

Multivariate Option Pricing Using Copulae

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Abstract: The complexity of financial products significantly increased in the past ten years. In this paper we investigate the pricing of basket options and more generally of complex exotic contracts depending on multiple indices. Our approach assumes that the underlying assets evolve as dependent GARCH(1,1) processes and it involves to model the dependency among the assets using a copula based on pair-copula constructions. Unlike most previous studies on this topic, we do not assume that the dependence observed between historical asset prices is similar to the dependence under the risk-neutral probability needed for the pricing. The method is illustrated with US market data on basket options written on two or three international indices.

Keywords: Pair-copula construction, basket options, multivariate derivatives, pricing.

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1 Introduction

There has been significant innovation in the financial industry in the past ten years. More and more basket options and complex exotic contracts depending on multiple indices are issued. This paper proposes to evaluate derivatives written possibly on more than two indices. Our approach assumes that each underlying asset evolves as a GARCH(1,1) process but unlike most previous studies on this topic as we will show, we do not necessarily assume that the dependence observed between historical prices is similar to the dependence under the risk-neutral probability. The method is implemented with market data on basket options written on three international indices. Most of the contracts written on several indices are over-the-counter options but we found price data of exchange-listed options linked to several indices on the New York Stock Exchange to illustrate the methodology.

We now review the existing approaches to price options written on several possibly dependent indices. The first approach is to consider a Black and Scholes model in a multivariate framework. The setting consists of n assets respectively modeled by geometric Brownian motions with constant volatility, constant interest rates and a Gaussian dependence measured by the correlation between the respective Brownian motions. Galichon [2006] extends the idea of the local volatility model developed by Dupire [1994] to build a stochastic correlation model (see also Langnau [2009]). Rosenberg [2003], Cherubini and Luciano [2002] propose a non-parametric estimation of the marginal risk-neutral densities (using option prices written on each asset). Van den Goorbergh, Genest and Werker [2005] adopt a parametric approach. They estimate a GARCH(1,1) to model each asset, then they make use of the transformation by Duan [1995] to obtain the risk-neutral distribution. Cherubini and Luciano [2002] and van den Goorbergh et al. [2005] model the dependency between the different underlying assets using historical data on the joint distribution of the two underlying assets. The same dependency is then assumed under the risk-neutral probability. Cherubini and Luciano [2002] study digital binary options and van den Goorbergh et al. [2005] apply their techniques on some hypothetical contracts written on the maximum or the minimum of two assets. Both papers study financial derivatives written on only two indices and have no empirical examples.

In this paper, we extend the paper of van den Goorbergh et al. [2005] in several directions. First, we consider contracts with possibly more than two underlying indices and show how it is possible to evaluate with a multivariate distribution using pair copula construction (Aas et al. [2009]). Second, we investigate a dataset of basket options prices and check that this approach can

be used to price these contracts accurately. Third, we study the sensitivity of basket option prices to the choice of the parameters for the GARCH(1,1) processes, the copula family and of its parameters to understand the impact of dependence misspecification. Finally, we investigate whether the multivariate copula of the underlying assets is the same under the objective measure P and under the risk-neutral measure Q . As far as we know all the previous studies using copulae make this assumption, except Galichon [2006] and Langnau [2009]. The latter authors indeed discuss and model the dynamics of the assets directly under the risk-neutral probability, and their model fits perfectly the market prices by construction.

There are arguments to believe that the copula under the objective measure P is similar to the copula under the risk-neutral measure Q . For example, in the multivariate Black and Scholes, the change of measure between P and Q does not influence the dependence between the two stocks. The covariance matrix stays the same, only the drift term of the geometric Brownian motions have changed. Rosenberg [2003] and also Cherubini and Luciano [2002] argue that under some assumptions, it is reasonable to think that the dependence of the underlying assets under P is similar to the one under Q . Rosenberg [2003] shows that the dependence structure under Q will be the same as under P when risk-neutral returns are increasing functions of the objective returns. Galichon [2006] argues in a very different way. To him, “it is an extreme assumption to make only in the extreme hypothesis where the market does not provide any supplemental information on the dependence structure, which is usually not the case (the price of basket options, for instance, contains information on the market price of the dependence structure)”. However in his study, he does not explore this direction. In our context, there is another reason that could also justify that the dependence may be different in the historical world and in the risk-neutral world, is that the change of measure has a particular effect on the GARCH(1,1) process. After this change, not only the drift is changed but also the volatilities. This transformation is not monotonic which suggests that the dependence may change as well.

In this paper, we explain how to price an option written on more than two assets in a dynamic-copula setting. Dependence problems with more than two assets are significantly more difficult. However using pair-copula constructions, the problem comes back to study the dependence between two variables at a time. We illustrate the study with a concrete example using data from the North American financial market. We can check the accuracy of the model prices compared to the market prices from the historical quotes that we have. This set of data is also used to discuss how the dependence

structure under the objective measure and the risk-neutral world may be different. In the first section, we explain how to proceed in the bivariate case. In the second section, we extend the study to an option written on three indices and we implement this methodology on examples of basket options in the third section.

2 Bivariate Option Pricing

In this section, we first recall how to price a European option that depends on the final value at maturity of two assets using a similar approach as van den Goorbergh, Genest and Werker [2005].

2.1 Distribution of the underlying assets under P

Denote by $S_i(t)$ the closing price of index i for the trading day t , and define the log-return on asset i for the t^{th} trading day as

$$r_{i,t+1} = \log(S_i(t+1)/S_i(t)) \quad (1)$$

where $i = 1$ or $i = 2$. Let $\mathcal{F}_t = \sigma((r_{1,s}, r_{2,s}), s \leq t)$ denote all returns information available at time t . Similar to van den Goorbergh, Genest and Werker [2005], we assume that the distribution of (S_1, S_2) under the objective measure P can be described as follows. The marginal distributions of $S_1(\cdot)$ and $S_2(\cdot)$ respectively follow GARCH(1,1) processes with Gaussian innovations. The dependence structure between the standardized innovations up to time t is given by a copula $C_t^P(\cdot, \cdot)$ that may depend on time t and is defined under the probability measure P . This model is quite general and allows for time-varying dependence as well as time-varying volatilities in a non-deterministic way. Indeed the dependence can change with the volatility in the financial market (see van den Goorbergh et al. [2005] for an example). However, for the ease of exposition, we restrict ourselves to the case when the dependence is not time-varying. Our model could easily be extended to time-varying copulae by adding more parameters to the model.

Under the objective measure P , the log-returns of each asset S_i for $i = 1$ and $i = 2$ evolve as follows:

$$\begin{cases} r_{i,t+1} = \mu_i + \eta_{i,t+1}, \\ \sigma_{i,t+1}^2 = w_i + \beta_i \sigma_{i,t}^2 + \alpha_i (r_{i,t+1} - \mu_i)^2, \\ \eta_{i,t+1} | \mathcal{F}_t \sim_P N(0, \sigma_{i,t}^2) \end{cases} \quad (2)$$

where $w_i > 0, \beta_i > 0$ and $\alpha_i > 0$, and where \sim_P refers to the distribution under P . μ_i is the expected daily log-return for S_i . The GARCH parameters are estimated by maximum likelihood, using the unconditional variance level $\frac{w_i}{1-\beta_i-\alpha_i}$ as starting value $\sigma_{i,0}^2$. Denote the standardized innovations by

$$(Z_{1,s}, Z_{2,s})_{s \leq t} := \left(\frac{\eta_{1,s}}{\sigma_{1,s}}, \frac{\eta_{2,s}}{\sigma_{2,s}} \right)$$

The standardized innovations $(Z_{1,s})_s$ and $(Z_{2,s})_s$ are respectively i.i.d. with a standard normal distribution $N(0, 1)$, but in general Z_1 is not independent of Z_2 . Let F_1^P be the cdf of Z_1 and F_2^P be the cdf of Z_2 under the objective measure P . Note that F_1^P and F_2^P are $N(0, 1)$ -distributed in our specific case. In general, using Sklar [1959]'s theorem, the joint distribution F^P of Z_1 and Z_2 can be written as a function of its marginals. Precisely there exists a unique copula C^P , such that

$$F^P(z_1, z_2) = C^P(F_1^P(z_1), F_2^P(z_2)) \quad (3)$$

for all $z_i \in \mathbb{R}$, $i = 1, 2$. We then assume that the copula $C^P(., .)$ is a parametric copula and θ_P corresponds to the parameter(s) of the copula used to model the dependence. We propose to look at three types of parametric copulae, the Gaussian copula, the Clayton copula and the Gumbel copula but it is straightforward to extend our study to other families of copulas. For example see Joe [1997] for many other bivariate copula families including two parameter families such as the student copula or the BB1 and BB7. The respective formulas and properties of the Gaussian, Clayton and Gumbel copulae are given in Appendix A. We have chosen the Gaussian as a benchmark, which has no tail dependence, while the Clayton allows for lower tail dependence and Gumbel upper tail dependence.

2.2 Pricing of a bivariate option

Consider first an option whose payoff depends only on the terminal values of two indices S_1 and S_2 . Let us denote by $g(S_1(T), S_2(T))$ the payoff of this bivariate financial derivative. In an arbitrage-free market, a price for this derivative can be obtained as the discounted conditional expected value of the option's payoff under a risk-neutral distribution. Let Q be the chosen risk-neutral probability. Then the price at time t of this derivative is given by

$$g_t = e^{-r_f(T-t)} E_Q [g(S_1(T), S_2(T)) | \mathcal{F}_t] \quad (4)$$

where E_Q denotes the expectation taken under the risk-neutral probability Q and r_f denotes the constant daily risk-free rate. Here $T - t$ corresponds to the time to maturity calculated in number of remaining trading days. The price (4) can also be expressed as a double integral

$$g_t = e^{-r_f(T-t)} \int_0^{+\infty} \int_0^{+\infty} g(s_1, s_2) f^Q(s_1, s_2) ds_1 ds_2$$

where f^Q denotes the joint density of $S_1(T)$ and $S_2(T)$ under the risk-neutral probability Q . Similar to (3), it is possible to express the joint density using the marginal densities f_1 and f_2 of respectively $S_1(T)$ and $S_2(T)$ as follows:

$$f^Q(x_1, x_2) = c_{12}^Q(F_1^Q(x_1), F_2^Q(x_2)) f_1^Q(x_1) f_2^Q(x_2)$$

where $c_{12}^Q = \frac{\partial^2 C^Q(y_1, y_2)}{\partial y_1 \partial y_2}$ and the superscript Q recalls that it corresponds to the distribution under the risk-neutral probability Q . To value the option and calculate its price (4), one needs to know the *joint* distribution of $S_1(T)$ and $S_2(T)$ under Q , that is their respective marginal risk-neutral distribution functions F_1^Q and F_2^Q , as well as the copula $C^Q(., .)$ that captures the dependence between $S_1(T)$ and $S_2(T)$ under Q .

Following the idea of Duan [1995] and van den Goorbergh, Genest and Werker [2005], and assuming that the conditions needed for the change of measure of Duan [1995] are satisfied, the log-returns under the risk-neutral probability measure Q are given as follows

$$\begin{cases} r_{i,t+1} = r_f - \frac{1}{2}\sigma_{i,t}^2 + \eta_{i,t+1}^*, \\ \sigma_{i,t+1}^2 = w_i + \beta_i \sigma_{i,t}^2 + \alpha_i (r_{i,t+1} - \mu_i)^2, \\ \eta_{i,t+1}^* | \mathcal{F}_t \sim_Q N(0, \sigma_{i,t}^2) \end{cases} \quad (5)$$

where r_f is the daily risk-free rate on the market that we assume constant. Note that the change of measure consists simply of a change in the drift of the GARCH process (as Duan [1995]). For this change of measure to be valid, the conditional distribution of each asset to the information \mathcal{F}_t at time t is similar to the conditional information to the information generated solely by this asset up to time t (see Duan [1995] for more information). This must hold at any time t .

Note that the daily risk free rate r_f plays a critical role in the simulation of the process in the risk-neutral world, and therefore in the pricing of the security. We need to control for the influence of significant changes in the level of the risk-free rate over the last years (see discussion in section 4).

Dependence Modelling

To model the dependence under Q , there are also two possible approaches. The first approach consists of assuming that it is similar to the dependence under P . As far as we know, this assumption has been made by most authors who propose methods to price options on multiple indices, including Cherubini and Luciano [2002], Chiou and Tsay [2008], Rosenberg [2003], and van den Goorbergh, Genest and Werker [2005]. The approach we adopt is quite different. We would like to infer from market prices of bivariate options the joint distribution of assets under Q , and therefore the copula under Q . We assume that the copula under Q belongs to the same family as the copula used under P but we do not impose that they have the same parameter. Our approach is therefore parametric.

Assume for example that the copula under P is a Gaussian, Gumbel or Clayton which are characterized by only one parameter θ_P . We are now looking to see if the same copula with a possibly different parameter θ_Q could better reflect market movements in the prices of the option. We observe at time t , g_t^M the market price of the option, and we can calculate using formula (4), a Monte Carlo estimate of this price, denoted by $\hat{g}_t^{mc}(\theta_Q)$. We then solve for the best parameter $\theta_Q(t)$ of the copula under Q such that the estimated price $\hat{g}_t^{mc}(\theta_Q(t))$ is as close as possible to the market price g_t^M . We are able to calculate θ_Q at each time t when one observes a market price. We can then compare θ_P with θ_Q to see whether they are significantly different.

Extension to time-varying dependence

In practice the dependence changes over time, in particular with the level of volatility on the market. When the volatility is high, the dependence is usually higher. But it is not difficult to extend our approach to the case when the parameters of the copula are time-varying, precisely are function of the volatility observed in the market. This could introduce additional parameters to estimate under P and under Q by assuming a relationship between the volatilities and the parameter of the copula (and we would need sufficient data). For example van den Goorbergh, Genest and Werker assume that

$$\theta_P(t) = f(\gamma_0 + \gamma_1 \log(\max(\sigma_{1,t}, \sigma_{2,t}))) \quad (6)$$

where f is a given function. Then there are two additional parameters γ_0 and γ_1 to fit and a specific study is needed each time to determine the best relationship (6) to assume between the volatilities in the market at time t and the copula parameter. Other time-varying copula models might involve GARCH components as in Ausin and Lopes [2009] or stochastic volatility components as in Haffner and Manner [2008] or Almeida and Czado [2010].

While Ausin and Lopes [2009] and Almeida and Czado [2010] use a Bayesian approach for estimation, the approach taken by Haffner and Manner [2008] involves efficient importance sampling.

We now extend the idea developed in this section to trivariate options in Section 3, and finally illustrate the study in Section 4 with examples of basket options written on three indices.

3 Multivariate option pricing when there are more than two indices

We first describe pair-copula construction in the case of three indices. It is then illustrated with an example of a trivariate option.

3.1 Multivariate Dependence Modeling

To model multivariate dependency, there are many different approaches. In this paper we continue to follow a copula approach. While there are many bivariate copulae the choice for multivariate copulae tended to be limited, especially with regard to asymmetric tail dependence among pairs of variables. Joe [1996] gave a construction method for multivariate copulae in terms of distribution functions requiring only bivariate copulae as building blocks. The bivariate building blocks represent bivariate margins as well as bivariate conditional distributions. Graphical methods to identify the necessary building blocks were subsequently developed by Bedford and Cooke ([2001] and [2002]). Their full potential to model different dependence structures for different pairs of variables were recognized by Aas, Czado, Frigessi and Bakken [2009] and applied to financial return data using maximum likelihood for estimation. They denote this construction approach the pair-copula construction method for multivariate copulae. For three dimensions the construction method is simple and proceeds as follows. Let $f(x_1, x_2, x_3)$ denote the joint density, which is decomposed for example by conditioning as

$$f(x_1, x_2, x_3) = f(x_1|x_2, x_3) \times f_{2|3}(x_2|x_3) \times f_3(x_3). \quad (7)$$

Now by Sklar's theorem we have $f(x_2, x_3) = c_{23}(F_2(x_2), F_3(x_3))f_2(x_2)f_3(x_3)$ and therefore

$$f_{2|3}(x_2|x_3) = c_{23}(F_2(x_2), F_3(x_3))f_2(x_2).$$

Similarly we have $f_{1|2}(x_1|x_2) = c_{12}(F_1(x_1), F_2(x_2))f_1(x_1)$. Finally we use Sklar's theorem for the conditional bivariate density

$$f(x_1, x_3|x_2) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))f_{1|2}(x_1|x_2)f_{3|2}(x_3|x_2)$$

and therefore

$$f(x_1|x_2, x_3) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))f_{1|2}(x_1|x_2).$$

Putting these expressions into (7) it follows that

$$\begin{aligned} f(x_1, x_2, x_3) &= c_{12}(F_1(x_1), F_2(x_2))c_{23}(F_2(x_2), F_3(x_3)) \\ &\times c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))f_1(x_1)f_2(x_2)f_3(x_3). \end{aligned}$$

The corresponding copula density is therefore given by

$$c_{123}(u_1, u_2, u_3) = c_{12}(u_1, u_2)c_{23}(u_2, u_3).c_{13|2}(F_{1|2}(u_1|u_2), F_{3|2}(u_3|u_2)) \quad (8)$$

The copula with density given by (8) is called a D-vine in three dimensions and involves only bivariate copulae. More general pair copula constructions are contained in Aas, Czado, Frigessi and Bakken [2009] and a recent survey on such constructions is given by Czado [2010].

We now show how to apply this technique to the valuation of options linked to three market indices.

3.2 Trivariate Option Pricing

In this section, we describe each step needed to obtain the price of an option written on three market indices. This methodology will then be applied to an example in Section 4.

Let us denote by $S_i(t)$ the closing value of the index i on day t . The first step consists of fitting a GARCH(1,1) process on each marginal using historical data on index prices. Then we study the dependence between the indices and estimate the type of copula and its parameters under the objective probability P . The third step corresponds to the simulation of the joint distribution of (S_1, S_2, S_3) under the risk-neutral probability. The last step explains how to obtain the price of the trivariate option, and therefore estimate the risk-neutral parameter.

Step 1: Calibration of the GARCH(1,1) processes.

At time t (valuation date of the option, say 3rd of November 2009), we calibrate a GARCH process using Δ past informations, corresponding to the Δ trading days prior to t . We then use the observations of $S_i(\cdot)$ from time $t - \Delta$ to time t to fit a GARCH(1,1) for each underlying asset S_i , $i = 1, 2, 3$, we find $\hat{\mu}_i$, \hat{w}_i , $\hat{\alpha}_i$ and $\hat{\beta}_i$ as well as the daily volatilities $\hat{\sigma}_{i,s}$, for each asset i and each time $t - \Delta < s \leq t$. Then we have Δ estimated standardized innovations

$$(Z_{1,s}, Z_{2,s}, Z_{3,s})_{s \in]t-\Delta, t]} := \left(\frac{\hat{\eta}_{1,s}}{\hat{\sigma}_{1,s}}, \frac{\hat{\eta}_{2,s}}{\hat{\sigma}_{2,s}}, \frac{\hat{\eta}_{3,s}}{\hat{\sigma}_{3,s}} \right).$$

In the GARCH(1,1) model used to calibrate the marginals, the standardized innovations are $N(0, 1)$. We obtained the corresponding estimated standardized innovations in the interval $(0, 1)$ by applying Φ , the cdf of the standard normal distribution $N(0, 1)$. Let us denote by U the corresponding variables

$$(U_{1,s}, U_{2,s}, U_{3,s})_s := \left(\Phi \left(\frac{\hat{\eta}_{1,s}}{\hat{\sigma}_{1,s}} \right), \Phi \left(\frac{\hat{\eta}_{2,s}}{\hat{\sigma}_{2,s}} \right), \Phi \left(\frac{\hat{\eta}_{3,s}}{\hat{\sigma}_{3,s}} \right) \right)_s. \quad (9)$$

The dependence structure between $(Z_{i,\cdot})_i$ is the same as between $(U_{i,\cdot})_i$ because a copula is invariant by a change by an increasing function (this is a standard result, see Joe [1997] for instance).

Step 2: Dependence under P .

At time t , we fit a copula on the joint distribution of $(U_{1,s}, U_{2,s}, U_{3,s})_{t-\Delta < s \leq t}$ as follows. The copula density $c^P(u_1, u_2, u_3)$ is equal to

$$c_{12}(u_1, u_2, \theta_{12})c_{23}(u_2, u_3, \theta_{23})c_{13|2}(F_{1|2, \theta_{12}}(u_1|u_2, \theta_{12}), F_{3|2, \theta_{23}}(u_3|u_2, \theta_{23}), \theta_{13|2}) \quad (10)$$

where c_{12} , c_{23} and $c_{13|2}$ are three parametric copula densities with respective parameters θ_{12} , θ_{23} and $\theta_{13|2}$, and where $F_{1|2, \theta_{12}}$ denotes the conditional cdf of U_1 given $U_2 = u_2$. The estimates of the parameters $\hat{\theta}_{12}$, $\hat{\theta}_{23}$ and $\hat{\theta}_{13|2}$ depend on the time t at which the estimation is done and also on the size of the time window. Note that θ_α where $\alpha = 12, 23$ or $13|2$ is a generic notation for the parameter(s) of the copula and may represent a vector of parameters if the parametric copula depends on more than one parameter.

To simplify, we also restrict ourselves to well-known classes of one parameter copulae. In the example, we investigate the Gaussian copula, the Gumbel copula and the Clayton copula. Since they are parametric copulae, it amounts to find the respective parameters that best fit the initial dependence structure.

Notice also that the decomposition (10) depends on the order of the variables. We choose c_{12} and c_{23} to correspond to the two pairs of variables

that exhibit the most dependence. We then estimate by Maximum Likelihood estimation $\widehat{\theta}_{12}^P$ and $\widehat{\theta}_{23}^P$. We then examine $c_{13|2}$ which is the dependence between S_1 and S_3 conditional to S_2 . The conditional distribution cannot be observed directly. To obtain random variables that are distributed along the conditional distribution, we use (9) and calculate for each observation $s \in [t - \Delta, t]$,

$$\begin{aligned} u_{1|2s} &= F_{1|2, \theta_{12}}(u_{1s} | u_{2s}, \widehat{\theta}_{12}^P) \\ u_{3|2s} &= F_{3|2, \theta_{23}}(u_{3s} | u_{2s}, \widehat{\theta}_{23}^P) \end{aligned} \quad (11)$$

where the conditional distribution $F_{1|2, \theta_{12}^P}$ is obtained by

$$F(u_1 | u_2, \widehat{\theta}_{12}^P) = \frac{\partial}{\partial u_2} C_{12}(u_1 | u_2, \widehat{\theta}_{12}^P) =: h(u_1, u_2, \widehat{\theta}_{12}^P) \quad (12)$$

and

$$F_{3|2, \theta_{23}^P}$$

similarly. Appendix A gives the expressions of the function h for each type of copula.

Step 2 is completed when the dependence structure is chosen and that the estimated parameters are calculated ($\widehat{\theta}_{12}^P$, $\widehat{\theta}_{23}^P$ and $\widehat{\theta}_{13|2}^P$). The superscript P recalls that the estimation of the dependence structure is obtained from historical data of the assets' prices, therefore it corresponds to the dependence structure under the objective measure P .

Step 3: Simulation of (S_1, S_2, S_3) under Q .

We assume that the copula under Q belongs to the same family as the one determined under P but may have different parameters. Given the risk-neutral parameters θ_{12}^Q , θ_{23}^Q and $\theta_{13|2}^Q$ that we will explain later how to obtain them, we simulate observations from the D-vine specification (8)

$$\left(U_{1,s}^Q, U_{2,s}^Q, U_{3,s}^Q \right)_{s < t \leq T} \quad (13)$$

with the dependence structure identified in Step 2 but with a new parameter set

$$\Theta^Q := \left(\theta_{12}^Q, \theta_{23}^Q, \theta_{13|2}^Q \right).$$

To simulate from a D-vine, we refer to Algorithm 2 on page 187 of Aas, Czado, Frigessi and Bakken [2009].

From the D-vine data (13), we can obtain recursively the standardized residuals such that

$$\Phi^{-1} \left(U_{i,s}^Q \right) = \frac{r_{i,s} - r_f + \frac{\sigma_{i,s-1}^2}{2}}{\sigma_{i,s-1}}. \quad (14)$$

This is a consequence of the dynamics of the assets under Q given by (5). Indeed given initial volatilities $\sigma_{i,0}$ for $i = 1, 2, 3$ (equal for instance to the square root of the unconditional variance level $\frac{w_i}{1-\beta_i-\alpha_i}$), one can calculate the first innovation $r_{i,1}$ using (14). Then using the second equation of (5), one computes $\sigma_{i,1}^2$ for $i = 1, 2, 3$. Recursively, it is possible to construct the full process.

Step 4: Pricing the option.

Using the step 3, we obtain simulations of $(S_1(T), S_2(T), S_3(T))$ based on a copula with parameter set Θ^Q , and a price of the option is obtained as

$$\text{Price at } t = e^{-r_f(T-t)} E_Q [g(S_1(T), S_2(T), S_3(T)) | \mathcal{F}_t], \quad (15)$$

where T denotes the number of days between the issuance date and the maturity of the option, and t is the number of days since the inception of the contract.

The remaining question is about the choice of the parameter set Θ^Q . Recall that it depends on time. One needs past observations of prices of the trivariate option, say at dates t_i , $i = 1..n$, the set of parameters Θ_Q needed to characterize the copula C^Q is calculated at time t such that it minimizes the sum of quadratic errors

$$\min_{\Theta_Q} \sum_{i=1}^n (\hat{g}_{t_i}^{mc}(\Theta_Q) - g_{t_i}^M)^2 \quad (16)$$

where $g_{t_i}^M$ denotes the market price of the trivariate option observed in the market at the date t_i , and $\hat{g}_{t_i}^{mc}$ is the Monte Carlo estimate of its price obtained by the procedure described above.

Remark 3.1 *In the case of two indices, this analysis is simpler, because the number of parameters is significantly reduced. For example, there could be only one parameter in formula (3) (i.e. when the copula is of Gumbel, Clayton or Gaussian type).*

Remark 3.2 *In the case of a path-dependent contract, the technique is similar, with*

$$\text{Price at } t = e^{-r_f(T-t)} E_Q [g(\{S_i(s)_{s \in [0,T]}\}_i) | \mathcal{F}_t]$$

instead of (15). The inception date is $t = 0$.

4 Empirical Analysis

We first describe the data, then present our numerical results.

4.1 Description of the Data

There are a few retail investment products linked to a basket of indices that are traded in North America (see Bernard, Boyle and Gornall [2010] for more information about these exchange-listed structured products). In May 2008, there were 24 index-linked notes written on multiple market indices (for a total volume of US\$590 million). Precisely, the data that we use in this paper, come from the secondary market for exchange-listed structured products on the New York Stock Exchange. We selected two structured products to illustrate our study: they were traded in the secondary market, linked to three indices, and we could obtain daily quotes for these two products. They are index linked notes listed on the New York stock exchange¹.

We now describe these trivariate basket options for which we have daily quotes of the prices from their respective issuance date to November 2, 2009. We also have the daily log-returns of each index involved in these structured products over the period under study. *MIB* and *IIL* are “Capital Protected Notes Based on the Value of a Basket of Three Indices”. *MIB* and *IIL* are both issued by Morgan Stanley and are very similar, therefore we only describe one of them. The notes *IIL* are linked to the *Dow Jones EURO STOXX 50SM* Index, the *S&P500* Index, and the *Nikkei 225* Index (let us denote them respectively by S_1 , S_2 and S_3). They were issued on July 31st, 2006 at an initial price of \$10 and matured on July 20, 2010 (which correspond roughly to 1,006 trading days). Their final payoff is given by

$$\text{\$10} + \text{\$10} \max \left(\frac{m_1 S_1(T) + m_2 S_2(T) + m_3 S_3(T) - 10}{10}, 0 \right) \quad (17)$$

where $m_i = \frac{10}{3S_i(0)}$ such that $m_1 S_1(0) + m_2 S_2(0) + m_3 S_3(0) = 10$ and the percentage weighting in the basket is 33.33% for each index. On July, 2006, $m_1 = 0.000917803$, $m_2 = 0.002643329$, and $m_3 = 0.000222122$.

¹All information about these products is contained in the official prospectus supplements that were publicly available on www.amex.com and now listed on www.nyse.com. They can also be obtained upon request from the authors.

4.2 Estimation of r_f

We now describe how we estimate the risk-free rate and how the GARCH(1,1) model developed by Duan [1995] is appropriate to price a similar contract written on one index. Actually Duan [1995] already showed that a GARCH(1,1) model could reflect well the implied volatility surface (the smile with respect to the strike and the decay with respect to the time to maturity). Unsurprisingly it gives reasonable estimates of the implied risk-free rate to price structured products written on one index.

The contracts *MIB* and *IIL* were issued in North America by Morgan Stanley, they are five year contracts. To check whether the GARCH(1,1) model may give reasonable prices, we consider other contracts issued by Morgan Stanley, with similar long-term maturity, but written on a single index so that the estimate of the risk-free rate is not influenced by the modeling of the dependence. We found two contracts of this type: the contract *PDJ* written on the Dow Jones Industrial Average, *DJIA* (issued on Feb 25, 2004 with maturity date of Dec 30, 2011), and the contract *PEL* written on the *S&P500* (issued on March 25th, 2004 with maturity date of Dec 30, 2011). Both of these contracts pay semi-annual coupons of respectively 0.4% and 0.5% (at the end of June and end of December) and their final payoff is calculated as

$$\$10 + \$10 \max \left(0, \frac{\frac{1}{8} \sum_{i=1}^8 S_{t_i} - S_0}{S_0} \right)$$

where S_0 is the initial value of the underlying at the issuing date, and where $t_i = 30^{th}$ December of each year (starting in 2004 and ending in 2011).

For each of these contracts, we fit a GARCH(1,1) process based on a 250 days window (one year of data because there are about 250 trading days per year) on respectively historical data of *S&P500* and *DJIA*. We then use Duan's [1995] change of measure given by (5) and simulate the price using different values for a continuously compounded risk-free rate $r \in (1\%, 10\%)$. We then solve for the value of r such that the model price coincides with the market price. We did this calculation for 5 dates for each contract, 31th December 2004, 2005, 2006, 2007 and 2008. More precisely the calculation was done on the following day after the payment of coupons, so it could also be 2nd or 3rd of January depending on when the next trading day is.

For example, the contract *PEL* was quoted on the 31st of December 2004 at \$9.60. At that time, the time to maturity is about 1754 days. In Figure 1, we represent the price of the contract *PEL* with respect to the risk free rate. It illustrates the importance of controlling the effect of the risk-free rate if we

want to discuss the change in the dependence structure under P and under Q . Here the “implied risk free rate” r is about 4.25% on December 31, 2004.

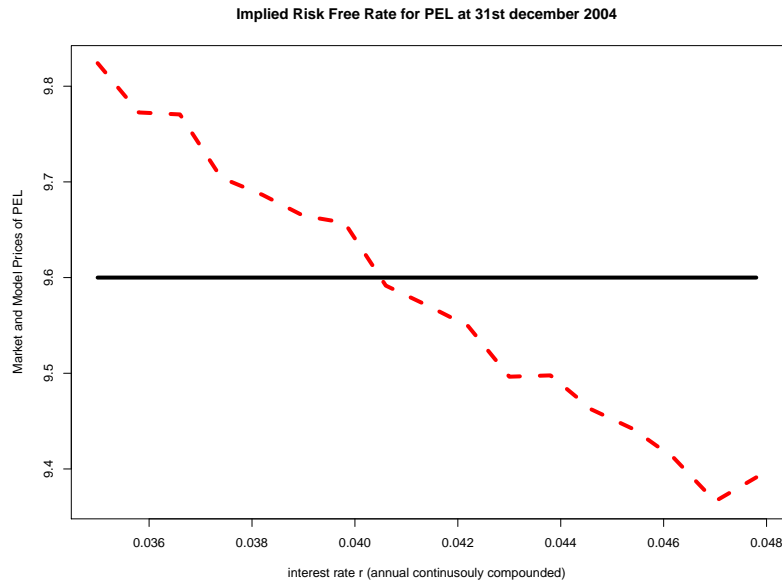


Figure 1: Price obtained by the model of Duan (1995) for the contract PEL as a function of the risk-free rate r_f (with 10,000 Monte Carlo simulations to draw this graph). On 31st dec 2004, it is quoted at \$9.60.

Date	12/31/04	12/31/05	12/31/06	12/31/07	12/31/08
r for PDJ	4.3%	5%	6.1%	4.1%	2%
r for PEL	4%	4.6%	5.3%	3.4%	2.6%
ZC yield	4.05%	4.82%	5.03%	3.85%	2.3%

Table 1: Implied risk-free rate used to price the contracts PDJ and PEL , expressed as a continuously compounded annual rate. We also report the continuously compounded rates of the US zero-coupon yield curve (the daily rate is obtained by $r_f = r/250$). Note that it is not exactly a “daily” rate but the time step is a “trading day”.

In Table 1 we report the values for r_f such that the respective market values of the contracts are approximately equal to the values obtained by Monte Carlo simulations. In Table 1, we compare the results for the “implied risk free rates” corresponding to the two contracts with the US yield curve. For example, at the end of December 2004, the time to maturity of the

contract is about 1754 days (with an approximation of 250 trading days per year for the period after November 2009). The US yield curve at the end of December 2004 tells us that for a zero-coupon bond of such maturity, the interest rate was 4.05% per annum expressed as a continuously compounded rate. A higher interest rate makes the contract less valuable. Note that the differences between the US yield curve and the implied risk-free rate are small. In addition these contracts are subject to default risk, therefore there could be a risk premium embedded in the interest rate used for discounting the future cash-flows that we neglect.

We are now confident that the GARCH(1,1) model is able to reproduce accurately the market prices. The risk-free rate used in the pricing could either be obtained as an implied interest rate or directly from the US yield curve. At any time t from 2004 to November 2007, given a maturity in days and the pricing date, we are able to get r the continuously compounded annual interest rate from the US zero-coupon yield curve, and therefore the corresponding daily rate $r_f = r/250$.

Note that prices at issue are hard to reproduce and to fit because they include commissions. On purpose, we choose to evaluate the contracts at several dates posterior to the issuance date by several months.

4.3 Contract *III*

We now apply the four steps that we described in Section 3 to study the contract *III* in details.

Step 1: Calibration of the GARCH(1,1) processes.

We first fit a GARCH(1,1) over the entire period from July 2006 to November 2009 on the daily log-returns of S_1 (*Dow Jones EURO STOXX 50SM*), S_2 (*S&P500* index), and S_3 (*NIKKEI 225* index). We then split the period into 3 subperiods of 290 trading days (from March 2006 to May 2007, from May 2007 to July 2008, and from August 2008 to November 2009) and fit a GARCH(1,1) for each subperiod.

In Table 2, the estimates of the GARCH(1,1) are reported. The first column corresponds to the full period of observations (March 2006 to November 2009). The three other columns correspond to the three subperiods previously described. We note that in all cases, $\alpha_i + \beta_i$ is close to 1 but strictly less than 1, as it should be. Note also that in the second subperiod, the daily log-returns are on average negative or very close to 0, because it includes the recent financial crisis.

	Full sample	period 1	period 2	period 3
$\hat{\mu}_1$	0.000350	0.000907	-0.000494	0.000743
$\hat{\omega}_1$	3.76e-06	9.88e-06	1.027e-05	7.57e-06
$\hat{\alpha}_1$	0.1343	0.1598	0.1482	0.1062
$\hat{\beta}_1$	0.8575	0.7275	0.8063	0.8854
$\hat{\mu}_2$	0.000414	0.000664	-0.000513	0.000593
$\hat{\omega}_2$	1.85e-06	2.72e-06	8.95e-06	5.42e-06
$\hat{\alpha}_2$	0.0932	0.0338	0.0513	0.119
$\hat{\beta}_2$	0.900	0.903	0.899	0.876
$\hat{\mu}_3$	0.000107	0.000525	-0.000594	0.0000213
$\hat{\omega}_3$	4.63e-06	4.75e-06	6.09e-06	1.83e-05
$\hat{\alpha}_3$	0.127	0.0643	0.142	0.197
$\hat{\beta}_3$	0.863	0.896	0.851	0.782
$\bar{\sigma}_{1,t}\sqrt{250}$	24.8%	14.5%	21.2%	38.7%
$\bar{\sigma}_{2,t}\sqrt{250}$	23.2%	10.3%	20.6%	38.8%
$\bar{\sigma}_{3,t}\sqrt{250}$	27.4%	17.5%	25.8%	39.1%

Table 2: Estimated parameters of GARCH(1,1) for S_1 (*STOXX50*), S_2 (*S&P500*) and S_3 (*NIK225*). $\bar{\sigma}_{i,t}$ denotes the average of the daily volatilities over the period under study (July 2006 to November 2009).

In addition to the parameters of the GARCH process, we also give the average daily volatilities for the full sample and for each subperiods. It is calculated as

$$\bar{\sigma}_{i,t} := \frac{1}{K} \sum_{s=t}^{t+K} \hat{\sigma}_{i,s}$$

multiplied by $\sqrt{250}$ to obtain an annual volatility (rather than a daily volatility), and where K denotes the number of days in the period of observation. It is striking how the volatility changes throughout the three periods. The dependence is also changing drastically such as it is described in the second step.

Step 2: Dependence under P .

We first investigated the dependence between each pair out of $\{S_1, S_2, S_3\}$. The strongest dependence appears to be between S_1 (European index) and S_2 (US Index), then between S_1 (European index) and S_3 (Asian index). This can be seen from the values of Kendall's tau reported in Table 3.

Table 3 also shows that Kendall's tau depends on the period. We then study the tail dependence by looking at the empirical contour plots for each

period (Figure 2, Panel A, B and C). The empirical contours are compared with theoretical contours. All parameter estimates to produce the graphs in Panels A, B and C of Figure 2 are obtained by maximum likelihood estimation.

	$S_1 - S_2$	$S_1 - S_3$	$S_2 - S_3$
Full	0.404	0.202	0.079
Period 1	0.314	0.197	0.104
Period 2	0.384	0.239	0.075
Period 3	0.495	0.181	0.062

Table 3: Overall dependence measured by Kendall’s tau for the full sample and then for each of the 3 periods.

Insert Figure 2 here.

From these contour plots, it seems that the dependence between S_1 and S_2 could be best modeled by a Clayton copula for period 2 and period 3. For the first period the Gauss copula seems more appropriate. The Clayton copula captures the tail dependence for the losses that occurred in 2008 and in 2009. During the first period (prior the crisis), one observes very few losses and almost no tail dependence for the losses. It shows that the Gauss copula is not appropriate to model periods of stress. The dependence between the US index S_2 and the Asian index S_3 is weak and shows less tail dependence. In the pair-copula construction, we do not need to model it directly: we need to study the conditional dependence after conditioning with respect to S_1 .

We compute the copula $C_{23|1}$ between the conditional distributions of S_2 given S_1 and S_3 given S_1 as follows. We first transform the observations into conditional observations by using the transformation given by (11) and (12). Assuming a Gauss copula (for the first period) and a Clayton copula (for the second and third period) between S_1 and S_2 , we calculate the conditional observations by (12) where the functions h are given in Appendix A by (21) (in the case of the Clayton distribution), by (23) (in the case of the Gumbel distribution), and by (19) (in the case of the Gaussian distribution). Note that the order of the variables is important and we are now conditioning with respect to S_1 instead of S_2 as it was done in (11) and (12).

Insert Figure 3 here.

Figure 3 shows that for each subperiod the dependence between the conditional distributions is very weak, it could even be slightly negative. Note that the Gumbel and Clayton copulas cannot be used to model negative dependence. S_2 and S_3 were already weakly dependent (as could be seen from Figure 2). Conditionally to S_1 , they are slightly negatively dependent and almost independent before the crisis. This last dependence is modeled with a Gauss copula.

The choice of copula is based on the comparison between empirical contour plots and theoretical contour plots as well as using the p -values of Cramér-von-Mises goodness of fit test (Appendix B). See the p -values reported directly on Panels A, B and C of Figure 2 and on Figure 3.

Step 3 and 4: Pricing the basket option

For the purpose of illustration, the pricing of the contract *IIL* is done on the 4th of May 2007 (about 200 days after issuance and when it is quoted at 10.60). Figure 4 displays the results.

To understand the impact of the choice of the copula and its parameters on the price of a basket option similar to the contracts *MIB* and *IIL* described previously, we run Monte Carlo simulations. Most of the dependence between the three indices comes from the dependence between S_1 and S_2 . Therefore we study the impact of the choice of the family for C_{12}^Q and of its parameter. We compare three cases, when the copula for the dependence between S_1 and S_2 is Clayton, when it is Gumbel and when it is Gaussian. To compare the three copulas, we adjust their respective parameters such that Kendall's tau between the standardized innovations of S_1 and of S_2 are equal. We use the bijection between τ and the parameter of the Gaussian, Gumbel and Clayton given respectively in Appendix A in (18), (22) and (20).

Parameters for the GARCH(1,1) are given by the ones reported in Table 1. Panel A corresponds to the second column, Panel B corresponds to the third column and Panel C corresponds to the fourth column. The maturity is 1,005 trading days and the interest rate $r_f = 6\%$.

For the dependence between S_1 and S_3 (respectively between $S_2|S_1$ and $S_3|S_1$), we assume a Clayton copula (respectively a Gaussian copula). We use the parameters for the copulas c_{13} and $c_{23|1}$ that we fitted in Step 2.

Insert Figure 4 here.

For the ease of comparison, the scale is the same for the three panels of Figure 4. It is then clear that the choice of the GARCH(1,1) parameters has

a more significant impact on the pricing of this option than the choice of the copula. Table 1 reports the parameters for the GARCH(1,1) process used in the different panels. Panel A displays a “calm” period where the volatility was quite small (10% to 17% for the three indices), Panel B displays the pricing in a crisis time where the volatility is higher (20 to 26%) but the expected daily log-return is negative. Finally Panel C reports the results when the volatility is very high and expected returns are positive. Thus it appears that the volatility plays an important role in the pricing of basket options, more important than the choice of the copula.

Note that the Clayton and Gumbel dependences tend to give higher prices for the basket option than the Gaussian dependence. We also observe that the sensitivity of the price with respect to Kendall’s tau could be very different with different assumptions on the dependence structure.

The paper presents a methodology and approach to price multivariate derivatives with dependent GARCH(1,1) processes. To draw firm conclusions more data are needed. In addition the contracts *MIB* and *IIL* are retail investment products. The secondary market for these markets has been criticized for being not liquid and for which issuers “choose” market prices. This may explain why these contract may appear underpriced in Figure 4. Another factor that we neglect is the presence of credit risk which would significantly decrease the value of the guarantee in the product and lower its price.

5 Conclusion

In a dynamic copula setting, it is not clear why the dependence under the objective measure (in the actual world) should be the same as the dependence under the risk-neutral measure. We describe the steps to price a multivariate derivative in this setting and illustrate the study with a dataset of multivariate derivatives prices sold in the US. It is hard to draw firm conclusions from the only example of this paper. It provides an illustration of how the pair copula construction methodology can be applied to model dependency and price multivariate derivatives. It also shows that the main risk in basket options may not be the dependence structure but the modelling of volatility as well as shifts of regimes. Our empirical analysis indeed highlights different periods: regime switching models may be more appropriate for pricing long-term derivatives as the ones studied in the paper.

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A Appendix: Copulae

Let τ denote Kendall's tau. We give the formulas for each parametric copula studied in the paper as well as how to get the parameter from an estimation of Kendall's tau. Proofs and more details on each class of copulas can be found in Aas et al. [2009].

A.1 Gaussian copula

Given a correlation parameter $\rho \in (-1, 1)$, denote by Φ , the cdf of $N(0, 1)$ and Φ_ρ^2 the cdf of a bivariate normal distribution. With these definitions the Gaussian copula cdf can be expressed as:

$$C(u, v, \rho) = \Phi_{2,\rho}(\Phi^{-1}(u), \Phi^{-1}(v)),$$

where $\Phi_{2,\rho}$ denotes the cdf of a standard bivariate distribution with correlation coefficient ρ . For the Gaussian copula, $\theta_P := \rho$, and it is equal to

$$\rho = \frac{2}{\pi} \arcsin(\tau). \quad (18)$$

The h function needed to simulate conditional observations (see (12)) is given by

$$h(u, v, \rho) = \frac{\partial}{\partial v} C(u, v, \rho) = \Phi \left(\frac{\Phi^{-1}(u) - \rho \Phi^{-1}(v)}{\sqrt{1 - \rho^2}} \right). \quad (19)$$

A.2 Clayton copula

The Clayton copula cdf is given by

$$C(u, v, \delta) = (u^{-\delta} + v^{-\delta} - 1)^{-\frac{1}{\delta}},$$

where $\delta \in (-1, 0) \cup (0, +\infty)$. The parameter of the Clayton copula δ is obtained from Kendall's tau τ as

$$\delta = \frac{2\tau}{1 - \tau}. \quad (20)$$

$$h(u, v, \delta) = \frac{\partial}{\partial v} C(u, v, \delta) = \frac{1}{v^{1+\delta}} (u^{-\delta} + v^{-\delta} - 1)^{-1-1/\delta} \quad (21)$$

A.3 Gumbel copula

For $1 \leq \lambda < \infty$,

$$C(u, v, \lambda) = \exp \left(- \left((-\ln(u))^\lambda + (-\ln(v))^\lambda \right)^{1/\lambda} \right)$$

Then the parameter $\lambda \geq 1$ verifies

$$\lambda = 1/(1 - \tau). \quad (22)$$

Furthermore $h(u, v, \lambda) = \frac{\partial}{\partial v} C(u, v, \lambda)$ can be expressed as

$$h(u, v, \lambda) = C(u, v, \lambda) \frac{(-\ln(v))^{\lambda-1}}{v} \left[(-\ln(u))^\lambda + (-\ln(v))^\lambda \right]^{\frac{1}{\lambda}-1} \quad (23)$$

B Cramér-von-Mises test

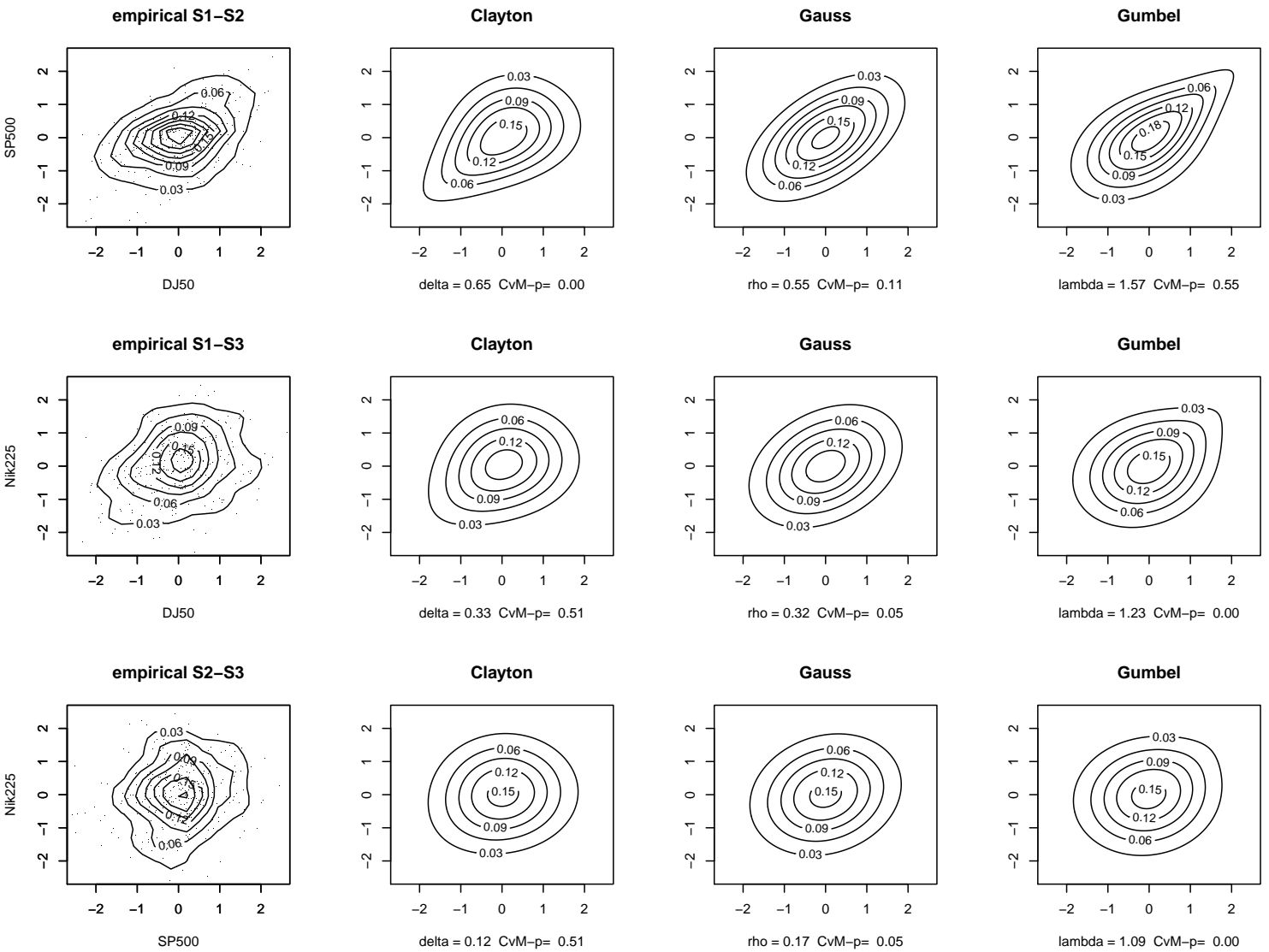
In copula goodness-of-fit testing we are interested in the copula alone and therefore do not want to make any assumptions with respect to the marginal distributions. Hence, a sensible approach is to base goodness-of-fit tests on ranks and pseudo-observations (the unknown marginal distribution functions F_j are replaced by their empirical versions $\hat{F}_j(t)$). The so-called *pseudo-observations* are defined as $U_{ij} = \frac{n}{n+1} \hat{F}_j(X_{ij})$, for all $i = 1, \dots, n$ and $j = 1, \dots, d$ (where n is the number of observations and d the dimension). The scaling factor $\frac{n}{n+1}$ is used to avoid numerical problems in the boundaries of $[0, 1]^d$. The information contained in pseudo-observations $\mathbf{U}_1, \dots, \mathbf{U}_n$ is naturally summarized in the corresponding *empirical copula* (empirical distribution of the observed sample introduced by Deheuvels [1979]) $C_n(u_1, \dots, u_d) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{U_{i1} \leq u_1, \dots, U_{id} \leq u_d\}}$, where $u_1, \dots, u_d \in [0, 1]$. Monte Carlo studies in Berg [2009] and Genest, Remillard and Beaudoin [2009] show that the following test based on the Cramér-von Mises test statistic

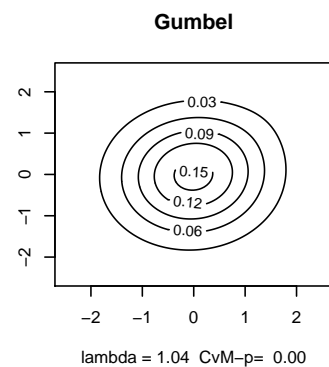
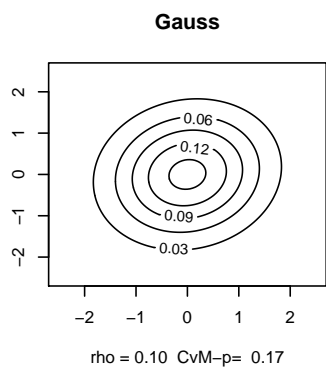
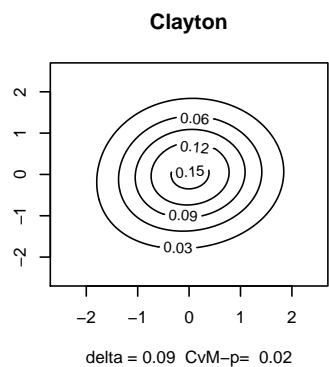
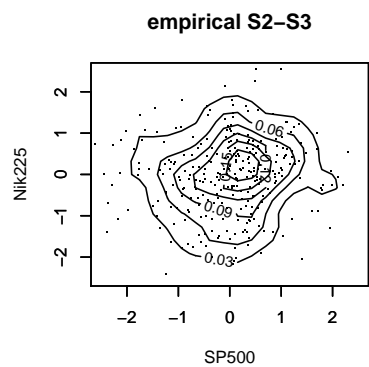
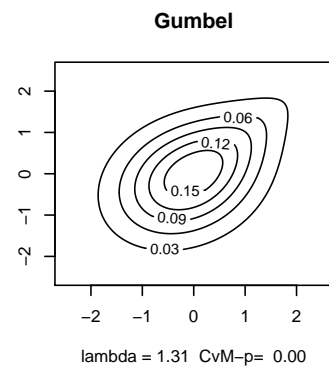
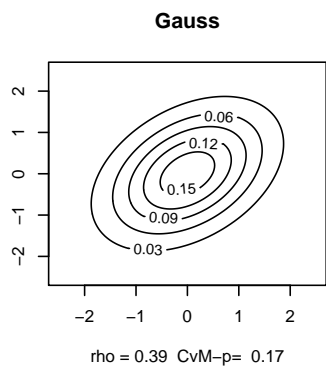
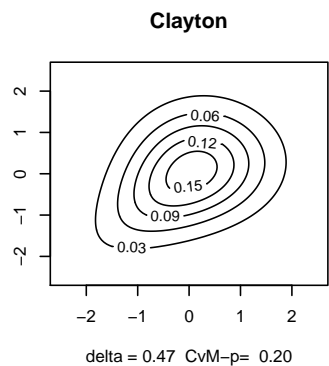
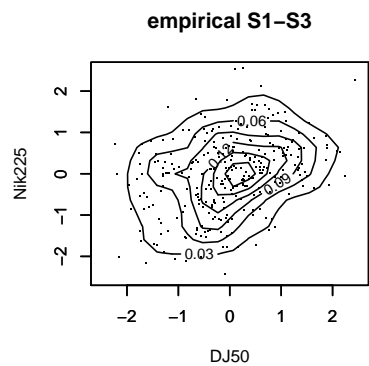
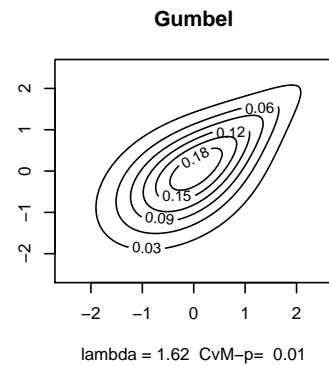
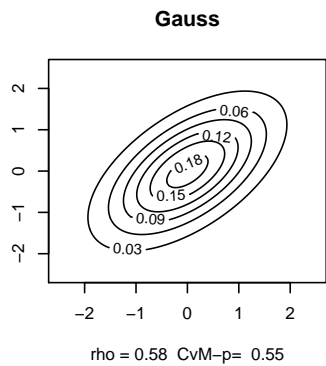
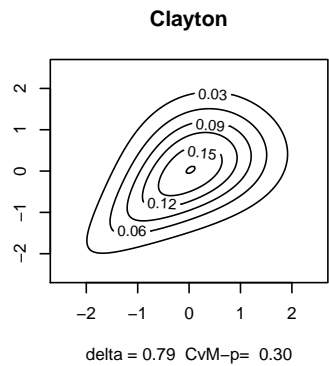
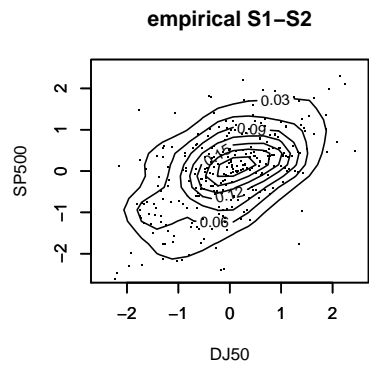
$$S_n = n \int_{[0,1]^d} [C_n(\mathbf{u}) - C_{\theta_n}(\mathbf{u})]^2 dC_n(\mathbf{u}) = \sum_{i=1}^n [C_n(\mathbf{U}_i) - C_{\theta_n}(\mathbf{U}_i)]^2 \quad (24)$$

performs very well. The test can be performed in arbitrary dimensions although the computational complexity increases quickly. However we only need the test in two dimensions thanks to the pair-copula construction. The limiting distribution of S_n as defined in (24) is unknown in practice and depends on the hypothesized copula. P-values therefore have to be calculated using a parametric bootstrap procedure as described in Genest et al. [2009].

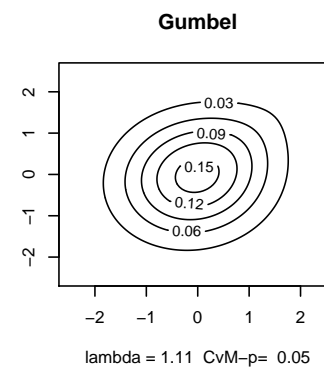
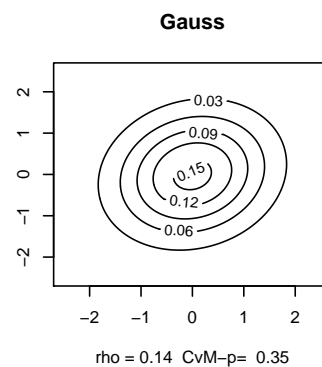
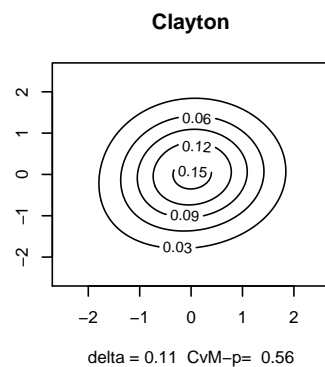
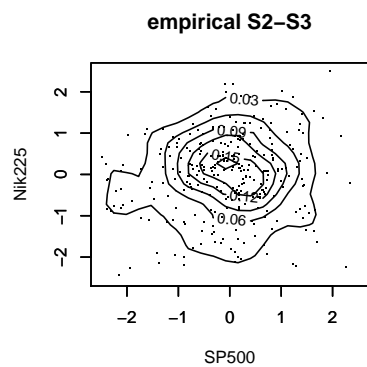
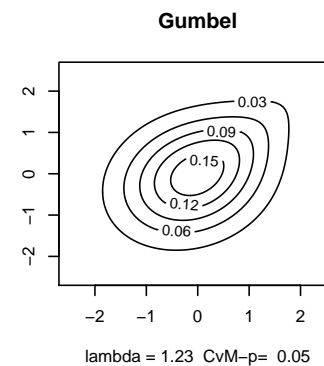
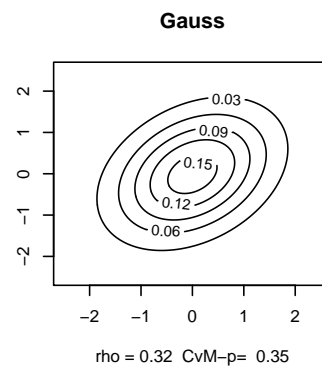
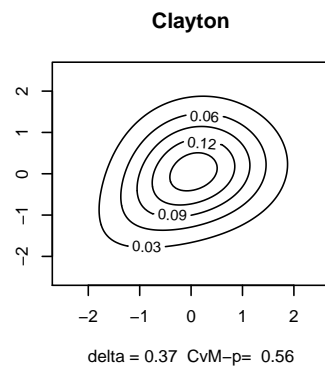
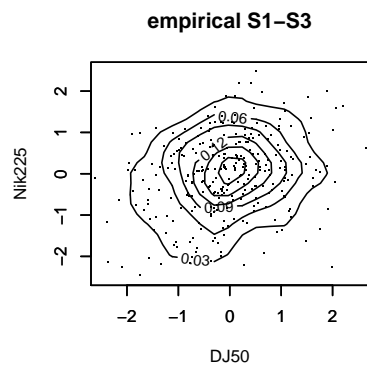
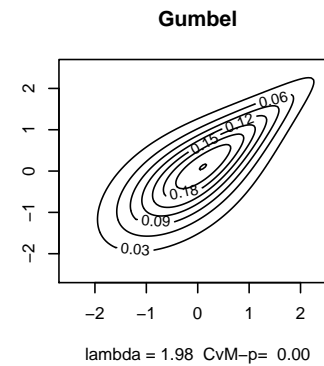
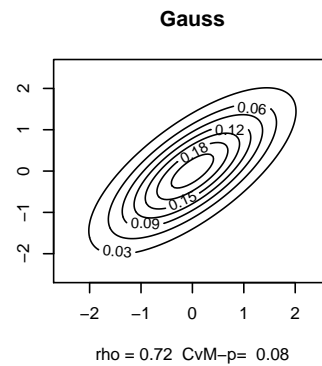
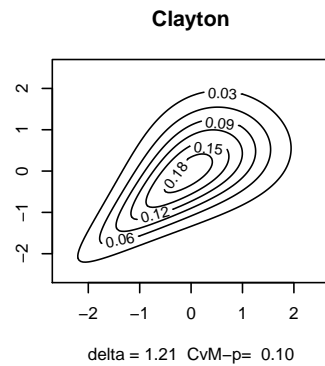
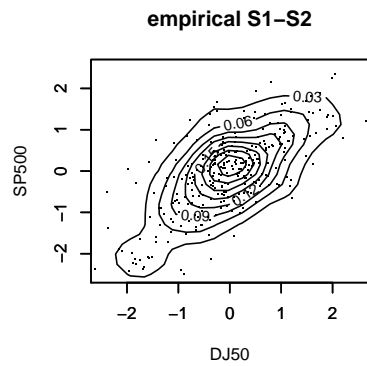
Figure 2: Scatterplot and normalized contour plots of pairs of standardized innovations for each pair [First row: empirical, 2nd-4th column: fitted Clayton, Gaussian and Gumbel copula contours]. $CvM - p$ is the p -value of the Gramér Von Mises Goodness of fit test (see appendix B). Panel A,B,C correspond to 3 sub-periods respectively from March 2006 to May 2007, from May 2007 to July 2008, and from August 2008 to November 2009. Finally “delta” is Clayton’s parameter δ in A.2 and “lambda” is Gumbel’s parameter λ in A.3.

Panel A: First Period



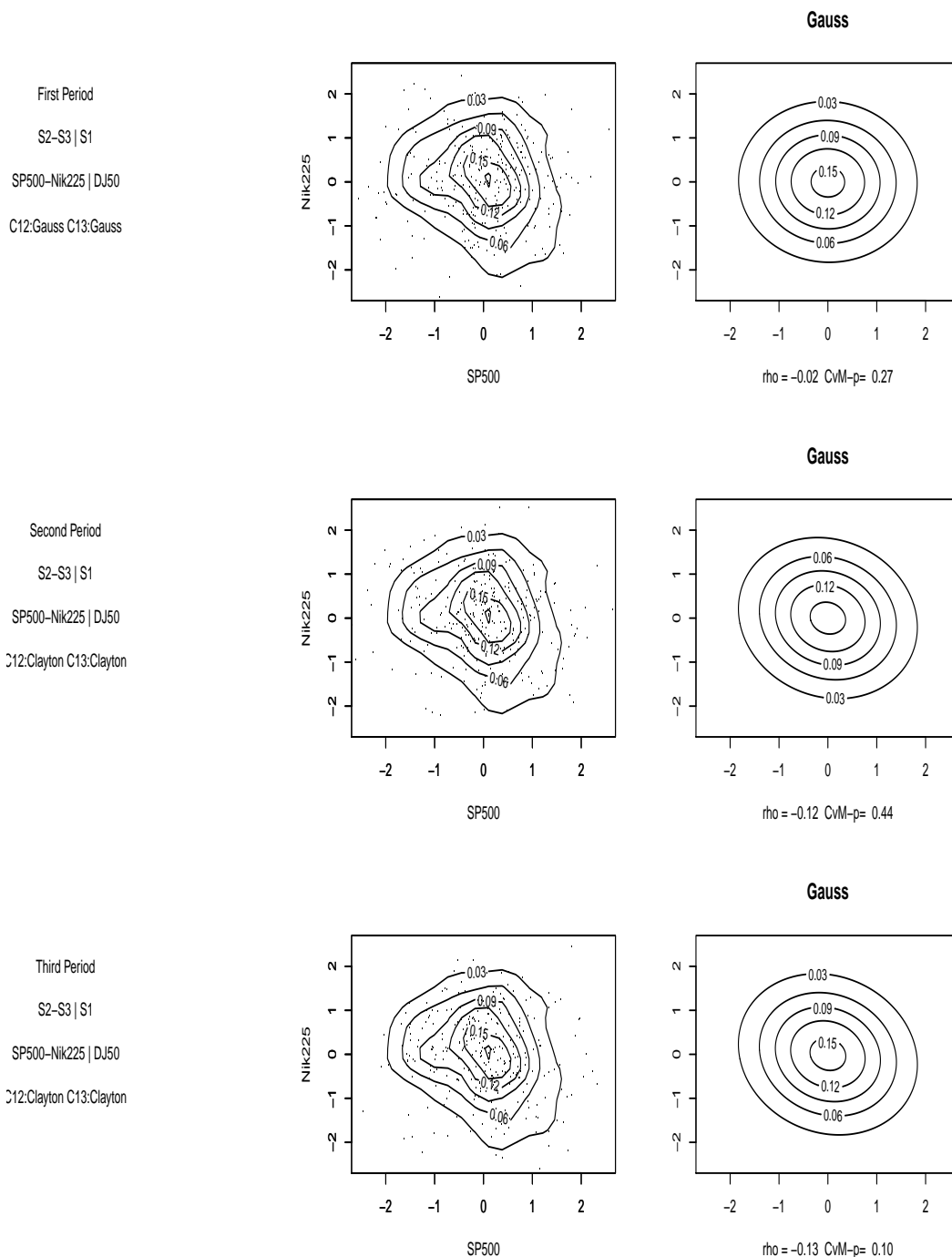


Panel B: Second Period



Panel C: Third Period

Figure 3: Scatterplot of the standardized innovations for the conditional standardized innovations of S_2 and S_3 conditional to S_1 . Each row corresponds to one period. $CvM - p$ denotes the p -value of the Cramér-Von-Mises Goodness-of-fit test for the Gaussian copula (see appendix B). The dependence is negative, Clayton and Gumbel are not appropriate. Rotated Gumbel or Clayton could be used but the Gaussian copula is not rejected.



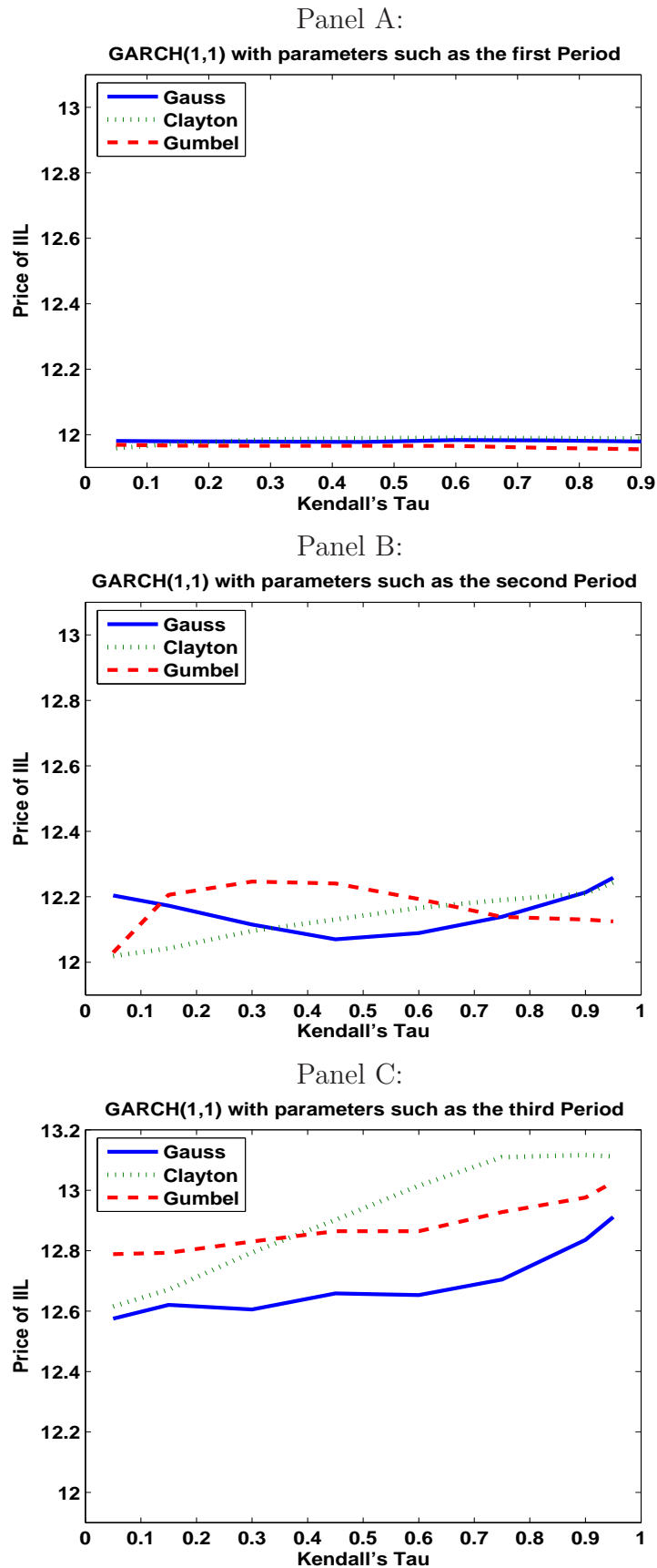


Figure 4: Price for the contract *ILL* on May 4th, 2007 as a function of Kendall's Tau. Each point is obtained with 50,000 Monte Carlo simulations. The market price quoted on that day was 10.9. Parameters of the GARCH(1,1) processes are given in Table 1