The Boolean Hierarchy of NP-Partitions*

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Abstract

We introduce the boolean hierarchy of k-partitions over NP for $k \geq 3$ as a generalization of the boolean hierarchy of sets (i.e., 2-partitions) over NP. Whereas the structure of the latter hierarchy is rather simple the structure of the boolean hierarchy of k-partitions over NP for $k \geq 3$ turns out to be much more complicated. We establish the Embedding Conjecture which enables us to get a complete idea of this structure. This conjecture is supported by several partial results.

Keywords. Computational complexity theory, classification problems, entailment, partitions, boolean hierarchy, polynomial hierarchy, completeness, orders and lattices.

1 Introduction

To divide the real world into two parts like big and small, black and white, or good and bad usually oversimplifies things. In most cases a partition into many parts is more appropriate. For example, take marks in school, scores for papers submitted to a conference, salary groups, or classes of risk. In mathematics, k-valued logic is just a language for dealing with k-valent objects, and in the computer science field of artificial intelligence, this language has become a powerful tool for reasoning about incomplete knowledge. In computational complexity for instance, proper partitions, although not mentioned explicitly, emerge in connection with locally definable acceptance types (cf. [23]).

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Nevertheless, complexity theoreticians mainly investigate the complexity of sets, i.e., partitions into two parts, or the complexity of functions, i.e., partitions into usually infinitely many parts. Both extremes seem not appropriate for studying the computational complexity of problems inherently being partitions into finitely many parts. If we study partitions into at least three parts by means of encoding the components of partitions (e.g., as $\{\ (x,i) \mid x \text{ is in the } i\text{-th component }\}$) then we may assume that many interesting phenomenons vanish by the encoding. On the other side, though partitions can be considered as functions with finite range, even the finite range allows combinatorial arguments because each component depends only on the other finitely many components of the partition. We would lose this feature when simply subsuming partitions under functions.

This paper studies, for the first time, systematically the computational complexity of partitions. Herein we will follow the approach to collect "similar" problems in complexity classes and to investigate relations among these classes. While complexity classes of sets represent decision problems our complexity classes of partitions represent classification problems. Very important classes of classification problems originate from questions concerning relations.

1.1 Classification and Decision Problems for Relations

Suppose that $\sim$ is any binary relation on a basic set M. When giving an explicit definition of $\sim$, we specify $\sim$ in the following way: For two elements $x, y \in M$, $x \sim y$ if and only if some definitional conditions hold for x and y. Thus the explicit specification of a relation has the form of a decision problem. But once the relation $\sim$ is fixed, the more natural question is to determine for any given x and y how they behave with respect to $\sim$: Is it true that both $x \sim y$ and $y \sim x$ hold or only $x \sim y$ holds or only $y \sim x$ holds or is even nothing true? Questions of this kind are significant in connection with, e.g., entailment issues as studied in automated reasoning, database theory, and constraint programming, or congruence and isomorphism problems equally of broad interest.

For a concrete example let us consider the entailment relation $\models$ for formulas of (two-valued) propositional logic. For propositional formulas H and H' it is defined as

$$H \models H' \iff_{\mathrm{def}} \text{ each satisfying assignment for } H \text{ is a satisfying assignment for } H'.$$

Given two arbitrary formulas there are the above four possible cases to classify according to the behavior the formulas show with respect to $\sim_R$. We translate this into the partition Entailment. The most natural way to define a partition is to fix its characteristic function. For any partition A the characteristic function c_A says for every x to which component of A this x belongs. So for any pair (H, H') of formulas we define

$$c_{\textsc{Entailment}}(H, H') =_{\mathrm{def}} \begin{cases} 1 & \text{if } H \not\models H' \text{ and } H' \not\models H, \\ 2 & \text{if } H \not\models H' \text{ and } H' \models H, \\ 3 & \text{if } H \models H' \text{ and } H' \not\models H, \\ 4 & \text{if } H \models H' \text{ and } H' \models H. \end{cases}$$

We should bring to mind that though the numbering of the cases to be distinguished is not essential for the classification itself yet it leads to different partitions. We also should be aware that for a collection of sets to be a partition it is not only necessary to have the pairwise disjointness of all sets but also that each possible element must be contained in one of these sets.

Apparently there exist very close connections between ENTAILMENT and the decision problem of whether $H \models H'$ for given H and H'. Let us explain this in more detail. For we consider two sets A and B that describe the decision problem formally: A is the set of all pairs (H, H') such that H entails H' and B is the set of all pairs (H, H') such that H' entails H. The partition ENTAILMENT and the sets A and B are intimately related in at least the following two ways:

1. Using the sets A and B the partition ENTAILMENT can be easily rewritten. So the first component of ENTAILMENT, denoted by ENTAILMENT_1, consists of all pairs of propositional formulas that do not belong to A or B. Opposite to this the fourth component of ENTAILMENT, denoted by ENTAILMENT_4, is nothing else than $A \cap B$. Since obviously A and B are coNP-complete (note that H is a tautology if and only if $H \vee \neg H \models H$) we easily observe that ENTAILMENT_4 is coNP-complete, whereas ENTAILMENT_1 is NP-complete. Equally it is not hard to verify that both the second and the third component of the entailment classification problem are complete for DP where DP [32] is the class of all set differences of NP sets with NP sets.

2. The following generation principle is more fundamental. Let f be the function defined as

$$f(0,0) = 1, \quad f(0,1) = 2, \quad f(1,0) = 3, \quad \text{and} \quad f(1,1) = 4. \tag{1}$$

   We immediately see that ENTAILMENT is exactly the partition being generated when f is applied to the characteristic pair of the sets A and B. That means that for all propositional formulas H and H' it holds that $c_{\text{ENTAILMENT}}(H, H') = f(c_A(H, H'), c_B(H, H'))$. In this manner the function f generates a whole class of partitions which we denote by $\text{coNP}(f)$. So ENTAILMENT belongs to the class $\text{coNP}(f)$. In fact, it is one of the hardest among all partitions in this class; it is in a sense complete for $\text{coNP}(f)$.

Both junctures of the entailment classification problem with the entailment decision problem make the boolean hierarchy over NP be involved in the study of complexity classes of partitions. On the one hand, the classes NP, coNP, and DP occurring as classes reflecting the computational difficulty of the projections of ENTAILMENT represent just the lowest levels of this complexity-theoretic hierarchy. On the other hand, the generation principle we described above is precisely the same as the one that generates the boolean hierarchy over NP at all.

1.2 The Boolean Hierarchy (of Sets) over NP

The boolean hierarchy over NP has been very extensively investigated in a series of papers, e.g., in [39, 10, 27, 8, 9, 7, 24, 38, 33]. Purely set-theoretically, the boolean hierarchy over a set class is a very fundamental structure providing a detailed view on the closure of this class under the boolean operations intersection, union, and complementation. The roots of such hierarchies go back to Hausdorff [18] who observed normal forms of sets belonging to the boolean closure of a set class. Underlining their great significance for computation theory, boolean hierarchies have been studied for much more classes than NP such as for 1NP (or US) [17], UP [22], $\text{C}_=\text{P}$ [16, 3], RP [4, 6], and partly for $\text{C}_=\text{L}$ [1] in complexity theory, for the recursively enumerable sets [12, 13] in recursion theory, or for classes occurring in automata theory [37, 5, 14].

The most general way to define the boolean hierarchy over NP is as follows (see [39]): For a boolean function $f : \{0,1\}^m \to \{0,1\}$, which represents combinations of boolean operations, and sets $B_1, \ldots, B_m$ let $f(B_1, \ldots, B_m)$ denote the set whose characteristic function satisfies that $c_{f(B_1,\ldots,B_m)}(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x))$ for all x. The class $\mathrm{NP}(f)$ consists of all sets $f(B_1, \ldots, B_m)$ when varying the sets B_i over NP. Up to the different ranges of functions and the different base classes this is just the generation principle we have used above to obtain a partition class capturing the complexity of ENTAILMENT. The boolean hierarchy over NP consists of all these classes $\mathrm{NP}(f)$. Note that for the definition of the boolean hierarchy over NP it does not make a difference if we take NP or coNP as the base class; we clearly prefer NP. Wagner and Wechsung [39] have proved that every class $\mathrm{NP}(f)$ coincides with one of the classes $\mathrm{NP}(i)$ or $\mathrm{coNP}(i)$ where $\mathrm{NP}(i)$ is the class of all sets which are the symmetric difference of i NP sets and $\mathrm{coNP}(i)$ is the class of all complements of $\mathrm{NP}(i)$ sets. The family of these classes is also known as the difference hierarchy [27]. Evidently, $\mathrm{DP} = \mathrm{NP}(2)$.

It is not known whether the boolean hierarchy over NP is finite or equivalently, whether $\mathrm{NP}(i) = \mathrm{coNP}(i)$ for some $i \geq 1$. However, Kadin [24] succeeded to prove that a finite boolean hierarchy over NP implies the finiteness of Meyer and Stockmeyer's polynomial hierarchy [31, 36]; an event which most researchers in computational complexity consider to be highly improbable.

1.3 The Boolean Hierarchy of k-Partitions over NP

Motivated by our example ENTAILMENT it is natural to introduce and to study the generalization of the boolean hierarchy of sets over NP to the case of partitions into k parts (k-partitions) for $k \geq 3$. Any set A is identified with the 2-partition $(A, \overline{A})$. For a function $f : \{1, 2\}^m \to \{1, 2, \ldots, k\}$ and sets $B_1, \ldots, B_m$ we define a k-partition $A = f(B_1, \ldots, B_m)$ by the defining condition that $c_A(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x))$ for all x. Note that the characteristic functions here are characteristic functions of partitions (for a formal definition and explanation of differences, see Section 2). The boolean hierarchy of k-partitions over NP consists of the classes $\mathrm{NP}(f) =_{\mathrm{def}} \{ f(B_1, \ldots, B_m) \mid B_1, \ldots, B_m \in \mathrm{NP} \}$. As we have seen by ENTAILMENT, this hierarchy enables to measure the computational complexity of classification problems based on relations for which the decision problems is in NP or coNP. The boolean hierarchy of sets now appears in this hierarchy as the special case $k = 2$.

Whereas the boolean hierarchy of sets over NP has a very simple structure (note that $\mathrm{NP}(i) \cup \mathrm{coNP}(i) \subseteq \mathrm{NP}(i+1) \cap \mathrm{coNP}(i+1)$ for all $i \geq 1$), the situation is much more complicated for the boolean hierarchy of k-partitions in the case $k \geq 3$. The main question is: Can we get an overview on the structure of this hierarchy? This question is not answered completely so far, but we will give partial answers, and we will establish a conjecture.

A function $f : \{1, 2\}^m \to \{1, 2, \ldots, k\}$ which defines the class $\mathrm{NP}(f)$ of k-partitions corresponds to the finite boolean lattice $(\{1, 2\}^m, \leq)$ with the labeling function f where $\leq$ means the vector-ordering on the set of all m-tuples of $\{1, 2\}$. Generalizing this idea we define for every finite lattice G with labeling function $f : G \to \{1, 2, \ldots, k\}$ (for short: the k-lattice (G, f)) a class $\mathrm{NP}(G, f)$ of k-partitions. This does not result in more classes: For every k-lattice (G, f) there exists a finite function f' such that $\mathrm{NP}(G, f) = \mathrm{NP}(f')$. However, the use of arbitrary lattices instead of only boolean lattices simplifies many considerations. In particular every class in the boolean hierarchy of k-partitions has a (essentially) unique description in terms of minimal k-lattices. The above-mentioned difference hierarchy is just a special case of this description for the boolean hierarchy of 2-partitions.

To get an idea of the structure of the boolean hierarchy of k-partitions over NP it is very important to have a criterion to decide whether $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$ for k-lattices (G, f) and (G', f'). For that we define a relation $\leq$ as follows: $(G, f) \leq (G', f')$ if and only if there is a monotonic $\varphi : G \to G'$ such that $f(x) = f'(\varphi(x))$ for all $x \in G$. The Embedding Lemma says that $(G, f) \leq (G', f')$ implies $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$, and the Embedding Conjecture expresses our conviction that the converse is also true unless the polynomial hierarchy is finite.

For the Embedding Conjecture there exists much evidence. For $k = 2$ we can, not surprisingly, confirm this conjecture to be true. Moreover, we will give a theorem which enables us to verify the Embedding Conjecture for $k \geq 3$ for a large class of k-lattices including all k-chains. The proof of this theorem uses a new chain-technique that extends Kadin's easy-hard arguments (cf. [24]), developed for establishing the boolean and polynomial connection (for sets), to the case of partitions. Further the conjecture holds true for two subclasses of k-lattices where the chain-technique does not work. Here, two different proof techniques are needed that both are inspired by results from the theory of selective sets in [20, 26, 21].

There is a machine-based approach to the boolean hierarchy of k-partitions over NP. Each partition belonging to some class $\mathrm{NP}(f)$ can be accepted in a natural way by nondeterministic polynomial-time machines with a notion of acceptance that depends on the function f. As a consequence one can show that all these classes possess complete partitions with respect to an appropriate many-one reduction. This reduction offers a translation of completeness from the whole partition onto the components. For instance, since ENTAILMENT is complete for $\mathrm{coNP}(f)$ with f as described in (1) we immediately obtain that each component of the partition ENTAILMENT is complete for the component classes of $\mathrm{coNP}(f)$, i.e., ENTAILMENT_1 is NP-complete, ENTAILMENT_2 and ENTAILMENT_3 are NP(2)-complete, and ENTAILMENT_4 is coNP-complete, all as we have already discussed. However, there exists a partition, say A, which is complete for another partition class such that all components of A are complete for the same classes as the components of ENTAILMENT are, but A does not reduce to ENTAILMENT unless NP is closed under complements (see Figure 15). This nicely illustrates that the study of partitions allows finer distinctions between classification problems as in case of restricting investigations to set encodings only.

1.4 Organization of the Paper

Section 2 contains the complexity-theoretical notions and notations that will be tacitly adopted in the paper. In Section 3 we give a formal definition and some basic facts about the classes of the boolean hierarchy of k-partitions over NP. The main goal of this paper is to gain an overview on the structure of this hierarchy. To this end, in Section 4 we alternatively characterize partition classes generated by finite functions in terms of labeled lattices. In Section 5 we study the relation $\leq$ on labeled lattices. In particular, it is shown that $\leq$ induces a sufficient condition for inclusions of partition classes. We further show in Section 6 that all classes in the boolean hierarchy of k-partitions have (essentially) unique descriptions by minimal lattices. Section 7 contains the derivation and discussion of the Embedding Conjecture which states that for k-lattices, being in relation $\leq$ is not only sufficient for inclusion but also necessary unless the polynomial hierarchy is finite. A large part of this section is devoted to supporting the conjecture. Assuming the Embedding Conjecture is true we give in Section 8 an instructive example of how complicated the boolean hierarchy of k-partitions is already in the case $k = 3$. Finally, in Section 9 we present a way to characterize partition classes generated by labeled lattices in terms of acceptance types for nondeterministic machines. This leads to reducibility notions and completeness concepts. This will be exemplified for ENTAILMENT.

2 Preliminaries

Sets. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \{1, 2, \dots\}$. The cardinality of an arbitrary finite set A is denoted by $\|A\|$. For sets A and B we use $A \setminus B$ to denote the set-difference of A with B, and we use $A \triangle B$ to denote the symmetric difference of A and B. For $m \geq 1$ let A^m denote the m-fold cartesian product of A with itself.

Let $\mathcal{P}(M)$ be the power set of a fixed basic set M. For a set $A \subseteq M$, its complement in the basic set M is denoted by $\overline{A}$, i.e., $\overline{A} = M \setminus A$. The characteristic function $c_A : M \to \{0, 1\}$ is defined for all $x \in M$ as $c_A(x) = 1 \iff_{\text{def}} x \in A$. Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of subsets of M, i.e., $\mathcal{K}, \mathcal{K}' \subseteq \mathcal{P}(M)$. We define

$$\begin{aligned} \text{co}\mathcal{K} &=_{\text{def}} \{\, \overline{A} \mid A \in \mathcal{K} \,\}, & \mathcal{K} \wedge \mathcal{K}' &=_{\text{def}} \{\, A \cap B \mid A \in \mathcal{K}, B \in \mathcal{K}' \,\}, \\ \mathcal{K} \vee \mathcal{K}' &=_{\text{def}} \{\, A \cup B \mid A \in \mathcal{K}, B \in \mathcal{K}' \,\}, & \mathcal{K} \oplus \mathcal{K}' &=_{\text{def}} \{\, A \triangle B \mid A \in \mathcal{K}, B \in \mathcal{K}' \,\}. \end{aligned}$$

The classes $\mathcal{K}(i)$ and $\text{co}\mathcal{K}(i)$ defined by $\mathcal{K}(0) =_{\text{def}} \{\emptyset\}$ and $\mathcal{K}(i+1) = \mathcal{K}(i) \oplus \mathcal{K}$ build the *boolean hierarchy over* $\mathcal{K}$ that has many equivalent definitions (see [39, 10, 27, 8]).[1] Some of them can be found in the following theorem.

Theorem 1 *Let $\emptyset, M \in \mathcal{K}$, let $\mathcal{K}$ be closed under intersection and union, and let $m \in \mathbb{N}_+$.*

1. $\mathcal{K}(2m-1) = \{\, A_1 \cup \bigcup_{j=1}^{m-1} (A_{2j+1} \setminus A_{2j}) \mid A_1, \dots, A_{2m-1} \in \mathcal{K}$ *and* $A_1 \subseteq \cdots \subseteq A_{2m-1} \,\}$.
2. $\mathcal{K}(2m) = \{\, \bigcup_{j=1}^{m} (A_{2j} \setminus A_{2j-1}) \mid A_1, \dots, A_{2m} \in \mathcal{K}$ *and* $A_1 \subseteq \cdots \subseteq A_{2m} \,\}$.
3. $\mathcal{K}(2m) = \mathcal{K}(2m-1) \wedge \text{co}\mathcal{K}$.
4. $\mathcal{K}(2m+1) = \mathcal{K}(2m) \vee \mathcal{K}$.
5. $\mathcal{K}(m+1) = \text{co}\mathcal{K}(m) \wedge \mathcal{K}$.
6. $\mathcal{K}(m+2) = \mathcal{K}(m) \vee (\mathcal{K} \wedge \text{co}\mathcal{K}) = \mathcal{K}(m) \wedge (\mathcal{K} \vee \text{co}\mathcal{K})$.
7. $\mathcal{K}(m) \cup \text{co}\mathcal{K}(m) \subseteq \mathcal{K}(m+1) \cap \text{co}\mathcal{K}(m+1)$.

$\text{BC}(\mathcal{K})$ is the boolean closure of $\mathcal{K}$, i.e., the smallest class which contains $\mathcal{K}$ and which is closed under intersection, union, and complements.

Orders and Lattices. We need some notions from lattice theory and order theory (see e.g., [15, 11]). A pair $(G, \leq)$ is a poset if $\leq$ is a partial order on the set G. Usually, we talk about the poset G. Where it is necessary we write $(G, \leq)$ to specify the order. For a poset $(G, \leq)$ the poset $(G, \geq)$ is the dual poset and is denoted by G^{∂}. A poset G is a chain if for all $x, y \in G$

[1] Usually for $\mathcal{K} = \text{NP}$, a level 0 is not considered in the way we do. The zero-level there is P. However for our purposes it is more helpful to regard P not as an element of the boolean hierarchy (unless $\text{P} = \text{NP}$).

it holds that $x \leq y$ or $y \leq x$ (i.e., any two elements are comparable with respect to $\leq$), and a poset G is an antichain if for all $x, y \in G$ it holds that $x \leq y$ implies $x = y$ (i.e., all elements are pairwise incomparable with respect to $\leq$). A finite poset $(G, \leq)$ is a lattice if for all $x, y \in G$ there exist (a) exactly one maximal element $z \in G$ such that $z \leq x$ and $z \leq y$ (which will be denoted by $x \wedge y$), and (b) exactly one minimal element $z \in G$ such that $z \geq x$ and $z \geq y$ (which will be denoted by $x \vee y$). For a finite lattice G we denote by 1_G the unique element greater than or equal to all $x \in G$ and by 0_G the unique element less than or equal to all $x \in G$. An element $x \neq 1_G$ is said to be *meet-irreducible* iff $x = a \wedge b$ implies $x = a$ or $x = b$ for all $a, b \in G$.

Functions. Let M and M' be any sets, and let $f : M \to M'$ be any function. The domain of f is denoted by D_f. For a set $A \subseteq D_f$, let $f(A) = \{f(x) \mid x \in A\}$ and let $f|_A$ denote the restriction of f to A. In particular, the range of f which is denoted by R_f is $f(D_f)$. The inverse of f is denoted by f^{-1}, i.e, $f^{-1} : B \to \mathcal{P}(M)$ such that for all $y \in B$, $f^{-1}(y) = \{x \in M \mid f(x) = y\}$. If $f^{-1}(y)$ is a singleton then we omit the braces. We use id_M to denote the identity map on M given by $\mathrm{id}_M(x) = x$ for all $x \in M$. Our use of the composition $f \circ f'$ is $(f \circ f')(x) =_{\mathrm{def}} f(f'(x))$. If f maps M to itself, then for $m \in \mathbb{N}_+$, $f^m : M \to M$ is the m-fold composition of f with itself. Let $M = \{a, b\}$ with $a \neq b$. Define $\overline{a} = b$ and $\overline{b} = a$. For any function $f : M^m \to M'$ with $m \in \mathbb{N}_+$, let f^∂ denote its dual function, that is, that function defined for all $x = (x_1, \ldots, x_m) \in M^m$ as $f^\partial(x_1, \ldots, x_m) =_{\mathrm{def}} f(\overline{x_1}, \ldots, \overline{x_m})$. The vector $(\overline{x_1}, \ldots, \overline{x_m})$ is denoted by $\overline{x}$.

Words. We will make no distinction between m-tuples $(x_1, \ldots, x_m)$ over a finite set (alphabet) M and words $x_1 \ldots x_m$ of length m over M. We fix the finite alphabet $\Sigma = \{0, 1\}$ for considerations about the input-output behavior of machines. More generally, let Δ be any finite alphabet. Δ^* is the set of all finite words that can be built with letters from Δ. For $x, y \in \Delta^*$, $x \cdot y$ (or xy for short) denotes the concatenation of x and y. The empty word is denoted by ε. For a given word $x = x_1 \ldots x_m$ the reversed word $x_m \ldots x_1$ is denoted by x^R. For $x \in \Delta^*$, $|x|$ denotes the length of x. For $n \in \mathbb{N}$, $\Delta^{\leq n}$ is the set of all words $x \in \Delta^*$ with $|x| \leq n$, and $\Delta^{=n}$ is the set of all words $x \in \Delta^*$ with $|x| = n$. If the alphabet Δ is ordered by $\leq$, then let $\leq_{\mathrm{lex}}$ denote the standard lexicographical order on Δ^*, that is, for each $x, y \in \Delta^*$, $x \leq_{\mathrm{lex}} y$ if and only if (a) $x = y$, (b) $|x| < |y|$, or (c) $|x| = |y|$ and there is an i with $x_j = y_j$ for all $j \in \{1, \ldots, i-1\}$ but $x_i < y_i$. Usually we consider words x and y of the same length n to be partially ordered by the vector-ordering, that is, $x \leq y$ iff $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$.

Basic Complexity Theory. The computational model we refer to is the standard Turing machine (for a formal description see, e.g., [40, 2]). We consider nondeterministic and deterministic versions of Turing machines. A Turing machine that can produce outputs on a special output tape is called a Turing transducer. We also consider Turing machines that have access to an oracle. The notions translate accordingly to such oracle Turing machines. If we consider an oracle Turing machine M accessing an oracle A then this is denoted by M^A.

Polynomial-time Turing machines are Turing machines that for a fixed polynomial p, make on every input x at most $p(|x|)$ computation steps before reaching a final state. In case of a nondeterministic polynomial-time Turing machine M, the set of all words accepted by M, denoted by $L(M)$, is the set of all words $x \in \Sigma^*$ for which M, on input x, has at least one computation path of at most $p(|x|)$ steps of running, that ends in an accepting final state. NP (P) is the class of all sets that are accepted by nondeterministic (deterministic) polynomial-time Turing machines. NP^B is the class of all sets that are accepted by nondeterministic polynomial-time Turing machine accessing the set B. For a class $\mathcal{K}$, $\mathrm{NP}^{\mathcal{K}}$ consists of all sets that belong to

NP^B for some $B \in \mathcal{K}$. The polynomial hierarchy [31, 36] is inductively defined as follows.

$$\Sigma_0^p =_{\mathrm{def}} \mathrm{P}, \quad \Sigma_{m+1}^p =_{\mathrm{def}} \mathrm{NP}^{\Sigma_m^p}, \quad \text{and} \quad \mathrm{PH} =_{\mathrm{def}} \bigcup_{m \in \mathbb{N}} \Sigma_m^p.$$

Let REC denote the class of all recursive sets, i.e., those sets that can be decided by deterministic Turing machines. RE denotes the class of all recursively enumerable sets, i.e., the class of all sets that are ranges of deterministic Turing transducers.

FP denotes the class of all functions that are computable by a deterministic polynomial-time Turing transducer. We say that a set $A \subseteq \Sigma^*$ is polynomial-time many-one reducible to a set $B \subseteq \Sigma^*$, in symbols $A \leq_m^p B$, if and only if there exists a function $f \in \mathrm{FP}$ such that for all $x \in \Sigma^*$, $x \in A \iff f(x) \in B$. A class $\mathcal{K} \subseteq \mathcal{P}(\Sigma^*)$ is closed under $\leq_m^p$ if for all $A, B \subseteq \Sigma^*$ it holds that $A \leq_m^p B$ and $B \in \mathcal{K}$ imply that $A \in \mathcal{K}$. All classes in the polynomial hierarchy are closed under $\leq_m^p$. A set A is $\leq_m^p$-complete for $\mathcal{K}$ if $A \in \mathcal{K}$ and $B \leq_m^p A$ for all $B \in \mathcal{K}$. SATISFIABILITY, denoting the set of all (encodings of) satisfiable propositional formulas, is an example of a set $\leq_m^p$-complete for NP.

We implicitly use the following correspondence val between Σ^* and $\mathbb{N}$: For $x \in \Sigma^*$, define $\mathrm{val}(x) =_{\mathrm{def}} \|\{y \in \Sigma^* \mid y <_{\mathrm{lex}} x\}\|$. Note that val is polynomial-time computable and invertible.

It is often needed to encode tuples of words of Σ^* into one word of Σ^*. Let $\langle \cdot, \cdot \rangle_2$ denote a standard polynomial-time computable and polynomial-time invertible pairing function on finite words (e.g., based on self-delimiting words; cf. [30]). This pairing function is used to define encodings of arbitrary m-tuples as $\langle x_1, \ldots, x_m \rangle =_{\mathrm{def}} \langle m, \langle x_1, \langle \ldots, \langle x_{m-1}, x_m \rangle_2 \ldots \rangle_2 \rangle_2 \rangle_2$. Conversely, if a word $\langle x_1, \ldots, x_m \rangle \in \Sigma^*$ is given then the function π_j^m denotes the projection to the j-th component of the m-tuple, i.e., $\pi_j^m(\langle x_1, \ldots, x_m \rangle) = x_j$. If h is any function mapping from Δ^* to Σ^*, then we define the function $\langle \pi_{i_1}^m, \ldots, \pi_{i_n}^m \rangle \circ h : \Delta^* \to \Sigma^*$ with $n \leq m$ to be for all $x \in \Delta^*$, $(\langle \pi_{i_1}^m, \ldots, \pi_{i_n}^m \rangle \circ h)(x) =_{\mathrm{def}} \langle \pi_{i_1}^m(h(x)), \ldots, \pi_{i_n}^m(h(x)) \rangle$.

Let poly denote the class of all functions $f : \mathbb{N} \to \Sigma^*$ such that there exists a polynomial p with $|f(n)| \leq p(n)$ for all $n \in \mathbb{N}$. For a class $\mathcal{K} \subseteq \mathcal{P}(\Sigma^*)$, the class $\mathcal{K}/\mathrm{poly}$ [25] is the class of all sets A for which there exist a set $B \in \mathcal{K}$ and a function $f \in \mathrm{poly}$ (the advice function) such that for all $x \in \Sigma^*$, $x \in A \iff \langle x, f(|x|) \rangle \in B$.

Partitions. Finally, let us make some notational conventions about partitions. For any set M, a k-tuple $A = (A_1, \ldots, A_k)$ with $A_i \subseteq M$ for each $i \in \{1, \ldots, k\}$ is said to be a *k-partition of M* if and only if $A_1 \cup A_2 \cup \cdots \cup A_k = M$ and $A_i \cap A_j = \emptyset$ for all i, j with $i \neq j$. The set A_i is said to be the *i-th component of A*. For two k-partitions A and B to be equal it is sufficient that $A_i \subseteq B_i$ for all $i \in \{1, \ldots, k\}$. Let $c_A : M \to \{1, \ldots, k\}$ be the characteristic function of a k-partition $A = (A_1, \ldots, A_k)$ of M, that is, $c_A(x) = i$ if and only if $x \in A_i$. For $\mathcal{K}_1, \ldots, \mathcal{K}_k \subseteq \mathcal{P}(M)$ let

$$(\mathcal{K}_1, \ldots, \mathcal{K}_k) =_{\mathrm{def}} \left\{ A \mid A \text{ is } k\text{-partition of } M \text{ and } A_i \in \mathcal{K}_i \text{ for all } i \in \{1, \ldots, k\} \right\}$$

and for $i \in \{1, \ldots, k\}$,

$$(\mathcal{K}_1, \ldots, \mathcal{K}_{i-1}, \cdot, \mathcal{K}_{i+1}, \ldots, K_k) =_{\mathrm{def}} (\mathcal{K}_1, \ldots, \mathcal{K}_{i-1}, \mathcal{P}(M), \mathcal{K}_{i+1}, \ldots, \mathcal{K}_k).$$

For a class $\mathcal{K}$ of k-partitions, let $\mathcal{K}_i =_{\mathrm{def}} \left\{ A_i \mid A \in \mathcal{K} \right\}$ be the *i-th projection of $\mathcal{K}$*. Obviously, $\mathcal{K} \subseteq (\mathcal{K}_1, \ldots, \mathcal{K}_k)$. In what follows we identify a set A with the 2-partition $(A, \overline{A})$. We thus use a characteristic function which on the complement of A, differs to the usual one for sets.

However, using 2 on the complement instead of 0 has the advantage of corresponding well with the vector-ordering as becomes clearer later in the paper. We identify a class $\mathcal{K}$ of sets with the class $(\mathcal{K}, \mathrm{co}\mathcal{K}) = (\mathcal{K}, \cdot) = (\cdot, \mathrm{co}\mathcal{K})$ of 2-partitions.

3 Partition Classes Defined by Finite Functions

Let $\mathcal{K}$ be a class of subsets of M such that $\emptyset, M \in \mathcal{K}$ and $\mathcal{K}$ is closed under intersection and union. In the literature, one way to define the classes of the boolean hierarchy of sets over $\mathcal{K}$ is as follows (see [39]). Let $f : \{1,2\}^m \to \{1,2\}$ be a boolean function. For $B_1, \ldots, B_m \in \mathcal{K}$ the set $f(B_1, \ldots, B_m)$ is defined by $c_{f(B_1,\ldots,B_m)}(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x))$. Then the classes $\mathcal{K}(f) =_{\mathrm{def}} \{\, f(B_1, \ldots, B_m) \mid B_1, \ldots, B_m \in \mathcal{K} \,\}$ form the boolean hierarchy over $\mathcal{K}$. Using finite functions $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$ we generalize this definition (remember in which sense sets are 2-partitions) to obtain the classes of the boolean hierarchy of k-partitions over $\mathcal{K}$ as follows.

Definition 2 *Let $k \geq 2$.*

1. *For any function $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$ with $m \geq 1$ and for sets $B_1, \ldots, B_m \in \mathcal{K}$, the k-partition $f(B_1, \ldots, B_m)$ is defined such that for all $x \in M$,*

$$c_{f(B_1,\ldots,B_m)}(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x)).$$

2. *For any function $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$ with $m \geq 1$, the* class of k-partitions over $\mathcal{K}$ defined by f *is given by the class*

$$\mathcal{K}(f) =_{\mathrm{def}} \{\, f(B_1, \ldots, B_m) \mid B_1, \ldots, B_m \in \mathcal{K} \,\}.$$

3. *The* boolean hierarchy of k-partitions over $\mathcal{K}$ *is defined to be the family*

$$\mathrm{BH}_k(\mathcal{K}) =_{\mathrm{def}} \{\, \mathcal{K}(f) \mid f : \{1,2\}^m \to \{1, 2, \ldots, k\} \text{ and } m \geq 1 \,\}.$$

4. $\mathrm{BC}_k(\mathcal{K}) =_{\mathrm{def}} \bigcup \mathrm{BH}_k(\mathcal{K})$.

Obviously if $i \in \{1, 2, \ldots, k\}$ is not a value of $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$ then $\mathcal{K}(f)_i = \{\emptyset\}$, that is $\mathcal{K}(f)$ does not really have an i-th component. Therefore we assume in what follows that f is surjective.

The following proposition shows that every partition in $\mathcal{K}(f)$ consists of sets from the boolean hierarchy over $\mathcal{K}$. This also justifies the use of the term *boolean* in the above definition.

Proposition 3 *Let $k \geq 2$ and let $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$ be any function with $m \geq 1$.*

1. $(\mathcal{K}, \ldots, \mathcal{K}) \subseteq \mathcal{K}(f) \subseteq (\mathrm{BC}(\mathcal{K}), \ldots, \mathrm{BC}(\mathcal{K}))$.

2. *If $\mathcal{K}$ is closed under complements then $\mathcal{K}(f) = (\mathcal{K}, \ldots, \mathcal{K})$.*

3. $\mathrm{BC}_k(\mathcal{K}) = (\mathrm{BC}(\mathcal{K}), \ldots, \mathrm{BC}(\mathcal{K}))$.

Proof.

1. We first show that $\mathcal{K}(f) \subseteq (\mathrm{BC}(\mathcal{K}), \dots, \mathrm{BC}(\mathcal{K}))$. Let $B_1, \dots, B_m$ be sets in $\mathcal{K}$, and consider the k-partition $A = f(B_1, \dots, B_m)$. For each $i \in \{1, 2, \dots, k\}$, we obtain

$$x \in A_i \iff \bigvee_{f(a_1 \dots a_m)=i} \bigwedge_{j=1}^{m} c_{B_j}(x) = a_j$$

and consequently

$$A_i \quad = \quad \bigcup_{f(a_1 \dots a_m)=i} \left[\left(\bigcap_{a_j=1} B_j \right) \setminus \left(\bigcup_{a_j=2} B_j \right) \right]. \tag{2}$$

Clearly, this gives $A_i \in \mathcal{K}(2 \cdot \|f^{-1}(i)\|)$.

Now we prove $(\mathcal{K}, \dots, \mathcal{K}) \subseteq \mathcal{K}(f)$. Let A be a k-partition in $(\mathcal{K}, \dots, \mathcal{K})$. For every $i \in \{1, 2, \dots, k\}$, fix some $v_i \in \{1, 2\}^m$ such that $f(v_i) = i$. Define for all $j \in \{1, 2, \dots, m\}$, sets B_j as

$$B_j =_{\text{def}} \bigcup_{v_i \leq 2^{j-1} 1 2^{m-j}} A_i.$$

It is easily observed that for all $a_1 \dots a_m \in \{1, 2\}^m$,

$$\bigcap_{a_j=1} B_j = \bigcup_{v_l \leq a_1 \dots a_m} A_l \quad \text{and} \quad \bigcup_{a_j=2} B_j = \bigcup_{v_l < a_1 \dots a_m} A_l.$$

By Equation (2) we obtain $A = f(B_1, \dots, B_m)$.

2. This statement is an immediate consequence of the first one.

3. The inclusion $\mathrm{BC}_k(\mathcal{K}) \subseteq (\mathrm{BC}(\mathcal{K}), \dots, \mathrm{BC}(\mathcal{K}))$ follows directly from 1. For the converse inclusion let $A \in (\mathrm{BC}(\mathcal{K}), \dots, \mathrm{BC}(\mathcal{K}))$, i.e., there exists an $r \geq 1$ such that for all $i \in \{1, 2, \dots, k\}$, $A_i \in \mathcal{K}(r)$. Hence there exist sets $B_1, \dots, B_{k \cdot r} \in \mathcal{K}$ such that for all $i \in \{1, 2, \dots, k\}$,

$$A_i = B_{(i-1) \cdot r+1} \triangle B_{(i-1) \cdot r+2} \triangle \cdots \triangle B_{i \cdot r}.$$

Observe that for every $a_1 \dots a_{k \cdot r}$, there exists an $i \in \{1, 2, \dots, k\}$ such that

$$\left(\bigcap_{a_j=1} B_j \right) \cap \left(\bigcap_{a_j=2} \overline{B_j} \right) \subseteq A_i.$$

Thus, we can define $f : \{1, 2\}^{k \cdot r} \to \{1, 2, \dots, k\}$ such that for all $a_1 \dots a_{k \cdot r} \in \{1, 2\}^{k \cdot r}$,

$$f(a_1 \dots a_{k \cdot r}) = i \iff_{\text{def}} \left(\bigcap_{a_j=1} B_j \right) \cap \left(\bigcap_{a_j=2} \overline{B_j} \right) \subseteq A_i,$$

and we obtain $A = f(B_1, \dots, B_{k \cdot r})$.

❑

For $k = 2$ the classes $\mathcal{K}(f)$ of the boolean hierarchy $\mathrm{BH}_2(\mathcal{K})$ of sets (2-partitions) have been completely characterized. For $f : \{1,2\}^m \to \{1,2\}$ let $\mu(f)$ be the maximum number of alternations of f-labels which can occur in a $\leq$-chain in $(\{1,2\}^m, \leq)$.

Theorem 4 [39] *For* $f : \{1,2\}^m \to \{1,2\}$,

$$\mathcal{K}(f) = \begin{cases} \mathcal{K}(\mu(f)) & \text{if } f(2^m) = 2, \\ \mathrm{co}\mathcal{K}(\mu(f)) & \text{if } f(2^m) = 1. \end{cases}$$

Consequently, $\mathrm{BH}_2(\mathcal{K}) = \{\ \mathcal{K}(m) \mid m \in \mathbb{N}_+\ \} \cup \{\ \mathrm{co}\mathcal{K}(m) \mid m \in \mathbb{N}_+\ \}$, and given a function $f : \{1,2\}^m \to \{1,2\}$ it is easy to determine the class $\mathcal{K}(m)$ or $\mathrm{co}\mathcal{K}(m)$ which coincides with $\mathcal{K}(f)$. As already mentioned above, the classes of $\mathrm{BH}_2(\mathcal{K})$ form a simple structure with respect to set inclusion. There do not exist three classes in $\mathrm{BH}_2(\mathcal{K})$ which are incomparable in this sense.

It is the goal of this chapter to get insights into the structure of the boolean hierarchy $\mathrm{BH}_k(\mathrm{NP})$ of k-partitions over NP for $k \geq 3$. What we can say at this point is, that already for $k = 3$ the structure of $\mathrm{BH}_k(\mathrm{NP})$ with respect to set inclusion is not as simple as for $k = 2$ (unless $\mathrm{NP} = \mathrm{coNP}$). This is shown by the following example.

Example 5 *For* a, b, c *such that* $\{a, b, c\} = \{1, 2, 3\}$ *define the function* $f_{abc} : \{1,2\}^2 \to \{1,2,3\}$ *by* $f_{abc}(11) = a$, $f_{abc}(12) = f_{abc}(21) = b$, *and* $f_{abc}(22) = c$. *Obviously,* $\mathrm{NP}(f_{abc})_a = \mathrm{NP}$, $\mathrm{NP}(f_{abc})_b = \mathrm{NP}(2)$, *and* $\mathrm{NP}(f_{abc})_c = \mathrm{coNP}$. *Now let* $abc \neq a'b'c'$. *If* $\mathrm{NP}(f_{abc}) = \mathrm{NP}(f_{a'b'c'})$ *then* $\mathrm{NP} = \mathrm{NP}(2)$ *or* $\mathrm{NP} = \mathrm{coNP}$, *or* $\mathrm{NP}(2) = \mathrm{coNP}$. *In each of these cases we obtain* $\mathrm{NP} = \mathrm{coNP}$. *Consequently, if* $\mathrm{NP} \neq \mathrm{coNP}$ *the six classes* $\mathrm{NP}(f_{abc})$ *are pairwise incomparable with respect to set inclusion.*

Definition 2 refers to a set class $\mathcal{K}$ with $\emptyset, M \in \mathcal{K}$ and which is closed under intersection and union. As $\mathcal{K}$ so $\mathrm{co}\mathcal{K}$ easily satisfies these conditions as well. Thus, all the definitions can be applied to $\mathrm{co}\mathcal{K}$. The following theorem shows that there is a very close connection between classes from $\mathrm{BH}_k(\mathcal{K})$ and classes from $\mathrm{BH}_k(\mathrm{co}\mathcal{K})$.

Theorem 6 $\mathcal{K}(f) = \mathrm{co}\mathcal{K}(f^\partial)$ *for all* $f : \{1,2\}^m \to \{1,2,\ldots,k\}$ *with* $m \geq 1$ *and* $k \geq 2$.

Proof. By symmetry, it suffices to show $\mathcal{K}(f) \subseteq \mathrm{co}\mathcal{K}(f^\partial)$. Therefore, consider a partition $A \in \mathcal{K}(f)$. Then there are sets $B_1, \ldots, B_m \in \mathcal{K}$ such that $A = f(B_1, \ldots, B_m)$. Since for all $a_1 \ldots a_m \in \{1,2\}^m$, $f(a_1 \ldots a_m) = f^\partial(\overline{a_1} \ldots \overline{a_m})$, we obtain that for all $x \in M$,

$$f(c_{B_1}(x), \ldots, c_{B_m}(x)) = f^\partial(c_{\overline{B_1}}(x), \ldots, c_{\overline{B_m}}(x)).$$

This gives $A = f(B_1, \ldots, B_m) = f^\partial(\overline{B_1}, \ldots, \overline{B_m})$. Hence, $A \in \mathrm{co}\mathcal{K}(f^\partial)$. ❑

In particular, $\mathrm{BH}_k(\mathcal{K})$ and $\mathrm{BH}_k(\mathrm{co}\mathcal{K})$ coincide even if $\mathcal{K}$ is not closed under complements.

Corollary 7 $\mathrm{BH}_k(\mathcal{K}) = \mathrm{BH}_k(\mathrm{co}\mathcal{K})$ *for all* $k \geq 2$.

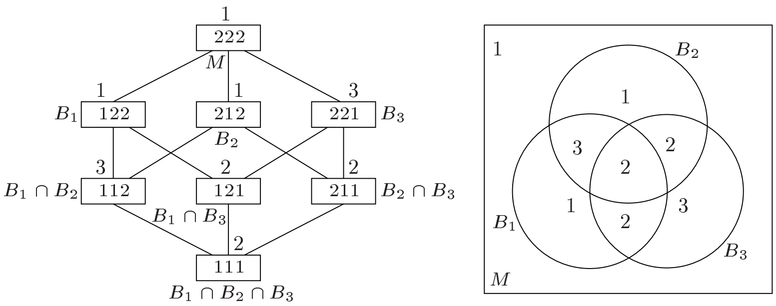

Figure 1: Partition defined by a boolean 3-lattice

4 Partition Classes Defined by Lattices

It turns out that, for $f : \{1,2\}^m \to \{1,2,\dots,k\}$, a k-partition $f(B_1,\dots,B_m)$ has a very natural equivalent lattice-theoretical definition. Consider the boolean lattice $\{1,2\}^m$ with the partial vector-ordering $\le$, and consider the function $S : \{1,2\}^m \to \mathcal{K}$ defined by

$$S(a_1,\dots,a_m) =_{\text{def}} \bigcap_{a_i=1} B_i,$$

where we define an intersection over an empty index set to be M. For an example see Figure 1. Note that $S(2,\dots,2) = M$ and $S(a \wedge b) = S(a) \cap S(b)$ for all $a, b \in \{1,2\}^m$. Defining

$$T_S(a) =_{\text{def}} S(a) \setminus \bigcup_{b<a} S(b)$$

we obtain the i-th component of $f(B_1,\dots,B_m)$ as

$$f(B_1,\dots,B_m)_i = \bigcup_{f(a)=i} T_S(a),$$

i.e., $f(B_1,\dots,B_m)$ can also be given by the function $S : \{1,2\}^m \to \mathcal{K}$.

On the other side, if we have any function $S : \{1,2\}^m \to \mathcal{K}$ such that $S(2,\dots,2) = M$ and $S(a \wedge b) = S(a) \cap S(b)$ for all $a, b \in \{1,2\}^m$ we can define

$$B_j =_{\text{def}} S(2^{j-1}12^{m-j}) \qquad \text{for } j \in \{1,2,\dots,m\},$$

and we obtain for $i \in \{1,2,\dots,k\}$

$$f(B_1,\dots,B_m)_i = \bigcup_{f(a)=i} T_S(a).$$

In this manner the class $\mathcal{K}(f)$ of k-partitions is completely characterized by the labeled boolean lattice $((\{1,2\}^m, \le), f)$.

In this section we will see that classes of k-partitions can also be defined by weaker structures than boolean algebras. Again we always suppose $\mathcal{K}$ to be a class such that $\emptyset, M \in \mathcal{K}$ and which is closed under intersection and union.

Definition 8 *Let G be a lattice.*

1. *A mapping $S : G \to \mathcal{K}$ is said to be a* $\mathcal{K}$-homomorphism on G *if and only if*
   (a) *$S(1_G) = M$ and*
   (b) *$S(a \wedge b) = S(a) \cap S(b)$ for all $a, b \in G$.*
2. *For a $\mathcal{K}$-homomorphism S on G and $a \in G$, let*

$$T_S(a) =_{\text{def}} S(a) \setminus \bigcup_{b<a} S(b).$$

Lemma 9 *Let G be a lattice, and let S be a $\mathcal{K}$-homomorphism on G.*

1. *$T_S(a) \in \mathcal{K} \wedge \text{co}\mathcal{K}$ for every $a \in G$.*
2. *$S(a) = \bigcup_{b \leq a} T_S(b)$ for every $a \in G$.*
3. *The set of all $T_S(a)$ for $a \in G$ yields a partition of M.*
4. *S is completely determined by its values for the meet-irreducible elements. That is, if S and S' are two $\mathcal{K}$-homomorphisms on G such that $S(a) = S'(a)$ for all meet-irreducible $a \in G$ then $S(a) = S'(a)$ for all $a \in G$.*

Proof.

1. Observe $T_S(a) = S(a) \cap \overline{\bigcup_{b<a} S(b)} \in \mathcal{K} \wedge \text{co}\mathcal{K}$ since $\mathcal{K}$ is closed under union.
2. The direction "$\supseteq$" is obvious since $T_S(b) \subseteq S(b) \subseteq S(a)$ for $b \leq a$. The converse inclusion can be verified by induction on $<$. Obviously, $S(0_G) = T_S(0_G)$. For $a > 0_G$ we obtain

$$S(a) = T_S(a) \cup \bigcup_{b<a} S(b) = T_S(a) \cup \bigcup_{b<a} \bigcup_{c \leq b} T_S(c) = T_S(a) \cup \bigcup_{c<a} T_S(c) = \bigcup_{c \leq a} T_S(c).$$

3. We have to show that every $x \in M$ is contained in exactly one $T_S(a)$. Proving the existence of such an $a \in G$, define
$$H =_{\text{def}} \{\ a \mid x \in S(a)\ \}$$
which is non-empty since $\bigcup_{a \in G} S(a) = M$. Since G is finite it follows that $x \in S(\bigwedge H)$. Let $b < \bigwedge H$. Then $b \notin H$, and hence $x \notin S(b)$. So, $x \in S(\bigwedge H) \setminus \bigcup_{b < \bigwedge H} S(b) = T_S(\bigwedge H)$. To show the uniqueness assume that there is an $a \neq \bigwedge H$ such that $x \in T_S(a)$. Then $x \in S(a)$ and hence $a \in H$. Consequently, $a > \bigwedge H$ and we obtain $x \notin S(a) \setminus \bigcup_{b<a} S(b) = T_S(a)$, a contradiction.
4. This is an immediate consequence of the definition of meet-irreducible elements and the condition $S(a \wedge b) = S(a) \cap S(b)$ for $\mathcal{K}$-homomorphisms.

❑

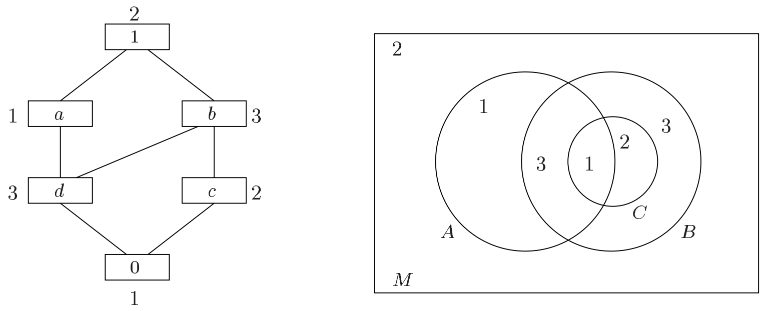

Figure 2: Partition defined by a 3-lattice

Any pair (G, f) of an arbitrary finite poset G and a function $f : G \to \{1, 2, \ldots, k\}$ is called a *k-poset*. A k-poset which is a lattice (boolean lattice) is called a *k-lattice* (*boolean k-lattice*, resp.).

Lemma 9 provides the soundness of the following definition.

Definition 10 *Let (G, f) be a k-lattice, $k \geq 2$.*

1. *For a $\mathcal{K}$-homomorphism S on G, the* k-partition defined by (G, f) and S *is given by*

$$(G, f, S) =_{\text{def}} \left(\bigcup_{f(a)=1} T_S(a) \ , \ldots , \ \bigcup_{f(a)=k} T_S(a) \right).$$

2. *The* class of k-partitions defined by (G, f) *is given by*

$$\mathcal{K}(G, f) =_{\text{def}} \left\{ \ (G, f, S) \ \middle| \ S \text{ is } \mathcal{K}\text{-homomorphism on } G \ \right\}.$$

Example 11 *Consider the* 3*-lattice (G, f) in Figure 2. The meet-irreducible elements of G are a, b, and c. By point* 4 *of Lemma 9 every $\mathcal{K}$-homomorphism $S : G \to \mathcal{K}$ is determined by fixing $S(a) = A$, $S(b) = B$, and $S(c) = C$. By the definition of $\mathcal{K}$-homomorphisms we get $S(1) = M$, $S(d) = S(a \wedge b) = S(a) \cap S(b) = A \cap B$, and $S(0) = S(d \wedge c) = S(d) \cap S(c) = A \cap B \cap C$. Furthermore, $C = S(c) = S(c \wedge b) = S(c) \cap S(b) = C \cap B$, i.e., $C \subseteq B$. We obtain*

$$\begin{array}{rclcl}
T_S(1) & = & M \setminus (A \cup B) & = & \overline{A} \cap \overline{B}, \\
T_S(a) & = & A \setminus (A \cap B) & = & A \cap \overline{B}, \\
T_S(b) & = & B \setminus ((A \cap B) \cup C) & = & \overline{A} \cap B \cap \overline{C}, \\
T_S(c) & = & C \setminus (A \cap B \cap C) & = & \overline{A} \cap C, \\
T_S(d) & = & (A \cap B) \setminus (A \cap B \cap C) & = & A \cap B \cap \overline{C}, \\
T_S(0) & = & (A \cap B \cap C) & = & A \cap C.
\end{array}$$

Hence

$$\begin{array}{rcl}
(G, f, S) & = & (T_S(a) \cup T_S(0), T_S(1) \cup T_S(c), T_S(b) \cup T_S(d)) \\
& = & (A \cap (\overline{B} \cup C), \overline{A} \cap (\overline{B} \cup C), B \cap \overline{C}),
\end{array}$$

and

$$\begin{array}{rcl}
\mathcal{K}(G, f) & = & \left\{ \ (A \cap (\overline{B} \cup C), \overline{A} \cap (\overline{B} \cup C), B \cap \overline{C}) \ \middle| \ A, B, C \in \mathcal{K} \text{ and } C \subseteq B \ \right\} \\
& \subseteq & (\mathcal{K}(3), \text{co}\mathcal{K}(3), \mathcal{K}(2)).
\end{array}$$

The discussion at the beginning of the section yields the following proposition.

Proposition 12 $\mathcal{K}(f) = \mathcal{K}((\{1,2\}^m, \leq), f)$ *for all* $f : \{1,2\}^m \to \{1,2,\ldots,k\}$ *with* $m \geq 1$ *and* $k \geq 2$.

So, if (G,f) is a boolean k-lattice then $\mathcal{K}(G,f) = \mathcal{K}(f)$. But if (G,f) is an arbitrary k-lattice, is $\mathcal{K}(G,f)$ also of the form $\mathcal{K}(f')$ for a suitable function f'? The following theorem says that this is generally true. This turns out to be very important for the further study of the structure of the boolean hierarchy of k-partitions because instead of large boolean k-lattices one can deal with usually much smaller equivalent k-lattices.

Theorem 13 *For every k-lattice (G,f) there is an $f' : \{1,2\}^m \to \{1,2,\ldots,k\}$ with $\mathcal{K}(G,f) = \mathcal{K}(f')$, where m is the number of meet-irreducible elements of G.*

We postpone the proof of this theorem to Section 5 where we can make use of the Embedding Lemma (Lemma 16).

Corollary 14 $\mathrm{BH}_k(\mathcal{K}) = \left\{ \mathcal{K}(G,f) \;\middle|\; (G,f) \text{ is a } k\text{-lattice} \right\}$ *for all* $k \geq 2$.

5 Comparing Partition Classes

To study the structure of the boolean hierarchy of k-partitions over $\mathcal{K}$ it would be important to have a criterion to decide whether $\mathcal{K}(G,f) \subseteq \mathcal{K}(G',f')$ for any two k-lattices (G,f) and (G',f'). To this end we establish, more generally, a relation $\leq$ between k-posets.

Definition 15 *Let (G,f) and (G',f') be k-posets with $k \geq 2$.*

1. *$(G,f) \leq (G',f')$ if and only if there is a monotonic mapping $\varphi : G \to G'$ such that for every $x \in G$, $f(x) = f'(\varphi(x))$.*
2. *$(G,f) \equiv (G',f')$ if and only if $(G,f) \leq (G',f')$ and $(G',f') \leq (G,f)$.*

The following lemma gives a sufficient condition for $\mathcal{K}(G,f) \subseteq \mathcal{K}(G',f')$.

Lemma 16 (Embedding Lemma.) *Let (G,f) and (G',f') be k-lattices with $k \geq 2$. If $(G,f) \leq (G',f')$, then $\mathcal{K}(G,f) \subseteq \mathcal{K}(G',f')$.*

Proof. Let (G,f) and (G',f') be k-lattices with $(G,f) \leq (G',f')$. Let $\varphi : G \to G'$ be a monotonic mapping such that $f(a) = f'(\varphi(a))$ for every $a \in G$. For a $\mathcal{K}$-homomorphism S on G define the mapping $S' : G' \to \mathcal{K}$ for all $a \in G'$ by

$$S'(a) =_{\mathrm{def}} \bigcup_{\varphi(b) \leq' a} S(b).$$

It is sufficient to prove that S' is a $\mathcal{K}$-homomorphism on G' with $(G,f,S) = (G',f',S')$, i.e., that

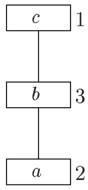

Figure 3: A 3-chain equivalent to the boolean 3-lattice in Figure 1

1. $S'(1_{G'}) = M$,
2. $S'(a \wedge' b) = S'(a) \cap S'(b)$ for all $a, b \in G$,
3. $T_S(a) \subseteq T_{S'}(\varphi(a))$ for all $a \in G$.

This can be shown as follows:

1. We conclude $S'(1_{G'}) = \bigcup_{\varphi(b) \leq' 1_{G'}} S(b) \supseteq S(1_G) = M$.
2. The inclusion "$\subseteq$" is valid because of the monotonicity of S'. For the converse inclusion consider $x \in S'(a) \cap S'(b)$. There exist $c, d \in G$ such that $\varphi(c) \leq' a$, $\varphi(d) \leq' b$, $x \in S(c)$, and $x \in S(d)$. We obtain $\varphi(c \wedge d) \leq' \varphi(c) \wedge' \varphi(d) \leq' a \wedge' b$ and $x \in S(c) \cap S(d) = S(c \wedge d)$, and consequently $x \in S'(a \wedge' b)$.
3. For $a \in G$ and $x \in T_S(a)$ we obtain $x \in S(a) \subseteq S'(\varphi(a))$. Assume that $x \notin T_{S'}(\varphi(a))$. Then there exists a $c <' \varphi(a)$ such that $x \in S'(c)$. Consequently, there exists a $b \in G$ such that $\varphi(b) \leq' c$ and $x \in S(b)$. Hence $x \in S(a) \cap S(b) = S(a \wedge b)$. Because of $x \in T_S(a)$ we get $a \wedge b \not< a$ and thus $a \leq b$. We conclude $\varphi(a) \leq' \varphi(b) \leq' c$, a contradiction.

❑

Example 17 *The 3-lattice (G, f) shown in Figure 1 and the 3-lattice (G', f') shown in Figure 3 are equivalent. This can be seen as follows: Define the functions $\varphi : G \to G'$ and $\psi : G' \to G$ by*

$$\begin{aligned} \varphi(111) = \varphi(121) = \varphi(211) &= a, \\ \varphi(112) = \varphi(221) &= b, \\ \varphi(122) = \varphi(212) = \varphi(222) &= c, \end{aligned}$$

and

$$\psi(a) = 111, \quad \psi(b) = 112, \quad \textit{and } \psi(c) = 222.$$

It is easy to see that φ and ψ are monotonic, $f(x) = f'(\varphi(x))$ for all $x \in G$, and $f'(x) = f(\psi(x))$ for all $x \in G'$. By the Embedding Lemma we obtain $\mathcal{K}(G, f) = \mathcal{K}(G', f')$ for all $\mathcal{K}$. Obviously,

$$\mathcal{K}(G', f') = \left\{ \ (\overline{B}, A, B \setminus A) \ \middle| \ A, B \in \mathcal{K} \textit{ and } A \subseteq B \ \right\} = (\mathrm{co}\mathcal{K}, \mathcal{K}, \cdot) = (\mathrm{co}\mathcal{K}, \mathcal{K}, \mathcal{K}(2)).$$

Now we are able to prove Theorem 13 from Section 4.

Proof. (Theorem 13) Let (G, f) be an arbitrary k-lattice, let I be the set of meet-irreducible elements of G, and let

$$I_a =_{\text{def}} \left\{ \, b \;\middle|\; b \geq a \text{ and } b \text{ meet-irreducible} \, \right\}$$

for every $a \in G$. It is well known (cf. [15]) that $\bigwedge I_a = a$ for every $a \in G$. We define the boolean k-lattice $((\mathcal{P}(I), \supseteq), h)$ by

$$h(U) =_{\text{def}} f(\bigwedge U) \qquad \text{for } U \subseteq I.$$

The function $\varphi : G \to \mathcal{P}(I)$ defined by $\varphi(a) =_{\text{def}} I_a$ is monotonic, and we get

$$h(\varphi(a)) = h(I_a) = f(\bigwedge I_a) = f(a).$$

By the Embedding Lemma we obtain $\mathcal{K}(G, f) \subseteq \mathcal{K}((\mathcal{P}(I), \supseteq), h)$. On the other hand, the function $\psi : \mathcal{P}(I) \to G$ defined by $\psi(U) =_{\text{def}} \bigwedge U$ is monotonic, and we get

$$f(\psi(U)) = f(\bigwedge U) = h(U).$$

Again by the Embedding Lemma we obtain $\mathcal{K}((\mathcal{P}(I), \supseteq), h) \subseteq \mathcal{K}(G, f)$. So we get $\mathcal{K}(G, f) = \mathcal{K}((\mathcal{P}(I), \supseteq), h)$, but $(\mathcal{P}(I), \supseteq)$ and $(\{1, 2\}^{|I|}, \leq)$ are isomorphic. ❑

Combining this proof of Theorem 13 and the Embedding Lemma one can generalize Theorem 6 to the following theorem.

Theorem 18 $\mathcal{K}(G, f) = \text{co}\mathcal{K}(G^\partial, f)$ *for all k-lattices (G, f) with $k \geq 2$.*

Proof. Let (G, f) be any k-lattice. By Theorem 13 there is a function $f' : \{1, 2\}^m \to \{1, 2, \ldots, k\}$ with $\mathcal{K}(G, f) = \mathcal{K}(f')$. In fact, the proof of Theorem 13 shows that $(G, f) \equiv (\{1, 2\}^m, f')$. Regarding the dual function f'^∂ we obtain that $(G^\partial, f) \equiv (\{1, 2\}^m, f'^\partial)$. By Theorem 6 and the Embedding Lemma, $\mathcal{K}(G, f) = \mathcal{K}(f') = \text{co}\mathcal{K}(f'^\partial) = \text{co}\mathcal{K}(G^\partial, f)$. ❑

6 Minimal Descriptions of Partition Classes

From Proposition 12 and Theorem 13 we know that the boolean hierarchy of k-partitions is precisely the family of all partition classes over $\mathcal{K}$ generated by k-lattices. The advantage of this characterization is that k-lattices allow often smaller descriptions of partition classes than functions (as shown by Example 17). The usage of labeled lattices provides also another advantage over functions: The minimal representations of partition classes using k-lattices are essentially unique, i.e., unique *up to isomorphism.*

Definition 19 *For k-posets (G, f) and (G', f') we write $(G, f) \cong (G', f')$ and we say that (G, f) and (G', f') are* isomorphic *if there exists a bijective function $\varphi : G \to G'$ such that φ and φ^{-1} are monotonic and $f'(\varphi(a)) = f(a)$ for every $a \in G$.*

Obviously, isomorphic k-lattices are equivalent, but there are equivalent k-lattices that are not isomorphic. For example, add to any k-lattice (G, f) a new element a which is less than all elements of G, and define $f(a) = f(0_G)$. The new k-lattice is equivalent but not isomorphic to (G, f).

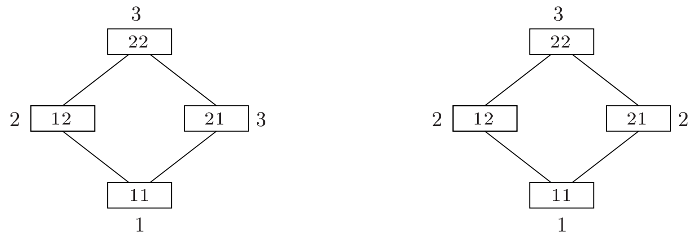

Figure 4: Non-isomorphic minimal equivalent boolean 3-lattices

Definition 20 *A finite k-lattice (k-poset) (G, f) is said to be* minimal *if there does not exist a k-lattice (k-poset, resp.) (G', f') such that $(G, f) \equiv (G', f')$ and $\|G'\| < \|G\|$.*

In this section we will prove that equivalent minimal k-lattices are isomorphic. This is a basic difference between k-lattices and k-valued functions (boolean k-lattices). Say that a function $f : \{1, 2\}^m \to \{1, 2, \ldots, k\}$ is minimal if there is no function of arity less than that of f, such that the corresponding boolean k-lattices are equivalent. The simple example in Figure 4 shows that minimal equivalent functions (boolean k-lattices) need not be isomorphic.

In order to prove our isomorphism theorem (Theorem 24) it seems to be easier to show this first for the case of posets.

Lemma 21 *Let (G, f) be a minimal k-poset, and let $\varphi : G \to G$ be a monotonic function such that $f(\varphi(a)) = f(a)$ for all $a \in G$. Then there exists an $m \geq 1$ with $\varphi^m = \mathrm{id}_G$.*

Proof. For every $a \in G$ let i_a be the smallest number such that there exists a $j > i_a$ with $\varphi^{i_a}(a) = \varphi^j(a)$, and let j_a be the smallest such j. Obviously,

$$\varphi^{i_a}\left(\{a, \varphi(a), \varphi^2(a), \ldots, \varphi^{j_a-1}(a)\}\right) = \{\varphi^{i_a}(a), \varphi^{i_a+1}(a), \ldots, \varphi^{j_a-1}(a)\}.$$

Note that the set $\{a, \varphi(a), \varphi^2(a), \ldots, \varphi^{j_a-1}(a)\}$ has exactly j_a elements and note also that the set $\{\varphi^{i_a}(a), \varphi^{i_a+1}(a), \ldots, \varphi^{j_a-1}(a)\}$ has exactly $j_a - i_a$ elements. Now assume $i_a > 0$. Then $\|\varphi^{i_a}(G)\| < \|G\|$ and $(\varphi^{i_a}(G), f) \equiv (G, f)$ which contradicts the minimality of (G, f). Hence $i_a = 0$ and $\varphi^{j_a}(a) = a$. Now let $m = \prod_{a \in G} j_a$ and get $\varphi^m = \mathrm{id}_G$. ❑

Lemma 22 *Equivalent minimal k-posets are isomorphic.*

Proof. Let (G, f) and (G', f') be equivalent minimal k-posets. There exist monotonic functions $\varphi : G \to G'$ and $\psi : G' \to G$ such that $f'(\varphi(a)) = f(a)$ for all $a \in G$ and $f(\psi(a)) = f'(a)$ for all $a \in G'$. Hence $\psi \circ \varphi$ is monotonic and $f(\psi(\varphi(a)) = f(a)$ for all $a \in G$. By Lemma 21 there exists an $m \geq 1$ such that $(\psi \circ \varphi)^m = \mathrm{id}_G$. Also $\varphi \circ \psi$ is monotonic and $f'(\varphi(\psi(a)) = f'(a)$ for all $a \in G'$, and there exists an $n \geq 1$ such that $(\varphi \circ \psi)^n = \mathrm{id}_{G'}$. Hence, $\psi \circ (\varphi \circ (\psi \circ \varphi)^{mn-1}) = \mathrm{id}_G$, $(\varphi \circ (\psi \circ \varphi)^{mn-1}) \circ \psi = \mathrm{id}_{G'}$, $\varphi \circ (\psi \circ \varphi)^{mn-1} : G \to G'$ is monotonic, $\psi : G' \to G$ is monotonic, and $f'(\varphi \circ (\psi \circ \varphi)^{mn-1}(a)) = f(a)$ for all $a \in G$. Thus $(G, f) \cong (G', f')$. ❑

Lemma 23 *A minimal k-poset, which is equivalent to a k-lattice, is a k-lattice.*

Proof. Let (G, f) be a minimal k-poset, and let (G', f') be a k-lattice such that $(G, f) \equiv (G', f')$ via $\varphi : G \to G'$ and $\psi : G' \to G$. By Lemma 21 there exists an $m \geq 1$ such that $(\psi \circ \varphi)^m = \mathrm{id}_G$. We define

$$\xi =_{\mathrm{def}} \varphi \circ (\psi \circ \varphi)^{m-1}.$$

Then we obtain $\psi \circ \xi = id_G$. To prove that G is a lattice it suffices to verify that

1. G has a supremum 1_G,
2. $a \wedge b$ exists for all $a, b \in G$.

This can be done as follows:

1. For $a \in G$ we get $\xi(a) \leq 1_{G'}$ and hence $a = \psi(\xi(a)) \leq \psi(1_{G'})$. Consequently, $1_G = \psi(1_{G'})$.
2. For $a, b, c \in G$ such that $c \leq a, b$ we get $\xi(c) \leq \xi(a), \xi(b)$ and hence $\xi(c) \leq \xi(a) \wedge \xi(b) \leq \xi(a), \xi(b)$. Consequently, $c = \psi(\xi(c)) \leq \psi(\xi(a) \wedge \xi(b)) \leq \psi(\xi(a)) = a, \psi(\xi(b)) = b$. That means $a \wedge b = \psi(\xi(a) \wedge \xi(b))$.

❑

From the preceding two lemmas we obtain immediately:

Theorem 24 *Equivalent minimal k-lattices are isomorphic. In other words, for every k-lattice there exists a (up to isomorphism) unique minimal equivalent k-lattice.*

This theorem ensures that we can always choose a unique starting point for investigations involving classes of the boolean hierarchy of k-partitions. Moreover, when restricting to the minimal k-lattices our relation $\leq$ becomes a partial order (however, this is merely a fact based on the selection of the minimal k-lattices as representatives of the equivalence classes with respect to $\leq$).

7 The Embedding Conjecture

Let us come back to the Embedding Lemma which shows that $(G, f) \leq (G', f')$ implies $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$. Thus we have a sufficient criterion for inclusion of partition classes. It would be, however, very useful if the criterion would be also necessary. In this section we pose the conjecture that this holds true for NP unless the polynomial hierarchy is finite. We support this conjecture with several results.

7.1 On Inverting the Embedding Lemma

We are interested in proving the following theorem for the case $\mathcal{K} = \mathrm{NP}$. Note that for the general formulation $\mathcal{K}$ is assumed to be such that $\emptyset, M \in \mathcal{K}$ and $\mathcal{K}$ is closed under intersection and union.

Definition 25 *We say that the* Embedding Theorem for $\mathcal{K}$ *holds if for all k-lattices* (G, f) *and* (G', f') *it is true that* $(G, f) \leq (G', f') \iff \mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$.

The difficult part of such theorems is the inversion of the Embedding Lemma, that is, the direction from right to left. If once proven for a class $\mathcal{K}$ the Embedding Theorem gives the complete information about $\mathrm{BH}_k(\mathcal{K})$. The following theorem shows that Embedding Theorems are in principle not out of reach:[2]

Theorem 26 *Let* (G, f) *and* (G', f') *be k-lattices with* $k \geq 2$. *If* $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$ *for every class* $\mathcal{K}$ *with* $\emptyset, M \in \mathcal{K}$ *and which is closed under intersection and union, then* $(G, f) \leq (G', f')$.

Proof. Let (G, f) and (G', f') be k-lattices. For each set $S \subseteq G$, define $D(S)$ as

$$D(S) =_{\text{def}} \left\{\, a \in G \;\middle|\; (\exists b \in S)[a \leq b] \,\right\}.$$

Let $\mathcal{K}$ be the set of all $D(S)$ for $S \subseteq G$. Clearly, $\emptyset, G \in \mathcal{K}$ and $\mathcal{K}$ is closed under finite union and intersection. Let S be the $\mathcal{K}$-homomorphism on G defined for every $a \in G$ as

$$S(a) =_{\text{def}} D(\{a\}).$$

Obviously, $T_S(a) = \{a\}$ and consequently $\left(f^{-1}(1), \ldots, f^{-1}(k)\right) \in \mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$. Hence, a $\mathcal{K}$-homomorphism $S' : G' \to \mathcal{K}$ on G' exists such that $\bigcup_{f'(d)=i} T_{S'}(d) = f^{-1}(i)$ for every $i \in \{1, 2, \ldots, k\}$. Define $h : G \to G'$ to be the function which assigns to each $a \in G$ the uniquely determined $d \in G'$ such that $a \in T_{S'}(d)$, i.e., $h^{-1}(d) = T_{S'}(d)$. Obviously, $a \in T_{S'}(h(a))$ and $f'(h(a)) = f(a)$. It remains to show that h is monotonic. Let $a, b \in G$ with $a \leq b$. Then $b \in T_{S'}(h(b)) \subseteq S'(h(b))$, so $a \in S'(h(b))$. From Lemma 9.2 there follows the existence of $c \in G'$ with $c \leq h(b)$ and $a \in T_{S'}(c)$. Thus $c = h(a)$, hence $h(a) \leq h(b)$. ❑

Because of the second item of Proposition 3, we cannot hope to invert the Embedding Lemma without an additional assumption to $\mathcal{K}$. A plausible one might be a strict boolean hierarchy of *sets* over $\mathcal{K}$. And indeed, for many subclasses of k-lattices, assuming the strictness of $\mathrm{BH}_2(\mathcal{K})$ is strong enough to show the Embedding Theorem for $\mathcal{K}$ and for these subclasses of labeled lattices.

For instance, we can prove that the Embedding Theorem for 2-lattices holds if we assume an infinite $\mathrm{BH}_2(\mathcal{K})$. To this end we first prove an analogue to Theorem 4 for 2-lattices. For a 2-lattice (G, f) let $\mu(G, f)$ be the maximum number of alternations of f-labels which can occur in a $\leq$-chain in the lattice G.

Theorem 27 *For every 2-lattice* (G, f),

$$\mathcal{K}(G, f) = \begin{cases} \mathcal{K}(\mu(G, f)) & \text{if } f(1_G) = 2, \\ \mathrm{co}\mathcal{K}(\mu(G, f)) & \text{if } f(1_G) = 1. \end{cases}$$

[2] Note that a disproof of Theorem 26 would imply that for every reasonable $\mathcal{K}$, there exists a pair of k-lattices that contradicts the Embedding Theorem for $\mathcal{K}$.

Proof. Let (G, f) be a 2-lattice. In the proof of Theorem 13 we defined a function $h : \{1,2\}^{|I|} \rightarrow \{1,2\}$ (remember that I is the set of meet-irreducible elements of G and that $(\mathcal{P}(I), \supseteq)$ and $(\{1,2\}^{|I|}, \leq)$ are isomorphic) such that $(G, f) \equiv (\{1,2\}^{|I|}, h)$. Thus, $\mathcal{K}(G, f) = \mathcal{K}(\{1,2\}^{|I|}, h) = \mathcal{K}(h)$, $\mu(G, f) = \mu(\{1,2\}^{|I|}, h) = \mu(h)$, and $f(1_G) = h(2^{|I|})$. By Theorem 4 we obtain the statement. ❑

Corollary 28 *Assume that* $\mathrm{BH}_2(\mathcal{K})$ *is infinite.*

1. *The minimal* 2*-lattice* (G, f) *such that* $\mathcal{K}(G, f) = \mathcal{K}(i)$ *is a chain with* $i + 1$ *elements with alternating labels* 1 *and* 2 *such that the maximum of the chain has label* 2*.*
2. *The minimal* 2*-lattice* (G, f) *such that* $\mathcal{K}(G, f) = \mathrm{co}\mathcal{K}(i)$ *is a chain with* $i + 1$ *elements with alternating labels* 1 *and* 2 *such that the maximum of the chain has label* 1*.*

As a consequence of Theorem 27 we get the validity of the (conditional) Embedding Theorem for 2-lattices.

Theorem 29 *Assume that* $\mathrm{BH}_2(\mathcal{K})$ *is infinite. For* 2*-lattices* (G, f) *and* (G', f') *the following statements are equivalent:*

1. $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$*.*
2. $\mu(G, f) < \mu(G', f')$ *or* $\big(\mu(G, f) = \mu(G', f')$ *and* $f(1_G) = f'(1_{G'})\big)$*.*
3. $(G, f) \leq (G', f')$*.*

Proof.

- (1) $\Rightarrow$ (2) is a consequence of Theorem 27.
- (3) $\Rightarrow$ (1) follows from the Embedding Lemma.
- For (2) $\Rightarrow$ (3) take a $\leq$-chain $(c_0, c_1, \ldots, c_r)$ in G' with maximum number of alternations between f'-labels, i.e., $r = \mu(G', f')$ and $f'(c_{i-1}) \neq f'(c_i)$ for $i \in \{1, \ldots, r\}$. For $a \in G$ define $\varphi(a)$ as follows:

$$\varphi(a) =_{\mathrm{def}} \begin{cases} c_i & \text{if } f(1_G) = f'(1_{G'}), \\ c_{i+1} & \text{if } f(1_G) \neq f'(1_{G'}). \end{cases}$$

  Here i is the maximum number of alternations between f-labels in a chain from a to 1_G. Obviously, φ is monotonic and $f'(\varphi(a)) = f(a)$.

❑

We now establish a theorem which shows that the Embedding Theorem for $\mathcal{K}$ holds for a large subclass of k-lattices (unless $\mathrm{BH}_2(\mathcal{K})$ is finite). At this, we make use of the following simple principle.

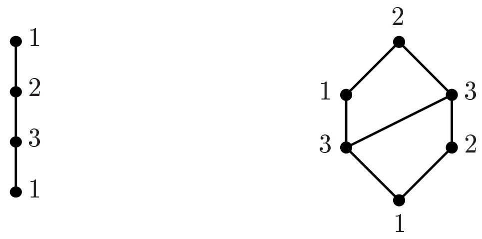

Figure 5: The 3-lattices of Example 32

Proposition 30 *Let (G, f) and (G', f') be k-lattices with $k \geq 2$. Let h be a function mapping $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, m\}$. If $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$, then $\mathcal{K}(G, h \circ f) \subseteq \mathcal{K}(G', h \circ f')$. Moreover, if h is injective, then the equivalence holds.*

Let (G, f) be a k-lattice. For $I, J \subseteq \{1, 2, \ldots, k\}$ with $I \cap J = \emptyset$, define $\mu_{I,J}(G, f)$ to be the maximum number of alternations between f-labels from I and f-labels from J in a chain of G whose minimum has an f-label from I.

Theorem 31 *Assume that $\mathrm{BH}_2(\mathcal{K})$ is infinite. For k-lattices (G, f) and (G', f'), if $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$, then $\mu_{I,J}(G, f) \leq \mu_{I,J}(G', f')$ for all $I, J \subseteq \{1, 2, \ldots, k\}$ with $I \cap J = \emptyset$.*

Proof. If $I = \emptyset$ or $J = \emptyset$, then $\mu_{I,J}(G, f) = 0$ for all (G, f). So, suppose I and J to be non-empty and $I \cap J = \emptyset$. Consider the function h mapping elements from I to $\min I$, elements from J to $\min J$, and elements not in I or J to themselves. Then, for all k-lattices (G, f), it holds $\mu_{I,J}(G, f) = \mu_{h(I),h(J)}(G, h \circ f)$. Therefore and because of Proposition 30, without loss of generality, we can assume that I and J are singletons; $I = \{i\}, J = \{j\}$, and $i \neq j$. For convenience, we write $\mu_{ij}(G, f)$ instead of $\mu_{\{i\},\{j\}}(G, f)$.

Let (G, f) and (G', f') be k-lattices. Let C be a maximal chain in G such that $\mu_{ij}(C, f|_C) = \mu_{ij}(G, f)$. Hence, $\mathcal{K}(C, f|_C) \subseteq \mathcal{K}(G', f')$. Since $f|_C : C \to \{i, j\}$ we have also $\mathcal{K}(C, f|_C) \subseteq \mathcal{K}(G', h)$ for all $h : G' \to \{1, 2, \ldots, k\}$ such that $h(a) = f'(a)$ if $f'(a) \in \{i, j\}$.

If there is no $b \in G'$ with $f'(b) \notin \{i, j\}$, then the claim is just the same already proven in Theorem 29. So, fix some $b \in G'$ such that $f'(b) \notin \{i, j\}$. For each $a \in G'$, let G'_a be the set $\{c \in G' \mid c \leq a\}$. Define for $a \in G'$

$$h(a) =_{\mathrm{def}} \begin{cases} f'(a) & \text{if } a \neq b, \\ i & \text{if } a = b \text{ and } \mu_{ij}(G'_b, f'|_{G'_b}) \text{ is even}, \\ j & \text{if } a = b \text{ and } \mu_{ij}(G'_b, f'|_{G'_b}) \text{ is odd}. \end{cases}$$

Hence, $\mathcal{K}(C, f|_C) \subseteq \mathcal{K}(G', h)$ and $\mu_{ij}(G', f') \leq \mu_{ij}(G', h)$. Consider a maximal chain $a_0 < a_1 < \cdots < a_r$ in G' such that $r = \mu_{ij}(G', h)$, $h(a_s) \in \{i, j\}$, $h(a_0) = i$, and $h(a_{s-1}) \neq h(a_s)$ for $s \in \{1, \ldots, r\}$. If $b \notin \{a_0, \ldots, a_r\}$ then $h(a_s) = f'(a_s)$ for all $s = \{0, 1, \ldots, r\}$ and hence $\mu_{ij}(G', f') \geq \mu_{ij}(G', h)$, thus $\mu_{ij}(G', f') = \mu_{ij}(G', h)$. Now let $b = a_s$ for some $s \in \{0, 1, \ldots, r\}$. Since $f'(a_{s-1}) = h(a_{s-1}) \neq h(a_s)$ and, by definition, $h(b) = h(a_s)$, the chain $a_0 < a_1 < \cdots < a_{s-1}$ cannot be a maximum chain in G'_b with alternating f'-labels starting with f'-label i. Hence there exists such a chain $b_0 < b_1 < \cdots < b_{s-1} < b_s$ in $(G'_b, f'|_{G'_b})$ and consequently such a chain

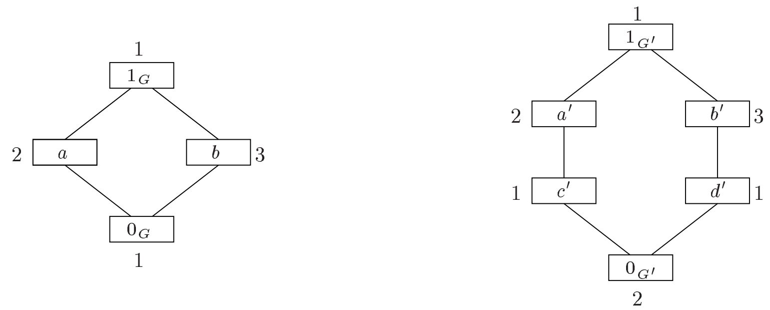

Figure 6: The 3-lattices critical for RE

$b_0 < b_1 < \cdots < b_{s-1} < b_s < a_{s+1} < \cdots < a_r$ in (G', f'). This means $\mu_{ij}(G', f') \geq r = \mu_{ij}(G', h)$ and hence, $\mu_{ij}(G', f') = \mu_{ij}(G', h)$.

Repeating this construction we obtain finally a function $g : G' \to \{i, j\}$ such that $\mathcal{K}(C, f|_C) \subseteq \mathcal{K}(G', g)$, $\mu_{ij}(C, f|_C) = \mu_{ij}(G, f)$, and $\mu_{ij}(G', g) = \mu_{ij}(G', f')$. In fact, $\mathcal{K}(C, f|_C)$ and $\mathcal{K}(G', g)$ are classes of 2-partitions. By Theorem 29, we obtain $\mu(C, f|_C) < \mu(G', g)$ or, $\mu(C, f|_C) = \mu(G', g)$ and $f(1_C) = g(1_{G'})$, from which we can conclude $\mu_{ij}(C, f|_C) \leq \mu_{ij}(G', g)$. ❑

Example 32 *Let (G, f) be the 3-lattice on the left-hand side and (G', f') be the 3-lattice on the right-hand side of Figure 5. To show $\mathcal{K}(G, f) \not\subseteq \mathcal{K}(G', f')$ if $\mathrm{BH}_2(\mathcal{K})$ is infinite, let $I = \{1\}$ and $J = \{2\}$. Then we have $\mu_{I,J}(G, f) = 2$ and $\mu_{I,J}(G', f') = 1$. Hence, by Theorem 31, $\mathcal{K}(G, f) \not\subseteq \mathcal{K}(G', f')$ unless $\mathrm{BH}_2(\mathcal{K})$ is finite. Reversely, let $I = \{1\}$ and $J = \{2, 3\}$. Then, $\mu_{I,J}(G', f') = 3$ and $\mu_{I,J}(G, f) = 2$. Thus, again by Theorem 31, $\mathcal{K}(G', f') \not\subseteq \mathcal{K}(G, f)$ unless $\mathrm{BH}_2(\mathcal{K})$ is finite.*

Theorem 29 and Theorem 31 suggest that a strict boolean hierarchy of sets is sufficient to establish Embedding Theorems. However, there are classes for which the Embedding Theorem does not hold though they have a strict boolean hierarchy. A very prominent example is the class RE. Clearly, the recursively enumerable sets are closed under intersection and union and contain $\emptyset$ and Σ^*. The strictness of the boolean hierarchy of the recursively enumerable sets goes back to Ershov [12].

Theorem 33 *The Embedding Theorem for the recursively enumerable sets does not hold.*

Proof. Let (G, f) be the left 3-lattice and (G', f') be the right 3-lattice in Figure 6. Obviously, $(G, f) \not\leq (G', f')$. However, it holds that $\mathrm{RE}(G, f) \subseteq \mathrm{RE}(G', f')$. To prove this we use the following well-known property of the recursively enumerable sets (cf., e.g., [35]): For all recursively enumerable sets A and B there are r ecursively enumerable sets $C \subseteq A$ and $D \subseteq B$ with $C \cup D = A \cup B$ and $C \cap D = \emptyset$.

Now let $(G, f, S) \in \mathrm{RE}(G, f)$. By the claim above there are sets $C, D \in \mathrm{RE}$ with $C \cup D = S(a) \cup S(b)$, $C \cap D = \emptyset$, $C \subseteq S(a)$, and $D \subseteq S(b)$. Since a RE-homomorphism on lattices only depends on its values on the meet-irreducible elements, it is enough to define S' on G' as

$$S'(a') =_{\mathrm{def}} C, \quad S'(b') =_{\mathrm{def}} D, \quad S'(c') =_{\mathrm{def}} C \cap S(b), \quad \text{and} \quad S'(d') =_{\mathrm{def}} D \cap S(a).$$

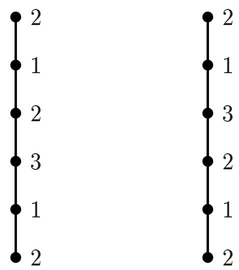

Figure 7: Counterexample to the mind-change technique

Clearly, it holds that $S'(c') \subseteq S'(a')$, $S'(d') \subseteq S'(b')$, and $S'(a') \cap S'(b') = \emptyset = S'(c') \cap S'(d')$. Moreover we have the following:

$$\begin{aligned}(G', f', S')_2 &= T_{S'}(a') \cup T_{S'}(0_{G'}) = T_{S'}(a') = S'(a') \setminus S'(c') = C \setminus (C \cap S(b)) \\ &= (C \cup S(b)) \setminus S(b) = (S(a) \cup S(b)) \setminus S(b) = S(a) \setminus (S(a) \cap S(b)) \\ &= S(a) \setminus S(0_G) = T_S(a) = (G, f, S)_2\end{aligned}$$

The remaining equalities can be shown similarly to the equality of the second component. This gives $(G, f, S) = (G', f', S')$. Hence, $(G, f, S) \in \mathrm{RE}(G', f')$. ❑

Most recently Selivanov [34] gave a complete characterization of the boolean hierarchy of partitions over recursively enumerable sets which is based on a coarser embedding relation $\leq'$ than we consider. With respect to that relation $\leq'$, for the 3-lattices in Figure 6 it holds $(G, f) \leq' (G', f')$.

Up to this theorem, all results so far hold for arbitrary classes with some simple closure properties. The forthcoming now makes use of the very nature of the class NP. As we have seen even an infinite boolean hierarchy of sets is not sufficient to invert the Embedding Lemma. Since the collapse of the boolean hierarchy over NP implies the collapse of the polynomial hierarchy (cf. [24]) the following conjecture seems to be reasonable.

Embedding Conjecture. *Assume the polynomial hierarchy is infinite. Let (G, f) and (G', f') be k-lattices. Then $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$ if and only if $(G, f) \leq (G', f')$.*

To provide evidence for the Embedding Conjecture we formulate in Subsection 7.2 a theorem (Theorem 50) which shows that the conjecture is true for a much larger subclass of k-lattices than touched by Theorem 31 including all 2-lattices (Corollary 49) and moreover, all k-chains (Theorem 47). Furthermore, the 3-lattices in Figure 6 turn out to be not a counterexample for the class NP. This is proven in Subsection 7.3.

7.2 Evidence I: The Case of k-Chains

We establish theorems that show that the Embedding Conjecture is true for a very large subclass of k-lattices based on differences in the chain structure of the lattices. In Theorem 31 differences concerning the mind changes in k-chains are considered. However, the theorem is not general enough to cover all k-chains. As an example consider the two 3-chains in Figure 7. Let (G, f)

be the left and (G', f') be the right 3-chain. On the one hand, it is easy to calculate that $\mu_{I,J}(G, f) = \mu_{I,J}(G', f')$ for all $I, J \subseteq \{1, 2, 3\}$ with $I \cap J = \emptyset$. On the other hand, obviously $(G, f) \not\leq (G', f')$ and $(G', f') \not\leq (G, f)$. So in order to support the Embedding Conjecture we have to prove that $\mathrm{NP}(G, f) \not\subseteq \mathrm{NP}(G', f')$ as well as $\mathrm{NP}(G', f') \not\subseteq \mathrm{NP}(G, f)$ unless the polynomial hierarchy is finite. In this subsection we will see how to do this. Proving such theorems, we detect some normal forms of (hypothetical) inclusions between partition classes enabling us a generalization of the easy-hard arguments developed by Kadin (cf. [24]) to the context of partition classes.

7.2.1 Partition Classes Defined by Chains

We first emphasize some simplifications and peculiarities of partition classes over labeled chains. As long as no further conditions are needed we consider general classes $\mathcal{K}$ with $\emptyset, M \in \mathcal{K}$ and that are closed under intersection and finite union. Partition classes over labeled chains are characterized by ascending chains of sets from $\mathcal{K}$.

We identify a k-chain (G, f) in a natural way with a word in $\{1, 2, \ldots, k\}^{|G|}$, namely with $f(a_1)f(a_2)\ldots f(a_n)$ when $a_1 < a_2 < \cdots < a_n$, $a_i \in G$, and $n = \|G\|$. Words representing k-chains are called k-words.

The relation $\leq$ over k-lattices translates to a subword relation between k-words. For that, we say that a k-word a is *repetition-free* if and only if $a_i \neq a_{i+1}$ for all $1 \leq i < n$. For an arbitrary k-word a its repetition-free version a_* is the word emerging from a when repeatedly replacing each occurrence of ss to s, where $s \in \{1, 2, \ldots, k\}$. Now, we can say that $a \preceq b$ for k-words a, b if and only if a_* is a subword of b. We say $a \equiv b$ whenever $a \preceq b$ and $b \preceq a$. If a and b are repetition-free k-words then $a \equiv b$ is equivalent to $a = b$. Obviously, the relation $\preceq$ for k-words corresponds with the relation $\leq$ for k-chains. Repetition-free k-words correspond to minimal k-chains. Dual k-chains correspond to reverse words.

There are some notations to bed adapted to k-words. Let a k-word a be given. Then a $\mathcal{K}$-homomorphism S on a is just a $\mathcal{K}$-homomorphism on $(\{1, 2, \ldots, |a|\}, a)$, the partition (a, S) generated by S is the partition $(\{1, 2, \ldots, |a|\}, a, S)$, and, finally, $\mathcal{K}(a) = \mathcal{K}(\{1, 2, \ldots, |a|\}, a)$. Here we have identified the k-word a with the function $a : \{1, 2, \ldots, |a|\} \rightarrow \{1, 2, \ldots, k\}$ given by $a(i) = a_i$.

If two k-words are comparable with respect to $\preceq$, there are possibly many monotonic mappings witnessing the relation. This ambiguity is often disadvantageous. So we consider the canonical embedding, mapping every letter of a k-word to the least possible letter in the other k-word.

Definition 34 *Let a and a' be k-words, $k \geq 1$. The* canonical embedding $\kappa[a, a']$ of a into a' *is a mapping from $\{0, 1, 2, \ldots, |a|\}$ to $\{0, 1, 2, \ldots, |a'|\}$ inductively defined as $\kappa[a, a'](0) =_{\mathrm{def}} 0$ and for $j > 0$ as*

$$\kappa[a, a'](j) =_{\mathrm{def}} \min\left\{ \, r \;\middle|\; r \geq \kappa[a, a'](j-1) \wedge a_j = a'_r \, \right\}$$

where $\min \emptyset$ is considered to be undefined.

If there is no reason for misunderstanding, then we omit $[a, a']$ in the description of the canonical embedding.

Proposition 35 *Let a and a' be k-words. Then, $a \preceq a'$ if and only if the canonical embedding κ of a into a' is total.*

Canonical embeddings make it possible to determine normal forms for $\mathcal{K}$-homomorphisms witnessing inclusions between partition classes.

Lemma 36 *Let a and a' be repetition-free k-words. Let κ be the canonical embedding of a into a'. If $\mathcal{K}(a) \subseteq \mathcal{K}(a')$, then for every $\mathcal{K}$-homomorphism S on a there is a $\mathcal{K}$-homomorphism S' on a' such that $(a, S) = (a', S')$ and $S(j) \subseteq S'(\kappa(j))$ for all $j \in D_\kappa$.*

Proof. Since $\mathcal{K}(a) \subseteq \mathcal{K}(a')$, there is a $\mathcal{K}$-homomorphism V on a' with $(a, S) = (a', V)$. We meet the convention that $S(0) = \emptyset$ and $V(0) = \emptyset$. Define S' for all $j \leq |a'|$ as

$$S'(j) =_{\text{def}} V(j) \cup S\left(\max_{\kappa(s) \leq j} s\right).$$

Obviously, S' is an $\mathcal{K}$-homomorphism on a' with $S(j) \subseteq S'(\kappa(j))$ for $j \in D_\kappa$. It remains to show $(a, S) = (a', S')$. We consider the partition (a', S') individually for every component $i \in \{1, 2, \ldots, k\}$. Fix a component i, and consider $T_{S'}(j)$ for $j \leq |a'|$ with $a'_j = i$. We have two different cases.

- *Case 1.* Suppose $\kappa(s) < j < \kappa(s+1)$ for an appropriate s, or $\kappa(\max D_\kappa) < j$. Then,

  $$\begin{aligned} T_{S'}(j) &= S'(j) \backslash S'(j-1) = (V(j) \cup S(s)) \setminus (V(j-1) \cup S(s)) = (V(j) \backslash V(j-1)) \backslash S(s) \\ &\subseteq T_V(j). \end{aligned}$$

  Hence, $T_{S'}(j) \subseteq T_V(j) \subseteq (a', V)_i = (a, S)_i$.

- *Case 2.* Suppose $j = \kappa(s)$ for an appropriate s. Then,

  $$\begin{aligned} T_{S'}(j) &= S'(j) \backslash S'(j-1) = (V(j) \cup S(s)) \setminus (V(j-1) \cup S(s-1)) \\ &= [(V(j) \backslash V(j-1)) \backslash S(s-1)] \cup [(S(s) \backslash S(s-1)) \backslash V(j-1)] \subseteq T_V(j) \cup T_S(s). \end{aligned}$$

  Since $a_s = a'_{\kappa(s)} = a'_j = i$, we obtain $T_{S'}(j) \subseteq T_V(j) \cup T_S(s) \subseteq (a', V)_i \cup (a, S)_i = (a, S)_i$.

Overall, we have shown $(a', S')_i \subseteq (a, S)_i$ for every i. Since (a', V) and (a, S) are partitions, we get the equalities $(a', S')_i = (a, S)_i$. Thus, $(a', S') = (a, S)$. ❑

7.2.2 Hardest Inclusions

It is our goal to prove the finiteness of the polynomial hierarchy in case of having an inclusion between partition classes which should not be true if the Embedding Conjecture would hold. For the boolean hierarchy $\text{BH}_2(\text{NP})$ it suffices to consider the inclusion $\text{NP}(m) \subseteq \text{coNP}(m)$ for $m \in \mathbb{N}_+$ or, in terms of 2-words,

$$\text{NP}(\underbrace{1212\ldots}_{m+1}) \subseteq \text{NP}(\underbrace{2121\ldots}_{m+1}).$$

The very simple structure of $\mathrm{BH}_2(\mathrm{NP})$, trivially, yields the following: If for any $m \in \mathbb{N}_+$ there is an $n < m$ with $\mathrm{NP}(m) \subseteq \mathrm{NP}(n)$, or there is an $l \leq m$ with $\mathrm{NP}(m) \subseteq \mathrm{coNP}(l)$, then $\mathrm{NP}(m) \subseteq \mathrm{coNP}(m)$. Again, in terms of 2-words, that means: Let a be a repetition-free 2-word. If for a there is an a' with $a \not\leq a'$ and $\mathrm{NP}(a) \subseteq \mathrm{NP}(a')$, then $\mathrm{NP}(a) \subseteq \mathrm{NP}(\overline{a})$. Note that for such a' it holds $|a'| \leq |a|$. For k-words with $k > 2$ this length condition is not true. For instance, consider 123 and $1(31)^m2$ for arbitrary $m \in \mathbb{N}_+$. Then, $123 \not\leq 1(31)^m2$, but $|1(31)^m2|$ can be arbitrarily large. Can we nevertheless identify short k-words with hardest inclusions to be considered?

In the following we give a positive answer to this question. To do that we need two lemmas.

Lemma 37 *$\mathcal{K}(a) = \mathrm{co}\mathcal{K}(a^R)$ for all k-words a.*

Proof. Follows from Theorem 18. ❑

Lemma 38 *Let a and a' be repetition-free k-words, $k \geq 2$. Let κ be the canonical embedding of a into a'. Let $r \in D_\kappa$ so that $a_i \neq a_r$ for all $i > r$. If $\mathcal{K}(a) \subseteq \mathcal{K}(a')$, then $\mathcal{K}(a) \subseteq \mathcal{K}(a'')$ where a'' emerges from a' when deleting from a' all the letters a'_j with $j > \kappa(r)$ and $a'_j = a_r$.*

Proof. Let $(a, S) \in \mathcal{K}(a)$ for $\mathcal{K}$-homomorphism S on f. By Lemma 36, there is a $\mathcal{K}$-homomorphism S' on a' with $(a, S) = (a', S')$ and $S(j) \subseteq S'(\kappa(j))$ for all $j \in D_\kappa$. It suffices to show $T_{S'}(j) = \emptyset$ for all $j > \kappa(r)$ with $a'_j = a_r$. Let $a_r = b$. Since $a'_j = a_r = b$, it holds $T_{S'}(j) \subseteq (a', S')_b = (a, S)_b \subseteq S(r)$. Hence, $S'(j) \subseteq S(r) \cup S'(j-1) \subseteq S'(\kappa(r)) \cup S'(j-1) \subseteq S'(j-1)$. The latter holds because $j > \kappa(r)$. Thus, $S'(j) = S'(j-1)$, and consequently, $T_{S'}(j) = \emptyset$. ❑

Now we are able to prove the theorem which identifies short k-words of at most the double of the length of a given k-word, but with a hard inclusion property.

Theorem 39 *Let a be any repetition-free k-word of length n, $k \geq 2$. If there is a repetition-free k-word a' with $a \not\leq a'$ and $\mathcal{K}(a) \subseteq \mathcal{K}(a')$ then $\mathcal{K}(a_1a_2 \dots a_n) \subseteq \mathcal{K}(a_2a_1a_3a_2 \dots a_na_{n-1})$.*

Proof. Let a' be a k-word such that $a \not\leq a'$ and $\mathcal{K}(a) \subseteq \mathcal{K}(a')$. First we will transform a' into a k-word of a certain structure preserving the inclusion. Note that inserting new letters in a' preserves $\mathcal{K}(a) \subseteq \mathcal{K}(a')$. Since $a \not\leq a'$, it holds that

$$a' = w_1a_1w_2a_2w_3 \dots w_ia_iw_{i+1} \quad \text{with } w_j \in \big(\{1, 2, \dots, k\} \setminus \{a_j\}\big)^* \text{ and } i < n.$$

Define the k-word b' by appending $a_{i+1}a_{i+2} \dots a_{n-1}$ to a' and then inserting $a_2, a_3, \dots, a_n$ into the new k-word as follows:

$$b' =_{\mathrm{def}} w_1a_2a_1w_2a_3a_2w_3 \dots w_{n-1}a_na_{n-1}w_n.$$

Note that it holds that $a \not\leq b'$. By Lemma 38 we can simplify the words w_j. We can set

$$b'' =_{\mathrm{def}} v_1a_2a_1v_2a_3a_2v_3 \dots v_{n-1}a_na_{n-1}v_n \quad \text{with } v_i \in \{a_{i+1}, a_{i+2}, \dots a_n\}^* \text{ and } v_n = \varepsilon,$$

i.e., for all i, v_i is defined to be w_i without the letters from $\{1,2,\dots,k\}\setminus\{a_i,a_{i+1},\dots,a_n\}$. Using Lemma 37 and again Lemma 38, we can also simplify the words v_i. Let b''' be defined as

$$b''' =_{\mathrm{def}} u_1 a_2 a_1 u_2 a_3 a_2 u_3 \dots u_{n-1} a_n a_{n-1}$$
$$\text{with } u_i \in \big(\{a_1,a_2,\dots,a_{i-1}\}\cap\{a_{i+1},a_{i+2},\dots,a_n\}\big)^* \text{ and } u_1=\varepsilon.$$

Making all subwords $a_{i-1}u_i a_{i+1}$ repetition-free (note that this implies $a_1u_2a_3 \equiv a_1a_3$ and $a_{n-2}u_{n-1}a_n \equiv a_{n-2}a_n$), we get the repetition-free k-word b defined as

$$b =_{\mathrm{def}} a_2a_1a_3a_2z_3a_4a_3z_4\dots z_{n-2}a_{n-1}a_{n-2}a_na_{n-1}$$
$$\text{with } z_i \in \big(\{a_1,a_2,\dots,a_{i-1}\}\cap\{a_{i+1},a_{i+2},\dots,a_n\}\big)^* \text{ for } i\in\{3,4,\dots,n-2\}.$$

In the remainder we will always suppose this k-word b. Note that b satisfies the conditions that $a \not\preceq b$ and $\mathcal{K}(a)\subseteq\mathcal{K}(b)$. Let κ be the canonical embedding of a into b. Let $m=|b|$. It holds that $\kappa(1)=2$ and $\kappa(n-1)=m$. We define κ' as $\kappa'(j)=\kappa(j-1)-1$ for all $j\in\{2,\dots,n\}$. Let S be any $\mathcal{K}$-homomorphism on a. Since $\mathcal{K}(a)\subseteq\mathcal{K}(b)$, and due to Lemma 36, there exists a $\mathcal{K}$-homomorphism V on b such that $(a,S)=(b,V)$ and $S(j)\subseteq V(\kappa(j))$ for all $j\in\{1,2,\dots,n-1\}$. Define a mapping S' for $j\in\{1,2,\dots,m\}$ as

$$S'(j) =_{\mathrm{def}} \begin{cases} V(j) & \text{if } j\in\{1,2,m-1,m\},\\ (V(j)\cap S(r))\cup V(2) & \text{if } j>2 \text{ and } \kappa'(r)\le j<\kappa'(r+1).\end{cases}$$

It holds that $S':\{1,2,\dots,m\}\to\mathcal{K}$ and $S'(j)\subseteq S'(j+1)$ for $1\le j<m$, i.e., S' is a $\mathcal{K}$-homomorphism on b. Moreover, S' satisfies the following conditions:

1. For all $j\in\{1,\dots,m\}$, if $j\notin R_\kappa\cup R_{\kappa'}$, then $T_{S'}(j)=\emptyset$,
2. $(a,S)=(b,S')$.

Note that proving these two facts is sufficient for the theorem because of the equalities $\kappa'(j)=\kappa(j-1)-1$ for all $j\in\{2,3,\dots,n\}$.

1. Let $j\notin R_\kappa\cup R_{\kappa'}$. Then, $2=\kappa(1)<j<\kappa'(n)$, i.e., there is an r such that $\kappa'(r)<j<\kappa'(r+1)$. Consequently,

$$\begin{aligned} T_{S'}(j) &= S'(j)\setminus S'(j-1) = ((V(j)\cap S(r))\cup V(2))\setminus((V(j-1)\cap S(r))\cup V(2))\\ &= ((V(j)\setminus V(j-1))\cap S(r))\setminus V(2)\subseteq T_V(j)\cap S(r).\end{aligned}$$

   Let q be maximal with $\kappa(q)<j$ and $a_q=b_j$. Let s be minimal with $j<\kappa'(s)$ and $a_s=b_j$. The existence of both q and s is assured due to the structure of b. Then, we have $T_{S'}(j)\subseteq T_V(j)\cap S(r)\subseteq T_V(j)\cap S(s-1)$. Moreover, $a_{s-1}\neq b_j$ since a is repetition-free. The statement would be proven if we would know the following:

   (∗) *There is no t with $q<t<s$ and $b_j=a_q=a_t=a_s$.*

   Using (∗) we can conclude: If $x\in T_{S'}(j)$, i.e., $x\in S(s-1)$ and $x\notin V(j-1)$, then $x\notin T_S(i)$ for all $q<i\le s-1$. Hence $x\in S(q)\subseteq V(\kappa(q))\subseteq V(j-1)$. This is a contradiction. Thus, $T_{S'}(j)=\emptyset$.

It remains to prove $(*)$. Assume the contrary to be true, i.e., there exists a t with $q < t < s$ and $b_j = a_q = a_t = a_s$. Then we have three cases yielding contradictions. The case $j \geq \kappa(t)$ contradicts the maximality of q and $q \neq t$. The case $j \leq \kappa'(t)$ contradicts the minimality of s and $s \neq t$. In the case $\kappa'(t) < j < \kappa(t)$ we conclude $\kappa(t-1) - 1 < j < \kappa(t)$ and, since $j \notin R_\kappa$, $\kappa(t-1) < j < \kappa(t)$. But now, it holds that $b_j \neq b_{\kappa(t)} = a_t$, contradicting $b_j = a_t$. Hence the assumption is false, i.e., such a t does not exist.

2. It suffices to show $T_{S'}(j) \subseteq (a,S)_i$ for every j with $b_j = i$. So, let j be so that $b_j = i$. There are two cases, $j \in R_{\kappa'}$ and $j \notin R_{\kappa'}$.

   - *Case* $j \in R_{\kappa'}$. If $j = \kappa'(2) = \kappa(1) - 1 = 1$, then $T_{S'}(j) = T_V(j) \subseteq (b,V)_i = (a,S)_i$. So, let $j = \kappa'(r)$ for $r > 2$, i.e., $j > 2$ and $i = b_j = a_r$. Then,

$$\begin{aligned} T_{S'}(j) &= S'(j) \setminus S'(j-1) = ((V(j) \cap S(r)) \cup V(2)) \setminus ((V(j-1) \cap S(r-1)) \cup V(2)) \\ &= ((V(j) \cap S(r)) \setminus (V(j-1) \cap S(r-1))) \ \setminus \ V(2) \\ &\subseteq ((V(j) \setminus V(j-1)) \cap S(r)) \cup ((S(r) \setminus S(r-1)) \cap V(j-1)) \subseteq T_V(j) \cup T_S(r) \\ &\subseteq (b,V)_i \cup (a,S)_i = (a,S)_i. \end{aligned}$$

   - *Case* $j \notin R_{\kappa'}$. If additionally $j \notin R_\kappa$, then by 1., $T_{S'}(j) = \emptyset \subseteq (a,S)_i$. So, let $j \in R_\kappa$. If $j = 2 = \kappa(1)$ or $j = m = \kappa(n-1)$, then $T_{S'}(j) = T_V(j) \subseteq (b,V)_i = (a,S)_i$. It remains to argue for $2 = \kappa(1) < j < \kappa(n-1)$. Then we have,

$$\begin{aligned} T_{S'}(j) &= S'(j) \setminus S'(j-1) = ((V(j) \cap S(r)) \cup V(2)) \setminus ((V(j-1) \cap S(r)) \cup V(2)) \\ &= ((V(j) \setminus V(j-1)) \cap S(r)) \ \setminus \ V(2) \ \subseteq \ T_V(j) \ \subseteq \ (b,V)_i \ = \ (a,S)_i. \end{aligned}$$

❑

Note that $a_1a_2\ldots a_n \not\leq a_2a_1a_3a_2\ldots a_na_{n-1}$ for every repetition-free k-word $a = a_1\ldots a_n$. Theorem 39 gives, e.g., that for the 3-word 123 it is enough to collapse the polynomial hierarchy from $\mathrm{NP}(123) \subseteq \mathrm{NP}(2132)$. Moreover, Theorem 39 is in some sense optimal. For repetition-free 2-words a, it holds $a_i = a_{i+2}$. Hence, for $a = a_1\ldots a_n$, we have $a_2a_1a_3a_2\ldots a_na_{n-1} \equiv \overline{a}$.

7.2.3 The Embedding Theorem for k-Chains

We now prove the Embedding Conjecture true for k-words. First, we determine complete NP-partitions for partition classes over k-words with a useful inductive structure.

Definition 40 *Let $L \subseteq \Sigma^*$. For any k-word a with $|a| = n \geq 2$ and $a_{n-1} \neq a_n$, the partition L^a is defined as follows*

1. *If $n = 2$, then for all $i \in \{1,2,\ldots,k\}$,*

$$L_i^a =_{\mathrm{def}} \begin{cases} L & \text{if } i = a_1, \\ \overline{L} & \text{if } i = a_2, \\ \emptyset & \text{if } i \notin \{a_1,a_2\}. \end{cases}$$

2. If $n > 2$, then for all $i \in \{1, 2, \ldots, k\}$,

$$L_i^a =_{\text{def}} \begin{cases} \{ \langle x_1, x_2, \ldots, x_{n-1} \rangle \mid x_1 \in L \vee \langle x_2, x_3, \ldots, x_{n-1} \rangle \in L_i^{a_2 a_3 \ldots a_n} \} & \text{if } i = a_1, \\ \{ \langle x_1, x_2, \ldots, x_{n-1} \rangle \mid x_1 \notin L \wedge \langle x_2, x_3, \ldots, x_{n-1} \rangle \in L_i^{a_2 a_3 \ldots a_n} \} & \text{if } i \neq a_1. \end{cases}$$

Easy inductive arguments show that L^a is really a partition. We need the definition of $\leq_m^p$-reduction for partitions: For k-partitions A and B it holds $A \leq_m^p B$ iff there is a function $f \in \mathrm{FP}$ such that $c_A(x) = c_B(f(x))$ for all $x \in \Sigma^*$.

Theorem 41 *Let L be a $\leq_m^p$-complete problem for* NP*. For any k-word a with $|a| = n \geq 2$ and $a_{n-1} \neq a_n$, the partition L^f is $\leq_m^p$-complete for the partition class* NP(a)*.*

Proof. It is obvious that L^a is in NP(a). The proof of hardness is by induction over the length n of k-words. The base of induction $n = 2$ is obvious. So suppose the proposition is true for all k-words of length n and consider an arbitrary partition $A \in \mathrm{NP}(a)$ for a k-word a of length $n+1$, i.e., there is an NP-homomorphism S on a such that

$$A_{a_1} = S(1) \cup \bigcup_{\substack{a_j = a_1 \\ j > 2}} S(j) \backslash S(j-1) \quad \text{and for } i \neq a_1, \quad A_i = \bigcup_{a_j = i} S(j) \backslash S(j-1).$$

Clearly, S is also an NP-homomorphism on $a_2 a_3 \ldots a_{n+1}$, and the defined partition A' belongs to NP$(a_2 a_3 \ldots a_{n+1})$. Thus, since $a_2 a_3 \ldots a_{n+1}$ is a k-word of length n, by the assumption of the induction, $A' \leq_m^p L^{a_2 a_3 \ldots a_{n+1}}$ via $\varphi \in \mathrm{FP}$. Further, $S(1) \leq_m^p L$ via $t \in \mathrm{FP}$. Define ψ as

$$\psi(x) =_{\text{def}} \langle t(x), (\pi_1^{n-1} \circ \varphi)(x), (\pi_2^{n-1} \circ \varphi)(x), \ldots, (\pi_{n-1}^{n-1} \circ \varphi)(x) \rangle.$$

Clearly, $\psi \in \mathrm{FP}$, and taking into account that $S(1) \subseteq S(2) \subseteq \cdots \subseteq S(n+1)$, it holds that

$$\begin{aligned} x \in A_{a_1} &\iff x \in S(1) \text{ or } x \in \bigcup_{\substack{a_j = a_1 \\ j > 2}} S(j) \backslash S(j-1) \\ &\iff t(x) \in L \vee \varphi(x) \in L_{a_1}^{a_2 a_3 \ldots a_{n+1}} \\ &\iff \psi(x) \in L_{a_1}^a \end{aligned}$$

and for $i \neq a_1$,

$$\begin{aligned} x \in A_i &\iff x \notin S(1) \text{ and } x \in \bigcup_{a_j = i} S(j) \backslash S(j-1) \\ &\iff t(x) \notin L \wedge \varphi(x) \in L_i^{a_2 a_3 \ldots a_{n+1}} \\ &\iff \psi(x) \in L_i^a. \end{aligned}$$

Hence, ψ shows $A \leq_m^p L^a$. This completes the induction. ❑

We apply the easy-hard technique invented by Kadin [24] to collapse the polynomial hierarchy from a collapse of the boolean hierarchy $\mathrm{BH}_2(\mathrm{NP})$. The proof consists of two parts that can be isolated.

In the first part of the proof, an inclusion $\mathrm{NP}(m) \subseteq \mathrm{coNP}(m)$ for some $m \in \mathbb{N}_+$ is translated downwards to the previous level $m - 1$ using a special polynomial advice called hard word.

Inductively, this can even be translated to the lowest level NP $\subseteq$ coNP/poly where the polynomial advice is just a tuple of hard words. The second part of the proof uses this inclusion NP $\subseteq$ coNP/poly to collapse the polynomial hierarchy to its third level [41]. This part has been improved many times in sophisticated ways to a deeper collapse (cf. [19, 33]) by a direct use of hard words.

Both parts of the proof are differently reflected by definitions. The concept of hard sequences plays the crucial role for the first part.

Definition 42 [24] *Let $L \subseteq \Sigma^*$. Let $m \in \mathbb{N}$, $n \in \mathbb{N}_+$, and $h : \Sigma^* \to \Sigma^*$. A tuple $\langle \omega_1, \ldots, \omega_j \rangle$ is said to be a* hard sequence for (L, m, n, h) *if and only if either $j = 0$ or*

1. $1 \leq j \leq n - 1$,
2. $|\omega_j| \leq m$,
3. $\omega_j \notin L$,
4. $(\pi^n_{j+1} \circ h)(\langle \omega_1, \ldots, \omega_j, x_{j+1}, \ldots, x_n \rangle) \notin L$ *for all* $x_{j+1}, \ldots, x_n \in \Sigma^{\leq m}$,
5. $\langle \omega_1, \ldots, \omega_{j-1} \rangle$ *is a hard sequence for* (L, m, n, h).

We call j the order *of a hard sequence $\langle \omega_1, \ldots, \omega_j \rangle$. A hard sequence $\langle \omega_1, \ldots, \omega_j \rangle$ for (L, m, n, h) is said to be a* maximal hard sequence for (L, m, n, h) *if and only if for all $\omega_{j+1} \in \Sigma^*$, the tuple $\langle \omega_1, \ldots, \omega_j, \omega_{j+1} \rangle$ is not a hard sequence for (L, m, n, h).*

Note that hard sequences do always exist independently from the parameters chosen, namely, at least hard sequences of order 0. Hence, maximal hard sequences do always exist as well.

A second concept central to collapsing the polynomial hierarchy in the context of the easy-hard technique is that of a twister. The definition of a twister builds up on the concept of maximal hard sequences.

Definition 43 *Let $L \subseteq \Sigma^*$ and let $n \in \mathbb{N}_+$. A function $h : \Sigma^* \to \Sigma^*$ is said to be an (L, n)-* twister *if and only if $h \in$ FP and for all $m \in \mathbb{N}$ and for all $x \in \Sigma^{\leq m}$, if $\langle \omega_1, , \ldots, \omega_j \rangle$ is a maximal hard sequence for (L, m, n, h), then there are $x_{j+2}, \ldots, x_n \in \Sigma^{\leq m}$ such that*

$$x \notin L \iff (\pi^n_{j+1} \circ h)(\langle \omega_1, \ldots, \omega_j, x, x_{j+2}, \ldots, x_n \rangle) \in L.$$

The following result is the deepest collapse of the polynomial hierarchy currently known to follow from the existence of some twisters. Note that twisters appear only implicitely in the literature [19, 33].

Lemma 44 [19, 33] *Let L be $\leq^p_m$-complete for* NP. *Let $n \in \mathbb{N}_+$. If there exists an (L, n)-twister then* PH $= \Sigma^p_2(n-1) \oplus \mathrm{NP}(n)$.

The next theorem generalizes the easy-hard technique to the case of partitions. This theorem is the key to the Embedding Theorem for k-chains.

Theorem 45 *Let $k \geq 2$. Let a and a' be k-words with $|a| = |a'| = n \geq 2$, $a_{n-1} \neq a_n$, $a'_{n-1} \neq a'_n$, and $a_i \neq a'_i$ for all $i \leq n$. If $\mathrm{NP}(a) \subseteq \mathrm{NP}(a')$, then $\mathrm{PH} = \Sigma_2^p(n-2) \oplus \mathrm{NP}(n-1)$.*

Proof. Let L be a $\leq_m^p$-complete set for NP. Thus, by assumption $\mathrm{NP}(a) \subseteq \mathrm{NP}(a')$, there is a polynomial-time computable function h which witnesses the reduction $L^a \leq_m^p L^{a'}$. We will show that h is an $(L, n-1)$-twister. For that, we first have to prove the following claim.

Claim. *If $\langle \omega_1, \ldots, \omega_j \rangle$ is a hard sequence for $(L, m, n-1, h)$, then for all $x_{j+1}, \ldots, x_{n-1} \in \Sigma^{\leq m}$ and for all $a \in \{1, 2, \ldots, k\}$,*

$$\langle x_{j+1}, \ldots, x_{n-1} \rangle \in L_a^{a_{j+1} \ldots a_n}$$
$$\iff \quad (\langle \pi_{j+1}^{n-1}, \ldots, \pi_{n-1}^{n-1} \rangle \circ h)(\langle \omega_1, \ldots, \omega_j, x_{j+1}, \ldots, x_{n-1} \rangle) \in L_a^{a'_{j+1} \ldots a'_n}.$$

This claim can be proven inductively on the order j of hard sequences. The base of induction $j = 0$ is just our given situation $\mathrm{NP}(a) \subseteq \mathrm{NP}(a')$. So, let $\langle \omega_1, \ldots, \omega_j, \omega_{j+1} \rangle$ be a hard sequence for $(L, m, n-1, h)$. Thus, $\omega_{j+1} \notin L$ and for all $x_{j+2}, \ldots, x_{n-1} \in \Sigma^{\leq m}$ it holds that $(\pi_{j+1}^{n-1} \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \notin L$. Hence, for $b = a_{j+1}$,

$$\begin{aligned}
&\langle x_{j+2}, \ldots, x_{n-1} \rangle \in L_b^{a_{j+2} \ldots a_n} \\
&\iff \omega_{j+1} \in L \text{ or } \langle x_{j+2}, \ldots, x_{n-1} \rangle \in L_b^{a_{j+2} \ldots a_n} \\
&\iff \langle \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle \in L_b^{a_{j+1} \ldots a_n} && \text{(since } b = a_{j+1}) \\
&\iff (\langle \pi_{j+1}^{n-1}, \ldots, \pi_{n-1}^{n-1} \rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \in L_b^{a'_{j+1} \ldots a'_n} \\
& && \text{(by induction hypothesis)} \\
&\iff (\pi_{j+1}^{n-1} \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \notin L \text{ and} \\
&\qquad (\langle \pi_{j+2}^{n-1}, \ldots, \pi_{n-1}^{n-1} \rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \in L_b^{a'_{j+2} \ldots a'_n} \\
& && \text{(since } b \neq a'_{j+1}) \\
&\iff (\langle \pi_{j+2}^{n-1}, \ldots, \pi_{n-1}^{n-1} \rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \in L_b^{a'_{j+2} \ldots a'_n}.
\end{aligned}$$

Now, consider $b = a'_{j+1}$. Then we conclude

$$\begin{aligned}
&\langle x_{j+2}, \ldots, x_{n-1} \rangle \in L_b^{a_{j+2} \ldots a_n} \\
&\iff \omega_{j+1} \notin L \text{ and } \langle x_{j+2}, \ldots, x_{n-1} \rangle \in L_b^{a_{j+2} \ldots a_n} \\
&\iff \langle \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle \in L_b^{a_{j+1} \ldots a_n} && \text{(since } b \neq a_{j+1}) \\
&\iff (\langle \pi_{j+1}^{n-1}, \ldots, \pi_{n-1}^{n-1} \rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \in L_b^{a'_{j+1} \ldots a'_n} \\
& && \text{(by induction hypothesis)} \\
&\iff (\pi_{j+1}^{n-1} \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \in L \text{ or} \\
&\qquad (\langle \pi_{j+2}^{n-1}, \ldots, \pi_{n-1}^{n-1} \rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \in L_b^{a'_{j+2} \ldots a'_n} \\
& && \text{(since } b = a'_{j+1}) \\
&\iff (\langle \pi_{j+2}^{n-1}, \ldots, \pi_{n-1}^{n-1} \rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1} \rangle) \in L_b^{a'_{j+2} \ldots a'_n}.
\end{aligned}$$

For the remaining case, let $b \notin \{a_{j+1}, a'_{j+1}\}$. Then

$$
\begin{aligned}
&\langle x_{j+2}, \ldots, x_{n-1}\rangle \in L_b^{a_{j+2}\ldots a_n} \\
&\iff \omega_{j+1} \notin L \text{ and } \langle x_{j+2}, \ldots, x_{n-1}\rangle \in L_b^{a_{j+2}\ldots a_n} \\
&\iff \langle \omega_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle \in L_b^{a_{j+1}\ldots a_n} && \text{(since } b \neq a_{j+1}) \\
&\iff (\langle \pi_{j+1}^{n-1}, \ldots, \pi_{n-1}^{n-1}\rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle) \in L_b^{a'_{j+1}\ldots a'_n} \\
& && \text{(by induction hypothesis)} \\
&\iff (\pi_{j+1}^{n-1} \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle) \notin L \text{ and} \\
&\qquad (\langle \pi_{j+2}^{n-1}, \ldots, \pi_{n-1}^{n-1}\rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle) \in L_b^{a'_{j+2}\ldots a'_n} \\
& && \text{(since } b \neq a'_{j+1}) \\
&\iff (\langle \pi_{j+2}^{n-1}, \ldots, \pi_{n-1}^{n-1}\rangle \circ h)(\langle \omega_1, \ldots, \omega_j, \omega_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle) \in L_b^{a'_{j+2}\ldots a'_n}.
\end{aligned}
$$

This completes the induction, and thus, the claim is proved.

Now, we prove that h is an $(L, n-1)$-twister, i.e., we have to show: If $\langle \omega_1, \ldots, \omega_j\rangle$ is a maximal hard sequence for $(L, m, n-1, h)$, then for all $x_{j+1} \in \Sigma^{\leq m}$ there are $x_{j+2}, \ldots, x_{n-1} \in \Sigma^{\leq m}$ such that

$$x_{j+1} \notin L \iff (\pi_{j+1}^{n-1} \circ h)(\langle \omega_1, \ldots, \omega_j, x_{j+1}, \ldots, x_{n-1}\rangle) \in L.$$

There are different cases depending on the order j of the maximal hard sequence. If $j = n-2 > 0$, then the assertion reduces exactly to the claim above, having in mind that $a_{n-1} \neq a'_{n-1}$. If $j < n-2$, then for every $x_{j+1} \in \Sigma^{\leq m}$, the sequence $\langle \omega_1, \ldots, \omega_j, x_{j+1}\rangle$ is not a hard sequence, since $\langle \omega_1, \ldots, \omega_j\rangle$ is maximal. Consequently, $x_{j+1} \in L$ or there are $x_{j+2}, \ldots, x_{n-1} \in \Sigma^{\leq m}$ with $(\pi_{j+1}^{n-1} \circ h)(\langle \omega_1, \ldots, \omega_j, x_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle) \in L$. Hence, $x_{j+1} \notin L$ implies the latter case. This proves the direction from left to right. Conversely, the claim shows for all $x_{j+2}, \ldots, x_{n-1} \in \Sigma^{\leq m}$ and $b = a'_{j+1} \neq a_{j+1}$

$$
\begin{aligned}
&x_{j+1} \notin L \text{ and } \langle x_{j+2}, \ldots, x_{n-1}\rangle \in L_b^{a_{j+2}\ldots a_n} \\
&\iff (\pi_{j+1}^{n-1} \circ h)(\langle \omega_1, \ldots, \omega_j, x_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle) \in L \text{ or} \\
&\qquad (\langle \pi_{j+2}^{n-1}, \ldots, \pi_{n-1}^{n-1}\rangle \circ h)(\langle \omega_1, \ldots, \omega_j, x_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle) \in L_b^{a'_{j+2}\ldots a'_n}.
\end{aligned}
$$

Now, if there are $x_{j+2}, \ldots, x_{n-1} \in \Sigma^{\leq m}$ with $(\pi_{j+1}^{n-1} \circ h)(\langle \omega_1, \ldots, \omega_j, x_{j+1}, x_{j+2}, \ldots, x_{n-1}\rangle) \in L$, then $x_{j+1} \notin L$. Thus, h is an $(L, n-1)$-twister, and using Lemma 44 we obtain the statement desired. ❑

Theorem 46 merges hardest inclusions and the preceding theorem, yielding a upper bound for the polynomial hierarchy collapse in case of unlikely inclusions of partition classes over k-words.

Theorem 46 *Let a be any repetition-free k-word with $k \geq 2$. Let $\delta_a = \|\{i \mid a_i = a_{i+2}\}\|$. If there is a k-word a' with $a \not\leq a'$ and* $\mathrm{NP}(a) \subseteq \mathrm{NP}(a')$, *then* $\mathrm{PH} = \Sigma_2^p(2|a| - \delta_a - 4) \oplus \mathrm{NP}(2|a| - \delta_a - 3)$.

Proof. For any k-word $z = z_1 \ldots z_n$, define the k-word $\hat{z}$ to be the repetition-free version of the word $z_2 z_1 z_3 z_2 \ldots z_n z_{n-1}$. Clearly, it holds $|\hat{z}| = 2|z| - \delta_z - 2$.

Let w be a shortest repetition-free k-subword of a with $w \not\leq a'$. Then, it holds $|\hat{w}| \leq |\hat{a}|$. This can be seen as follows: Assume that w emerges from a when only deleting the j-th letter in a and making the remainder repetition-free. Then, $\delta_w \geq \delta_a - 2$ (by considering the worst case $a_{j-2} = a_j$, $a_{j-1} = a_{j+1}$, and $a_j = a_{j+2}$). Thus,

$$|\hat{w}| \leq 2(|a|-1) - \delta_w - 2 \leq 2|a| - (\delta_a - 2) - 4 = 2|a| - \delta_a - 2 = |\hat{a}|.$$

By induction, we obtain $|\hat{w}| \leq |\hat{a}|$ for arbitrary repetition-free k-subwords of a.

Because of $w \not\leq a'$ and $\mathrm{NP}(w) \subseteq \mathrm{NP}(a) \subseteq \mathrm{NP}(a')$, it holds $\mathrm{NP}(w) \subseteq \mathrm{NP}(\hat{w})$ by Theorem 39. Let κ be the canonical embedding of w into $\hat{w}$. Let $|w| = n$ and $|\hat{w}| = m$. Then, it holds $|D_\kappa| = n-1$. Consider the k-word w' defined for all $j \leq m$ by

$$w'_j =_{\mathrm{def}} \begin{cases} w_r & \text{if } \kappa(r-1) \leq j < \kappa(r), \\ w_n & \text{if } j \geq \kappa(n-1). \end{cases}$$

Since $|w| \leq |\hat{w}|$, the k-word w' is well-defined. Moreover, the following facts are clearly true.

1. $|w'| = |\hat{w}| = m$,
2. $w' \equiv w$,
3. $w'_m \neq w'_{m-1}$ (for $\hat{w}$ this is true due to repetition-freeness).

In order to meet the assumptions of Theorem 45, it remains to prove $w'_j \neq \hat{w}_j$ for all $j \leq m$. Assume the contrary to be true, i.e., there is a $j \leq m$ such that $w'_j = \hat{w}_j$. Let s be maximal with $\kappa(s-1) \leq j$. Then, $w'_j = w_s$ and consequently, $\kappa(s) = j$. But this is a contradiction to the repetition-freeness of w, if $j = \kappa(s-1)$, or to the definition of the canonical embedding κ, if $j > \kappa(s-1)$ and $s \in D_\kappa$, or to $w \not\leq \hat{w}$, if $j > \kappa(s-1)$ and $s = n$. Hence, $w'_j \neq \hat{w}_j$ for all $j \leq m$. Now we can apply Theorem 45. Consequently, from our assumption $\mathrm{NP}(w') = \mathrm{NP}(w) \subseteq \mathrm{NP}(\hat{w})$, we obtain $\mathrm{PH} = \Sigma_2^p(|\hat{w}|-2) \oplus \mathrm{NP}(|\hat{w}|-1) \subseteq \Sigma_2^p(|\hat{a}|-2) \oplus \mathrm{NP}(|\hat{a}|-1)$. ❑

Summarizing all we have done so far we state the Embedding Theorem for k-chains as the formal confirmation of the Embedding Conjecture for k-chains.

Theorem 47 (Embedding Theorem for NP with respect to k-chains.) *Assume that the polynomial hierarchy is infinite. Let (G, f) and (G', f') be k-chains with $k \geq 2$. Then, $(G, f) \leq (G', f')$ if and only if $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$.*

Proof. Without loss of generality, let a and a' be repetition-free k-words representing (G, f) and (G', f'). The direction from left to right is just the Embedding Lemma. For the other direction, let $a \not\leq a'$. Suppose $\mathrm{NP}(a) \subseteq \mathrm{NP}(a')$. Then by Theorem 46, the polynomial hierarchy is finite contradicting our assumption. Hence, $\mathrm{NP}(a) \not\subseteq \mathrm{NP}(a')$. ❑

We get once more that the Embedding Conjecture is generally true for 2-lattices. This is a consequence of Theorem 47 and the following simple proposition.

Proposition 48 *Every 2-lattice is equivalent to its longest chain with alternating labels 1 and 2.*

Corollary 49 *Assume the polynomial hierarchy is infinite. For 2-lattices (G, f) and (G', f') it holds that $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$ if and only if $(G, f) \leq (G', f')$.*

7.2.4 An Extension to k-Lattices

In the preceding we have proved the Embedding Theorem for k-chains. Now we apply this theorem in order to get validity of the Embedding Conjecture for a large subclass of general k-lattices.

Theorem 50 *Assume that the polynomial hierarchy is infinite. Let (G, f) and (G', f') be k-lattices. If $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$, then every minimal k-subchain of (G, f) occurs as a k-subchain of (G', f').*

Proof. Let $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$. Assume there is k-subchain (C, c), identified with the k-word c, such that $(C, c) \not\leq (G', f')$. Let $d^1, \ldots, d^m$ be all k-words representing longest repetition-free k-subchains of (G', f'), and let κ_j be the canonical embedding of c into d^j. Let r denote the maximum of $D_{\kappa_1} \cup \cdots \cup D_{\kappa_m}$. Define z to be the following k-word

$$\begin{aligned} z =_{\mathrm{def}} \;& d^1_{\kappa_1(0)+1} \cdots d^1_{\kappa_1(1)-1} d^2_{\kappa_2(0)+1} \cdots d^2_{\kappa_2(1)-1} \cdots d^m_{\kappa_m(0)+1} \cdots d^m_{\kappa_m(1)-1} c_1 \cdot \\ & \cdot\, d^1_{\kappa_1(1)+1} \cdots d^1_{\kappa_1(2)-1} d^2_{\kappa_2(1)+1} \cdots d^2_{\kappa_2(2)-1} \cdots d^m_{\kappa_m(1)+1} \cdots d^m_{\kappa_m(2)-1} c_2 \cdot \\ & \cdot \;\ldots\; \cdot \\ & \cdot\, d^1_{\kappa_1(r-1)+1} \cdots d^1_{\kappa_1(r)-1} d^2_{\kappa_2(r-1)+1} \cdots d^2_{\kappa_2(r)-1} \cdots d^m_{\kappa_m(r-1)+1} \cdots d^m_{\kappa_m(r)-1} c_r . \end{aligned}$$

Clearly, $c \not\preceq z$ and $d^j \preceq z$ for all $j \in \{1, 2, \ldots, m\}$. We prove $\mathrm{NP}(G', f') \subseteq \mathrm{NP}(z)$. For that, it suffices to show $(G', f') \leq (\{1, 2, \ldots, |z|\}, z)$. We define a mapping $\varphi : G' \to \{1, 2, \ldots, |z|\}$ for $x \in G'$ as follows

$$\varphi(x) =_{\mathrm{def}} \bigvee_{e \text{ represents a chain through } x} \; \bigwedge_{j \text{ with } e \preceq d^j} (\kappa[d^j, z] \circ \kappa[e, d^j])(x).$$

We have to prove that φ is monotonic and $f'(x) = z_{\varphi(x)}$. The latter is obviously true by construction of φ. For the monotonicity, let $x, y \in G'$ with $x \leq y$. Consider e representing a chain through x. Since the value $\varphi(x)$ only depends on chain up to x, without loss of generality we can suppose e to represent a chain additionally going through y and we can suppose j to be so that $(\kappa[d^j, z] \circ \kappa[e, d^j])(y)$ is minimal for all $(\kappa[d^i, z] \circ \kappa[e, d^i])(y)$ with $e \preceq d^i$. Hence, $\varphi(x) \leq (\kappa[d^j, z] \circ \kappa[e, d^j])(y) \leq \varphi(y)$, and thus, φ is monotonic. Now we have a situation $\mathrm{NP}(c) \subseteq \mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f') \subseteq \mathrm{NP}(z)$ but $c \not\preceq z$. Consequently, by Theorem 47, this is contradiction to the strictness of the polynomial hierarchy. Hence, our assumption was false, and every repetition-free k-subchain of (G, f) is also a k-subchain of (G', f'). ❑

As an example, Theorem 50 easily gives that the 3-lattices in Figure 2 and Figure 3 define incomparable partition classes over NP, unless the polynomial hierarchy is finite.

7.3 Evidence II: Beyond Chains

Assume that the polynomial hierarchy does not collapse. By Theorem 50, if the k-lattice (G, f) has a minimal k-subchain which is not a k-subchain of the k-lattice (G', f') then $\mathrm{NP}(G, f) \not\subseteq \mathrm{NP}(G', f')$.

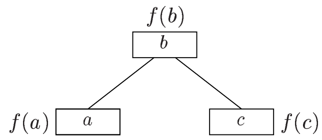

Figure 8: The upper triangle

But what about k-lattices which have the same minimal k-subchains? For example, take the 3-lattices (G, f) and (G', f') represented in Figure 6, that have been used to vitiate the Embedding Theorem for recursively enumerable sets. Since $(G, f) \not\leq (G', f')$ the Embedding Conjecture says that $\mathrm{NP}(G, f) \not\subseteq \mathrm{NP}(G', f')$. However, Theorem 50 does not help to show this because each subchain of (G, f) occurs in (G', f').

In the following we will see that we can prove theorems similar to Theorem 50 for some simple substructures other than subchains. In particular, we get from Theorem 54 that for the 3-lattices (G, f) and (G', f') in Figure 6, $\mathrm{NP}(G, f) \not\subseteq \mathrm{NP}(G', f')$ unless the polynomial hierarchy is finite.

7.3.1 The Upper Triangle

The first structure we investigate is the upper triangle as presented in Figure 8. The main result with respect to upper triangles is Theorem 52. The key to prove this theorem is the following lemma. The proof of this lemma is inspired by a work of Hemaspaandra *et al.* [20].

Lemma 51 *If for all sets $A, B \in \mathrm{NP}$ there exist sets $C, D \in \mathrm{NP}$ such that $C \cup D = \Sigma^*$, $C \subseteq \overline{B \setminus A}$, and $D \subseteq \overline{A \setminus B}$, then* $\mathrm{NP} = \mathrm{coNP}$.

Proof. Suppose that the premise of the lemma is true. Consider the sets A and B defined as

$$\begin{aligned} A &=_{\mathrm{def}} \left\{ \langle F_1, F_2 \rangle \mid F_1 \in \textsc{Satisfiability} \right\} \\ B &=_{\mathrm{def}} \left\{ \langle F_1, F_2 \rangle \mid F_2 \in \textsc{Satisfiability} \right\} \end{aligned}$$

Obviously, A and B belong to NP. The supposition implies that there are NP sets C and D with $C \cup D = \Sigma^*$, $C \subseteq \overline{B \setminus A}$, and $D \subseteq \overline{A \setminus B}$. Let M_1 and M_2 be nondeterministic polynomial-time Turing machines accepting C and D, i.e., $L(M_1) = C$ and $L(M_2) = D$.

Recall that for a formula H, $H \in \textsc{Satisfiability}$ if and only if $H_0 \in \textsc{Satisfiability}$ or $H_1 \in \textsc{Satisfiability}$.

Let $M_1 \times M_2$ be that machine that on an input $\langle F_1, F_2 \rangle$ first simulates M_1 on F_1 (ending with result α) and then simulates M_2 on F_2 (ending with result β). Consider $M_1 \times M_2$ on an input $\langle H_0, H_1 \rangle$ for a propositional formula H along an arbitrary computation path.

- *Case* $(\alpha, \beta) = (1, 1)$. That is $\langle H_0, H_1 \rangle \in C \cap D \subseteq \overline{B \setminus A} \cap \overline{A \setminus B} = (A \cap B) \cup \overline{A \cup B}$.
  - If $\langle H_0, H_1 \rangle \in A \cap B$, then $H, H_0, H_1 \in \textsc{Satisfiability}$.
  - If $\langle H_0, H_1 \rangle \in \overline{A \cup B}$, then $H, H_0, H_1 \notin \textsc{Satisfiability}$.

All in all,

$$H \in \text{SATISFIABILITY} \Longleftrightarrow H_0 \in \text{SATISFIABILITY}.$$

- *Case* $(\alpha, \beta) = (1, 0)$. That is, we know $\langle H_0, H_1 \rangle \in C$ and we assume moreover, $\langle H_0, H_1 \rangle \in C \setminus D = E \cup F \cup G$, where $E \subseteq \overline{A \cup B}$, $F = A \setminus B$, and $G \subseteq A \cap B$.
  - If $\langle H_0, H_1 \rangle \in E \subseteq \overline{A \cup B}$, then $H, H_0, H_1 \notin \text{SATISFIABILITY}$.
  - If $\langle H_0, H_1 \rangle \in G \subseteq A \cap B$, then $H, H_0, H_1 \in \text{SATISFIABILITY}$.
  - If $\langle H_0, H_1 \rangle \in F = A \setminus B$, then $H, H_0 \in \text{SATISFIABILITY}$.

  All in all,

$$H \in \text{SATISFIABILITY} \Longleftrightarrow H_0 \in \text{SATISFIABILITY}.$$

- *Case* $(\alpha, \beta) = (0, 1)$. Analogous arguments as for $(\alpha, \beta) = (1, 0)$ show

$$H \in \text{SATISFIABILITY} \Longleftrightarrow H_1 \in \text{SATISFIABILITY}.$$

- *Case* $(\alpha, \beta) = (0, 0)$. Since $C \cup D = \Sigma^*$ there is always an accepting path. Thus this case is irrelevant.

Define M to be a machine that on input H works in the following way: M simulates $M_1 \times M_2$ on $\langle H_0, H_1 \rangle$ to answer the question $H \in \overline{\text{SATISFIABILITY}}$. M rejects along computation paths with result $(0, 0)$. Along a computation path with result $(1, 1)$ or $(1, 0)$, M simulates $M_1 \times M_2$ on input $\langle H_{00}, H_{01} \rangle$ to answer the question $H_0 \in \overline{\text{SATISFIABILITY}}$. Along paths with $(0, 1)$, M simulates $M_1 \times M_2$ on (H_{10}, H_{11}) to answer the question $H_1 \in \overline{\text{SATISFIABILITY}}$. Continuing in this way we obtain after n simulations of $M_1 \times M_2$ where n is number of variables in H a question $H_{\alpha_0 \alpha_1 \dots \alpha_n} \in \overline{\text{SATISFIABILITY}}$. Answer this question with negation of $H_{\alpha_0 \alpha_1 \dots \alpha_n}$. Clearly, M runs in polynomial time and $L(M) = \overline{\text{SATISFIABILITY}}$. Hence, $\text{SATISFIABILITY} \in \text{coNP}$. ❑

Theorem 52 *Assume that* NP $\neq$ coNP. *Let* (G, f) *and* (G', f') *be* k*-lattices with* $k \geq 3$. *If* $\text{NP}(G, f) \subseteq \text{NP}(G', f')$ *then all* k*-subposets in* (G, f) *having the form as in Figure 8 with pairwise different labels* $f(a)$, $f(b)$, *and* $f(c)$ *do also occur in* (G', f').

Proof. Let (G, f) and (G', f') be k-lattices. Suppose that $\text{NP}(G, f) \subseteq \text{NP}(G', f')$. Suppose that there exists a k-subposet of (G, f) as described in Figure 8. So let $\{a, b, c\} \subseteq G$ be such that $a < b$, $c < b$, a and c are incomparable, and $\|\{f(a), f(b), f(c)\}\| = 3$. Because of Proposition 30, without loss of generality we can assume that $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$. The proof is by contradiction. That is, we assume to the contrary that there exist no $a', b', c' \in G'$ with $a' < b'$, $c' < b'$, $f'(a') = 1$, $f'(b') = 2$, and $f'(c) = 3$.

Let A and B be arbitrary sets in NP. Define a mapping $S : G \to \text{NP}$ for all $z \in G$ as

$$S(z) =_{\text{def}} \begin{cases} \Sigma^* & \text{if } z \geq b, \\ A \cup B & \text{if } z \geq a, z \geq c, \text{ and } z \not\geq b, \\ A & \text{if } z \geq a \text{ and } z \not\geq c, \\ B & \text{if } z \not\geq a \text{ and } z \geq c, \\ A \cap B & \text{if } z \not\geq a \text{ and } z \not\geq c. \end{cases}$$

It is easily seen that S is an NP-homomorphism on G and that $T_S(0_G) = A \cap B$, $T_S(a) = A \setminus B$, $T_S(c) = B \setminus A$, and $T_S(b) = \overline{A \cup B}$. Depending on the value $f(0_G)$ we have several k-partitions defined by (G, f) and S. Without loss of generality, we can assume that $f(0_G) \in \{1, 2, 3, 4\}$. This gives the following four k-partitions:

$$(G, f, S) = \begin{cases} \big(\ \ A, & \overline{A \cup B}, & B \setminus A, & \emptyset, & \emptyset, \ldots, \emptyset\big) & \text{if } f(0_G) = 1 \\ \big(A \setminus B, & (A \cap B) \cup \overline{A \cup B}, & B \setminus A, & \emptyset, & \emptyset, \ldots, \emptyset\big) & \text{if } f(0_G) = 2 \\ \big(A \setminus B, & \overline{A \cup B}, & B, & \emptyset, & \emptyset, \ldots, \emptyset\big) & \text{if } f(0_G) = 3 \\ \big(A \setminus B, & \overline{A \cup B}, & B \setminus A, & A \cap B, & \emptyset, \ldots, \emptyset\big) & \text{if } f(0_G) = 4 \end{cases}$$

Since $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$ there is an NP-homomorphism S' on G' with $(G, f, S) = (G', f', S')$. We consider the following sets of elements of G':

$$\begin{aligned} U_1 &=_{\mathrm{def}} \left\{ z \in G' \;\middle|\; f'(z) = 2 \wedge (\forall x, x \leq z)[f'(x) \neq 3] \right\}, \\ U_3 &=_{\mathrm{def}} \left\{ z \in G' \;\middle|\; f'(z) = 2 \wedge (\forall x, x \leq z)[f'(x) \neq 1] \right\}. \end{aligned}$$

Since there exist no $a', b', c' \in G'$ with $a' < b'$, $c' < b'$, $f'(a') = 1$, $f'(b') = 2$, and $f'(c) = 3$, it holds that $U_1 \cup U_3 = \left\{ z \in G' \;\middle|\; f'(z) = 2 \right\}$. Define sets C and D as

$$C =_{\mathrm{def}} A \cup \bigcup_{z \in U_1} S'(z) \quad \text{and} \quad D =_{\mathrm{def}} B \cup \bigcup_{z \in U_3} S'(z).$$

Clearly, $C, D \in \mathrm{NP}$. Moreover the following is true:

1. $C \cup D = \Sigma^*$,
2. $C \subseteq \overline{B \setminus A}$,
3. $D \subseteq \overline{A \setminus B}$.

This can be verified as follows:

1. Let $x \notin \left(\bigcup_{z \in U_1} S'(z)\right) \cup \left(\bigcup_{z \in U_3} S'(z)\right)$. Then $x \notin (G', f', S')_2$. We conclude

   $$\begin{aligned} \overline{(G', f', S')_2} &= (G', f', S')_1 \cup (G', f', S')_3 \cup (G', f', S')_4 \\ &= (G, f, S)_1 \cup (G, f, S)_3 \cup (G, f, S)_4 \ \subseteq\ A \cup B. \end{aligned}$$

   Thus, $x \in A \cup B$. Hence, for all $x \in \Sigma^*$ we have that $x \in C \cup D$.

2. Obviously, $A \subseteq \overline{B \setminus A}$. Furthermore,

   $$\begin{aligned} \bigcup_{z \in U_1} S'(z) &\subseteq (G', f', S')_1 \cup (G', f', S')_2 \cup (G', f', S')_4 \\ &= (G, f, S)_1 \cup (G, f, S)_2 \cup (G, f, S)_4 \ =\ \overline{(G, f, S)_3} \ \subseteq\ \overline{B \setminus A}. \end{aligned}$$

   Consequently, $C \subseteq \overline{B \setminus A}$.

3. Analogous argumentation as for the second statement.

Since A and B were arbitrarily chosen, we can apply Lemma 51. This implies that $\mathrm{NP} = \mathrm{coNP}$. Hence, a contradiction. ❑

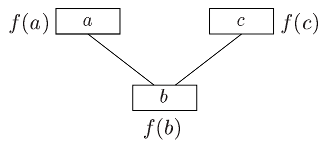

Figure 9: The lower triangle

7.3.2 The Lower Triangle

The structure dual to the upper triangle is the lower triangle presented in Figure 9. Although the proof of Theorem 54 which is here the main result similar to Theorem 52 uses the duality of the structures, the key lemma for establishing the theorem works different to Lemma 51. Interestingly, we are not able to prove the strong consequence that NP is closed under complementation as in Lemma 8 but only by taking polynomial advice. The proof involves techniques of Ko [26] and Hemaspaandra *et al.* [21].

Lemma 53 *If for all sets* $A, B \in \mathrm{NP}$ *there exist sets* $C, D \in \mathrm{NP}$ *such that* $A \backslash B \subseteq C$, $B \backslash A \subseteq D$, *and* $C \cap D = \emptyset$, *then* $\mathrm{NP} \subseteq \mathrm{coNP/poly}$.

Proof. Suppose that the premise of the lemma is true. Let $L \in \mathrm{NP}$. Define the sets A and B as follows:

$$\begin{aligned} A &=_{\mathrm{def}} \{\, \langle x, y\rangle \mid \min\{x,y\} \in L \,\} \\ B &=_{\mathrm{def}} \{\, \langle x, y\rangle \mid \max\{x,y\} \in L \,\} \end{aligned}$$

The supposition implies that there are NP sets C and D with $A \setminus B \subseteq C$, $B \setminus A \subseteq D$, and $C \cap D = \emptyset$. On an intuitive level, if $x \leq y$, then "$\langle x, y\rangle \in C$" means "if $y \in L$ then $x \in L$", and "$\langle x, y\rangle \in D$" means "if $x \in L$ then $y \in L$".

Let $n_0 \in \mathbb{N}$ be the smallest number such that $L \cap \Sigma^{\leq n_0}$ is non-empty. Let $n \geq n_0$ be an arbitrary natural number. We construct a set S_n that will serve as an advice for strings of length $\leq n$. Define for $z \in \Sigma^{\leq n}$ the set $B(z)$ as

$$\begin{aligned} B(z) =_{\mathrm{def}} \{\, x \in \Sigma^{\leq n} \mid\, & [x \neq z \wedge (x < z \rightarrow \langle x, z\rangle \in C) \wedge (z < x \rightarrow \langle x, z\rangle \in D)] \vee \\ & (x < z \wedge \langle x, z\rangle \notin C \cup D) \,\}. \end{aligned}$$

If $G \subseteq L \cap \Sigma^{\leq n}$, then for all $x, z \in G$ with $x \neq z$ either $x \in B(z)$ or $z \in B(x)$. This gives

$$\sum_{z \in G} \|B(z) \cap G\| = \binom{\|G\|}{2} \qquad \text{for all } G \subseteq L \cap \Sigma^{\leq n}. \tag{3}$$

For a set $G \subseteq \Sigma^{\leq n}$, let y_G be a word in G such that $\|B(y_G) \cap G\| \geq \|B(x) \cap G\|$ for all $x \in G$. We consider a certain sequence of sets $\{G_1, G_2, \dots\}$. In particular, we are interested in the words y_{G_j}. Let y_j denote y_{G_j}. Then for all $j \in \mathbb{N}_+$, the sets G_j are inductively defined as follows:

$$\begin{aligned} G_1 &=_{\mathrm{def}} L \cap \Sigma^{\leq n} && \text{if } j = 1 \\ G_j &=_{\mathrm{def}} G_{j-1} \setminus \big(\{y_{j-1}\} \cup B(y_{j-1})\big) && \text{if } j \geq 2. \end{aligned}$$

The following can be shown by inductive arguments:

$$\|G_j\| \leq \frac{\|G_1\|}{2^{j-1}} \qquad \text{for all } j \in \mathbb{N}_+. \tag{4}$$

For $j = 1$, this obvious. For $j \geq 2$, using Equation (3) we easily observe that

$$\|B(y_{j-1}) \cap G_{j-1}\| \geq \frac{\|G_{j-1}\| - 1}{2}.$$

Thus we can conclude

$$\|G_j\| \leq \|G_{j-1}\| - \left(1 + \frac{\|G_{j-1}\| - 1}{2}\right) \leq \frac{\|G_{j-1}\|}{2} \leq \frac{\|G_1\|}{2^{j-1}}.$$

From Equation (4) it immediately follows that there is a smallest r such that for all $s \geq r$, $G_s = \emptyset$. It holds that $r \leq 2 + \log_2 \|G_1\| \leq 2 + \log_2 2^{n+1} \leq n + 3$. Now let S_n be the set

$$S_n =_{\text{def}} \{y_1, y_2, \ldots, y_{r-1}\}.$$

Thus, $\|S_n\| \leq n + 2$. Moreover, we obtain that $S_n \subseteq L$ and that for all $x \in \Sigma^{\leq n}$, it holds:

- If $x \in L$ then there is an $y \in S_n$ such that exactly *one* of the following statements is true:
  - $x = y$ or
  - if $x < y$ then $\langle x, y\rangle \in C$, and if $y < x$ then $\langle x, y\rangle \in D$, or
  - $x < y$ and $\langle x, y\rangle \notin C \cup D$.
- If $x \notin L$ then it holds that for all $y \in S_n$, *all* of the following statements are true:
  - $x \neq y$ and
  - if $x < y$ then $\langle x, y\rangle \in D$ and
  - if $y < x$ then $\langle x, y\rangle \in C$.

From this we can conclude that for all $x \in \Sigma^{\leq n}$,

$$x \in L \iff \begin{array}{l}\text{there exists an } y \in S_n \text{ such that } x = y \text{ or the following is true:} \\ \text{if } x < y \text{ then } \langle x, y\rangle \notin D, \text{ and if } y < x \text{ then } \langle x, y\rangle \notin C.\end{array}$$

Define a set A' as follows:

$$A' =_{\text{def}} \Big\{ \langle x, T\rangle \;\Big|\; |x| \geq n_0 \wedge T \subseteq \Sigma^{\leq n} \wedge \|T\| \leq n + 1 \wedge \\ (\exists y \in T)\big[x = y \vee [(x < y \rightarrow \langle x, y\rangle \notin D) \wedge (y < x \rightarrow \langle x, y\rangle \notin C)]\big] \Big\}$$

It is easily seen that A' is in coNP. Define the advice function h as

$$h(n) =_{\text{def}} \begin{cases} S_n & \text{if } n \geq n_0, \\ \emptyset & \text{if } n < n_0. \end{cases}$$

Clearly, h has polynomial length in n, i.e., $h \in \text{poly}$. Furthermore, we have that for all $x \in \Sigma^*$,

$$x \in L \iff \langle x, h(|x|)\rangle \in A'.$$

Hence, $L \in \text{coNP/poly}$. ❑

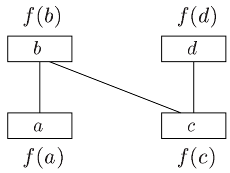

Figure 10: A next step towards resolution of the Embedding Conjecture

Theorem 54 *Assume that the polynomial hierarchy is infinite. Let* (G, f) *and* (G', f') *be k-lattices with* $k \geq 3$. *If* $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$ *then all k-subposets in* (G, f) *having the form as in Figure 9 with pairwise different labels* $f(a)$, $f(b)$, *and* $f(c)$ *do also occur in* (G', f').

Proof. Let (G, f) and (G', f') be k-lattices. Suppose that $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$. Suppose that there exists a k-subposet of (G, f) as described in Figure 9. So let $\{a, b, c\} \in G$ be such that $a > b$, $c > b$, a and c are incomparable, and $\|\{f(a), f(b), f(c)\}\| = 3$. We assume to the contrary that there exist no $a', b', c' \in G'$ with $a' > b'$, $c' > b'$, $f'(a') = f(a)$, $f'(b') = f(b)$, and $f'(c) = f(c)$.

Theorem 18 implies that $\mathrm{coNP}(G^{\partial}, f) \subseteq \mathrm{coNP}(G'^{\partial}, f')$. Thus, our situation translates exactly to the situation in Theorem 52 with respect to coNP. Following the proof of Theorem 52 we obtain that for all sets $A, B \in \mathrm{coNP}$, there exist sets $C, D \in \mathrm{coNP}$ with $C \cup D = \Sigma^*$, $C \subseteq \overline{B \setminus A}$, and $D \subseteq \overline{A \setminus B}$. This easily implies that for all sets $A, B \in \mathrm{NP}$, there exist sets $C, D \in \mathrm{NP}$ such that $C \cap D = \emptyset$, $A \setminus B \subseteq C$, and $B \setminus A \subseteq D$. By Lemma 53, it follows that $\mathrm{NP} \subseteq \mathrm{coNP/poly}$, hence the polynomial hierarchy is finite. Thus we have a contradiction. ❑

From Theorem 54 we easily see that, assuming an infinite polynomial hierarchy, $\mathrm{NP}(G, f) \not\subseteq \mathrm{NP}(G', f')$ for (G, f) being the left 3-lattice and (G', f') being the right 3-lattice in Figure 6. So the counterexample to the Embedding Theorem for recursively enumerable sets is not a counterexample to the Embedding Conjecture.

7.4 Next Steps Towards Resolution

All the theorems we proved in the last subsections to support the Embedding Conjecture are of the following shape:

> *Assume the polynomial hierarchy is infinite. Let* (G, f) *and* (G', f') *be k-lattices. If* $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$ *then all k-subposets of* (G, f) *having a certain pattern* **P** *do also occur in* (G', f').

The patterns for which the according theorem holds are chains, lower, and upper triangles. Progress towards an affirmative resolution of the conjecture means to enlarge this class of patterns. Because the previous theorems all need different proof techniques we have not been able to learn very much from these solutions. It will be important to prove new patterns step by step. The pattern which is the next candidate to be resolved is pictured in Figure 10. The difficult case is $f(b) = f(c)$ and $f(b) \notin \{f(a), f(d)\}$. Reference issues can be found in the following section.

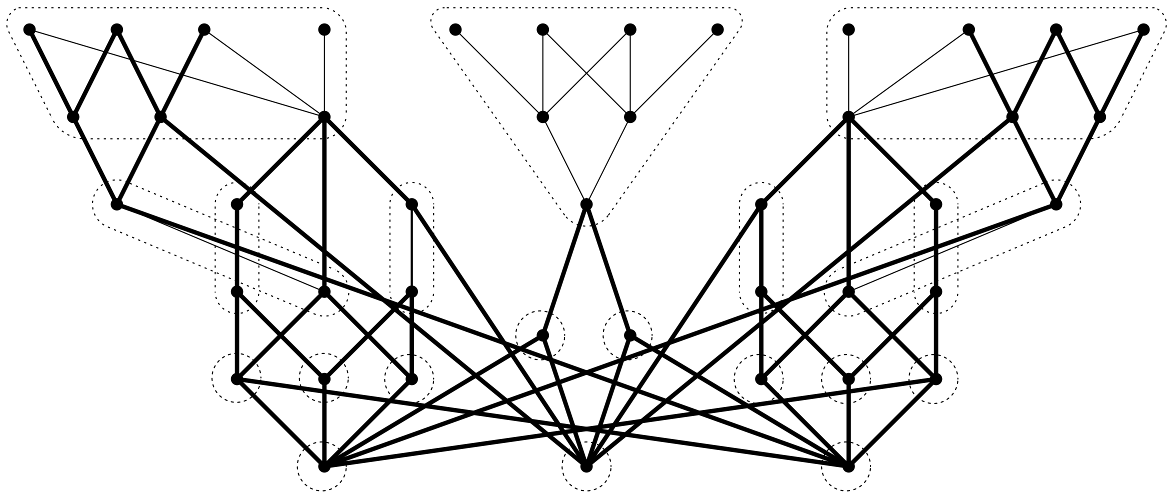

Figure 11: Scheme of all boolean 3-lattices of the form $(\{1,2\}^3, f)$ with $f(1,1,1) = 1$

8 On the Structure of $BH_3(NP)$

Assume the Embedding Conjecture is true and an infinite polynomial hierarchy. Then the structure of the boolean hierarchy of k-partitions with respect to set inclusion is identical with the partial order of $\leq$-equivalence classes of k-lattices with respect to $\leq$. To get an idea of the complexity of the latter structure we will now present the partial order of all equivalence classes of 3-lattices which include a boolean 3-lattice of the form $(\{1,2\}^3, f)$ with surjective f (for non-surjective f these k-lattices do not really define 3-partitions). The 5796 different boolean 3-lattices of the form $(\{1,2\}^3, f)$ with surjective f are in 132 different equivalence classes.

Figure 11 shows the partial order of the 44 equivalence classes which contain boolean 3-lattices of the form $(\{1,2\}^3, f)$ such that $f(1,1,1) = 1$. The cases $f(1,1,1) = 2$ and $f(1,1,1) = 3$ yield isomorphic partial orders. A line from equivalence class **G** up to equivalence class **G'** means that $(G, f) < (G', f')$ for every $(G, f) \in$ **G** and $(G', f') \in$ **G'**. We emphasize that such a study would be intractable without the possibility to present boolean k-lattices by equivalent k-lattices. All 3-lattices in equivalence classes framed by the same dotted line have the same minimal labeled subchains.

Figure 12 shows the middle part and Figure 13 shows the right part of the partial order in Figure 11. In both diagrams, each equivalence class is represented by the minimal 3-lattice. The left part of the partial order in Figure 11 is symmetric to the right part where the labels 2 and 3 change their role.

Theorem 55 *Assume the polynomial hierarchy is infinite. If in Figure 12 and Figure 13 there is a thick line from class* **G** *up to class* **G'** *then* $\mathrm{NP}(G, f) \subset \mathrm{NP}(G', f')$ *for every* $(G, f) \in$ **G** *and* $(G', f') \in$ **G'**.

Every "thick line" in this theorem is an application of Theorem 50 besides the one's marked by $\wedge$ or $\vee$ which are just Theorem 52 (for $\wedge$) and Theorem 54 (for $\vee$).

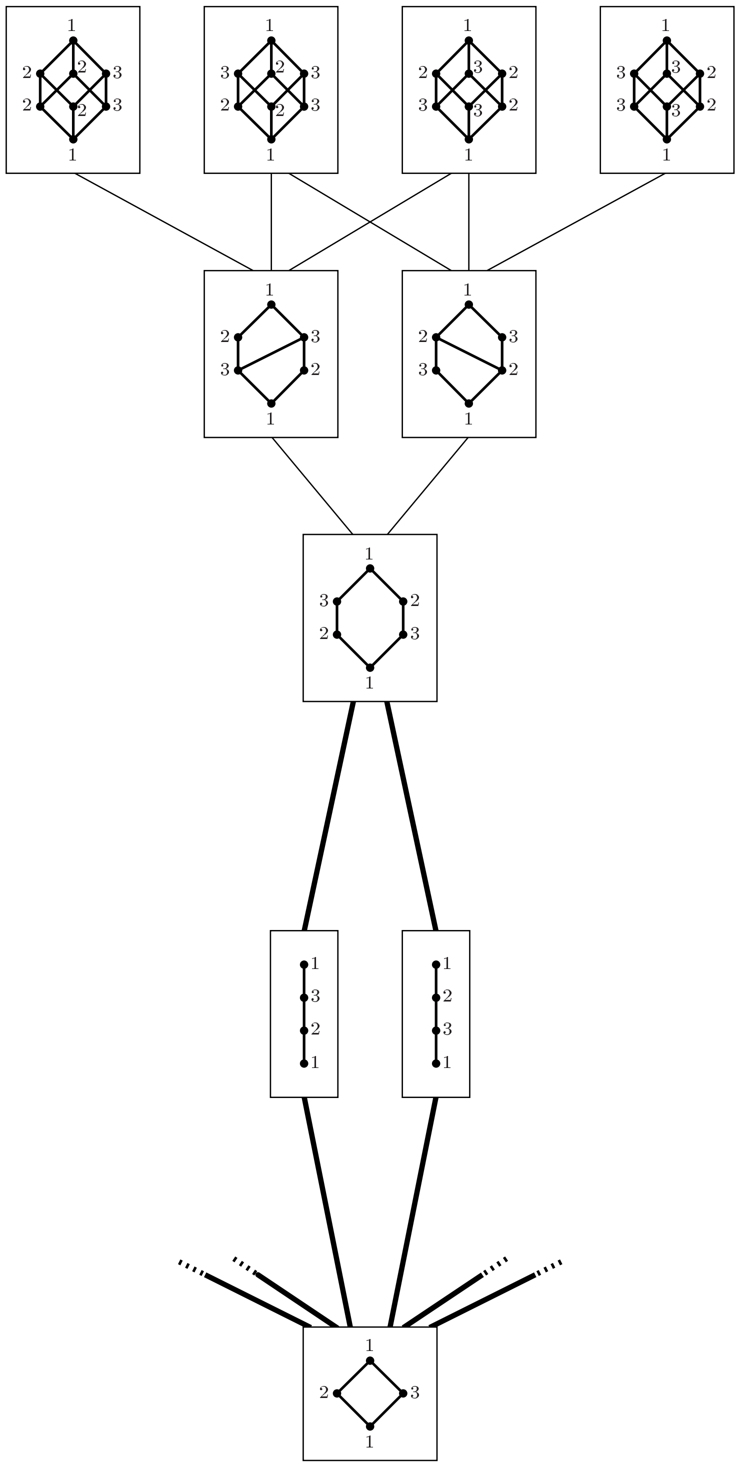

Figure 12: Closer look at the middle part of the scheme in Figure 11

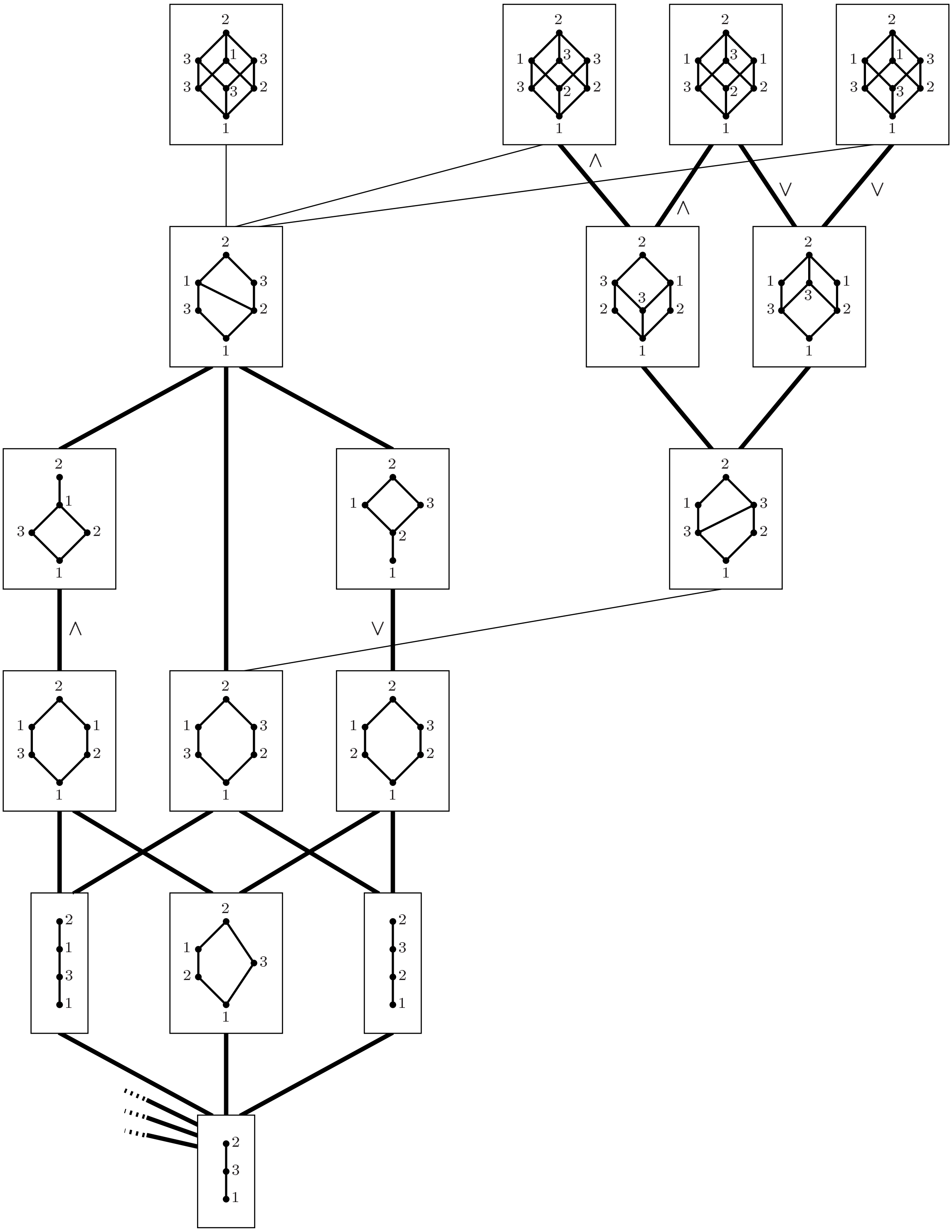

Figure 13: Closer look at the right part of the scheme in Figure 11

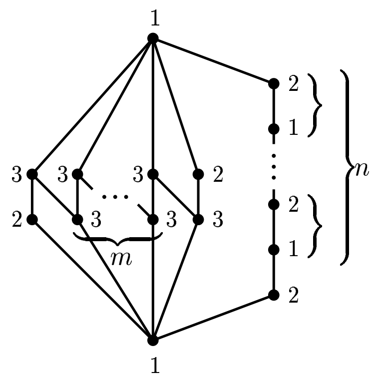

Figure 14: The 3-lattice $\mathfrak{L}(m,n)$ for $m,n \in \mathbb{N}$

At the end of this section we mention that the boolean hierarchy of 3-partitions over NP does not have bounded width with respect to set inclusion unless the polynomial hierarchy collapses.

Proposition 56 *Assume that the polynomial hierarchy is infinite. For every $m \in \mathbb{N}$ there exist at least m partition classes in* $\mathrm{BH}_3(\mathrm{NP})$ *that are incomparable with respect to set inclusion.*

Proof. Let $m \in \mathbb{N}$. We define m 3-chains that are incomparable with respect to $\leq$. Let $G_m = (\{1,2,\dots,m\},\leq)$ be the chain with the natural order on $\{1,2,\dots,m\}$. For every $i \in \{1,2,\dots,m\}$ let $f^i_m : G_m \to \{1,2,3\}$ be the function defined as

$$f^i_m(j) = \begin{cases} 1 & \text{if } (j < i \text{ and } j \text{ is odd}) \text{ or } (j > i \text{ and } j \text{ is even}), \\ 2 & \text{if } (j < i \text{ and } j \text{ is even}) \text{ or } (j > i \text{ and } j \text{ is odd}), \\ 3 & \text{if } j = i. \end{cases}$$

It is easy to see that for all $i,j \in G_m$ with $i \neq j$ the 3-lattices (G_m, f^i_m) and (G_m, f^j_m) are incomparable with respect to $\leq$. Since the polynomial hierarchy is supposed to be strict, by the Embedding Theorem for NP with respect to k-chains (Theorem 47) we obtain that all generated partition classes are pairwise incomparable with respect to set inclusion. ❑

In fact, if the Embedding Conjecture is true and the polynomial hierarchy is strict then the boolean hierarchy of 3-partitions has an infinite subfamily of partition classes that are pairwise incomparable with respect to set inclusion. Even worse, under this assumption, $\mathrm{BH}_3(\mathrm{NP})$ is not well founded with respect to set inclusion then there exist infinite descending chains of partition classes. For instance, consider the family of all 3-lattices $\mathfrak{L}(m,n)$ for $m,n \in \mathbb{N}$ as depicted in Figure 14. One can easily observe the following facts:

1. If an $n \in \mathbb{N}$ is fixed then for all $m \in \mathbb{N}$ it holds $\mathfrak{L}(m,n) > \mathfrak{L}(m+1,n)$. Hence we have an infinite descending chain of 3-lattices thus inducing an infinite descending chain of partition classes.

2. For all $m,n \in \mathbb{N}$ with $m \neq n$ it holds that $\mathfrak{L}(m,m) \not\leq \mathfrak{L}(n,n)$ and $\mathfrak{L}(n,n) \not\leq \mathfrak{L}(m,m)$. This gives the infinite antichain of 3-lattices, hence an infinite antichain of partition classes.

9 Machines That Accept Partitions

In this section we will see how the partitions of classes in the boolean hierarchy of k-partitions over NP can be accepted in a natural way by nondeterministic polynomial-time machines with a notion of acceptance which depends on the generating functions.

Definition 57 *For $m \in \mathbb{N}_+$ a* polynomial-time m-machine M *is a nondeterministic polynomial-time machine producing on every computation path an element from the set $\{0, 1, \ldots, m\}$. For an input x let $M(x) =_{\text{def}} \{\ i \neq 0 \mid \text{there exists a path of } M \text{ on } x \text{ with result } i\ \}$.*

Obviously, a polynomial-time 1-machine is an ordinary nondeterministic polynomial-time machine. All the sets $L_i(M) =_{\text{def}} \{\ x \mid \text{there exists a path of } M \text{ on } x \text{ with result } i\ \}$ are in NP and we obtain $M(x) = \{i \mid x \in L_i(M)\}$ and $c_{L_i(M)}(x) = c_{M(x)}(i)$ for all x.

Definition 58 *For a function $f : \mathcal{P}(\{1, \ldots, m\}) \to \{1, \ldots, k\}$ and a polynomial-time m-machine M let (M, f) be the k-partition defined by $c_{(M,f)}(x) = f(M(x))$ for all $x \in \Sigma^*$.*

Note that every function $f : \mathcal{P}(\{1, 2, \ldots, m\}) \to \{1, 2, \ldots, k\}$ can also be considered to be the function $f : \{1, 2\}^m \to \{1, 2, \ldots, k\}$ and vice versa by the relationships $f(a_1, \ldots, a_m) = f(\{i \mid a_i = 1\})$ for $a_1, \ldots, a_m \in \{1, 2\}$ and $f(A) = f(c_A(1), \ldots, c_A(m))$ for $A \subseteq \{1, 2, \ldots, m\}$.

Theorem 59 $\mathrm{NP}(f) = \{\ (M, f) \mid M$ *is a polynomial-time m-machine* $\}$ *for all $m \in \mathbb{N}_+$ and all functions $f : \{1, 2\}^m \to \{1, 2, \ldots, k\}$.*

Proof. To show the forwards inclusion let $B_1, \ldots, B_m \in$ NP. There are nondeterministic polynomial-time machines $M_1, \ldots, M_m$ such that M_i accepts B_i for $i \in \{1, 2, \ldots, m\}$. Define M to be a nondeterministic polynomial-time machine which simulates $M_1, \ldots, M_m$ in parallel but when simulating M_i it outputs i rather than 1. Obviously, for all $i \in \{1, 2, \ldots, m\}$, $L_i(M) = B_i$ and we conclude

$$\begin{aligned} c_{f(B_1,\ldots,B_m)}(x) &= f(c_{B_1}(x), \ldots, c_{B_m}(x)) = f(c_{L_1(M)}(x), \ldots, c_{L_m(M)}(x)) \\ &= f(c_{M(x)}(1), \ldots, c_{M(x)}(m)) = f(M(x)) = c_{(M,f)}(x). \end{aligned}$$

For the inclusion "$\supseteq$" consider a polynomial-time m-machine M and conclude

$$\begin{aligned} c_{(M,f)}(x) &= f(M(x)) = f(c_{M(x)}(1), \ldots, c_{M(x)}(m)) = f(c_{L_1(M)}(x), \ldots, c_{L_m(M)}(x)) \\ &= c_{f(L_1(M),\ldots,L_m(M))}(x). \end{aligned}$$

❑

Finally, we discuss completeness for the partition classes $\mathrm{NP}(f)$. We will see that it is easy to construct from an arbitrary NP-complete problem a problem which is complete for $\mathrm{NP}(f)$.

We already used the notion of many-one reductions for partitions. We say that the k-partition A is polynomial-time many-one reducible to the k-partition B (for short $A \leq_m^p B$) if and only if there exists a polynomial-time computable function g such that $c_A(w) = c_B(g(w))$ for all w. Note that in the case $k = 2$ this yields exactly the classical notion of polynomial-time many-one reducibility for sets.

From Theorem 59 we easily obtain the following:

Proposition 60 *Let $k \geq 2$. All classes in* $\mathrm{BH}_k(\mathrm{NP})$ *and* $\mathrm{BC}_k(\mathrm{NP})$ *are closed under* $\leq_m^p$.

A k-partition A is $\leq_m^p$-complete for a partition class $\mathcal{C}$ (which is closed under $\leq_m^p$) if and only if $A \in \mathcal{C}$ and $B \leq_m^p A$ for every k-partition $B \in \mathcal{C}$. Recall that π_j^m denote projections of an encoded word $w = \langle w_1, \ldots, w_m \rangle$. For a set $A \subseteq \Sigma^*$ and a function $f : \{1,2\}^m \to \{1,2,\ldots,k\}$ define the k-partition $A(f)$ by

$$c_{A(f)}(w) =_{\mathrm{def}} f((c_A \circ \pi_1^m)(w)(c_A \circ \pi_2^m)(w) \ldots (c_A \circ \pi_m^m)(w)) \qquad \text{for all } w \in \Sigma^*.$$

Theorem 61 *Let $f : \{1,2\}^m \to \{1,2,\ldots k\}$ with $k \geq 2$. Let A be $\leq_m^p$-complete for* NP. *Then $A(f)$ is $\leq_m^p$-complete for* $\mathrm{NP}(f)$.

Proof. Defining $A_i =_{\mathrm{def}} \{\, w \mid \pi_i^m(w) \in A \,\}$ for $i \in \{1,2,\ldots,m\}$ we obtain $A_i \in \mathrm{NP}$. For every $w \in \Sigma^*$ we conclude

$$c_{A(f)}(w) = f((c_A \circ \pi_1^m)(w) \ldots (c_A \circ \pi_m^m)(w)) = f(c_{A_1}(w) \ldots c_{A_m}(w)) = c_f(A_1, \ldots, A_m).$$

Consequently, $A(f) = f(A_1, \ldots, A_m) \in \mathrm{NP}(f)$.

Now take any $B_1, \ldots, B_m \in \mathrm{NP}$. Since A is $\leq_m^p$-complete for NP there exist polynomial-time computable functions $g_1, \ldots, g_m$ such that for every $i \in \{1,2,\ldots,m\}$, $w \in B_i \Leftrightarrow g_i(w) \in A$. Defining $g(w) =_{\mathrm{def}} \langle g_1(w), \ldots, g_m(w) \rangle$ for every $w \in \Sigma^*$, we can conclude

$$\begin{aligned} c_{f(B_1,\ldots,B_m)}(w) &= f(c_{B_1}(w), \ldots, c_{B_m}(w)) = f((c_A \circ g_1)(w), \ldots, (c_A \circ g_m)(w)) \\ &= f((c_A \circ \pi_1^m \circ g)(w), \ldots, (c_A \circ \pi_m^m \circ g)(w)) = c_{A(f)}(g(w)). \end{aligned}$$

Hence $f(B_1, \ldots, B_m) \leq_m^p A(f)$. ❑

As a natural example of complete partition, consider the classification problem Entailment we have extensively discussed in the introductory chapter.

Theorem 62 Entailment *is $\leq_m^p$-complete for* $\mathrm{NP}(f)$ *where $f : \{1,2\}^2 \to \{1,2,3,4\}$ is the function defined as $f(1,1) = 1$, $f(1,2) = 2$, $f(2,1) = 3$, and $f(2,2) = 4$.*

Proof. Obviously, Entailment is in $\mathrm{NP}(f)$. Consider the partition Satisfiability(f) which is $\leq_m^p$-complete for $\mathrm{NP}(f)$ by Theorem 61. More explicitly:

$$\begin{aligned} \textsc{Satisfiability}(f)_1 &= \{\, \langle F_1, F_2 \rangle \mid H_1 \in \textsc{Satisfiability}, H_2 \in \textsc{Satisfiability} \,\}, \\ \textsc{Satisfiability}(f)_2 &= \{\, \langle F_1, F_2 \rangle \mid H_1 \in \textsc{Satisfiability}, H_2 \notin \textsc{Satisfiability} \,\}, \\ \textsc{Satisfiability}(f)_3 &= \{\, \langle F_1, F_2 \rangle \mid H_1 \notin \textsc{Satisfiability}, H_2 \in \textsc{Satisfiability} \,\}, \\ \textsc{Satisfiability}(f)_4 &= \{\, \langle F_1, F_2 \rangle \mid H_1 \notin \textsc{Satisfiability}, H_2 \notin \textsc{Satisfiability} \,\}. \end{aligned}$$

We have to show that Satisfiability$(f) \leq_m^p$ Entailment. This reduction is seen by the following algorithm. On input $\langle F_1, F_2 \rangle$, make the sets of variables in F_1 and in F_2 disjoint, take two new variables z_1 and z_2 not involved in F_1 or F_2, and output $\langle F_1', F_2' \rangle$ where $F_1' =_{\mathrm{def}} z_1 \wedge F_1$ and $F_2' =_{\mathrm{def}} z_2 \wedge F_2$. Obviously, the algorithm runs in polynomial time. Moreover, we have that

$$\begin{aligned} F_1' \models F_2' &\iff F_1' \notin \textsc{Satisfiability} \\ F_2' \models F_1' &\iff F_2' \notin \textsc{Satisfiability}. \end{aligned}$$

Thus $\langle F_1, F_2 \rangle \in \textsc{Satisfiability}(f)_i \Leftrightarrow \langle F_1', F_2' \rangle \in \textsc{Entailment}_i$ for all $i \in \{1,2,3,4\}$. ❑

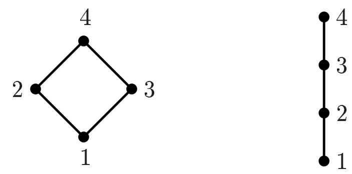

Figure 15: Classes with complete partitions having components of same complexities

Proving completeness results for entire partitions instead of only for the components allows finer distinguishing the complexity of classification problems. Obviously, completeness translates from the partition to the components: If the k-partition A is $\leq_m^p$-complete for the partition class $\mathcal{C}$ then for each $i \in \{1,\dots,k\}$, A_i is $\leq_m^p$-complete for the class $\mathcal{C}_i$. The converse direction need not to hold as can be seen for the partition classes that are described by the 4-lattices in Figure 15. Each class belongs to $\mathrm{BH}_4(\mathrm{NP})$, thus has complete partitions. Entailment is just a complete partition for the class generated by left 4-lattice in the figure. Let A be any $\leq_m^p$-complete partition for the class generated by the right 4-chain. Then for all $i \in \{1,2,3,4\}$ we have $\textsc{Entailment}_i \equiv_m^p A_i$ but A does not reduce to Entailment unless NP = coNP as follows easily from Theorem 31.

10 Conclusion

In the preceding sections, we have investigated the boolean hierarchy of k-partitions over NP for $k \geq 3$ as a generalization of the boolean hierarchy of sets (i.e., 2-partitions) over NP. Whereas the structure of the latter hierarchy is rather simple the structure of the boolean hierarchy of k-partitions over NP for $k \geq 3$ turned out to be much more complicated. We established the Embedding Conjecture which enables us to get an overview on this structure. This conjecture was supported by several partial results. A complete proof of or a counterexample to the Embedding Conjecture for NP are left to find. However, a counterexample—two k-lattices (G,f) and (G',f') with $(G,f) \not\leq (G',f')$, but $\mathrm{NP}(G,f) \subseteq \mathrm{NP}(G',f')$—may be hard to find since more recently, it has been proven in [28] that the relation $\leq$ induces a sufficient and necessary criterion for *relativizable* inclusions.

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