Decay of covariances, uniqueness of ergodic component and scaling limit for a class of $\nabla\phi$ systems with non-convex potential

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March 12, 2009

Abstract

We consider a gradient interface model on the lattice with interaction potential which is a non-convex perturbation of a convex potential. Using a technique which decouples the neighboring vertices sites into even and odd vertices, we show for a class of non-convex potentials: the uniqueness of ergodic component for $\nabla\phi$- Gibbs measures, the decay of covariances, the scaling limit and the strict convexity of the surface tension.

AMS 2000 Subject Classification. 60K35, 82B24, 35J15

Key words and phrases. effective non-convex gradient interface models, uniqueness of ergodic component, decay of covariances, scaling limit, surface tension

1 Introduction

Phase separation in $\mathbb{R}^{d+1}$ can be described by effective interface models. In this setting we ignore overhangs and for $x \in \mathbb{Z}^d$, we denote by $\phi(x) \in \mathbb{R}$ the height of the interface above or below the site x. Let Λ be a finite set in $\mathbb{Z}^d$ with boundary

$$\partial\Lambda := \{x \notin \Lambda,\ ||x-y|| = 1 \text{ for some } y \in \Lambda\},\ \text{where } \|x-y\|= \sum_{i=1}^{d} |x_i - y_i| \tag{1}$$

and with given boundary condition ψ such that $\phi(x) = \psi(x)$ for $x \in \partial\Lambda$. Let $\bar{\Lambda} := \Lambda \cup \partial\Lambda$ and let $\mathrm{d}\phi_\Lambda = \prod_{x\in\Lambda} \mathrm{d}\phi(x)$ be the Lebesgue measure over $\mathbb{R}^\Lambda$. For a finite region $\Lambda \subset \mathbb{Z}^d$, *the finite Gibbs measure* ν_Λ^ψ *on* $\mathbb{R}^{\mathbb{Z}^d}$ *with boundary condition* ψ for the field of height variables $(\phi(x))_{x\in\mathbb{Z}^d}$ over Λ is defined by

$$\nu_\Lambda^\psi(\mathrm{d}\phi) = \frac{1}{Z_\Lambda^\psi} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} \mathrm{d}\phi_\Lambda \delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda}),\ \text{with } Z_\Lambda^\psi = \int_{\mathbb{R}^{\mathbb{Z}^d}} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} \mathrm{d}\phi_\Lambda \delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda}) \tag{2}$$

where $\delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda}) = \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_{\psi(x)}(d\phi(x))$; ν_Λ^ψ is characterized by the inverse temperature $\beta > 0$ and the Hamiltonian H_Λ^ψ on Λ, which we assume to be of gradient type:

$$H_\Lambda^\psi(\phi) = \sum_{i\in I} \sum_{\substack{x\in\Lambda \\ x+e_i\in\Lambda\cup\partial\Lambda}} U(\nabla_i\phi(x)), \tag{3}$$

[*]Supported by the DFG-Forschergruppe 718 'Analysis and stochastics in complex physical systems'

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where

$$I = \{-d, -d+1, \dots, -1, 1, 2, \dots, d\}$$

and where we introduced for each $x \in \mathbb{Z}^d$ and each $i \in I$, the discrete gradient

$$\nabla_i \phi(x) = \phi(x + e_i) - \phi(x),$$

that is, the interaction depends only on the differences of neighboring heights. Note that $e_i, i = 1, 2, \dots d$ denote the unit vectors and $e_{-i} = -e_i$. We thus have a massless model with a continuous symmetry. $U \in C^2(\mathbb{R})$ is a function with quadratic growth at infinity:

$$U(\eta) \geq A|\eta|^2 - B, \qquad \eta \in \mathbb{R} \tag{4}$$

for some $A > 0, B \in \mathbb{R}$. Our state space $\mathbb{R}^{\mathbb{Z}^d}$ being unbounded, such models are facing delocalization in lower dimensions $d = 1, 2$, and no infinite Gibbs state exists in these dimensions. Instead of looking at the Gibbs measures of the $(\phi(x))_{x \in \mathbb{Z}^d}$, Funaki and Spohn proposed to consider the distribution of the gradients $(\nabla_i \phi(x))_{i \in I, x \in \mathbb{Z}^d}$ under ν (see Definition 2 below) in the so-called **gradient Gibbs** measures, which in view of the Hamiltonian (3), can also be given in terms of a Dobrushin-Landford-Ruelle description.

Assuming strict convexity of U:

$$0 < C_1 \leq U'' \leq C_2 < \infty \tag{5}$$

Funaki and Spohn showed in [14], the existence and uniqueness of ergodic gradient Gibbs measures for every tilt $u \in \mathbb{R}^d$, see also Sheffield [21]. Moreover, they also proved that the corresponding free energy, or surface tension, $\sigma \in C^1(\mathbb{R}^d)$ is convex. Both results are essential for the derivation of the hydrodynamical limit of the Ginzburg Landau model.

In fact under the strict convexity assumption (5) of U, much is known for the gradient field. At large scales it behaves much like the harmonic crystal or gradient free fields which is a Gaussian field with quadratic U. In particular Naddaf and Spencer [20] showed that the rescaled gradient field converges weakly as $\epsilon \searrow 0$ to a continuous homogeneous Gaussian field, that is

$$S_\epsilon(f) = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \sum_{i \in I} (\nabla_i \phi(x) - u_i) f_i(\epsilon x) \to N(0, \sigma_u^2(f)) \ \text{ as } \ \epsilon \to 0, \qquad f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$$

where the convergence takes place under ergodic ν with tilt u (see also Giacomin et al. [16] and Biskup and Spohn [3] for similar results). This scaling limit theorem derived at standard scaling $\epsilon^{d/2}$, is far from trivial, since, as shown in Delmotte and Deuschel [8], the gradient field has slowly decaying, non absolutely summable covariances, of the algebraic order

$$|\text{cov}_\nu(\nabla_i \phi(x), \nabla_j \phi(y))| \sim \frac{C}{1 + \|x - y\|^d}. \tag{6}$$

The aim of this paper is to relax the strict convexity assumption (5). Our potential is of the form

$$U(\nabla_i \phi(x)) = V(\nabla_i \phi(x)) + g(\nabla_i \phi(x))$$

where $V, g \in C^2(\mathbb{R})$ are such that

$$C_1 \leq V'' \leq C_2, \ 0 < C_1 < C_2 \quad \text{ and } \quad -C_0 \leq g'' \leq 0, \text{ with } C_0 > C_2 \tag{7}$$

and

$$\|g''\|_{L^1(\mathbb{R})} < \infty \text{ or } \|g''\|_{L^2(\mathbb{R})} < \infty \text{ or } \|g'\|_{L^1(\mathbb{R})} < \infty. \tag{8}$$

(For the case of a non-convex perturbation g with compact support, see Remark 24).
Our main result shows that if the inverse temperature β is sufficiently small, that is if:

$$\sqrt{\frac{\beta}{C_1}} \|g''\|_{L^1(\mathbb{R})} \leq \frac{C_1}{2C_2\sqrt{d}}, \tag{9}$$

or

$$(\beta)^{1/4}||g''||_{L^2(\mathbb{R})} < \frac{(C_1)^{3/2}}{2(C_2)^{3/4}d^{1/4}} \tag{10}$$

or

$$(\beta)^{3/4}||g'||_{L^2(\mathbb{R})} \leq \frac{(C_1)^{3/2}}{2(C_2)^{5/4}}\frac{1}{(2d)^{3/4}}, \tag{11}$$

then the results known in the strict convex case hold. In particular we have uniqueness of the ergodic component at every tilt $u \in \mathbb{R}^d$, strict convexity of the surface tension, scaling limit theorem and decay of covariances. As stated above, the hydrodynamical limit for the corresponding Ginzburg-Landau model, should then essentially follow from these results.

Note that uniqueness of the ergodic measures is not true at any β for this type of models: Biskup and Kotecky give an example of non convex U which can be described as the mixture of two Gaussians with two different variances, where two ergodic gradient Gibbs measures coexists at $u = 0$ tilt, cf. Biskup and Kotecky [2] (see also Figure 4: Example (a) below). For similar results for discrete models, see [12]. The situation at lower temperature (i.e. large β) is again quite different: using renormalization group techniques, Adams et al. show the strict convexity for small tilt u, cf. [1].

In a previous paper with S. Mueller, cf. [7], we have proved strict convexity of the surface tension for moderate β in a regime similar to (9). The method used in [7], based on two scale decomposition of the free field, gives less sharp estimates for the temperature, however it is more general and could be applied to non bipartite graphs. In this paper we use a different technique, which relies on the bipartite property of our model. We consider the distribution of the even gradient (that is of $\phi(y) - \phi(x)$ where both x, y are even): which is again a gradient field and show that under the condition (9), that the resulting Hamiltonian is strictly convex. The main idea, similar to [7], is that convexity can be gained via integration (see also Brascamp et al. [5] for previous use of the even/odd representation). In fact we show more: the Hamiltonian associated to the even variables admits an random walk representation, cf. Helffer and Sjöstrand [17] or Deuschel [10], which is the key tool in deriving covariance estimates such as (6) and scaling limit theorems. The other ingredient is the fact, that given the even gradients, the conditional law of the odd variables is simply a product law. Of course this is a special feature of our bipartite model, in particular it would be quite challenging to iterate the procedure, a scheme which could possibly lower the temperature towards the transition β_c. Note that iterating the scheme is an interesting open problem.

The rest of the paper is presented as follows: in Section 2 we define the model and recall the definition of gradient Gibbs measures. Section 3 presents the odd/even characterization of the gradient field, in particular our main result, Theorem 10, shows that the random walk representation holds for the even sites under the condition (5). Section 3 also presents a few examples, in particular we show that our criteria gets very close to the Biskup-Kotecky transition, cf. example 3.3.2. In section 4, we give a proof of the uniqueness of the ergodic component. In view of the product law for conditional distribution of the odd sites given the even gradient, this follows immediately from the uniqueness of the even gradient ergodic measures. Here we adapt the dynamical coupling argument of [14] to our situation. Section 5 deals with the decay of covariances, the proof is based on the random walk representation for the even sites which allows us to use the result of [8]. Section 6 shows the scaling limit theorem, here again we focus on the even variables and apply the random walk representation idea of [16]. Finally section 7 proves the strict convexity of the surface tension, or free energy, which follows from the convexity of the Hamiltonian for the even gradient. We also show a few useful equalities dealing with the derivative of σ, since they play an important role for the hydrodynamic limits of the Ginzburg Landau model.

2 General Definitions and Notations

2.1 ϕ-Gibbs Measures

For $A \subset \mathbb{Z}^d$, we shall denote by $\mathcal{F}_A$ the σ-field generated of $\mathbb{R}^{\mathbb{Z}^d}$ generated by $\{\phi(x) : x \in A\}$.

Definition 1 *The probability measure $\nu \in P(\mathbb{R}^{\mathbb{Z}^d})$ is called a Gibbs measure for the ϕ-field (ϕ-Gibbs measure for short), if its conditional probability of $\mathcal{F}_{\Lambda^c}$ satisfies the DLR equation*

$$\nu(\,\cdot\,|\mathcal{F}_{\Lambda^c})(\psi) = \nu_\Lambda^\psi(\cdot), \quad \nu - a.e.\ \psi,$$

for every finite $\Lambda \subset \mathbb{Z}^d$.

It is known that the ϕ-Gibbs measures exist under condition (4) when the dimension $d \geq 3$, but not for $d = 1, 2$, where the field "delocalizes" as $\Lambda \nearrow \mathbb{Z}^d$, c.f. [13]. An infinite volume limit (thermodynamic limit) for ν_Λ^ψ and $\Lambda \nearrow \mathbb{Z}^d$ exists only when $d \geq 3$.

2.2 $\nabla\phi-$Gibbs Measures

2.2.1 Notation on $\mathbb{Z}^d$

Let $(\mathbb{Z}^d)^* := \{b = (x_b, y_b) \mid x_b, y_b \in \mathbb{Z}^d, \|x_b - y_b\| = 1, b$ directed from x_b to $y_b\}$; note that each undirected bond appears twice in $(\mathbb{Z}^d)^*$. Let $\Lambda^* := (\mathbb{Z}^d)^* \cap (\Lambda \times \Lambda)$, $\partial\Lambda^* := \{b = (x_b, y_b) \mid x_b \in \mathbb{Z}^d \setminus \Lambda, y_b \in \Lambda, \|x_b - y_b\| = 1\}$ and $\overline{\Lambda^*} := \{b = (x_b, y_b) \in (\mathbb{Z}^d)^* \mid x_b \in \Lambda$ or $y_b \in \Lambda\}$.

The height variables $\phi = \{\phi(x); x \in \mathbb{Z}^d\}$ on $\mathbb{Z}^d$ automatically determines a field of height differences $\nabla\phi = \{\nabla\phi(b); b \in (\mathbb{Z}^d)^*\}$. One can therefore consider the distribution μ of $\nabla\phi$-field under the ϕ-Gibbs measure μ. We shall call μ the $\nabla\phi$-Gibbs measure. In fact, it is possible to define the $\nabla\phi$-Gibbs measures directly by means of the DLR equations and, in this sense, $\nabla\phi$-Gibbs measures exist for all dimensions $d \geq 1$.

A sequence of bonds $\mathcal{C} = \{b^{(1)}, b^{(2)}, \ldots, b^{(n)}\}$ is called a chain connecting y and x, $x, y \in \mathbb{Z}^d$, if $y_{b_1} = y, x_{b^{(i)}} = y_{b^{(i+1)}}$ for $1 \leq i \leq n-1$ and $x_{b^{(n)}} = x$. The chain is called a closed loop if $x_{b^{(n)}} = y_{b^{(1)}}$. A plaquette is a closed loop $\mathcal{A} = \{b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}\}$ such that $\{x_{b^{(i)}}, i = 1, \ldots, 4\}$ consists of 4 different points.

The field $\eta = \{\eta(b)\} \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ is said to satisfy the plaquette condition if

$$\eta(b) = -\eta(-b) \text{ for all } b \in (\mathbb{Z}^d)^* \text{ and } \sum_{b \in \mathcal{A}} \eta(b) = 0 \text{ for all plaquettes } \mathcal{A} \text{ in } \mathbb{Z}^d, \tag{12}$$

where $-b$ denotes the reversed bond of b. Let χ be the set of all $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ which satisfy the plaquette condition and let $L_r^2, r > 0$ be the set of all $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ such that

$$|\eta|_r^2 := \sum_{b \in (\mathbb{Z}^d)^*} |\eta(b)|^2 e^{-2r|x_b|} < \infty.$$

We denote $\chi_r = \chi \cap L_r^2$ equipped with the norm $|\cdot|_r$. For $\phi = (\phi(x))_{x \in \mathbb{Z}^d}$ and $b \in (\mathbb{Z}^d)^*$, we define the height differences $\eta^\phi(b) := \nabla\phi(b) = \phi(y_b) - \phi(x_b)$. Then $\nabla\phi = \{\nabla\phi(b)\}$ satisfies the plaquette condition. Conversely, the heights $\phi^{\eta,\phi(0)} \in \mathbb{R}^{\mathbb{Z}^d}$ can be constructed from height differences η and the height variable $\phi(0)$ at $x = 0$ as

$$\phi^{\eta,\phi(0)}(x) := \sum_{b \in \mathcal{C}_{0,x}} \eta(b) + \phi(0), \tag{13}$$

where $\mathcal{C}_{0,x}$ is an arbitrary chain connecting 0 and x. Note that $\phi^{\eta,\phi(0)}$ is well-defined if $\eta = \{\eta(b)\} \in \chi$.

2.2.2 Definition of $\nabla\phi$-Gibbs measures

We next define the finite volume $\nabla\phi$-Gibbs measures. For every $\xi \in \chi$ and finite $\Lambda \subset \mathbb{Z}^d$ the space of all possible configurations of height differences on $\overline{\Lambda^*}$ for given boundary condition ξ is defined as

$$\chi_{\overline{\Lambda^*},\xi} = \{\eta = (\eta(b))_{b\in\overline{\Lambda^*}}; \eta \vee \xi \in \chi\},$$

where $\eta \vee \xi \in \chi$ is determined by $(\eta \vee \xi)(b) = \eta(b)$ for $b \in \overline{\Lambda^*}$ and $= \xi(b)$ for $b \notin \overline{\Lambda^*}$.

Remark 2 Note that $\chi_{\overline{\Lambda^*},\xi}$ is an affine space such that $\dim \chi_{\overline{\Lambda^*},\xi} = |\Lambda|$ (at least when $\mathbb{Z}^d \setminus \Lambda$ is connected). Indeed, fixing a point $x_0 \notin \Lambda$, we consider the map $J_\Lambda : \chi_{\overline{\Lambda^*},\xi} \ni\rightarrow \phi = \{\phi(x)\} \in \mathbb{R}^\Lambda$ defined by

$$\phi(x) = \sum_{b\in C_{x_0,x}} (\eta \vee \xi)(b)$$

for a chain $C_{x_0,x}$ connecting x_0 and $x \in \Lambda$. J_Λ is then well-defined and diffeomorphic.

The finite volume $\nabla\phi$-Gibbs measure in Λ (or more precisely, in Λ^*) with boundary condition ξ is defined by

$$\mu_{\Lambda,\xi}(\mathrm{d}\eta) = \frac{1}{Z_{\Lambda.\xi}} \exp\left\{ -\frac{\beta}{2} \sum_{b\in\overline{\Lambda^*}} U(\eta(b)) \right\} \mathrm{d}\eta_{\Lambda,\xi} \in P(\chi_{\overline{\Lambda^*},\xi}),$$

where $\mathrm{d}\eta_{\Lambda,\xi}$ denotes a uniform measure on the affine space $\chi_{\overline{\Lambda^*},\xi}$ and $Z_{\Lambda,\xi}$ is the normalization constant. Let $P(\chi)$ be the set of all probability measures on χ and let $P_2(\chi)$ be those $\mu \in P(\chi)$ satisfying $E^\mu[|\eta(b)|^2] < \infty$ for each $b \in (\mathbb{Z}^d)^*$.

Remark 3 For every $\xi \in \chi$ and $a \in \mathbb{R}$, let $\psi = \phi^{\xi,a}$ be defined by (13) and consider the measure ν_Λ^ψ. Then $\mu_{\Lambda,\xi}$ is the image measure of ν_Λ^ψ under the map $J'_\Lambda : \{\phi(x)\}_{x\in\Lambda} \to \{\eta(b) := \nabla(\phi\vee\psi)(b)\},\ b \in \overline{\Lambda^*}$. Note that the image measure is determined only by ξ and is independent of the choice of a. Similarly, let $\tilde{J'}_\Lambda : \{\phi(x)\}_{x\in\mathbb{Z}^d} \to \{\eta(b)\}_{b\in\chi} := \nabla(\phi \vee \psi)(b),\ b \in \overline{\Lambda^*}$ and $= \nabla\psi(b)$ otherwise.

Definition 4 *The probability measure $\mu \in P(\chi)$ is called a Gibbs measure for the height differences ($\nabla\phi$-Gibbs measure for short), if it satisfies the DLR equation*

$$\mu(\,\cdot\,|\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\Lambda^*}})(\xi) = \mu_{\Lambda,\xi}(\cdot), \quad \mu - a.e.\ \xi,$$

for every finite $\Lambda \subset \mathbb{Z}^d$, where $\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\Lambda^*}}$ stands for the σ-field of χ generated by $\{\eta(b); b \in (\mathbb{Z}^d)^*\setminus\overline{\Lambda^*}\}$.

We will define by

$$\mathcal{G}(H) := \{\mu \in P_2(\chi) : \mu \text{ is } \nabla\phi - \text{Gibbs measure such that } \mu_{\Lambda,\xi} \text{ has Hamiltonian } H_\Lambda^\xi\}.$$

3 Even/Odd Representation

3.1 Notation on the Even Subset of $\mathbb{Z}^d$

As $\mathbb{Z}^d$ is a bipartite graph, we will label the vertices of $\mathbb{Z}^d$ as **even** and **odd** vertices, such that every **even** vertex has only **odd** nearest neighbor vertices and vice-versa. Let $\mathcal{E}^d := \{a = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d \mid \sum_{i=1}^d a_i = 2p, p \in \mathbb{Z}\}$, $\mathcal{O}^d := \{a = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d \mid \sum_{i=1}^d a_i = 2p + 1, p \in \mathbb{Z}\}$ and $\mathcal{O}^d_\Lambda := \mathcal{O}^d \cap \Lambda$. Let $\Lambda^{\mathcal{E}} \subset \mathcal{E}^d$ finite. We will next define the bonds in $\mathcal{E}^d$ in a similar fashion to the definitions for bonds on $\mathbb{Z}^d$. Let $(\mathcal{E}^d)^* := \{b = (x_b, y_b) \mid x_b, y_b \in \mathcal{E}^d, \|x_b - y_b\| = 2, b \text{ directed from } x_b \text{ to } y_b\}$, $(\Lambda^{\mathcal{E}})^* := (\mathcal{E}^d)^* \cap (\Lambda^{\mathcal{E}} \times \Lambda^{\mathcal{E}})$, $\overline{(\Lambda^{\mathcal{E}})^*} := \{b = (x_b, y_b) \in (\mathcal{E}^d)^* \mid x_b \in \Lambda^{\mathcal{E}} \text{ or } y_b \in \Lambda^{\mathcal{E}}\}$ $\partial(\Lambda^{\mathcal{E}})^* := \{b = (x_b, y_b) \mid x_b \in \mathcal{E}^d \setminus \Lambda^{\mathcal{E}}, y_b \in \Lambda^{\mathcal{E}}, \|x_b - y_b\| = 2\}$, $\partial^-\Lambda^{\mathcal{E}} := \{y \in \Lambda^{\mathcal{E}} \mid y = y_b \text{ for some } b \in \partial(\Lambda^{\mathcal{E}})^*\}$ and $\partial\Lambda^{\mathcal{E}} := \{y \notin \Lambda^{\mathcal{E}} \mid y = y_b \text{ for some } b \in \partial(\Lambda^{\mathcal{E}})^*\}$.

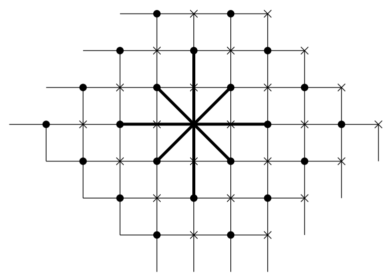

Figure 1: The bonds of 0 in $\mathcal{E}^2$

An **even** plaquette is a closed loop $\mathcal{A}^{\mathcal{E}} = \{b^{(1)}, b^{(2)}, \ldots, b^{(n)}\}$, where $b^{(i)} \in (\mathcal{E}^d)^*$, $n \in \{3,4\}$, such that $\{x_{b^{(i)}}, i = 1, \ldots, n\}$ consists of n different points in $\mathcal{E}^d$. The field $\eta = \{\eta(b)\} \in \mathbb{R}^{(\mathcal{E}^d)^*}$ is said to satisfy the **even** plaquette condition if

$$\eta(b) = -\eta(-b) \text{ for all } b \in (\mathcal{E}^d)^* \text{ and } \sum_{b \in \mathcal{A}^{\mathcal{E}}} \eta(b) = 0 \text{ for all } \textbf{even} \text{ plaquettes in } \mathcal{E}^d. \tag{14}$$

Let $\chi^{\mathcal{E}}$ be the set of all $\eta \in \mathbb{R}^{(\mathcal{E}^d)^*}$ which satisfy the even plaquette condition. For each $b = (x_b, y_b) \in (\mathcal{E}^d)^*$ we define the even height differences $\eta^{\mathcal{E}}(b) := \nabla^{\mathcal{E}}\phi(b) = \phi(y_b) - \phi(x_b)$. The heights $\phi^{\eta^{\mathcal{E}}, \phi(0)}$ can be constructed from the height differences $\eta^{\mathcal{E}}$ and the height variable $\phi(0)$ at $a = 0$ as

$$\phi^{\eta^{\mathcal{E}}, \phi(0)}(a) := \sum_{b \in C^{\mathcal{E}}_{0,a}} \eta^{\mathcal{E}}(b) + \phi(0), \tag{15}$$

where $a \in \mathcal{E}^d$ and $C^{\mathcal{E}}_{0,a}$ is an arbitrary path in $\mathcal{E}^d$ connecting 0 and a. Note that $\phi^{\eta, \phi(0)}(a)$ is well-defined if $\eta^{\mathcal{E}} = \{\eta^{\mathcal{E}}(b)\} \in \chi^{\mathcal{E}}$. We also define $\chi^{\mathcal{E}}_r$ similarly as we define χ_r. As on $\mathbb{Z}^d$, let $P(\chi^{\mathcal{E}})$ be the set of all probability measures on $\chi^{\mathcal{E}}$ and let $P_2(\chi^{\mathcal{E}})$ be those $\mu \in P(\chi^{\mathcal{E}})$ satisfying $E^{\mu}[|\eta^{\mathcal{E}}(b)|^2] < \infty$ for each $b \in (\mathcal{E}^d)^*$.

Remark 5 *Let $\eta \in \chi$. Using the plaquette condition property of η, we will define $\eta_{|(\mathcal{E}^d)^*}$ from η thus: if $b_1 = (x, x+e_i)$, $b_2 = (x+e_j, x)$ and $b^{\mathcal{E}} = (x+e_i, x+e_j)$, we define $\eta_{|(\mathcal{E}^d)^*}(b^{\mathcal{E}}) = \eta(b_1) + \eta(b_2)$. Note that $\eta_{|(\mathcal{E}^d)^*} \in \chi^{\mathcal{E}}$ for $\eta_{|(\mathcal{E}^d)^*}$ thus defined.*

3.2 Definition of $\nabla^{\mathcal{E}}\phi$-Gibbs measures

For every $\xi \in \chi^{\mathcal{E}}$ and finite $\Lambda^{\mathcal{E}} \subset \mathcal{E}^d$ the space of all possible configurations of height differences on $\overline{(\Lambda^{\mathcal{E}})^*}$ for given boundary condition $\xi^{\mathcal{E}}$ is defined as

$$\chi^{\mathcal{E}}_{\overline{(\Lambda^{\mathcal{E}})^*}, \xi^{\mathcal{E}}} = \{\eta^{\mathcal{E}} = (\eta^{\mathcal{E}}(b))_{b \in \overline{(\Lambda^{\mathcal{E}})^*}}; \eta^{\mathcal{E}} \vee \xi^{\mathcal{E}} \in \chi^{\mathcal{E}}\},$$

where $\eta^{\mathcal{E}} \vee \xi^{\mathcal{E}} \in \chi^{\mathcal{E}}$ is determined by $(\eta^{\mathcal{E}} \vee \xi^{\mathcal{E}})(b) = \eta^{\mathcal{E}}(b)$ for $b \in \overline{(\Lambda^{\mathcal{E}})^*}$ and $= \xi^{\mathcal{E}}(b)$ for $b \notin \overline{(\Lambda^{\mathcal{E}})^*}$.

The $\phi^{\mathcal{E}}$-Gibbs measure $\nu^{(2)}$ and the $\nabla^{\mathcal{E}}\phi$-Gibbs measure $\mu^{(2)}$ can be defined similarly to the ϕ-Gibbs measure and the $\nabla\phi$-Gibbs measure in Subsections 2.1 and 2.2.2.

Remark 6 Note that $\chi^{\mathcal{E}}_{\overline{(\Lambda^{\mathcal{E}})^*}, \xi^{\mathcal{E}}}$ is an affine space such that $\dim \chi^{\mathcal{E}}_{\overline{(\Lambda^{\mathcal{E}})^*}, \xi^{\mathcal{E}}} = |\Lambda^{\mathcal{E}}|$ (at least when $\mathcal{E}^d \setminus \Lambda^{\mathcal{E}}$ is connected). Indeed, fixing a point $x_0 \notin \Lambda^{\mathcal{E}}$, we consider the map $K_{\Lambda^{\mathcal{E}}} : \chi^{\mathcal{E}}_{\overline{(\Lambda^{\mathcal{E}})^*}, \xi^{\mathcal{E}}} \ni \to \phi^{\mathcal{E}} = \{\phi(x)\} \in \mathbb{R}^{\Lambda^{\mathcal{E}}}$ defined by

$$\phi^{\mathcal{E}}(x) = \sum_{b \in C_{x_0,x}} (\eta^{\mathcal{E}} \vee \xi^{\mathcal{E}})(b)$$

for a chain $C_{x_0,x}$ connecting x_0 and $x \in \Lambda$. $K_{\Lambda^{\mathcal{E}}}$ is then well-defined and diffeomorphic. Similarly, let $\tilde{K}_{\Lambda^{\mathcal{E}}} : \chi^{\mathcal{E}} \to \phi^{\mathcal{E}} = \{\phi(x)\} \in \mathbb{R}^{\mathcal{E}^d} := \sum_{b \in C_{x_0,x}} (\eta^{\mathcal{E}} \vee \xi^{\mathcal{E}})(b), \eta \in \chi^{\mathcal{E}}_{\overline{(\Lambda^{\mathcal{E}})^*}, \xi^{\mathcal{E}}}$ and $= \psi^{\xi^{\mathcal{E}}, a}(x)$ similarly defined as in (15) otherwise.

Remark 7 For every $\xi^{\mathcal{E}} \in \chi^{\mathcal{E}}$ and $a \in \mathbb{R}$, let $\psi^{\mathcal{E}} = \phi^{\xi^{\mathcal{E}},a}$ be defined by (15) and consider the measure $\nu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}$. Then $\mu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}$ is the image measure of $\nu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}$ under the map $K'_{\Lambda^{\mathcal{E}}} : \{\phi(x)\}_{x\in\Lambda^{\mathcal{E}}} \to \{\eta^{\mathcal{E}}(b) := \nabla^{\mathcal{E}}(\phi^{\mathcal{E}} \vee \psi^{\mathcal{E}})(b)\},\ b \in \overline{(\Lambda^{\mathcal{E}})^*}$. Note that the image measure is determined only by $\xi^{\mathcal{E}}$ and is independent of the choice of a.

3.3 Restriction of a $\nabla\phi$-Gibbs measure to $\mathcal{E}^d$

Let
$$\theta(x) = (\phi(x+e_1),\ldots,\phi(x+e_d),\phi(x-e_1),\ldots,\phi(x-e_d)) \text{ and } \phi^{\mathcal{E}} = (\phi(x))_{x\in\mathcal{E}^d}. \tag{16}$$

Definition 8 *Let $\Lambda^{\mathcal{E}}$ be a finite set in $\mathcal{E}^d$. We define a finite set $\Lambda_{\Lambda^{\mathcal{E}}} \subset \mathbb{Z}^d$ associated to $\Lambda^{\mathcal{E}}$ as follows: if $x \in \Lambda^{\mathcal{E}}$, then $x \in \Lambda_{\Lambda^{\mathcal{E}}}$ and $x+e_i \in \Lambda_{\Lambda^{\mathcal{E}}}$ for all $i \in I$. Note that by definition, $\partial\Lambda_{\Lambda^{\mathcal{E}}} = \partial\Lambda^{\mathcal{E}}$ (see Figures 2 and 3).*

Lemma 9 *Let ν be a ϕ-Gibbs measure with finite Gibbs measure ν^{ψ}_{Λ}, with Hamiltonian H^{ψ}_{Λ} as in (3). Then $\nu|_{\mathcal{E}^d} := \nu^{(2)} \in P(\mathbb{R}^{\mathcal{E}^d})$ is a ϕ-Gibbs measure with finite Gibbs measure $\nu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}$, such that $\nu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}$ has Hamiltonian $H^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}$, where*

$$H^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}(\phi^{\mathcal{E}}) := \sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} F_x(\theta(x)),\ \text{with } F_x(\theta(x)) = -\log\int_{\mathbb{R}} e^{-2\beta\sum_{i\in I}U(\nabla_i\phi(x))}\,\mathrm{d}\phi(x) \tag{17}$$

and F_x are functions of the even gradients (see Remark 11).

Proof. Let
$$\mathcal{F}_{\mathbb{Z}^d} := \sigma\left(\phi(x), x\in\mathbb{Z}^d\right) \text{ and } \mathcal{F}_{\mathcal{E}^d} := \sigma\left(\phi(x), x\in\mathcal{E}^d\right).$$

Set
$$H_x(\phi) = \sum_{i\in I} U(\nabla_i\phi(x)).$$

To prove the statement of the lemma, we will use the fact that ν is a Gibbs measure, which means that for all Λ finite sets in $\mathbb{Z}^d$ and for all $A \in \mathcal{F}_{\mathbb{Z}^d}$ we have

$$\nu(A|\mathcal{F}_{\Lambda^c})(\psi) \quad = \quad \nu^{\psi}_{\Lambda}(A) = \frac{1}{Z^{\psi}_{\Lambda}}\int_A e^{-\beta H^{\psi}_{\Lambda}(\phi)}\,\mathrm{d}\phi_{\Lambda}\delta_{\psi}(d\phi_{\mathbb{Z}^d\setminus\Lambda}), \tag{18}$$

where $\delta_{\psi}(d\phi_{\mathbb{Z}^d\setminus\Lambda}) = \prod_{x\in\mathbb{Z}^d\setminus\Lambda}\delta_{\psi(x)}(d\phi(x))$. Note first that (cf (3))

$$H^{\psi}_{\Lambda}(\phi) = \sum_{x\in\bar{\Lambda}} H_x(\phi) = 2\sum_{x\in\mathcal{O}^d_{\Lambda}} H_x(\phi).$$

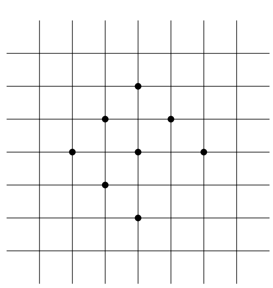

Figure 2: The graph of $\Lambda^{\mathcal{E}}$

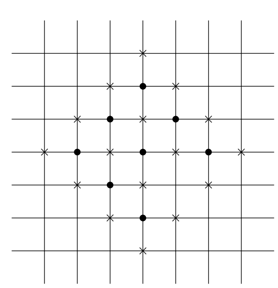

Figure 3: The graph of $\Lambda_{\Lambda^{\mathcal{E}}}$

Let $\Lambda^{\mathcal{E}}$ be a finite set in $\mathcal{E}^d$; we define $\delta_\psi(d\phi_{\mathcal{E}^d\setminus\Lambda^{\mathcal{E}}}) = \prod_{x\in\mathcal{E}^d\setminus\Lambda^{\mathcal{E}}} \delta_{\psi(x)}(d\phi(x))$. Let $A \in \mathcal{F}_{\mathcal{E}^d} \subset \mathcal{F}_{\mathbb{Z}^d}$, $\mathrm{d}\phi_{\Lambda^{\mathcal{E}}} = \prod_{x\in\Lambda^{\mathcal{E}}} \mathrm{d}\phi(x)$ and $\mathrm{d}\phi_{\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} = \prod_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} \mathrm{d}\phi(x)$. Then, by integrating out the odd height variables and due to (18) and $\partial\Lambda_{\Lambda^{\mathcal{E}}} = \partial\Lambda^{\mathcal{E}}$, we have for every $\psi \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\psi_{|\mathcal{E}^d} = \psi^{\mathcal{E}}$

$$
\begin{aligned}
\nu^{\psi}_{\Lambda_{\Lambda^{\mathcal{E}}}}(A) &= \mathbf{E}_\nu(1_A|\mathcal{F}_{(\Lambda_{\Lambda^{\mathcal{E}}})^c})(\psi) = \frac{1}{Z^{\psi}_{\Lambda_{\Lambda^{\mathcal{E}}}}} \int_{\mathbb{R}^{\overline{\Lambda_{\Lambda^{\mathcal{E}}}}}} 1_A(\phi) e^{-\beta H^{\psi}_{\Lambda_{\Lambda^{\mathcal{E}}}}(\phi)} d\phi_{\Lambda_{\Lambda^{\mathcal{E}}}} \delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda_{\Lambda^{\mathcal{E}}}}) \\
&= \frac{1}{Z^{\psi}_{\Lambda_{\Lambda^{\mathcal{E}}}}} \int_{\mathbb{R}^{\overline{\Lambda_{\Lambda^{\mathcal{E}}}}}} 1_A(\phi) e^{-\sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} 2\beta H_x(\phi)} \mathrm{d}\phi_{\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} \mathrm{d}\phi_{\Lambda^{\mathcal{E}}} \delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda}) \\
&= \frac{1}{Z^{\psi}_{\Lambda_{\Lambda^{\mathcal{E}}}}} \int_{\mathbb{R}^{\overline{\Lambda^{\mathcal{E}}}}} 1_A(\phi) \left(\int_{\mathbb{R}^{\overline{\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}}}} e^{-2\beta\sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} H_x(\phi)} \mathrm{d}\phi_{\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} \right) \mathrm{d}\phi_{\Lambda^{\mathcal{E}}} \delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda_{\Lambda^{\mathcal{E}}}}) \\
&= \frac{1}{Z^{\psi}_{\Lambda_{\Lambda^{\mathcal{E}}}}} \int_{\mathbb{R}^{\overline{\Lambda^{\mathcal{E}}}}} 1_A(\phi) \prod_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} \left(\int_{\mathbb{R}} e^{-2\beta H_x(\phi)} \mathrm{d}\phi(x) \right) \mathrm{d}\phi_{\Lambda^{\mathcal{E}}} \delta_\psi(d\phi_{\mathcal{E}^d\setminus\Lambda^{\mathcal{E}}}) \\
&= \frac{1}{Z^{\psi}_{\Lambda_{\Lambda^{\mathcal{E}}}}} \int_A e^{-\sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} F^{\Lambda}_x(\phi(x+e_1),\ldots,\phi(x-e_d))} \prod_{i\in I} \prod_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}} \mathrm{d}\phi(x+e_i) \delta_\psi(d\phi_{\mathcal{E}^d\setminus\Lambda^{\mathcal{E}}}) \\
&= \nu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}(A),
\end{aligned} \tag{19}
$$

where $Z^{\psi}_{\Lambda_{\Lambda^{\mathcal{E}}}} = Z^{\psi^{\mathcal{E}}}_{\Lambda^{\mathcal{E}}}$ is the normalizing constant and $\overline{\Lambda^{\mathcal{E}}}$ and $\overline{\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}}$ are the restrictions of $\overline{\Lambda_{\Lambda^{\mathcal{E}}}}$ to the set of the evens, respectively of the odds . It follows from formula (19) that $\mathbf{E}_\nu(1_A|\mathcal{F}_{(\Lambda_{\Lambda^{\mathcal{E}}})^c}) \in \mathcal{F}_{(\Lambda^{\mathcal{E}})^c}$. Since for every $\Lambda^{\mathcal{E}} \subset \mathcal{E}^d$ we have that $\mathcal{F}_{(\Lambda^{\mathcal{E}})^c} \subset \mathcal{F}_{(\Lambda_{\Lambda^{\mathcal{E}}})^c}$, we have by using (19) that for every $A \in \mathcal{F}_{\mathcal{E}^d}$

$$
\begin{aligned}
\nu^{(2)}(A|\mathcal{F}_{(\Lambda^{\mathcal{E}})^c})(\psi^{\mathcal{E}}) &= \nu(A|\mathcal{F}_{(\Lambda^{\mathcal{E}})^c})(\psi^{\mathcal{E}}) = \mathbf{E}_\nu\left(\mathbf{E}_\nu(1_A|\mathcal{F}_{(\Lambda_{\Lambda^{\mathcal{E}}})^c})|\mathcal{F}_{(\Lambda^{\mathcal{E}})^c}\right)(\psi^{\mathcal{E}}) \\
&= \mathbf{E}_\nu\left(\nu_{\Lambda_{\Lambda^{\mathcal{E}}}}(A)|\mathcal{F}_{(\Lambda^{\mathcal{E}})^c}\right)(\psi^{\mathcal{E}}) = \nu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}(A).
\end{aligned}
$$

Therefore $\nu|_{\mathcal{E}^d}$ satisfies the DLR equations. □

Remark 10 Note that the new Hamiltonian $H^{(2)}$ depends on β through the functions F_x.

Remark 11 Note that for any constant $C_{2d} = (C, C, \ldots, C) \in \mathbb{R}^{2d}$, by using the change of variables $\phi(x) \to \phi(x) + C$ in the integral formula for F_x in (17), we have

$$F_x(\theta(x)) = F_x(\theta(x) + C_{2d}).$$

In particular, this means that for any $i \in I$

$$F_x(\theta(x)) = F_x(\phi(x+e_1) - \phi(x+e_i), \ldots, \phi(x-e_d) - \phi(x+e_i)). \tag{20}$$

Therefore we are still dealing with a gradient system, even though this is no longer a two-body gradient system. F_x, and consequently $H^{(2)}$, are functions of the even height differences by (20) and (17).

Lemma 12 *Let G be a $\mathcal{F}_{\mathbb{Z}^d}$-measurable and bounded function. Then for all Gibbs measures ν and all $\psi \in \mathbb{R}^{\mathbb{Z}^d}$, we have*

$$\mathbf{E}_\nu\left(G|\mathcal{F}_{\mathcal{E}^d}\right)(\psi) = \int_{\mathbb{R}^{\mathbb{Z}^d}} G(\phi) \prod_{x\in\mathcal{O}^d} \nu^{\psi}_x(\mathrm{d}\phi(x))\delta_\psi(d\phi_{\mathcal{E}^d}), \ \textit{with}\ \delta_\psi(d\phi_{\mathcal{E}^d}) = \prod_{x\in\mathcal{E}^d} \delta_{\psi(x)}(d\phi(x)), \tag{21}$$

$\nu^{\psi}_x(\mathrm{d}\phi(x)) = \frac{e^{-2\beta\sum_{i\in I} U(\psi(x+e_i)-\phi(x))}\,\mathrm{d}\phi(x)}{Z(\theta(x))}$ *and* $Z(\theta(x)) = \int_{\mathbb{R}} e^{-2\beta\sum_{i\in I} U(\psi(x+e_i)-\phi(x))}\,\mathrm{d}\phi(x)$.

PROOF. Using a standard argument, it is enough to consider G with finite support. From the DLR equations for ν, we have

$$\mathbf{E}_\nu\left(G|\mathcal{F}_{\mathbb{Z}^d\setminus\mathcal{O}_n^d}\right)(\psi) = \int_{\mathbb{R}^{\bar{\Lambda}}} G(\phi) \prod_{x\in\mathcal{O}_n^d} \nu_x^\psi(\,\mathrm{d}\phi(x))\delta_\psi(d\phi_{\mathbb{Z}^d\setminus\mathcal{O}_n^d}), \text{ with } \mathcal{O}_n^d = \{x\in\mathcal{O}^d : ||x|| \le n\}.$$

Since $\mathcal{F}_{\mathbb{Z}^d\setminus\mathcal{O}_{n+1}^d} \subseteq \mathcal{F}_{\mathbb{Z}^d\setminus\mathcal{O}_n^d}$ for all $n\in N$ and $\cap_{n=1}^\infty \mathcal{F}_{\mathbb{Z}^d\setminus\mathcal{O}_n^d} = \mathcal{F}_{\mathcal{E}^d}$, we have by the convergence of conditional expectations

$$\mathbf{E}_\nu\left(G|\mathcal{F}_{\mathcal{E}^d}\right)(\psi) = \lim_{n\to\infty}\mathbf{E}_\nu\left(G|\mathcal{F}_{\mathbb{Z}^d\setminus\mathcal{O}_n^d}\right).$$

Let us denote by $P_n^\psi \in P(\mathbb{R}^{\mathbb{Z}^d})$ the measure defined by

$$P_n^\psi(A) := \int_A \prod_{x\in\mathcal{O}_n^d} \nu_x^\psi(\,\mathrm{d}\phi(x))\delta_\psi(d\phi_{\mathbb{Z}^d\setminus\mathcal{O}_n^d}) = \otimes_{x\in\mathcal{O}_n^d}\nu_x^\psi \otimes_{x\in\mathbb{Z}^d\setminus\mathcal{O}_n^d} \delta_{\psi(x)}(\,\mathrm{d}\phi(x)).$$

Then by the Kolmogorov's extension of measures for infinite product measures, $\nu \in P(\mathbb{R}^{\mathbb{Z}^d})$ defined by

$$\nu(A) = \int_A \prod_{x\in\mathcal{O}^d} \nu_x^\psi(\,\mathrm{d}\phi(x))\delta_\psi(d\phi_{\mathcal{E}^d})$$

is the unique extension of P_n such that $P_n(A) = \nu(pr_n^{-1}(A))$, with $pr(\phi) := \phi' \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\phi'(x) = \phi(x)$, if $x\in\mathcal{O}_n^d$ and $\phi'(x) = \psi(x)$ otherwise. We also have $\lim_{n\to\infty} P_n(A) = \nu(A)$. The claim follows. □

We will define by

$$\mathcal{G}^{\mathcal{E}}(H) := \{\mu^{(2)} \in P_2(\chi^{\mathcal{E}}) : \mu^{(2)} \text{ is } \nabla^{\mathcal{E}}\phi - \text{Gibbs measure such that } \mu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}} \text{ has Hamiltonian } H^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}\}.$$

Lemma 13 *Let $\mu\in\mathcal{G}(H)$. Then $\mu|_{(\mathcal{E}^d)^*} := \mu^{(2)} \in \mathcal{G}^{\mathcal{E}}(H^{(2)})$, where $H^{(2)}$ is defined as in (17).*

PROOF. Let
$$\mathcal{F}_{(\mathbb{Z}^d)^*} := \sigma\left(\eta(b), b\in(\mathbb{Z}^d)^*\right) \text{ and } \mathcal{F}_{(\mathcal{E}^d)^*} := \sigma\left(\eta^{\mathcal{E}}(b), b\in(\mathcal{E}^d)^*\right).$$

Fixing $a\in\mathbb{R}$, for all Λ finite sets in $\mathbb{Z}^d$ with $\mathbb{Z}^d\setminus\Lambda$ connected and for all $A\in\mathcal{F}_{(\mathbb{Z}^d)^*}$, we have by Remark 3 that

$$\mu_{\Lambda,\xi}(A) = \mathbf{E}_{\nu_\Lambda^\psi}(1_A \circ \tilde{J}'_\Lambda), \text{ with } \psi \text{ given by (13).} \tag{22}$$

For all $B\in\mathcal{F}_{\mathcal{E}^d}$ and $\Lambda^{\mathcal{E}}$ finite sets in $\mathcal{E}^d$ with $\mathcal{E}^d\setminus\Lambda^{\mathcal{E}}$ connected, we have by Remark 6

$$\nu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}(B) = \mathbf{E}_{\mu^{(2)}_{\Lambda^{\mathcal{E}},\xi^{\mathcal{E}}}}(1_B\circ\tilde{K}_\Lambda), \text{ with } \xi^{\mathcal{E}}(b) = \nabla^{\mathcal{E}}\psi(b),\ b\in\chi^{\mathcal{E}}. \tag{23}$$

Let $\Lambda^{\mathcal{E}}$ be a finite set in $\mathcal{E}^d$ and let $A\in\mathcal{F}_{(\mathcal{E}^d)^*}\subset\mathcal{F}_{(\mathbb{Z}^d)^*}$; then since $\mu\in\mathcal{G}(H)$, by using Lemma 9, (22) and (23), we have for every $\xi\in\chi$ such that $\xi_{|(\mathcal{E}^d)^*} = \xi^{\mathcal{E}}\in\chi^{\mathcal{E}}$ (recall Remark 5)

$$\mu_{\Lambda_{\Lambda^{\mathcal{E}}},\xi}(A) = \mathbf{E}_{\nu_{\Lambda_{\Lambda^{\mathcal{E}}},\psi}}(1_A\circ\tilde{J}'_\Lambda) = \mathbf{E}_{\nu^{(2)}_{\Lambda^{\mathcal{E}},\psi^{\mathcal{E}}}}(1_A\circ\left(\tilde{J}'_\Lambda\right)_{|\mathcal{E}^d}) = \mathbf{E}_{\mu^{(2)}_{\Lambda^{\mathcal{E}},\xi^{\mathcal{E}}}}(1_A\circ\left(\tilde{J}'_\Lambda\right)_{|\mathcal{E}^d}\circ\tilde{K}_{\Lambda^{\mathcal{E}}}) = \mu^{(2)}_{\Lambda^{\mathcal{E}},\xi^{\mathcal{E}}}(A), \tag{24}$$

where for the last equality we used the fact that $\left(\tilde{J}'_\Lambda\right)_{|\mathcal{E}^d}\circ\tilde{K}_{\Lambda^{\mathcal{E}}} = Id$. It follows from (24) that $\mu_{\Lambda_{\Lambda^{\mathcal{E}}},\xi}\in\mathcal{F}_{(\mathcal{E}^d)^*\setminus\overline{(\Lambda^{\mathcal{E}})}^*}$. Then using $\mathcal{F}_{(\mathcal{E}^d)^*\setminus\overline{(\Lambda^{\mathcal{E}})}^*}\subset\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{(\Lambda_{\Lambda^{\mathcal{E}}})}^*}$ and (24), we have

$$\mu^{(2)}(A|\mathcal{F}_{(\mathcal{E}^d)^*\setminus\overline{(\Lambda^{\mathcal{E}})}^*})(\xi^{\mathcal{E}}) = \mu(A|\mathcal{F}_{(\mathcal{E}^d)^*\setminus\overline{(\Lambda^{\mathcal{E}})}^*})(\xi^{\mathcal{E}}) = \mathbf{E}_\mu\left(\mathbf{E}_\mu\left(A|\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{(\Lambda_{\Lambda^{\mathcal{E}}})}^*}\right)|\mathcal{F}_{(\mathcal{E}^d)^*\setminus\overline{(\Lambda^{\mathcal{E}})}^*}\right)(\xi^{\mathcal{E}}) = \mu^{(2)}_{\Lambda^{\mathcal{E}},\xi^{\mathcal{E}}}(A).$$

□

Remark 14 Let ν be a ϕ-Gibbs measure as in Lemma 9 and let G be a $\mathcal{F}_{(\mathbb{Z}^d)^*}$-measurable and bounded function. Then in view of Lemma 12 and Remark 14, $\mathbf{E}_\nu\left(G|\mathcal{F}_{\mathcal{E}^d}\right)$ is $\mathcal{F}_{(\mathcal{E}^d)^*}$-measurable and

$$\mathbf{E}_\nu\left(G|\mathcal{F}_{\mathcal{E}^d}\right) = \mathbf{E}_\nu\left(G|\mathcal{F}_{(\mathcal{E}^d)^*}\right).$$

Lemma 15 *Let G be a $\mathcal{F}_{(\mathbb{Z}^d)^*}$-measurable and bounded function. Then for all $\mu \in \mathcal{G}(H)$ and all $\xi \in \chi$, we have*

$$\mathbf{E}_\mu\left(G|\mathcal{F}_{(\mathcal{E}^d)^*}\right)(\xi) = \int_{\mathbb{R}^{\mathbb{Z}^d}} G(\nabla\phi) \prod_{x\in\mathcal{O}^d} \nu_x^\psi(\,\mathrm{d}\phi(x))\delta_\psi(d\phi_{\mathcal{E}^d}), \tag{25}$$

where ν_x^ψ have been defined in Lemma 12 and ψ is given by (13).

Proof. First note that for $\Lambda = \{x\} \in \mathcal{O}^d$, from the DLR conditions for μ we have $\mu(\,\cdot\,|\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*})(\xi) = \mu_{x,\xi}(\cdot)$. Note now that $\mathcal{F}_{(\mathcal{E}^d)^*} = \cap_{x\in\mathcal{O}^d}\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*}, \forall x \in \mathcal{O}^d$. Then for arbitrary $x \in \mathcal{O}_n^d$

$$\mathbf{E}_\mu(G|\cap_{x\in\mathcal{O}_n^d}\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*}) = \mathbf{E}_\mu(\mathbf{E}_\mu(G|\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*})|\cap_{x\in\mathcal{O}_n^d}\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*}) = \mathbf{E}_\mu(\mathbf{E}_{\mu_{x,\xi}}(G)|\cap_{x\in\mathcal{O}_n^d}\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*}),$$

where $\mathbf{E}_{\mu_x^\xi}(G)$ is $\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*}$-measurable. Repeating the above reasoning for all $x \in \mathcal{O}_n^d$ and noting the fact that $\mathbf{E}_{\otimes_{x\in\mathcal{O}_n^d}\mu_{x,\xi}\otimes_{b\in(\mathbb{Z}^d)^*\setminus(\overline{\mathcal{O}_n^d})^*}\delta_{\xi(b)}(\eta(b))}(G)$ is $\cap_{x\in\mathcal{O}_n^d}\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*}$-measurable, it follows that

$$\mu(\cdot|\cap_{x\in\mathcal{O}_n^d}\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*})(\xi) = \otimes_{x\in\mathcal{O}_n^d}\mu_{x,\xi}\otimes_{b\in(\mathbb{Z}^d)^*\setminus(\overline{\mathcal{O}_n^d})^*}\delta_{\xi(b)}(\eta(b)).$$

Therefore, by Kolmogorov's extension theorem applied to product measures

$$\mu(\,\cdot\,|\mathcal{F}_{(\mathcal{E}^d)^*})(\xi) = \lim_{n\to\infty}\mu(\,\cdot\,|\cap_{x\in\mathcal{O}_n^d}\mathcal{F}_{(\mathbb{Z}^d)^*\setminus\overline{\{x\}}^*})(\xi) = \otimes_{x\in\mathcal{O}^d}\mu_{x,\xi}. \tag{26}$$

The statement of the lemma follows now from (26) and Remark 3. □

Corollary 16 *Let $l \in I$ be a chosen fixed element in I and let G be a $\mathcal{F}_{(\mathbb{Z}^d)^*}$-measurable and bounded function. Then for all $\mu \in \mathcal{G}(H)$ and all $\xi \in \chi$, $\mathbf{E}_\mu\left(G|\mathcal{F}_{(\mathcal{E}^d)^*}\right)$ can be written as a function of the even gradients. More precisely*

$$\mathbf{E}_\mu\left(G|\mathcal{F}_{(\mathcal{E}^d)^*}\right)(\xi) = \int G(\nabla\phi^l)\prod_{x\in\mathcal{O}^d}\mu_x^{\xi,l}(\,\mathrm{d}\phi(x)), \tag{27}$$

where ϕ^l is obtained from ϕ by making in (21) for all $x \in \mathcal{O}^d$, the change of variables $\phi(x) \to \phi(x) + \phi(x+e_l)$, that is for all $i \in I$, $\nabla_i\phi(x) \to \phi(x+e_i) - \phi(x+e_l) - \phi(x)$. We defined by

$$\mu_x^{\xi,l}(\,\mathrm{d}\phi(x)) = \frac{1}{Z(\nabla_l^{\mathcal{E}}\theta(x))}\exp\left(-2\beta\sum_{i\in I}U(\xi(b_{(x+e_i,x+e_l)}) - \phi(x))\right)\mathrm{d}\phi(x), \tag{28}$$

where $b_{(x+e_i,x+e_l)}$ is the bond $(x+e_i, x+e_l)$ and which depends only on the even gradients $\nabla^{\mathcal{E}}\phi$, with

$$Z(\nabla_l^{\mathcal{E}}\theta(x)) := Z(\phi(x+e_1) - \phi(x+e_l), \ldots, \phi(x-e_d) - \phi(x+e_l)).$$

Proof. The proof is a simple consequence of Lemma 15 and Remark 14. □

3.4 Random Walk Representation

3.4.1 Definition and Theorems

For $i \in I$, let

$$D^i F(y_1, \dots, y_d, y_{-1}, \dots, y_{-d}) := \frac{\partial}{\partial y_i} F(y_1, \dots y_d, y_{-1}, \dots, y_{-d}).$$

Definition 17 *Let $x \in \mathcal{O}^d$. We say that F_x satisfies* ***the random walk representation****, if there exists $\underline{c}, \bar{c} > 0$ such that for all $i, j \in I$*

$$D^{i,i} F_x = - \sum_{j \in I, j \neq i} D^{i,j} F_x \text{ and } \underline{c} \leq -D^{i,j} F_x \leq \bar{c} \text{ for } i \neq j.$$

The main result of this section is:

Theorem 18 *(Random Walk Representation) Let $U \in C^2(\mathbb{R})$ be such that it satisfies (4). We also assume that $V, g \in C^2(\mathbb{R})$ satisfy (7). Then, if*

$$\sqrt{\frac{\beta}{C_1}} ||g''||_{L^1(\mathbb{R})} < \frac{C_1}{2C_2\sqrt{d}}, \tag{29}$$

there exists $\underline{c}, \bar{c} > 0$ such that for all $x \in \mathcal{O}^d$, F_x satisfies the random walk representation.

Lemma 19 *Suppose $x \in \mathcal{O}^d$. Then for all $j \in I$, we have*

$$D^j F_x(\theta(x)) = - \sum_{i \in I, i \neq j} D^i F_x(\theta(x)), \quad D^{j,j} F_x(\theta(x)) = - \sum_{i \in I, i \neq j} D^{i,j} F_x(\theta(x)), \tag{30}$$

and for all $i \in I, i \neq j$

$$D^{i,j} F_x(\theta(x)) = -4\beta^2 \text{cov}_{\nu_x} \left(U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x)) \right), \tag{31}$$

where ν_x have been defined in Lemma 12 and $\mathbf{E}_{\nu_x}$ and cov_{ν_x} are respectively the expectation and the covariance with respect to the measure ν_x.

Proof. For all $j \in I$, from (20) we have

$$\begin{aligned} D^j F_x(\theta(x)) &= \frac{\partial}{\phi(x+e_j)} F_x(\phi(x+e_1) - \phi(x+e_j), \dots, \phi(x-e_d) - \phi(x+e_j)) \\ &= - \sum_{i \in I, i \neq j} D^i F_x(\phi(x+e_1) - \phi(x+e_j), \dots, \phi(x-e_d) - \phi(x+e_j)) \end{aligned} \tag{32}$$

and for $i \neq j$

$$\begin{aligned} D^i F_x(\theta(x)) &= \frac{\partial}{\phi(x+e_i)} F_x(\phi(x+e_1) - \phi(x+e_j), \dots, \phi(x-e_d) - \phi(x+e_j)) \\ &= D^i F_x(\phi(x+e_1) - \phi(x+e_j), \dots, \phi(x-e_d) - \phi(x+e_j)). \end{aligned} \tag{33}$$

It follows now from (32) and (33) that

$$D^j F_x(\theta(x)) = - \sum_{i \in I, i \neq j} D^i F_x(\theta(x)). \tag{34}$$

By differentiating with respect to $\phi(x+e_i)$ and $\phi(x+e_j)$ in F_x, we have for for all $i, j \in I, i \neq j$

$$D^{i,j} F_x(\theta(x)) = -4\beta^2 \text{cov}_{\nu_x} \left(U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x)) \right). \tag{35}$$

The second assertion in (30) follows now immediately from (34) and (35). □

The following lemma is elementary to prove by using Taylor expansion and will be needed for the proof of Theorem 18:

Lemma 20 *(Representation of Covariances)*

Let $k \in \mathbb{N}$. For all L^2-functions $F, G \in C^1(\mathbb{R}^k; \mathbb{R})$, with bounded partial derivatives and for all measures $\nu \in P(\mathbb{R}^k)$ such that $\phi \in L^2(\nu)$ and with bounded derivatives, we have

$$\begin{aligned}\mathrm{cov}_\nu(F, G) &= \frac{1}{2}\iint [F(\phi) - F(\psi)]\,[G(\phi) - G(\psi)]\,\nu(\,\mathrm{d}\phi)\nu(\,\mathrm{d}\psi) \\ &= \frac{1}{2}\iint [(\phi-\psi)\cdot DF(\phi,\psi)]\,[(\phi-\psi)\cdot DG(\phi,\psi)]\,\nu(\,\mathrm{d}\phi)\nu(\,\mathrm{d}\psi)\end{aligned}$$

where we denote by

$$DF(\phi,\psi) := \int_0^1 DF\left(\psi + t(\phi-\psi)\right)\,\mathrm{d}t, \quad DG(\phi,\psi) := \int_0^1 DG\left(\psi + s(\phi-\psi)\right)\,\mathrm{d}s$$

and by

$$DF(\phi) := \left(D^1F(\phi), \dots, D^kF(\phi)\right).$$

Proof of Theorem 18 It follows from Definition 17 and Lemma 19 that, in order to prove that the random walk representation holds for F_x, all we need is to show that there exist $c_l, c_u > 0$ such that

$$c_l \le \mathrm{cov}_{\nu_x}\left(U'(\nabla_i\phi(x)), U'(\nabla_j\phi(x))\right) \le c_u.$$

We have $U = V + g$, where $C_1 \le V'' \le C_2$. Then using Lemma 20 for $V'(\nabla_i\phi(x))$ and $V'(\nabla_j\phi(x))$, we see that

$$\begin{aligned}0 \le C_1^2\mathrm{var}_{\nu_x}(\phi(x)) \le C_1\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x))) \le \mathrm{cov}_{\nu_x}(V'(\nabla_i\phi(x)), V'(\nabla_j\phi(x))) \\ \le C_2\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x))). \quad (36)\end{aligned}$$

Since $g'' < 0$, we have

$$\mathrm{cov}_{\nu_x}(g'(\nabla_i\phi(x)), g'(\nabla_j\phi(x))) = \mathrm{cov}_{\nu_x}(-g'(\nabla_i\phi(x)), -g'(\nabla_j\phi(x))),$$

and we can use Lemma 20 to obtain

$$0 \le \mathrm{cov}_{\nu_x}(g'(\nabla_i\phi(x)), g'(\nabla_j\phi(x))) \le C_0^2\mathrm{var}_{\nu_x}(\phi(x)). \quad (37)$$

By using Lemma 20 for $\mathrm{cov}_{\nu_x}(V'(\nabla_i\phi(x)), g'(\nabla_j\phi(x)))$ and similarly for $\mathrm{cov}_{\nu_x}(V'(\nabla_j\phi(x)), g'(\nabla_i\phi(x)))$, we have

$$-C_0\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x))) \le \mathrm{cov}_{\nu_x}(V'(\nabla_j\phi(x)), g'(\nabla_i\phi(x))) < 0, \quad (38)$$

From (36), (37) and (38), it follows that to find an upper bound for $\mathrm{cov}_{\nu_x}\left(U'(\nabla_i\phi(x)), U'(\nabla_j\phi(x))\right)$, we need to find an upper bound for $\mathrm{cov}_{\nu_x}(\phi(x)), V'(\nabla_j\phi(x)))$ and for $\mathrm{var}_{\nu_x}(\phi(x))$; to find a lower bound, we need to find a lower bound for $\mathrm{cov}_{\nu_x}(\phi(x)), V'(\nabla_j\phi(x)))$ and $\mathrm{cov}_{\nu_x}(g'(\nabla_i\phi(x)), V'(\nabla_j\phi(x)))$. Note now that from (36), we have

$$\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x))) \le \frac{1}{2dC_1}\mathrm{cov}_{\nu_x}\left(V'(\nabla_j\phi(x)), \sum_{i\in I} V'(\nabla_i\phi(x))\right). \quad (39)$$

Using integration by parts, we have

$$\begin{aligned}\mathrm{cov}_{\nu_x}\left(V'(\nabla_j\phi(x)), \sum_{i\in I} V'(\nabla_i\phi(x)\right) = \frac{1}{\beta}\mathbf{E}_{\nu_x}\left(V''(\nabla_j\phi(x))\right) \\ - \mathrm{cov}_{\nu_x}\left(V'(\nabla_j\phi(x)), \sum_{i\in I} g'(\nabla_i\phi(x))\right). \quad (40)\end{aligned}$$

By using the Cauchy-Schwarz inequality and (36), we have

$$\begin{aligned}0 \leq -\mathrm{cov}_{\nu_x}(V'(\nabla_j\phi(x)), g'(\nabla_i\phi(x))) &\leq \sqrt{\mathrm{var}_{\nu_x}(V'(\nabla_j\phi(x)))}\sqrt{\mathrm{var}_{\nu_x}(g'(\nabla_i\phi(x)))} \\ &\leq \sqrt{C_2\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x)))}\sqrt{\mathrm{var}_{\nu_x}(g'(\nabla_i\phi(x)))}. \quad (41)\end{aligned}$$

Then we estimate $\mathrm{var}_{\nu_x}(g'(\nabla_i\phi(x)))$ by applying Lemma 20 to get

$$\begin{aligned}&\mathrm{var}_{\nu_x}(g'(\nabla_i\phi(x))) \\ &= \frac{1}{2}\iint(\phi(x)-\psi(x))^2\left[\int_0^1 g''\left(\psi(x)-\phi(x+e_i)+t(\phi(x)-\psi(x))\right)\,\mathrm{d}t\right]^2\nu_x(\,\mathrm{d}\phi)\nu_x(\,\mathrm{d}\psi) \\ &= \frac{1}{2}\iint\left[\int_{\psi(x)-\phi(x+e_i)}^{\phi(x)-\phi(x+e_i)} g''(s)\,\mathrm{d}s\right]^2\nu_x(\,\mathrm{d}\phi)\nu_x(\,\mathrm{d}\psi) \leq \frac{1}{2}||g''||^2_{L^1(\mathbb{R})}, \quad (42)\end{aligned}$$

where for the second equality we made the change of variable $s = \psi(x) - \phi(x+e_i) + t(\phi(x)-\psi(x))$. From (39), (40) and (42), we now get the upper bound

$$\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x))) \leq \frac{C_2}{2d\beta C_1} + \frac{\sqrt{C_2}}{C_1\sqrt{2}}||g''||_{L^1(\mathbb{R})}\sqrt{\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x)))}, \quad (43)$$

from which we get

$$\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x))) \leq \left(\frac{\sqrt{C_2}}{2\sqrt{2}C_1}||g''||_{L^1(\mathbb{R})} + \frac{1}{2}\sqrt{\frac{C_2}{2C_1^2}||g''||^2_{L^1(\mathbb{R})} + 2\frac{C_2}{d\beta C_1}}\right)^2. \quad (44)$$

Also, by using (36), we get from (44)

$$\mathrm{var}_{\nu_x}(\phi(x))) \leq \frac{1}{C_1}\left(\frac{\sqrt{C_2}}{2\sqrt{2}C_1}||g''||_{L^1(\mathbb{R})} + \frac{1}{2}\sqrt{\frac{C_2}{2C_1^2}||g''||^2_{L^1(\mathbb{R})} + 2\frac{C_2}{d\beta C_1}}\right)^2 := \sigma^2. \quad (45)$$

The upper bound now follows from (36), (37), (44) and (45). To find a lower bound, note now that from (36) we get

$$\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x))) \geq \frac{1}{2dC_2}\mathrm{cov}_{\nu_x}\left(V'(\nabla_j\phi(x)), \sum_{i\in I} V'(\nabla_i\phi(x))\right).$$

By using (40) and (38), we get

$$\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x))) \geq \frac{C_1}{2dC_2\beta}. \quad (46)$$

From (36), (37), (41) and (42), we get

$$\begin{aligned}&\mathrm{cov}_{\nu_x}\left(U'(\nabla_i\phi(x)), U'(\nabla_j\phi(x))\right) \\ &\geq \sqrt{\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x)))}\left[C_1\sqrt{\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_j\phi(x)))} - \sqrt{C_2/2}||g''||_{L^1(\mathbb{R})}\right]. \quad (47)\end{aligned}$$

The lower bound now follows from (46) and (47). □

Remark 21 In order to get the random walk representation, the condition (29) is not unique. The condition can be replaced by other conditions on the perturbation g by estimating the bound on $\mathrm{cov}_{\nu_x}(V'(\nabla_i\phi(x)), g'(\nabla_j\phi(x)))$ by a different method. For example

- To prove (10), we use the Cauchy-Schwartz inequality and (36)

$$\begin{aligned}\left|\mathrm{cov}_{\nu_x}(V'(\nabla_i\phi(x)), g'(\nabla_j\phi(x)))\right| &\le \sqrt{\mathrm{var}_{\nu_x}(V'(\nabla_i\phi(x)))}\sqrt{\mathrm{var}_{H_x}(g'(\nabla_j\phi(x)))} \\ &\le \sqrt{C_2\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_i\phi(x)))}\sqrt{\mathrm{var}_{\nu_x}(g'(\nabla_j\phi(x)))}.\end{aligned}$$

But

$$\begin{aligned}&\mathrm{var}_{\nu_x}(g'(\nabla_j\phi(x)) \\ &= \frac{1}{2}\iint(\phi(x)-\psi(x))^2\left[\int_0^1 g''\left(\psi(x)+t(\phi(x)-\psi(x))-\phi(x+e_j)\right)\,\mathrm{d}t\right]^2\nu_x(\,\mathrm{d}\phi)\nu_x(\,\mathrm{d}\psi) \\ &\le \frac{1}{2}\iint(\phi(x)-\psi(x))^2\int_0^1\left[g''\left(\psi(x)+t(\phi(x)-\psi(x))-\phi(x+e_j)\right)\right]^2\,\mathrm{d}t\;\nu_x(\,\mathrm{d}\phi)\nu_x(\,\mathrm{d}\psi) \\ &= \frac{1}{2}\iint(\phi(x)-\psi(x))\int_{\psi(x)-\phi(x+e_j)}^{\phi(x)-\phi(x+e_j)}\left[g''(s)\right]^2\,\mathrm{d}s\;\nu_x(\,\mathrm{d}\phi)\nu_x(\,\mathrm{d}\psi) \\ &\le \frac{1}{2}||g''||^2_{L^2(\mathbb{R})}\iint|\phi(x)-\psi(x)|\,\nu_x(\,\mathrm{d}\phi)\nu_x(\,\mathrm{d}\psi) \\ &\le \frac{1}{2}||g''||^2_{L^2(\mathbb{R})}\sqrt{\iint(\phi(x)-\psi(x))^2\,\nu_x(\,\mathrm{d}\phi)\nu_x(\,\mathrm{d}\psi)} \\ &= \frac{1}{\sqrt{2}}||g''||^2_{L^2(\mathbb{R})}\sqrt{\mathrm{var}_{H_x}(\phi(x))} \le ||g''||^2_{L^2(\mathbb{R})}\sqrt{\frac{\mathrm{cov}_{\nu_x}(\phi(x), V'(\nabla_i\phi(x)))}{2C_1}},\end{aligned}$$

where we used Lemma 20 for the first and third equality, Jensen's inequality for the first and third inequality and (36) for the last inequality. The rest of the argument to obtain the bound in (10) follows the same steps as the proof of Theorem 18.

(b) Another possible condition, condition (11), is obtained using the same steps as in the proof of Theorem 18. To obtain it, we estimate by yet another different method

$$\begin{aligned}\left|\mathrm{cov}_{\nu_x}(V'(\nabla_i\phi(x)), g'(\nabla_j\phi(x)))\right| &\le \sqrt{\mathrm{var}_{\nu_x}(V'(\nabla_i\phi(x))}\sqrt{\mathrm{var}_{\nu_x}(g'(\nabla_j\phi(x))} \\ &\le (C_2)^{3/4}(2d\beta)^{1/4}\sqrt{\mathrm{cov}_{\nu_x}(\phi(x)), V'(\nabla_i\phi(x)))||(g')^2||_{L^1(\mathbb{R})}},\end{aligned}$$

where we used the Cauchy-Schwarz inequality for the first inequality, (36) and Lemma 22 below, for the second inequality.

Lemma 22 *If $h \in L^1(\mathbb{R})$, then we have*

$$|\mathbf{E}_{\nu_x}(h)| \le \sqrt{2d\beta C_2}||h||_{L^1(\mathbb{R})}.$$

Proof. Using integration by parts and Cauchy-Schwartz, we have

$$\begin{aligned}|\mathbf{E}_{\nu_x}(h)| &= \left|\mathbf{E}_{\nu_x}\left(\frac{\partial}{\partial y}\left(\int_{-\infty}^y h(z)\,\mathrm{d}z\right)\right)\right| = \left|\mathbf{E}_{\nu_x}\left(H'_x(y)\left(\int_{-\infty}^y h(z)\,\mathrm{d}z\right)\right)\right| \\ &\le \mathbf{E}^{1/2}_{\nu_x}\left((H'_x)^2\right)\mathbf{E}^{1/2}_{\nu_x}\left(\left(\int_{-\infty}^y h(z)\,\mathrm{d}z\right)^2\right) = \mathbf{E}^{1/2}_{\nu_x}(H''_x)\mathbf{E}^{1/2}_{\nu_x}\left(\int_{-\infty}^y h(z)\,\mathrm{d}z\right)^2 \\ &\le \sqrt{2d\beta C_2}||h||_{L^1(\mathbb{R})}.\end{aligned}$$

Note that we also used property (4) and integration by parts in the above formula. □

Remark 23 Note that if we consider the case where U is strictly convex such that $C_1 \le U'' \le C_2$, in view of (36), (46) and (43), the one step integration preserves the strict convexity of the new Hamiltonian

$$\frac{C_1^2}{2d\beta C_2} \le \mathrm{cov}_{\nu_x}\left(U'(\nabla_i\phi(x)), U'(\nabla_j\phi(x))\right) \le \frac{C_2^2}{2d\beta C_1}.$$

Remark 24 Note that we can extend the results to the case where we have a perturbation with compact support (See also Example (b) and the graph below). More precisely, assume that $U = Y + h$, where U satisfies (4), $D_1 \le Y'' \le D_2$ and $-D_0 \le h'' \le 0$ on $[a,b]$ and $0 < h'' < D_3$ on $\mathbb{R} \setminus [a,b]$, with $a, b \in \mathbb{R}$ and $h''(a) = h''(b) = 0$. Set

$$g(s) = h(s)1_{\{s\in[a,b]\}} + \left[h(b) + h'(b)(s-b)\right]1_{\{s>b\}} + \left[h(a) + h'(a)(s-a)\right]1_{\{s<a\}}$$

and

$$V(s) = Y(s) + h(s)1_{\{s\notin[a,b]\}} - \left[h(b) + h'(b)(s-b)\right]1_{\{s>b\}} - \left[h_i(a) + h'(a)(s-a)\right]1_{\{s<a\}}.$$

Thus, we have $V, g \in C^2(\mathbb{R})$, with $-D_0 \le h''(s) = g''(s) \le 0$ for $s \in [a,b]$ and $g''(s) = 0$ for $s \in \mathbb{R} \setminus [a,b]$ and $D_1 \le V''(s) = Y''(s) + h''(s)1_{\{s\notin[a,b]\}} \le D_2 + D_3$. Note that this procedure can also be extended to the case where h'' changes sign more than once.

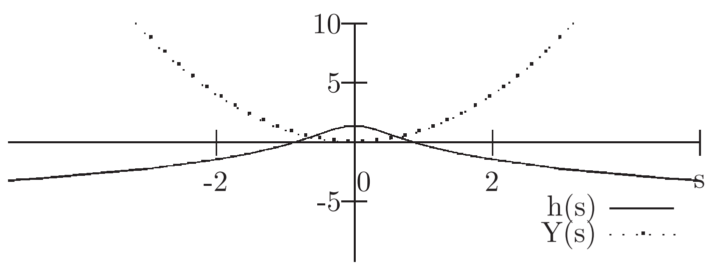

Figure 4: Example of Y and h for Remark 24

3.4.2 Examples

(a) Let $p \in (0,1)$ and $0 < k_2 < k_1$. Let

$$U(s) = -\log\left(pe^{-k_1\frac{s^2}{2}} + (1-p)e^{-k_2\frac{s^2}{2}}\right).$$

Set $a = \frac{k_1}{k_2}$. Take $p > a^{-1}$ in order that the potential U is non-convex. If

$$0 < (\beta)^{3/4}p(1-p)^{1/4}(a-1)^{1/4} \le \frac{1}{2(2d)^{3/4}(\pi)^{1/4}},$$

then (11) is satisfied and the RW representation holds. If $\beta = 1$ and $k_1 \gg k_2$, the above condition is equivalent to $p < p_0$, where $p_0 \approx \frac{1}{2(2d)^{3/4}\pi^{1/4}}a^{-1/4}$. This is close to, for $d = 2$, the critical point p_c, such that $\frac{p_c}{1-p_c} = a^{-1/4}$, of [2], where uniqueness of ergodic states is violated for this example of potential U.

The computations follow. Take

$$V''(s) = \frac{pk_1e^{-k_1\frac{s^2}{2}} + (1-p)k_2e^{-k_2\frac{s^2}{2}}}{pe^{-k_1\frac{s^2}{2}} + (1-p)e^{-k_2\frac{s^2}{2}}}$$

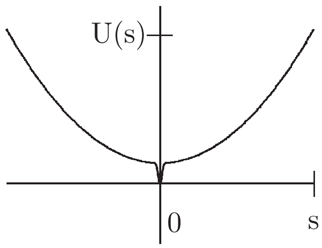

Figure 5: Example (a)

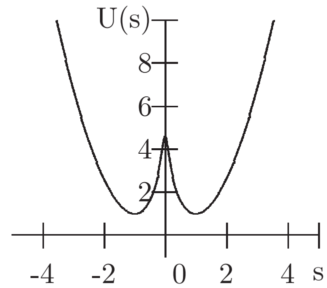

Figure 6: Example (b)

and

$$g''(s) = -\frac{p(1-p)(k_1-k_2)^2 s^2}{p^2 e^{-(k_1-k_2)\frac{s^2}{2}} + 2p(1-p) + (1-p)^2 e^{(k_1-k_2)\frac{s^2}{2}}}.$$

We have

$$k_2 \le V''(s) \le pk_1 + (1-p)k_2 \quad \text{and} \quad -\frac{p(k_1-k_2)}{1-p} \le g''(s) \le 0,$$

where the lower bound inequality for $g''(s)$ follows from the fact that $g''(s)$ attains its minimum for $s \ge \sqrt{\frac{2}{k_1-k_2}}$. Then

$$||g'(s)||_{L^2(\mathbb{R})} \le \frac{2p}{1-p}(k_1-k_2)^{1/4}(\pi)^{1/4} + o\left(\frac{2p}{1-p}(k_1-k_2)^{1/4}(\pi)^{1/4}\right).$$

By using condition (11), the RW representation holds.

(b) $U(s) = s^2 + a - \log(s^2+a)$, where $0 < a < 1$. Let $0 < \beta < \frac{a}{4d\left(2+\frac{2}{25a}\right)^2}$. Then the RW representation holds.

Then, using the notation from Remark 24, take $Y(s) = s^2$ and $h(s) = -\log(s^2+a)$. We have $Y''(s) = 2$, so $D_1 = D_2 = 2$; also $h''(s) = 2\frac{s^2-a}{(s^2+a)^2}$, with $-\frac{2}{a} \le h''(s) \le 0$ for $s \in [-\sqrt{a}, \sqrt{a}]$ and $0 < h''(s) \le \frac{2}{25a}$ otherwise. Then $C_0 = \frac{2}{a}$, $C_1 = 2$,$C_2 = 2 + \frac{2}{25a}$ and $||g''(s)||_{L^1(\mathbb{R})} = \frac{2}{\sqrt{a}}$. By using condition (29), the RW representation holds.

4 Uniqueness of ergodic component

In this section, we extend to a class of non-convex potentials, the uniqueness of ergodic component result, proved for strictly convex potentials in [14].

We denote by S the class of all shift invariant $\mu \in P_2(\chi)$ which are stationary and by ext S those $\mu \in S$ which are ergodic with respect to shifts (for definitions of shift-invariance and ergodicity, see for example page 122 in [15]). For each $u \in \mathbb{R}^d$, we denote by $(\text{ext } S)_u$ the family of all $\mu \in \text{ext } S$ such that $\mathbf{E}_\mu\left(\eta(b_{e_i})\right) = u_i, i = 1, 2 \ldots d$, where we denoted by b_{e_i} the bond $(e_i, 0)$. We will prove that

Theorem 25 *Let $U = V + g$, where U satisfy (4) and V and g satisfy (7) and (29). Then for every $u \in \mathbb{R}^d$, there exists at most one ergodic and shift-invariant $\mu_u \in \mathcal{G}(H)$ such that $\mathbf{E}_{\mu_u}\left(\eta(b_{e_i})\right) = u_i, i = 1, 2 \ldots d$.*

The proof will be done in 2 steps: first, we will prove the uniqueness of ergodic, shift-invariant $\mu_{\mathcal{E}^d} := \mu^{(2)} \in \mathcal{G}^{\mathcal{E}}(H^{(2)})$ and then we will use this result combined with the properties of the $\nabla^{\mathcal{E}}\phi$-Gibbs measure to extend the result to μ.

4.1 Uniqueness of ergodic component for the even

Let $F \in C^2(\mathbb{R}^{2d};\mathbb{R})$ be such that for all $(a_1, a_2, \ldots, a_d, a_{-1}, \ldots, a_{-d}) \in \mathbb{R}^d$ and for all $c \in \mathbb{R}$

$$F(a_1, \ldots, a_d, a_{-1}, \ldots, a_{-d}) = F(a_1 + c, \ldots, a_d + c, a_{-1} + c, \ldots, a_{-d} + c). \tag{48}$$

Note that from property (48), by the same reasoning as in Lemma 19 we have that for all $j \in I$, (30) holds. Assume that there exist $c_- > 0$ and $c_+ > 0$ such that for all $(a_1, a_2, \ldots, a_d, a_{-1}a_{-d}) \in \mathbb{R}^{2d}$

$$c_- \leq D^{i,j}F(a_1, a_2, \ldots, a_d, a_{-1}, \ldots, a_{-d}) \leq c_+. \tag{49}$$

Let

$$\mathcal{L} = \{F \in C^2(\mathbb{R}^{2d};\mathbb{R}) \mid F \text{ satisfies (48) and (49)}\}.$$

The proofs in this section follow very closely the arguments from [14]. To make the current paper self-contained, we will sketch proofs for all the theorems in the section. There are three main ingredients necessary in proving uniqueness of ergodic component for a Hamiltonian satisfying (48) and (49). These steps are: the study of the dynamics of the height variables (which dynamics are generated by SDE), a coupling argument and the use of the ergodicity.

4.1.1 Dynamics

Suppose the dynamics of the **even** height variables $\phi_t = \{\phi_t(a)\} \in \mathbb{R}^{\mathcal{E}^d}$ are generated by the SDE

$$\mathrm{d}\phi_t(a) = -\sum_{x\in\mathcal{O}^d, |x-a|=1} \frac{\partial}{\partial\phi(a)} F_x(\phi_t(x+e_1), \ldots, \phi_t(x+e_{-d}))\,\mathrm{d}t + \sqrt{2}dW_t(a), \quad a \in \mathcal{E}^d \tag{50}$$

where for all $x \in \mathcal{O}^d$, $F_x \in \mathcal{L}$ and $\{W_t(a), a \in \mathcal{E}^d\}$ is a family of independent Brownian motions. Note that in (50), for each $x \in \mathcal{O}^d$ such that $|x - a| = 1$, there exists $i \in I$ such that $a = x + e_i$.

The dynamics for the **even** height differences $\eta^{\mathcal{E}} = \{\eta_t^{\mathcal{E}}(b)\} \in (\mathcal{E}^d)^*$ are determined by the SDE

$$\begin{aligned}
\mathrm{d}\eta_t^{\mathcal{E}}(b) \;=\; & \mathrm{d}\phi_t(x_b) - \mathrm{d}\phi_t(y_b) = -\sum_{x\in\mathcal{O}^d, |x-x_b|=1} \frac{\partial}{\partial\phi(x_b)} F_x(\phi_t(x+e_1), \ldots, \phi_t(x+e_{-d}))\,\mathrm{d}t \\
& + \sum_{x\in\mathcal{O}^d, |x-y_b|=1} \frac{\partial}{\partial\phi(y_b)} F_x(\phi_t(x+e_1), \ldots, \phi_t(x+e_{-d}))\,\mathrm{d}t \\
& + \sqrt{2}d[W_t(x_b) - W_t(y_b)],
\end{aligned} \tag{51}$$

where $b = (x_b, y_b) \in (\mathcal{E}^d)^*$.

Lemma 26 *(a) The solution of (51) satisfies $\eta_t^{\mathcal{E}} \in \chi^{\mathcal{E}}$ for all $t > 0$.*

(b) If ϕ_t is a solution of (50), then $\eta_t^{\mathcal{E}} := \nabla^{\mathcal{E}}\phi_t$ is a solution of (51).

(c) If $\eta_t^{\mathcal{E}}$ is a solution of (51) and we define $\phi_t(0)$ through (50) for $x = 0$ and $\nabla^{\mathcal{E}}\phi_t(b) = \eta_t^{\mathcal{E}}(b)$, with $\phi_0(0) \in \mathbb{R}$, then $\phi_t := \phi^{\eta_t^{\mathcal{E}}, \phi_t(0)}$ is a solution of (50).

(d) For each $\eta^{\mathcal{E}} \in \chi_r^{\mathcal{E}}, r > 0$ the SDE (51) has a unique $\chi_r^{\mathcal{E}}$-valued continuous solution $\eta_t^{\mathcal{E}}$ starting at $\eta_0^{\mathcal{E}} = \eta^{\mathcal{E}}$.

Proof. The proofs for (a), (b) and (c) are immediate, so we will concentrate on the proof for (d). For every $\theta_t(x)$ and $\bar{\theta}_t(x)$, by expanding $D^j F_x(\theta_t(x))$ in Taylor series around $\bar{\theta}_t(x)$ to get

$$D^j F_x(\theta_t(x)) - D^j F_x(\bar{\theta}_t(x)) = \sum_{k\in I} \tilde{\phi}_t(x+e_k) \int_0^1 D^{j,k} F_x \left(\bar{\theta}_t(x) + y\left(\theta_t(x) - \bar{\theta}_t(x)\right)\right) \mathrm{d}y. \tag{52}$$

By using now the fact that $F_x \in \mathcal{F}$, we obtain global Lipschitz continuity in $\chi_r^{\mathcal{E}}$ of the drift term of the SDE in (51), from which a standard method of of successive approximations gives existence and uniqueness of the solution in (51). □

First, we will prove

Lemma 27 *Let ϕ_t and $\bar{\phi}_t$ be two solutions for (50) and set $\tilde{\phi}_t(a) := \phi_t(a) - \bar{\phi}_t(a)$, where $a \in \mathcal{E}^d$. Then for every finite $\Lambda^{\mathcal{E}} \subset \mathcal{E}^d$, we have*

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{a\in\Lambda^{\mathcal{E}}}(\tilde{\phi}_t(a))^2 &= -2\sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}}\sum_{\substack{\{j\in I|\\ x+e_j\in\Lambda^{\mathcal{E}}\}}}\Big[D^jF_x(\phi_t(x+e_1),\ldots,\phi_t(x+e_{-d}))\\ &\quad -D^jF_x(\bar{\phi}_t(x+e_1),\ldots,\bar{\phi}_t(x+e_{-d}))\Big]\tilde{\phi}_t(x+e_j),\end{aligned} \tag{53}$$

and

$$\frac{\partial}{\partial t}\sum_{a\in\Lambda^{\mathcal{E}}}(\tilde{\phi}_t(a))^2 \le -c_-\sum_{b\in(\Lambda^{\mathcal{E}})^*}\left[\nabla^{\mathcal{E}}\tilde{\phi}_t(b)\right]^2 + 2c_+\sum_{b\in\partial(\Lambda^{\mathcal{E}})^*}|\phi_t(y_b)|\left|\nabla^{\mathcal{E}}\tilde{\phi}_t(b)\right|. \tag{54}$$

PROOF. Let $a \in \Lambda^{\mathcal{E}}$. Then from (50), we have

$$\begin{aligned}\frac{\partial}{\partial t}(\tilde{\phi}_t(a))^2 &= -2\sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}},|x-a|=1}\Big[\frac{\partial}{\partial\phi(a)}F_x(\phi_t(x+e_1),\ldots,\phi_t(x+e_{-d}))\\ &\quad -\frac{\partial}{\partial\phi(a)}F_x(\bar{\phi}_t(x+e_1),\ldots,\bar{\phi}_t(x+e_{-d}))\Big]\tilde{\phi}_t(a).\end{aligned} \tag{55}$$

By summing now over all $a \in \Lambda^{\mathcal{E}}$ in (55), we get (53). For simplicity of notation, we will denote as before by $\theta_t(x) := (\phi_t(x+e_1),\ldots,\phi_t(x+e_{-d}))$ and by $\bar{\theta}_t(x) := \big(\bar{\phi}_t(x+e_1),\ldots,\bar{\phi}_t(x+e_{-d})\big)$. To find an upper bound for the sum, we expand now $D^jF_x(\theta_t(x))$ in Taylor series around $\bar{\theta}_t(x)$ to get

$$D^jF_x(\theta_t(x)) - D^jF_x(\bar{\theta}_t(x)) = \sum_{k\in I}\tilde{\phi}_t(x+e_k)\int_0^1 D^{j,k}F_x\left(\bar{\theta}_t(x)+y\left(\theta_t(x)-\bar{\theta}_t(x)\right)\right)\,\mathrm{d}y. \tag{56}$$

Then

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_{a\in\Lambda^{\mathcal{E}}}(\tilde{\phi}_t(a))^2\\ &= -2\sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}}\sum_{\substack{\{j\in I,\\ x+e_j\in\Lambda^{\mathcal{E}}\}}}\sum_{k\in I}\tilde{\phi}_t(x+e_k)\tilde{\phi}_t(x+e_j)\int_0^1 D^{j,k}F_x\left(\bar{\phi}_t(x)+y\left(\phi_t(x)-\bar{\phi}_t(x)\right)\right)\,\mathrm{d}y\\ &\quad = 2\sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}}\sum_{\substack{\{j\in I,\\ x+e_j\in\Lambda^{\mathcal{E}}\}}}\sum_{k\in I,k\neq j}\left[\tilde{\phi}_t^{\,2}(x+e_j)-\tilde{\phi}_t(x+e_k)\tilde{\phi}_t(x+e_j)\right]\\ &\qquad\qquad \int_0^1 D^{j,k}F_x\left(y\phi_t(x)+(1-y)\bar{\phi}_t(x)\right)\,\mathrm{d}y\\ &\quad = \sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}}\sum_{\substack{\{j,k\in I,j\neq k|\\ x+e_j,x+e_k\in\Lambda^{\mathcal{E}}\}}}\left[\tilde{\phi}_t(x+e_j)-\tilde{\phi}_t(x+e_k)\right]^2\int_0^1 D^{j,k}F_x\left(y\phi_t(x)+(1-y)\bar{\phi}_t(x)\right)\,\mathrm{d}y\\ &\quad +2\sum_{x\in\mathcal{O}^d_{\Lambda_{\Lambda^{\mathcal{E}}}}}\sum_{\substack{\{j\in I,\\ x+e_j\in\Lambda^{\mathcal{E}}\}}}\sum_{\substack{\{k\in I|\\ x+e_k\in\partial\Lambda^{\mathcal{E}}\}}}\left[\tilde{\phi}_t^{\,2}(x+e_j)-\tilde{\phi}_t(x+e_k)\tilde{\phi}_t(x+e_j)\right]\\ &\qquad\qquad \int_0^1 D^{j,k}F_x\left(y\phi_t(x)+(1-y)\bar{\phi}_t(x)\right)\,\mathrm{d}y\\ &\quad \le -c_-\sum_{b\in(\Lambda^{\mathcal{E}})^*}\left[\nabla^{\mathcal{E}}\tilde{\phi}_t(b)\right]^2 + 2c_+\sum_{b\in\partial(\Lambda^{\mathcal{E}})^*}|\phi_t(y_b)|\left|\nabla^{\mathcal{E}}\tilde{\phi}_t(b)\right|,\end{aligned} \tag{57}$$

where we used (56) in the first equality, (30) in the second equality, symmetry and the fact that $D^{j,k}F_x = D^{k,j}F_x$ in the third equality and (49) in the inequality on the last line.

$\square$

4.1.2 Coupling Argument

Let $\mathcal{N}_+ = \{f^0_{ij} \mid f^0_{ij} = e_i + e_j, \text{ where } i, j \in I, j \neq -i, i \leq j, j \geq 1\}$. Let us define now a generator set in $\mathcal{E}^d$:

$$e^{\mathcal{E}}_i = e_i + e_{i+1},\ i = 1, 2, \ldots d-1 \text{ and } e^{\mathcal{E}}_d = \begin{cases} e_d - e_1 & d \text{ even} \\ e_d + e_1 & d \text{ odd} \end{cases}$$

For each $u \in \mathbb{R}^d$, let $\tilde{u}_i, i = 1, 2, \ldots, d$ be defined as follows:

$$\tilde{u}_i = u_i + u_{i+1},\ i = 1, 2, \ldots, d-1 \text{ and } \tilde{u}_d = \begin{cases} u_d - u_1 & d \text{ even} \\ u_d + u_1 & d \text{ odd} \end{cases}$$

For $x \in (\mathcal{E}^d)^*$, we define the **even** shift operators $\sigma^{\mathcal{E}}_x : \mathbb{R}^{\mathcal{E}^d} \to \mathbb{R}^{\mathcal{E}^d}$, for **even** heights by $\sigma^{\mathcal{E}}_x(y) = \phi(y - x)$ for $y \in \mathcal{E}^d$ and $\phi \in \mathbb{R}^{\mathcal{E}^d}$ and for **even** height differences by $(\sigma^{\mathcal{E}}_x \eta)(b) = \eta(b - x)$, for $b \in (\mathcal{E}^d)^*$ and $\eta \in \chi^{\mathcal{E}}$. Then shift-invariance and ergodicity for $\mu^{(2)}$ are defined in the usual way. We denote by $S^{\mathcal{E}}$ the class of all shift invariant (with respect to the **even** shifts) $\mu \in P_2(\chi^{\mathcal{E}})$ which are stationary for the SDE (50) and by ext $S^{\mathcal{E}}$ those $\mu^{\mathcal{E}} \in S^{\mathcal{E}}$ which are ergodic with respect to the **even** shifts. For each $u \in \mathbb{R}^d$, we denote by $\left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{u}}$ the family of all $\mu^{\mathcal{E}} \in \text{ext } S^{\mathcal{E}}$ such that $\mathbf{E}_{\mu^{\mathcal{E}}}\left(\eta^{\mathcal{E}}_t\left(b_{e^{\mathcal{E}}_i}\right)\right) = \tilde{u}_i, i = 1, 2 \ldots d$, where $b_{e^{\mathcal{E}}_i}$ is the even bond $(e^{\mathcal{E}}_i, 0)$. We will prove that $\mu^{\mathcal{E}}$ is unique.

For clarity purposes, we will sketch the coupling argument used in [14] to prove uniqueness of $\mu^{\mathcal{E}}$. Suppose that there exist $\mu^{\mathcal{E}} \in \left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{u}}$ and $\bar{\mu}^{\mathcal{E}} \in \left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{v}}$ for $u, v \in \mathbb{R}^d$. Let us construct two independent-$\chi^{\mathcal{E}}_r$ valued random variables $\eta^{\mathcal{E}} = \{\eta^{\mathcal{E}}(b)\}$ and $\bar{\eta}^{\mathcal{E}} = \{\bar{\eta}^{\mathcal{E}}(b)\}$ on a common probability space (Ω, F, P) in such a manner that $\eta^{\mathcal{E}}$ and $\bar{\eta}^{\mathcal{E}}$ are distributed by $\mu^{\mathcal{E}}$ and $\bar{\mu}^{\mathcal{E}}$ respectively. We define $\phi_0 = \phi^{\eta,0}$ and $\bar{\phi}_0 = \phi^{\bar{\eta},0}$.Let ϕ_t and $\bar{\phi}_t$ be two solutions of the SDE (50) with common Brownian motions having initial data ϕ_0 and $\bar{\phi}_0$. Since $\mu^{\mathcal{E}}, \bar{\mu}^{\mathcal{E}} \in S^{\mathcal{E}}$, we conclude that $\eta^{\mathcal{E}}_t = \left(\eta^{\mathcal{E}}\right)^{\phi_t}$ and $\bar{\eta}^{\mathcal{E}} = \left(\eta^{\mathcal{E}}\right)^{\bar{\phi}_t}$ are distributed by $\mu^{\mathcal{E}}$ and $\bar{\mu}^{\mathcal{E}}$ respectively, for all $t \geq 0$. Let $\tilde{u}, \tilde{v} \in \mathbb{R}^d$ be such that $\tilde{u}_i = \mathbf{E}_{\mu^{\mathcal{E}}}\left(\eta^{\mathcal{E}}_t\left(b_{e^{\mathcal{E}}_i}\right)\right)$ and $\tilde{v}_i = \mathbf{E}_{\bar{\mu}^{\mathcal{E}}}\left(\bar{\eta}^{\mathcal{E}}_t\left(b_{e^{\mathcal{E}}_i}\right)\right)$. We claim that:

Lemma 28 *There exists a constant $C > 0$ independent of $\tilde{u}, \tilde{v} \in \mathbb{R}^d$ such that*

$$\bar{\lim}_{T \to \infty} \frac{1}{T} \int_0^T \sum_{i=1}^d \mathbf{E}^P\left[\left(\eta^{\mathcal{E}}_t(e^{\mathcal{E}}_i) - \bar{\eta}^{\mathcal{E}}_t(e^{\mathcal{E}}_i)\right)^2\right] \mathrm{d}t \leq C|\tilde{u} - \tilde{v}|^2. \tag{58}$$

PROOF. **Step 1.** For simplicity of notation, we will label for this proof the d^2 elements of $\mathcal{N}_+$ as $f^0_1, f^0_2, \ldots, f^0_{d^2}$. By applying Lemma 27 to the differences $\{\tilde{\phi}_t(x) := \phi_t(x) - \bar{\phi}_t(x)\}$, where $x \in \mathcal{E}^d$ and by using the fact that the distribution of $(\eta^{\mathcal{E}}, \bar{\eta}^{\mathcal{E}}) = (\nabla^{\mathcal{E}}\phi_t, \nabla^{\mathcal{E}}\bar{\phi}_t)$ on $\chi^{\mathcal{E}}_r \times \chi^{\mathcal{E}}_r$ is shift-invariant on the evens , one obtains just as in [14] for every $T > 0$, $\Lambda^{\mathcal{E}}_N := [-N, N]^d \cap \mathcal{E}^d \subset \mathcal{E}^d$, where $N \in \mathbb{N}$

$$\int_0^T g(t)\,\mathrm{d}t \leq \frac{2d^2}{c_-|\Lambda^{\mathcal{E}}_N|}\mathbf{E}^P\left[\sum_{x \in \Lambda^{\mathcal{E}}_N} (\tilde{\phi}_0(x))^2\right] + \frac{(2c_+c_0)^2 d^2}{(c_-N)^2}\int_0^T \sup_{y \in \partial\Lambda^{\mathcal{E}}_N} \|\tilde{\phi}_t(y)\|^2|_{L^2(P)}\,\mathrm{d}t, \tag{59}$$

where

$$g(t) = \sum_{i=1}^{d^2} \mathbf{E}^P\left[(\nabla\tilde{\phi}_t(f_i))^2\right] \text{ and } c_0 := \sup_{\{N \geq 1\}}\left\{N\frac{|\partial(\Lambda^{\mathcal{E}})^*|}{|(\Lambda^{\mathcal{E}})^*|}\right\} < \infty.$$

Step 2. Next we derive, just as in [14], the following bound on the boundary term: For each $\epsilon > 0$ there exists an $l_0 \in \mathbb{N}$ such that

$$\sup_{y \in \partial\mathcal{E}^d_\Lambda} \|\tilde{\phi}_t(y)\|^2_{L^2(P)} \leq C_1\left(\epsilon^2 N^2 + N^2|\tilde{u} - \tilde{v}|^2 + N^{-2}t\int_0^t g(s)\,\mathrm{d}s\right) \tag{60}$$

for every $t > 0$ and $l \geq l_0$, where $C_1 > 0$ is a constant independent of ϵ, l, and t.

The main ingredient necessary for us to to be able to reproduce the proof in Step 2 of [14] is *the mean ergodic theorem for co-cycles*, which we can use because $\mathcal{E}^d$ is a sub-algebra (see for example [4], [19] or [18]) and apply it to $\mu^{\mathcal{E}} \in (\text{ext}\, \mathcal{S}^{\mathcal{E}})_u$ to obtain

$$\lim_{\substack{\|x\|\to\infty,\\ x\in\mathcal{E}^d}} \frac{1}{\|x\|}\|\phi^{\nu,0}(x) - x\cdot\tilde{u}\|_{L^2(\mu^{\mathcal{E}})} = 0.$$

In order to use the same reasoning as in Step 2 of [14], we also need to prove

$$\left|\left|\frac{1}{|\Lambda^{\mathcal{E}}_{[N/2]}|}\sum_{x\in\Lambda^{\mathcal{E}}_{[N/2]}}\tilde{\phi}_t(x)\right|\right|_{L^2(P)} \le \left|\left|\frac{1}{|\Lambda^{\mathcal{E}}_{[N/2]}|}\sum_{x\in\Lambda^{\mathcal{E}}_{[N/2]}}\tilde{\phi}_0(x)\right|\right|_{L^2(P)} + \frac{c_+|\partial(\Lambda^{\mathcal{E}}_{[N/2]})^*|}{d^2|\Lambda^{\mathcal{E}}_{[N/2]}|}\int_0^t\sum_{i=1}^{d^2}||\nabla\tilde{\phi}_t(f_i)||_{L^2(P)} \tag{61}$$

To prove the above statement, note first that by using (50) we have

$$\begin{aligned}\frac{\partial}{\partial t}\left\{\sum_{a\in\Lambda^{\mathcal{E}}_{[N/2]}}\tilde{\phi}_t(a)\right\} &= -2\sum_{x\in\Lambda_{\Lambda^{\mathcal{E}}_{[N/2]}}\cap\mathcal{O}^d}\ \sum_{\substack{\{j\in I|\\ x+e_j\in\Lambda^{\mathcal{E}}_{[N/2]}\}}}\left[D^jF_x\left(\theta_t(x)\right) - D^jF_x\left(\bar{\theta}_t(x)\right)\right]\\ &= -2\sum_{x\in\Lambda_{\Lambda^{\mathcal{E}}_{[N/2]}}\cap\mathcal{O}^d}\ \sum_{\substack{\{j\in I|\\ x+e_j\in\partial^-\Lambda^{\mathcal{E}}_{[N/2]}\}}}\left[D^jF_x\left(\theta_t(x)\right) - D^jF_x\left(\bar{\theta}_t(x)\right)\right],\end{aligned} \tag{62}$$

where for the second equality we used (30). By using Taylor's expansion and (33) in (62), we get

$$|D^jF_x\left(\theta_t(x)\right) - D^jF_x\left(\bar{\theta}_t(x)\right)| \le c_+\sum_{k\in I|x+e_k\in\partial^-\Lambda^{\mathcal{E}}_{[N/2]},x+e_i\in\partial\Lambda^{\mathcal{E}}_{[N/2]}}\left|\tilde{\phi}_t(x+e_k) - \tilde{\phi}_t(x+e_i)\right|. \tag{63}$$

Then, applying (62) and (63) to the left-hand side of (61), we have

$$\left|\left|\frac{1}{|\Lambda^{\mathcal{E}}_{[N/2]}|}\sum_{x\in\Lambda^{\mathcal{E}}_{[N/2]}}\tilde{\phi}_t(x)\right|\right|_{L^2(P)} \le \left|\left|\frac{1}{|\Lambda^{\mathcal{E}}_{[N/2]}|}\sum_{x\in\Lambda^{\mathcal{E}}_{[N/2]}}\tilde{\phi}_0(x)\right|\right|_{L^2(P)} + \frac{c_+}{|\Lambda^{\mathcal{E}}_{[N/2]}|}\sum_{b\in\partial(\Lambda^{\mathcal{E}}_{[N/2]})^*}\int_0^t||\nabla^{\mathcal{E}}\phi_s(b)||_{L^2(P)}\,\mathrm{d}s.$$

(61) follows immediately now by using the shift-invariance property on the evens in the above equation. With these estimates, the proof from Step 2 in [14] can now be immediately reproduced.

Step 3 The desired estimate (58) follows now by using the same arguments as in Step 3 of [14] and by using the fact that

$$\int_0^T\sum_{i=1}^d\mathbf{E}^P\left[\left(\eta^{\mathcal{E}}_t(e^{\mathcal{E}}_i) - \bar{\eta}^{\mathcal{E}}_t(e^{\mathcal{E}}_i)\right)^2\right]\mathrm{d}t \le \int_0^T g(t)\,\mathrm{d}t.$$

□

Theorem 29 *For every $u \in \mathbb{R}^d$, there exists at most one $\mu^{\mathcal{E}}_u \in \left(\text{ext}\, S^{\mathcal{E}}\right)_{\tilde{u}}$.*

Proof. By using Lemma 28, the proof follows the same arguments as in [14], so it will be omitted. □

4.2 Proof of Theorem 25

Note first that any $\mu \in \mathcal{G}(H)$ is reversible under the dynamics η_t defined by the (51). In particular, $\mathcal{G} \subset \mathcal{S}$.

Suppose now that there exist $\mu, \bar{\mu} \in \mathcal{G}(H)$ ergodic and shift-invariant such that $\mathbf{E}_\mu\left(\eta_t(b_{e_i})\right) = u_i, i = 1, 2 \ldots d$ for $u \in \mathbb{R}^d$. Note now that $\mathbf{E}_\mu\left(\eta^{\mathcal{E}}_t\left(b_{e^{\mathcal{E}}_i}\right)\right) = \mathbf{E}_{\bar{\mu}}\left(\eta^{\mathcal{E}}_t\left(b_{e^{\mathcal{E}}_i}\right)\right) = \tilde{u}_i, i = 1, 2 \ldots d.$

Hence, from Lemma 13, we get that $\mu|_{(\mathcal{E}^d)^*}, \bar{\mu}|_{(\mathcal{E}^d)^*} \in \mathcal{G}^{\mathcal{E}}(H^{(2)})$. Since for all $\eta^{\mathcal{E}} \in \chi^{\mathcal{E}}$, with $\eta^{\mathcal{E}}(b) = \phi(y_b) - \phi(x_b)$, $b = (x_b, y_b) \in (\mathcal{E}^d)^*$, we can write $\eta^{\mathcal{E}}(b) = \eta(b_1) + \eta(b_2)$, $b_1, b_2 \in (\mathbb{Z}^d)^*$, shift-invariance and ergodicity under the even shifts for $\mu|_{(\mathcal{E}^d)^*}, \bar{\mu}|_{(\mathcal{E}^d)^*}$ follow immediately from the similar properties for $\mu, \bar{\mu}$. We also have reversibility for the even (see for example [15]).

Therefore $\mu|_{(\mathcal{E}^d)^*}, \bar{\mu}|_{(\mathcal{E}^d)^*} \in \left(\text{ext } S^{\mathcal{E}}\right)_{\tilde{u}}$, so we can apply Theorem 29 to get $\mu|_{(\mathcal{E}^d)^*} = \bar{\mu}|_{(\mathcal{E}^d)^*}$. Then for any $A \in \mathcal{F}_{(\mathbb{Z}^d)^*}$, we have $\mathbf{E}_\mu(1_A|\mathcal{F}_{(\mathcal{E}^d)^*}), \mathbf{E}_{\bar{\mu}}(1_A|\mathcal{F}_{(\mathcal{E}^d)^*}) \in \mathcal{F}_{(\mathcal{E}^d)^*}$. From Lemma 15 we have $\mathbf{E}_\mu(1_A|\mathcal{F}_{(\mathcal{E}^d)^*}) = \mathbf{E}_{\bar{\mu}}(1_A|\mathcal{F}_{(\mathcal{E}^d)^*})$ and thus

$$\mu(A) = \mathbf{E}_\mu(1_A) = \mathbf{E}_\mu(\mathbf{E}_\mu(1_A|\mathcal{F}_{(\mathcal{E}^d)^*})) = \mathbf{E}_{\bar{\mu}}(\mathbf{E}_\mu(1_A|\mathcal{F}_{(\mathcal{E}^d)^*})) = \mathbf{E}_{\bar{\mu}}(\mathbf{E}_{\bar{\mu}}(1_A|\mathcal{F}_{(\mathcal{E}^d)^*})) = \mathbf{E}_{\bar{\mu}}(A) = \bar{\mu}(A).$$

□

5 Covariance

We will extend in this section the covariance estimates of [8] to the class of non-convex potentials $U = V + g$ which satisfy (4) such V and g satisfy (7) and (29).

Let $F \in C_b^1(\chi_r)$, where $C_b^1(\chi_r)$ denotes the set of differentiable functions with bounded derivatives depending on finitely many coordinates. For every $b = (x, x + e_i) \in (\mathbb{Z}^d)^*$, let

$$\partial_b F = \frac{\partial}{\partial \nabla_{e_i} \phi(x)} F(\nabla \phi) \text{ and } ||\partial_b F||_\infty = \sup_{\nabla \phi} |\partial_b F(\nabla \phi)|.$$

We define $\partial_{b^{\mathcal{E}}} F$ and $||\partial_{b^{\mathcal{E}}} F||_\infty$ similarly for $b^{\mathcal{E}} \in (\mathcal{E}^d)^*$.

Remark 30 Let $k \in I$ fixed. Take $b^{\mathcal{E}} = (x + e_j, x + e_l) \in (\mathcal{E}^d)^*$. By the change of variables $\phi(x + e_l) + \phi(x + e_j) = \alpha_1$ and $\phi(x + e_l) - \phi(x + e_j) = \alpha_2$, we have $\phi(x + e_l) = \frac{\alpha_1 + \alpha_2}{2}$ and $\phi(x + e_j) = \frac{\alpha_1 - \alpha_2}{2}$. Let $F(\nabla \phi^k) = F((\phi(z + e_s) - \phi(z + e_k) - \phi(z)))_{s \in I, z \in \mathcal{O}^d}$. Using the chain rule

$$\partial_{b^{\mathcal{E}}} F(\nabla \phi^k) = \frac{\partial}{\partial \alpha_2} F(\nabla \phi^k) = \frac{\text{sgn}(\phi(x + l))}{2} \sum_{\{y \in \mathcal{O}^d : ||y - (x + e_l)|| = 1,\ ||y - (x + e_j)|| \neq 1\}} \frac{\partial F(\nabla \phi^k)}{\partial^{(y+s, y+k)} \phi}$$
$$- \frac{\text{sgn}(\phi(x + j))}{2} \sum_{\{y \in \mathcal{O}^d : ||y - (x + e_j)|| = 1,\ ||y - (x + e_l)|| \neq 1\}} \frac{\partial F(\nabla \phi^k)}{\partial^{(y+s, y+k)} \phi} + \text{sgn}(\phi(x + l)) \delta_{(l,j)}(s, k) \frac{\partial F(\nabla \phi^k)}{\partial^{(y+s, y+k)} \phi}, \tag{64}$$

where for all $s \in I$, $\frac{\partial F(\nabla \phi)}{\partial^{(y+s, y+k)}}$ denotes the partial derivatives $D^m F$ such that m is the index which gives the position in F of $\phi(y + e_s) - \phi(y + e_k) - \phi(y)$, $\text{sgn}(\phi(x + s))$ denotes the sign of $\phi(x + e_s)$ in that term and $\delta_{(l,j)}(s, k) = 1$ if $\{l, j\} = \{s, k\}$ and 0 otherwise.

Remark 31 With the same notation as the one from Remark 30, we have

$$\left| \sup_{\nabla \phi^k} \partial_{b^{\mathcal{E}}} F(\nabla \phi^k) \right| \leq \sum_{b : b \sim b^{\mathcal{E}}} \sup_{\nabla \phi} |\partial_b F(\nabla \phi)| = \sum_{b : b \sim b^{\mathcal{E}}} ||\partial_b F||_\infty,$$

where $b \sim b^{\mathcal{E}}$ are $b = (x, x + e_s) \in (\mathbb{Z}^d)^*, x \in \mathcal{O}^d$ such that $s \in \{l, j\}$. The remark is easy to prove, by using Remark 30 and by noting that, using a similar approach to calculating $\partial_{b^{\mathcal{E}}} F(\nabla \phi^k)$ as for Remark 30, we get for $b_1 = (x, x + e_l)$ and $b_2 = (x + e_j, x)$

$$\partial_b F(\nabla \phi^k) = \partial_{b_1} F(\nabla \phi^k) + \partial_{b_2} F(\nabla \phi^k).$$

Theorem 32 *Let $u \in \mathbb{R}^d$. Assume $U = V + g$, where U satisfies (4) and V and g satisfy (7) and (29). Let $F, G \in C_b^1(\chi_r)$. Then there exists $C > 0$ such that*

$$|\text{cov}_{\mu_u}(F(\nabla \phi), G(\nabla \phi))| \leq C \sum_{b, b' \in (\mathbb{Z}^d)^*} \frac{||\partial_b F||_\infty ||\partial_{b'} G||_\infty}{1 + ||b_1 - b'_1||^d}.$$

PROOF. We have

$$\begin{aligned}\mathrm{cov}_{\mu_u}(F(\nabla\phi),G(\nabla\phi)) &= \mathbf{E}_{\mu_u}\left[\mathrm{cov}_{\mu_u}(F(\nabla\phi),G(\nabla\phi)|\mathcal{F}_{(\mathcal{E}^d)^*})\right] \\ &\quad +\mathrm{cov}_{\mu_u}\left(\mathbf{E}_{\mu_u}[F(\nabla\phi)|\mathcal{F}_{(\mathcal{E}^d)^*}],\mathbf{E}_{\mu_u}[G(\nabla\phi)|\mathcal{F}_{(\mathcal{E}^d)^*}]\right),\end{aligned}$$

where by Lemma 15 and Corollary 28 with $l=k$, we have

$$\mathbf{E}_{\mu_u}(F(\nabla\phi)|\mathcal{F}_{(\mathcal{E}^d)^*})(\xi) = \int F(\nabla\phi^k)\prod_{y\in\mathcal{O}^d}(\mu_u)_y^{\xi,k}\,\mathrm{d}\phi(y), \tag{65}$$

where $(\mu_u)_y^{\xi,k}$ are defined as in (28); a similar formula holds for G. Note that under $\mu_u(\,\cdot\,|\mathcal{F}_{(\mathcal{E}^d)^*})$, the gradients $(\nabla\phi_i(x), x\in\mathcal{O}^d, i\in I)$ are independent. Thus, there exists $c'>0$ such that

$$\begin{aligned}\left|\mathrm{cov}_{\mu_u}(F(\nabla\phi),G(\nabla\phi)|\mathcal{F}_{(\mathcal{E}^d)^*})\right| &\le c'\sum_{b\in(\mathbb{Z}^d)^*}||\partial_bF||_\infty||\partial_bG||_\infty\mathrm{var}_{\mu_u}(\nabla\phi(b)|\mathcal{F}_{(\mathcal{E}^d)^*}) \\ &\le c'\sigma^2\sum_{b\in(\mathbb{Z}^d)^*}||\partial_bF||_\infty||\partial_bG||_\infty, \end{aligned}\tag{66}$$

where for the first inequality we used Lemma 3.1 in [9] and for the last inequality we used (45). Next, in view of Lemma 13, Theorem 18 and the fact that Theorem 6.2 in [8] can be adapted to the case of the infinite even lattice with strictly convex potential, there exists $c''>0$ such that

$$\left|\mathrm{cov}_{\mu_u}(\hat F,\hat G)\right| \le c''\sum_{b^{\mathcal{E}},\ \tilde b^{\mathcal{E}}\in(\mathcal{E}^d)^*}\frac{||\partial_{b^{\mathcal{E}}}\hat F||_\infty||\partial_{\tilde b^{\mathcal{E}}}\hat G||_\infty}{1+\|b_1^{\mathcal{E}}-\tilde b_1^{\mathcal{E}}\|^d},\ \hat F=\mathbf{E}_{\mu_u}[F(\nabla\phi)|\mathcal{F}_{(\mathcal{E}^d)^*}]\text{ and }\hat G=\mathbf{E}_{\mu_u}[G(\nabla\phi)|\mathcal{F}_{(\mathcal{E}^d)^*}]. \tag{67}$$

We need to estimate now $\partial_{b^{\mathcal{E}}}\hat F$ and $\partial_{b^{\mathcal{E}}}\hat G$. But

$$\begin{aligned}\partial_{b^{\mathcal{E}}}\hat F &= \partial_{b^{\mathcal{E}}}\mathbf{E}_{\mu_u}[F(\nabla\phi)|\mathcal{F}_{(\mathcal{E}^d)^*}] = \partial_{b^{\mathcal{E}}}\left[\int F(\nabla\phi^k)\prod_{y\in\mathcal{O}^d}(\mu_u)_y^{\xi,k}\,\mathrm{d}\phi(y)\right] \\ &= \partial_{b^{\mathcal{E}}}\left[\int F(\nabla\phi^k)\prod_{y\in\mathcal{O}^d}\frac{1}{Z(\nabla_k^{\mathcal{E}}\theta(y))}e^{-\sum_{i\in I}U(\phi(y+e_i)-(\phi(y+e_k)+\phi(y)))}\,\mathrm{d}\phi(y)\right] \\ &= \int\partial_{b^{\mathcal{E}}}F(\nabla\phi^k)\prod_{y\in\mathcal{O}^d}\frac{1}{Z(\nabla_k^{\mathcal{E}}\theta(y))}e^{-\sum_{i\in I}U(\phi(y+e_i)-(\phi(y+e_k)+\phi(y)))}\,\mathrm{d}\phi(y) \\ &\quad -\mathrm{cov}_{\mu_u}\left(F(\nabla\phi^k),\partial_{b^{\mathcal{E}}}\left(\sum_{y\in\mathcal{O}^d}\sum_{i\in I}U(\phi(y+e_i)-(\phi(y+e_k)+\phi(y)))\right)\Bigg|\mathcal{F}_{(\mathcal{E}^d)^*}\right), \end{aligned}\tag{68}$$

from which, by using also Remark 31

$$|\partial_{b^{\mathcal{E}}}\hat F| \le \sum_{b:b\sim b^{\mathcal{E}}}||\partial_bF||_\infty+\left|\mathrm{cov}_{\mu_u}\left(F(\nabla\phi),\partial_{b^{\mathcal{E}}}\left(\sum_{y\in\mathcal{O}^d}\sum_{i\in I}U(\phi(y+e_i)-(\phi(y+e_k)+\phi(y)))\right)\Bigg|\mathcal{F}_{(\mathcal{E}^d)^*}\right)\right|. \tag{69}$$

By Remark 30, we have for $b^{\mathcal{E}}=(x+e_l,x+e_j)\in(\mathcal{E}^d)^*$

$$\partial_{b^{\mathcal{E}}}\left(\sum_{y\in\mathcal{O}^d}\sum_{i\in I}U(\phi(y+e_i)-(\phi(y+e_k)+\phi(y)))\right)$$

$$
\begin{aligned}
= \quad & \frac{\operatorname{sgn}(\phi(x+l))}{2}\sum_{i\in I}\sum_{\{y\in\mathcal{O}^d:||y-(x+e_l)||=1,\ ||y-(x+e_j)||\neq 1\}} U'(\phi(y+e_i)-(\phi(y+e_k)+\phi(y)))\\
& -\frac{\operatorname{sgn}(\phi(x+j))}{2}\sum_{i\in I}\sum_{\{y\in\mathcal{O}^d:||y-(x+e_j)||=1,\ ||y-(x+e_l)||\neq 1\}} U'(\phi(y+e_i)-(\phi(y+e_k)+\phi(y)))\\
& +\operatorname{sgn}(\phi(x+l))\delta_{(l,j)}(s,k)U'(\phi(y+e_i)-(\phi(y+e_k)+\phi(y))). \qquad (70)
\end{aligned}
$$

By applying (70) and then (66) to the covariance in (69), coupled with another application of Remark 30 to the resulting $\partial_{b^{\mathcal{E}}}U'$ terms and using $|U''|\le C_0+C_2$, we get for some $c'''>0$

$$
\begin{aligned}
&\left|\operatorname{cov}_{\mu_u}\left(F(\nabla\phi), \partial_{b^{\mathcal{E}}}\left(\sum_{y\in\mathcal{O}^d}\sum_{i\in I}U(\phi(y)-(\phi(y+e_k)-\phi(y+e_i)))\right)\Big|\mathcal{F}_{(\mathcal{E}^d)^*}\right)\right|\\
&\le c'''(C_0+C_2)\sum_{\substack{b^{\mathcal{E}}\in(\mathcal{E}^d)^*:\ y+e_k\in b^{\mathcal{E}}\\ or\ y+e_i\in b^{\mathcal{E}}}} ||\partial_{b^{\mathcal{E}}}F||_\infty \operatorname{var}_{\mu_u}(\nabla\phi(b)|\mathcal{F}_{(\mathcal{E}^d)^*}) \le c'''\sigma^2(C_0+C_2)\sum_{\substack{b\in(\mathcal{E}^d)^*:\ y+e_k\in b^{\mathcal{E}}\\ or\ y+e_i\in b^{\mathcal{E}}}} ||\partial_{b^{\mathcal{E}}}F||_\infty. \qquad (71)
\end{aligned}
$$

The statement of the theorem follows now from (69), (71), (66), (67) and Remark 31. □

6 Scaling Limit

We will extend next the scaling limit results results from [16] to a class of non-convex potentials.

Theorem 33 *Let $u\in\mathbb{R}^d$. Assume $U=V+g$, where U satisfies (4) and V and g satisfy (7) and (29). Set*

$$S_\epsilon(f)=\epsilon^{d/2}\sum_{x\in\mathbb{Z}^d}\sum_{i\in I}(\nabla_i\phi(x)-u_i)f_i(\epsilon x),$$

where $f\in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Then

$$S_\epsilon(f)\to N(0,\sigma_u^2(f)) \quad as \quad \epsilon\to 0, \ where\ \sigma_u^2(f)>0.$$

Proof. For simplicity, we will only prove that for all $i\in I$

$$S_{\epsilon,i}(f)\to N(0,\sigma_u^2(f)) \ \text{ as } \ \epsilon\to 0, \ \text{where } S_{\epsilon,i}(f)=\epsilon^{d/2}\sum_{x\in\mathbb{Z}^d}f(x\epsilon)(\nabla_i\phi(x)-u_i).$$

$$
\begin{aligned}
S_{\epsilon,i}(f) \quad = \quad & \epsilon^{d/2}\sum_{x\in\mathbb{Z}^d}f(x\epsilon)\left[\phi(x+e_i)-\phi(x)-u_i\right]=\epsilon^{d/2}\sum_{x\in\mathcal{E}^d}f(x\epsilon)\left[\phi(x+2e_i)-\phi(x)-2u_i\right]\\
& -\epsilon^{d/2}\sum_{x\in\mathcal{E}^d}f(x\epsilon)\left[\phi(x+2e_i)-\phi(x+e_i)-u_i\right]+\epsilon^{d/2}\sum_{x\in\mathcal{O}^d}f(x\epsilon)\left[\phi(x+e_i)-\phi(x)-u_i\right]\\
= \quad & \epsilon^{d/2}\sum_{x\in\mathcal{E}^d}f(x\epsilon)\left[\phi(x+2e_i)-\phi(x)-2u_i\right]\\
& +\epsilon^{d/2}\sum_{x\in\mathcal{E}^d}\Big[f((x+e_i)\epsilon)-f(x\epsilon)\Big]\left[\phi(x+2e_i)-\phi(x+e_i)-u_i\right]=S_\epsilon^e(f)+R_\epsilon(f).
\end{aligned}
$$

We can show CLT for $S_{\epsilon,i}^e(f)$ since the summation is concentrated on the even sites; the proof uses the same arguments as in [16] and is based on the Random Walk Representation. Also, since by Theorem 32

$$\left|\operatorname{cov}_{\mu_u}(\nabla_i\phi(x),\nabla_j\phi(y))\right|\le\frac{C}{(\|x-y\|+1)^d},$$

we have

$$\begin{aligned}\mathrm{var}_{\mu_u}(R_{\epsilon,i}(f)) &\le \epsilon^d \sum_{x,y\in\mathcal{E}^d} |\nabla_i f(x\epsilon)||\nabla_i f(y\epsilon)||\mathrm{cov}_{\mu_u}(\phi(x+e_i)-\phi(x),\phi(y+e_i)-\phi(y))| \\ &\le \epsilon^d \sum_{x,y\in\mathcal{E}^d} |\nabla_i f(x\epsilon)||\nabla_i f(y\epsilon)|\frac{C}{(\|x-y\|+1)^d},\end{aligned}$$

where $\nabla_i f(x\epsilon) = f((x+e_i)\epsilon) - f(x\epsilon)$. Expanding $f((x+e_i)\epsilon)$ in Taylor expansion around $x\epsilon$, we have $\nabla_i f(x\epsilon) = D^i f(a)\epsilon$, for some $a \in \mathbb{R}^d$. As $f \in C_0^\infty(\mathbb{R}^d)$, there exist $M, N > 0$ such that for all $x \in \mathbb{R}^d$ with $|\epsilon x| \le N$ we have $f(\epsilon x) \le M$, $|D^i f(\epsilon x)| \le M$ and both functions equal to 0 for $|\epsilon x| > N$. Therefore

$$\begin{aligned}\mathrm{var}_{\mu_u}(R_{\epsilon,i}(f)) &\le \sum_{\substack{x,y\in\mathcal{E}^d,\\ |\epsilon x|\le N, |\epsilon y|\le N}} \frac{\epsilon^{d+2}M^2C}{(\|x-y\|+1)^d} \le \epsilon^{d+2}M^2C \sum_{\substack{y\in\mathcal{E}^d,\\ |\epsilon y|\le N}} \int_{-\frac{N}{\epsilon}}^{\frac{N}{\epsilon}} \cdots \int_{-\frac{N}{\epsilon}}^{\frac{N}{\epsilon}} \frac{\mathrm{d}x_1\,\mathrm{d}x_2\ldots\,\mathrm{d}x_d}{\left(\sum_{i=1}^d |x_i-y_i|+1\right)^d} \\ &\le \epsilon^2 C(d,N,M)\log(1+2dN/\epsilon) \le 2dNC(d,N,M)\epsilon,\end{aligned}$$

where $C(d,N,M)$ is a positive constant depending on d, M and N. It follows that $R_{\epsilon,i}(f) \to 0$ as $\epsilon \to 0$ in probability. □

7 Surface tension

We will extend here to the family of non-convex potentials satisfying (4), (7) and (29), the surface tension strict convexity result from [14] and [11]. Additionally, in Theorem 37 we prove a series of surface tension equalities, which are important for the derivation of the hydrodynamic limit.

Let $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d = \mathbb{Z}^d \bmod (N)$ be the lattice torus in $\mathbb{Z}^d$ and let $u \in \mathbb{R}^d$. Then, we define the surface tension on the torus $\mathbb{T}_N^d$ as

$$\sigma_{\mathbb{T}_N}^\beta(u) = -\frac{1}{|\mathbb{T}_N^d|^d}\log\frac{Z_{\mathbb{T}_N}^\beta(u)}{Z_{\mathbb{T}_N}^\beta(0)}, \quad Z_{\mathbb{T}_N}^\beta(u) = \int_{\mathbb{R}^{\mathbb{T}_N^d}} \exp(-\beta H_{\mathbb{T}_N}(\phi + \langle\cdot,u\rangle)) \prod_{x\in\mathbb{T}_N^d\setminus\{0\}} \mathrm{d}\phi(x)$$

and where $H_{\mathbb{T}_N}$ is the Hamiltonian on the torus $\mathbb{T}_N^d$ given by

$$H_{\mathbb{T}_N}(\phi) = \sum_{i\in I}\sum_{x\in\mathbb{T}_N^d} U(\nabla_i\phi(x)) = \sum_{i\in I}\sum_{x\in\mathbb{T}_N^d} \left[V(\nabla_i\phi(x)) + g(\nabla_i\phi(x))\right].$$

Note that we define $u_{-i} = -u_i$ for $i = 1, 2, \ldots, d$. Just as in the previous sections, let us label the vertices of the torus as odd and even; let the set of odd vertices on the torus be $\mathbb{O}_N^d$ and the set of even vertices be $\mathbb{E}_N^d$. Then we can of course first integrate all the odd coordinate first and then:

$$\begin{aligned}Z_{\mathbb{T}_N}^\beta(u) &= \int_{\mathbb{R}^{\mathcal{E}_N^d}} \left(\int_{\mathbb{R}^{\mathbb{O}_N^d}} \exp(-\beta H_{\mathbb{T}_N}(\phi + \langle\cdot,u\rangle) \prod_{x\in\mathbb{O}_N^d} \mathrm{d}\phi(x) \right) \prod_{x\in\mathbb{E}_N^d\setminus\{0\}} \mathrm{d}\phi(x) \\ &= \int_{\mathbb{R}^{\mathbb{E}_N^d}} \exp(-\beta H_{\mathbb{E}_N^d}(\phi,u)) \prod_{x\in\mathbb{E}_N^d\setminus\{0\}} d\phi(x),\end{aligned}$$

where $H_{\mathbb{E}_N}(\phi,u)$ is the induced Hamiltonian on the even. It is easy to see that

$$H_{\mathbb{E}_N}(\phi,u) = H_{\mathbb{E}_N}(\phi + \langle\cdot,u\rangle, 0).$$

Then, defining the **even** surface tension on $\mathbb{E}_N^d$ as

$$\sigma_{\mathbb{E}_N}^\beta(u) = -\frac{1}{|\mathbb{E}_N^d|}\log\frac{Z_{\mathbb{E}_N}^\beta(u)}{Z_{\mathbb{E}_N}^\beta(0)}, \text{ with } Z_{\mathbb{E}_N}^\beta(u) = \int_{\mathbb{R}^{\mathbb{E}_N^d}} \exp(-\beta H_{\mathbb{E}_N}(\phi + \langle\cdot,u\rangle, 0)) \prod_{x\in\mathbb{E}_N^d\setminus\{0\}} \mathrm{d}\phi(x),$$

we obtain the following result by integrating out the odds

Lemma 34

$$\sigma^{\beta}_{\mathbb{E}_N}(u) = \frac{1}{2}\sigma^{\beta}_{\mathbb{T}_N}(u).$$

We will next prove strict convexity for the **even** surface tension, uniformly in N.

Lemma 35 *Suppose that $V, g \in C^2(\mathbb{R})$ such that they satisfy (7) and (29). Then, for all $N = 2k$, we have*

$$D^2\sigma^{\beta}_{\mathbb{E}_N}(u) \geq C|\mathbb{E}^d_N| Id, \ \forall \ u \in \mathbb{R}^d. \tag{72}$$

That is, the **even** *surface tension is uniformly convex, uniformly in N.*

Proof. First note that if $N = 2k$, we can write $H_{\mathbb{E}_N}(\phi, u)$ as

$$H_{\mathbb{E}_N}(\phi, u) = \sum_{x \in \mathbb{O}^d_N} F_x(\theta, u), \text{ with } F_x(\theta(x), u) = -\log \int_{\mathbb{R}} e^{-2\beta \sum_{i \in I} U(\nabla_i \phi(x) + u_i)} \, \mathrm{d}\phi(x) \tag{73}$$

and where, just as in (16), $\theta(x) = (\phi(x + e_1), \ldots, \phi(x - e_d))$. Note that for all $i \in I$, we have $u_{-i} = -u_i$. Then

$$\sigma^{\beta}_{\mathbb{E}_N}(u) = -\frac{1}{|\mathbb{E}^d_N|} \log \frac{\int_{\mathbb{E}^d_N} e^{-\sum_{x \in \mathbb{O}^d_N} F_x(\theta(x), u)} \prod_{i \in I} \prod_{x + e_i \in \mathbb{E}^d_N} \mathrm{d}\phi(x + e_i)}{\int_{\mathcal{E}^d_N} e^{-\sum_{x \in \mathbb{O}^d_N} F_x(\theta(x), 0)} \prod_{i \in I} \prod_{x + e_i \in \mathbb{E}^d_N} \mathrm{d}\phi(x + e_i)}.$$

As the denominator of $\sigma_{\mathbb{E}_N}(u)$ doesn't depend on u, it is enough to focus on the term

$$R_{\mathbb{E}_N}(u) := \log \int_{\mathbb{E}^d_N} e^{-\sum_{x \in \mathbb{O}^d_N} F_x(\theta(x), u)} \prod_{i \in I} \prod_{x + e_i \in \mathbb{E}^d_N} \mathrm{d}\phi(x + e_i). \tag{74}$$

Note now that by Theorem 18, we have that for each $x \in \mathcal{O}^d_\Lambda$, F_x is convex, that is

$$\left(D^2 F_x(\theta)(\bar\theta)\right)(\bar\theta) \geq c_1 \sum_{\substack{i,j \in I, \\ i \neq j}} \left|\bar\theta(x + e_i) - \bar\theta(x + e_j)\right|^2. \tag{75}$$

Because by Theorem 18 the F_x fulfill the random walk representation, we can apply to R_N Lemma 3.2 in [7], (75) and the fact that for all $i \in I$, we have $u_{-i} = -u_i$, to get the statement of the lemma. □

We consider the finite volume Gibbs measures $\tilde\mu_{N,u} \in P(\chi_{\mathbb{T}^d_N})$ with periodic boundary conditions which, for each $u \in \mathbb{R}^d$, are defined by

$$\tilde\mu_{N,u}(\mathrm{d}\tilde\eta) = Z^{-1}_{N,u} e^{-\frac{1}{2}\sum_{b \in (\mathbb{T}^d_N)^*} V(\tilde\eta(b) + u_b)} \, \mathrm{d}\tilde\eta_N \in P(\chi_{\mathbb{T}^d_N}).$$

Here $\mathrm{d}\tilde\eta_N$ is the uniform measure on the affine space $\chi_{\mathbb{T}^d_N}$ and $Z_{N,u}$ is the normalizing constant. The law of $\{\eta(b) := \tilde\eta(b) + u_b\}$ under $\tilde\mu_{N,u}$ is denoted by $\mu_{N,u}$.

Lemma 36 *$\mu_{N,u}$ converges weakly to $\mu_u \in \mathrm{ext}\, \mathcal{G}$.*

Proof. Tightness of the family $\{\mu_{N,u}\}_N$ is known for non-convex potentials with quadratic growth at ∞ (see Remark 4.4 page 152 in [15]). Therefore a limiting measure exists by taking $N \to \infty$ along a suitable sub-sequence. Note now that Theorem 32 can be also adapted to the torus case; this is due to the fact that for N even, the F_x fulfill the random walk representation on $\mathbb{T}^d_N$ and that Theorem 6.2 in [8] can be also proved for the torus, because the torus is translation invariant. Using Theorems 25, the proof follows now the same reasoning as the proof of Theorem 3.2 in [14]. In particular, because of the uniqueness of ergodic gradient Gibbs measures for each u, $\mu_{N,u}$ converges weakly to μ_u. □

Let

$$\nabla\sigma_{\mathbb{T}^d_N} = \left(D^1\sigma_{\mathbb{T}^d_N}, \ldots, D^d\sigma_{\mathbb{T}^d_N}\right), \text{ where } D^i\sigma_{\mathbb{T}^d_N} = \frac{\partial\sigma_{\mathbb{T}^d_N}}{\partial u_i}, i = 1, \ldots d.$$

Theorem 37 *Suppose that $V, g \in C^2(\mathbb{R})$ are such that they satisfy (7) and (29) and such that for all $i \in I$, U is symmetric. Then we have*

(a) $\lim_{N\to\infty} \sigma^{\beta}_{\mathbb{T}_N}(u) = \sigma_T(u), \quad \sigma_T \in C^1(\mathbb{R}^d)$*;*

(b) σ_T *is strictly convex as a function of* u*;*

(c) $\mathbf{E}_{\mu_u}[\eta(b)] = u_b$*;*

(d) $\mathbf{E}_{\mu_u}[U'(\eta(e_i)] = D^i\sigma_T(u)$*, for all* $i = 1, \dots d$*;*

(e) $\mathbf{E}_{\mu_u}[\sum_{i=1}^d \eta(e_i)U'(\eta(e_i)] = u \cdot \nabla\sigma_T(u) + 1$*, for all* $i = 1, \dots d$*;*

(f) $|\nabla\sigma(u) - \nabla\sigma(v)| \leq C|u - v|$ *for some* $C > 0$*.*

Proof.

(a) Using that $\limsup_{N\to\infty} \frac{1}{|\mathbb{T}_N^d|} \log \mathbf{E}_{\mu_{N,u}}\left[e^{\alpha\sum_{b\in(\mathbb{T}_N^d)^*}\eta^2(b)}\right] < \infty$ for some $\alpha > 0$ (see Remark 4.4 page 152 in [15]) and noting from Lemma 36 that $\tilde{\mu}_{N,u}$ converges weakly to $\tilde{\mu}_u$ as $N \to \infty$, the proof now follows the same steps as the proof of Theorem 3.4.(0) in [14].

(b) Since by (a), $\lim_{N\to\infty} \sigma^{\beta}_{\mathbb{T}_N}(u) = \sigma_T(u)$, every sub-sequence of $\sigma^{\beta}_{\mathbb{T}_N}(u)$ will converge to $\sigma_T(u)$, in particular for $N = 2k$. The statement of the theorem follows immediately by using now Lemma 34 and Lemma 35 applied to the sub-sequence $\left(\sigma^{\beta}_{\mathbb{T}_N}(u)\right)_N$, with $N = 2k$.

(c) , (d) and (e) follow just as in [14], so their proofs will be omitted.

(f) Let $N = 2k$. Define

$$\mu^{\mathbb{E}}_{N,u}(\,\mathrm{d}\phi^{\mathbb{E}}) = \frac{1}{Z^{\mathbb{E}}_{N,u}} e^{-\sum_{x\in\mathbb{O}_N} F_x(\theta(x),u)}\,\mathrm{d}\phi^{\mathbb{E}_N},$$

where $\mathrm{d}\phi^{\mathbb{E}_n} = \prod_{x\in\mathbb{E}_N^d\setminus\{0\}} \phi(x)$ and $Z^{\mathbb{E}}_{N,u}$ is the normalizing constant. Due to the fact that the random walk representation holds on the set of the evens and to Theorem 29, one can show as in [14] that for $N = 2k$, $\mu^{\mathbb{E}}_{N,u}$ converges weakly to $\mu^{\mathbb{E}}_u \in \left(\mathrm{ext}\, S^{\mathbb{E}}\right)_{\tilde{u}}$, where the same notations as in the uniqueness of ergodic component section apply. Note now from (73) that

$$\mathbf{E}_{\mu_{N,u}}\left[U'(\nabla_i\phi(x)\right] = \mathbf{E}_{\mu^{\mathbb{E}}_{N,u}}\left[D^iF_x(\theta(x),u)\right], \text{ where } x \in \mathbb{O}_N^d.$$

Using now (d), the weak convergence of $\mu_{N,u}$ to μ_u and the weak convergence of $\mu^{\mathbb{E}}_{N,u}$ to $\mu^{\mathbb{E}}_u$, we get

$$\mathbf{E}_{\mu_u}[U'(\eta(e_i)] = \mathbf{E}_{\mu^{\mathbb{E}}_u}\left[D^iF_x(\theta(x),u)\right] = D^i\sigma_T(u). \tag{76}$$

Using the random walk representation and Taylor expansion, we have

$$\left|D^iF_x(\theta(x)) - D^iF_x(\bar{\theta}(x))\right| \leq c_+ \sum_{k\in I}\left|\phi(x+e_k) - \bar{\phi}(x+e_k)\right|. \tag{77}$$

The bound in (f) is now a simple consequence of (76), (77) and Lemma 28.

□

Acknowledgment

The authors thank Elliot Lieb for suggesting to them the use of the even/odd representation.

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