A fluid cluster Poisson input process can look like a fractional Brownian motion even in the slow growth aggregation regime

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Abstract

We show that, contrary to the common wisdom, the cumulative input process in a fluid queue with cluster Poisson arrivals can converge, in the slow growth regime, to a fractional Brownian motion, and not to a Lévy stable motion. This emphasizes the lack of robustness of Lévy stable motions as "birds-eye" descriptions of the traffic in communication networks.

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1 Introduction

This paper concerns the asymptotic behavior of certain fluid random streams of the type that have often been taken as natural models of input to fluid queues and queuing networks. We are specifically interested in the effect of heavy tails on such asymptotic behavior. We are not considering the actual queues in this paper, in the sense that we only investigate the potential input process to a queue, and not what happens when the service starts. However, our task, which lies in understanding how the input process deviates from its completely regular and linear average behavior, will help understanding how an actual queue with such input behaves. Indeed, it is precisely the deviations from the average behavior that build the queue!

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The motivation for our interest in deviations of input processes from their average (as well as the motivation in the many other papers on this subject) lies in the fact that networks with heavy tailed inputs are difficult to analyze, since they are not well suited to Brownian or Poisson approximations. Nonetheless, it is believed that heavy tails cause unusual (and often negative) effects. For example, it is believed that infinite variance in the distribution of the file sizes or bandwidth requests in communication networks causes long range dependence and self-similar structure in the network (see e.g. Park and Willinger (2000)).

Since queues with heavy tailed input are difficult to analyze directly, the hope has been to get insight into their behavior by approximating the input by something standard, specifically by the average, linear, stream plus a certain deviation from that average. In the influential paper of Mikosch et al. (2002) they showed that for the so-called ON/OFF model and the infinite source Poisson model, the properly compensated and normalized cumulative input in a fluid queue looks like a fractional Brownian motion in the *fast growth regime* and like a Lévy stable motion in the *slow growth regime*. This result was later extended to networks of fluid queues by D'Auria and Samorodnitsky (2005). A random field version of such results is in Kaj et al. (2007). The terms *fast growth regime* and *slow growth regime* are important. They refer to the relative magnitude of the time scale at which the input process is considered and the number of independent streams the input consists of. We will return to this important point shortly. Boundary regimes have been discovered as well; see e.g. Gaigalas and Kaj (2003).

These previous results appear to indicate that the deviations from the average in a heavy tailed input process could look, in the limit, as either a fractional Brownian motion or a Lévy stable motion. These are two very different stochastic processes, one with light tails but strongly dependent increments, while the other with independent increments but heavy tails. One expects very different performance in a queue with such different inputs. What was needed, therefore, was a study of robustness of these two possible limits under departures from the very specific model assumptions of Mikosch et al. (2002). Such study was undertaken by Mikosch and Samorodnitsky (2007) in a general setup, described below. Consider a stationary marked point process

$$((\mathcal{T}_m, Z_m))_{m\in\mathbb{Z}},$$

where a possible interpretation in the language of communication systems views $\ldots < \mathcal{T}_{-1} < \mathcal{T}_0 < 0 < \mathcal{T}_1 < \ldots$ as the arrival times of data packets at a server and Z_m as the file size of the data packet transmitted at $\mathcal{T}_m$. Each arrival corresponds to a "*source*", and it transmits its data at a unit rate. Thus, Z_m denotes also the transmission period. The number of active sources at time t is given by the process

$$U(t) = \sum_{m\in\mathbb{Z}} \mathbf{1}_{\{\mathcal{T}_m \le t < \mathcal{T}_m + Z_m\}} \qquad \text{for } t \ge 0, \tag{1.1}$$

and the amount of data transmitted to the server in the interval $[0, t]$ is given by the *input process*

$$A(t) = \int_0^t U(y)\, dy = \sum_{m\in\mathbb{Z}} [Z_m \wedge (t - \mathcal{T}_m) - Z_m \wedge (-\mathcal{T}_m)], \quad t \ge 0, \tag{1.2}$$

which has continuous sample paths and thus, reflects the fluid queue. Under the assumption that the marks (Z_m) have, under the Palm measure, a finite mean, $A(t)$ has a finite mean with $\mathbb{E}(A(t)) = \mu t$ where $\mu > 0$ is the expected amount of data arriving at the server in $[0, 1]$.

Let $(A_i)_{i\in\mathbb{N}}$ be iid copies of the process A. We view each (A_i) as the input process of data generated by the ith "user", with different users having nothing to do with each other, hence the independence assumption. With n such independent input processes and at a time scale M, the deviation of the *cumulative input process* from its mean is the stochastic process

$$D_{n,M}(t) = \sum_{i=1}^{n} (A_i(tM) - \mu tM) \quad \text{for } t \geq 0. \tag{1.3}$$

One is interested in the limits of the sequence of processes $(D_{n,M})$ as n and M grow to infinity. It is here where the idea of "fast growth" and "slow growth" appears.

The terms *fast growth regime* and *slow growth regime* were introduced in Mikosch et al. (2002) for the ON/OFF and infinite source Poisson model, and they describe the relative rates at which n and M in (1.3) grow to infinity. Intuitively, the fast growth regime is the situation where the number n of the input processes is relatively large in comparison to the time scale M, while the slow growth regime means the opposite situation. In fact, the paper Mikosch et al. (2002) used a very specific boundary, $n(M) \uparrow \infty$ as $M \uparrow \infty$ such that the fast growth regime meant $n \gg n(M)$ while the slow growth regime meant $n \ll n(M)$. There are substantial reasons to separate the regimes. Indeed, if the number of sources n is very large, then the process $\big(D_{n,M}(t),\, t \geq 0\big)$ in (1.3) is the sum of a very large number of iid terms that change relatively slowly. If the number of active sources in (1.1) has a finite variance (as is the case in most systems considered in literature), the same would be the case for the input process in (1.2). Then one would expect a Gaussian limit for the deviation from the mean of the cumulative input process as in a classical central limit theorem. On the other hand, if the time scale M is very large, then the main phenomenon in (1.3) is, actually, in the deviations of the individual input processes from their means, $A_i(t) - \mu t$, for large t. *A priori*, there is no reason to expect these latter deviations to be Gaussian-looking, unless very specific assumptions are imposed on the generic input process $\big(A(t),\, t \geq 0\big)$. Those are assumptions that are not of the kind that is usually imposed in the literature on the input to communication systems. Correspondingly, Mikosch et al. (2002) discovered that, in the ON/OFF and infinite source Poisson model, the deviation from the mean of the cumulative input process have a stable limit in the slow growth regime.

We take a related, but somewhat more general point of view on the notions of fast growth regime and slow growth regime, which was already used in Mikosch and Samorodnitsky (2007). Specifically, if there is a function $n(M) \uparrow \infty$ as $M \uparrow \infty$ such that a limit theorem holds when $n \gg n(M)$, we say that this limit theorem holds in the fast growth regime because the main effect in (1.3) is the averaging over the many independent input streams. In fact, this regime typically allows an *iterated* limiting procedure: first let $n \to \infty$ and then let $M \to \infty$.

Similarly, if there is a function $n(M) \uparrow \infty$ as $M \uparrow \infty$ such that a limit theorem holds when $n \ll n(M)$, we say that this limit theorem holds in the slow growth regime because the main effect in (1.3) are the deviations of the individual input processes from their means, and the limit will typically hold if we let first $M \to \infty$ and then $n \to \infty$.

It has become a part of the folklore that in the former scenario a fractional Gaussian limit is likely to arise, while in the latter scenario a Lévy stable limit can be expected. What Mikosch and Samorodnitsky (2007) discovered was that the fractional Brownian limits of Mikosch et al. (2002) in the fast growth regime were very robust, and held under very general assumptions on the underlying stationary marked point process. On the other hand, the Lévy stable limits turned out to be non-robust, and very special conditions were needed to ensure such limits. One of the conclusions of Mikosch and Samorodnitsky (2007) was, in certain circumstances of a very irregular arrival process, a fractional Brownian limit was possible even in the slow growth regime. They provided a somewhat artificial example of such situation, and conjectured that the same was true in the important case of a cluster Poisson arrival process. It is the purpose of this paper to consider that case and establish the fractional Brownian limit. Once this is accomplished, we understand that the appearance of a fractional Brownian limit in the slow growth regime is not exotic but, in fact, can be possible under very natural and common assumptions. This emphasizes how robust the fractional Brownian motion limiting behavior is. In a related work, a reflected version of the fractional Brownian limit was established in Delgado (2007) for the workload in fluid queuing networks in a heavy traffic regime.

We would like to mention, at this point, that for certain input point processes, changing the number n of independent input streams as above is equivalent to changing the intensity λ_0 of an underlying Poisson process. This is true for the $M/G/\infty$ model of Mikosch et al. (2002), and it is also true for the model considered in the present paper. For such point processes, it is possible (and natural) to distinguish between different situations according to the relative rates at which the Poisson intensity and the time scale grow to infinity. Accordingly, if there exists a function $\lambda(M) \uparrow \infty$ such that a limit theorem holds if $\lambda_0 \gg \lambda(M)$, one will say that the limit holds in the fast growth regime, and if a limit theorem holds under the assumption $\lambda_0 \ll \lambda(M)$, one will say that the limit holds in the slow growth regime. In this paper we will use, nonetheless a description via a discrete number n of independent input streams, because this is a language in which a key result of Mikosch and Samorodnitsky (2007), which we use in this paper, is stated (see Theorem 3.3 below). We will use, correspondingly, the fast and slow growth regime terminology introduced above, that compares the rates of growth of the number n and time scale M.

This paper is arranged as follows. The arrival cluster Poisson model we are working with is formally described in Section 2. The main result of the paper is stated and discussed in Section 3. The arguments required to prove the main result uses a number of renewal theoretical and extreme value results, some of which may be of independent interest. These appear in Section 4. Section 5 presents the proof of the main theorem. Finally, Section 6 contains additional lemmas and other technical results needed for the proof of the main theorem.

2 The cluster Poisson model

We assume that the data files sizes $(Z_m)_{m\in\mathbb{Z}}$ form an iid sequence independent of the arrival process $(\mathcal{T}_m)_{m\in\mathbb{Z}}$. Let the number of sources arriving at the server in the interval $(s,t]$ be described by

$$N(s,t] = \sum_{m\in\mathbb{Z}} \mathbf{1}_{\{s<\mathcal{T}_m\le t\}} \qquad \text{for } s<t.$$

Furthermore, we assume that this arrival point process is a *cluster Poisson process*. Specifically:

(i) initial cluster points, denoted by $\ldots < \Gamma_{-1} < 0 < \Gamma_1 < \Gamma_2 < \ldots$ form a homogeneous Poisson process $\widetilde{N}$ with rate λ_0;

(ii) at each initial cluster center Γ_m an independent copy of a randomly stopped renewal point process N_c starts.

A generic point process N_c has the form

$$N_c[0,t] = N_0[0,t] \wedge (K+1),$$

where N_0 is a renewal point process with arrival times $0 = T_0 < T_1 < \ldots$, and K is a nonnegative integer valued random variable independent of N_0. The interarrival times $X_k = T_k - T_{k-1}$ for $k \ge 1$ are iid random variables, with a common distribution F, and the cluster size K has distribution F_K. The cluster with the initial point Γ_m will have the points $\Gamma_{m,k} = \Gamma_m + T_{k,m}$ for $k = 0,\ldots,K_m$ where $(T_{k,m})_{k\in\mathbb{N}_0}$ are independent copies of $(T_k)_{k\in\mathbb{N}_0}$, independent of $(\Gamma_m)_{m\in\mathbb{Z}}$.

The within-cluster interarrival times and the cluster sizes are assumed to satisfy the following conditions.

Assumption A

(a) The interarrival distribution function satisfies

$$\overline{F} \in \mathcal{R}_{-1/\beta} \qquad \text{with } \beta > 1.$$

(b) The cluster size distribution function satisfies

$$\overline{F}_K \in \mathcal{R}_{-\alpha} \qquad \text{with } 1 < \alpha < \min(2,\beta).$$

(c) The marks (Z_m) form a sequence of iid random variables, independent of the underlying point process. Further, we will assume that $\mathbb{E}|Z_m|^2 < \infty$.

Notice that Assumption A*(a)* assures that the within-cluster interarrival times have infinite mean; it also makes the arrival process sufficiently irregular for our result. The Assumption A*(b)* makes sure that the data files transmitted within each cluster have infinite variance. Note that the intensity of N is

$$\lambda = \lambda_0(1+\mathbb{E}(K)). \tag{2.1}$$

For our main result, Theorem 3.1 below, we will introduce an additional assumption on the interarrival distribution function F, as follows.

Assumption B

Assume that either

1. $\beta < 2$ *and*
$$\limsup_{x\to\infty} x\frac{\overline{F}(x)-\overline{F}(x+1)}{\overline{F}(x)} < \infty, \tag{2.2}$$
or

2. F *is arithmetic, with step size* $\Delta > 0$, *and*
$$\limsup_{n\geq 0} n\frac{F\big(\{n\Delta\}\big)}{\overline{F}(n\Delta)} < \infty\,. \tag{2.3}$$

Remark 2.1 We need the technical Assumption B above to obtain a local renewal theorem; see Lemma 4.3 below or Theorem 3 in Doney (1997). In fact, if the local renewal theorem is known to hold (if only in the form of an upper bound), then Assumption B is unnecessary. We conjecture that the local renewal theorem holds under (2.2) for any $\beta > 1$, regardless of whether or not F is arithmetic. □

We denote by
$$h(u) = \overline{F}^{\leftarrow}(1/u) = u^{\beta} l(u) \quad \text{for } u > 1 \tag{2.4}$$
the generalized tail inverse function of the within-cluster interarrival time distribution (see Resnick (2006), Section 2.1.2). Here, l is a slowly varying function. One implication of Assumption A*(a)* is the weak convergence
$$\big(T_{\lfloor nt\rfloor}/h(n)\big)_{t\geq 0} \Longrightarrow (S_{1/\beta}(t))_{t\geq 0} \tag{2.5}$$
in $\mathcal{D}\,[0,\infty)$ as $n\to\infty$; see Kallenberg (2002), Theorem 16.14. Here $(S_{1/\beta}(t))_{t\geq 0}$ is an $1/\beta$-stable subordinator. We will use the notation
$$I(u) = \inf\{t\geq 0 : S_{1/\beta}(t) > u\} \quad \text{for } \ u > 0, \tag{2.6}$$
for its inverse process. Then the weak convergence
$$\big(\overline{F}(r)N_0\,(0,ru]\big)_{u\geq 0} \Longrightarrow (I(u))_{u\geq 0} \tag{2.7}$$
in $\mathcal{D}\,[0,\infty)$ as $r\to\infty$ holds. In particular, the process $(I(u))_{u\geq 0}$ is self-similar of index $1/\beta$; cf. Meerschaert and Scheffler (2004).

We will continue using the notation $\Longrightarrow$ for weak convergence, $\stackrel{\mathbb{P}}{\longrightarrow}$ for convergence in probability, $\stackrel{\nu}{\Longrightarrow}$ for vague convergence, and $\stackrel{\text{fidi}}{\Longrightarrow}$ for weak convergence of the finite dimensional distributions. For $x\in\mathbb{R}$ we write $x_+ = \max(0,x)$. For two random variables X, Y the symbol $X \stackrel{d}{=} Y$ means that X has the same distribution as Y.

We will also adopt the following convention. We will use the notation $\alpha_1, \alpha_2, \beta_1$ and β_2 for positive numbers satisfying $\alpha_1 < \alpha < \alpha_2$ and $\beta_1 < \beta < \beta_2$, in the sense that the statements in the text where this notation appears hold for any choice of numbers satisfying the above conditions with, perhaps, different multiplicative constants.

3 The Main Result

Below is the main result of this paper. It describes a slow growth regime under which the properly normalized deviations from the mean process (1.3) converge to a fractional Brownian motion. For a positive sequence $M_n \uparrow \infty$ serving as the time scale for a system with n input processes we define

$$b_n = \sqrt{nM_n\overline{F}(M_n)^{-2}\mathbb{P}(K > \overline{F}(M_n)^{-1})} \quad \text{for } n \geq 1\,. \tag{3.1}$$

The sequence $(b_n)_{n\in\mathbb{N}}$ turns out to be the right normalization for process (1.3).

Theorem 3.1 *Let the Poisson cluster model satisfy Assumption A and B. Further, let M_n be a sequence of positive constants such that $M_n \uparrow \infty$ and such that b_n in (3.1) satisfies*

$$\lim_{n\to\infty} n b_n^{-\frac{\alpha-1}{\beta}+\rho} = 0 \tag{3.2}$$

for some $\rho > 0$. Then the cumulative input process $S_n(t) = b_n^{-1}D_{n,M_n}(t)$, $n \geq 1$, $t \geq 0$, satisfies

$$(S_n(t))_{t\geq 0} \overset{fidi}{\Longrightarrow} (\mathbb{E}(Z)B_H(t))_{t\geq 0} \quad \text{as } n \to \infty,$$

and the limiting process B_H is a fractional Brownian motion with

$$H = \frac{2+\beta-\alpha}{2\beta} \in (0.5, 1) \tag{3.3}$$

and

$$\begin{aligned} \mathrm{Var}(B_H(1)) = & \frac{2\lambda_0}{2+\beta-\alpha}\int_0^\infty y^{-(2+\beta-\alpha)/\beta}\mathbb{P}(S_{1/\beta}(1) \leq y)\,dy+ \\ & +\lambda_0\int_0^\infty \mathbb{E}\left(\frac{2}{2-\alpha}I(w+1)^{2-\alpha} + \frac{2}{\alpha-1}I(w)I(w+1)^{1-\alpha} - \frac{2}{(2-\alpha)(\alpha-1)}I(w)^{2-\alpha}\right)dw \end{aligned}$$

where $(S_{1/\beta}(t))_{t\geq 0}$ and $(I(w))_{w\geq 0}$ are as in (2.5) and (2.6), respectively.

Remark 3.2 Note for any $\epsilon > 0$ there exist $C > 1$ such that

$$C^{-1}n^{\frac{1}{2}}M_n^{H-\epsilon} \leq b_n \leq Cn^{\frac{1}{2}}M_n^{H+\epsilon} \quad \text{for } n \geq 1\,,$$

with H given by (3.3). Hence, a necessary and sufficient condition for (3.2) is that for some $\epsilon > 0$

$$M_n \gg n^{\frac{2\beta-\alpha+1}{2H(\alpha-1)}+\epsilon}.$$

This identifies (3.2) as a slow growth condition and explains the appearance of this term in the title of the paper. □

We will prove Theorem 3.1 by showing that the assumptions of Theorem 5.9 in Mikosch and Samorodnitsky (2007) are satisfied. For convenience, we state that theorem below, in a form simplified for the situation where the marks are independent of the arrival process.

Theorem 3.3 (Mikosch and Samorodnitsky (2007)) *Consider a marked stationary point process, where the marks (Z_m) are independent of the arrival process N (whose intensity is λ), and have a finite first moment. Let M_n be a sequence of positive constants with $M_n \uparrow \infty$. Suppose that there exists a sequence $b_n \uparrow \infty$ such that the following conditions are satisfied:*

(a) Let N_i be iid copies of N. Then

$$\left(b_n^{-1} \sum_{i=1}^{n} (N_i\,(0, M_n t] - \lambda M_n t) \right)_{t \geq 0} \stackrel{\text{fidi}}{\Longrightarrow} (\xi(t))_{t \geq 0},$$

where $(\xi(t))$ is some non-degenerate at zero stochastic process.

(b) Let $(Z_m^{(i)})_{m \in \mathbb{Z}}$ for $i \in \mathbb{N}$ be iid copies of $(Z_m)_{m \in \mathbb{Z}}$. Then

$$b_n^{-1} \sum_{i=1}^{n} \sum_{m=1}^{\lfloor M_n \rfloor} (Z_m^{(i)} - \mathbb{E}(Z)) \stackrel{\mathbb{P}}{\longrightarrow} 0 \quad \text{as } n \to \infty.$$

(c) Let $I_i^*(0)$ be the total amount of data, of the ith input process, in the session arriving by time 0 which are not finished by that time. Then

$$b_n^{-1} \sum_{i=1}^{n} I_i^*(0) \stackrel{\mathbb{P}}{\longrightarrow} 0 \quad \text{as } n \to \infty.$$

Under these conditions the normalized process $S_n(t) = b_n^{-1} D_{n,M_n}(t)$, $n \in \mathbb{N}$, $t \geq 0$, satisfies

$$(S_n(t))_{t \geq 0} \stackrel{\text{fidi}}{\Longrightarrow} (\mathbb{E}(Z)\xi(t))_{t \geq 0} \quad \text{as } n \to \infty.$$

As we will see, the slow growth condition (3.2) is needed only for verification of condition *(c)* in Theorem 3.3.

4 Some Renewal and Extreme Value Theory

Our first proposition in this section deals with the tails of randomly stopped random sums when both the individual terms and the number of terms have infinite means. It complements the existing results dealing with the situations where at least one of these means is finite; see e.g. Faÿ et al. (2006).

Proposition 4.1 *Let (X_k) be iid random variables independent of the positive integer-valued random variable K with distribution function F_K, and let $T_K = \sum_{k=1}^{K} X_k$ with distribution function F_{T_K}. Let G be the distribution function of $|X_1|$. Assume that*

$$\overline{F}_K \in \mathcal{R}_{-\kappa} \quad \text{for some } 0 < \kappa < 1 \tag{4.1}$$

and

$$\overline{G} \in \mathcal{R}_{-\gamma} \ \text{ for some } 0 < \gamma < 1, \qquad \lim_{x \to \infty} \frac{\mathbb{P}(X_1 > x)}{\overline{G}(x)} = p \in (0, 1]\,. \tag{4.2}$$

Then

$$\lim_{x\to\infty} \frac{\mathbb{P}(T_K > x)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} = \mathbb{E}\big((S_\gamma)_+^{\gamma\kappa}\big)\,, \tag{4.3}$$

where S_γ is a strictly γ-stable random variable such that

$$\mathbb{P}(S_\gamma > x) \sim px^{-\gamma} \quad \text{and} \quad \mathbb{P}(S_\gamma < -x) \sim (1-p)x^{-\gamma} \quad \text{as } x \to \infty.$$

In particular, $\overline{F}_{T_K} \in \mathcal{R}_{-\kappa\gamma}$.

Proof. For $k \geq 1$, let $a_k := \overline{G}^{\leftarrow}(1/k)$, and note that

$$\frac{1}{a_k}\big(X_1 + \ldots + X_k\big) \overset{k\to\infty}{\Longrightarrow} S_\gamma \tag{4.4}$$

(cf. (2.5)). For large $M > 1$ we write

$$\begin{aligned} \mathbb{P}(T_K > x) &= \mathbb{P}\big(T_K > x,\, K > M\overline{G}(x)^{-1}\big) + \mathbb{P}\big(T_K > x,\, K \leq M^{-1}\overline{G}(x)^{-1}\big) \\ &\quad + \mathbb{P}\big(T_K > x,\, M^{-1}\overline{G}(x)^{-1} < K \leq M\overline{G}(x)^{-1}\big) \\ &=: E_{1,M}(x) + E_{2,M}(x) + E_{3,M}(x). \end{aligned} \tag{4.5}$$

Note that, as $x \to \infty$,

$$E_{1,M}(x) \leq \mathbb{P}\big(K > M\overline{G}(x)^{-1}\big) \sim M^{-\kappa}\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)\,,$$

and so

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{E_{1,M}(x)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} = 0\,. \tag{4.6}$$

We claim, further, that for any $\kappa < \kappa_1 < 1$, for all M large enough,

$$\limsup_{x\to\infty} \frac{E_{2,M}(x)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} \leq M^{-(1-\kappa_1)}\,. \tag{4.7}$$

Indeed, suppose that (4.7) fails for some $\kappa < \kappa_1 < 1$. Then there is a sequence $x_j \uparrow \infty$ such that $j\overline{G}(x_j) \to 0$ as $j \to \infty$ and

$$E_{2,j}(x_j) \geq \frac{1}{2} j^{-(1-\kappa_1)}\mathbb{P}\big(K > \overline{G}(x_j)^{-1}\big) \tag{4.8}$$

for $j \in \mathbb{N}$. Let $p_k := \mathbb{P}(K = k)$ for $k \geq 1$. Note that

$$E_{2,j}(x_j) = \sum_{k=1}^{\lfloor j^{-1}\overline{G}(x_j)^{-1}\rfloor} p_k \mathbb{P}(X_1 + \ldots + X_k > x_j)\,.$$

Theorem 9.1 in Denisov et al. (2008) shows that

$$\mathbb{P}(X_1 + \ldots + X_k > x_j) \sim kp\overline{G}(x_j)$$

as $j \to \infty$ uniformly in $k \leq j^{-1}\overline{G}(x_j)^{-1}$. Therefore, for large j, by Karamata's theorem,

$$E_{2,j}(x_j) \leq 2 \sum_{k=1}^{\lfloor j^{-1}\overline{G}(x_j)^{-1}\rfloor} k\, p_k p\overline{G}(x_j) \leq \frac{4p}{1-\kappa} j^{-1}\mathbb{P}\big(K > j^{-1}\overline{G}(x_j)^{-1}\big)\,,$$

and by Potters's inequalities (cf. Resnick (2006), p. 36), for any $\kappa < \kappa_2 < \kappa_1$ there is $C_1 > 0$ such that for large j

$$E_{2,j}(x_j) \le C_1 j^{-(1-\kappa_2)} \mathbb{P}\big(K > \overline{G}(x_j)^{-1}\big).$$

This, clearly, contradicts (4.8), and so (4.7) has to hold. We conclude that

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{E_{2,M}(x)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} = 0\,. \tag{4.9}$$

We now consider the term $E_{3,M}(x)$ in (4.5). For $M^{-1}\overline{G}(x)^{-1} < k \le M\overline{G}(x)^{-1}$ we denote $r = xa_k^{-1}$. Since $\overline{G}(a_k) \sim k^{-1}$ as $k \to \infty$, we see that, for all x large enough, and $M^{-1}\overline{G}(x)^{-1} < k \le M\overline{G}(x)^{-1}$,

$$(2M)^{-1}\overline{G}(x)^{-1} \le \overline{G}(a_k)^{-1} \le 2M\overline{G}(x)^{-1}\,,$$

which implies that for the same range of x and k, $(4M)^{-1} \le r^\gamma \le 4M$. In particular, $\mathbb{P}(S_\gamma > r)$ is bounded away from 0. Since by (4.4)

$$\mathbb{P}(X_1 + \ldots + X_k > x) = \mathbb{P}\left(\frac{1}{a_k}(X_1 + \ldots + X_k) > r\right) \longrightarrow \mathbb{P}(S_\gamma > r) \quad \text{as } k \to \infty$$

(if r is kept fixed), we conclude that

$$\lim_{x\to\infty} \sup_{M^{-1}\overline{G}(x)^{-1} < k \le M\overline{G}(x)^{-1}} \left| \frac{\mathbb{P}(X_1 + \ldots + X_k > x)}{\mathbb{P}(S_\gamma > x/a_k)} - 1 \right| = 0.$$

Therefore,

$$E_{3,M}(x) \sim \sum_{k=\lfloor M^{-1}\overline{G}(x)^{-1}\rfloor+1}^{\lfloor M\overline{G}(x)^{-1}\rfloor} p_k \mathbb{P}\big(S_\gamma > x/a_k\big) \qquad \text{as } x \to \infty.$$

If f denotes the density of S_γ, this statement translates by Fubini into

$$\begin{aligned}
E_{3,M}(x) \;&\sim\; \int_0^\infty \sum_{k=\lfloor M^{-1}\overline{G}(x)^{-1}\rfloor+1}^{\lfloor M\overline{G}(x)^{-1}\rfloor} p_k\, \mathbf{1}_{\{a_k > x/y\}}\, f(y)\, dy \\
&\sim\; \int_0^\infty \sum_{k=\lfloor M^{-1}\overline{G}(x)^{-1}\rfloor+1}^{\lfloor M\overline{G}(x)^{-1}\rfloor} p_k\, \mathbf{1}_{\{k > \overline{G}(x/y)^{-1}\}}\, f(y)\, dy \\
&=\; \int_0^\infty \Big[\mathbb{P}\Big(\max\big(M^{-1}\overline{G}(x)^{-1}, \overline{G}(x/y)^{-1}\big) < K \le M\overline{G}(x)^{-1}\Big)\Big] f(y)\, dy\,.
\end{aligned}$$

Now, for every $y > 0$, as $x \to \infty$,

$$\frac{\mathbb{P}\Big(\max\big(M^{-1}\overline{G}(x)^{-1}, \overline{G}(x/y)^{-1}\big) < K \le M\overline{G}(x)^{-1}\Big)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} \longrightarrow \Big[\min\big(M^\kappa, y^{\gamma\kappa}\big) - M^{-\kappa}\Big]_+\,,$$

while the same ratio on the left hand side is bounded from above, for large x uniformly in $y > 0$ by

$$\frac{\mathbb{P}\big(K > M^{-1}\overline{G}(x)^{-1}\big)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} \le 2M^{\kappa}\,.$$

Therefore, by the dominated convergence theorem,

$$\lim_{x\to\infty} \frac{E_{3,M}(x)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} = \int_0^\infty \Big[\min\big(M^\kappa, y^{\gamma\kappa}\big) - M^{-\kappa}\Big]_+ f(y)\,dy. \tag{4.10}$$

As $M \to \infty$, the right hand side of (4.10) converges to $\mathbb{E}\big((S_\gamma^+)^{\gamma\kappa}\big)$, and so the statement of the proposition follows from (4.5), (4.6), (4.9) and (4.10). □

The next two results are renewal theorems needed in the proof of the main theorem.

Proposition 4.2 *Let (X_k) be an iid sequence of positive random variables with distribution function F, such that $\overline{F} \in \mathcal{R}_{-1/\beta}$, $0 < 1/\beta < 1$. Let $T_j = \sum_{k=1}^j X_k$, $j \in \mathbb{N}_0$. Suppose that $(c(t))_{t\ge 0}$ is a non-negative eventually non-increasing function, regularly varying of index $-\eta$ in infinity, $1 < \eta < 2$. Then*

$$\sum_{j=0}^\infty c(j)\mathbb{P}(T_j > x) \sim \frac{1}{\eta - 1} C_{\eta,\beta} \overline{F}(x)^{-1} c(\overline{F}(x)^{-1}) \in \mathcal{R}_{(1-\eta)/\beta} \qquad \text{as } x \to \infty,$$

where $C_{\eta,\beta} = \mathbb{E}((S_{1/\beta}(1))^{(\eta-1)/\beta})$, and $(S_{1/\beta}(t))_{t\ge 0}$ is the positive strictly $1/\beta$-stable stochastic process in (2.5).

Proof. Let H_β be the distribution function of $S_{1/\beta}(1)$. Then by the weak convergence in (2.5)

$$\lim_{n\to\infty} \sup_{r\in\mathbb{R}} |H_\beta(r) - \mathbb{P}(T_n \le a_n r)| = 0,$$

where $a_n = \overline{F}^{\leftarrow}(1/n)$ (cf. Petrov (1975), Theorem 11, p.15, and Theorem 10, p. 88). Thus, there exist a positive sequence $(\epsilon_j)_{j\ge 0}$ with $\epsilon_j \downarrow 0$ as $j \to \infty$ such that for any $r > 0$

$$\mathbb{P}(T_j > r) \le \overline{H}_\beta(a_j^{-1} r) + \epsilon_j.$$

Let $\delta_1, \delta_2 > 0$, $\delta_1 < \delta_2$ and $\delta = (\delta_1, \delta_2)$. Then

$$\sum_{j=\lceil \delta_1 \overline{F}(r)^{-1}\rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1}\rfloor} c(j)\mathbb{P}(T_j > r) \le \sum_{j=\lceil \delta_1 \overline{F}(r)^{-1}\rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1}\rfloor} c(j)\overline{H}_\beta(a_j^{-1} r) + \sum_{j=\lceil \delta_1 \overline{F}(r)^{-1}\rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1}\rfloor} c(j)\epsilon_j =: J_1(\delta, r) + J_2(\delta, r).$$

First, we study the first summand. Let $x_j^{(r)} := j\overline{F}(r)$ and ℓ be a slowly varying function such that $\overline{F}(x) = \ell(x)x^{-1/\beta}$. Then, as $n \to \infty$,

$$n\overline{F}(a_n) = n\ell(a_n)a_n^{-1/\beta} \longrightarrow 1. \tag{4.11}$$

Since $\delta_1 \overline{F}(r)^{-1} \le j \le \delta_2 \overline{F}(r)^{-1}$ we have for some $C_1, C_2 > 0$, for all r large enough,

$$C_1 r \le \overline{F}^{\leftarrow}(j) = a_j \le C_2 r.$$

By Theorem 1.5.2 of Bingham et al. (1987) we obtain $\ell(a_j) \sim \ell(r)$ as $r \to \infty$ uniformly for $\delta_1 \overline{F}(r)^{-1} \le j \le \delta_2 \overline{F}(r)^{-1}$. Thus, (4.11) gives $\ell(r) \sim j^{-1} a_j^{1/\beta}$ as $r \to \infty$ uniformly for $\delta_1 \overline{F}(r)^{-1} \le j \le \delta_2 \overline{F}(r)^{-1}$, and

$$(x_j^{(r)})^{-\beta} = \left(j\ell(r) r^{-1/\beta} \right)^{-\beta} \sim a_j^{-1} r \quad \text{as } r \to \infty.$$

Hence, as $r \to \infty$,

$$\begin{aligned}
\sum_{j=\lceil \delta_1 \overline{F}(r)^{-1} \rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1} \rfloor} c(j) \overline{H}_\beta(a_j^{-1} r) &\sim \sum_{j=\lceil \delta_1 \overline{F}(r)^{-1} \rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1} \rfloor} c\left(x_j^{(r)} \overline{F}(r)^{-1} \right) \overline{H}_\beta((x_j^{(r)})^{-\beta}) \\
&= \overline{F}(r)^{-1} \sum_{j=\lceil \delta_1 \overline{F}(r)^{-1} \rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1} \rfloor} (x_{j+1}^{(r)} - x_j^{(r)}) c\left(x_j^{(r)} \overline{F}(r)^{-1} \right) \overline{H}_\beta((x_j^{(r)})^{-\beta}).
\end{aligned}$$

Since $c \in \mathcal{R}_{-\eta}$ we obtain by Theorem 1.5.2 of Bingham et al. (1987) as $r \to \infty$,

$$\begin{aligned}
\frac{1}{\overline{F}(r)^{-1} c(\overline{F}(r)^{-1})} \sum_{j=\lceil \delta_1 \overline{F}(r)^{-1} \rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1} \rfloor} c(j) \overline{H}_\beta(a_j^{-1} r) &\sim \sum_{j=\lceil \delta_1 \overline{F}(r)^{-1} \rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1} \rfloor} (x_{j+1}^{(r)} - x_j^{(r)}) (x_j^{(r)})^{-\eta} \overline{H}_\beta((x_j^{(r)})^{-\beta}) \\
&\sim \int_{\delta_1}^{\delta_2} y^{-\eta} \overline{H}_\beta(y^{-\beta})\, dy,
\end{aligned}$$

and so

$$J_1(\delta, r) \sim \overline{F}(r)^{-1} c(\overline{F}(r)^{-1}) \int_{\delta_1}^{\delta_2} y^{-\eta} \overline{H}_\beta(y^{-\beta})\, dy \quad \text{as } r \to \infty.$$

On the other hand,

$$\begin{aligned}
J_2(\delta, r) &\le \epsilon_{\delta_1 \overline{F}(r)^{-1}} \sum_{j \ge \delta_1 \overline{F}(r)^{-1}} c(j) \\
&\sim \epsilon_{\delta_1 \overline{F}(r)^{-1}} (\eta - 1)^{-1} \delta_1 \overline{F}(r)^{-1} c(\delta_1 \overline{F}(r)^{-1}) \\
&\sim \epsilon_{\delta_1 \overline{F}(r)^{-1}} (\eta - 1)^{-1} \delta_1^{1-\eta} \overline{F}(r)^{-1} c(\overline{F}(r)^{-1}) \quad \text{as } r \to \infty \qquad (4.12)
\end{aligned}$$

by Bingham et al. (1987), Proposition 1.5.10. Since δ_1 is arbitrary and $\epsilon_{\delta_1 \overline{F}(r)^{-1}} \to 0$ as $r \to \infty$ we obtain

$$\lim_{\delta_2 \to \infty} \lim_{\delta_1 \downarrow 0} \lim_{r \to \infty} \overline{F}(r) c(\overline{F}(r)^{-1})^{-1} (J_1(\delta, r) + J_2(\delta, r)) = \frac{1}{\eta - 1} \mathbb{E}((S_{1/\beta}(1))^{(\eta-1)/\beta}). \qquad (4.13)$$

Next, Proposition 1.5.8 in Bingham et al. (1987) and Lemma 6.4 result in

$$\begin{aligned}
\sum_{j \le \delta_1 \overline{F}(r)^{-1}} c(j) \mathbb{P}(T_j > r) &\le C_3 \sum_{j \le \delta_1 \overline{F}(r)^{-1}} c(j) j \overline{F}(r) \\
&\sim C_4 \delta_1^{2-\eta} \overline{F}(r)^{-1} c(\overline{F}(r)^{-1})
\end{aligned}$$

as $r \to \infty$ for some $C_3, C_4 > 0$. Hence,

$$\lim_{\delta_1 \downarrow 0} \lim_{r \to \infty} \overline{F}(r) c(\overline{F}(r)^{-1})^{-1} \sum_{j \leq \delta_1 \overline{F}(r)^{-1}} c(j) \mathbb{P}(T_j > r) = 0. \tag{4.14}$$

Also, by Bingham et al. (1987), Proposition 1.5.10,

$$\begin{aligned} \overline{F}(r) c(\overline{F}(r)^{-1})^{-1} \sum_{j \geq \delta_2 \overline{F}(r)^{-1}} c(j) \mathbb{P}(T_j > r) &\leq \overline{F}(r) c(\overline{F}(r)^{-1})^{-1} \sum_{j \geq \delta_2 \overline{F}(r)^{-1}} c(j) \\ &\sim \frac{1}{\eta - 1} \delta_2^{1-\eta} \stackrel{\delta_2 \to \infty}{\longrightarrow} 0. \end{aligned} \tag{4.15}$$

By (4.13), (4.14) and (4.15) the result follows. □

The following result is a local renewal theorem.

Lemma 4.3 *Let the conditions of Proposition 4.2 hold, and assume additionally Assumption B. Then*

$$\sum_{j=0}^{\infty} c(j) \left[\mathbb{P}(T_j > x) - \mathbb{P}(T_j > x + 1) \right] \sim \frac{1}{\beta} C_{\eta,\beta} x^{-1} \overline{F}(x)^{-1} c(\overline{F}(x)^{-1}) \in \mathcal{R}_{(1-\eta)/\beta - 1} \quad \text{as } x \to \infty.$$

Proof. Under the first scenario of Assumption B, the proof, using Proposition 4.2, is the same as the proof of Theorem 2 in Anderson and Athreya (1988), which in particular requires $\beta < 2$. Under the second scenario of Assumption B, the statement is Theorem 3 of Doney (1997). □

5 Verification of the conditions of Theorem 3.3

The main result of this paper, Theorem 3.1, is proved in this section via verifying the conditions of Theorem 3.3.

5.1 Verification of condition (a) of Theorem 3.3

Proposition 5.1 *Let the Poisson cluster model satisfy Assumption A and B. Further, let M_n, b_n be a sequence of positive constants such that $M_n \uparrow \infty$ and $b_n \uparrow \infty$, respectively. Let N_i be iid copies of N. Then*

$$\left(b_n^{-1} \sum_{i=1}^{n} (N_i\,(0, M_n t] - \lambda M_n t) \right)_{t \geq 0} \stackrel{\text{fidi}}{\Longrightarrow} (B_H(t))_{t \geq 0},$$

with $(B_H(t))_{t \geq 0}$ as given in Theorem 3.1.

Proof. We can write

$$\begin{aligned} b_n^{-1} \sum_{i=1}^{n} [N_i\,(0, M_n t] - \lambda M_n t] &= b_n^{-1} \sum_{i=1}^{n} \left[N_i^{(0,M_n t]}\,(0, M_n t] - \mathbb{E}(N_i^{(0,M_n t]}\,(0, M_n t]) \right] \\ &\quad + b_n^{-1} \sum_{i=1}^{n} \left[N_i^{(-\infty,0]}\,(0, M_n t] - \mathbb{E}(N_i^{(-\infty,0]}\,(0, M_n t]) \right] \\ &=: \xi_n^{+}(t) + \xi_n^{-}(t), \end{aligned}$$

where $N_i^A(B) = \#\{\Gamma_{m,k}^{(i)} : m \in \mathbb{Z}, k \in \{0, \dots, K_m^{(i)}\}, \Gamma_{m,k}^{(i)} = \Gamma_m^{(i)} + T_{m,k}^{(i)} \in B \text{ and } \Gamma_m^{(i)} \in A\}$. We will show in Lemma 5.2 and Lemma 5.3 that

$$(\xi_n^+(t))_{t\geq 0} \stackrel{\text{fidi}}{\Longrightarrow} (B_H^+(t))_{t\geq 0} \quad \text{and} \quad (\xi_n^-(t))_{t\geq 0} \stackrel{\text{fidi}}{\Longrightarrow} (B_H^-(t))_{t\geq 0}, \tag{5.1}$$

where $(B_H^+(t))_{t\geq 0}$ and $(B_H^-(t))_{t\geq 0}$ are independent fractional Brownian motions of index H with time 1 variances

$$\sigma_+^2 = \frac{2\lambda_0}{2+\beta-\alpha} \int_0^\infty y^{-(2+\beta-\alpha)/\beta} \mathbb{P}(S_{1/\beta}(1) \leq y)\, dy \tag{5.2}$$

and

$$\sigma_-^2 = \lambda_0 \int_0^\infty \mathbb{E}\left(\frac{2}{2-\alpha} I(w+1)^{2-\alpha} + \frac{2}{\alpha-1} I(w) I(w+1)^{1-\alpha} - \frac{2}{(2-\alpha)(\alpha-1)} I(w)^{2-\alpha} \right) dw. \tag{5.3}$$

By the independence of $(\xi_n^+(t))$ and $(\xi_n^-(t))$, (5.1) implies that

$$(\xi_n(t))_{t\geq 0} := (\xi_n^+(t) + \xi_n^-(t))_{t\geq 0} \stackrel{\text{fidi}}{\Longrightarrow} (B_H^+(t) + B_H^-(t))_{t\geq 0} =: (B_H(t))_{t\geq 0}, \tag{5.4}$$

where $(B_H(t))_{t\geq 0}$ is a fractional Brownian motion with time 1 variance $\sigma^2 = \sigma_+^2 + \sigma_-^2$. □

In order to prove (5.1) we notice that $\xi_n^+(t)$ and $\xi_n^-(t)$ are infinitely divisible random variables whose characteristic function can be written in the form

$$\mathbb{E}(\exp(i\theta\xi_n^\pm(t))) = \exp\left\{ \int_0^\infty \left(\mathrm{e}^{i\theta x} - 1 - i\theta x \right) \nu_{n,t}^\pm(dx) \right\},$$

where $\nu_{n,t}^\pm$ are the corresponding Lévy measures. These can be represented in the form

$$\nu_{n,t}^\pm = n\lambda_0(\mathbb{P}_1 \times \text{Leb}) \circ \zeta_\pm^{-1},$$

with the following notation. Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ be a probability space on which a generic cluster process $(N_c\,[0,u])_{u\geq 0}$ is defined. The maps ζ_+ and ζ_- are defined as follows: $\zeta_+ : \Omega_1 \times (0, M_n t] \to [0, \infty)$ is given by $\zeta_+(\omega_1, u) = N_c\,[0,u]\,(\omega_1)/b_n$, and $\zeta_- : \Omega_1 \times \mathbb{R}_+ \to [0,\infty)$ is given by $\zeta_-(\omega_1, u) = N_c\,(u, u + M_n t]\,(\omega_1)/b_n$. To see this write $N_i^{(0,M_n t]}\,(0, M_n t]$ and $N_i^{(-\infty,0]}\,(0, M_n t]$ as integrals with respect to a Poisson random measure and use, for example, Lemma 12.2 *(i)* in Kallenberg (2002) (cf. proof of Proposition 3.5 in Faÿ et al. (2006)). For the notational simplicity below we often drop the subscript in $\mathbb{P}_1$ and, hence, write for $A \in \mathcal{B}(\mathbb{R})$

$$\nu_{n,t}^+(A) = n\lambda_0 \int_0^{M_n t} \mathbb{P}(N_c\,[0,u]\,/b_n \in A)\, du, \quad \nu_{n,t}^-(A) = n\lambda_0 \int_0^\infty \mathbb{P}(N_c\,(u, u+M_n t]\,/b_n \in A)\, du.$$

Since the Lévy measures are concentrated on the positive half line, we can apply standard results for the weak convergence of infinitely divisible distributions, see e. g. Theorem 15.14 in Kallenberg (2002). Without loss of generality we will assume $\lambda_0 = 1$ in the following.

Lemma 5.2 *Let Assumption A hold and let $t \geq 0$, $\epsilon > 0$. Then*

(a) $\nu_{n,t}^+ \stackrel{\nu}{\Longrightarrow} 0$ *on* $(0, \infty]$ *as* $n \to \infty$.

(b) $\lim_{n\to\infty}\int_{\{|x|\le\epsilon\}} x^2\nu^+_{n,t}(dx) = t^{2H}\sigma_+^2$ *with* σ_+^2 *as in (5.2) and* H *as in (3.3).*

(c) $\lim_{n\to\infty}\int_{\{|x|>\epsilon\}} x\nu^+_{n,t}(dx) = 0.$

In particular,

$$(\xi_n^+(t))_{t\ge0} \overset{\text{fidi}}{\Longrightarrow} (B_H^+(t))_{t\ge0}$$

where $(B_H^+(t))_{t\ge0}$ *is a fractional Brownian motion with Hurst index* H *and time* 1 *variance* σ_+^2*.*

Proof. We use the decomposition

$$\begin{aligned}
&\int_{\{|x|\le\epsilon\}} x^2\nu^+_{n,t}(dx)\\
&= \frac{n}{b_n^2}\mathbb{E}\left(\mathbf{1}_{\{K+1\le\epsilon b_n\}}\int_0^{M_nt} N_c\,[0,u]^2\;du\right) + \frac{n}{b_n^2}\mathbb{E}\left(\mathbf{1}_{\{K+1>\epsilon b_n\}}\int_0^{M_nt\wedge T_{\lfloor\epsilon b_n-1\rfloor}} N_c\,[0,u]^2\;du\right)\\
&=: I_{1,1}(n)+I_{1,2}(n), \qquad (5.5)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\{|x|>\epsilon\}} x\nu^+_{n,t}(dx) &= n\epsilon\int_0^{M_nt}\mathbb{P}(N_c\,[0,u]>\epsilon b_n)\,du + n\int_0^{M_nt}\int_\epsilon^\infty \mathbb{P}(N_c\,[0,u]>xb_n)\,dx\;du\\
&=: I_{2,1}(n)+I_{2,2}(n). \qquad (5.6)
\end{aligned}$$

The claim *(b)* now follows from Lemma 6.5 and Lemma 6.6, while the claim *(c)* follows from Lemma 6.7 and Lemma 6.8. Then *(a)* is a conclusion of

$$0\le\limsup_{n\to\infty}\nu^+_{n,t}(\epsilon,\infty)\le\limsup_{n\to\infty}\epsilon^{-1}\int_{\{|x|>\epsilon\}} x\nu^+_{n,t}(dx) = 0 \quad \forall\,\epsilon>0.$$

Hence, *(a)-(c)* and Theorem 15.14 in Kallenberg (2002) result in

$$\xi_n^+(t)\Longrightarrow B_H^+(t) \quad \text{as } n\to\infty, \quad \forall\,t\ge0. \qquad (5.7)$$

Applying Lemma 4.8 of Kallenberg (2002) we see that for every $k\ge1$ and $0\le t_1<t_2<\ldots<t_k<\infty$, the family of the laws of the random vectors

$$(\xi_n^+(t_1),\ldots,\xi_n^+(t_k))_{n\in\mathbb{N}}, \qquad (5.8)$$

are tight. Let $\widetilde{\mathbf{B}}_H=(\widetilde{B}_H(t_1),\ldots,\widetilde{B}_H(t_k))$ be a weak subsequential limit of this family, i.e. there exist a subsequence (n_i) such that

$$(\xi_{n_i}^+(t_1),\ldots,\xi_{n_i}^+(t_k))\Longrightarrow\widetilde{\mathbf{B}}_H \quad \text{as } i\to\infty.$$

On the one hand, $\widetilde{\mathbf{B}}_H$ is infinitely divisible (because of the Poisson arrivals of clusters). On the other hand, the one-dimensional marginal distributions of $\widetilde{\mathbf{B}}_H$ are Gaussian with $\widetilde{B}_H(t_i)\overset{d}{=}B_H^+(t_i)$ by (5.7). Hence, $\widetilde{\mathbf{B}}_H$ is zero mean multivariate Gaussian. We will compute now its covariance matrix. The stationarity of the N_i^+'s and, hence, that of the ξ_n^+'s imply by (5.7) that for $1 \leq j \leq i \leq k$,

$$\xi_n^+(t_i) - \xi_n^+(t_j) \Longrightarrow B_H^+(t_i - t_j) \quad \text{as } n \to \infty.$$

Thus, $\widetilde{B}_H(t_i) - \widetilde{B}_H(t_j) \stackrel{d}{=} B_H^+(t_i - t_j)$, and so

$$\begin{aligned}
\mathrm{Cov}(\widetilde{B}_H(t_i), \widetilde{B}_H(t_j)) &= \frac{1}{2}\left(\mathbb{E}(\widetilde{B}_H(t_i)^2) + \mathbb{E}(\widetilde{B}_H(t_j)^2) - \mathbb{E}((\widetilde{B}_H(t_i) - \widetilde{B}_H(t_j))^2)\right) \\
&= \frac{1}{2}\left(\mathbb{E}(B_H^+(t_i)^2) + \mathbb{E}(B_H^+(t_j)^2) - \mathbb{E}(B_H^+(t_i - t_j)^2)\right) \\
&= \frac{\sigma_+^2}{2}(t_i^{2H} + t_j^{2H} - (t_i - t_j)^{2H}).
\end{aligned}$$

This implies that the random vectors in (5.8) converge weakly to the corresponding finite dimensional distributions of the appropriate fractional Brownian motion, and this verifies the statement. □

Lemma 5.3 *Let Assumption A and B hold and let $t \geq 0$, $\epsilon > 0$. Then*

(a) $\nu_{n,t}^- \stackrel{\nu}{\Longrightarrow} 0$ *on* $(0, \infty]$ *as* $n \to \infty$.

(b) $\lim_{n\to\infty} \int_{\{|x|\leq\epsilon\}} x^2 \nu_{n,t}^-(dx) = t^{2H}\sigma_-^2$ *with* σ_-^2 *as in (5.3) and* H *as in (3.3).*

(c) $\lim_{n\to\infty} \int_{\{|x|>\epsilon\}} x \nu_{n,t}^-(dx) = 0$.

In particular,

$$(\xi_n^-(t))_{t\geq 0} \stackrel{\text{fidi}}{\Longrightarrow} (B_H^-(t))_{t\geq 0}$$

where $(B_H^-(t))_{t\geq 0}$ *is a fractional Brownian motion with Hurst index* H *and time* 1 *variance* σ_-^2.

Proof. *(b)* The argument is similar to that of Lemma 5.2, but somewhat more involved technically. We start with introducing some notation. Let

$$\begin{aligned}
H_n^{(1)}(w) &:= \mathbb{E}\left(N_c\left(M_n w, M_n(w+t)\right]^2 \mathbf{1}_{\{N_c(M_n w, M_n(w+t)]\leq \epsilon b_n\}} \mathbf{1}_{\{K > N_0(0, M_n(w+t)]\}}\right), \qquad (5.9) \\
H_n^{(2)}(w) &:= \mathbb{E}\left(N_c\left(M_n w, M_n(w+t)\right]^2 \mathbf{1}_{\{N_c(M_n w, M_n(w+t)]\leq \epsilon b_n\}} \mathbf{1}_{\{N_0(0, M_n w] < K \leq N_0(0, M_n(w+t)]\}}\right),
\end{aligned}$$

so that

$$\mathbb{E}\left(N_c\left(M_n w, M_n(w+t)\right]^2 \mathbf{1}_{\{N_c(M_n w, M_n(w+t)]\leq \epsilon b_n\}}\right) = H_n^{(1)}(w) + H_n^{(2)}(w).$$

By Lemma 6.9, Lemma 6.10 and Theorem 6.11 we can use the dominated convergence theorem so that

$$\begin{aligned}
\lim_{n\to\infty} \int_{\{|x|\leq\epsilon\}} x^2 \nu_{n,t}^-(dx) &= \int_0^\infty \lim_{n\to\infty} \frac{n M_n}{b_n^2}(H_n^{(1)}(w) + H_n^{(2)}(w))\, dw \\
&= \int_0^\infty \Big[\mathbb{E}((I(w+t) - I(w))^2 I(w+t)^{-\alpha}) \\
&\quad + \mathbb{E}\left(\frac{\alpha}{2-\alpha} I(w+t)^{2-\alpha} + \frac{2\alpha}{\alpha-1} I(w) I(w+t)^{1-\alpha}\right) \\
&\quad - \mathbb{E}\left(I(w)^2 I(w+t)^{-\alpha} + \frac{2}{(2-\alpha)(\alpha-1)} I(w)^{2-\alpha}\right)\Big]\, dw.
\end{aligned}$$

Now, by substituting w by tz and using the self-similarity of I of index $1/\beta$ we obtain *(b)*.

(c) Let $M > 0$. Then

$$\begin{aligned}\int_{\{|x|>\epsilon\}} x\nu_{n,t}^{-}(dx) &= \frac{nM_n}{b_n}\int_M^\infty \mathbb{E}\Big(N_c\left(M_n w, M_n(w+t)\right] \mathbf{1}_{\{N_c(M_n w, M_n(w+t)]>\epsilon b_n\}}\Big)\, dw \\ &\quad + \frac{nM_n}{b_n}\int_0^M \mathbb{E}\Big(N_c\left(M_n w, M_n(w+t)\right] \mathbf{1}_{\{N_c(M_n w, M_n(w+t)]>\epsilon b_n\}}\Big)\, dw \\ &=: I_{3,1}(n) + I_{3,2}(n). \end{aligned} \tag{5.10}$$

Therefore, *(c)* follows by Lemma 6.12. The remainder of the proof is the same as in Lemma 5.2. □

5.2 Verification of condition (b) of Theorem 3.3

This is an immediate consequence of the Chebyshev inequality and (6.2).

5.3 Verification of condition (c) of Theorem 3.3

It is in this part of the argument that the slow growth condition (3.2) plays a role. Condition *(c)* of Theorem 3.3 is a direct conclusion of (3.2) and the next lemma since then $\lim_{n\to\infty} n\mathbb{P}(I^*(0) > b_n) = 0$.

Lemma 5.4 *Let the Poisson cluster model satisfy Assumption A. Then there exists a constant $C > 0$ such that*

$$\mathbb{P}(I^*(0) > z) \le C z^{-\frac{\alpha_1 - 1}{\beta_2}} \quad \forall\, z > 0.$$

Proof. The initial step is to show that we may, without loss of generality, assume that the interarrival times of the cluster process N_0 are bounded from below by a positive number. To this end we modify the renewal point process N_0 into a different renewal point process, $\widetilde{N}_0$, as follows. Let $\delta > 0$ be such that $\mathbb{P}(X \ge \delta) > 0$.

Let $\widetilde{T}_1 := \min\{T_j : T_j \ge \delta\}$. Define $\widetilde{Z}_0 := Z_0 + \sum_{i=1}^{U_1 - 1}(Z_i + \delta)$, where $U_1 := \min\{j : T_j \ge \delta\}$. We view $\widetilde{Z}_0$ as the amount of data in the single arrival at time $\widetilde{T}_0 := 0$.

In general, given $\widetilde{T}_m$ and U_m, we define the next arrival by $\widetilde{T}_{m+1} := \min\{T_j : T_j - \widetilde{T}_m \ge \delta\}$, and the amount of data brought in by the arrival at time $\widetilde{T}_m$ as $\widetilde{Z}_m := Z_{U_m} + \sum_{i=U_m+1}^{U_{m+1}-1}(Z_i + \delta)$, where $U_{m+1} := \min\{j : T_j - \widetilde{T}_m \ge \delta\}$.

Note that with this (sample path) modification, every arrival point of the original process N_0 will arrive, in the new process, not later than before (but it may be aggregated with other points of N_0 into a single new arrival), and its transmission will last in the new process for at least as long as in the original process. We will still take K of the new aggregated arrivals, so this modification can only increase the random variable $I^*(0)$.

For the new process the random amount of data brought in with any arrival has the representation $\widetilde{Z}_0 = Z_0 + \sum_{i=1}^{U_1 - 1}(Z_i + \delta)$, and since U_1 is stochastically dominated by a geometric random variable, we see that $\mathbb{E}(\widetilde{Z}_0^2) < \infty$. Furthermore, the interarrival times of the new process satisfy $\widetilde{X}_m \geq \delta$ a.s. and $\mathbb{P}(X_1 > x) \leq \mathbb{P}(\widetilde{X}_m > x) \leq \mathbb{P}(X_1 + \delta > x) \sim \mathbb{P}(X_1 > x)$ as $x \to \infty$. Hence, $\mathbb{P}(\widetilde{X}_1 > x) \sim \mathbb{P}(X_1 > x)$ as $x \to \infty$.

Therefore, for the purpose of obtaining an upper bound, we may work with the new renewal process, and we will simply assume that the original renewal process N_0 has interarrival times that are bounded from below by a positive constant.

We observe that $I^*(0)$ is an infinitely divisible random variable with Lévy measure given by

$$\mu(B) = \lambda_0 \int_0^\infty \mathbb{P}(A^{(c)}(x) \in B)\, dx \quad \text{for } B \in \mathcal{B}(\mathbb{R}),$$

where $A^{(c)}(x)$ is the total amount of data in a session belonging to a single cluster, initiated at zero, that does not finish by time $x > 0$, i.e. $A^{(c)}(x) = \sum_{j=1}^{N_c(0,x]} (T_j + Z_j - x)_+$. To see this write $I^*(0)$ with respect to a Poisson random measure and use, for example, Lemma 2.2 *(i)* in Kallenberg (2002). Without loss of generality let $\lambda_0 = 1$. We have, therefore, the decomposition

$$\mu(z,\infty) = \int_0^\infty \mathbb{P}(A^c(x) > z)\, dx \leq I_{4,0} + I_{4,1} + I_{4,2} + I_{4,3}, \tag{5.11}$$

where

$$\begin{aligned}
I_{4,0} &= \int_0^z \mathbb{P}(A^{(c)}(x) > z)\, dx, \\
I_{4,1} &= \int_z^\infty \mathbb{P}(Z_{N_c(0,x]} > z + (x - T_{N_c(0,x]}), T_K > z)\, dx, \\
I_{4,2} &= \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \int_z^\infty \mathbb{P}\left(\bigcup_{j=0}^{N_c(0,x]-1} \{Z_j > x - T_j\} \,\middle|\, \mathcal{F} \right) dx \right), \\
I_{4,3} &= \mathbb{E}\left(\mathbf{1}_{\{T_K \leq z\}} \int_z^\infty \mathbb{P}\left(\bigcup_{j=0}^{K} \{Z_j > x - T_j\} \,\middle|\, \mathcal{F} \right) dx \right),
\end{aligned}$$

where $\mathcal{F}$ is the σ-field generated by the cluster point process N_c.

Let $z \geq 1$. Then Proposition 4.1 in Faÿ et al. (2006) gives us

$$I_{4,0} \leq z\mathbb{P}\left(\sum_{j=0}^{K} Z_j > z \right) \leq C_0 z^{1-\alpha_1}. \tag{5.12}$$

Next,

$$\begin{aligned}
I_{4,1} \leq\ & \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \sum_{k=N_0(0,z]}^{K-1} \int_{T_k}^{T_{k+1}} \mathbb{P}(Z_k > z + (x - T_k) | \mathcal{F})\, dx \right) \\
& + \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \int_{T_K}^\infty \mathbb{P}(Z_K > z + (x - T_K) | \mathcal{F})\, dx \right).
\end{aligned}$$

By Markov's inequality we obtain

$$
\begin{aligned}
I_{4,1} &\leq C_1\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\sum_{k=N_0(0,z]}^{K-1}\int_{T_k}^{T_{k+1}}(z+(x-T_k))^{-2}\,dx\right)\\
&\quad +C_2\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\int_{T_K}^{\infty}(z+(x-T_K))^{-2}\,dx\right)\\
&= C_1\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\sum_{k=N_0(0,z]}^{K-1}[z^{-1}-(z+X_{k+1})^{-1}]\right)+C_2\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\,z^{-1}\right)\\
&\leq C_3z^{-1}\mathbb{E}(\mathbf{1}_{\{T_K>z\}}\,K). \qquad (5.13)
\end{aligned}
$$

Note that

$$\mathbb{E}(\mathbf{1}_{\{T_K>z\}}\,K)=\mathbb{E}(K)\sum_{k=1}^{\infty}\mathbb{P}\big(\widetilde{K}=k\big)\mathbb{P}\big(X_1+\ldots+X_k>z\big)=\mathbb{E}(K)\mathbb{P}(T_{\widetilde{K}}>z)\,,$$

where $\widetilde{K}$ is a positive integer valued random variable with $\mathbb{P}\big(\widetilde{K}=k\big)=k\mathbb{P}(K=k)/\mathbb{E}(K),\ k\in\mathbb{N}$. Further, by Karamata's Theorem

$$\mathbb{P}\big(\widetilde{K}>n\big)\sim\frac{1}{\mathbb{E}(K)}\frac{\alpha}{\alpha-1}n\mathbb{P}(K>n)\qquad\text{as } n\to\infty.$$

Hence, by Proposition 4.1

$$\mathbb{E}(\mathbf{1}_{\{T_K>z\}}\,K)\sim C_4\overline{F}(z)^{-1}\mathbb{P}\left(K>\overline{F}(z)^{-1}\right)\leq C_5z^{-\frac{\alpha_1-1}{\beta_2}}, \qquad (5.14)$$

and thus,

$$I_{4,1}\leq C_6z^{-1}z^{-\frac{\alpha_1-1}{\beta_2}}\leq C_7z^{-\frac{\alpha_1-1}{\beta_2}}. \qquad (5.15)$$

Next, we decompose $I_{4,2}$ into

$$I_{4,2}=I_{4,2,1}+I_{4,2,2} \qquad (5.16)$$

where

$$
\begin{aligned}
I_{4,2,1} &= \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\int_{z}^{T_{K+1}}\mathbb{P}\left(\bigcup_{j=0}^{N_0(0,x]-1}\{Z_j>x-T_j\}\,\middle|\,\mathcal{F}\right)\,dx\right),\\
I_{4,2,2} &= \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\int_{T_{K+1}}^{\infty}\mathbb{P}\left(\bigcup_{j=0}^{K}\{Z_j>x-T_K\}\,\middle|\,\mathcal{F}\right)\,dx\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
I_{4,2,1} &= \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\int_{z}^{T_{K+1}}\sum_{j=0}^{N_0(0,x]-1}\mathbb{P}(Z_j>x-T_j|\mathcal{F})\,dx\right)\\
&\leq \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\sum_{k=N_0(0,z]}^{K}\int_{T_k}^{T_{k+1}}\sum_{j=0}^{k-1}\mathbb{P}(Z_j>x-T_j|\mathcal{F})\,dx\right).
\end{aligned}
$$

Again applying Markov's inequality and (5.14) lead to

$$\begin{aligned} I_{4,2,1} &\le C_8\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\sum_{k=N_0(0,z]}^{K}\sum_{j=0}^{k-1}\int_{T_k}^{T_{k+1}}(x-T_j)^{-2}\,dx\right) \\ &= C_8\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\sum_{j=0}^{K-1}\left[(T_{N_0(0,z]\vee(j+1)}-T_j)^{-1}-(T_{K+1}-T_j)^{-1}\right]\right) \\ &\le C_9\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\sum_{j=1}^{K}X_j^{-1}\right) \\ &\le C_9\delta^{-1}\mathbb{E}(\mathbf{1}_{\{T_K>z\}}\,K) \\ &\sim C_{10}z^{-\frac{\alpha_1-1}{\beta_2}} \end{aligned} \tag{5.17}$$

as $z\to\infty$. Further, by Markov's inequality, as above and (5.14)

$$I_{4,2,2}\le\mathbb{E}(Z^2)\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\,K\int_{T_K+1}^{\infty}(x-T_K)^{-2}\,dx\right)\le C_{11}\mathbb{E}(\mathbf{1}_{\{T_K>z\}}\,K)\le C_{12}z^{-\frac{\alpha_1-1}{\beta_2}} \tag{5.18}$$

for z large enough. We decompose $I_{4,3}$ into

$$\begin{aligned} I_{4,3} &= \mathbb{E}\left(\mathbf{1}_{\{z/2\le T_K\le z\}}\int_z^{z+1}\mathbb{P}\left(\bigcup_{j=0}^{K}\{Z_j>x-T_j\}\middle|\,\mathcal{F}\right)dx\right) \\ &\quad+\mathbb{E}\left(\mathbf{1}_{\{z/2\le T_K\le z\}}\int_{z+1}^{\infty}\mathbb{P}\left(\bigcup_{j=0}^{K}\{Z_j>x-T_j\}\middle|\,\mathcal{F}\right)dx\right) \\ &\quad+\mathbb{E}\left(\mathbf{1}_{\{T_K<z/2\}}\int_{z}^{\infty}\mathbb{P}\left(\bigcup_{j=0}^{K}\{Z_j>x-T_j\}\middle|\,\mathcal{F}\right)dx\right) \\ &=: I_{4,3,1}+I_{4,3,2}+I_{4,3,3}. \end{aligned} \tag{5.19}$$

On the one hand, by Proposition 4.1 in Faÿ et al. (2006),

$$I_{4,3,1}\le\mathbb{P}(T_K>z/2)\le C_{13}z^{-\frac{1}{\beta_2}}\,. \tag{5.20}$$

On the other hand, by Markov's inequality and (5.14) we obtain

$$I_{4,3,2}\le C_{14}\mathbb{E}\left(\mathbf{1}_{\{z/2\le T_K\le z\}}\,K\int_{z+1}^{\infty}(x-T_K)^{-2}\,dx\right)\le C_{15}\mathbb{E}(\mathbf{1}_{\{T_K>z/2\}}\,K)\le C_{16}z^{-\frac{\alpha_1-1}{\beta_2}}. \tag{5.21}$$

Finally, another application of Markov's inequality gives us

$$I_{4,3,3}\le\mathbb{E}\left(\mathbf{1}_{\{T_K<z/2\}}\,K\int_{z}^{\infty}(x-T_K)^{-2}\,dx\right)\le C_{17}\mathbb{E}(K)z^{-1}. \tag{5.22}$$

A conclusion of (5.11)-(5.22) is

$$\mu(z,\infty)\le C_{18}z^{1-\alpha_1}+C_{19}z^{-\frac{\alpha_1-1}{\beta_2}}+C_{20}z^{-\frac{1}{\beta_2}}+C_{21}z^{-1}\le C_{22}z^{-\frac{\alpha_1-1}{\beta_2}}.$$

Hence, a stochastic domination argument and the fact that the tail of a regularly varying Lévy measure is equivalent to the tail of its distribution function give the result. □

6 Auxiliary Results

A number of lemmas and other auxiliary results are collected in this section. We start with a lemma that clarifies the behavior of the normalizing sequence (b_n) in Theorem 3.1.

Lemma 6.1 *Let Assumption A hold and (b_n) be defined as in (3.1). Then*

$$\lim_{n\to\infty} (\overline{F}(M_n)b_n)^{-1} = 0, \tag{6.1}$$
$$\lim_{n\to\infty} nM_n b_n^{-2} = 0. \tag{6.2}$$

Proof. For n large we have by Potter's theorem

$$\overline{F}(M_n)b_n = n^{\frac{1}{2}} M_n^{\frac{1}{2}} \mathbb{P}(K > \overline{F}(M_n))^{-\frac{1}{2}} \geq n^{\frac{1}{2}} M_n^{\frac{1}{2}} \overline{F}(M_n)^{\frac{\alpha_2}{2}} \geq n^{\frac{1}{2}} M_n^{\frac{1}{2}} M_n^{-\frac{\alpha_2}{2\beta_1}}.$$

Since $(\beta_1 - \alpha_2)/(2\beta_1) > 0$ and $M_n \to \infty$ as $n \to \infty$ we obtain

$$(\overline{F}(M_n)b_n)^{-1} \leq n^{-\frac{1}{2}} M_n^{-\frac{\beta_1-\alpha_2}{\beta_1}} \overset{n\to\infty}{\longrightarrow} 0.$$

Finally, (6.2) results from

$$nM_n b_n^{-2} = \overline{F}(M_n)^2 \mathbb{P}(K > \overline{F}(M_n)^{-1})^{-1} \leq \overline{F}(M_n)^{2-\alpha_2} \overset{n\to\infty}{\longrightarrow} 0. \qquad \square$$

The next result is a simple consequence of the strong Markov property which is useful in various places in our arguments.

Lemma 6.2 *Let f, g be measurable functions and f be increasing. Suppose N_0 is a renewal process. Then for $w, \delta > 0$*

$$\begin{aligned}
&\mathbb{E}(f(N_0\,(w, w+\delta])g(N_0\,(0, w])\,\mathbf{1}_{\{N_0(0,w] \neq N_0(0,w+\delta]\}}) \\
&\quad \leq \mathbb{E}(f(1 + N_0\,(0,\delta]))\mathbb{E}(g(N_0\,(0,w])\,\mathbf{1}_{\{N_0(0,w] \neq N_0(0,w+\delta]\}}).
\end{aligned}$$

Proof. Condition on the time and the number of the first arrival after w and use the iid assumption of the interarrival times. $\square$

The next lemma gives a simple estimate on the probability of having "too many" arrivals within a time interval.

Lemma 6.3 *Let (X_k) be an iid sequence of positive random variables with distribution function F, such that $\overline{F} \in \mathcal{R}_{-1/\beta}$, $0 < 1/\beta < 1$ and let h be the generalized tail inverse function (2.4). Let $T_m = \sum_{k=1}^m X_k$, $m \in \mathbb{N}$. For any $\delta > 0$ such that $\overline{F}(\delta) > 0$ and $m \geq 1$,*

(i) we have

$$\mathbb{P}\big(T_m \leq \delta\big) \leq F(\delta)^m \leq \mathrm{e}^{-m\overline{F}(\delta)}\,; \tag{6.3}$$

(ii) if $x \geq \delta/h(m)$, then for any $\beta_1 < \beta < \beta_2$ we have

$$\mathbb{P}\big(T_m \leq h(m)x\big) \leq \mathrm{e}^{-C\min(x^{-\frac{1}{\beta_1}}, x^{-\frac{1}{\beta_2}})} \tag{6.4}$$

for some $C = C(\delta, \beta_1, \beta_2)$;

Proof. Trivially, for $\delta > 0$,

$$\mathbb{P}\big(T_m \leq \delta\big) \leq \big[1 - \overline{F}(\delta)\big]^m .$$

Now (6.3) follows from the fact that $(1-a^{-1})^a \leq \mathrm{e}^{-1}$ for $a \geq 1$, and Potter's bounds (cf. Resnick (2006), p. 36) give (6.4). □

The following simple result on convolution tails of random variables with infinite mean is often useful.

Lemma 6.4 *Let (X_k) be an iid sequence of positive random variables with distribution function F, $\overline{F} \in \mathcal{R}_{-1/\beta}$ and $0 < 1/\beta < 1$. Then there exist $K > 0$ and $n_0 \in \mathbb{N}$ such that for any $x > 0$ and $n \geq n_0$,*

$$\overline{F^{n*}}(x) \leq Kn\overline{F}(x). \tag{6.5}$$

Proof. Suppose that the statement is not true. Then for each $j \geq 1$ there exist a $n_j \geq j$ and a $x_j > 0$ such that

$$\overline{F^{n_j*}}(x_j) \geq jn_j\overline{F}(x_j). \tag{6.6}$$

Let h be the generalized tail inverse function (2.4). Assume first that there is a sequence $j_k \uparrow \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \frac{x_{j_k}}{h(n_{j_k})} = \infty.$$

This implies $\lim_{k\to\infty} n_{j_k}\overline{F}(x_{j_k}) = 0$. Therefore, by Theorem 9.1 in Denisov et al. (2008) we obtain

$$\lim_{k\to\infty} \left| \frac{\overline{F^{n_{j_k}*}}(x_{j_k})}{n_{j_k}\ \overline{F}(x_{j_k})} - 1 \right| = 0,$$

which contradicts (6.6).

Next, we suppose that there is $M > 0$ such that

$$x_j \leq Mh(n_j) \quad \text{ for all } j \in \mathbb{N}. \tag{6.7}$$

Then

$$n_j\overline{F}(x_j) \geq n_j\overline{F}(Mh(n_j)) \overset{j\to\infty}{\longrightarrow} M^{-1/\beta}$$

by the regular variation of $\overline{F}$. Thus, (6.6) results in

$$\overline{F^{n_j*}}(x_j) \geq jn_j\overline{F}(x_j) \overset{j\to\infty}{\longrightarrow} \infty.$$

Since $\overline{F^{n_j*}}$ is bounded by 1, this is impossible. Hence, the claim follows. □

6.1 Auxiliary Results for the Proof of Lemma 5.2

The next series of lemmas provides estimates needed to prove the convergence of $\boldsymbol{\xi}_{\boldsymbol{n}}^{+}$ in Lemma 5.2. We are using the same notation.

Lemma 6.5 *Let Assumption A hold and let $I_{1,1}(n)$ be as in (5.5). Then*

$$\lim_{n\to\infty} I_{1,1}(n) = \frac{2}{2+\beta-\alpha} t^{\frac{2+\beta-\alpha}{\beta}} \int_0^\infty y^{-\frac{2+\beta-\alpha}{\beta}} \mathbb{P}(S_{1/\beta}(1) \le y)\, dy.$$

Proof. We have by the independence of K and N_0

$$\begin{aligned}\mathbb{E}(N_c\,[0,u]^2\,\mathbf{1}_{\{K+1\le\epsilon b_n\}}) &= \int_0^{\epsilon^2 b_n^2} \mathbb{P}((N_0\,[0,u]^2 \wedge (K+1)^2)\,\mathbf{1}_{\{K+1\le\epsilon b_n\}} > x)\, dx \\ &= 2\int_0^{\epsilon b_n} y\mathbb{P}(N_0\,[0,u] > y)\mathbb{P}(y < K+1 \le \epsilon b_n)\, dy.\end{aligned}$$

Hence,

$$\begin{aligned} I_{1,1}(n) &= 2\frac{n}{b_n^2}\int_0^{M_n t}\int_0^{\epsilon b_n} y\mathbb{P}(N_0\,[0,u] > y)\mathbb{P}(y < K+1 \le \epsilon b_n)\, dy\, du \\ &= 2\frac{n}{b_n^2}\int_0^{\epsilon b_n} y\mathbb{P}(y < K+1 \le \epsilon b_n)\int_0^{M_n t} \mathbb{P}(T_{\lfloor y\rfloor} \le u)\, du\, dy \\ &= 2\frac{n}{b_n^2}\int_0^{\epsilon b_n} y\mathbb{P}(y < K+1 \le \epsilon b_n)\mathbb{E}(M_n t - T_{\lfloor y\rfloor})_+\, dy \\ &= 2\frac{n}{b_n^2}\overline{F}(M_n)^{-2}\int_0^{\epsilon b_n \overline{F}(M_n)} z\mathbb{P}(z\overline{F}(M_n)^{-1} < K+1 \le \epsilon b_n)\mathbb{E}(M_n t - T_{\lfloor z\overline{F}(M_n)^{-1}\rfloor})_+\, dz \\ &= 2\int_0^{\epsilon} z\frac{\mathbb{P}(z\overline{F}(M_n)^{-1} < K+1 \le \epsilon b_n)}{\mathbb{P}(K > \overline{F}(M_n)^{-1})}\mathbb{E}\left(t - \frac{T_{\lfloor z\overline{F}(M_n)^{-1}\rfloor}}{M_n}\right)_+ dz \\ &\quad +2\int_\epsilon^{\epsilon b_n\overline{F}(M_n)} z\frac{\mathbb{P}(z\overline{F}(M_n)^{-1} < K+1 \le \epsilon b_n)}{\mathbb{P}(K > \overline{F}(M_n)^{-1})}\mathbb{E}\left(t - \frac{T_{\lfloor z\overline{F}(M_n)^{-1}\rfloor}}{M_n}\right)_+ dz \\ &=: J_1(n,\epsilon) + J_2(n,\epsilon). \end{aligned} \tag{6.8}$$

By Karamata's theorem

$$\begin{aligned} J_1(n,\epsilon) &\le \frac{2t}{\mathbb{P}(K > \overline{F}(M_n)^{-1})}\int_0^\epsilon z\mathbb{P}(K+1 > z\overline{F}(M_n)^{-1})\, dz \\ &= \frac{\mathbb{P}(K+1 > \overline{F}(M_n)^{-1})}{\mathbb{P}(K > \overline{F}(M_n)^{-1})}\frac{2t}{\overline{F}(M_n)^{-2}\mathbb{P}(K+1 > \overline{F}(M_n)^{-1})}\int_0^{\epsilon\overline{F}(M_n)^{-1}} z\mathbb{P}(K+1 > z)\, dz \\ &\overset{n\to\infty}{\longrightarrow} \frac{2t\alpha}{2-\alpha}\epsilon^{2-\alpha}, \end{aligned} \tag{6.9}$$

and we conclude that $\lim_{\epsilon\downarrow 0}\lim_{n\to\infty} J_1(n,\epsilon) = 0$. We estimate $J_2(n,\epsilon)$ as follows. By Potter's inequality there exists $C_1 > 0$ such that for $z \ge \epsilon$ and n large,

$$\frac{\mathbb{P}(K+1 > z\overline{F}(M_n)^{-1})}{\mathbb{P}(K > \overline{F}(M_n)^{-1})} \le C_1 z^{-\alpha_1}.$$

Similarly, Potter's inequality leads to

$$\frac{h(\lfloor z\overline{F}(M_n)^{-1}\rfloor)}{M_n} \geq \frac{h(z\overline{F}(M_n)^{-1}-1)}{h(\overline{F}(M_n)^{-1}+1)} \geq C_2 z^{\beta_1} \quad \text{for } z \geq \epsilon.$$

If we define $m_n = \lfloor z\overline{F}(M_n)^{-1}\rfloor$, then for $\delta > 0$ such that $F(\delta) < 1$,

$$\begin{aligned}
\mathbb{E}\left(t - \frac{T_{\lfloor z\overline{F}(M_n)^{-1}\rfloor}}{M_n}\right)_+ &= \mathbb{E}\left(t - \frac{T_{m_n}}{h(m_n)}\frac{h(m_n)}{M_n}\right)_+ \\
&\leq \mathbb{E}\left(t - \frac{T_{m_n}}{h(m_n)} C_2 z^{\beta_1}\right)_+ \\
&= C_2 z^{\beta_1}\left[\int_0^{\delta/h(m_n)} + \int_{\delta/h(m_n)}^{C_2^{-1}tz^{-\beta_1}}\right] \mathbb{P}\left(\frac{T_{m_n}}{h(m_n)} \leq x\right) dx \\
&=: C_2 z^{\beta_1}[V_1(n,z) + V_2(n,z)].
\end{aligned}$$

We have by (6.3) for large n,

$$V_1(n,z) \leq \big(\delta/h(m_n)\big)\mathbb{P}(T_{m_n} \leq \delta) \leq \delta \mathrm{e}^{-m_n\overline{F}(\delta)} \leq C_3^{-1}\mathrm{e}^{-C_3 z}$$

for some $C_3 > 0$, since $m_n \geq z$ for n large. Further, by (6.4)

$$V_2(n,z) \leq \int_0^{C_2^{-1}tz^{-\beta_1}} \mathrm{e}^{-C_4 \min(x^{-1/\beta_1}, x^{-1/\beta_2})}\, dx \leq C_5^{-1} t z^{-\beta_1}\mathrm{e}^{-C_5 z} \leq C_6^{-1}\mathrm{e}^{-C_6 z}$$

for some $C_4, C_5, C_6 > 0$. Hence, we have

$$\mathbb{E}\left(t - \frac{T_{\lfloor z\overline{F}(M_n)^{-1}\rfloor}}{M_n}\right)_+ \leq C_2 z^{\beta_1}[V_1(n,z) + V_2(n,z)] \leq C_7^{-1} z^{\beta_1}\mathrm{e}^{-C_7 z},$$

and so by the dominated convergence theorem, (2.5) and the regular variation of $\overline{F}_K$,

$$\lim_{n\to\infty} J_2(n,\epsilon) = 2\int_\epsilon^\infty z^{1-\alpha}\mathbb{E}(t - z^\beta S_{1/\beta}(1))_+\, dz.$$

Therefore, by (6.8),

$$\begin{aligned}
\lim_{n\to\infty} I_{1,1}(n) &= 2\int_0^\infty z^{1-\alpha}\,\mathbb{E}(t - z^\beta S_{1/\beta}(1))_+\, dz \\
&= \frac{2}{\beta} t^{\frac{2+\beta-\alpha}{\beta}} \int_0^\infty x^{-\frac{2+\beta-\alpha}{\beta}}\,\mathbb{E}(1 - x^{-1}S_{1/\beta}(1))_+\, dx \\
&= \frac{2}{\beta} t^{\frac{2+\beta-\alpha}{\beta}} \int_0^\infty x^{-\frac{2+\beta-\alpha}{\beta}-1} \int_0^x \mathbb{P}(S_{1/\beta}(1) \leq z)\, dz\, dx \\
&= \frac{2}{2+\beta-\alpha} t^{\frac{2+\beta-\alpha}{\beta}} \int_0^\infty z^{-\frac{2+\beta-\alpha}{\beta}}\,\mathbb{P}(S_{1/\beta}(1) \leq z)\, dz.
\end{aligned}$$

□

Lemma 6.6 *Let Assumption A hold and let $I_{1,2}(n)$ be as in (5.5). Then*

$$\lim_{n\to\infty} I_{1,2}(n) = 0.$$

Proof. By the independence of K and N_0 we have

$$I_{1,2}(n) \le \frac{n}{b_n^2}\mathbb{E}\left(\mathbf{1}_{\{K+1>\epsilon b_n\}}\int_0^{M_n t} N_0\,[0,u]^2\;du\right) = \frac{n}{b_n^2}\mathbb{P}(K+1>\epsilon b_n)\int_0^{M_n t}\mathbb{E}(N_0\,[0,u]^2)\,du.$$

Thus,

$$\begin{aligned} I_{1,2}(n) &\le \frac{n}{b_n^2}\mathbb{P}(K+1>\epsilon b_n)\int_0^{M_n t}\int_0^\infty \mathbb{P}(N_0\,[0,u]^2>x)\,dx\,du \\ &\le 2\frac{n}{b_n^2}M_n t\mathbb{P}(K+1>\epsilon b_n)\int_0^\infty z\mathbb{P}(T_{\lfloor z\rfloor}\le M_n t)\,dz. \end{aligned}$$

By (6.3), Potter's inequality and (6.1),

$$\begin{aligned} I_{1,2}(n) &\le C_1\frac{n}{b_n^2}M_n t\mathbb{P}(K+1>\epsilon b_n)\int_0^\infty z\mathrm{e}^{-\lfloor z\rfloor\overline{F}(M_n t)}\,dz \\ &\le C_2\frac{n}{b_n^2}M_n t\mathbb{P}(K+1>\epsilon b_n)\overline{F}(M_n t)^{-2} \\ &= C_3 t^{1-\alpha_1}\frac{\mathbb{P}(K>\epsilon b_n-1)}{\mathbb{P}(K>\overline{F}(M_n)^{-1})}\stackrel{n\to\infty}{\longrightarrow}0. \end{aligned}$$

□

Lemma 6.7 *Let Assumption A hold and let $I_{2,1}(n)$ be as in (5.6). Then*

$$\lim_{n\to\infty} I_{2,1}(n) = 0.$$

Proof. Suppose $\epsilon = 1$. The independence of K and N_0 results in

$$I_{2,1}(n) = n\mathbb{P}(K+1>b_n)\int_0^{M_n t}\mathbb{P}(N_0\,[0,u]>b_n)\,du = n\mathbb{P}(K+1>b_n)\int_0^{M_n t}\mathbb{P}(T_{\lfloor b_n\rfloor}\le u)\,du.$$

As in (6.3) we obtain

$$\begin{aligned} I_{2,1}(n) &\le n\mathbb{P}(K+1>b_n)\int_0^{M_n}\mathrm{e}^{-\lfloor b_n\rfloor\overline{F}(u)}\,du \\ &\le n\mathbb{P}(K+1>b_n)M_n\mathrm{e}^{-(b_n-1)\overline{F}(M_n)} \\ &= \frac{\mathbb{P}(K+1>b_n)}{\mathbb{P}(K>\overline{F}(M_n)^{-1})}(\overline{F}(M_n)b_n)^2\mathrm{e}^{-(b_n-1)\overline{F}(M_n)}\stackrel{n\to\infty}{\longrightarrow}0, \end{aligned}$$

since $b_n\overline{F}(M_n)\stackrel{n\to\infty}{\longrightarrow}\infty$ by Lemma 6.1. □

Lemma 6.8 *Let Assumption A hold and let $I_{2,2}(n)$ be as in (5.6). Then*

$$\lim_{n\to\infty} I_{2,2}(n) = 0.$$

Proof. Suppose $\epsilon = 1$ and $t = 1$. Then

$$\begin{aligned} I_{2,2}(n) &= n\int_1^\infty \mathbb{P}(K+1>b_n x)\int_0^{M_n}\mathbb{P}(N_0\,[0,u]>b_n x)\,du\,dx \\ &= n\int_1^\infty \mathbb{P}(K+1>b_n x)\int_0^{M_n}\mathbb{P}(T_{\lfloor b_n x\rfloor}\le u)\,du\,dx. \end{aligned}$$

Again as in (6.3) we obtain by (6.1),

$$
\begin{aligned}
I_{2,2}(n) &\leq n\int_1^\infty \mathbb{P}(K+1>b_nx)\int_0^{M_n} \mathrm{e}^{-\lfloor b_nx\rfloor \overline{F}(u)}\,du\,dx \\
&\leq nM_n\int_1^\infty \mathbb{P}(K+1>b_nx)\mathrm{e}^{-(b_nx-1)\overline{F}(M_n)}\,dx \\
&\leq \mathrm{e}^1 nM_n\int_1^\infty \mathrm{e}^{-b_nx\overline{F}(M_n)}\,dx \\
&\leq \mathrm{e}^1 nM_n(b_n\overline{F}(M_n))^{-1}\mathrm{e}^{-b_n\overline{F}(M_n)} \\
&\leq \mathrm{e}^1 (n^{\frac{1}{2}}M_n^{H-\frac{1}{\beta}})^{\max\{2,\frac{1}{H-1/\beta}\}}(b_n\overline{F}(M_n))^{-1}\mathrm{e}^{-b_n\overline{F}(M_n)} \\
&\leq C_1(b_n\overline{F}(M_n))^{\max\{2,\frac{1}{H-1/\beta}\}-1}\mathrm{e}^{-b_n\overline{F}(M_n)} \overset{n\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

which is the result. □

6.2 Auxiliary Results for the Proof of Lemma 5.3

The next several results deal with the convergence of $\boldsymbol{\xi}_{\boldsymbol{n}}^{-}$ in Lemma 5.3.

Lemma 6.9 *Let Assumption A hold and let $H_n^{(1)}(w)$ be as in (5.9). Then*

$$
\lim_{n\to\infty}\frac{nM_n}{b_n^2}H_n^{(1)}(w) = \mathbb{E}((I(w+t)-I(w))^2 I(w+t)^{-\alpha}).
$$

Proof. We divide $H_n^{(1)}$ in three parts and define

$$
A_{n,w} = \{N_0\left(M_nw, M_n(w+t)\right] \leq \epsilon b_n, K > N_0\left(0, M_n(w+t)\right]\}.
$$

For $M>0$ let

$$
\begin{aligned}
H_n^{(1,1,M)}(w) &= \mathbb{E}\left(N_0\left(M_nw, M_n(w+t)\right]^2 \mathbf{1}_{\{M^{-1}\leq \overline{F}(M_n)N_0(0,M_n(w+t)]\leq M\}}\mathbf{1}_{A_{n,w}}\right), \\
H_n^{(1,2,M)}(w) &= \mathbb{E}\left(N_0\left(M_nw, M_n(w+t)\right]^2 \mathbf{1}_{\{\overline{F}(M_n)N_0(0,M_n(w+t)]<M^{-1}\}}\mathbf{1}_{A_{n,w}}\right), \\
H_n^{(1,3,M)}(w) &= \mathbb{E}\left(N_0\left(M_nw, M_n(w+t)\right]^2 \mathbf{1}_{\{\overline{F}(M_n)N_0(0,M_n(w+t)]>M\}}\mathbf{1}_{A_{n,w}}\right),
\end{aligned}
$$

so that

$$
H_n^{(1)}(w) = H_n^{(1,1,M)}(w) + H_n^{(1,2,M)}(w) + H_n^{(1,3,M)}(w). \tag{6.10}
$$

Regularly varying functions converge uniformly on compact sets (cf. Bingham et al. (1987), Theorem 1.5.2). Thus, (2.7) gives

$$
\begin{aligned}
\frac{nM_n}{b_n^2}H_n^{(1,1,M)}(w) &= \mathbb{E}\Bigg(\frac{N_0\left(M_nw, M_n(w+t)\right]^2}{\overline{F}(M_n)^{-2}}\mathbf{1}_{\{N_0(M_nw,M_n(w+t)]\leq\epsilon b_n, M^{-1}\leq\overline{F}(M_n)N_0(0,M_n(w+t)]\leq M\}} \\
&\qquad \times\frac{\mathbb{P}(K>N_0\left(0,M_n(w+t)\right]|\mathcal{F}_0)}{\mathbb{P}(K>\overline{F}(M_n)^{-1})}\Bigg) \\
&\overset{n\to\infty}{\longrightarrow} \mathbb{E}((I(w+t)-I(w))^2\,\mathbf{1}_{\{M^{-1}\leq I(w+t)\leq M\}}\,I(w+t)^{-\alpha}),
\end{aligned} \tag{6.11}
$$

where $\mathcal{F}_0 = \sigma(N_0)$. For the second summand of $H_n^{(1)}$ we have for large n by Potter's inequality

$$
\begin{aligned}
&\frac{nM_n}{b_n^2} H_n^{(1,2,M)}(w) \\
&\quad \le \mathbb{E}\left(\left(\frac{N_0\left(M_n w, M_n(w+t)\right]}{\overline{F}(M_n)^{-1}} \right)^2 \frac{\mathbf{1}_{\{K > N_0(0, M_n(w+t)]\}}}{\mathbb{P}(K > \overline{F}(M_n)^{-1})} \mathbf{1}_{\{N_0(0,M_n w] \neq N_0(0, M_n(w+t)] < \overline{F}(M_n)^{-1} M^{-1}\}} \right) \\
&\quad \le C_1 \mathbb{E}\left(\left(\frac{N_0\left(0, M_n(w+t)\right]}{\overline{F}(M_n)^{-1}} \right)^{2-\alpha_2} \mathbf{1}_{\{N_0(0,M_n w] \neq N_0(0, M_n(w+t)] < \overline{F}(M_n)^{-1} M^{-1}\}} \right) \\
&\quad \le C_1 M^{\alpha_2 - 2} \stackrel{M\to\infty}{\longrightarrow} 0.
\end{aligned} \tag{6.12}
$$

By Potter's inequality the last term of $H_n^{(1)}$ has the upper bound

$$
\begin{aligned}
&\frac{nM_n}{b_n^2} H_n^{(1,3,M)}(w) \\
&\quad \le \frac{nM_n}{b_n^2} \mathbb{E}\left(N_0\left(0, M_n(w+t)\right]^2 \mathbf{1}_{\{\overline{F}(M_n) N_0(0, M_n(w+t)] \ge M\}} \mathbb{P}(K > N_0\left(0, M_n(w+t)\right] | \mathcal{F}_0) \right) \\
&\quad \le C_2 \mathbb{E}\left(\left(\frac{N_0\left(0, M_n(w+t)\right]}{\overline{F}(M_n)^{-1}} \right)^{2-\alpha_1} \mathbf{1}_{\{N_0(0, M_n(w+t)] \ge \overline{F}(M_n)^{-1} M\}} \right) \\
&\quad = C_2 \int_{M^{2-\alpha_1}}^{\infty} \mathbb{P}\left(\left(\frac{N_0\left(0, M_n(w+t)\right]}{\overline{F}(M_n)^{-1}} \right)^{2-\alpha_1} > y \right) dy \\
&\qquad + C_2 M^{2-\alpha_1} \mathbb{P}\left(\left(\frac{N_0\left(0, M_n(w+t)\right]}{\overline{F}(M_n)^{-1}} \right)^{2-\alpha_1} > M^{2-\alpha_1} \right).
\end{aligned}
$$

The first term on the right hand side above can be bounded as follows. For some constant $C_3 > 0$ we obtain as in (6.3) that

$$
\begin{aligned}
&\int_{M^{2-\alpha_1}}^{\infty} \mathbb{P}\left(\left(\frac{N_0\left(0, M_n(w+t)\right]}{\overline{F}(M_n)^{-1}} \right)^{2-\alpha_1} > y \right) dy \\
&\quad = (2-\alpha_1) \int_M^{\infty} z^{1-\alpha_1} \mathbb{P}(N_0\left(0, M_n(w+t)\right] > z\overline{F}(M_n)^{-1})\, dz \\
&\quad \le (2-\alpha_1) \int_M^{\infty} z^{1-\alpha_1} \mathbb{P}(T_{\lfloor z\overline{F}(M_n)^{-1} \rfloor + 1} \le M_n(w+t))\, dz \\
&\quad \le (2-\alpha_1) \int_M^{\infty} z^{1-\alpha_1} \exp(-z\overline{F}(M_n)^{-1} \overline{F}(M_n(w+t)))\, dz \\
&\quad \le C_3^{-1} \int_M^{\infty} z^{1-\alpha_1} \mathrm{e}^{-C_3 (\omega+t)^{-\frac{1}{\beta_1}} z}\, dz \longrightarrow 0 \quad \text{as } M \to \infty.
\end{aligned}
$$

Similarly,

$$
M^{2-\alpha_1} \mathbb{P}\left(\left(\frac{N_0\left(0, M_n(w+t)\right]}{\overline{F}(M_n)^{-1}} \right)^{2-\alpha_1} > M^{2-\alpha_1} \right) \longrightarrow 0 \quad \text{as } M \to \infty.
$$

Hence, the result follows. □

Lemma 6.10 *Let Assumption A hold and let $H_n^{(2)}(w)$ be as in (5.9). Then*

$$\begin{aligned}\lim_{n\to\infty}\frac{nM_n}{b_n^2}H_n^{(2)}(w) &= \mathbb{E}\left(\frac{\alpha}{2-\alpha}I(w+t)^{2-\alpha}+\frac{2\alpha}{\alpha-1}I(w)I(w+t)^{1-\alpha}\right)\\ &\quad -\mathbb{E}\left(I(w)^2I(w+t)^{-\alpha}+\frac{2}{(2-\alpha)(\alpha-1)}I(w)^{2-\alpha}\right).\end{aligned}$$

Proof. We define

$$A_{n,M}=\{M^{-1}\le \overline{F}(M_n)N_0\left(0,M_nw\right]\le \overline{F}(M_n)N_0\left(0,M_n(w+t)\right]\le M, N_0\left(M_nw,M_n(w+t)\right]\le \epsilon b_n\}$$

and

$$A_M=\{M^{-1}\le I(w)\le I(w+t)\le M\}.$$

By Karamata's theorem and the uniform convergence of regularly varying functions on compact sets we have

$$\begin{aligned}&\mathbb{E}\left(K^2\frac{\mathbf{1}_{\{K\le N_0(0,M_n(w+t)]\}}-\mathbf{1}_{\{K\le N_0(0,M_nw]\}}}{\overline{F}(M_n)^{-2}\mathbb{P}(K>\overline{F}(M_n)^{-1})}\mathbf{1}_{A_{n,M}}\right)\\ &\quad\overset{n\to\infty}{\longrightarrow}\frac{\alpha}{2-\alpha}\mathbb{E}\left(\left(I(w+t)^{2-\alpha}-I(w)^{2-\alpha}\right)\mathbf{1}_{A_M}\right),\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}&\mathbb{E}\left(\frac{N_0\left(0,M_nw\right]}{\overline{F}(M_n)^{-1}}K\frac{\mathbf{1}_{\{K>N_0(0,M_nw]\}}-\mathbf{1}_{\{K>N_0(0,M_n(w+t)]\}}}{\overline{F}(M_n)^{-1}\mathbb{P}(K>\overline{F}(M_n)^{-1})}\mathbf{1}_{A_{n,M}}\right)\\ &\quad\overset{n\to\infty}{\longrightarrow}\frac{\alpha}{\alpha-1}\mathbb{E}\left(I(w)(I(w)^{1-\alpha}-I(w+t)^{1-\alpha})\mathbf{1}_{A_M}\right).\end{aligned} \tag{6.14}$$

Further,

$$\begin{aligned}&\mathbb{E}\left(\frac{N_0\left(0,M_nw\right]^2}{\overline{F}(M_n)^{-2}}\frac{\mathbf{1}_{\{K>N_0(0,M_nw]\}}-\mathbf{1}_{\{K>N_0(0,M_n(w+t)]\}}}{\mathbb{P}(K>\overline{F}(M_n)^{-1})}\mathbf{1}_{A_{n,M}}\right)\\ &\quad\overset{n\to\infty}{\longrightarrow}\mathbb{E}\left(I(w)^2(I(w)^{-\alpha}-I(w+t)^{-\alpha})\mathbf{1}_{A_M}\right).\end{aligned} \tag{6.15}$$

Thus, (6.13)-(6.15) give us

$$\begin{aligned}&\lim_{M\to\infty}\lim_{n\to\infty}\frac{nM_n}{b_n^2}\mathbb{E}\left(N_c\left(M_nw,M_n(w+t)\right]^2\mathbf{1}_{\{N_0(0,M_nw]<K\le N_0(0,M_n(w+t)]\}}\mathbf{1}_{A_{n,M}}\right)\\ &=\frac{\alpha}{2-\alpha}\mathbb{E}\left(I(w+t)^{2-\alpha}-I(w)^{2-\alpha}\right)-2\frac{\alpha}{\alpha-1}\mathbb{E}\left(I(w)(I(w)^{1-\alpha}-I(w+t)^{1-\alpha})\right)\\ &\quad+\mathbb{E}\left(I(w)^2(I(w)^{-\alpha}-I(w+t)^{-\alpha})\right).\end{aligned}$$

The integral over the complement of the event $A_{n,M}$ vanishes in the limit, as $M\to\infty$, in the same way as in Lemma 6.9. □

The following theorem is needed to apply dominated convergence to establish the convergence of $\boldsymbol{\xi}_{\boldsymbol{n}}^{-}$ in Lemma 5.3.

Theorem 6.11 *Let Assumption A and B hold. Then there exists a non-negative measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty g(w)\,dw < \infty$ and for every $n \in \mathbb{N}$*

$$\frac{nM_n}{b_n^2}\mathbb{E}(N_c\,(M_n w, M_n(w+1)]^2) \le g(w) \quad \forall\, w > 0.$$

Proof. The existence of the required function on the interval $(0, M]$ for an arbitrary $M > 0$ follows from Lemma 6.16 below, so we only need to construct a required function on the interval (M, ∞). We define

$$\begin{aligned} A_{n,w} &= \{N_0\,(0, M_n w] \neq N_0\,(0, M_n(w+1)]\}, \\ B_{n,w} &= \{N_0\,(0, M_n] = N_0\,(0, M_n w]\} \cap A_{n,w}, \\ C_{n,w} &= \{N_0\,(0, M_n] \neq N_0\,(0, M_n w]\} \cap A_{n,w}. \end{aligned}$$

We have for $w > M$

$$\begin{aligned} \frac{nM_n}{b_n^2}\mathbb{E}(N_c\,(M_n w, M_n(w+1)]^2) &\le \frac{nM_n}{b_n^2}\mathbb{E}(N_c\,(M_n w, M_n(w+1)]^2\,\mathbf{1}_{B_{n,w}}) \\ &\quad + \frac{nM_n}{b_n^2}\mathbb{E}(N_0\,(M_n w, M_n(w+1)]^2\,\mathbf{1}_{\{K > N_0(0,M_n w] > 0\}}\,\mathbf{1}_{C_{n,w}}) \\ &=: J_{2,1}(n,w) + J_{2,2}(n,w). \end{aligned} \tag{6.16}$$

Potter's inequality and Lemma 6.2 result in

$$\begin{aligned} &J_{2,2}(n,w) \\ &\quad \le \mathbb{E}\left(\frac{N_0\,(M_n w, M_n(w+1)]^2}{\overline{F}(M_n)^{-2}}\left[C_1\left(\frac{N_0\,(0, M_n w]}{\overline{F}(M_n)^{-1}}\right)^{-\alpha_1} + C_2\left(\frac{N_0\,(0, M_n w]}{\overline{F}(M_n)^{-1}}\right)^{-\alpha_2}\right]\mathbf{1}_{C_{n,w}}\right) \\ &\quad \le \mathbb{E}\left(\frac{(N_0\,(0, M_n] + 1)^2}{\overline{F}(M_n)^{-2}}\right)\mathbb{E}\left(\left[C_1\left(\frac{N_0\,(0, M_n w]}{\overline{F}(M_n)^{-1}}\right)^{-\alpha_1} + C_2\left(\frac{N_0\,(0, M_n w]}{\overline{F}(M_n)^{-1}}\right)^{-\alpha_2}\right]\mathbf{1}_{C_{n,w}}\right). \end{aligned}$$

By (6.3) we have for large n

$$\begin{aligned} \mathbb{E}\left(\frac{N_0\,(0, M_n]^2}{\overline{F}(M_n)^{-2}}\right) &= \frac{1}{\overline{F}(M_n)^{-2}}\int_0^\infty \mathbb{P}(N_0\,(0, M_n]^2 > x)\,dx \\ &\le \frac{2}{\overline{F}(M_n)^{-2}}\int_0^\infty y\mathbb{P}(T_{\lfloor y\rfloor+1} \le M_n)\,dy \\ &\le \frac{2}{\overline{F}(M_n)^{-2}}\int_0^\infty y\mathrm{e}^{-y\overline{F}(M_n)}\,dy \\ &= 2\int_0^\infty z\mathrm{e}^{-z}\,dz < \infty. \end{aligned} \tag{6.17}$$

Hence, (6.16), (6.17) and Proposition 6.13 below show that $J_{2,2}(n,w)$ is uniformly in n bounded from above by an integrable on $[M,\infty)$ function. The fact that the same is true for $J_{2,1}(n,w)$ follows from Lemma 6.17 below. □

The last piece needed to prove Lemma 5.3 is the next lemma.

Lemma 6.12 *Let Assumption A and B hold and let $I_{3,1}(n)$ and $I_{3,2}(n)$, respectively, be as in (5.10). Then*

(a) $\lim_{n\to\infty} I_{3,1}(n) = 0$.

(b) $\lim_{n\to\infty} I_{3,2}(n) = 0$.

Proof. *(a)* We assume, once again for the ease of notation, that $\epsilon = 1$. Let $\theta > 0$. We have

$$\begin{aligned} I_{3,1}(n) &\leq b_n^{-\theta} \frac{nM_n}{b_n^2} \int_M^\infty \mathbb{E}\Big(N_c\left(M_n w, M_n(w+t)\right]^{2+\theta} \mathbf{1}_{\{N_c(M_n w, M_n(w+t)]>b_n\}}\Big)\, dw \\ &\leq b_n^{-\theta} \overline{F}(M_n)^{-\theta} \int_M^\infty \frac{1}{\mathbb{P}(K > \overline{F}(M_n)^{-1})} \mathbb{E}\Big(\Big(\frac{N_c\left(M_n w, M_n(w+t)\right]}{\overline{F}(M_n)}\Big)^{2+\theta}\Big)\, dw. \end{aligned}$$

As in the proof of Theorem 6.11 we have that the integral is bounded above by $C_1 \int_M^\infty w^{-r}\, dw$ for some $C_1 > 0, r > 1$. Then by (6.1) we conclude that $I_{3,1}(n) \to 0$ as $n \to \infty$.
(b) For $I_{3,2}$, notice that by Lemma 6.3

$$\begin{aligned} &\mathbb{E}\left(\frac{N_0\left(0, M_n(M+t)\right]}{\overline{F}(M_n)^{-1}} \mathbf{1}_{\{N_0(0,M_n(M+t)]>b_n\}}\right) \\ &= b_n \overline{F}(M_n) \mathbb{P}(N_0\left(0, M_n(M+t)\right] > b_n) + \overline{F}(M_n) \int_{b_n}^\infty \mathbb{P}(N_0\left(0, M_n(M+t)\right] > x)\, dx \\ &\leq b_n \overline{F}(M_n) \mathrm{e}^{-b_n \overline{F}(M_n(M+t))} + \overline{F}(M_n) \int_{b_n}^\infty \mathrm{e}^{-x\overline{F}(M_n(M+t))}\, dx \\ &= b_n \overline{F}(M_n) \mathrm{e}^{-b_n \overline{F}(M_n(M+t))} + \overline{F}(M_n(M+t))^{-1} \overline{F}(M_n) \mathrm{e}^{-b_n \overline{F}(M_n(M+t))}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} I_{3,2}(n) &\leq \frac{nM_n}{b_n} M \mathbb{E}(N_0\left(0, M_n(M+t)\right] \mathbf{1}_{\{N_0(0,M_n(M+t)]>b_n\}} \mathbf{1}_{\{K>b_n\}}) \\ &\leq M \overline{F}(M_n) b_n \mathbb{E}\left(\frac{N_0\left(0, M_n(M+t)\right]}{\overline{F}(M_n)^{-1}} \mathbf{1}_{\{N_0(0,M_n(M+t)]>b_n\}}\right) \frac{\mathbb{P}(K > b_n)}{\mathbb{P}(K > \overline{F}(M_n)^{-1})} \\ &\leq C_1 \big[(\overline{F}(M_n) b_n)^2 + \overline{F}(M_n(M+t))^{-1} \overline{F}(M_n) \overline{F}(M_n) b_n\big] \mathrm{e}^{-b_n \overline{F}(M_n(M+t))} (b_n \overline{F}(M_n))^{-\alpha_1} \\ &\overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

by (6.1). □

6.3 Auxiliary Results for the Proof of Theorem 6.11

The following proposition is the first ingredient in the proof of Theorem 6.11.

Proposition 6.13 *Let $\eta > 1$ and $M > 1$, and suppose that Assumption A and B hold. Then there exists a non-negative measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_M^\infty g(w)\, dw < \infty$ and for every $n \in \mathbb{N}$*

$$\mathbb{E}\left(\left(\frac{N_0\left(0, M_n w\right]}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \mathbf{1}_{\{N_0(0,M_n]\neq N_0(0,M_n w]\neq N_0(0,M_n(w+1)]\}}\right) \leq g(w) \quad \forall\, w \geq M.$$

The statement follows from Lemma 6.14 and Lemma 6.15 below.

Lemma 6.14 *Let $\eta > 1$ and $M > 1$, and suppose that Assumption A and B hold. Then there exists a non-negative measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_M^\infty g(w)\, dw < \infty$ and for every $n \in \mathbb{N}$*

$$\mathbb{E}\left(\left(\frac{N_0\,(0, M_n w]}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \mathbf{1}_{\{N_0(0,M_n(w-1)] \neq N_0(0,M_n w]\}}\right) \leq g(w) \quad \forall\, w \geq M.$$

Proof. Let $w \geq M$ and n so large such that $M_n^{-1} \leq 2^{-1}$. We have

$$\begin{aligned} J_1(n, w) &:= \mathbb{E}\left(\left(\frac{N_0\,(0, M_n w]}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \mathbf{1}_{\{N_0(0,M_n(w-1)] \neq N_0(0,M_n w]\}}\right) \\ &= \int_{M_n(w-1)}^{M_n w} \overline{F}(M_n w - y) \sum_{j=1}^{\infty} \left(\frac{j}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \mathbb{P}(T_j \in dy) \\ &= \left[\int_{M_n(w-1)}^{M_n w - 2} + \int_{M_n w-2}^{M_n w}\right] \overline{F}(M_n w - y) \sum_{j=1}^{\infty} \left(\frac{j}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \mathbb{P}(T_j \in dy) \\ &=: J_{1,1}(n, \omega) + J_{1,2}(n, \omega). \end{aligned}$$

Now,

$$J_{1,1}(n, \omega) \leq \sum_{k=\lfloor M_n(w-1)\rfloor -1}^{\lceil M_n w - 2\rceil -1} \overline{F}\left(M_n\left(w - \frac{k+1}{M_n}\right)\right) \sum_{j=1}^{\infty} \left(\frac{j}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \left[\mathbb{P}(T_j \leq k+1) - \mathbb{P}(T_j \leq k)\right].$$

Since $\overline{F}$ is regularly varying of index $-1/\beta$, by Potter's inequality there exists a constant $0 \leq C_1 < \infty$ such that

$$J_{1,1}(n, \omega) \leq C_1 \sum_{k=\lfloor M_n(w-1)\rfloor}^{\lceil M_n w-2\rceil -1} \left(w - \frac{k+1}{M_n}\right)^{-\frac{1}{\beta_1}} \frac{\overline{F}(M_n)^{1-\eta}}{k\overline{F}(k)^{1-\eta}} k\overline{F}(k)^{1-\eta} \sum_{j=1}^{\infty} j^{-\eta} \left[\mathbb{P}(T_j \leq k+1) - \mathbb{P}(T_j \leq k)\right].$$

Using Lemma 4.3, we obtain

$$J_{1,1}(n, \omega) \;\leq\; C_2 \sum_{k=\lfloor M_n(w-1)\rfloor}^{\lceil M_n w-2\rceil -1} \left(w - \frac{k+1}{M_n}\right)^{-\frac{1}{\beta_1}} \frac{\overline{F}(M_n)^{1-\eta}}{k\overline{F}(k)^{1-\eta}}.$$

Taking again the regular variation of $\overline{F}$ and Potter's Theorem into account yields

$$\begin{aligned} J_{1,1}(n, \omega) &\leq C_3 \sum_{k=\lfloor M_n(w-1)\rfloor}^{\lceil M_n w-2\rceil -1} \left(w - \frac{k+1}{M_n}\right)^{-\frac{1}{\beta_1}} \frac{1}{k} \left(\frac{k}{M_n}\right)^{\frac{1-\eta}{\beta_1}} \\ &= C_3 \sum_{k=\lfloor M_n(w-1)\rfloor}^{\lceil M_n w-2\rceil -1} \frac{1}{M_n} \left(w - \frac{k+1}{M_n}\right)^{-\frac{1}{\beta_1}} \left(\frac{k}{M_n}\right)^{\frac{1-\eta}{\beta_1}-1} \\ &\leq C_4 \int_{w-1}^{w} (w-z)^{-\frac{1}{\beta_1}} z^{\frac{1-\eta}{\beta_1}-1}\, dz \\ &\leq C_5 w^{\frac{1-\eta}{\beta_1}-1}, \end{aligned} \tag{6.18}$$

which is an integrable function on $[M, \infty)$ since $\eta > 1$.

Finally, using, once again, Lemma 4.3, we obtain

$$\begin{aligned} J_{1,2}(n,\omega) &\le \overline{F}(M_n)^{-\eta} \sum_{j=1}^{\infty} j^{-\eta} \left[\mathbb{P}(T_j \le M_n w) - \mathbb{P}(T_j \le M_n w - 2)\right] \\ &\le C_6 \overline{F}(M_n)^{-\eta} \frac{1}{M_n w} \overline{F}(M_n w)^{\eta-1} \\ &\le C_7 \frac{1}{M_n \overline{F}(M_n)} w^{\frac{1-\eta}{\beta_1}-1}, \end{aligned}$$

which is uniformly bounded by an integrable function. □

Lemma 6.15 *Let $\eta > 1$ and $M > 1$, and suppose that Assumption A and B hold. Then there exists a non-negative measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_M^\infty g(w)\,dw < \infty$ and for every $n \in \mathbb{N}$*

$$\mathbb{E}\left(\left(\frac{N_0\,(0, M_n w]}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \mathbf{1}_{\{N_0(0,M_n] \neq N_0(0,M_n(w-1)] = N_0(0,M_n w] \neq N_0(0,M_n(w+1)]\}}\right) \le g(w) \quad \forall\, w \ge M.$$

Proof. As in the previous lemma,

$$\begin{aligned} J_1(n,w) &:= \mathbb{E}\left(\left(\frac{N_0\,(0, M_n w]}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \mathbf{1}_{\{N_0(0,M_n] \neq N_0(0,M_n(w-1)] = N_0(0,M_n w] \neq N_0(0,M_n(w+1)]\}}\right) \\ &= \int_{M_n}^{M_n(w-1)} [\overline{F}(M_n w - y) - \overline{F}(M_n(w+1) - y)] \sum_{j=1}^{\infty} \left(\frac{j}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \mathbb{P}(T_j \in dy) \\ &\le \sum_{k=\lfloor M_n \rfloor}^{\lceil M_n(w-1) \rceil - 1} \left[\overline{F}(M_n w - k - 1) - \overline{F}(M_n w + M_n - k)\right] \\ &\quad \times \sum_{j=1}^{\infty} \left(\frac{j}{\overline{F}(M_n)^{-1}}\right)^{-\eta} \left[\mathbb{P}(T_j \le k+1) - \mathbb{P}(T_j \le k)\right]. \end{aligned}$$

By Lemma 4.3 we have for n large

$$J_1(n,w) \le C_1 \sum_{k=\lfloor M_n \rfloor}^{\lceil M_n(w-1) \rceil - 1} \frac{\overline{F}(M_n w - k - 1) - \overline{F}(M_n w + M_n - k)}{\overline{F}(M_n)} \frac{\overline{F}(M_n)^{1-\eta}}{k \overline{F}(k)^{1-\eta}}.$$

Note that for every k in the above sum by Assumption B

$$\begin{aligned} \overline{F}(M_n w - k - 1) - \overline{F}(M_n w + M_n - k) &\le \sum_{j=-1}^{\lceil M_n \rceil - 1} \left[\overline{F}(M_n w - k + j) - \overline{F}(M_n w - k + j + 1)\right] \\ &\le C_2 \sum_{j=-1}^{\lceil M_n \rceil - 1} \frac{\overline{F}(M_n w - k + j)}{M_n w - k + j} \le C_3 M_n \frac{\overline{F}(M_n w - k - 1)}{M_n w - k - 1}. \end{aligned}$$

We conclude by Potter's Theorem that for large n and all k as above

$$\frac{\overline{F}(M_n w - k - 1) - \overline{F}(M_n w + M_n - k)}{\overline{F}(M_n)} \le C_4 \left(w - \frac{k+1}{M_n}\right)^{-\frac{1}{\beta_2}-1}.$$

Hence, we obtain

$$J_1(n,w) \le C_4 \sum_{k=\lfloor M_n \rfloor}^{\lceil M_n(w-1)\rceil -1} \left(w - \frac{k+1}{M_n} \right)^{-\frac{1}{\beta_2}-1} \frac{\overline{F}(M_n)^{1-\eta}}{k\overline{F}(k)^{1-\eta}}.$$

Similar calculations as in (6.18) result in

$$J_1(n,w) \le C_5 \int_1^{w-1} (w-z)^{-\frac{1}{\beta_2}-1} z^{\frac{1-\eta}{\beta_2}-1}\, dz \le C_6 w^{-\frac{\eta-1}{\beta_2}-1},$$

as an easy computation shows. This is an integrable on $[M,\infty)$ function. □

The final two lemmas needed for the proof of Theorem 6.11 follow.

Lemma 6.16 *Let $M>0$, and suppose that Assumption A hold. Then there exists a positive constant $C<\infty$ such that for every $n \in \mathbb{N}$*

$$\frac{nM_n}{b_n^2} \mathbb{E}(N_c\,(M_n w, M_n(w+1)]^2) \le C \quad \forall\, w \le M.$$

Proof. It is clearly enough to establish the required bound for n large enough. By Potter's inequality and Karamata's theorem, we obtain for all n large enough and $0<w\le M$

$$\begin{aligned}
&\frac{nM_n}{b_n^2} \mathbb{E}(N_c\,(M_n w, M_n(w+1)]^2) \\
&\le \frac{nM_n}{b_n^2} \mathbb{E}(N_c\,(0, M_n(M+1)]^2) \\
&= \frac{nM_n}{b_n^2} \mathbb{E}\left(N_0\,(0, M_n(M+1)]^2\, \mathbf{1}_{\{K > N_0(0,M_n(M+1)]>0\}} \right) + \frac{nM_n}{b_n^2} \mathbb{E}\left(K^2\, \mathbf{1}_{\{K \le N_0(0,M_n(M+1)]\}} \right) \\
&\le C_1 \mathbb{E}\left(\left(\frac{N_0\,(0, M_n(M+1)]}{\overline{F}(M_n)^{-1}} \right)^{2-\alpha_1} \right) + C_2 \mathbb{E}\left(\left(\frac{N_0\,(0, M_n(M+1)]}{\overline{F}(M_n)^{-1}} \right)^{2-\alpha_2} \right).
\end{aligned}$$

The right hand side is bounded for n large enough by computations similar to (6.17). □

Lemma 6.17 *Let $M>1$ and suppose that Assumption A and B hold. Then there exists a non-negative measurable function $g:\mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_M^\infty g(w)\, dw < \infty$ and for every $n \in \mathbb{N}$*

$$\frac{nM_n}{b_n^2} \mathbb{E}\left(N_c\,(M_n w, M_n(w+1)]^2 \;\; 1_{\{N_0(0,M_n] = N_0(0,M_n w] \ne N_0(0,M_n(w+1)]\}} \right) \le g(w) \quad \forall\, w \ge M.$$

Proof. We define $B_{n,w} := \{N_0\,(0,M_n] = N_0\,(0,M_n w] \ne N_0\,(0, M_n(w+1)]\}$. Notice that by Lemma 6.2,

$$\frac{nM_n}{b_n^2} \mathbb{E}(N_c\,(M_n w, M_n(w+1)]^2\, \mathbf{1}_{B_{n,w}}) \le \frac{nM_n}{b_n^2} \mathbb{E}\left(\min\left(N_0\,(0,M_n]\,, K\right)^2 \right) \mathbb{P}(B_{n,w}). \qquad (6.19)$$

Further,

$$\begin{aligned}
\mathbb{E}\left(\min\left(N_0\,(0,M_n]\,, K\right)^2 \right) &= 2\int_0^\infty t\, \mathbb{P}\big(N_0\,(0,M_n] > t\big) \mathbb{P}(K>t)\, dt \\
&= 2\Big[\int_0^{\overline{F}(M_n)^{-1}} + \int_{\overline{F}(M_n)^{-1}}^\infty \Big] t\, \mathbb{P}\big(N_0\,(0,M_n] > t\big) \mathbb{P}(K>t)\, dt.
\end{aligned}$$

Since by Karamata's theorem, as $n \to \infty$,

$$\begin{aligned}
\int_0^{\overline{F}(M_n)^{-1}} t\,\mathbb{P}\big(N_0\,(0, M_n] > t\big)\mathbb{P}(K > t)\,dt &\leq \int_0^{\overline{F}(M_n)^{-1}} t\mathbb{P}(K > t)\,dt \\
&\sim C_1\big(\overline{F}(M_n)^{-1}\big)^2\mathbb{P}\big(K > \overline{F}(M_n)^{-1}\big)
\end{aligned}$$

and by (6.17)

$$\begin{aligned}
\int_{\overline{F}(M_n)^{-1}}^{\infty} t\,\mathbb{P}\big(N_0\,(0, M_n] > t\big)\mathbb{P}(K > t)\,dt &\leq C_2\mathbb{P}\big(K > \overline{F}(M_n)^{-1}\big)\mathbb{E}\left(N_0\,(0, M_n]^2\right) \\
&\leq C_3\big(\overline{F}(M_n)^{-1}\big)^2\mathbb{P}\big(K > \overline{F}(M_n)^{-1}\big),
\end{aligned}$$

we have the bound

$$\mathbb{E}\left(\min\big(N_0\,(0, M_n]\,, K\big)^2\right) \leq C_4\big(\overline{F}(M_n)^{-1}\big)^2\mathbb{P}\big(K > \overline{F}(M_n)^{-1}\big).$$

On the other hand, by Assumption B and the same arguments as in (6.17),

$$\begin{aligned}
\mathbb{P}(B_{n,w}) &= \mathbb{P}\left(N_0\,(0, M_n] = N_0\,(0, M_n w] \neq N_0\,(0, M_n(w+1)]\right) \\
&= \sum_{j=0}^{\infty}\int_0^{M_n}\Big[\overline{F}(M_n w - y) - \overline{F}(M_n w + M_n - y)\Big]\mathbb{P}(T_j \in\, dy) \\
&\leq \Big[\overline{F}(M_n w - M_n) - \overline{F}(M_n w + M_n)\Big]\mathbb{E}(N_0\,(0, M_n]) \\
&\leq \sum_{k=0}^{\lfloor M_n\rfloor}\Big[\overline{F}(M_n w - M_n + k) - \overline{F}(M_n w + k + 1)\Big]\Big[C_5\overline{F}(M_n)^{-1}\Big] \\
&\leq C_6\overline{F}(M_n)^{-1}\sum_{k=0}^{\lfloor M_n\rfloor}\frac{\overline{F}(M_n w - M_n + k)}{M_n w - M_n + k} \\
&\leq C_7\overline{F}(M_n)^{-1}M_n\frac{\overline{F}(M_n w - M_n)}{M_n w - M_n} \\
&\leq C_8 w^{-1}\frac{\overline{F}(M_n w)}{\overline{F}(M_n)}. \qquad (6.20)
\end{aligned}$$

We conclude that

$$\frac{nM_n}{b_n^2}\mathbb{E}(N_c\,(M_n w, M_n(w+1)]^2\,\mathbf{1}_{B_{n,w}}) \leq C_9 w^{-1}\frac{\overline{F}(M_n w)}{\overline{F}(M_n)} \leq C_{10}w^{-1-1/\beta_2},$$

which is an integrable function on $[M, \infty)$. □

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