evaluation instead of passing the biased result of (5.30) on to the next iteration. Thereby, we exploit the truth of the looking ahead in the algorithm of Longstaff and Schwartz for the case that the guess was good, i.e. $\overline{\xi}(m, r_m^n, l_m^n(i), i) = \hat{\imath}$, and hazard the consequences of the Tsitsiklis van Roy scheme to obtain a solution for bad guesses, i.e. $\overline{\xi}(m, r_m^n, l_m^n(i), i) \neq \hat{\imath}$. Hence, the Longstaff and Schwartz part helps us to reduce error accumulation [Carmona and Ludkovski, 2010, p. 366]. Figure 5.3 gives an overview of the successive steps in the scheme described above.

This scheme is computational efficient by the same arguments as in 5.4.1, since we use the same scheme to obtain the estimate for the conditional expectation. But the forecasting of the charging level by a regime guess involves a not in generally controllable source of numerical instability.

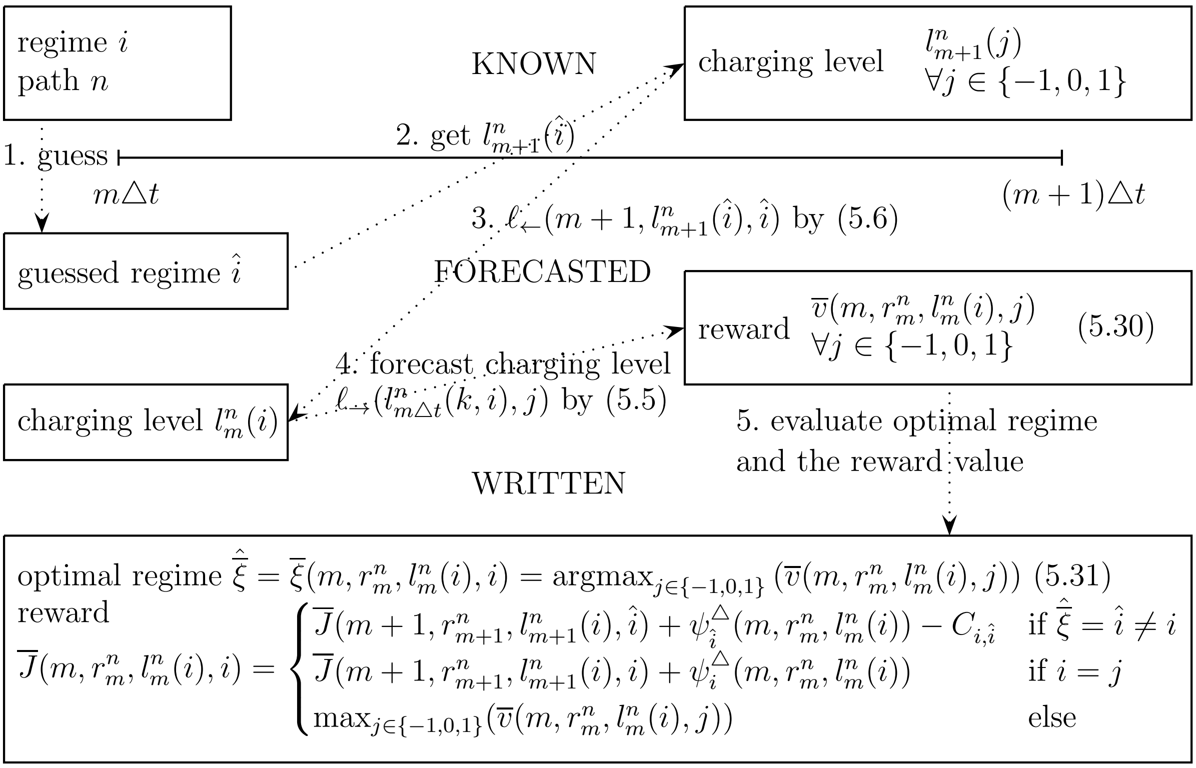

Figure 5.3: Visualization of the steps in the path dependent BLSM scheme.

5.5 Function Selection

The selection of the functions in the set $\mathcal{B}$ (or respective $\overline{\mathcal{B}}$) is quite heuristic. We first motivate the univariate functions case of $\mathcal{B}$ and subsequently the bivariate case $\overline{\mathcal{B}}$.

From a theoretical point of view a subset of Hilbert basis functions is consequential, from the motivation of the approach given in Section 5.2. For example Longstaff and Schwartz [2001] used the Laguerre polynomials

$$D_j(x) = \exp^{-\frac{x}{2}} \frac{\exp^x}{j!} \frac{d^j(d^j(x^j \exp^x)}{dx^j}, \tag{5.34}$$

where $j = 0, \ldots, (N_{\mathcal{B}} - 1)$ are the specifications of the respective function in the set of all basis functions. On the other hand, since we select just a small number of functions, any set of linear independent functions is sufficient. For example with monomials $1,\ x,\ x^2,\ \ldots,\ x^{N_{\mathcal{B}}}$ the same space is spanned as with the Laguerre polynomials (5.34). Both sets have in common that they grow very fast in x, since the $\{R_m\}_{m\in\{0,\ldots,M\}}$ is of large scale it is numerical advisable to scale down to the interval $[0, 2]$. Moreover, Tsitsiklis and van Roy [2000] give the advice that these functions should represent the most salient properties of the function to be estimated. This means in our case that a well guess would be functions of the form $1,\ x,\ x^2,\ \ldots$ or $e^{\zeta x}$.

For the bivariate regression, required bivariate functions can be obtained by the tensor product of the univariate functions. Let N_{b_1} be the number of functions for the renewable energy part and N_{b_2} for the charging level part, hence $N_{\overline{\mathcal{B}}} = N_{b_1} N_{b_2}$. For the example given in (5.34), the bivariate functions are:

$$B_j(\hat{r}, \hat{l}) = D_{j_1}(\hat{r}) D_{j_2}(\hat{l}), \tag{5.35}$$

where $j = 0, \ldots, N_{\overline{\mathcal{B}}}$, $j_2 = j \mod N_{b_1}$ and $j_1 = \frac{j-j_2}{N_{b_1}}$. Furthermore, $\hat{r}$ and $\hat{l}$ are the values of the renewable energy process realization r and the charging level l rescaled to the interval $[0, 2]$. The use of the normalized values is necessary since the Laguerre polynomials get volatile for large function values, see Figure 5.4. But by the same arguments as in the

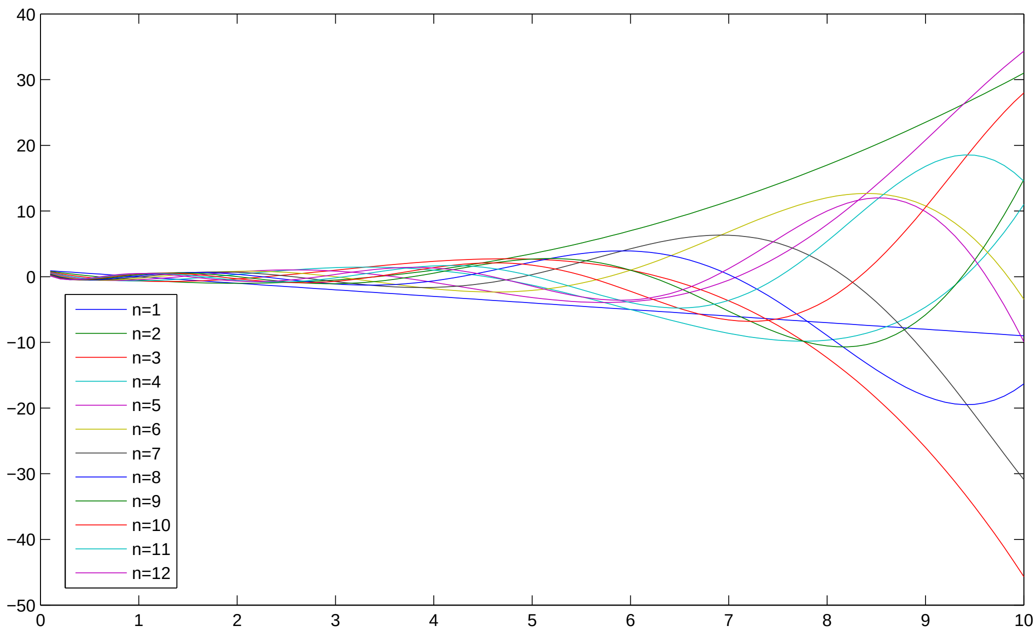

Figure 5.4: Laguerre polynomials for $n = 1, \ldots, 12$.

univariate case we use the tensor product of the functions 1, x, x^2, ... and $e^{\zeta x}$.

Using empirical tests Ludkovski [2005] showed that a large number of basis functions reduce the accuracy due to over fitting. We stay with this result and take a set of $N_{\overline{\mathcal{B}}} = 13$ functions for the bivariate case and $N_{\mathcal{B}} = 6$ for the univariate part. We use this simple set of functions since more complex functions we tried did not perform better. But for example Gobet et al. [2005] made studies using approaches, based on Voronoi partitions, hypercubes and global polynomials, to abandon parts of the heuristic in the basis selection as well as allowing for a time dependent selection of the basic function set.

Chapter 6

Numeric Example for Germany

In this chapter we consider a concrete example problem and solve it with the numerical algorithms as introduced in Chapter 5. This means that we compute the optimal charging-storing-discharging strategy of an energy storage and the respective expected maximal reward. For the problem setup in Section 6.1 we follow parts of Chapter 3, namely the price fixing in Section 3.1 and the storage characteristics in Section 3.2. Also in Section 6.1 we compare the rewards obtained by the two algorithms described in Sections 5.3 and 5.4 and give some insight considering the convergence of these algorithms by the standard error. In Section 6.2 we analyze the influence of the different parameters in the problem setup on the reward. Finally in Section 6.3 we explore the switching regions depending on the current regime, charging level and the contribution from renewable sources.

6.1 Problem setup

We assume that the agent is a leaser of an energy storage in Germany for one week, i.e. $T = 1$. The storage is handed over to the agent in the regime $i = 0$, i.e. keeping the charging level constant, and with initial level $L_0 = 2$. The respective buy back provision of the lending contract is mirrored by

$$V(R_T, D_T, L_T) = 2\ \rho(T, R_T, D_T)\ \max(0, L_0 - L_T),$$

the penalty function introduced in (3.12). The storage has the following characteristics:

- $\mathsf{c}_{max} = 4$ [MWh], maximal storage capacity,
- $\mathsf{c}_{min} = 0$ [MWh], minimal storage charging level,
- $\mathsf{a}_{in} = 0.2 \cdot 7 \cdot 24$ [MWh], maximal weekly charging rate,
- $\mathsf{a}_{out} = 0.5 \cdot 7 \cdot 24$ [MWh], maximal weekly discharging rate,
- $\mathsf{b} = 0.1$ [Euro/MWh], costs to store one MWh over one time unit,
- $\mathsf{K}_i = 0$ [Euro], managing and operation costs per time unit and
- $C_{i,j} = 0.25$ [Euro] for $i, j \in \{-1, 0, 1\}$, $j \neq i$, switching costs.

Additionally we assume that the storage charging and discharging rate is independent of the current charging level. Hence the change function (3.4) is given by:

$$a_i(t, L_t) = \begin{cases} 0.2 \cdot 7 \cdot 24 \cdot \mathbf{1}_{L_t > 0} & \text{if } i = -1, \\ 0 & \text{if } i = 0, \\ 0.5 \cdot 7 \cdot 24 \cdot \mathbf{1}_{L_t < 4} & \text{if } i = +1. \end{cases}$$

As derived in (3.2), for calculating the price in the German electricity market we need the energy contribution from renewable sources R_t, the demand D_t, the number of conventional power technologies N and the marginal costs of conventional power technologies $\{\mathsf{p}_n\}_{n \in \{0,\dots,N\}}$. First of all we assume a fixed demand $D > 0$ (as introduced in Chapter 4) from the respective fraction of the yearly demand in Germany. According to the "Statistische Bundesamt" (http://www.destatis.de/) the German energy demand in the year 2008 was 614.8 TWh, which we take as our fixed yearly demand. Hence, using the one week time horizon and the definition $\triangle t = T/M$ for the resulting size of the time intervals, the fixed demand is given by $D = 614.8 \cdot 10^6 \cdot (7/365) \cdot \triangle t$. Since we have not found any better model up to now, we allow for a myopic approach regarding the contribution of renewable energies. We assume that the renewable energy process follows an exponential Ornstein Uhlenbeck process, given by:

$$d \log(R_t) = 17.1(\log(0.7D) - \log(R_t))\, dt + 1.33 dW_t. \tag{6.1}$$

The asymptotic mean of the renewable energy production process is 70% of the respective demand and there is a strong mean reversion of 17.1. See Appendix A for a more detailed definition of the parameters and an introduction to the numerical implementation for the simulation of such a process.
In accordance with von Roon and Huck [2010] we summarize the conventional power technologies and respective marginal costs to obtain the merit order curve. The relevant extracted prices and capacities are

- 8 [Euro/MWh] and 16000 [MW] for nuclear energy,
- 38 [Euro/MWh] and 13000 [MW] for lignite-fired power plants,
- 52 [Euro/MWh] and 4000 [MW] for efficient combined cycle gas turbine plants,
- 60 [Euro/MWh] and 18000 [MW] for coal-fired plants,
- 78 [Euro/MWh] and 6000 [MW] for old combined cycle gas turbine plants,
- 100 [Euro/MWh] and 2000 [MW] for gas turbine power stations and
- 125 [Euro/MWh] and 2000 [MW] for medium fuel oil fired plants.

Furthermore, we assume that renewable technologies have marginal costs of 6 [Euro/MWh], i.e. $\mathsf{p}_0 = 6$. Hence, the respective price function (as defined in (4.1)) is given by:

$$\rho(t,r) = \begin{cases} 6 & \text{if } (D-r) < 0, \\ 8 & \text{if } 0 \leq (D-r) < 16000(7 \cdot 24)\triangle t, \\ 38 & \text{if } 16000(7 \cdot 24)\triangle t \leq (D-r) < 29000(7 \cdot 24)\triangle t, \\ 52 & \text{if } 29000(7 \cdot 24)\triangle t \leq (D-r) < 33000(7 \cdot 24)\triangle t, \\ 60 & \text{if } 33000(7 \cdot 24)\triangle t \leq (D-r) < 51000(7 \cdot 24)\triangle t, \\ 78 & \text{if } 51000(7 \cdot 24)\triangle t \leq (D-r) < 57000(7 \cdot 24)\triangle t, \\ 100 & \text{if } 57000(7 \cdot 24)\triangle t \leq (D-r) < 59000(7 \cdot 24)\triangle t, \\ 125 & \text{if } 59000(7 \cdot 24)\triangle t < (D-r). \end{cases} \tag{6.2}$$

Moreover, the risk free interest rate is given by $\mathsf{r}_m = 0.06$. In summary the problem setup values are:

$\mathsf{c}_{max} =$	4	$\mathsf{c}_{min} =$	0
$\mathsf{a}_{in} =$	$0.2 \cdot 7 \cdot 24$	$\mathsf{a}_{out} =$	$0.5 \cdot 7 \cdot 24$
$\mathsf{b} =$	0.1	$\mathsf{K_i} =$	0
$C_{i,j} =$	$0.25\ i,j \in \{-1,0,1\},\ j \neq i$	$\mathsf{r}_m =$	0.06

Table 6.1: Problem setup values for the storage example

Within this setting we evaluate the BLSM and MiTvR scheme. In the following the number of equidistant time intervals M is defined as 168, which corresponds to an hourly spacing.
The random mesh BLSM (as described in Section 5.4.1) is evaluated for a different number of Monte Carlo paths, namely $N \in \{8000,\ 16000,\ 24000,\ 32000,\ 40000\}$. To obtain an approximation of the respective Monte Carlo sampling error the BLSM scheme was run 50 times for each N. The results are shown in Table 6.2. In this table we can observe convergence of the form $\mathcal{O}((\triangle t\ N)^{-1/2})$, similar to the results in Carmona and Ludkovski [2010]. Analytically this error is hard to analyze due to the pathwise-maximum and the cross-path regression coupled with the charging level dependence. For simpler applications detailed studies are available, see for example Tsitsiklis and van Roy [2000] and Gobet et al. [2005]. The respective value of the BLSM scheme for $N = 40000$ Monte Carlo paths is 414.52 and the evaluation takes on average 235 seconds.
The reward evaluation was also done using the MiTvR scheme (described in Subsection 5.3.1) for $N = 10000$ paths and different charging level spacing of $G \in \{20,\ 40,\ 80\}$. The results for this scheme are shown in Table 6.3. For $G = 80$ the scheme takes on average 1269 seconds for the evaluation and returns a reward of 457.33.
The monetary storage values only differ by 9% but the BLSM scheme evaluation time is less than 1/6 of the time the MiTvR scheme requires. The difference in computation time is caused by the slow lattice scheme implemented in MiTvR. From now on we focus on the BLSM scheme due to its advantages.

paths	mean	standard deviation	computational time
8000	414.8322	1.6336	43.4097
16000	414.5005	1.1127	88.2305
24000	414.6244	0.9957	136.4711
32000	414.2998	0.7695	183.3401
40000	414.5241	0.7305	235.4589

Table 6.2: Convergence of the Monte Carlo sampling error for the BLSM scheme based on the storage given by Table 6.1 and the market price obtained by (6.2) conjoint with renewable energy process (6.1)

spacing	mean	standard deviation	computational time
456.675	20	0.4635	115.0461
457.1781	40	0.4363	335.3286
457.3314	80	0.4147	1269.769348

Table 6.3: Convergence of the Monte Carlo sampling error for the MiTvR scheme with $N = 10000$ paths based on the storage given by Table 6.1 and the market price obtained by (6.2) conjoint with renewable energy process (6.1)

6.2 Sensitivity analysis of the different parameters

Now we study the influence of the different storage characteristics on the expected reward. The storage flexibility given by the charging and discharging rates, a_{in} and a_{out}, has the greatest influence on the reward. In particular this means that if the charging and discharging rates are halved the expected reward is nearly halved. The influence of the doubled charging and discharging rate is weaker, as shown in Table 6.4. For a higher charging rate and equal discharging rate the expected reward soars in comparison to a higher discharging rate with equal charging rate, see Table 6.4. Thus, we obtain that the main restriction is the slow charging rate. In particular this means that, we cannot charge the storage fast enough to realize all possible rewards. A maximal reward is obtained when the charging and discharging rate is nearly equal, since then we can react on changes with an equal rate. Charging and discharging characteristics of this type are common for pumped water storages (an example the characteristic is shown on www.enbw.com/.../pumpspeicherkraftwerk/index.jsp) and sodium sulphur batteries, thus they are able to compensate the high acquisition costs.

a_{in}	a_{out}	reward
0.5	0.5	640.2108
0.4	0.5	534.0965
0.1	0.5	277.4365
0.2	0.7	430.4399
0.1	0.25	246.9002
0.4	1	648.1473

Table 6.4: Influence of different charging a_{in} and discharging rates a_{out} on the expected reward evaluated by the BLSM scheme based on the storage given by Table 6.1 and the market price obtained by (6.2) conjoint with the renewable energy process in (6.1)

The reward is rising with the storage size, but the size of this trend flattens out. So the additional reward for a rise in the storage size by 2 MWh is about 60 Euro and for further extension by 2 MWh we just gain about 30 Euro, see first and last column in Table 6.5. This result confirms, that we cannot charge the storage fast enough to realize all possible rewards. A shift of the capacity about one MWh leaves the reward unchanged, see Table 6.5.

c_{min}	c_{max}	reward
0	6	470.4096
0	3	373.5098
1	5	414.5276
0.5	6	459.2759
0	8	501.9667

Table 6.5: Influence of different maximal storage capacities c_{max} and minimal storage levels c_{min} on the expected reward evaluated by the BLSM scheme based on the storage given by Table 6.1 and the market price obtained by (6.2) conjoint with the renewable energy process in (6.1) .

The influence of the switching costs, $C_{i,j}$, is shown in Table 6.6. A rise in the switching costs of 0.05 has an influence of about 3 Euro. This is the expected influence but due to the size this influences is marginal. A small variation in the costs to store one MWh, b,

$C_{i,j}$	0.05	0.1	0.2	0.25	0.4	0.5
reward	427.0505	424.1018	418.4974	415.8048	408.0835	403.201

Table 6.6: Influence of different switching costs $C_{i,j}$ on the expected reward evaluated by the BLSM scheme based on the storage given by Table 6.1 and the market price obtained by (6.2) conjoint with the renewable energy process in (6.1) .

has almost no influence on the reward, see Table 6.7.

b	0	0.05	0.1	0.2	0.3	0.5
reward	415.9987	415.9018	415.8048	415.6109	415.4172	415.0301

Table 6.7: Influence of different switching costs $C_{i,j}$ on the expected reward evaluated by the BLSM scheme based on the storage given by Table 6.1 and the market price obtained by (6.2) conjoint with the renewable energy process in (6.1) .

As well we take a closer look at the influence of the current charging level and the energy contribution of renewable sources on the reward at time $m\triangle t = 0.49405$, which is one hour before the middle of our lending contract over one week. With a rising current charging level the expected reward also rises, as shown in Figure 6.1. The reward depicts the minimum for an energy contribution of renewable sources around the long term mean $0.7D$, i.e. round 49000 MWh. This demonstrates that a deviation from the long term mean implies arbitrage opportunities, namely to sell the already charged energy for high prices or buy electricity for low prices.

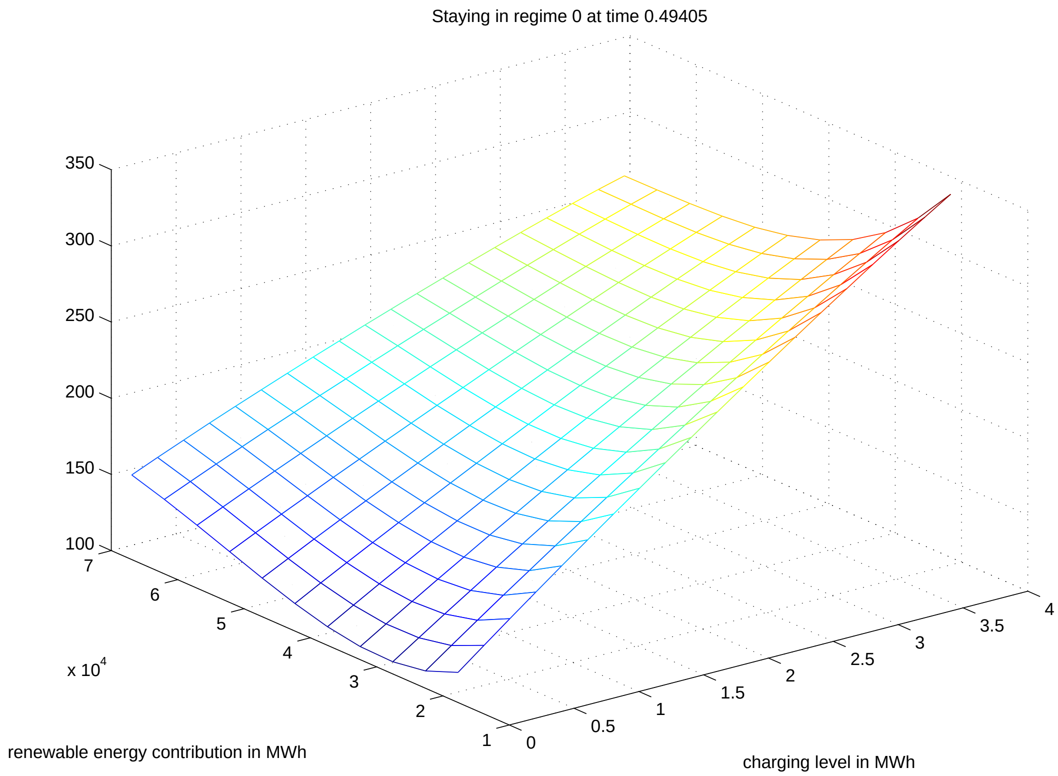

Figure 6.1: The maximal reward at time $t = 0.49405$ (one hour before the middle of the lending contract) and in the regime keeping level constant, $i = 0$, for the current charging levels and contributions from renewable sources based on the storage given by Table 6.1 and the market price obtained by (6.2) conjoint with the renewable energy process in (6.1).

Finally the influence of a nuclear energy plant phase-out is evaluated. To obtain the respective price function without the nuclear technology, we erase the marginal costs of 8 Euro and the capacity of 16000 MWh from the price function (6.2). Hence the new price function is given by:

$$\rho(t,r) = \begin{cases} 6 & \text{if } (D-r) < 0, \\ 38 & \text{if } 16000(7\cdot 24)\triangle t \leq (D-r) < 13000(7\cdot 24)\triangle t, \\ 52 & \text{if } 29000(7\cdot 24)\triangle t \leq (D-r) < 17000(7\cdot 24)\triangle t, \\ 60 & \text{if } 33000(7\cdot 24)\triangle t \leq (D-r) < 35000(7\cdot 24)\triangle t, \\ 78 & \text{if } 51000(7\cdot 24)\triangle t \leq (D-r) < 41000(7\cdot 24)\triangle t, \\ 100 & \text{if } 57000(7\cdot 24)\triangle t \leq (D-r) < 43000(7\cdot 24)\triangle t, \\ 125 & \text{if } 59000(7\cdot 24)\triangle t < (D-r). \end{cases} \tag{6.3}$$

Under given storage characteristics the new setting reduces the reward to 300.578 Euro. This effect on the reward is caused by the fact that we have less low price periods. We have to bear in mind that the next expensive technology lignite-fired power plants has already marginal costs of 38 Euro. Thus in short run a phase-out of this technology is not beneficial for the agent. Following the energy plan formulated by the German government we extend the renewable energy sector. Consequently we raise the average mean of energy produced by renewable sources up to $0.9D$, i.e. round 63000 MWh. The resulting reward in this setting is 473.662 Euro, which is higher than the reward in the current energy market.

6.3 Switching Regions

The collection of regions with optimal regime $j \in \{-1, 0, 1\}$ conditional on a previous given regime $i \in \{-1, 0, 1\}$ at time $m\triangle t$ is called a switching region. Here j runs over all possible values whereas i is fixed. The optimal regime depends on the current energy contribution from renewable sources and the current charging level. The regions are determined by $\overline{\xi}(m, r_m^n, l_m^n, i)$ in (5.31) (or $\hat{\xi}(m, r_m^n, g, i)$ in (5.26)) and depend on the realization of the renewable energy contribution r_m^n and the random charging level l_m^n (or the current grid point $c_g = \mathsf{c}_{min} + g \triangle g$). Examples for the resulting regions are shown in Figure 6.2 and Figure 6.4. Each of those figures exhibit the respective switching regions at given time and different previous regimes, namely for the previous regime $i = -1$ in the plot on the left hand side, $i = 0$ in the plot in the middle and $i = +1$ in the plot on the right hand side. Moreover, the respective plots display the regions, where staying in the current regime is optimal in black and the switching regions to another regime in blue and green. In the following we examine the influence of the energy contribution by renewable sources and the current charging level on the optimal switching decisions. Another interesting point is studying the influence of the remaining contract time.

First we take a closer look at the switching regions in the middle of the lending contract. Figure 6.2 for the BLSM scheme and Figure 6.3 for the MiTvR show the switching regions at time $m\triangle t = 0.537$, namely a bit less than 4 days, for each regime $i \in \{-1, 0, 1\}$, i.e. on the left hand side for charging $i = -1$, in the middle for keeping the level constant ($i = 0$) and the discharging regime $i = 1$. The regime $i = 0$ is more or less only advisable, if we already are in this regime. This makes sense since we neither obtain more flexibility by more energy in the storage nor more reward due to discharging. For the case that we are already in regime $i = 0$ the holding out in this regime is advantageous due to the switching costs. Besides, we see that the switching regions are ruled by the current contribution from renewable sources. This means for high contributions, i.e. low prices, the regime $i = -1$ is favorable, since we might be able to sell the electricity for higher prices later on. On the other hand for low contributions, consequently high prices, we sell the stored energy and realize the difference between the current price and the price at charging time. Only at the boundary of storage the size current charging level plays a role.

When the time comes closer to the expiry date the current charging level gets more and more important due to the fact that the agent is charged twice the market price, if he returns the storage at a lower level than the initial level, $L_0 = 2$. This can be seen in Figure 6.4 for the BLSM scheme and in Figure 6.5 for the MiTvR scheme, which shows the switching level at time $m\triangle t = 0.9523$, i.e. 8 hours before the expiry date. At this time it is advantageous to charge if the current charging level is below L_0, since it is cheaper

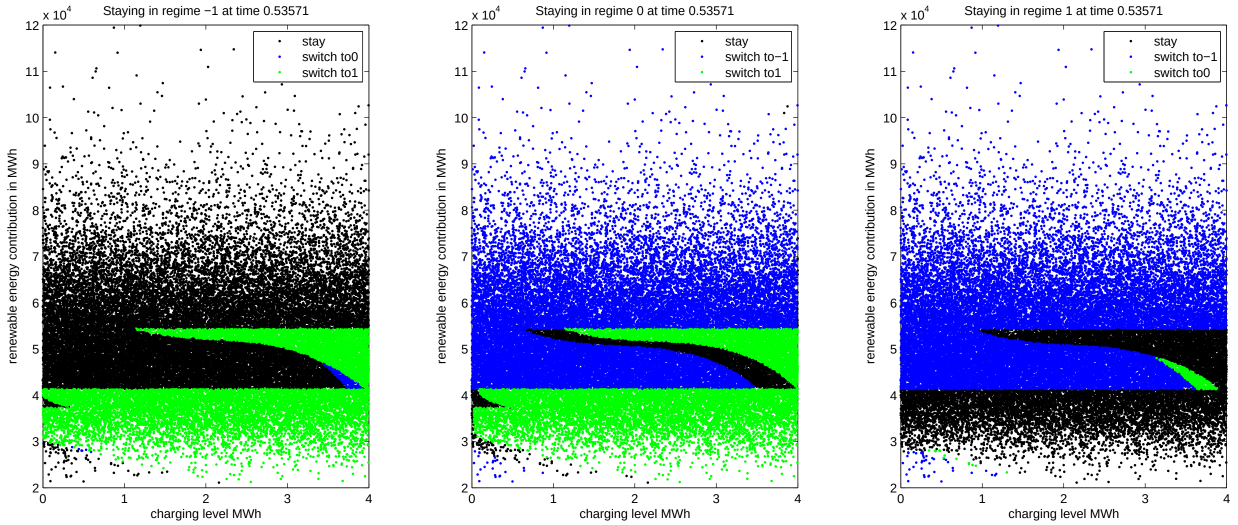

Figure 6.2: Optimal switching regions at $m\triangle t = 0.5$ depending on the current regime, $i = -1$ charging on the left hand side, $i = 0$ keeping the storage level constant in the middle and $i = 1$ discharging on the left hand side, the current renewable energy contribution and the charging level. (Generated by the BLSM random mesh scheme 5.4.2 for N=40000 and M=168 for the example given in Section 6.1)

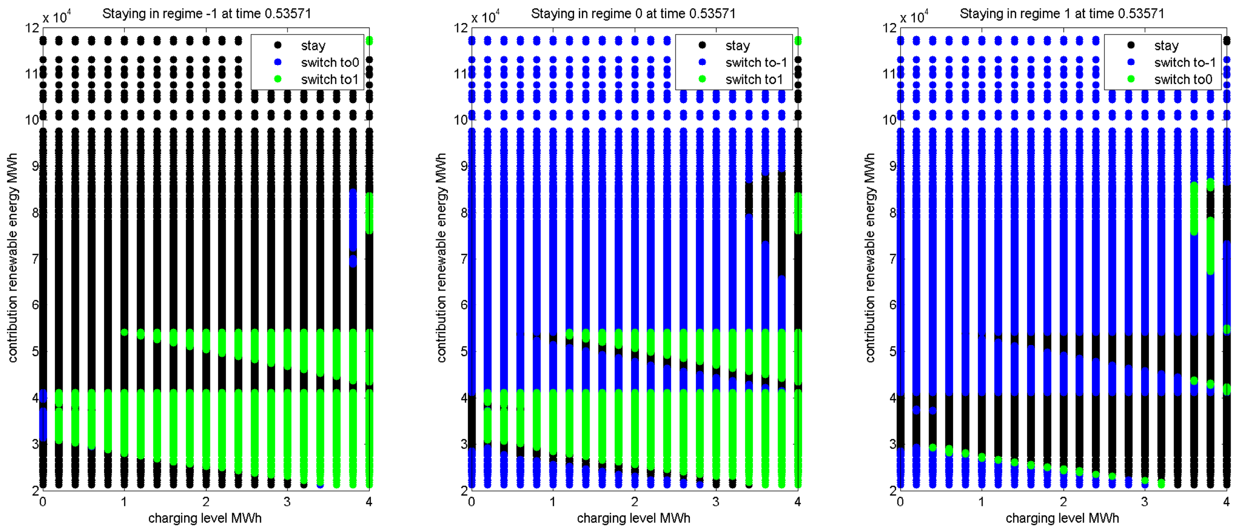

Figure 6.3: Optimal switching regions at $m\triangle t = 0.5$ depending on the current regime, $i = -1$ charging on the left hand side, $i = 0$ keeping the storage level constant in the middle and $i = 1$ discharging on the left hand side, the current renewable energy contribution and the charging level. (Generated by the MiTvR scheme 5.3.1 for N=40000, M=168 and G=20 for the example given in Section 6.1)

than paying the double price. On the other hand it is advisable to discharge if the current level is greater than L_0, since we get no compensation for a higher charging level than the initial one. But still the regime $i = 0$ (keeping the charging level constant) is a minority. This is mainly due to the fact that the charging rate of 20 hours and the discharging rate of 8 hours are fast.

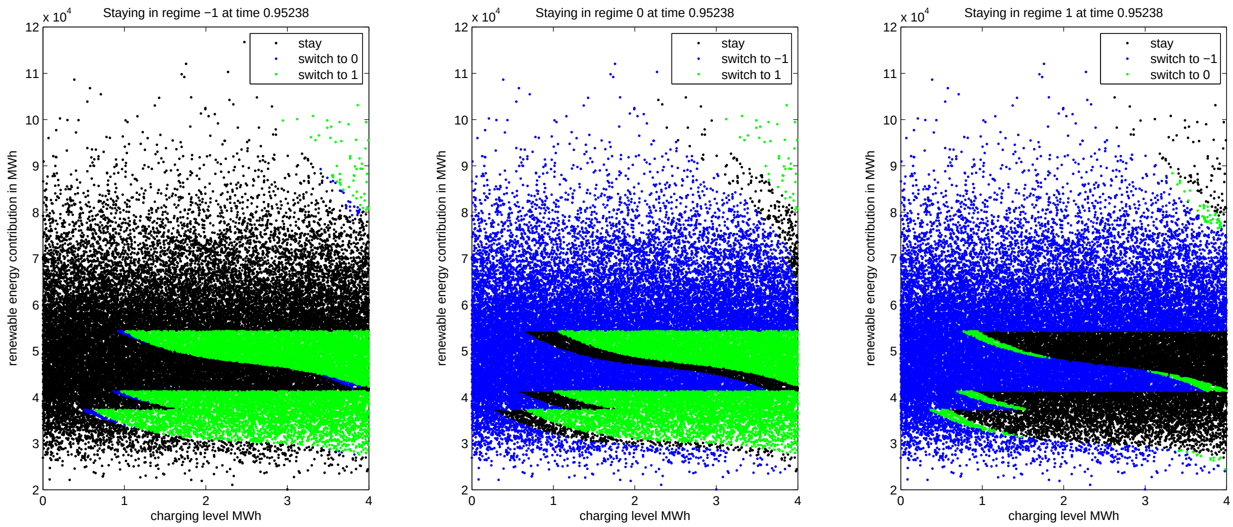

Figure 6.4: Optimal switching regions at $m\triangle t = 0.95$ depending on the current regime, $i = -1$ charging on the left hand side, $i = 0$ keeping the storage level constant in the middle and $i = 1$ discharging on the left hand side, the current renewable energy contribution and the charging level. (Generated by the BLSM random mesh scheme 5.4.2 for N=40000 and M=168 for the example given in Section 6.1)

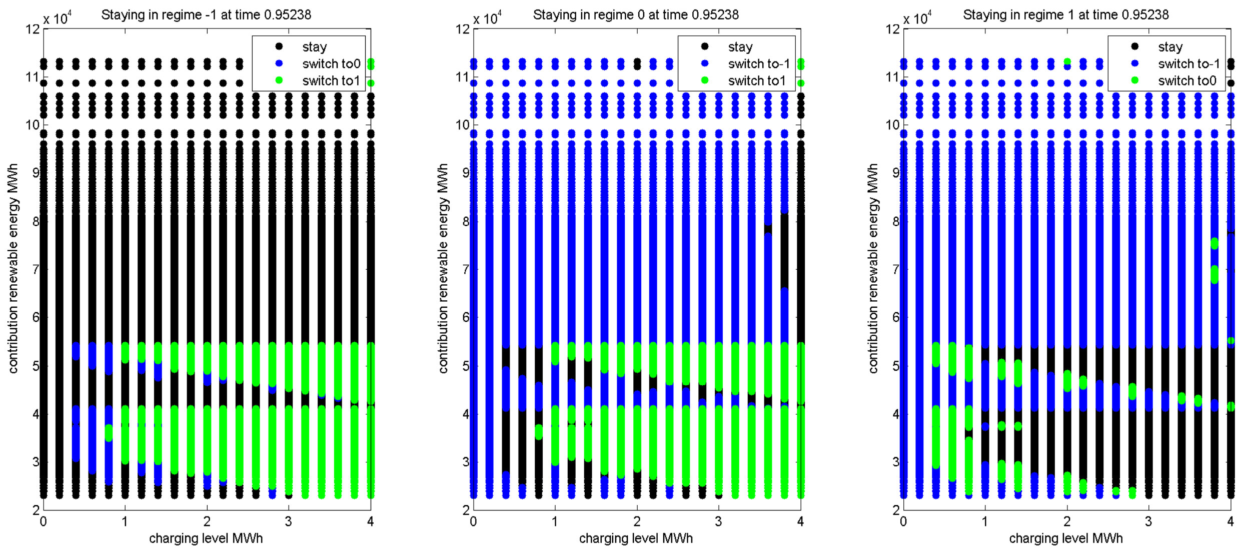

Figure 6.5: Optimal switching regions at $m\triangle t = 0.5$ depending on the current regime, $i = -1$ charging on the left hand side, $i = 0$ keeping the storage level constant in the middle and $i = 1$ discharging on the left hand side, the current renewable energy contribution and the charging level. (Generated by the MiTvR scheme 5.3.1 for N=40000, M=168 and G=20 for the example given in Section 6.1)

Chapter 7

Conclusion

In this thesis we answered the questions how to control an energy storage and what the economic value of such a storage is. Based on historical data we confirmed the need of energy storages for mixed energy markets, i.e. markets with renewable and conventional production plants, in Chapter 2. In Chapter 3 we developed a general model for a storage facility and the price in the electricity market based on the merit order curve. For this setting we showed that a solution of the respective optimal multiple switching problem exists and we derived the characteristics of such an optimal strategy in Chapter 4. Since an analytical derivation of these two parameters is not feasible, we implemented two algorithms in Chapter 5 to obtain the solution numerically. The thesis is completed by a concrete example stated in Chapter 6, focused on the German electricity market.

In Chapter 2 we derived a linear optimization problem for minimizing the required energy generated by conventional technologies based on historical time series of the load and renewable energy production relative to the installed capacities. The flexible parameters, namely installed capacity of wind plants, of PV plants and of storage facilities, are used as regulation screws to obtain the influences of different development scenarios. To simplify the examination of the scenarios, a GUI was implemented, which gives a visual overview as well as detailed figures. The interaction of wasted energy, required backup by conventional plants and the resulting storage streams, altering two variable inputs of this problem, have been examined. The main result is that the capacities of storages have the most significant influence on all parameters, especially for a large installed capacity of PV power plants. In the rest of this thesis we dealt with a more elaborate problem of optimizing the reward based on the market price derived on an energy market with renewable energy.

We stated the characteristics of an energy storage as well as the derivation of the market price, in Chapter 3. The price was obtained by a version of the merit order curve, which gives us great flexibility to incorporate market developments. Thus, we have been able to evaluate the value of the storage and the strategy after a phase-out of nuclear energy as well as under the assumption of an increase in the renewable energy plants installed capacity. In Chapter 4 we showed that the multiple switching problem – between charging, discharging and keeping the storage level constant – derived from the problem setup in Chapter 3 has a theoretical solution. This solution is accomplished with an optimal strategy, equivalent to the single process Snell strategy. The theory introduced in this chapter can be rounded off by rewriting the results in the framework of BSDE. Due to the complexity resulting from charging level dependence this is still an open point. An other interesting point would be to derive the set of processes where a solution of the multiple switching problem exists, aside from the set of processes purely driven by a Brownian motion we considered.

Due to the fact that the reward of the optimal stopping problem is analytically not tractable, we introduced two algorithms based on Monte Carlo methods in Chapter 6. The common basic idea for both algorithms is the approximation of the conditional expectation by a regression and the strategy to go pathwise backwards in time for the evaluation. Thereby given the knowledge about the current state, represented by functions of the current state variables, and the known future values we obtained an estimation of the future reward conditioned on the current knowledge. On the one hand we implemented the MiTvR scheme. The consecutive steps in this algorithm are shown in Subsection 5.3.1. This scheme is based on a lattice in the interval of possible storage levels. Thereby the scheme is numerically stable, but computational time and storage intensive. On the other hand two versions of the BLSM scheme were implemented. One develops the charging level pathwise backwards in time and the other one is using a stochastic mesh in the charging levels. The pathwise BLSM scheme is more intuitive, but not stable under different schemes to guess the optimal regime on the first hand. We discussed this scheme in Subsection 5.4.3. The stochastic mesh BLSM, as developed in Subsection 5.4.1, is stable and needs less computational time and storage in comparison to the MiTvR scheme. For the BLSM schemes the question of convergence is still on the schedule, even if the numerical test suggests the convergence rate of Monte Carlo methods. This numerical convergence result is also shown in Chapter 5. Furthermore, a good insight would be to show why the path dependent BLSM value is strongly dependent on the chosen guessing scheme. An additional point would be to upgrade the implementation by altering sets of bases functions over time used for the regression.

Finally, we showed in Chapter 6 the numeric results of the different schemes and conducted a sensitivity analysis of the storage characteristics parameters. The charging and discharging rates have the greatest impact on the reward, especially asymmetric rates imply less reward. This is due to the fact that we cannot react equally flexible on the energy price for the different regimes. Another result in this section is that the optimal regimes, depending on time and the current regime, are mainly ruled by the current energy contribution. Just at times close to the expiry date the charging level becomes relevant for the decision of optimal regime. Furthermore, we showed how the decision rules for the optimal regime can be derived from the numerical schemes by time and regime dependent optimal switching regions characterized by the contribution of renewable sources and the charging level.

Appendix A

Ornstein Uhlenbeck process

The Ornstein-Uhlenbeck (OU) process $\{X_t\}_{t\in[0,T]}$ with the asymptotic mean μ, the mean reversion θ and the variance σ^2 satisfies the following stochastic differential equation [Gasserman, 2004, Eq. (3.39)]:

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \quad \text{for } t \in [0, T], \tag{A.1}$$

where $\{W_t\}_{t\in[0,T]}$ is a standard Brownian motion. The drift in (A.1) changes algebraic sign, i.e. positive when $\mu - X_t > 0$ and negative when $\mu - X_t < 0$. This means if the X_t is smaller than μ the process tends to drift upwards and if X_t at time $t \in [0, T]$ is greater than μ the process tends to drift downwards to the asymptotic mean, a property generally referred to as mean reversion. Moreover the OU process is Gaussian and Markov.

Solving the (A.1) for a given initial condition X_0, satisfying that $E[|X_0|^2] < \infty$ we obtain:

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma e^{-\theta t} \int_0^t e^{\theta s} dW_s. \tag{A.2}$$

From (A.2) it follows, that the random variable X_t for a fixed $t \in [0, T]$ is normally distributed with expectation

$$\mathbb{E}[X_t] = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) \tag{A.3}$$

and the variance

$$Var(X_t) = \mathbb{E}\left[(\sigma e^{-\theta t} \int_0^t e^{\theta s} dW_s)^2\right] = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}). \tag{A.4}$$

To simulate the process at time $(m+1)\triangle t,\ m \in \{0, \dots, (M-1)\}$, we use the results (A.2), (A.3) and (A.4), and set [Gasserman, 2004, Eq. (3.46)]:

$$X_{(m+1)\triangle t} = X_{m\triangle t} e^{-\theta\triangle t} + \mu(1 - e^{-\theta\triangle t}) + \sqrt{\frac{\sigma^2}{2\theta}(1 - e^{-2\theta\triangle t})} Z_{m+1}, \tag{A.5}$$

where Z_{m+1} are standard normal distributed samples and $\triangle t = \frac{T}{M}$.

Moreover we take a closer look at the exponential Ornstein Uhlenbeck process, given by:

$$dX_t = X_t[\theta(\mu - \log(X_t))dt + \sigma dW_t]. \tag{A.6}$$

The stochastic differential equation (A.6) can be equivalently written in the following form

$$d\log(X_t) = \left[\theta\left(\mu - \log(X_t)\right) - \frac{1}{2}\sigma^2\right] + \sigma dW_t,$$

by applying the Itô formulae for $f(x) = \log(x)$. Bearing the second formulation in mind the simulation of the process can be analogously to (A.5) be done by

$$X_{(m+1)\triangle t} = \exp\left(\log(X_{m\triangle t})e^{-\theta\triangle t} + \mu_1(1 - e^{-\theta\triangle t}) + \sqrt{\frac{\sigma^2}{2\theta}(1 - e^{-2\theta\triangle t})}Z_{m+1}\right),$$

where $\mu_1 = \mu - \frac{\sigma^2}{2\theta}$.

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