GENERATING LONG MEMORY MODELS BASED ON CARMA PROCESSES

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Abstract. Starting from short memory (Gaussian) processes we present various approaches to construct long memory processes and generalize these concepts to the Lévy setting. Moreover, Ornstein-Uhlenbeck processes are replaced by more general moving average (MA) processes, e.g. CARMA processes, thus allowing to model a broader class of autocorrelation functions, for instance oscillating autocorrelations. We obtain superposititions of MA processes, in particular supCARMA processes, as well as, by randomizing the time scale of short memory processes, a rather large class of long memory MA processes. Finally, Lévy-driven Gamma-mixed moving average processes exhibiting long memory are introduced. The latter model has the nice property that its integrated process can be calculated explicitly and converges to a fractional Lévy process.

1. Introduction. In modern mathematical finance continuous time models play a crucial role because they allow handling unequally spaced data and even high frequency data, which are realistic for liquid markets. In this context Lévy-driven processes of Ornstein-Uhlenbeck (OU) type have been extensively studied over the last recent years and widely used in applications. Several examples of univariate non-Gaussian OU processes can be found in Barndorff-Nielsen & Shephard (2001a), where OU processes are used to model stochastic volatility. In this paper we replace OU processes by the more general CARMA(p, q) processes driven by a two-sided Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$, defined as

$$L(t) = \begin{cases} L_1(t), & t \geq 0 \\ -L_2(-t-), & t < 0 \end{cases}, \tag{1.1}$$

where $L_1 = \{L_1(t)\}_{t\geq 0}$ is a Lévy process and $L_2 = \{L_2(t)\}_{t\geq 0}$ is an independent copy of L_1. The virtue of CARMA(p, q) processes is that a much larger class of autocorrelations can be modeled. In particular, the autocorrelation functions of CARMA processes are not necessarily monotone decreasing as that of OU processes. Moreover, it has been shown in an econometric analysis by Todorov & Tauchen (2004) that CARMA and in particular CARMA(2, 1) processes are reasonable processes to model stochastic volatility. Lévy-driven CARMA processes, have been studied and applied during the last years (see e.g. Brockwell (2001a), Brockwell (2001b), Todorov & Tauchen (2004) and the references therein), but like OU processes belong to the class of short memory processes, due to the fact that their autocorrelation functions show an exponential rate of decay.

Recently, Brockwell (2004) (see also Brockwell & Marquardt (2005)) defined fractionally integrated CARMA (FICARMA) processes by a fractional integration of the kernel function of the short memory CARMA process. These FICARMA processes exhibit long memory properties in the sense that their autocorrelations are hyperbolically decreasing. However,

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due to the slow decay of the fractionally integrated kernel function, simulation algorithms for FICARMA processes have been very slow and expensive and so far no stimulating estimation techniques are available.

An alternative approach to generate long memory processes, namely infinite superposition of Ornstein-Uhlenbeck (supOU) processes, was studied in Barndorff-Nielsen (2001). SupOU processes provide a flexible class of models which incorporate long-range dependence as well as self-similarity-like properties. Furthermore, supOU processes have the potential to describe some of the other key distributional features of typical data in finance, turbulence and other fields of application. In particular, empirical volatility has tails heavier than normal, long-range dependence in the sense that the empirical autocorrelation function decreases slower than exponential and exhibits volatility clusters on high levels. Barndorff-Nielsen & Shephard (2001b) investigate supOU processes as volatility models and show that supOU processes are capable to model these so-called stylized facts. Moreover Fasen & Klüppelberg (2007) study the extremal behaviour of supOU processes. However, supOU processes are defined in terms of integrals with respect to an independently scattered random measure - often referred to as Lévy basis - and thus the supOU framework adds another non-trivial layer of complexity to the problem of simulation and parameter estimation.

This is our motivation to propose in this paper several alternative approaches to generate long memory processes leading to models which are easy to simulate and estimate. After having introduced the necessary preliminaries in Section 2, we consider in Section 3 superpositions of general moving average (MA) processes. To the best of our knowledge, this is the first approach to construct an infinite superposition of CARMA (supCARMA) processes. By randomizing the time scale of a (short memory) MA process we construct in Section 4 a MA model which allows for modeling long memory situations and which has the same autocovariance functions as the supOU (or supCARMA) processes when its kernel function equals that of an OU (CARMA) process. In Section 5 we discuss mixing models by transfer functions. In particular, we come up with a Lévy-driven Gamma-mixed OU process which on the one hand is the limiting process of centered m-factor models, as well as closely related to the fractional Lévy processes considered in Marquardt (2006b). In fact, the latter aggregation model has the nice property that its integrated process can be calculated explicitly and converges to a fractional Lévy process. Finally, we calculate integrated volatility and show how the aggregation idea can be generalized.

2. Preliminaries. In this section we recall some basic defintions and notions which we will need in the following sections.

2.1. CARMA and OU Processes. As the name already suggests, CARMA processes belong the class of continuous-time moving average (MA) processes.

Definition 2.1 (Stationary MA Process). *A stationary continuous time moving average (MA) process is a process of the form*

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)\, L(du), \qquad t \in \mathbb{R}, \tag{2.1}$$

where $g : \mathbb{R} \to \mathbb{R}$, called kernel function, is measurable and the driving process $L = \{L(t)\}_{t\in\mathbb{R}}$ is a two-sided Lévy process on $\mathbb{R}$ having generating triplet (γ, σ^2, ν). We call L the back-

ground driving Lévy process (BDLP) of the MA process $Y = \{Y(t)\}_{t\in\mathbb{R}}$.

Definition 2.2 (CARMA(p,q) Process). *A Lévy-driven continuous time autoregressive moving average CARMA(p,q) process* $\{Y(t)\}_{t\geq 0}$ *of order* (p,q) *with* $p,q \in \mathbb{N}_0, p > q$ *is defined to be the stationary solution of the formal p-th order linear differential equation,*

$$p(D)Y(t) = q(D)DL(t), \quad t \geq 0, \tag{2.2}$$

where D *denotes differentiation with respect to* t, $\{L(t)\}_{t\geq 0}$ *is a Lévy process with Lévy measure* ν *satisfying* $\int_{|x|>1} \log|x|\nu(dx) < \infty$,

$$p(z) := z^p + a_1 z^{p-1} + ... + a_p \ \ \text{and} \ \ q(z) := b_0 z^q + b_1 z^{q-1} + ... + b_q, \tag{2.3}$$

where $a_p \neq 0$, $b_q \neq 0$. *The polynomials* $p(\cdot)$ *and* $q(\cdot)$ *are referred to as the autoregressive and moving average polynomial, respectively.*

Since in general the derivative of a Lévy process does not exist, (2.2) is interpreted as being equivalent to the observation and state equations

$$Y(t) = b^T Z(t) \qquad \text{and} \tag{2.4}$$

$$dZ(t) = AZ(t)dt + e\,L(dt), \quad t \geq 0, \tag{2.5}$$

where $A = \left[\begin{array}{c|ccc} 0 & & I_{p-1} & \\ \hline -a_p & -a_{p-1} & \dots & -a_1 \end{array}\right]$, $e^T = [0,\dots,0,1]$, $b^T = \left[\ b_q, b_{q-1}, \dots, b_{q-p+1}\ \right]$ with $b_{-1} = b_{-2} = \ldots = b_{q-p+1} = 0$, if $q < p-1$ and $I_{p-1} \in M_{p-1}(\mathbb{R})$ denotes the identity matrix.

Remark 2.3. It is easy to check that the eigenvalues $\vartheta_1, \dots, \vartheta_p$ of the matrix A are the zeros of the autoregressive polynomial $p(z)$.

Proposition 2.4 (Brockwell (2004, Section 2)). *If all eigenvalues* $\vartheta_1, \dots, \vartheta_p$ *of* A*, i.e. the roots of* $p(z)$*, have negative real parts, the process* $\{Z(t)\}_{t\in\mathbb{R}}$ *defined by* $Z(t) = \int_{-\infty}^{t} e^{A(t-u)} e\,L(du)$, $t \in \mathbb{R}$, *is the strictly stationary solution of (2.5) for* $t \in \mathbb{R}$ *with corresponding CARMA process*

$$Y(t) = \int_{-\infty}^{t} b^T e^{A(t-u)} e\,L(du), \quad t \in \mathbb{R}, \tag{2.6}$$

where L *is a two-sided Lévy process as defined in (1.1).*

As it drives the CARMA process, we refer to L as the BDLP (see Definition 2.1).

From (2.6) it is obvious that $Y = \{Y(t)\}_{t\in\mathbb{R}}$ is a moving average process, since it has the form

$$Y(t) = \int_{-\infty}^{t} g(t-u)\,L(du), \quad t \in \mathbb{R},$$

with kernel

$$g(t) = b^T e^{At} e 1_{[0,\infty)}(t) \tag{2.7}$$

satisfying $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Replacing e^{At} by its spectral representation, the kernel g given by (2.7) can be expressed as (Brockwell (2004))

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\vartheta} \frac{q(i\vartheta)}{p(i\vartheta)} d\vartheta, \quad t \in \mathbb{R}. \tag{2.8}$$

From now on we assume that the condition on the eigenvalues of A in Proposition 2.4 is always satisfied.

Obviously, the CARMA(1,0) process is an Ornstein-Uhlenbeck (OU) process

$$Y(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} L(ds), \quad t \in \mathbb{R}, \tag{2.9}$$

where $\lambda > 0$. In the literature the OU process is often written in the form of an Itô stochastic differential equation (SDE) $dY(t) = -\lambda Y(t)dt + L(dt)$, where it is understood that Y has to be the stationary solution of the SDE.

As long as $E[L(1)^2] < \infty$, the autocorrelation functions of CARMA processes exist and show an exponential rate of decay, i.e. CARMA processes belong to the class of short memory models. In particular, for the OU process we have $r_Y(h) = \mathrm{corr}(Y(t+h), Y(t)) = e^{-\lambda h}$, $h \in \mathbb{R}$. This and the Markov property of CARMA processes are often too restrictive and recently various approaches have been made to generalize the Lévy-driven OU and CARMA model, respectively.

In order to expand the class of OU processes one can for instance construct a process as the sum, or superposition, of independent OU processes, each indexed by different parameter values, i.e.

$$\tilde{Y}^m(t) = \sum_{j=1}^{m} Y_j(t), \tag{2.10}$$

where Y_j is an OU process as defined in (2.9). In econometrics this finite superposition of OU processes is often called an m-factor model. This model can be extended to the infinite dimensional case by allowing m, the number of components of Y, to go off to infinity and by replacing summation with integration (see Section 2.2 below).

2.2. Lévy Basis and SupOU Processes. This section contains a brief review of the definition and the main properties of supOU processes. Throughout this section we write $\tau(x) = 1_{\{|x| \le 1\}}$ and work with an infinitely divisible independently scattered random measure, which we will refer to as *Lévy basis*, $\Lambda = \{\Lambda(A);\ A \in \mathcal{T}\}$, on some probability space $(\Omega, \mathcal{F}, P)$, as defined in Rajput & Rosinski (1989), where $\mathcal{T}$ is a σ-ring on $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$. This means that for every sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{T}$ of disjoint sets the random variables $\Lambda(A_n)$, $n = 1, 2, \ldots$, are independent and $\Lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Lambda(A_n)$ *a.s.*, whenever $\cup_{n=1}^{\infty} A_n \in \mathcal{T}$. Furthermore, for every $A \in \mathcal{T}$, $\Lambda(A)$ is an infinitely divisible random variable whose characteristic function can be written in the form (see Rajput & Rosinski (1989, Proposition 2.4)),

$$E[\exp\{iu\Lambda(A)\}] = \exp\{\Psi(u)\}, \quad u \in \mathbb{R}, \tag{2.11}$$

where

$$\Psi(u) = ium_0(A) - \frac{u^2}{2} m_1(A) + \int_{\mathbb{R}} (e^{iux} - 1 - iu\tau(x))\, Q(A, dx). \tag{2.12}$$

Here m_0 is a signed measure, m_1 is a positive measure and Q is a generalized Lévy measure, which means by definition that for every fixed $A \in \mathcal{T}$ with points $\omega = (s, \lambda)$, $Q(A, \cdot)$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$, i.e. $Q(A, \{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2)\, Q(A, dx) < \infty$. In this paper, like in Barndorff-Nielsen (2001), our discussion is restricted to the case where

$$Q(d\omega, dx) = ds\, \pi(d\lambda)\, \nu(dx) \tag{2.13}$$

factorizes as the product of some probability measure π on $\mathbb{R}_+$, the Lebesgue measure (*Leb*) and some Lévy measure ν on $\mathbb{R}$. Moreover, we assume

$$m_0(d\omega) = \gamma\, ds\, \pi(d\lambda) \quad \text{and} \quad m_1(d\omega) = \sigma^2\, ds\, \pi(d\lambda),$$

for $\gamma \in \mathbb{R}$ and $\sigma^2 \geq 0$. We shall refer to $(\gamma, \sigma^2, \nu, \pi)$ as the *Lévy characteristics* of Λ, as the distribution of Λ is completely determined by $(\gamma, \sigma^2, \nu, \pi)$. Writing

$$\psi(u) = iu\gamma - \frac{u^2}{2}\sigma^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iu\tau(x))\, \nu(dx), \quad u \in \mathbb{R}, \tag{2.14}$$

we can rewrite (2.11) as

$$E[\exp\{iu\Lambda(A)\}] = \exp\left\{ \int_{\mathbb{R}_+} \int_{\mathbb{R}} \psi(u 1_A(\lambda, s))\, ds\, \pi(d\lambda) \right\}, \quad u \in \mathbb{R}. \tag{2.15}$$

Notice that for $A = \mathbb{R}_+ \times [0, t]$, $t \geq 0$, we obtain the corresponding Lévy process $\Lambda(\mathbb{R}_+ \times [0, t]) = L(t)$ with characteristic triplet (γ, σ^2, ν).

Integrals with respect to a Lévy basis Λ are defined as limits in probability of integrals over simple functions. In what follows we will work with the following proposition and refer to Rajput & Rosinski (1989, Theorem 2.7) for details and proofs.

Proposition 2.5. *A measurable function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is Λ-integrable if and only if*

(i) $\displaystyle\int_{\mathbb{R}_+} \int_{\mathbb{R}} \left| f(\lambda, s)\gamma + \int_{\mathbb{R}} (\tau(x f(\lambda, s)) - f(\lambda, s)\tau(x))\, \nu(dx) \right| ds\, \pi(d\lambda) < \infty,$

(ii) $\displaystyle\sigma^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}} |f(\lambda, s)|^2\, ds\, \pi(d\lambda) < \infty,$

(iii) $\displaystyle\int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}} (1 \wedge |x f(\lambda, s)|^2)\, \nu(dx)\, ds\, \pi(d\lambda) < \infty.$

If (i)-(iii) hold, $\int f\, d\Lambda$ is infinitely divisible with characteristic function

$$E\left[\exp\left\{ iu \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(\lambda, s)\, d\Lambda(\lambda, s) \right\} \right] = \exp\left\{ \int_{\mathbb{R}_+} \int_{\mathbb{R}} \psi(u f(\lambda, s))\, ds\, \pi(d\lambda) \right\}. \tag{2.16}$$

Now we recall the definition of supOU processes (Barndorff-Nielsen (2001)) which can serve as long memory models.

We say that a stationary process $X = \{X(t)\}_{t\in\mathbb{R}}$ exhibits long-range dependence (long memory) if the autocovariance (or autocorrelation) function γ_X of X behaves as

$$\gamma_X(h) \sim \tilde{L}(h)h^{-2d} \tag{2.17}$$

for $h \to \infty$, where $\tilde{L}$ is a slowly varying function and $d \in (0, 0.5)$.

Suppose that the above assumptions are satisfied, i.e. $\Omega = \mathbb{R}_+ \times \mathbb{R}$ with points $\omega = (s, \lambda)$ and let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbb{R}_+ \times \mathbb{R}$. Furthermore let Λ be a Lévy basis on $(\Omega, \mathcal{B})$ with characteristics $(\gamma, 0, \nu, \pi)$, where π is a probability measure on $\mathbb{R}_+$. Moreover, we assume

$$\gamma = -\int_{\{|x|>1\}} |x|\,\nu(dx). \tag{2.18}$$

Definition 2.6 (supOU Process). *Let Λ be a Lévy basis which satisfies the above conditions. We call a stochastic process $X = \{X(t)\}_{t\in\mathbb{R}}$ of the form*

$$X(t) = \int_{\mathbb{R}_+} e^{-\lambda t} \int_{-\infty}^{\lambda t} e^{s}\,\Lambda(ds, d\lambda), \quad t \in \mathbb{R}, \tag{2.19}$$

a supOU process.

Remark 2.7. The process X defined in (2.19) is well-defined, stationary and infinitely divisible, where the finite-dimensional distributions of the stationary process X have the cumulant generating function

$$\begin{aligned} &\log E[\exp\{i(u_1 X_{t_1} + \ldots + u_m X_{t_m})\}] \\ &\quad = \int_{\mathbb{R}_+}\int_{\mathbb{R}} \psi\left(\sum_{j=1}^{m} u_j e^{-\lambda(t_j - s)} 1_{[0,\infty)}(t_j - s)\right) \lambda\, ds\, \pi(d\lambda), \end{aligned} \tag{2.20}$$

where $m \in \mathbb{N}$, $-\infty < t_1 < \ldots < t_m < \infty$, $u_1, \ldots, u_m \in \mathbb{R}$ and ψ is given as in (2.14).

Observe that we can conclude from (2.18) that $E[X(t)] = 0$ for all $t \in \mathbb{R}$. Furthermore, we have the following proposition.

Proposition 2.8. *Assuming the supOU process X is square integrable the autocorrelation function r_X of X is given by*

$$r_X(h) = \int_0^\infty e^{-\lambda h}\,\pi(d\lambda), \quad h \geq 0. \tag{2.21}$$

Example 2.9. *Suppose that π is the $\Gamma(1-2d, 1)$ law, where $0 < d < 0.5$, i.e.*

$$\pi(d\lambda) = \frac{1}{\Gamma(1-2d)}\lambda^{-2d} e^{-\lambda}\,d\lambda.$$

Then

$$r_X(h) = (1+h)^{2d-1},$$

i.e. X exhibits long-range dependence.

Now, if we interprete λ as a random variable with distribution π we can rewrite (2.21) as

$$r_X(h) = E[e^{-\lambda h}],$$

i.e. we can model the memory of a linear, nonnegative process through the moment generating function of the memory parameter and the choice of this moment generating function has no impact on the marginal distribution of λ.

Some choices of the moment generating function will deliver short memory models, others will deliver long memory. However, all of these models will have correlations which are nonnegative at all values of h. This fact is an unfortunate limitation of the class of supOU processes.

Based on this observation our motivation is to define superpositions of CARMA (supCARMA) processes, since CARMA processes allow for a much more general class of autocorrelation functions. In particular we will not only derive supCARMA processes but a superposition of general moving average processes.

3. Superposition of Moving Average Processes. This section is devoted to the construction of supCARMA processes and superposititions of moving average (supMA) processes, in general. In order to construct superpositions of MA processes we work with Lévy bases as introduced in Section 2.2. Moreover, we need the following "generalization" of MA processes.

Definition 3.1. *For $\lambda > 0$ define a moving average process $Y_\lambda = \{Y_\lambda(t)\}_{t\in\mathbb{R}}$ by*

$$Y_\lambda(t) := Y(\lambda t) = \int_{-\infty}^{t} g(\lambda(t-s))\, L(\lambda ds), \quad t \in \mathbb{R}, \tag{3.1}$$

where L is a Lévy processes and $g : \mathbb{R} \to \mathbb{R}$ is a kernel function satisfying $g(t) = 0$ for $t < 0$.

For simplicity from now on we assume that $E[L(1)] = 0$ and $E[L(1)^2] < \infty$. In particular, then (3.1) exists whenever $g \in L^2(\mathbb{R})$.

Proposition 3.2. *The process Y_λ defined in (3.1) is stationary. Moreover, we have for all $t \in \mathbb{R}$ and $\lambda > 0$ fixed*

$$Y_\lambda(t) \overset{d}{=} Y_\lambda(0) \overset{d}{=} Y(0).$$

Proof. Stationarity can be easily seen from

$$\begin{aligned}\log E\left[\exp\left\{i\sum_{j=1}^{m} u_j Y_\lambda(t_j)\right\}\right] &= \int_{\mathbb{R}} \psi\left(\sum_{j=1}^{m} u_j g(\lambda(t_j - s))1_{[0,\infty)}(t_j - s)\right) \lambda\, ds \\ &= \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \psi\left(\sum_{j=k}^{m} u_j g(\lambda(t_j - s))\right) \lambda\, ds,\end{aligned}$$

where $m \in \mathbb{N}$, $-\infty < t_0 < t_1 < \ldots < t_m$ and $u_1, \ldots, u_m \in \mathbb{R}$ and ψ is given as in (2.14). In particular,

$$\log E\left[e^{iuY_\lambda(t)}\right] = \int_{-\infty}^{t} \psi\left(ug(\lambda(t-s)\right)\lambda\, ds$$
$$= \int_{0}^{\infty} \psi\left(ug(\lambda s)\right)\lambda\, ds = \int_{0}^{\infty} \psi\left(ug(s)\right)\, ds = \log E\left[e^{iuY(0)}\right].$$

$\square$

We are now in the position to define superpositions of MA processes.

Definition 3.3. *Let Λ be a Lévy basis satisfying the conditions of Section 2.2. We call stochastic processes $X = \{X(t)\}_{t\in\mathbb{R}}$ of the form*

$$X(t) = \int_{0}^{\infty}\int_{-\infty}^{t} g(\lambda(t-s))\,\Lambda(\lambda\, ds, d\lambda), \quad t \in \mathbb{R}, \tag{3.2}$$

superpositions of moving average processes or *supMA processes, for short.*

The following proposition is straightforward, we therefore omit its proof.

Proposition 3.4. *The process X defined in (3.2) is well-defined, stationary and infinitely divisible, where the finite-dimensional distributions of the stationary process X have the cumulant generating function*

$$\log E\left[\exp\left\{i\sum_{j=1}^{m} u_j X(t_j)\right\}\right] = \int_{\mathbb{R}_+}\int_{\mathbb{R}} \psi\left(\sum_{j=1}^{m} u_j g(\lambda(t_j - s))1_{[0,\infty)}(t_j - s)\right)\lambda\, ds\, \pi(d\lambda),$$

where $m \in \mathbb{N}$, $-\infty < t_1 < \ldots < t_m < \infty$, $u_1, \ldots, u_m \in \mathbb{R}$ and ψ is given as in (2.14).

Remark 3.5.

(i) Substituting the OU kernel $g(t) = e^{-\lambda t}$ into (3.2) we obtain the supOU processes discussed in Section 2.2.

(ii) Substituting the CARMA kernel (2.8) into (3.2), we thus obtain a superposition of CARMA processes, which we will refer to as supCARMA processes. The supCARMA process is then given by

$$X(t) = \int_{\mathbb{R}_+}\int_{-\infty}^{t} b^T e^{A(t-s)} e\,\Lambda(\lambda ds, d\lambda) = b^T\left(\int_{\mathbb{R}_+} e^{A\lambda t} e \int_{-\infty}^{\lambda t} e^{-As} e\,\Lambda(ds, d\lambda)\right).$$

Moreover, a formal calculation shows that the supCARMA process is the stationary solution of

$$X(t) = b^T Z(t),$$
$$dZ(t) = \int_{\mathbb{R}_+}\left\{-\lambda Z(t, d\lambda)\, dt + \Lambda(dt, d\lambda)\right\},$$

where

$$Z(t, B) = \int_{B} e^{A\lambda t} e \int_{-\infty}^{\lambda t} e^{-As} e\,\Lambda(ds, d\lambda).$$

(iii) Obviously, $E[X(t)] = 0$ for all $t \in \mathbb{R}$.

(iv) Let us denote by $\gamma_{Y_\lambda}(h) := E[Y_\lambda(t+h)Y_\lambda(t)]$, $h \geq 0$, the autocovariance function of the MA process (3.1) and by $r_{Y_\lambda}(h)$ the corresponding autocorrelation function. Then, assuming that the supMA process X is square integrable, we obtain that the autocorrelation function r_X of the supMA process X is given by

$$r_X(h) = \int_0^\infty r_{Y_\lambda}(h)\, \pi(d\lambda), \quad h \geq 0.$$

We have already mentioned in the introduction that supOU processes and thus in particular supMA processes are often too complex models for simulations and parameter estimation. Therefore, in the following section we will discuss a simple and effective method of constructing long memory models which leads to a class of more tractable long memory models. For Gaussian processes a similar idea has been considered in Chunsheng (2003).

4. Long Memory Generation via Randomization of Moving Average Processes. We consider again the stationary MA process Y_λ introduced in (3.1). We have seen that the process Y_λ is stationary and that in distribution holds

$$Y_\lambda(t) \stackrel{d}{=} Y_\lambda(0) = Y(0),$$

which is (pathwise) independent of λ.

Now, assume that λ is a non-negative random variable with distribution π independent of the BDLP L. By this randomization we then define a process $\tilde{X} = \{\tilde{X}(t)\}_{t\in\mathbb{R}}$ such that

$$\tilde{X}(t) \,|\, \lambda \stackrel{d}{=} Y_\lambda(t), \tag{4.1}$$

i.e.

$$\tilde{X}(t,\omega) = Y_{\lambda(\omega)}(t,\omega) = \int_{-\infty}^{t} g(\lambda(\omega)(t-s))\, L(\lambda(\omega)ds,\, \omega), \quad t \in \mathbb{R}.$$

Proposition 4.1. *The process $\tilde{X}$ defined in (4.1) is well-defined (provided that Y_λ is well-defined) and stationary. Furthermore, the finite-dimensional distributions of the stationary process $\tilde{X}$ have the characteristic functions*

$$E\left[\exp\left\{i\sum_{j=1}^{m} u_j \tilde{X}(t_j)\right\}\right] = \int_0^\infty \exp\left\{\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \psi\left(\sum_{j=k}^{m} u_j g(\lambda(t_j - s))\right) \lambda\, ds\right\} \pi(d\lambda).$$

where $m \in \mathbb{N}$, $-\infty < t_1 < \ldots < t_m < \infty$, $u_1, \ldots, u_m \in \mathbb{R}$ and ψ is given in (2.14).

Proof. We have

$$E\left[\exp\left\{i\sum_{j=1}^{m}u_j\tilde{X}(t_j)\right\}\right] = E\left[E\left[\exp\left\{i\sum_{j=1}^{m}u_j\tilde{X}(t_j)\right\}\,\Big|\,\lambda\right]\right]$$

$$= \int_0^\infty E\left[\exp\left\{i\sum_{j=1}^{m}u_jY_\lambda(t_j)\right\}\right]\,\pi(d\lambda)$$

$$= \int_0^\infty \exp\left\{\int_{\mathbb{R}}\psi\left(\sum_{j=1}^{m}u_jg(\lambda(t_j-s))1_{[0,\infty)}(t_j-s)\right)\lambda\,ds\right\}\,\pi(d\lambda)$$

$$= \int_0^\infty \exp\left\{\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\psi\left(\sum_{j=k}^{m}u_jg(\lambda(t_j-s))\right)\lambda\,ds\right\}\,\pi(d\lambda),$$

from which follows stationarity. □

Note that the preceding theorem shows that the process X given by (3.2) and the process $\tilde{X}$ defined by (4.1) are not identical. In particular, the marginal distributions of $\tilde{X}$ are in general no longer infinitely divisible. However, we have the following result.

PROPOSITION 4.2. *The process $\tilde{X}$ defined by (4.1) has the same autocorrelation function $r_{\tilde{X}}$ as the supMA process, i.e.*

$$r_{\tilde{X}}(h) = \int_0^\infty r_{Y_\lambda}(h)\,\pi(d\lambda), \quad h \geq 0, \tag{4.2}$$

where $r_{Y_\lambda}(h) = r_Y(\lambda h)$ denotes the autocorrelation function of the process Y_λ defined in (3.1) and r_Y denotes the autocorrelation function of the MA process (2.1).

Proof. We have

$$\mathrm{cov}(\tilde{X}(t+h),\tilde{X}(t)) = E[\tilde{X}(t+h)\tilde{X}(t)] - E[\tilde{X}(t+h)]E[\tilde{X}(t)]$$

$$= \int_0^\infty E[\tilde{X}(t+h)\tilde{X}(t)\,|\,\lambda]\,\pi(d\lambda) - E[E[\tilde{X}(t+h)\,|\,\lambda]]\;E[E[\tilde{X}(t)\,|\,\lambda]]$$

$$= \int_0^\infty E[Y_\lambda(t+h)Y_\lambda(t)]\,\pi(d\lambda) - \int_0^\infty E[Y_\lambda(t+h)]E[Y_\lambda(t)]\,\pi(d\lambda)$$

$$= \int_0^\infty \gamma_{Y_\lambda}(h)\,\pi(d\lambda)$$

□

In order to randomize CARMA processes we define for $\lambda > 0$ a stationary CARMA process by

$$Y_\lambda(t) = \int_{-\infty}^{t} g(\lambda(t-s))\,L(\lambda ds) \tag{4.3}$$

with kernel function given by

$$g(\lambda t) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{i\lambda t\vartheta} \frac{q(i\vartheta)}{p(i\vartheta)}\, d\vartheta.$$

Note that $g(\lambda t) = 0$ whenever $t < 0$.

Proposition 4.3. *If the eigenvalues $\vartheta_1, ..., \vartheta_p$ of A have negative real parts, the autocovariance function of (4.3) is given by*

$$\gamma_{Y_\lambda}(h) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{ih\lambda\vartheta} \frac{|q(i\vartheta)|^2}{|p(i\vartheta)|^2}, \quad h \geq 0. \tag{4.4}$$

Proof. Let $\delta(h)$ be the Dirac function with density concentrated on 0. Thus

$$\int\limits_{\mathbb{R}} \delta(h) e^{-ih\omega} dh = 1 \quad \text{and} \quad \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{ih\omega} d\omega = \delta(h).$$

Now let $h \geq 0$, then

$$\begin{aligned}
\gamma_{Y_\lambda}(h) &= \int\limits_{-\infty}^{t} g(\lambda(t-u)) g(\lambda(t+h-u))\lambda\, du = \int\limits_{\mathbb{R}} g(\lambda u) g(\lambda(u+h))\, \lambda\, du \\
&= \int\limits_{\mathbb{R}} \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{iu\lambda\vartheta} \frac{q(i\vartheta)}{p(i\vartheta)} d\vartheta \int\limits_{\mathbb{R}} \frac{1}{2\pi} e^{i\lambda(u+h)\omega} \frac{q(i\omega)}{p(i\omega)} d\omega\, \lambda\, du \\
&= \frac{1}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} e^{i\lambda h\omega} \frac{q(i\vartheta) q(i\omega)}{p(i\vartheta) p(i\omega)} \left(\frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{i\lambda(\vartheta+\omega)u} du \right) d\vartheta\, \lambda\, d\omega \\
&= \frac{1}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} e^{i\lambda h\omega} \frac{q(i\vartheta) q(i\omega)}{p(i\vartheta) p(i\omega)} \lambda^{-1} \delta(\vartheta+\omega)\, d\vartheta\, \lambda\, d\omega \\
&= \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{i\lambda h\omega} \frac{q(-i\omega) q(i\omega)}{p(-i\omega) p(i\omega)}\, d\omega.
\end{aligned}$$

∎

Remark 4.4. *If we additionally assume that the eigenvalues $\vartheta_1, ..., \vartheta_p$ of A are distinct it can be shown (using results of Brockwell (2004)) that the autocovariance function simplifies to*

$$\gamma_{Y_\lambda}(h) = \sum_{r=1}^{p} \frac{q(\vartheta_r) q(-\vartheta_r)}{p'(\vartheta_r) p(-\vartheta_r)} e^{\vartheta_r \lambda h}, \quad h \geq 0. \tag{4.5}$$

Corollary 4.5. *From Proposition 4.2 follows that the randomized CARMA process $\tilde{X}$ defined by (4.1) with Y_λ given by (4.3) has autocorrelation function*

$$\gamma_{\tilde{X}}(h) = \int\limits_{0}^{\infty} \gamma_{Y_\lambda}(h)\, \pi(d\lambda), \quad h \geq 0,$$

where $\gamma_{Y_\lambda}(h)$ is given by (4.4) or (4.5), respectively.

Example 4.6 (Long Memory). *Assume that* λ *is* $\Gamma(1-2d,1)$ *distributed, i.e.*

$$\pi(d\lambda) = \frac{1}{\Gamma(1-2d)} e^{-\lambda} \lambda^{-2d}\, d\lambda, \quad \lambda > 0.$$

Then

$$\int_0^\infty e^{\vartheta\lambda h}\, \pi(d\lambda) = \frac{1}{\Gamma(1-2d)} \int_0^\infty e^{\vartheta\lambda h} e^{-\lambda} \lambda^{-2d}\, d\lambda = (1-\vartheta h)^{2d-1}.$$

Consequently, if we assume that the eigenvalues $\vartheta_1, \ldots, \vartheta_p$ *of* A *are distinct with negative real parts, we have*

$$\begin{aligned} \gamma_{\tilde{X}}(h) &= \int_0^\infty \gamma_{Y_\lambda}(h)\, \pi(d\lambda) = \sum_{r=1}^p \frac{q(\vartheta_r)q(-\vartheta_r)}{p'(\vartheta_r)p(-\vartheta_r)} \int_0^\infty e^{\vartheta_r \lambda h}\, \pi(d\lambda) \\ &= \sum_{r=1}^p \frac{q(\vartheta_r)q(-\vartheta_r)}{p'(\vartheta_r)p(-\vartheta_r)} (1-\vartheta_r h)^{2d-1}, \end{aligned}$$

i.e. we have long memory if $0 < d < 0.5$*.*

This example shows that randomization as in (4.1) can be used to derive long memory processes from a given (short memory) process Y_λ by selecting the distribution π of λ appropriately. The memory behaviour of the resulting process $\tilde{X}$ in (4.1) depends highly on the behaviour of π near the origin. In particular, the role of π is to slow down the process by randomizing its time scale. As the following result (see Chunsheng (2003, p. 1137)) shows, the resulting process $\tilde{X}$ then exhibits long memory.

Proposition 4.7. *Assume that the distribution* π *of* λ *has a density* f_π *with respect to the Lebesgue measure, i.e.* $\pi(d\lambda) = f_\pi(\lambda)\, d\lambda$*. Furthermore, suppose that this density* f_π *is monotone on the interval* $(0,1]$*, vanishes outside* $[0,1]$ *and that*

$$f_\pi(\lambda) \sim cL(\lambda^{-1})\lambda^{2d-1}, \quad \lambda \to 0, \tag{4.6}$$

where $0 < d < 0.5$*,* $c > 0$ *and* $\tilde{L}(\cdot)$ *is a slowly varying function. Then the autocorrelation function of* $\tilde{X}$ *takes the form*

$$r_{\tilde{X}}(h) = \int_0^1 r_Y(\lambda h) f_\pi(\lambda)\, d\lambda, \quad h \in \mathbb{R}$$

and as $h \to \infty$*,*

$$r_{\tilde{X}}(h) \sim c\tilde{L}(h)h^{-2d} \int_0^\infty r_Y(u) u^{2d-1}\, du,$$

provided that $\int_0^\infty r_Y(u)u^{2d-1}\, du$ *is convergent.*

Note that for an OU process Y_λ, (4.6) is a necessary and sufficient condition for the corresponding process $\tilde{X}$ to exhibit long memory in the sense of (2.17) (see Chunsheng (2003)).

Though the previous randomization technique leads to a class of statistically tractable long memory processes, it might in some practical applications be a disadvantage that processes $\tilde{X}$ generated by this approach do not have infinite divisible finite dimensional distributions, in general.

5. Long Memory by Aggregation. Our starting point in this section is the Lévy-driven OU process defined by (2.9).

It has been shown in Marquardt & Stelzer (2007) that for every (two-sided) square integrable Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ with $E[L(1)] = 0$ and $E[L(1)^2] = \Sigma_L$ there exists a random orthogonal measure Φ_L with spectral measure F_L such that $E[\Phi_L(\Delta)] = 0$ for any bounded Borel set Δ,

$$F_L(dt) = \frac{\Sigma_L}{2\pi} dt \tag{5.1}$$

and

$$L(t) = \int_{\mathbb{R}} \frac{e^{i\mu t} - 1}{i\mu} \Phi_L(d\mu). \tag{5.2}$$

Moreover, the random measure Φ_L is uniquely determined by

$$\Phi_L([a,b)) = \int_{\mathbb{R}} \frac{e^{-i\mu a} - e^{-i\mu b}}{2\pi i\mu} L(d\mu) \tag{5.3}$$

for all $-\infty < a < b < \infty$. An obvious consequence of these results is that the OU process (2.9) has spectral representation

$$Y(t) = \int_{\mathbb{R}} e^{it\omega} \frac{1}{i\omega + \lambda} \Phi_L(d\omega), \quad t \in \mathbb{R}. \tag{5.4}$$

Now, assume again that λ is a non-negative random variable. In particular, suppose λ has a $\Gamma(1-d, \beta)$ law for some $0 < d < 0.5$, $\beta > 0$. The next proposition follows by easy calculations. We omit the proof.

Proposition 5.1. *The time domain transfer function is given by*

$$E\left[e^{-\lambda(t-s)}\right] = \int_0^\infty e^{-x(t-s)} \frac{\beta^{1-d}}{\Gamma(1-d)} x^{-d} e^{-\beta x}\, dx = \left(\frac{\beta}{\beta + t - s}\right)^{1-d}, \tag{5.5}$$

the frequency domain transfer function takes the form

$$\begin{aligned} E\left[\frac{1}{i\omega + \lambda}\right] &= \int_0^\infty \frac{1}{i\omega + x} \frac{\beta^{1-d}}{\Gamma(1-d)} x^{-d} e^{-\beta x}\, dx \\ &= e^{i\beta\omega} \beta^{1-d} \int_\beta^\infty u^{d-1} e^{-iu\omega}\, du \\ &= e^{i\beta\omega} \beta^{1-d} (i\omega)^{-d} \bar{\Gamma}(i\beta\omega, d), \end{aligned}$$

where $\bar{\Gamma}(\alpha, \beta)$ denotes the incomplete Gamma function with complex-valued argument.

Definition 5.2. *We call a process $\tilde{Y} = \{\tilde{Y}_d^\beta(t)\}_{t\in\mathbb{R}}$ defined by*

$$\tilde{Y}(t) = \tilde{Y}_d^\beta(t) = \int_{-\infty}^t E\left[e^{-\lambda(t-s)}\right] L(ds) = \int_{-\infty}^t \left(\frac{\beta}{\beta + t - s}\right)^{1-d} L(ds) \tag{5.6}$$

a Lévy-driven Gamma-mixed OU process with parameters $d \in (0, 0.5)$ and $\beta > 0$, $L\Gamma OU(d, \beta)$ process for short.

In the Gaussian case (i.e. when the Lévy process is a Brownian motion) such processes have been studied in Iglói & Terdik (1999). They are of particular interest for us, since they can be seen as limiting processes of centered m-factor models having representation (2.10), as the following theorem shows.

THEOREM 5.3. *Define*

$$\tilde{Y}^m(t) = \frac{1}{m} \sum_{j=1}^{m} Y_j(t), \quad m \in \mathbb{N},$$

where Y_j are OU processes given by

$$Y_j(t) = \int_{-\infty}^{t} e^{-\lambda_j (t-s)} \, L(ds), \quad t \in \mathbb{R}.$$

Then

$$\tilde{Y}^m(t) \to \tilde{Y}(t), \qquad \text{in } L^2(\Omega, \mathcal{F}, P)$$

as $m \to \infty$.

Proof. We have a.s.

$$\tilde{Y}^m(t) = \int_{-\infty}^{t} \frac{1}{m} \sum_{j=1}^{m} e^{-\lambda_j (t-s)} \, L(ds).$$

By the law of large numbers

$$\lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} e^{-\lambda_j (t-s)} = E[e^{\lambda_1 (t-s)}] = \left(\frac{\beta}{\beta + t - s} \right)^{1-d}, \textit{i.e.}$$

$$\frac{1}{m} \sum_{j=1}^{m} e^{-\lambda s} - \left(\frac{\beta}{\beta + s} \right)^{1-d} \to 0 \ \textit{a.s.}$$

By an argument given in Iglói & Terdik (1999, Appendix), it follows that the series of functions

$$\left(\frac{1}{m} \sum_{j=1}^{m} e^{-\lambda_j s} - \left(\frac{\beta}{\beta + s} \right)^{1-d} \right)^2$$

is uniformly integrable. Therefore,

$$E[\tilde{Y}^m(t) - \tilde{Y}(t)]^2 = \int_{0}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} e^{-\lambda s} - \left(\frac{\beta}{\beta + s} \right)^{1-d} \right)^2 ds \to 0 \ \textit{a.s.}$$

as $m \to \infty$. The left-hand side does not depend on t, hence the assertion follows. □

Remark 5.4. *Obviously, the process $\tilde{Y}$ is a moving average process with kernel*

$$g(t) = \left(\frac{\beta}{\beta+t}\right)^{1-d}.$$

Therefore $\tilde{Y}$ is stationary, infinitely divisible, where the finite dimensional distributions have the cumulant generating function

$$\log E\left[\exp\left\{\sum_{j=1}^{m} u_j \tilde{Y}(t_j)\right\}\right] = \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \psi\left(\sum_{j=k}^{m} u_j \left(\frac{\beta}{\beta+t_j-s}\right)^{1-d}\right) ds.$$

Provided the driving Lévy process L has zero mean and is square integrable, we obtain the following second-order properties.

Proposition 5.5. *The process $\tilde{Y}$ defined in (5.6) has zero mean and autocorrelation function*

$$r_{\tilde{Y}}(h) = {}_2F_1\left(1-d, 1-2d; 2-2d, -\frac{h}{\beta}\right), \tag{5.7}$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function.

Proof. $E[\tilde{Y}(t)] = 0$ is a direct consequene of $E[L(1)] = 0$. Therefore,

$$\begin{aligned}
\gamma_{\tilde{Y}}(h) &= E[\tilde{Y}(t+h)\tilde{Y}(t)] \\
&= E[L(1)^2] \int_{-\infty}^{t} \frac{\beta^{2-2d}}{(\beta+t+h-s)^{1-d}(\beta+t-s)^{1-d}} ds \\
&= E[L(1)^2] \int_{0}^{\infty} \frac{\beta^{2-2d}}{(\beta+h+s)^{1-d}(\beta+s)^{1-d}} ds \\
&= E[L(1)^2]\beta \int_{0}^{1} s^{-2d}\left(1+\frac{sh}{\beta}\right)^{d-1} ds \\
&= \frac{\beta E[L(1)^2]}{1-2d}\, {}_2F_1\left(1-d, 1-2d; 2-2d, -\frac{h}{\beta}\right).
\end{aligned}$$

Hence, for all $t \in \mathbb{R}$, $E[\tilde{Y}(t)^2] = \frac{\beta E[L(1)^2]}{1-2d}$ and the assertion follows from

$$r_{\tilde{Y}}(h) = \frac{\gamma_{\tilde{Y}}(h)}{E[\tilde{Y}(t)^2]}.$$

□

In particular, the preceding proposition shows that the process $\tilde{Y}$ exhibits long memory, since

$$r_{\tilde{Y}}(h) \sim \beta^{2-2d} B(d, 1-2d)\, h^{2d-1}, \quad \text{as } h \to \infty,$$

where $B(\alpha, \beta)$ denotes the Beta function.

5.1. Convergence to a Fractional Lévy Process. A remarkable result is the relation between the process $\tilde{Y}$ and fractional Lévy processes. The name "fractional Lévy process" already suggests that it can be regarded as a generalization of fractional Brownian motion (FBM). Fractional Lévy processes have been studied in Marquardt (2006a) and Marquardt (2006b) and we refer to the latter works for details.

DEFINITION 5.6 (Fractional Lévy Process (FLP)). *Let* $L = \{L(t)\}_{t\in\mathbb{R}}$ *be a Lévy process on* $\mathbb{R}$ *with* $E[L(1)] = 0$, $E[L(1)^2] < \infty$ *and without Brownian component. For fractional integration parameter* $0 < d < 0.5$ *a stochastic process*

$$M_d(t) = \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} \left[(t-s)_+^d - (-s)_+^d\right] L(ds), \quad t \in \mathbb{R}, \tag{5.8}$$

is called a fractional Lévy process *(FLP).*

Note that the kernel (5.8) given by

$$f_t(s) = \frac{1}{\Gamma(1+d)}[(t-s)_+^d - (-s)_+^d], \quad s \in \mathbb{R}, \tag{5.9}$$

is square integrable. Thus, fractional Lévy processes are well-defined and belong to $L^2(\Omega)$ for fixed t.

In general, fractional Lévy processes are not differentiable. However, let us consider a process, which can be interpreted as the formal derivative of M_d (as $\beta \to 0$), namely

$$m_d^\beta(t) = \frac{1}{\Gamma(d)} \int_{-\infty}^{t-\beta} (t-s)^{d-1} L(ds), \quad t \in \mathbb{R}, \tag{5.10}$$

where we assume that $\beta > 0$ is small. Whereas for $\beta = 0$ the process m_d^0 does not exist in L^2, the process m_d^β is well-defined in an L^2-sense for $\beta \neq 0$. Furthermore, m_d^β exhibits long memory. This is a consequence of the fact that we cut off only the recent past and present, but not the long past, from which the long memory arises.

REMARK 5.7. Since,

$$m_d^\beta(t) \stackrel{d}{=} \frac{1}{\Gamma(d)} \int_{-\infty}^{t} (t+\beta-s)^{d-1} L(ds), \quad t \in \mathbb{R},$$

we have the relation

$$m_d^\beta(t) \stackrel{d}{=} \tilde{Y}_d^\beta(t) \frac{\beta^{d-1}}{\Gamma(d)}.$$

Now, applying Fubini's theorem for stochastic integrals we obtain

$$\int_0^t m_d^\beta(s)\,ds = \frac{1}{\Gamma(d)} \int_0^t \int_{-\infty}^s (s+\beta-u)^{d-1}\,L(du)\,ds$$

$$= \frac{1}{\Gamma(d)} \int_{-\infty}^0 \int_0^t (s+\beta-u)^{d-1}\,ds\,L(du) + \frac{1}{\Gamma(d)} \int_0^t \int_u^t (s+\beta-u)^{d-1}\,ds\,L(du)$$

$$= \frac{1}{\Gamma(d+1} \int_{-\infty}^0 [(t+\beta-u)^d - (\beta-u)^d]\,L(du) + \frac{1}{\Gamma(d+1)} \int_0^t (t+\beta-u)^d\,L(du) - \frac{\beta^d}{\Gamma(d+1)} L(t)$$

$$= \frac{1}{\Gamma(d+1)} \int_{-\infty}^{-\beta} [(t-s)^d - (-s)^d]\,L(ds) + \frac{1}{\Gamma(d+1)} \int_0^{t-\beta} (t-s)^d\,L(ds) - \frac{\beta^d}{\Gamma(d+1)}\,L(t). \qquad \blacksquare$$

In particular, the integral process of the properly scaled process $\tilde{Y}_d^\beta$ converges (in L^2) to the FLP, i.e.

$$\frac{\beta^{d-1}}{\Gamma(d)} \int_0^t \tilde{Y}_d^\beta(s)\,ds \to M_d(t)$$

as $\beta \to 0$ for all $t \in \mathbb{R}$.

5.2. Applications to Modelling Integrated Volatility. A key meaure in finance is integrated volatility. For the integrated OU process it takes the simple structure

$$IY_\lambda(t) := \int_0^t Y_\lambda(u)\,du = \lambda^{-1}\left[L(\lambda t) - Y_\lambda(t) + Y_\lambda(0)\right],$$

whereas for the integrated supOU process $IX(t) := \int_0^t X(u)\,du$ we do not have an explicit representation. However, we can calculate its cumulants generating function

$$\psi_{IX}(u) := \log E[\exp\{iuIX(t)\}] = u \int_0^\infty \int_0^t \psi_X'\left(\frac{u}{\lambda}(1-e^{-\lambda s})\right)\,ds\,\pi(d\lambda),$$

where ψ_X' denotes the derivative of the cumulants generating function ψ_X of the supOU process X (provided that $\psi_X(u)$ is differentiable for $u \neq 0$ and provided that $u\psi_X'(u) \to 0$ for $0 \neq u \to 0$). Note that it follows by (2.20) that

$$\psi_X(u) = \int_0^\infty \psi(e^{-s}u)\,du = \psi_{Y_\lambda}(u), \quad \psi(u) = u\,\psi_X'(u),$$

where ψ is given by (2.14).

Now, we consider the process $\tilde{Y} = \{\tilde{Y}_d^\beta(t)\}_{t\in\mathbb{R}}$ and wish to describe it integrated version. That is,

$$I\tilde{Y}_d^\beta(t) = \int_0^t \tilde{Y}_d^\beta(s)ds = \int_{-\infty}^\infty \int_0^{t-s} \left(\frac{\beta}{\beta+v}\right)^{1-d} dv\,L(ds) \tag{5.11}$$

In order to evaluate quantities such as

$$\int_0^{t-s} \left(\frac{\beta}{\beta+v}\right)^{1-d} dv,$$

it is better to use a double expectation argument. That is go back to the time transfer function in Proposition 5.1. Note this will work quite well for more general kernels based on the aggregation idea. Recall

$$E\left[e^{-\lambda(t-s)}\right] = \int\limits_0^\infty e^{-x(t-s)} \frac{\beta^{1-d}}{\Gamma(1-d)} x^{-d} e^{-\beta x}\, dx = \left(\frac{\beta}{\beta+t-s}\right)^{1-d}. \tag{5.12}$$

Hence, we have that

$$\begin{aligned}
\int_0^{t-s} \left(\frac{\beta}{\beta+v}\right)^{1-d} dv &= \int\limits_0^{t-s} E[e^{-\lambda v}]\, dv \\
&= \int\limits_0^{t-s}\int\limits_0^\infty e^{-xv} \frac{1}{\Gamma(1-d)} \beta^{1-d} x^{-d} e^{-\beta x}\, dx\, dv \\
&= \beta^{1-d} \int\limits_0^\infty \int\limits_0^{t-s} e^{-xv}\, dv \frac{x^{-d}}{\Gamma(1-d)} e^{-\beta x}\, dx \\
&= \beta^{1-d} \int_0^\infty (1 - e^{-x(t-s)}) \frac{1}{\Gamma(1-d)} x^{-d-1} e^{-\beta x}\, dx.
\end{aligned}$$

But

$$\frac{1}{\Gamma(1-d)} x^{-d-1} e^{-\beta x}$$

is the Lévy density of a generalized Gamma subordinator otherwise known as a tempered stable process. It follows that for $-\infty < s < t$

$$\int_0^{t-s} \left(\frac{\beta}{\beta+v}\right)^{1-d} dv = \beta^{1-d}\left[(t-s+\beta)^d - \beta^d\right].$$

Hence,

$$I\tilde{Y}_d^\beta(t) = \beta^{1-d} \int_{-\infty}^t \left[(t-s+\beta)^d - \beta^d\right] L(ds). \tag{5.13}$$

Thus, we can extend the aggregation idea by defining an arbitrary transfer function

$$E\left[\mathrm{e}^{-\lambda(t-s)}\right] = e^{-\Phi(t-s)},$$

where $-\Phi(x)$ is logarithm of the Laplace transform of a positive random variable λ with density (or probability mass function) denoted as $\pi(\lambda)$. Note here we, at this point, do not necessarily require infinite divisibility of π. Now provided that

$$\rho(\lambda) = \lambda^{-1}\pi(\lambda)$$

is a Lévy density of some subordinator, say S, it follows that

$$\psi_\pi(x) = -\log E\left[e^{-xS}\right] = \int_0^\infty (1 - e^{\lambda x})\rho(\lambda)d\lambda.$$

Thus, we have that a general aggregated process is of the form

$$\tilde{Y}_\pi(t) = \int_{-\infty}^t e^{-\Phi(t-s)} L(ds)$$

and its integrated form is given by

$$I\tilde{Y}_\pi(t) = \int_{-\infty}^t \psi_\pi(t-s)L(ds).$$

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