Extremes of Lévy Driven Mixed MA Processes with Convolution Equivalent Distributions

Vicky Fasen *

October 15, 2008

Abstract

We investigate the extremal behavior of stationary mixed MA processes $Y(t) = \int_{\mathbb{R}_+\times\mathbb{R}} f(r, t-s)\, d\Lambda(r,s)$ for $t \geq 0$, where f is a deterministic function and Λ is an infinitely divisible and independently scattered random measure. Particular examples of mixed MA processes are superpositions of Ornstein-Uhlenbeck processes, applied as stochastic volatility models in [2]. We assume that the finite dimensional distributions of Λ are in the class of convolution equivalent tails and in the maximum domain of attraction of the Gumbel distribution. On the one hand, we compute the tail behavior of $Y(0)$ and $\sup_{t\in[0,1]} Y(t)$. On the other hand, we study the extremes of Y by means of marked point processes based on maxima of Y in random intervals. A complementary result guarantees the convergence of the running maxima of Y to the Gumbel distribution.

AMS 2000 Subject Classifications: primary: 60G70
secondary: 60F05, 60G10, 60G55

Keywords: convolution equivalent distribution, extreme value theory, MA process, marked point process, mixed MA process, point process, random measure, shot noise process, subexponential distribution, supOU process

*Center for Mathematical Sciences, Technische Universität München, D-85747 Garching, Germany, email: fasen@ma.tum.de. Financial support from the Deutsche Forschungsgemeinschaft through a research grant is gratefully acknowledged.

1 Introduction

In this paper we consider *mixed moving average (MA) processes* Y which have the representation

$$Y(t) = \int_{\mathbb{R}_+ \times \mathbb{R}} f(r, t-s)\, d\,\Lambda(r,s) \quad \text{for } t \geq 0, \tag{1.1}$$

where $\mathbb{R}_+ = (0,\infty)$, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a measurable function called the *kernel function*, and $\Lambda = \{\Lambda(A) : A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\}$ is an *infinitely divisible, independently scattered random measure* (idisrm). Here $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ denotes the Borel sets in $\mathbb{R}_+ \times \mathbb{R}$.

First, we recall the definition of an idisrm. A stochastic process $\{\Lambda(A) : A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\}$ is an idisrm if for any $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ and any disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$:

– $\Lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Lambda(A_n)$ almost surely (a. s.).

– $(\Lambda(A_n))_{n\in\mathbb{N}}$ are independent (independently scattered (i. s.) property).

– $\Lambda(A)$ is infinitely divisible (i. d.).

Any stochastic process satisfying the first property is a random measure. A *completely random measure* as defined in Kingman [19] is a positive independently scattered random measure in the sense that all finite dimensional distributions of Λ are positive. The reader is referred to Rajput and Rosinski [23], Urbanik and Woyczyński [30] and Kwapień and Woyczyzński [20] for more details on idisrm and integrals as in (1.1). The expression *Lévy random field* is a synonym for idisrm which can be found in the literature, e. g., Barndorff-Nielsen and Shephard [2].

In this paper we confine our attention to idisrm whose characteristic function can be written in the form

$$\mathbb{E}\exp(iu\Lambda(A)) = \exp(\psi(u)\Pi(A)) \quad \text{for } u \in \mathbb{R}, \tag{1.2}$$

where Π is a measure on $\mathbb{R}_+ \times \mathbb{R}$, which is the product of a probability measure π on $\mathbb{R}_+$ and the Lebesgue measure λ on $\mathbb{R}$, i. e. $\Pi = \pi \times \lambda$, and

$$\psi(u) = ium - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{iux} - 1 - iu\,\mathbf{1}_{[-1,1]}(x)\right)\, \nu(dx) \; \text{ for } u \in \mathbb{R}.$$

The function ψ is the cumulant generating function of an i. d. random variable with *generating triplet* (m, σ^2, ν), where $m \in \mathbb{R}$, $\sigma^2 \geq 0$, and ν is a measure on $\mathbb{R}$, called *Lévy measure*, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\,\nu(dx) < \infty$. Especially

$$\mathbb{E}\exp(iu\Lambda(\mathbb{R}_+ \times [0,t])) = \exp(t\psi(u)) \quad \text{for } u \in \mathbb{R},$$

which is the characteristic function of a Lévy process with generating triplet (m, σ^2, ν). Thus, we denote by

$$L(t) = \Lambda(\mathbb{R}_+ \times [0,t]) \quad \text{for } t \geq 0, \tag{1.3}$$

the *underlying driving Lévy process.* The *generating quadruple* (m, σ^2, ν, π) determines completely the distribution of Λ.

In particular, if $f(r,s) = g(s)$ for $(r,s) \in \mathbb{R}_+ \times \mathbb{R}$, where $g : \mathbb{R} \to \mathbb{R}$ is measurable, then the MA process

$$Y(t) = \int_{\mathbb{R}} g(t-s)\, dL(s) \quad \text{for } t \in \mathbb{R}, \tag{1.4}$$

is a special case of a mixed MA process. A typical example of Y given in (1.1) is the supOU process defined by Barndorff-Nielsen [1] and used as a stochastic volatility model in Barndorff-Nielsen and Shephard [2,3]. This model has the potential to model long memory often observed in financial data (cf. the overview paper Fasen and Klüppelberg [14]). Continuous-time processes are particularly appropriate models for irregularly spaced and high frequency data, so that interest in these models is increasing.

The paper can be considered a continuation of Fasen [12,13] on the extremal behavior of regularly varying mixed MA processes and subexponential Lévy driven MA processes. In the present paper we investigate finite dimensional distributions of Λ, represented by $L(1)$, that have probability tails decreasing at most exponentially fast, but faster than polynomial. More precisely, we assume that the distribution function of $L(1)$ is in the class of *convolution equivalent distributions*, denoted by $\mathcal{S}(\gamma)$. We recall that a distribution function F on $\mathbb{R}$ with $F(x) < 1$ for every $x \in \mathbb{R}$ is in $\mathcal{S}(\gamma)$ for some $\gamma \geq 0$, if and only if the following conditions hold:

(i) For all $y \in \mathbb{R}$ locally uniformly $\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = \exp(-\gamma y)$, i.e. $F \in \mathcal{L}(\gamma)$.

(ii) $\lim_{x\to\infty} \overline{F * F}(x)/\overline{F}(x)$ exists and is finite.

Here $\overline{F}(\cdot) = 1 - F(\cdot)$. If $F \in \mathcal{S}(\gamma)$ and Z is a random variable with distribution function F, then we write $Z \in \mathcal{S}(\gamma)$. The class $\mathcal{S}(0) = \mathcal{S}$ is called *subexponential* because no exponential moments exist. Hence, the subexponential Lévy driven MA processes studied in [13] is a subclass of our general model. Most of the literature on convolution equivalent distributions is formulated for positive random variables, but we can think of $Z \in \mathcal{S}(\gamma)$ if and only if $Z^+ \in \mathcal{S}(\gamma)$. Moreover, we study only distributions of $L(1)$ in MDA($\mathcal{G}$), where MDA($\mathcal{G}$) denotes the *maximum domain of attraction* of the Gumbel distribution. The interested reader is referred to Embrechts et al. [10], Chapter 3, for more details on classical extreme value theory.

The paper is organized as follows. We start with preliminaries in Section 2, where, in Section 2.1, the precise assumptions on the family of kernel functions f, the sample paths of Y and the tail behavior of $L(1)$ are presented. Next, we conclude the preliminaries (Section 2.2) with the most important properties of the class of convolution equivalent distributions. In Section 3 and Section 4 we state the main results. Section 3 is devoted to the tail behavior of the stationary distribution of Y and its Lévy measure, and the tail behavior of $M(h) = \sup_{0\leq t\leq h} Y(t)$ for any $h > 0$. Here, the tail of $M(h)$ is again in $\mathcal{S}(\gamma)$. If the finite dimensional distributions of Λ are only subexponential, then the computation of the tail behavior of $M(h)$ results from a straightforward application of Rosinski and Samorodnitsky [26], Theorem 3.1. The case $\gamma > 0$ is much more involved. For this case, Braverman and Samorodnitsky [5] have only deduced upper and lower bounds for the tail behavior of functionals of i. d. stochastic processes with tails in $\mathcal{S}(\gamma)$, which also apply to the maxima of mixed MA processes. In contrast to them, we are able to present the explicit tail behavior. Section 4, on the other hand, deals with the behavior of large exceedances of Y as the size and the time of occurrence of exceedances. This is done by marked point processes. A complementary result guarantees the convergence of running maxima of Y to the Gumbel distribution, which is useful for statistical analysis.

Finally, Section 5 is devoted to the more technical proofs of our results.

2 Preliminaries

In this section we introduce the general conditions of this paper and state important properties of the class of convolution equivalent distributions for later reference.

First, we introduce the notation: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ we write $f^+ := \sup_{(r,t)\in\mathbb{R}_+\times\mathbb{R}} f^+(r,t)$ and $f^- := \sup_{(r,t)\in\mathbb{R}_+\times\mathbb{R}} f^-(r,t) < \infty$ with $f^+(r,t) := \max\{f(r,t), 0\}$ and $f^-(r,t) := \max\{-f(r,t), 0\}$. We define for $\delta > 0$,

$$\mathbb{L}^\delta(\Pi) := \left\{ f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \text{ measurable}, \int_{\mathbb{R}_+\times\mathbb{R}} |f(r,s)|^\delta \, d\,\Pi(r,s) < \infty \right\}.$$

If $f \in \mathbb{L}^\delta(\Pi)$ for some $\delta > 0$, then for every $y > 0$ the sets $B^+_{f,y} := \{(r,s) \in \mathbb{R}_+ \times \mathbb{R} : f(r,s) \geq y\}$, $B^-_{f,y} := \{(r,s) \in \mathbb{R}_+ \times \mathbb{R} : f(r,s) \leq -y\}$ and $A_{f,y} := B^+_{f,y} + B^-_{f,y}$ are Borel measurable. The symbol $\overset{n\to\infty}{\Longrightarrow}$ denotes weak convergence as $n \to \infty$, and $\overset{P}{\longrightarrow}$ denotes convergence in probability. $X \overset{d}{=} Y$, if the distributions of the random variables X and Y coincide. For real functions g and h we abbreviate $g(t) \sim h(t)$ as $t \to \infty$, if $g(t)/h(t) \to 1$ as $t \to \infty$.

2.1 Assumptions

The stochastic integral in (1.1) is defined for each fixed $t \geq 0$ as a limit in probability of integrals over simple functions; see Rajput and Rosinski [23], Theorem 2.7, who formulated conditions in terms of the kernel function f and the generating quadruple of Λ, such that the stochastic integral $\int_{\mathbb{R}_+\times\mathbb{R}} f(r,s)\, d\Lambda(r,s)$ is well-defined. In order to use the results of [23] we need the following condition.

Condition (M). *Let Y be a mixed MA process as in (1.1). The Lévy measure ν of Λ satisfies $\nu\,(1, \cdot \vee 1]\,/\nu(1,\infty) \in \mathrm{MDA}(\mathcal{G})$ with infinite right endpoint and the tail balance condition*

$$\lim_{x\to\infty} \frac{\nu\,(-\infty, -x)}{\nu\,(x,\infty)} = \frac{1-p}{p} \tag{2.1}$$

for some $p \in (0,1]$. *The kernel function* $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is bounded, and one of the following conditions holds:*

(M1) $f \in \mathbb{L}^1(\Pi)$.

(M2) $f \in \mathbb{L}^2(\Pi)$ *and* $\mathbb{E}(L(1)) = 0$.

If f *is positive, (2.1) can be dropped.*

If we apply [23] to our set-up, we obtain the following conclusion, analogous to Proposition 4 in Fasen [13] (for the proof see Section 5.1).

Proposition 2.1 *Let* Y *be a mixed MA process as in (1.1) satisfying condition (M). Then* Y *has a stationary version, and the stationary distribution* Y_0 *is i. d. The generating triplet of* Y_0 *is* (m_Y, σ_Y^2, ν_Y), *where for* $x > 0$,

$$\begin{aligned} m_Y &= \int_{\mathbb{R}_+\times\mathbb{R}} \left(mf(r,s) + \int_{\mathbb{R}} \left(\mathbf{1}_{[-1,1]}(xf(r,s)) - f(r,s)\,\mathbf{1}_{[-1,1]}(x)\right)\,\nu(dx)\right) d\,\Pi(r,s), \\ \sigma_Y^2 &= \sigma^2 \int_{\mathbb{R}_+\times\mathbb{R}} f(r,s)^2\, d\,\Pi(r,s), \qquad (2.2) \\ \nu_Y\,(x,\infty) &= \int_{\mathbb{R}_+\times\mathbb{R}} \nu\left(\frac{x}{f^+(r,s)},\infty\right) d\,\Pi(r,s) + \int_{\mathbb{R}_+\times\mathbb{R}} \nu\left(-\infty, \frac{-x}{f^-(r,s)}\right) d\,\Pi(r,s). \end{aligned}$$

In addition, we require conditions on the sample paths of Y. An important issue in our work is the decomposition of Λ into two independent idisrm according to the jump sizes of the underlying driving Lévy process L represented by ν. We define

$$\Lambda = \Lambda_1 + \Lambda_2, \tag{2.3}$$

where the idisrm Λ_1 has the generating quadruple $(0, 0, \nu_1, \pi)$ with

$$\nu_1\,(A) = \nu(A \cap (1,\infty)) \quad \text{for } A \in \mathcal{B}(\mathbb{R}),$$

and Λ_2 has the generating quadruple $(m, \sigma^2, \nu_2, \pi)$ with $\nu_2 = \nu - \nu_1$. If $f \in \mathbb{L}^1(\Pi)$ this decomposition of Λ induces a decomposition of Y in $Y = Y_1 + Y_2$, where

$$Y_i(t) = \int_{\mathbb{R}_+\times\mathbb{R}} f(r, t-s)\, d\Lambda_i(r,s) \qquad \text{for } t \in \mathbb{R},\ i = 1, 2, \tag{2.4}$$

are independent mixed MA processes, since Λ_1 and Λ_2 are independent.

Let $N = (N(t))_{t\in\mathbb{R}}$ be a Poisson process with intensity $\mu = \nu_1(\mathbb{R}) = \nu(1,\infty) > 0$ and jump times $\Gamma = (\Gamma_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $-\infty < \ldots < \Gamma_{-1} \le 0 < \Gamma_1 < \ldots < \infty$. We define $\Gamma_0 := 0$. Further, let $Z = (Z_k)_{k\in\mathbb{Z}}$ be an iid sequence with distribution function $\mathbb{P}(Z_1 \le x) = \nu_1\,(-\infty, x]\,/\mu$, $x \in \mathbb{R}$, and $R = (R_k)_{k\in\mathbb{Z}}$ be an iid sequence with distribution π. Finally, we assume Γ, Z, R and Λ_2 are independent. Then

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Z_k\,\mathbf{1}_{\{(R_k,\Gamma_k)\in A\}} \qquad \text{for } A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \tag{2.5}$$

is an idisrm with the generating quadruple $(0, 0, \nu_1, \pi)$, and the underlying driving Lévy process $L_1(t) = \sum_{k=1}^{N(t)} Z_k$ is a compound Poisson process.

Throughout the paper we use either condition (M) or the more restrictive condition (G) presented below. The choice of condition (M) or (G) is indicated clearly in every result.

Condition (G).

(i) Let Y be a separable (cf. Definition 9.2.3 in Samorodnitsky and Taqqu [27]) and a stationary version of the mixed MA process as in (1.1) on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Furthermore,

$$\mathbb{P}\left(\sup_{t\in[0,1]} |Y(t)| < \infty\right) = 1.$$

(ii) The kernel function f satisfies $f(r, 0) = 1$, $f^- < 1$ and $f(r, \cdot)$ is right continuous. Moreover, there exists a measurable function $f^* : \mathbb{R}_+ \times \mathbb{R} \to [0, 1]$ with $|f(r, s)| \leq f^*(r, s)$ for $(r, s) \in \mathbb{R}_+ \times \mathbb{R}$, $f^*(r, s) = 0$ for $(r, s) \in \mathbb{R}_+ \times (-\infty, 0)$, $f^*(r, \cdot)$ is non-increasing for any $r \in \mathbb{R}_+$, and $f^*(r, s) \to 0$ as $s \to \infty$ π-almost everywhere. Furthermore, we suppose for every $\epsilon > 0$ there exists a $y > 0$ such that $0 < \Pi(B^+_{f,y}) \leq \Pi(A_{f^*,y}) \leq \epsilon$.

(iii) The kernel function $f^* \in \mathbb{L}^1(\Pi)$.

(iv) Moreover, $L(1) \in \mathrm{MDA}(\mathcal{G}) \cap \mathcal{S}(\gamma)$ for some $\gamma \geq 0$, with $a_T > 0$, $b_T \in \mathbb{R}$ and $u_T = a_T x + b_T$ for $x \in \mathbb{R}$ such that

$$\lim_{T\to\infty} T\mathbb{P}(L(1) > u_T) = \exp(-x).$$

If $f^- < 0$ then we suppose also (2.1).

Remark 2.2 Assumption *(ii)* is a restriction on the family of kernel functions. The function f^* is an integrable upper bound of $|f|$. In particular, for a right continuous kernel function $f : [0, \infty) \to \mathbb{R}$ of a MA process *(ii)* means that there is a non-increasing integrable function $f^* : [0, \infty) \to [0, \infty)$ with $f^*(0) = f(0) = 1$ and $|f(s)| \leq f^*(s)$ for $s \geq 0$.

If $f^* \in \mathbb{L}^1(\Pi)$, as stated in *(iii)*, then $f \in \mathbb{L}^1(\Pi)$ as well. Thus, *(iii)* and *(iv)* imply that Y satisfies condition (M). □

In the following we take for Y_1, Y_2 as given in (2.4) the following versions:

(i') $Y_1(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{N(t)} f(R_k, t - \Gamma_k) Z_k,\ t \geq 0,$ pathwise a.s.

(ii') Y_1, Y_2 are separable.

And we define

(iii') $\widetilde{Y}(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{N(t)} f^*(R_k, t - \Gamma_k) Z_k, t \geq 0,$ pathwise a.s.

Remark 2.3 These versions of Y_1, Y_2 and $\widetilde{Y}$ exist for the following reason. We can take for Λ_1 the version given in (2.5). Since $\Lambda_1(\omega)$ is a measure for any $\omega \in \Omega$ and f^* is positive, the process $\widetilde{Y}$ exist pathwise. Let $\Omega_0 \in \mathcal{A}$ with $\mathbb{P}(\Omega_0) = 1$ such that $\widetilde{Y}(0)(\omega) < \infty$ for any $\omega \in \Omega_0$. Set Ω_0 exists since $f^* \in \mathbb{L}^1(\Pi)$. Next, let $\Omega_t \in \mathcal{A}$ with $\mathbb{P}(\Omega_t) = 1$ such that $L_1(t)(\omega) < \infty$ for any $\omega \in \Omega_t$. Then $|\widetilde{Y}(s)(\omega)| \leq L_1(t)(\omega) + \widetilde{Y}(0)(\omega) < \infty$ for any $s \in [0,t], \omega \in \Omega_0 \cap \Omega_t$. Hence, $\widetilde{Y}$ is finite on every compact set a.s. Thus, by dominated convergence we obtain that the version of Y_1 is well defined as well. Since $f(r,\cdot)$ is right continuous and $f^*(r,\cdot)$ is non-increasing, we obtain again by dominated convergence that Y_1 is right continuous as well. In particular, it follows that Y_1 is separable. Since Y is separable by assumption (G) we conclude that Y_2 is separable. □

Example 2.4 (SupOU process) A prominent example of a mixed MA process is the supOU process (superposition of Ornstein-Uhlenbeck processes) defined by

$$Y(t) = \int_{\mathbb{R}_+\times\mathbb{R}} \mathbf{1}_{[0,\infty)}(t-s)\mathrm{e}^{-r(t-s)}\, d\,\Lambda(r,s) \quad \text{for } t \in \mathbb{R} \tag{2.6}$$

with kernel function $f(r,s) = \mathrm{e}^{-rs}\,\mathbf{1}_{[0,\infty)}(s)$. An important special case of (2.6) is the OU (Ornstein-Uhlenbeck) process, for which π has support only in some point λ, where $\lambda > 0$, i.e. $\pi(\{\lambda\}) = 1$. For a probability measure π and for some $\delta > 0$ we have $f \in \mathbb{L}^\delta(\Pi)$ if and only if $\int_{\mathbb{R}_+\times\mathbb{R}_+} \mathrm{e}^{-rs\delta}\, d\,\Pi(r,s) = \delta^{-1}\int_{\mathbb{R}_+} r^{-1}\,\pi(dr) < \infty$. Thus, we assume $\lambda^{-1} := \int_{\mathbb{R}_+} r^{-1}\,\pi(dr) < \infty$. SupOU processes satisfy condition *(ii)* with $f^* = f$. □

2.2 The class $\mathcal{S}(\gamma)$ of convolution equivalent distributions

The following Proposition summarizes most known properties of $\mathcal{S}(\gamma)$ needed for this paper.

Proposition 2.5 *For a distribution function F we write $m_F(\gamma) = \int_{\mathbb{R}} \mathrm{e}^{\gamma y}\, F(dy)$.*

(i) If $F \in \mathcal{L}(\gamma)$, then $\lim_{x\to\infty} \mathrm{e}^{\epsilon x}\overline{F}(x) = \infty$ for any $\epsilon > \gamma$.
(ii) Suppose $F \in \mathcal{S}(\gamma)$, G is a distribution function with $\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x) = q \geq 0$ and $m_G(\gamma) < \infty$. Then $\lim_{x\to\infty} \overline{F * G}(x)/\overline{F}(x) = m_G(\gamma) + q m_F(\gamma)$ and $F * G \in \mathcal{S}(\gamma)$. If $q > 0$, then also $G \in \mathcal{S}(\gamma)$. Particularly, $\lim_{x\to\infty} \overline{F * F}(x)/\overline{F}(x) = 2m_F(\gamma)$.
(iii) Let F be an i. d. distribution function with Lévy measure ν. Then

$$F \in \mathcal{S}(\gamma) \iff \frac{\nu\,[1,\cdot\vee 1)}{\nu\,[1,\infty)} \in \mathcal{S}(\gamma) \iff \overline{F}(x) \sim m_F(\gamma)\nu(x,\infty) \text{ as } x \to \infty,$$

$$m_F(\gamma) < \infty \quad \text{and} \quad \frac{\nu\,[1,\cdot\vee 1)}{\nu\,[1,\infty)} \in \mathcal{L}(\gamma).$$

(iv) If X, Y are i. d., $X \in \mathcal{S}(\gamma)$ and $\nu_Y(x,\infty)/\nu_X(x,\infty) \stackrel{x\to\infty}{\longrightarrow} 0$. Then

$$\mathbb{P}(Y > x) = o(\mathbb{P}(X > x)) \quad \text{as } x \to \infty.$$

(v) Let $F \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\mathcal{G})$ with a_n, b_n sequences of constants such that $\lim_{n\to\infty} n\overline{F}(a_n x + b_n) = \exp(-x)$*. Then $b_n \to \infty$, $a_n \to \gamma^{-1}$, where for $\gamma = 0$ we interpret γ^{-1} to be infinity, and $b_n/a_n \to \infty$ as $n \to \infty$. Furthermore, the tail of F is* rapidly varying, *i. e.* $\lim_{x\to\infty} \overline{F}(xt)/\overline{F}(x) = 0$ *for* $t > 1$.

Proof. *(i)* Embrechts and Goldie [9], Lemma 2.4. *(ii)* Goldie and Resnick [16], Theorem 2.3. *(iii)* Watanabe [31], Theorem 1.1, who presents the correct proof of Cline [6] and Pakes [22], who stated this result earlier.
(iv) Step 1. Let the characteristic triplet of Y be of the form $(0, 0, \nu_Y)$ where

$$\nu_Y(x, \infty) = \mu G(x), \quad x \geq 1, \quad \text{and} \quad \nu_Y(-\infty, 1) = 0$$

for some $\mu > 0$ and a distribution function G on $[1, \infty)$. Then by Sato [28], Remark 27.3,

$$\mathbb{P}(Y > x) = \mathrm{e}^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \overline{G^{*n}}(x).$$

Since $X \in \mathcal{S}(\gamma)$, we have that $\overline{G}(x)/\overline{F}_X(x) \to 0$ as $x \to \infty$, where F_X denotes the distribution function of X. Then $\lim_{x\to\infty} \overline{G^{*n}}(x)/\overline{F_X^{*n}}(x) = 0$, for any $n \in \mathbb{N}$, and hence, by *(ii)* $\lim_{x\to\infty} \overline{G^{*n}}(x)/\overline{F}_X(x) = 0$. Further, by Watanabe [31], Theorem C, we have

$$\mathrm{e}^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \overline{F_X^{*n}}(x) \sim \mathrm{e}^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} n m_F(\gamma)^{n-1} \overline{F}_X(x) \quad \text{as } x \to \infty,$$

such that by dominated convergence

$$\mathbb{P}(Y > x) = \mathrm{e}^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \overline{G^{*n}}(x) = o(\overline{F}(x)) \quad \text{as } x \to \infty.$$

Hence, (iv) is satisfied.
Step 2. A decomposition of Y as in (2.4) and Step 1 give the result.
(v) Is a result of Karamata's representation for regularly varying functions and $\overline{F} \circ \log \in \mathcal{R}_{-\gamma}$. □

A typical example of a distribution function in $\mathcal{S} \cap \mathrm{MDA}(\mathcal{G})$ is the heavy tailed Weibull distribution and the lognormal distribution. Prominent examples of distribution functions in $\mathcal{S}(\gamma) \cap \mathrm{MDA}(\mathcal{G})$, $\gamma > 0$, are subclasses of the IG (inverse Gaussian), GIG (generalized inverse Gaussian distribution), GH (generalized hyperbolic) and CGMY (Carr, Geman, Madan and Yor) distributions, which are important in finance. We refer to Cline [6] for more details on distribution functions in $\mathcal{S}(\gamma) \cap \mathrm{MDA}(\mathcal{G})$, and to Schoutens [29] for applications of the above mentioned distribution functions to finance.

3 Tail behavior

If we use the representation of the Lévy measure of Y_0 in (2.2) and that the tails of the Lévy measure and the probability measure in $\mathcal{S}(\gamma)$ are equivalent (Proposition 2.5 *(iii)*), then we obtain the analogous results of Proposition 5 and Lemma 6 in Fasen [13] for mixed MA processes.

Proposition 3.1 *Let Y be a mixed MA process as in (1.1) satisfying condition (M). Let $y > 0$ and $\Pi(B^+_{f,y}) > 0$. Then as $x \to \infty$,*

$$\nu_Y(x, \infty) \sim \int_{B^+_{f,y}} \nu\left(\frac{x}{f^+(r,s)}, \infty\right) d\,\Pi(r,s) + \int_{B^-_{f,y}} \nu\left(-\infty, \frac{-x}{f^-(r,s)}\right) d\,\Pi(r,s). \tag{3.1}$$

Hence, $Y_0 \in \mathcal{S}(\gamma)$ if and only if the right hand side of (3.1) is in $\mathcal{S}(\gamma)$. In that case

$$\begin{aligned}\mathbb{P}(Y_0 > x) \;\sim\; & \frac{\mathbb{E}e^{\gamma Y_0}}{\mathbb{E}e^{\gamma L(1)}} \int_{B^+_{f,y}} \mathbb{P}\left(f^+(r,s)L(1) > x\right) d\,\Pi(r,s) \\ & + \frac{\mathbb{E}e^{\gamma Y_0}}{\mathbb{E}e^{-\gamma L(1)}} \int_{B^-_{f,y}} \mathbb{P}\left(-f^-(r,s)L(1) > x\right) d\,\Pi(r,s) \quad \text{as } x \to \infty.\end{aligned} \tag{3.2}$$

In some special cases (3.2) has a simpler structure. For example, let Y be an i. d. stationary MA process with kernel function $|f(s)| \to 0$ as $|s| \to \infty$, $\Pi(B^+_{f,y}) > 0$ for some $y > 0$ and $A_{f,y} \subseteq \mathbb{R}_+ \times (-s_0, s_0)$ for some $s_0 > 0$. Then we have $Y_0 \in \mathcal{S}(\gamma)$ if and only if as $x \to \infty$,

$$\mathbb{P}(Y_0 > x) \sim 2s_0 \frac{\mathbb{E}e^{\gamma Y_0}}{\mathbb{E}e^{\gamma L(1)}} \mathbb{P}(f^+(s_0U)L(1) > x) + 2s_0 \frac{\mathbb{E}e^{\gamma Y_0}}{\mathbb{E}e^{-\gamma L(1)}} \mathbb{P}(-f^-(s_0U)L(1) > x),$$

where U is a uniform random variable on $(-1, 1)$ independent of $L(1)$. This means that the tail of Y_0 is equivalent to the tail of a MA process with kernel function $f(s)\,\mathbf{1}_{(-s_0,s_0)}(s)$. If $L(1)$ is symmetric, the last equation reduces to

$$\mathbb{P}(Y_0 > x) \sim 2s_0 \frac{\mathbb{E}e^{\gamma Y_0}}{\mathbb{E}e^{\gamma L(1)}} \mathbb{P}(|f(s_0U)|L(1) > x) \quad \text{as } x \to \infty.$$

Lemma 3.2 *Let Y be a mixed MA process as in (1.1), satisfying condition (M) with $f^+, f^- \leq 1$. Suppose for every $\epsilon > 0$ there exists a $0 < y < 1$, such that $0 < \Pi(A_{f,y}) \leq \epsilon$. Then*

$$\nu_Y(x, \infty) = o(\nu(x, \infty)) \quad \text{and} \quad \nu_Y(-\infty, -x) = o(\nu(x, \infty)) \quad \text{as } x \to \infty.$$

If $L(1) \in \mathcal{S}(\gamma)$, then $\mathbb{E}e^{\gamma Y_0} < \infty$ and

$$\mathbb{P}(|Y_0| > x) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty.$$

We now present one of the basic results of this paper.

Theorem 3.3 *Let Y be a mixed MA process as in (1.1) satisfying condition (G). Define $M(h) = \sup_{0 \leq t \leq h} Y(t)$ for $h > 0$. Then*

$$\mathbb{P}(M(h) > x) \sim h \frac{\mathbb{E}e^{\gamma Y_0}}{\mathbb{E}e^{\gamma L(1)}} \mathbb{P}(L(1) > x) \quad \text{as } x \to \infty. \tag{3.3}$$

This theorem says that the tail of the maxima of any mixed MA processes decreases by the same order of magnitude if the driving idisrm have the same Lévy measure in $\mathcal{S}(\gamma)$. Only the constant $\mathbb{E}e^{\gamma Y_0}[\mathbb{E}e^{\gamma L(1)}]^{-1}$ in (3.3) is affected by the kernel function. The proof of Theorem 3.3 is very technical, although the idea is obvious. One reason is that, in contrast to $\gamma = 0$, by Proposition 2.5 *(ii)* small jumps of Λ, represented in Λ_2, and hence Y_2, influence the size of maxima. On the other hand, they do not have an impact on the location of local maxima. We will present a short sketch of the proof.

Local maxima of Y are achieved by large jumps of the underlying driving Lévy process $(L(t))_{t\geq 0}$, which happen at the jump time points $(\Gamma_k)_{k\in\mathbb{N}}$ of $(L_1(t))_{t\geq 0}$. Hence, the process $(Y(\Gamma_k))_{k\in\mathbb{N}}$ reflects the process of maxima of Y. A computational challenge of the proof is that $(Y(\Gamma_k))_{k\in\mathbb{N}}$ is not stationary. We will explain this statement.

On the one hand, we have, by the independence of Y_2 and Λ_2,

$$(Y(\Gamma_k + t) + f(R_0, \Gamma_k + t)Z_0)_{t\geq 0} \stackrel{d}{=} (Y_1(\Gamma_k + t) + Y_2(t) + f(R_0, \Gamma_k + t)Z_0)_{t\geq 0}, \tag{3.4}$$

and, on the other hand,

$$\begin{aligned}(Y_1(\Gamma_k + t) + f(R_0, \Gamma_k + t)Z_0 + Y_2(t))_{t\geq 0} \quad &\stackrel{d}{=} \left(\sum_{j=-\infty}^{\infty} f(R_j, t - \Gamma_j)Z_j + Y_2(t)\right)_{t\geq 0} \\ &= (Y(t) + f(R_0, t)Z_0)_{t\geq 0}. \end{aligned} \tag{3.5}$$

A conclusion of (3.4) and (3.5) is

$$(Y(\Gamma_k + t) + f(R_0, \Gamma_k + t)Z_0)_{t\geq 0} \stackrel{d}{=} (Y(t) + f(R_0, t)Z_0)_{t\geq 0}. \tag{3.6}$$

However, a slight modification implies that the sequence $(X_k)_{k\in\mathbb{N}_0}$ with

$$X_k = Y(\Gamma_k) + f(R_0, \Gamma_k)Z_0 \quad \text{ for } k \in \mathbb{N}_0$$

is stationary with

$$X_k \stackrel{d}{=} Y_0 + Z_0 \quad \text{ for } k \in \mathbb{N}_0, \tag{3.7}$$

and $(Y(\Gamma_k))_{k\in\mathbb{N}_0}$ is not stationary. Further, $Y(\Gamma_k) \stackrel{k\to\infty}{\Longrightarrow} X_0$. Let u_n be given as in condition (G). Then the proof has the outline

$$\begin{aligned}\lim_{n\to\infty} n\mathbb{P}(M(h) > u_n) \quad &= \quad \lim_{k\to\infty}\lim_{n\to\infty} n\mathbb{P}\left(\sup_{k\leq t\leq k+h} Y(t) > u_n\right) \\ &= \quad \lim_{k\to\infty}\lim_{n\to\infty} n\mathbb{P}\left(\max_{\Gamma_j\in[k,k+h]} Y(\Gamma_j) > u_n\right) \\ &= \quad \lim_{k\to\infty}\lim_{n\to\infty} n\mathbb{P}\left(\max_{\Gamma_j\in[k,k+h]} X_j > u_n\right) \\ &= \quad \mathbb{E}(N(h)) \lim_{n\to\infty} n\mathbb{P}(X_0 > u_n).\end{aligned}$$

4 Extremal behavior

In this section we study the extremal behavior of convolution-equivalent mixed MA processes. We use a similar notation as in Fasen [12, 13]. Let $\mathcal{T}_d$ be the space $(-\infty,\infty]\times[-\infty,\infty]^d$, $d\in\mathbb{N}_0$, and $M_P(\mathcal{T}_d)$ denotes the class of point measures on $[0,\infty)\times\mathcal{T}_d$. We abbreviate PRM($\vartheta$), where ϑ is a Radon measure on $[0,\infty)\times\mathcal{T}_d$, as a Poisson random measure with intensity measure ϑ. More about point processes can be found in the monographs of Daley and Vere-Jones [7], Kallenberg [18] and Resnick [24]. Let $t_1,\ldots,t_d\in\mathbb{R}$. We define $\mathbf{f}:\mathbb{R}_+\to\mathbb{R}^d$ by

$$\mathbf{f}(r)=(f(r,t_1),\ldots,f(r,t_d))\quad\text{ and }\quad\mathbf{Y}(\Gamma_k)=(Y(\Gamma_k+t_1),\ldots,Y(\Gamma_k+t_d)),\ k\in\mathbb{N}_0.$$

Theorem 4.1 *Let Y be a mixed MA process as in (1.1) satisfying condition (G). Define $M_k=\sup_{t\in[\Gamma_k,\Gamma_{k+1})}Y(t)$. Let $(W_k)_{k\in\mathbb{N}}$ be a sequence of iid random variables with $W_1\stackrel{d}{=}Y_0$, independent of the iid sequence $(R^{(k)})_{k\in\mathbb{N}_0}$, where $R^{(k)}$ has probability distribution π, and $\sum_{k=1}^\infty\varepsilon_{(s_k,P_k)}$, which is a* PRM($\vartheta$) *with $\vartheta(dt\times dx)=[\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\,dt\times\mathrm{e}^{-x}\,dx$. Then*

$$\zeta_n:=\sum_{k=1}^\infty\varepsilon_{(\Gamma_k/n,a_n^{-1}(M_k-b_n),b_n^{-1}\mathbf{Y}(\Gamma_k))}\stackrel{n\to\infty}{\Longrightarrow}\sum_{k=1}^\infty\varepsilon_{(s_k,P_k+\gamma W_k,\mathbf{f}(R^{(k)}))}=:\xi\quad\text{ in } M_P(\mathcal{T}_d).$$

The multivariate point process ζ_n can be interpreted as a *marked point process* (cf. Daley and Vere-Jones [7], Section 6.4) in the sense that $b_n^{-1}\mathbf{Y}(\Gamma_k)$ are the marks which describe the sample path behavior of the continuous-time process Y, if M_k has an exceedance over the threshold a_nx+b_n, $x\in\mathbb{R}$. Looking at the third coordinate of ζ_n and ξ, we conclude that local extremes of Y on high levels happen at the jump times Γ of the Lévy process L. Thus, the small jumps of L, modelled in Λ_2, have no influence on the location of extremes of Y on high levels. In contrast to the location of local extremes, small jumps affect the size of local extremes when $\gamma>0$, which follows from the second coordinate of ξ. There, P_k is a result of the jump size Z_k of the Lévy process at time Γ_k and γW_k of $Y(\Gamma_k)-Z_k$.

Corollary 4.2 *Let Y be a mixed MA process as in (1.1) satisfying condition (G), and define $M(T)=\sup_{0\le t\le T}Y(t)$ for $T>0$. Then*

$$\lim_{T\to\infty}\mathbb{P}(a_T^{-1}(M(T)-b_T)\le x)=\exp(-\mathbb{E}\mathrm{e}^{\gamma Y_0}[\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\mathrm{e}^{-x})\quad\text{ for } x\in\mathbb{R}.$$

Theorem 3.3 and Corollary 4.2 give that the extremal index (see Leadbetter et al. [21], pp. 67) of the stationary discrete-time sequence $(M_{k,h})_{k\in\mathbb{N}}$ with $M_{k,h}=\sup_{(k-1)h\le t\le kh}Y(t)$ for $h>0$ is equal 1. Thus, exceedances over high thresholds of Y do not tend to occur in clusters. Like the proof of Theorem 3.3, the proof of Theorem 4.1 will be based on the following point process result associated with the stationary sequence $(X_k)_{k\in\mathbb{N}_0}$.

Proposition 4.3 *Let Y be a mixed MA process as in (1.1) satisfying the assumptions of Theorem 4.1. Define $X_k=Y(\Gamma_k)+f(R_0,\Gamma_k)Z_0$ for $k\in\mathbb{N}_0$. Then*

$$\xi_n:=\sum_{k=1}^\infty\varepsilon_{(k/(\mu n),a_n^{-1}(X_k-b_n),b_n^{-1}\mathbf{Y}(\Gamma_k))}\stackrel{n\to\infty}{\Longrightarrow}\sum_{k=1}^\infty\varepsilon_{(s_k,P_k+\gamma W_k,\mathbf{f}(R^{(k)}))}=:\xi\quad\text{ in } M_P\left(\mathcal{T}_d\right).\quad(4.1)$$

Remark 4.4 *(a)* Recall from p. 10 that $(X_k)_{k\in\mathbb{N}_0}$ is a stationary sequence. Let $\widetilde{a}_n > 0$ and $\widetilde{b}_n \in \mathbb{R}$ be sequences of constants satisfying

$$\lim_{n\to\infty} n\mathbb{P}(X_0 > \widetilde{a}_n x + \widetilde{b}_n) = \exp(-x) \quad \text{for } x \in \mathbb{R}.$$

Hence, $\lim_{n\to\infty} n\mathbb{P}(L(1) > \widetilde{a}_n x + \widetilde{b}_n) = \mu\mathbb{E}e^{\gamma L(1)}[\mathbb{E}e^{\gamma Y_0}]^{-1}\exp(-x)$; see Lemma 5.3 below. Thus, Proposition 4.3 says that

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n,\widetilde{a}_n^{-1}(X_k-\widetilde{b}_n))} \overset{n\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \varepsilon_{(\widetilde{s}_k,\widetilde{P}_k)} \quad \text{in } M_P(\mathcal{T}_0),$$

where $\sum_{k=1}^{\infty} \varepsilon_{(\widetilde{s}_k,\widetilde{P}_k)}$ is a PRM(ϑ) with $\vartheta(dt\times dx) = dt \times e^{-x}\,dx$. This means that the extremal behavior of the stationary sequence $(X_k)_{k\in\mathbb{N}_0}$ is the same as the extremal behavior of the associated iid sequence with distribution of X_0.
(b) If we replace X_k by $Y(\Gamma_k)$ in (4.1) then the result is still true. □

5 Proofs

5.1 Proofs of Section 2

Proof of Proposition 2.1. We shall prove that the assumptions of Rajput and Rosinski [23], Theorem 2.7, are satisfied. W. l. o. g. we assume $f^+ = 1$.

First of all we will show

$$\int_{\mathbb{R}_+\times\mathbb{R}}\int_{-\infty}^{\infty} \min\{1, |f(r,s)|^2x^2\}\,\nu(dx)\,d\Pi(r,s) = \int_{\mathbb{R}_+\times\mathbb{R}} f(r,s)^2\,d\Pi(r,s)\int_{-1}^{1} x^2\,\nu(dx)$$
$$+ \int_{\{|x|>1\}}\left[\int_{\{f(r,s)^2x^2>1\}} 1\,d\Pi(r,s) + \int_{\substack{\{f(r,s)^2x^2\le 1,\\ |x|>1\}}} f(r,s)^2x^2\,d\Pi(r,s)\right]\nu(dx) < \infty. \tag{5.1}$$

By the standard property of Lévy measures $\int_{\mathbb{R}}(1\wedge|x|^2)\,\nu(dx) < \infty$. Then the quadratic integrability of f gives that the first summand of (5.1) is finite. Again as $f \in \mathbb{L}^2(\Pi)$ and $\int_{\{|x|>1\}}|x|^2\nu(dx) < \infty$ by $\mathbb{E}|L(1)|^2 < \infty$, we also conclude that the third term of (5.1) is finite. For the remaining part of the second integral we have with $\mu_1 := \nu(1,\infty)$ and $\mu_2 := \nu(-\infty,-1)$, respectively, and $Z^{(1)}$ with d.f. $\nu|_{(1,\infty)}/\mu_1$ and $Z^{(2)}$ with d.f. $\nu|_{(-\infty,-1)}/\mu_2$, respectively, the equality

$$\int_{\{|x|>1\}}\int_{\{f(r,s)^2x^2>1\}} 1\,d\Pi(r,s)\nu(dx)$$
$$= \int_{\mathbb{R}_+\times\mathbb{R}} \left[\mu_1\mathbb{P}(|f(r,s)|Z^{(1)} > 1) + \mu_2\mathbb{P}(|f(r,s)|Z^{(2)} > 1)\right]\,d\Pi(r,s).$$

Since $f \in \mathbb{L}^2(\Pi)$ there exist a set $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $\Pi(B) < \infty$, such that $|f(r,t)| \le 1/2$ for every $(r,t) \in B^c$. Using Davis and Resnick [8], Proposition 1.1, there exist $K_1, K_2 > 0$, such that the last term can be bounded above by

$$\Pi(B)[\mu_1 + \mu_2] + \left[K_1 \mathbb{P}(Z^{(1)} > 1/2) + K_2 \mathbb{P}(Z^{(2)} > 1/2)\right] \int_{B^c} |f(r,s)|^2 \, d\Pi(r,s) < \infty.$$

This results in (5.1). Next we prove

$$\int_{\mathbb{R}_+ \times \mathbb{R}} \left| m f(r,s) + \int_{-\infty}^{\infty} \kappa(x f(r,s)) - f(r,s)\kappa(x) \, \nu(dx) \right| d\Pi(r,s) < \infty, \tag{5.2}$$

where we have to distinguish between the assumptions *(M1)* and *(M2)*, respectively.
(M1): We have the upper bound

$$\begin{aligned} &\int_{\mathbb{R}_+ \times \mathbb{R}} \left| m f(r,s) + \int_{\mathbb{R}_+ \times \mathbb{R}} \kappa(x f(r,s)) - f(r,s)\kappa(x) \, \nu(dx) \right| d\Pi(r,s) \\ &\quad \le m \int_{\mathbb{R}_+ \times \mathbb{R}} |f(r,s)| \, d\Pi(r,s) + \int_{\mathbb{R}_+ \times \mathbb{R}} |f(r,s)| \, d\Pi(r,s) \int_{\{|x|>1\}} |x| \, \nu(dx) < \infty, \end{aligned}$$

where we used Fubini's theorem, $f \in \mathbb{L}^1(\Pi)$ and $\mathbb{E}|L(1)| < \infty$.
(M2): Sato [28], Example 25.12, says $m = -\int_{\{|x|>1\}} x \, \nu(dx)$, such that (5.2) is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}_+ \times \mathbb{R}} \left| \int_{\{|x|>1\}} f(r,s) x \, \mathbf{1}_{\{|f(r,s)x|>1\}} \, \nu(dx) \right| d\Pi(r,s) \\ &\quad \le \int_{\mathbb{R}_+ \times \mathbb{R}} |f(r,s)|^2 \, d\Pi(r,s) \int_{\{|x|>1\}} |x|^2 \, \nu(dx) < \infty, \end{aligned}$$

where we used $f \in \mathbb{L}^2(\Pi)$ and $\mathbb{E}|L(1)|^2 < \infty$. □

5.2 Proofs of Section 3

For the proof of Theorem 3.3 we require some Lemmatas (cf. the ideas on p. 10).

Lemma 5.1 *Let Y be a stationary separable mixed MA process as in (1.1), where $|f(r,s)| \le 1$ for $(r,s) \in \mathbb{R}_+ \times \mathbb{R}$, and for $\widehat{M}(h) = \sup_{t \in [0,h]} |Y(t)|$ for $h > 0$ we have $\mathbb{P}(\widehat{M}(1) < \infty) = 1$. Assume that the Lévy measure ν has support only on $[-c, c]$ for some $c > 0$. Let τ be a positive random variable on the same probability space as Y and be independent of Y with $\mathbb{E}(\tau) < \infty$. Then for every $\epsilon > 0$ there exists a $C > 0$ such that*

$$\mathbb{P}(\widehat{M}(\tau) > x) \le (\mathbb{E}(\tau) + 1)\mathbb{P}(\widehat{M}(1) > x) \le (\mathbb{E}(\tau) + 1) C \mathrm{e}^{-\epsilon x} \quad \text{for } x > 0. \tag{5.3}$$

Proof. Denote by F_τ the distribution function of τ. Then we have the uniform bound

$$\frac{\mathbb{P}(\widehat{M}(\tau) > x)}{\mathbb{P}(\widehat{M}(1) > x)} = \int_0^\infty \frac{\mathbb{P}(\widehat{M}(h) > x)}{\mathbb{P}(\widehat{M}(1) > x)} F_\tau(dh) \le \int_0^\infty (h+1) F_\tau(dh) \le \mathbb{E}(\tau + 1). \tag{5.4}$$

Hence, it remains to show that $\tau = 1$ satisfies (5.3).

Let S be a countable set such that $\widehat{M}(1) = \sup_{t\in[0,1]\cap S} Y(t)$ a. s. Such a set exists since Y is separable. Furthermore, since the Lévy measure of the i. d. process $(Y(t))_{t\in[0,1]\cap S}$ has support in $[-c, c]^S$ (cf. Proposition 2.1), the assumptions of Braverman and Samorodnitsky [5], Lemma 2.1, are satisfied, and we obtain

$$C := \mathbb{E}\exp(\epsilon\,\widehat{M}(1)) = \mathbb{E}\exp(\epsilon \sup_{t\in[0,1]\cap S} |Y(t)|) < \infty \qquad \text{for every } \epsilon > 0.$$

If we use Markov's inequality we can conclude

$$\mathbb{P}(\widehat{M}(1) > x) \leq \mathbb{E}\exp(\epsilon\widehat{M}(1))\,\mathrm{e}^{-\epsilon x} = C\mathrm{e}^{-\epsilon x} \qquad \text{for } x > 0. \qquad \square$$

Lemma 5.2 *Let the assumptions of Theorem 3.3 hold. Suppose τ is a positive random variable on the same probability space as Y_2, and τ is independent of Y_2 with $\mathbb{E}(\tau) < \infty$. Then*

$$\mathbb{P}\left(\sup_{t\in[0,\tau]} Y_2(t) > x\right) = o(\mathbb{P}(L(1) > x)) \qquad \text{as } x \to \infty.$$

Proof. Along the lines of (5.4) we have

$$\mathbb{P}\left(\sup_{t\in[0,\tau]} Y_2(t) > x\right) \leq (\mathbb{E}(\tau) + 1)\mathbb{P}\left(\sup_{t\in[0,1]} Y_2(t) > x\right) \qquad \text{for } x > 0. \tag{5.5}$$

We decompose Λ_2 into two independent idisrm such that $\Lambda_2 = \Lambda_2^{(1)} + \Lambda_2^{(2)}$, where $\Lambda_2^{(1)}$ has the generating quadruple $(0, 0, \nu_2^{(1)}, \pi)$ with $\nu_2^{(1)}(A) = \nu(A \cap (-\infty, -1))$ for $A \in \mathcal{B}(\mathbb{R})$, and $\Lambda_2^{(2)}$ has generating quadruple $(m, \sigma^2, \nu - \nu_1 - \nu_2^{(1)}, \pi)$. The underlying driving Lévy process of $\Lambda_2^{(1)}$ is denoted by $L_2^{(1)}$ and has generating triplet $(0, 0, \nu_2^{(1)})$. Hence, $Y_2(t) = Y_2^{(1)}(t) + Y_2^{(2)}(t)$, where

$$Y_2^{(1)}(t) = \int_{\mathbb{R}_+\times\mathbb{R}} f(r, t-s)\, d\,\Lambda_2^{(1)}(r, s) \quad \text{and} \quad Y_2^{(2)}(t) = \int_{\mathbb{R}_+\times\mathbb{R}} f(r, t-s)\, d\,\Lambda_2^{(2)}(r, s)$$

are independent stationary mixed MA processes. Further, there exists a countable set S such that $\sup_{t\in[0,1]} Y_2(t) = \sup_{t\in[0,1]\cap S} Y_2(t)$ a. s since Y_2 is separable. Thus,

$$\begin{aligned}
\sup_{t\in[0,1]\cap S} Y_2(t) &\leq -\int_{\mathbb{R}_+\times\mathbb{R}} \sup_{0\leq t\leq 1} f^-(r, t-s)\, d\,\Lambda_2^{(1)}(r, s) + \sup_{t\in[0,1]\cap S} Y_2^{(2)}(t) \\
&\leq -\int_{\mathbb{R}_+\times(0,1]} f^-\, d\,\Lambda_2^{(1)}(r, s) - \int_{\mathbb{R}_+\times(-\infty,0]} \sup_{0\leq t\leq 1} f^-(r, t-s)\, d\,\Lambda_2^{(1)}(r, s) \\
&\quad + \sup_{t\in[0,1]\cap S} Y_2^{(2)}(t) \\
&\leq -f^- L_2^{(1)}(1) - \int_{\mathbb{R}_+\times\mathbb{R}} f^*(r, -s)\, d\,\Lambda_2^{(1)}(r, s) + \sup_{t\in[0,1]\cap S} Y_2^{(2)}(t).
\end{aligned}$$

Next, the Lévy measure $\nu - \nu_1 - \nu_2^{(1)}$ of $\Lambda_2^{(2)}$ has support in $[-1,1]$. Thus, on the one hand, we have, by Lemma 5.1 and Proposition 2.5,

$$\mathbb{P}\left(\sup_{t\in[0,1]\cap S} |Y_2^{(2)}(t)| > x\right) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty. \tag{5.6}$$

On the other hand, if we use Lemma 3.2 and the tail balance condition (2.1), we obtain

$$\mathbb{P}\left(\int_{\mathbb{R}_+\times\mathbb{R}} -f^*(r,-s)\, d\,\Lambda_2^{(1)}(r,s) > x\right) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty. \tag{5.7}$$

Further, the rapidly varying tail of $L(1) \in \mathrm{MDA}(\mathcal{G}) \cap \mathcal{S}(\gamma)$ and the tail balance condition (2.1) result in

$$\mathbb{P}(-f^- L_2^{(1)}(1) > x) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty. \tag{5.8}$$

Hence, the conclusion of Lemma 5.2 follows from (5.5)-(5.8) and Proposition 2.5. □

Lemma 5.3 *Let the assumptions of Theorem 3.3 hold. Define the sequences*

$$M_k^* = \sup_{t\in[\Gamma_k,\Gamma_{k+1})} [Y(t) + f(R_0,t)Z_0] \quad \textit{and} \quad X_k = Y(\Gamma_k) + f(R_0,\Gamma_k)Z_0 \textit{ for } k \in \mathbb{N}_0.$$

Then

$$\lim_{n\to\infty} n\mathbb{P}\left(M_k^* > u_n\right) = \lim_{n\to\infty} n\mathbb{P}\left(X_0 > u_n\right) = \frac{\mathbb{E}\mathrm{e}^{\gamma Y_0}}{\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}}\mathrm{e}^{-x} \quad \textit{for } k \in \mathbb{N}_0.$$

Proof. Let $\widetilde{Y}_0$ be as distributed as the stationary distribution of $\widetilde{Y}$. We assume Y_0 and $\widetilde{Y}_0$ are independent of $(\Gamma_k)_{k\in\mathbb{N}_0}$, $(R_k)_{k\in\mathbb{N}_0}$, $(Z_k)_{\mathbb{N}_0}$.

First, by (3.7), Lemma 3.2 and Proposition 2.5 *(iii)* we have

$$\lim_{n\to\infty} n\mathbb{P}\left(X_k > u_n\right) = \lim_{n\to\infty} n\mathbb{P}\left(Y_0 + Z_0 > u_n\right) = \frac{\mathbb{E}\mathrm{e}^{\gamma Y_0}}{\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}}\mathrm{e}^{-x}. \tag{5.9}$$

Let $\epsilon > 0$. We define for $k \in \mathbb{N}_0$,

$$\begin{aligned} M_{k,\epsilon}^{(1)} &:= \sup_{t\in[\Gamma_k,\Gamma_k+\epsilon(\Gamma_{k+1}-\Gamma_k))} [Y(t) + f(R_0,t)Z_0], \\ M_{k,\epsilon}^{(2)} &:= \sup_{t\in[\Gamma_k+\epsilon(\Gamma_{k+1}-\Gamma_k),\Gamma_{k+1})} [Y(t) + f(R_0,t)Z_0]. \end{aligned}$$

Then we obtain

$$\mathbb{P}(M_k^* > u_n, X_k \le u_n) \le \mathbb{P}(M_{k,\epsilon}^{(1)} > u_n, X_k \le u_n) + \mathbb{P}(M_{k,\epsilon}^{(2)} > u_n). \tag{5.10}$$

We will prove that both terms on the right hand side tend to 0 as $n \to \infty, \epsilon \downarrow 0$.

Step 1. First, we investigate the first summand of (5.10). We have

$$
\begin{aligned}
M_{k,\epsilon}^{(1)} &\leq Z_k + \sup_{t\in[\Gamma_k,\Gamma_k+\epsilon(\Gamma_{k+1}-\Gamma_k))} [Y_1(t) + f(R_0,t)Z_0 - f(R_k, t-\Gamma_k)Z_k] \\
&\quad + \sup_{t\in[\Gamma_k,\Gamma_k+\epsilon(\Gamma_{k+1}-\Gamma_k))} Y_2(t) \\
&\stackrel{d}{=} Z_0 + \sup_{t\in[0,\epsilon\Gamma_1)} Y_1(t) + \sup_{t\in[0,\epsilon\Gamma_1)} Y_2(t). \qquad (5.11)
\end{aligned}
$$

Next, Lemma 5.2 results in

$$
\mathbb{P}\left(\sup_{t\in[0,\epsilon\Gamma_1)} Y_2(t) > x\right) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty. \qquad (5.12)
$$

Furthermore, by Lemma 3.2,

$$
\mathbb{P}\left(\sup_{t\in[0,\epsilon\Gamma_1)} Y_1(t) > x\right) \leq \mathbb{P}(\widetilde{Y}_0 > x) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty \qquad (5.13)
$$

holds. Hence, a conclusion of (5.9)-(5.13) and Proposition 2.5 *(ii)* is

$$
\begin{aligned}
&\limsup_{n\to\infty} n\mathbb{P}(M_{k,\epsilon}^{(1)} > u_n, X_k \leq u_n) = \limsup_{n\to\infty} n\left[\mathbb{P}(M_{k,\epsilon}^{(1)} > u_n) - \mathbb{P}(X_k > u_n)\right] \\
&\leq \limsup_{n\to\infty} n\left[\mathbb{P}(Z_0 + \sup_{t\in[0,\epsilon\Gamma_1)} Y_1(t) + \sup_{t\in[0,\epsilon\Gamma_1)} Y_2(t) > u_n) - \mathbb{P}(X_k > u_n)\right] \\
&= \left[\frac{\mathbb{E}\mathrm{e}^{\gamma \sup_{t\in[0,\epsilon\Gamma_1)} Y_1(t) + \gamma \sup_{t\in[0,\epsilon\Gamma_1)} Y_2(t)}}{\mathbb{E}\mathrm{e}^{\gamma Y_0}} - 1\right] \frac{\mathbb{E}\mathrm{e}^{\gamma Y_0}}{\mu \mathbb{E}\mathrm{e}^{\gamma L(1)}} \mathrm{e}^{-x} \xrightarrow{\epsilon\downarrow 0} 0. \qquad (5.14)
\end{aligned}
$$

Step 2. As in (5.11) we have

$$
\begin{aligned}
M_{k,\epsilon}^{(2)} &\leq \widetilde{Y}(\Gamma_k + \epsilon(\Gamma_{k+1}-\Gamma_k)) + f^*(R_0, \Gamma_k + \epsilon(\Gamma_{k+1}-\Gamma_k))Z_0 + \sup_{t\in[\Gamma_k,\Gamma_{k+1})} Y_2(t) \\
&\stackrel{d}{=} \sum_{j=-\infty}^{0} f^*(R_j, \epsilon\Gamma_1 - \Gamma_j)Z_j + \sup_{t\in[0,\Gamma_1)} Y_2(t) \\
&\leq f^*(R_0, \epsilon\Gamma_1)Z_0 + \widetilde{Y}_0 + \sup_{t\in[0,\Gamma_1)} Y_2(t). \qquad (5.15)
\end{aligned}
$$

Let $0 < y < 1$. Then

$$
\begin{aligned}
&n\mathbb{P}(f^*(R_0,\epsilon\Gamma_1)Z_0 + \widetilde{Y}_0 + \sup_{t\in[0,\Gamma_1)} Y_2(t) > u_n) \qquad (5.16) \\
&\leq n\mathbb{P}(Z_0 + \widetilde{Y}_0 > u_n - \epsilon a_n, (R_0,\epsilon\Gamma_1) \in A_{f^*,y}) \\
&\quad + n\mathbb{P}(Z_0 + \widetilde{Y}_0 + \sup_{t\in[0,\Gamma_1)} Y_2(t)\, \mathbf{1}_{\{(R_0,\epsilon\Gamma_1)\in A_{f^*,y}\}} > u_n, \sup_{t\in[0,\Gamma_1)} Y_2(t)\, \mathbf{1}_{\{(R_0,\epsilon\Gamma_1)\in A_{f^*,y}\}} > \epsilon a_n) \\
&\quad + n\mathbb{P}(yZ_0 + \widetilde{Y}_0 + \sup_{t\in[0,\Gamma_1)} Y_2(t) > u_n).
\end{aligned}
$$

The first summand of (5.16) is by Proposition 2.5 *(ii)*

$$\lim_{n\to\infty} n\mathbb{P}(Z_0 + \widetilde{Y}_0 > u_n - \epsilon a_n, (R_0, \epsilon\Gamma_1) \in A_{f^*,y}) \\ = \mathrm{e}^{-(x-\epsilon)} \frac{\mathbb{E}\mathrm{e}^{\gamma \widetilde{Y}_0}}{\mu \mathbb{E}\mathrm{e}^{\gamma L(1)}} \mathbb{P}((R_0, \epsilon\Gamma_1) \in A_{f^*,y}) \stackrel{y\uparrow 1}{\longrightarrow} 0. \quad (5.17)$$

The rapid variation of $\mathbb{P}(Z_0 > x)$, Lemma 3.2 and (5.12) result in

$$\limsup_{n\to\infty} n\mathbb{P}(yZ_0 + \widetilde{Y}_0 + \sup_{t\in[0,\Gamma_1)} Y_2(t) > u_n) = 0. \quad (5.18)$$

The second summand of (5.16) tends, as $n \to \infty$, to (similar to Lemma 2.4 in Goldie and Resnick [16])

$$\mathrm{e}^{-x} \frac{\mathbb{E}\mathrm{e}^{\gamma \widetilde{Y}_0}}{\mu \mathbb{E}\mathrm{e}^{\gamma L(1)}} \mathbb{E}\left(\mathrm{e}^{\gamma \sup_{t\in[0,\Gamma_1)} Y_2(t)} \mathbf{1}_{\{\sup_{t\in[0,\Gamma_1)} Y_2(t) > \epsilon\gamma^{-1}\}} \mathbf{1}_{\{(R_0,\epsilon\Gamma_1)\in A_{f^*,y}\}} \right) \stackrel{y\uparrow 1}{\longrightarrow} 0, \quad (5.19)$$

where we used dominated convergence. Thus, (5.15)-(5.19) give

$$n\mathbb{P}(M^{(2)}_{k,\epsilon} > u_n) \le n\mathbb{P}(f^*(R_0, \epsilon\Gamma_1) Z_0 + \widetilde{Y}_0 + \sup_{t\in[0,\Gamma_1)} Y_2(t) > u_n) \stackrel{n\to\infty}{\longrightarrow} 0. \quad (5.20)$$

The result of Lemma 5.3 is then a conclusion of (5.10), (5.14) and (5.20). □

Lemma 5.4 *Let Y be a mixed MA process as in (1.1) satisfying condition (G).*
(a) Define $M(h) = \sup_{t\in[0,h]} Y(t)$ for $h > 0$. Then

$$\mathbb{P}(M(h) > x) \sim h\mathbb{P}(M(1) > x) \quad \text{as } x \to \infty.$$

(b) Let $X_k = Y(\Gamma_k) + f(R_0, \Gamma_k) Z_0$ and $\widetilde{M}_{k,m} = \max_{j=k+1,\dots,k+m} X_j$, $k \in \mathbb{N}_0$, $m \in \mathbb{N}$. Then

$$\lim_{n\to\infty} n\mathbb{P}(\widetilde{M}_{k,m} > u_n) = m \lim_{n\to\infty} n\mathbb{P}(X_0 > u_n) = m \frac{\mathbb{E}\mathrm{e}^{\gamma Y_0}}{\mu \mathbb{E}\mathrm{e}^{\gamma L(1)}} \mathrm{e}^{-x}.$$

Proof. *(a).* Let $M_k(h) = \sup_{k\le t\le k+h} Y(t)$ for $k \in \mathbb{N}_0$. We suppose for a moment that

$$\lim_{n\to\infty} n\mathbb{P}(M(h) > u_n, M_{(j-1)h}(h) > u_n) = 0 \quad \text{for } j \ge 2 \quad (5.21)$$

holds. Since Y is stationary, we have for $m \in \mathbb{N}$,

$$\begin{aligned} m\mathbb{P}(M(1) > u_n) &\ge \mathbb{P}(M(m) > u_n) \\ &\ge m\mathbb{P}(M(1) > u_n) - \sum_{\substack{i,j=0 \\ i\ne j}}^{m-1} \mathbb{P}\left(M_i(1) > u_n, M_j(1) > u_n\right). \end{aligned}$$

Then (5.21) results in

$$\mathbb{P}(M(m) > u_n) \sim m\mathbb{P}(M(1) > u_n) \quad \text{as } n \to \infty.$$

Hence, for $h \in \mathbb{Q}$, where $h = m/k$ with $m, k \in \mathbb{N}$, we have

$$\mathbb{P}(M(h) > u_n) \sim k^{-1}\mathbb{P}(M(m) > u_n) \sim h\mathbb{P}(M(1) > u_n) \quad \text{as } n \to \infty.$$

The monotonicity of $M(\cdot)$ leads to *(a)* for $h > 0$.
We continue to prove (5.21). We use the following upper bounds (cf. (5.6)):

$$\begin{aligned} M(h) &\leq L_1(h) + \widetilde{Y}_0 + \sup_{t\in[0,h]} Y_2(t), && (5.22)\\ M_{(j-1)h}(h) &\leq [L_1(jh) - L_1((j-1)h)] + \int_{\mathbb{R}_+\times(h,(j-1)h]} f^*(r,(j-1)h-s)\, d\,\Lambda_1(r,s) \\ &\quad + \int_{\mathbb{R}_+\times(0,h]} f^*(r,h-s)\, d\,\Lambda_1(r,s) + \widetilde{Y}_0 + \sup_{t\in[(j-1)h,jh]} Y_2(t). \end{aligned}$$

Then

$$\begin{aligned} M(h) + M_{(j-1)h}(h) \leq{}& [L_1(jh) - L_1((j-1)h)] + \int_{\mathbb{R}_+\times(h,(j-1)h]} f^*(r,(j-1)h-s)\, d\,\Lambda_1(r,s) \\ &+ \int_{\mathbb{R}_+\times(0,h]} (1 + f^*(r,h-s))\, d\,\Lambda_1(r,s) + 2\widetilde{Y}_0 + 2\sup_{t\in[0,jh]} Y_2(t). \quad (5.23) \end{aligned}$$

Observe that the summands on the right hand side of (5.23) are independent. Moreover,

$$\mathbb{P}(L_1(jh) - L_1((j-1)h) > 2u_n) = \mathbb{P}(L_1(h) > 2u_n) = o(\mathbb{P}(L(1) > u_n)) \qquad (5.24)$$

as $n \to \infty$ due to the rapidly varying tail of the distribution function of $L(1)$. A consequence of Lemma 3.2 is as $n \to \infty$,

$$\mathbb{P}\left(\int_{\mathbb{R}_+\times(0,h]} (1 + f^*(r,h-s))\, d\,\Lambda_1(r,s) > 2u_n\right) = o(\mathbb{P}(L(1) > u_n)). \qquad (5.25)$$

Again with Lemma 3.2 we have for the second and the fourth summand of (5.23),

$$\begin{aligned} \mathbb{P}\left(\int_{\mathbb{R}_+\times(h,(j-1)h]} f^*(r,(j-1)h-s)\, d\,\Lambda_1(r,s) > 2u_n\right) &\leq \mathbb{P}(2\widetilde{Y}_0 > 2u_n) && (5.26)\\ &= o(\mathbb{P}(L(1) > u_n)) \end{aligned}$$

as $n \to \infty$. Finally, Lemma 5.2 yields

$$\mathbb{P}\left(2\sup_{t\in[0,jh]} Y_2(t) > 2u_n\right) = o(\mathbb{P}(L(1) > u_n)) \quad \text{as } n \to \infty. \qquad (5.27)$$

Thus, we obtain by (5.23)-(5.27),

$$\lim_{n\to\infty} n\mathbb{P}(M(h) > u_n, M_{(j-1)h}(h) > u_n) \leq \lim_{n\to\infty} n\mathbb{P}(M(h) + M_{(j-1)h}(h) > 2u_n) = 0.$$

(b). Along the lines of the proof of *(a)* we derive *(b)* by

$$\lim_{n\to\infty} n\mathbb{P}(X_0 > u_n, X_j > u_n) = 0 \quad \text{for } j \geq 1,$$

which can be proven as (5.21). □

Lemma 5.5 *Let Y be a mixed MA process as in (1.1) satisfying condition (G). We define $X_k = Y(\Gamma_k) + f(R_0, \Gamma_k)Z_0$, $M_k(m) = \sup_{k\leq t\leq k+m} Y(t)$, $\widetilde{M}_{k,m} = \max_{j=k+1,\ldots,k+m} X_j$ and*

$$A_{k,m}^{(n)} = \{|N(k) - \mu k| \leq \epsilon\mu k, |N(m+k) - N(k) - \mu m| \leq \epsilon\mu m, f^*(R_0, k)Z_0 \leq \epsilon a_n\},$$

for $k \in \mathbb{N}_0$, $n, m \in \mathbb{N}$. Then the following statements hold:

(a) $\lim_{n\to\infty} n\mathbb{P}(M(1) > u_n) = \lim_{\epsilon\downarrow 0}\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} \frac{n}{m}\mathbb{P}(M_k(m) > u_n, A_{k,m}^{(n)})$.

(b) $\lim_{n\to\infty} n\mathbb{P}(X_0 > u_n) = \lim_{\epsilon\downarrow 0}\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} \frac{n}{m\mu}\mathbb{P}(\widetilde{M}_{\lceil\mu k\rceil,\lfloor\mu m\rfloor} > u_n + \epsilon a_n, A_{k,m}^{(n)})$.

Proof. Without loss of generality we assume $\mu = 1$.

(a). Step 1. Note that since Y is stationary and Lemma 5.4 *(a)* holds,

$$\lim_{n\to\infty} n\mathbb{P}(M(1) > u_n) = \lim_{n\to\infty}\frac{n}{m}\mathbb{P}(M(m) > u_n) = \lim_{n\to\infty}\frac{n}{m}\mathbb{P}(M_k(m) > u_n) \tag{5.28}$$

is valid.

Step 2. Next, we will prove for $k \in \mathbb{N}$

$$\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\frac{n}{m}\mathbb{P}(M_k(m) > u_n, |N(k) - k| > \epsilon k) = 0. \tag{5.29}$$

Let $0 \leq j \leq m-1$. Note that for $t > k+j$,

$$\begin{aligned} Y_1(t) &\leq \left[\int_{\mathbb{R}_+\times(-\infty,k]} + \int_{\mathbb{R}_+\times(k,k+j]} + \int_{\mathbb{R}_+\times(k+j,t]}\right] f^*(r, t-s)\, d\Lambda_1(r,s) \\ &\leq \widetilde{Y}(k) + \widetilde{Y}_k(k+j) + [L_1(t) - L_1(k+j)], \end{aligned}$$

where $\widetilde{Y}(k)$, $\widetilde{Y}_k(k+j) := \int_{\mathbb{R}_+\times(k,k+j]} f^*(r, k+j-s)\, d\Lambda_1(r,s)$ and $L_1(t) - L_1(k+j)$ are independent, and $N(k)$, $\widetilde{Y}_k(k+j)$, $L_1(t) - L_1(k+j)$ are independent as well.

Define $M_l^{(2)} := \sup_{t\in[l,l+1]} Y_2(t)$ for $l \in \mathbb{N}_0$. Then we have

$$M_{k+j}(1) \leq M_{k+j}^{(2)} + \widetilde{Y}(k) + \widetilde{Y}_k(k+j) + [L_1(k+j+1) - L_1(k+j)].$$

Hence, with Y' independent of $\widetilde{Y}(k)$, $(L_1(t) - L_1(k+j))_{t\geq k+j}$, $(N(t))_{t\leq k}$ and Y_2, and $Y' \stackrel{d}{=} \widetilde{Y}_0$, we obtain $\mathbb{P}(\widetilde{Y}_k(k+j) > x) \leq \mathbb{P}(Y' > x)$ for $x \in \mathbb{R}$, and

$$\begin{aligned} &\mathbb{P}(M_{k+j}(1) > u_n, |N(k) - k| > \epsilon k) \\ &\quad\leq \mathbb{P}(M_0^{(2)} + Y' + L_1(1) > u_n - \epsilon a_n)\mathbb{P}(|N(k) - k| > \epsilon k) \\ &\qquad+ \mathbb{P}(M_0^{(2)} + \widetilde{Y}(k)\,\mathbf{1}_{\{|N(k)-k|>\epsilon k\}} + Y' + [L_1(k+j+1) - L_1(k+j)] > u_n, \\ &\qquad\quad \widetilde{Y}(k)\,\mathbf{1}_{\{|N(k)-k|>\epsilon k\}} > \epsilon a_n). \end{aligned} \tag{5.30}$$

Lemma 5.2 results in

$$\mathbb{P}(M_0^{(2)} > x) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty,$$

and a conclusion of Lemma 3.2 is

$$\mathbb{P}(Y' > x) = \mathbb{P}(\widetilde{Y}_0 > x) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty.$$

Thus, by Proposition 2.5 *(ii)* and the law of large numbers (LLN), we obtain for the first summand of inequality (5.30),

$$\begin{aligned} &\lim_{n\to\infty} n\mathbb{P}(M_0^{(2)} + Y' + L_1(1) > u_n - \epsilon a_n)\mathbb{P}\left(|N(k) - k| > \epsilon k\right) \\ &\quad = \mathrm{e}^{-x+\epsilon}\mathbb{E}\mathrm{e}^{\gamma M_0^{(2)}}\mathbb{E}\mathrm{e}^{\gamma \widetilde{Y}_0}\mathbb{E}\mathrm{e}^{\gamma L_1(1)}[\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\mathbb{P}\left(|N(k) - k| > \epsilon k\right) \stackrel{k\to\infty}{\longrightarrow} 0. \end{aligned} \tag{5.31}$$

For the second summand of (5.30) we have, similar to Lemma 2.4 in Goldie and Resnick [16],

$$\begin{aligned} &n\mathbb{P}(M_0^{(2)} + \widetilde{Y}(k)\,\mathbf{1}_{\{|N(k)-k|>\epsilon k\}} + Y' + L_1(k+j+1) - L_1(k+j) > u_n, \widetilde{Y}(k)\,\mathbf{1}_{\{|N(k)-k|>\epsilon k\}} > \epsilon a_n) \\ &\quad \stackrel{n\to\infty}{\longrightarrow} \mathrm{e}^{-x}\mathbb{E}\mathrm{e}^{\gamma M_0^{(2)}}\mathbb{E}\mathrm{e}^{\gamma \widetilde{Y}_0}\mathbb{E}\mathrm{e}^{\gamma L_1(1)}[\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\int_{\epsilon\gamma^{-1}}^{\infty} \mathrm{e}^{\gamma y}\, F_k(dy), \end{aligned} \tag{5.32}$$

where F_k is a distribution function with tail

$$\overline{F}_k(y) := \mathbb{P}(\widetilde{Y}(k)\,\mathbf{1}_{\{|N(k)-k|>\epsilon k\}} > y) \le \mathbb{P}(\widetilde{Y}_0 > y)^{1/2}\mathbb{P}(|N(k) - k| > \epsilon k)^{1/2} \stackrel{k\to\infty}{\longrightarrow} 0 \tag{5.33}$$

for $y > 0$ by Hölder's inequality. This means $F_k \stackrel{k\to\infty}{\Longrightarrow} \mathbf{1}_{[0,\infty)}$. For $\widetilde{\epsilon} > 0$ there exists a $y_0 > 0$ such that

$$\int_{y_0}^{\infty} \mathrm{e}^{\gamma y}\, F_k(dy) \le \mathbb{E}[\mathrm{e}^{\gamma \widetilde{Y}_0}\,\mathbf{1}_{\{\widetilde{Y}_0 > y_0\}}] \le \widetilde{\epsilon} \quad \text{for every } k \in \mathbb{N},$$

since $\mathbb{E}\mathrm{e}^{\gamma \widetilde{Y}_0} < \infty$ by Lemma 3.2. Hence, we have

$$\int_{\epsilon\gamma^{-1}}^{\infty} \mathrm{e}^{\gamma y}\, F_k(dy) \le \int_{\epsilon\gamma^{-1}}^{y_0} \mathrm{e}^{\gamma y}\, F_k(dy) + \widetilde{\epsilon} \stackrel{k\to\infty}{\longrightarrow} \widetilde{\epsilon}. \tag{5.34}$$

Thus, the right hand side of (5.32) tends to 0 as $k \to \infty$. Then

$$\frac{n}{m}\mathbb{P}(M_k(m) > u_n, |N(k) - k| > \epsilon k) \le \frac{n}{m}\sum_{j=0}^{m-1} \mathbb{P}\left(M_{k+j}(1) > u_n, |N(k) - k| > \epsilon k\right)$$

and (5.30)-(5.34) give the result that the right hand side tends to 0 as $n, m, k \to \infty$, which gives (5.29).

Step 3. Next, we will show

$$\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} \frac{n}{m}\mathbb{P}(M_k(m) > u_n, |N(m+k) - N(k) - m| > \epsilon m) = 0. \tag{5.35}$$

We define

$$B_{m,k,j:} = \{|N(m+k) - N(k+j+1) + N(k+j) - N(k) - (m-1)| > \epsilon m/2\},$$

and compute the upper bound

$$\mathbb{P}\left(M_k(m) > u_n, |N(m+k) - N(k) - m| > \epsilon m\right) \tag{5.36}$$
$$\leq \sum_{j=0}^{m-1} \mathbb{P}\left(M_{k+j}(1) > u_n, B_{m,k,j}\right) + \mathbb{P}\left(M_{k+j}(1) > u_n, |N(k+j+1) - N(k+j) - 1| > \epsilon m/2\right).$$

For the first summand of (5.36) we find an upper bound similar to (5.30),

$$\begin{aligned}
\mathbb{P}\left(M_{k+j}(1) > u_n, B_{m,k,j}\right) &\leq \mathbb{P}(M_{k+j}^{(2)} + \widetilde{Y}(k+j) + L_1(k+j+1) - L_1(k+j) > u_n, B_{m,k,j}) \\
&= \mathbb{P}(M_0^{(2)} + \widetilde{Y}(j) + L_1(j+1) - L_1(j) > u_n, B_{m,0,j}) \\
&\leq \mathbb{P}(M_0^{(2)} + L_1(1) > u_n - \epsilon a_n)\mathbb{P}(|N(m-1) - (m-1)| > \epsilon m/2) \\
&+\mathbb{P}(M_0^{(2)} + \widetilde{Y}(j)\,\mathbf{1}_{B_{m,0,j}} + L_1(j+1) - L_1(j) > u_n, \widetilde{Y}(j)\,\mathbf{1}_{B_{m,0,j}} > \epsilon a_n) \\
&=: I_0^{(n,m)} + I_j^{(n,m)}.
\end{aligned} \tag{5.37}$$

On the other hand, as in (5.32)–(5.34),

$$\lim_{n\to\infty} \frac{n}{m} \sum_{j=1}^{m} I_j^{(n,m)} = \mathrm{e}^{-x}\mathbb{E}\mathrm{e}^{\gamma M_0^{(2)}}\mathbb{E}\mathrm{e}^{\gamma L_1(1)}[\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1} \int_{\epsilon\gamma^{-1}}^{\infty} \mathrm{e}^{\gamma y}\, G_m(dy), \tag{5.38}$$

where G_m is a distribution function with tail

$$\begin{aligned}
\overline{G}_m(y) &= \frac{1}{m}\sum_{j=1}^{m} \mathbb{P}(\widetilde{Y}(j)\,\mathbf{1}_{B_{m,0,j}} > y) \leq \mathbb{P}(\widetilde{Y}_0 > y)^{1/2} \frac{1}{m}\sum_{j=1}^{m} \mathbb{P}(B_{m,0,j})^{1/2} \\
&= \mathbb{P}(\widetilde{Y}_0 > y)^{1/2}\mathbb{P}(|N(m-1) - (m-1)| > \epsilon m/2)^{1/2} \stackrel{m\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where we again used Hölder's inequality and the LLN. As in (5.34) we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty} \frac{n}{m}\sum_{j=1}^{m} I_j^{(n,m)} = 0. \tag{5.39}$$

Then (5.37)-(5.39) result in

$$\lim_{m\to\infty}\lim_{n\to\infty} \frac{n}{m}\sum_{j=1}^{m} \mathbb{P}\left(M_{k+j}(1) > u_n, B_{m,k,j}\right) = 0. \tag{5.40}$$

The second summand of (5.36) satisfies, by Watanabe [31], Theorem C,

$$\begin{aligned}
&\limsup_{n\to\infty} n\mathbb{P}\left(M_{k+j}(1) > u_n, |N(k+j+1) - N(k+j) - 1| > \epsilon m/2\right) \\
&= \limsup_{n\to\infty} n\mathbb{P}\left(M(1) > u_n, |N(1) - 1| > \epsilon m/2\right) \\
&\leq \limsup_{n\to\infty} n\mathbb{P}\left(M_0^{(2)} + \widetilde{Y}_0 + \sum_{j=1}^{N(1)\,\mathbf{1}_{\{|N(1)-1|>\epsilon m/2\}}} Z_j > u_n\right) \\
&= \mathbb{E}\mathrm{e}^{\gamma M_0^{(2)}}\mathbb{E}\mathrm{e}^{\gamma \widetilde{Y}_0}[\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1} \sum_{\substack{l\geq 0\\ |l-1|>\epsilon m/2}} \mathbb{P}(N(1) = l)l(\mathbb{E}\mathrm{e}^{\gamma Z_0})^{(l-1)} \\
&\leq \mathbb{E}\mathrm{e}^{\gamma M_0^{(2)}}\mathbb{E}\mathrm{e}^{\gamma \widetilde{Y}_0}[\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\mathbb{E}\left[\mathbf{1}_{\{N(1)>\epsilon m/2\}}\,\mathrm{e}^{\gamma L_1(1)}\right] \stackrel{m\to\infty}{\longrightarrow} 0.
\end{aligned} \tag{5.41}$$

Thus, by (5.36), (5.40) and (5.41) the conclusion (5.35) follows.

Step 4. In the last step we have

$$\lim_{k\to\infty}\lim_{m\to\infty}\limsup_{n\to\infty}\frac{n}{m}\mathbb{P}(M_k(m)>u_n, f^*(R_0,k)Z_0>\epsilon a_n)$$
$$=\lim_{k\to\infty}\limsup_{n\to\infty} n\mathbb{P}(M(1)>u_n)\mathbb{P}(f^*(R_0,k)Z_0>\epsilon a_n)=0. \tag{5.42}$$

Then Step 1−4 ((5.28), (5.29), (5.35), (5.42)) result in *(a)*.

(b). Notice that by the LLN we have

$$\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}(A^{(n)\,c}_{k,m})\le \lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}(|N(k)-k|>\epsilon k) \tag{5.43}$$
$$+\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}(|N(m+k)-N(k)-m|>\epsilon m)+\mathbb{P}(f^*(R_0,k)Z_0>\epsilon a_n)=0.$$

Then, given the independence of Z_{k+j} and $A^{(n)\,c}_{k,m}$, we obtain

$$\mathbb{P}(\widetilde{M}_{k,m}>u_n+\epsilon a_n, A^{(n)\,c}_{k,m}) \tag{5.44}$$
$$\le \sum_{j=1}^{m}\mathbb{P}(Z_{k+j}+|X_{k+j}-Z_{k+j}|>u_n+\epsilon a_n, A^{(n)\,c}_{k,m})$$
$$\le m\mathbb{P}(Z_1>u_n)\mathbb{P}(A^{(n)\,c}_{k,m})$$
$$+\sum_{j=1}^{m}\mathbb{P}(Z_{k+j}+|X_{k+j}-Z_{k+j}|\,\mathbf{1}_{A^{(n)\,c}_{k,m}}>u_n+\epsilon a_n, |X_{k+j}-Z_{k+j}|\,\mathbf{1}_{A^{(n)\,c}_{k,m}}>\epsilon a_n).$$

The first summand on the right hand side of (5.44) multiplied by n/m tends to 0 as $n,m,k\to\infty$ by (5.43). Applying Hölder's inequality we obtain

$$\overline{F}_{k,j,m}(y):=\mathbb{P}(|X_{k+j}-Z_{k+j}|\,\mathbf{1}_{A^c_{k,m}}>y)\le \mathbb{P}(\widetilde{Y}_0>y)^{1/2}\mathbb{P}(A^c_{k,m})=:g_{k,m}(y)$$

for $y>0$, where $A_{k,m}$ is defined as $A^{(n)}_{k,m}$ with a_n replaced by γ^{-1}. Then by the LLN

$$\limsup_{m\to\infty} g_{k,m}(y)\le \mathbb{P}(\widetilde{Y}_0>y)^{\frac{1}{2}}[\mathbb{P}(|N(k)-k|>\epsilon k)+\mathbb{P}(f^*(R_0,k)Z_0>\epsilon\gamma^{-1})]=:g_k(y)$$

for $y>0$. Moreover, $\lim_{k\to\infty} g_k(y)=0$ by the LLN. Hence, similar to (5.33), we get

$$\lim_{k\to\infty}\limsup_{m\to\infty}\lim_{n\to\infty}\frac{n}{m}\sum_{j=1}^{m}\mathbb{P}(Z_{k+j}+|X_{k+j}-Z_{k+j}|\,\mathbf{1}_{A^{(n)\,c}_{k,m}}>u_n, |X_{k+j}-Z_{k+j}|\,\mathbf{1}_{A^{(n)\,c}_{k,m}}>\epsilon a_n)$$
$$\le \lim_{k\to\infty}\limsup_{m\to\infty} \mathrm{e}^{-x}[\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\int_{\epsilon\gamma^{-1}}^{\infty}\mathrm{e}^{\gamma y}\,g_{k,m}(dy)=0. \tag{5.45}$$

Now, statement *(b)* is a conclusion drawn from (5.44)-(5.45). □

Proof of Theorem 3.3. On the one hand, invoking Lemma 5.4 *(b)* and Lemma 5.5 *(b)* we have

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\frac{n}{m}\mathbb{P}(M_k(m)>u_n,A_{k,m}^{(n)})\\
&\geq \lim_{\epsilon\downarrow 0}\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\frac{n}{m}\mathbb{P}(\widetilde{M}_{\lceil\mu k\rceil,\lfloor\mu m\rfloor}>u_n+\epsilon a_n,A_{k,m}^{(n)})-\mathbb{P}(\widetilde{M}_{0,\lceil\epsilon\mu(2k+m)+1\rceil}>u_n+\epsilon a_n)\\
&=\frac{\mathbb{E}\mathrm{e}^{\gamma Y_0}}{\mathbb{E}\mathrm{e}^{\gamma L(1)}}\mathrm{e}^{-x},
\end{aligned}\tag{5.46}$$

and, on the other hand, Lemma 5.3 with $M_j^*=\sup_{t\in[\Gamma_j,\Gamma_{j+1})}Y(t)$, $j\in\mathbb{N}_0$, results in

$$\begin{aligned}
\limsup_{n\to\infty}\frac{n}{m}\mathbb{P}(M_k(m)>u_n,A_{k,m}^{(n)}) &\leq \limsup_{n\to\infty}\frac{n}{m}\sum_{j=\lceil(1-\epsilon)\mu k\rceil}^{\lfloor(1+\epsilon)\mu(k+m)\rfloor}\mathbb{P}(M_j^*>u_n-\epsilon a_n)\\
&\leq \frac{[(1+\epsilon)m+2\epsilon k]\mu}{m}\frac{\mathbb{E}\mathrm{e}^{\gamma Y_0}}{\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}}\mathrm{e}^{-x+\epsilon}\\
&\overset{m\to\infty}{\longrightarrow}(1+\epsilon)\frac{\mathbb{E}\mathrm{e}^{\gamma Y_0}}{\mathbb{E}\mathrm{e}^{\gamma L(1)}}\mathrm{e}^{-x+\epsilon}\overset{\epsilon\downarrow 0}{\longrightarrow}\frac{\mathbb{E}\mathrm{e}^{\gamma Y_0}}{\mathbb{E}\mathrm{e}^{\gamma L(1)}}\mathrm{e}^{-x}.
\end{aligned}\tag{5.47}$$

Hence, we obtain the conclusion by Lemma 5.5 *(a)*, (5.46) and (5.47). □

5.3 Proofs of Section 4

The foundation for the proof of Proposition 4.3 is the following Proposition.

Proposition 5.6 *Let Y be a mixed MA process as in (1.1) satisfying the assumptions of Proposition 4.3. Define $X_k^*=\sum_{j=-\infty, j\neq k}^{\infty}f(R_j,\Gamma_k-\Gamma_j)Z_j+Y_2(\Gamma_k)$ for $k\in\mathbb{N}_0$. Then*

$$\kappa_n:=\sum_{k=1}^{\infty}\varepsilon_{(k/(\mu n),a_n^{-1}(Z_k-b_n),a_n^{-1}X_k^*,b_n^{-1}\mathbf{Y}(\Gamma_k))}\overset{n\to\infty}{\Longrightarrow}\sum_{k=1}^{\infty}\varepsilon_{(s_k,P_k,\gamma W_k,\mathbf{f}(R^{(k)}))}=:\kappa \quad \text{in } M_P(\mathcal{T}_{d+1}).$$

We use the next Lemma for the proof of Proposition 5.6, which is similar to Fasen [13], Lemma 11 *(a)*, where it is derived for subexponential distributions.

Lemma 5.7 *Let Y be a mixed MA process as in (1.1) satisfying condition (G) and $J=\{t_1,\ldots,t_d\}$. Then for any $T>0$*

$$\lim_{n\to\infty}\sum_{k=1}^{nT}\mathbb{P}\left(\sup_{t\in J}\left|\frac{Y(\Gamma_k+t)}{b_n}-f(R_k,t)\right|>\epsilon,Z_k>u_n\right)=0.$$

Proof of Proposition 5.6. This proof follows along the lines of the proof of Proposition 3.1 and Proposition 3.2 in Davis and Resnick [8]. Since we must consider that, on the one hand, $(Y_2(\Gamma_k))_{k\in\mathbb{N}}$ is a dependent sequence and, on the other hand, we are investigating marked point processes, we present this proof so that the paper will be self-contained.

Let the iid sequence $(W_k)_{k\in\mathbb{N}}$ have the representation

$$W_k = \int_{\mathbb{R}_+\times\mathbb{R}} f(r,s)\, d\Lambda^{(k)}(r,s) = \sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty} f(R_j^{(k)},\Gamma_j^{(k)})Z_j^{(k)} + \int_{\mathbb{R}_+\times\mathbb{R}} f(r,s)\, d\,\Lambda_2^{(k)}(r,s) \stackrel{d}{=} Y_0,$$

where $(\Lambda^{(k)})_{k\in\mathbb{N}}$ are i. i .d. idisrm with generating quadruple (m,σ^2,ν,π) and decomposition as given in (2.3).

Step 1. For fixed $m\in\mathbb{N}$ we define

$$X_k^{(m)} = \sum_{\substack{j=k-m\\ j\neq k}}^{k+m} f(R_j,\Gamma_k-\Gamma_j)Z_j + \int_{\mathbb{R}_+\times(\Gamma_k-m,\Gamma_k+m]} f(r,\Gamma_k-s)\, d\,\Lambda_2(r,s) \text{ for } k\in\mathbb{N},$$

and the iid sequence

$$W_k^{(m)} = \sum_{\substack{j=-m\\ j\neq 0}}^{m} f(R_j^{(k)},\Gamma_j^{(k)})Z_j^{(k)} + \int_{\mathbb{R}_+\times(-m,m]} f(r,s)\, d\,\Lambda_2^{(k)}(r,s) \stackrel{d}{=} X_k^{(m)}.$$

First, we shall show

$$\kappa_n^{(m)} := \sum_{k=1}^{\infty} \varepsilon_{(k/(\mu n),a_n^{-1}(Z_k-b_n),a_n^{-1}X_k^{(m)},\mathbf{f}(R_k))} \stackrel{n\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \varepsilon_{(s_k,P_k,\gamma W_k^{(m)},\mathbf{f}(R_k))} =: \kappa^{(m)} \qquad (5.48)$$

in $M_P(\mathcal{T}_{d+1})$. The proof is a conclusion of Theorem 2.1 in Davis and Resnick [8]. We start by proving the mixing condition $\boldsymbol{D^*}$ given in Davis and Resnick [8], p. 47, for the stationary sequence of random vectors

$$\mathbf{V}_{n,k}^{(m)} = (a_n^{-1}(Z_k-b_n),a_n^{-1}X_k^{(m)},\mathbf{f}(R_k)) \quad \text{for } k\in\mathbb{N}.$$

Let $l>2m+1$, $T>0$, $I_1=\{i_1,\dots,i_p\}$ and $I_2=\{j_1,\dots,j_q\}$, where $1\le i_1<\cdots<i_p<j_1<\cdots<j_q\le nT$ and $j_1-i_p\ge l$. Suppose $g_k\in C_K^+(\mathcal{T}_{d+1})$, where $C_K^+(\mathcal{T}_{d+1})$ is the space of positive continuous functions on $\mathcal{T}_{d+1}$ with compact support, and $g_k\le 1$ for $k\in I_1\cup I_2$. Then

$$\left|\mathbb{E}\left[\prod_{j=1}^{2}\prod_{k\in I_j} g_k(\mathbf{V}_{n,k}^{(m)})\right] - \prod_{j=1}^{2}\mathbb{E}\left[\prod_{k\in I_j} g_k(\mathbf{V}_{n,k}^{(m)})\right]\right| \le \mathbb{P}(\Gamma_l\le 2m+1) \stackrel{l\to\infty}{\longrightarrow} 0,$$

where inequality can be understood by conditioning under $(e_k)_{k\in\mathbb{Z}}$, where (e_k) is an iid sequence of exponentially distributed random variables independent of Λ_2, and $\Gamma_k = \sum_{i=1}^k e_i$.

The second step is to prove the anti-clustering condition (2.2) in Davis and Resnick [8]. Let $g\in C_K^+(\mathcal{T}_{d+1})$ and $g\le 1$. Let the support of g be contained in the set $A\times[-\infty,\infty]^{d+1}$, where A is a compact subset of $(-\infty,\infty]$. Then

$$\limsup_{n\to\infty} n\sum_{k=1}^{\lfloor n/i\rfloor} \mathbb{E}[g(\mathbf{V}_{n,1}^{(m)})g(\mathbf{V}_{n,k}^{(m)})] \le i^{-1}[\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-2}\left[\int_A \mathrm{e}^{-x}\,dx\right]^2 \stackrel{i\to\infty}{\longrightarrow} 0.$$

Finally, since $a_n \to \gamma^{-1}$ as $n \to \infty$ we have for $(x,\infty) \times [\alpha,\beta] \times [\boldsymbol{\gamma},\boldsymbol{\delta}] \subseteq \mathcal{T}_{d+1}$,

$$\lim_{n\to\infty} n\mathbb{P}(\mathbf{V}_{n,k}^{(m)} \in (x,\infty) \times [\alpha,\beta] \times [\boldsymbol{\gamma},\boldsymbol{\delta}]) = [\mu \mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\mathrm{e}^{-x}\mathbb{P}(\gamma W_1^{(m)} \in [\alpha,\beta], \mathbf{f}(R^{(1)}) \in [\boldsymbol{\gamma},\boldsymbol{\delta}]).$$

Thus, statement (5.48) follows from Theorem 2.1 in Davis and Resnick [8].

Step 2. To complete the proof, by Theorem 4.4 in Billingsley [4], it suffices to show that $\kappa^{(m)} \overset{m\to\infty}{\Longrightarrow} \kappa$ and that, by the definition of vague convergence, for every $\delta > 0$ and $h \in C_K^+([0,\infty) \times \mathcal{T}_{d+1})$ with $h \leq 1$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left[\left|\int h\, d\kappa_n^{(m)} - \int h\, d\kappa_n\right| > \delta\right] = 0. \tag{5.49}$$

We define the distribution functions $\mathbf{F}_\gamma^{(m)}$ of $(\gamma W_1^{(m)}, \mathbf{f}(R^{(1)}))$ and $\mathbf{F}_\gamma$ of $(\gamma W_1, \mathbf{f}(R^{(1)}))$, respectively. Then $\kappa^{(m)}$ and κ are PRM$(\widetilde{\vartheta}^{(m)})$ and PRM$(\widetilde{\vartheta})$, respectively, with

$$\begin{aligned}
\widetilde{\vartheta}^{(m)}(dt \times dx \times d\mathbf{u}) &= [\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\, dt \times \mathrm{e}^{-x}\, dx \times \mathbf{F}_\gamma^{(m)}(d\mathbf{u}), \\
\widetilde{\vartheta}(dt \times dx \times d\mathbf{u}) &= [\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\, dt \times \mathrm{e}^{-x}\, dx \times \mathbf{F}_\gamma(d\mathbf{u}).
\end{aligned}$$

Since $W_1^{(m)} \overset{P}{\longrightarrow} W_1$ as $m \to \infty$ for every set $I = [s,t) \times (x,\infty) \times [\alpha,\beta] \times [\boldsymbol{\gamma},\boldsymbol{\delta}] \subseteq [0,\infty) \times \mathcal{T}_{d+1}$ we obtain $\widetilde{\vartheta}^{(m)}(I) \overset{m\to\infty}{\longrightarrow} \widetilde{\vartheta}(I)$. This results in $\kappa^{(m)} \overset{m\to\infty}{\Longrightarrow} \kappa$ by Kallenberg [18], Proposition 16.16.

Next, we show (5.49). Suppose the support of h is contained in $[0,T] \times [x,\infty] \times [-\infty,\infty]^{d+1}$. By the uniform continuity of h, given $\epsilon > 0$, there exists a $\theta > 0$ such that

$$\sup_{\max_{i=1,\ldots,d+1}|u_{i1}-u_{i2}|<\theta} \{|h(t,y,\mathbf{u}_1) - h(t,y,\mathbf{u}_2)| : t \in [0,\infty], y \in (-\infty,\infty]\} < \epsilon.$$

Thus, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\left[\left|\int h\, d\kappa_n^{(m)} - \int h\, d\kappa_n\right| > \delta\right] \leq \limsup_{n\to\infty} \delta^{-1}\mathbb{E}\left|\int h\, d\kappa_n^{(m)} - \int h\, d\kappa_n\right| \\
&\quad \leq \limsup_{n\to\infty} nT\delta^{-1}\mathbb{P}(a_n^{-1}(Z_1 - b_n) \geq x)[\epsilon + \mathbb{P}(|a_n^{-1}X_1^{(m)} - a_n^{-1}X_1^*| > \theta)] \\
&\qquad + \limsup_{n\to\infty} \delta^{-1} \sum_{k=1}^{nT} \mathbb{P}\left(\sup_{i=1,\ldots,d}\left|\frac{Y(\Gamma_k + t_i)}{b_n} - f(R_k,t_i)\right| > \theta, Z_k > a_n x + b_n\right) \\
&\quad = T\delta^{-1}[\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\mathrm{e}^{-x}[\epsilon + \mathbb{P}(|\gamma W_1^{(m)} - \gamma W_1| > \theta)],
\end{aligned}$$

where we used Markov's inequality and Lemma 5.7. Invoking $W_1^{(m)} \overset{P}{\longrightarrow} W_1$ as $m \to \infty$, the last expression tends to $T\delta^{-1}[\mu\mathbb{E}\mathrm{e}^{\gamma L(1)}]^{-1}\mathrm{e}^{-x}\epsilon$ as $m \to \infty$. Since $\epsilon > 0$ is arbitrary, we conclude that the limit must be 0, which proves (5.49). □

Proof of Proposition 4.3. Since Z_k and X_k^* are independent with $X_k = Z_k + X_k^*$ and

$$\mathbb{P}(X_k^* > x) = \mathbb{P}(Y_0 > x) = o(\mathbb{P}(L(1) > x)) \quad \text{as } x \to \infty,$$

we obtain the result, similar to Theorem 3.3 in Davis and Resnick [8], by Proposition 5.6 and the continuous mapping theorem. □

Proof of Theorem 4.1. Let $M_k^* = \sup_{t\in[\Gamma_k,\Gamma_{k+1})}[Y(t) + f(R_0,t)Z_0]$ for $k \in \mathbb{N}_0$. First, we show for the stationary sequence $(M_k^*)_{k\in\mathbb{N}_0}$

$$\zeta_n^* := \sum_{k=1}^{\infty} \varepsilon_{(k/(\mu n), a_n^{-1}(M_k^* - b_n), b_n^{-1}\mathbf{Y}(\Gamma_k))} \overset{n\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \varepsilon_{(s_k, P_k + \gamma W_k, \mathbf{f}(R_k))} = \xi \quad \text{ in } M_P(\mathcal{T}_d).$$

Let $I = [s,t) \times (x,\infty) \times [\boldsymbol{\gamma},\boldsymbol{\delta}] \subseteq [0,\infty) \times \mathcal{T}_d$. Applying Lemma 5.3 and using the notation of Proposition 4.3 yield

$$\lim_{n\to\infty} \mathbb{P}(\zeta_n^*(I) \neq \xi_n(I)) \leq \lim_{n\to\infty} n\mu(t-s)[\mathbb{P}(M_0^* > u_n) - \mathbb{P}(X_0 > u_n)] = 0.$$

This implies $\zeta_n^* \overset{n\to\infty}{\Longrightarrow} \xi$ by Proposition 4.3 and Rootzén [25], Lemma 3.3. Now, we define the point processes

$$\zeta_n^{**} := \sum_{k=1}^{\infty} \varepsilon_{(k/(\mu n), a_n^{-1}(M_k - b_n), b_n^{-1}\mathbf{Y}(\Gamma_k))} \qquad \text{for } n \in \mathbb{N}.$$

Let $\epsilon > 0$ and $I_\epsilon = [s,t) \times (x-\epsilon, x+\epsilon) \times [\boldsymbol{\gamma},\boldsymbol{\delta}] \subseteq [0,\infty) \times \mathcal{T}_d$. Then, taking (B.3) in Fasen et al. [15] into account, we obtain

$$\begin{aligned}
&\mathbb{P}(\zeta_n^{**}(I) \neq \zeta_n^*(I)) \\
&\quad \leq \mathbb{P}(\zeta_n^*(I_\epsilon) > 0) + \mathbb{P}\left(\left|\Gamma_m - \frac{m}{\mu}\right| > \frac{m}{2\mu}\right) + \mathbb{P}\left(\zeta_n^*(I) \neq \zeta_n^{**}(I), \zeta_n^*(I_\epsilon) = 0, \Gamma_m \geq \frac{m}{2\mu}\right) \\
&\quad \leq \mathbb{P}(\zeta_n^*(I_\epsilon) > 0) + \frac{\widetilde{K}}{m^3} + \mathbb{P}(f^*(R_0, m/(2\mu))Z_0 > \epsilon a_n).
\end{aligned}$$

The last expression tends, as $n \to \infty$, to

$$\mathbb{P}(\xi(I_\epsilon) > 0) + \frac{\widetilde{K}}{m^3} + \mathbb{P}(f^*(R_0, m/(2\mu))Z_0 > \epsilon\gamma^{-1}). \tag{5.50}$$

Since $f^*(R_0, m/(2\mu)) \to 0$ as $m \to \infty$ a. s., we conclude that (5.50) tends to $\mathbb{P}(\xi(I_\epsilon) > 0)$ as $m \to \infty$. Since ϵ is arbitrary we obtain that $\lim_{n\to\infty} \mathbb{P}(\zeta_n^*(I) \neq \zeta_n^{**}(I)) = 0$. Again the conclusion $\zeta_n^{**} \overset{n\to\infty}{\Longrightarrow} \xi$ follows from Rootzén [25], Lemma 3.3. By a modification of an argument of Hsing and Teugels [17] (see the proofs of their Theorem 4.2 and Lemma 2.1, and for more details Fasen [11], Corollary 1.2.2) we have $\lim_{n\to\infty} \mathbb{P}(\zeta_n^{**}(I) \neq \zeta_n(I)) = 0$, and hence, $\zeta_n \overset{n\to\infty}{\Longrightarrow} \xi$. □

References

[1] Barndorff-Nielsen, O. E. (2001). Superposition of Ornstein–Uhlenbeck type processes. *Theory Probab. Appl.* **45**, 175–194.

[2] Barndorff-Nielsen, O. E. and Shephard, N. (2001). Modelling by Lévy processes for financial econometrics. In: O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick (Eds.), *Lévy Processes: Theory and Applications*, pp. 283–318. Birkhäuser, Boston.

[3] Barndorff-Nielsen, O. E. and Shephard, N. (2001). Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics (with discussion). *J. Roy. Statist. Soc. Ser. B* **63**, 167–241.

[4] Billingsley, P. (1999). *Convergence of Probability and Measures*. 2nd edn. Wiley, New York.

[5] Braverman, M. and Samorodnitsky, G. (1995). Functionals of infinitely divisible stochastic processes with exponential tails. *Stochastic Process. Appl.* **56**, 207–231.

[6] Cline, D. B. H. (1986). Convolution tails, product tails and domains of attraction. *Probab. Theory Related Fields* **72**, 529–557.

[7] Daley, D. J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes*, vol. I: Elementary Theory and Methods. 2nd edn. Springer, New York.

[8] Davis, R. and Resnick, S. (1988). Extremes of moving averages of random variables from the domain of attraction of the double exponential distribution. *Stochastic Process. Appl.* **30**, 41–68.

[9] Embrechts, P. and Goldie, C. M. (1982). On convolution tails. *Stochastic Process. Appl.* **13**, 263–278.

[10] Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.

[11] Fasen, V. (2004). *Extremes of Lévy Driven MA Processes with Applications in Finance*. Ph.D. thesis, Munich University of Technology.

[12] Fasen, V. (2005). Extremes of regularly varying mixed moving average processes. *Adv. in Appl. Prob.* **37**, 993–1014.

[13] Fasen, V. (2006). Extremes of subexponential Lévy driven moving average processes. *Stochastic Process. Appl.* **116**, 1066–1087.

[14] Fasen, V. and Klüppelberg, C. (2007). Extremes of SupOU processes. In: F. E. Benth, G. Di Nunno, T. Lindstrom, B. Oksendal, and T. Zhang (Eds.), *Stochastic Analysis and Applications: The Abel Symposium 2005*, pp. 340–359. Springer.

[15] Fasen, V., Klüppelberg, C., and Lindner, A. (2006). Extremal behavior of stochastic volatility models. In: A. N. Shiryaev, M. d. R. Grossinho, P. E. Oliviera, and M. L. Esquivel (Eds.), *Stochastic Finance*. Springer, New York.

[16] Goldie, C. M. and Resnick, S. (1988). Subexponential distribution tails and point processes. *Commun. Statist.-Stochastic Models* **4**, 361–372.

[17] Hsing, T. and Teugels, J. L. (1989). Extremal properties of shot noise processes. *Adv. Appl. Probability* **21**, 513–525.

[18] Kallenberg, O. (1997). *Foundations of Modern Probability*. Springer, New York.

[19] Kingman, J. F. C. (1993). *Poisson Processes*. Oxford University Press, Oxford.

[20] Kwapień, S. and Woyczyński, W. A. (1992). *Random Series and Stochastic Integrals: Single and Multiple*. Birkhäuser, Boston.

[21] Leadbetter, M. R., Lindgren, G., and Rootzén, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York.

[22] Pakes, A. G. (2004). Convolution equivalence and infinite divisibility. *J. Appl. Probab.* **41**, 407–424.

[23] Rajput, B. S. and Rosinski, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields* **82**, 453–487.

[24] Resnick, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.

[25] Rootzén, H. (1986). Extreme value theory for moving average processes. *Ann. Probab.* **14**, 612–652.

[26] Rosinski, J. and Samorodnitsky, G. (1993). Distributions of subadditive functionals of sample paths of infinitely divisible processes. *Ann. Probab.* **21**, 996–1014.

[27] Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York.

[28] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.

[29] Schoutens, W. (2003). *Lévy Processes in Finance*. Wiley, Chichester.

[30] Urbanik, K. and Woyczyński, W. A. (1967). Random integrals and Orlicz spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* pp. 161–169.

[31] Watanabe, T. (2008). Convolution equivalence and distributions of random sums. *Probab. Theory Relat. Fields* **142**, 367–397.