Semi-Parametric Models for the Multivariate Tail Dependence Function - the Asymptotically Dependent Case

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Summary

In general, the risk of joint extreme outcomes in financial markets can be expressed as a function of the tail dependence function of a high-dimensional vector after standardizing marginals. Hence it is of importance to model and estimate tail dependence functions. Even for moderate dimension, nonparametrically estimating a tail dependence function is very inefficient and fitting a parametric model to tail dependence functions is not robust. In this paper we propose a semi-parametric model for (asymptotically dependent) tail dependence functions via an elliptical copula. Based on this model assumption, we propose a novel estimator for the tail dependence function, which proves favorable compared to the empirical tail dependence function estimator, both theoretically and empirically.

Keywords: Asymptotic normality, Dependence modeling, Elliptical copula, Elliptical distribution, Regular variation, Semi-parametric model, Tail dependence function.

1 Introduction

Risk management is a discipline for living with the possibility that future events may cause adverse effects. An important issue for risk managers is how to quantify different types of risk such as market risk, credit risk, operational risk, etc. Due to the multivariate nature of risk, i.e., risk depending on high dimensional vectors of some underlying risk factors, a particular concern for a risk manager is how to model the dependence between extreme outcomes although those extreme outcomes occur rarely. A mathematical formulation of this question is as follows. Let $X = (X_1, \ldots, X_d)^T$ be a random vector with distribution function F and continuous marginals $F_1, \ldots, F_d$. Then the dependence is completely determined by the copula C of X given by Sklar's representation (see Nelsen (1998) or Joe (1997))

$$C(x_1, \ldots, x_d) = F(F_1^{\leftarrow}(x_1), \ldots, F_d^{\leftarrow}(x_d)), \quad x = (x_1, \ldots, x_d)^T \in [0,1]^d,$$

where $F_j^{\leftarrow}(y) := \inf\{x \in \mathbb{R} : F_j(x) \geq y\}$ denotes the generalized inverse function of F_j for $j = 1, \ldots, d$. To study the extreme dependence around the boundary point $(1, \ldots, 1)^T$, the *tail dependence function* of X is defined as

$$\lambda^X(x_1, \ldots, x_d) \quad = \quad \lim_{t \to 0} t^{-1} P\left(1 - F_1(X_1) \leq tx_1, \ldots, \; 1 - F_d(X_d) \leq tx_d\right), \tag{1.1}$$

where $x_1, \ldots, x_d \geq 0$. A similar definition can be given for studying the extreme dependence around the boundary point $(0, \ldots, 0)^T$. The function λ^X is related to the so-called *survival copula*

$$\widehat{C}(x_1, \ldots, x_d) = P(F_1(X_1) > x_1, \ldots, F_d(X_d) > x_d), \quad x = (x_1, \ldots, x_d)^T \in [0,1]^d,$$

which implies

$$\lambda^X\left(x_1, \ldots, x_d\right) = \lim_{t \to 0} t^{-1} \widehat{C}\left(1 - tx_1, \ldots, 1 - tx_d\right).$$

Note that λ^X is in general neither a copula nor a tail copula and should also not be confused with the extreme tail dependence copula introduced by Juri and Wüthrich (2002). Another related notion appears in the literature: Kallsen and Tankov (2005) investigate so-called Lévy copulas. For a d-dimensional Lévy process $(L(t))_{t \geq 0}$ a Lévy copula models dependence in the jump measures of the marginal Lévy processes. For $t > 0$ denote by $\widehat{C}_t$ the survival copula of $(L_1(t), \ldots, L_d(t))$, then the Lévy copula appears as $\lim_{t \to 0} t^{-1} \widehat{C}_t(1 - tx_1, \ldots, 1 - tx_d)$; see Theorem 5.1 of the above paper. Note that in general C_t changes with t, so the analogy to our extreme dependence function is rather vague. Extreme dependence for the bivariate case, when $d = 2$, has been thoroughly investigated and $\lambda^X(1,1)$ is called the *upper tail dependence coefficient* of X_1 and X_2, see Joe (1997). For $x, y \in [0,1]^2$ the function $x + y - \lambda^X(x,y)$ is called the stable tail dependence function of X_1 and X_2 by Huang (1992); such notions go back to Gumbel (1960), Pickands (1981) and Galambos (1987), and they represent the tail dependence structure of the model. For $d = 2$ the tail dependence function $\lambda^X(x,y)$ or the stable tail dependence function $x + y - \lambda^X(x,y)$ can be estimated nonparametrically (see Huang (1992) and Schmidt and Stadtmüller (2006)), or via other measures such as Pickands' dependence function and spectral measure (see Abdous, Ghoudi and Khoudraji (1999) and Einmahl, de Haan and Piterbarg (2001)). For more properties of the multivariate Pickands dependence function, we refer to Falk and Reiss (2005). Also parametric models for the tail dependence function have been suggested and estimated, see Tawn (1988), Ledford and Tawn (1997) and Coles (2001) for examples and further references. Most applications of both, nonparametric and parametric estimation of tail dependence functions are focused on the case $d = 2$, although theoretically both methods are applicable to the case $d > 2$. Recently Hsing, Klüppelberg and Kuhn (2004) estimate the tail dependence function nonparametrically in higher dimensions. Heffernan and Tawn (2004) propose a conditional approach to model multivariate extremes via investigating the limits of normalized conditional distributions. Obviously, when the dimension d is large, nonparametric estimation severely suffers from the curse of dimensionality, and fitting parametric models is not robust in general. In this paper, we concentrate on the case of asymptotically dependent tail structures only, i.e., $\lambda^X(x_1, \ldots, x_d) > 0$, and neither work with purely nonparametric estimates nor specify a parametric model.

Instead we propose to model the tail dependence function of observations via an elliptical copula. By doing this, some part of the tail dependence function can be estimated via the whole sample, and another part of it can be estimated by only employing the data in the tail region of the sample. Therefore, this novel approach avoids the difficulty of dimensional curse and provides a robust way in modeling tail structures, which may be viewed as a semi-parametric approach. Recently, there is an increasing interest of applying elliptical copulas to risk management; see McNeil, Frey and Embrechts (2005) and references therein.

With the help of elliptical copulas, we are able to derive explicit formulae for tail copulas (see section 2) and to easily simulate distributions with such tail copulas (see section 4). We organize this paper as follows. In section 2, we present our methodologies and main results by focusing on dimension $d = 2$. In section 3, we compare our new tail dependence function estimator with the empirical tail dependence function estimator in terms of both, asymptotic variances and asymptotic mean squared errors. A simulation study is given in section 4. Theoretical results and a real data analysis for higher dimensions are provided in section 5.

2 Methodologies and Main Results

Throughout we let $Z = (Z_1, \ldots, Z_d)^T$ denote an elliptical random vector satisfying

$$Z \quad \stackrel{\mathrm{d}}{=} \quad GAU, \tag{2.1}$$

where $G > 0$ is a random variable, A is a deterministic $d \times d$ matrix with $AA^T := \Sigma = (\sigma_{ij})_{1 \le i,j \le d}$ and $\mathrm{rank}(\Sigma) = d$, U is a d-dimensional random vector uniformly distributed on the unit hyper-sphere $\mathcal{S}_d := \{z \in \mathbb{R}^d : z^T z = 1\}$, and U is independent of G. Representation (2.1) implies that the elliptical random vector is uniquely defined by the matrix Σ and the random variable G. For a detailed description of elliptical distributions, we refer to Fang, Kotz and Ng (1987). Define the linear correlation between Z_i and Z_j as $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$ and denote by $R := (\rho_{ij})_{1\le i,j\le d}$ the linear correlation matrix. Throughout we assume that $\rho_{ii} > 0$ for all $i = 1, \ldots, d$ and $|\rho_{ij}| < 1$ for all $i \neq j$. Note that ρ_{ij} is defined for any elliptical distribution, and it coincides with the usual correlation if finite second moments exist. We also assume throughout regular variation of $P(G > \cdot\,)$ with index $\alpha > 0$ (notation: $P(G > \cdot\,) \in RV_{-\alpha}$), which implies that the tail dependence coefficient $\lambda_{ij}^Z(1,1)$ of Z_{ij} is positive for all $i \neq j$ (cf. Hult and Lindskog (2002), Theorem 4.3). For illustration of our methodology, we focus on the case $d = 2$ from now on and the extension to $d > 2$ is given in section 5. If $P(G > \cdot\,) \in RV_{-\alpha}$) Klüppelberg, Kuhn and Peng (2007) showed a version of the following representation (see Lemma 6.1)

$$\lambda^Z(x,y) = \left(x \int_{g\left((x/y)^{1/\alpha}\right)}^{\pi/2} (\cos\phi)^\alpha \,\mathrm{d}\phi + y \int_{g\left((x/y)^{-1/\alpha}\right)}^{\pi/2} (\cos\phi)^\alpha \,\mathrm{d}\phi \right) \times \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \,\mathrm{d}\phi \right)^{-1} := \lambda(\alpha; x, y, \rho), \tag{2.2}$$

where

$$g(t) := \arctan\left((t-\rho)/\sqrt{1-\rho^2} \right) \in [-\arcsin\rho, \pi/2], \quad t > 0, \tag{2.3}$$

which generalizes the corresponding result in Theorem 4.3 of Hult and Lindskog (2002), where only the expression of $\lambda^Z(1,1)$ is given. Therefore, by assuming that the observations have the same distribution as an elliptical random vector Z defined in (2.1) and $P(G > \cdot) \in RV_{-\alpha}$ for some $\alpha > 0$, Klüppelberg, Kuhn and Peng (2007) propose to estimate the tail dependence function via (2.2) and show that this new estimator is better than the empirical tail dependence function estimator both theoretically and empirically. Note that in this set-up α can be estimated from observations due to the fact that Z is multivariate regularly varying (see Theorem 4.3 of Hult and Lindskog (2002)). For the definition and properties of multivariate regular variation, we refer to Resnick (1987).

Since it may be too restrictive to assume that observations follow an elliptical distribution, we propose here to assume only that the copula C (not the full distribution) of the observation vector X is the same as the copula of Z given in (2.1) with $P(G > \cdot\,) \in RV_{-\alpha}$, i.e., we assume an *elliptical copula* for our data. This can be reformulated as

$$P\left(F_1(X_1) \le x, F_2(X_2) \le y\right) = P\left(F_1^Z(Z_1) \le x, F_2^Z(Z_2) \le y\right), \tag{2.4}$$

where F_1^Z and F_2^Z denote the marginal distributions of Z.

Then, immediately by definition (1.1) and (2.4), the observation vector X has the same tail dependence function (2.2) as Z. We can conclude that the tail dependence function depends on the copula only. Although X and Z share the same copula, their distributions may be different. As a matter of fact, the distribution of X may not be multivariate regularly varying although the distribution of Z is. Consequently, α can not be estimated from the observations X.

The natural measure of dependence in the context of a copula is Kendall's tau, which depends only on the ranks of the data and, hence, is independent of the marginals. Moreover, it also works, if first or second moments do not exist. It follows from Theorem 4.2 of Hult and Lindskog (2002) that, for a continuous positive random variable G, Kendall's tau is related to linear correlation by

$$\tau = \frac{2}{\pi} \arcsin \rho;$$

i.e., for an elliptical copula also ρ is independent of the marginals; cf. Klüppelberg and Kuhn (2006) for further details.

Recall that Kendall's tau is defined as

$$\tau \;=\; P\left(\left(X_{11}-X_{21}\right)\left(X_{12}-X_{22}\right)>0\right)-P\left(\left(X_{11}-X_{21}\right)\left(X_{12}-X_{22}\right)<0\right).$$

We propose the following estimation procedure. Given the empirical estimator of the tail dependence function and an estimator of ρ or, equivalently, τ we estimate α by inverting (2.2). Then we estimate the tail dependence function $\lambda^X(x,y)$ by plugging the estimators for α and ρ into (2.2); see below for details.

The empirical tail dependence function is defined as

$$\widehat{\lambda}^{\text{emp}}(x,y;k) \;=\; \frac{1}{k}\sum_{i=1}^{n} I\left(1-\widehat{F}_1(X_{i1}) \le \frac{k}{n}x,\; 1-\widehat{F}_2(X_{i2}) \le \frac{k}{n}y\right), \tag{2.5}$$

where $\widehat{F}_j$ denotes the empirical distribution of $\{X_{ij}\}_{i=1}^n$ for $j=1,2$ and we consider $k=k(n)\to\infty$ and $k/n\to 0$ as $n\to\infty$ since we deal with tail events.

Then we estimate the correlation ρ between X_{11} and X_{12} by $\widehat{\rho}=\sin\left(\frac{\pi}{2}\widehat{\tau}\right)$, where

$$\widehat{\tau} \;=\; \frac{2}{n(n-1)}\sum_{1\le i<j\le n} \operatorname{sign}\left((X_{i1}-X_{j1})(X_{i2}-X_{j2})\right). \tag{2.6}$$

Note that τ, i.e., ρ, is estimated by employing the whole sample and with convergence rate $n^{-1/2}$, a result which goes back to Hoeffding (1948)); cf. also Lee (1990) and Klüppelberg and Kuhn (2006), Theorem 4.6. This rate is much faster than that for any tail dependence function estimator. That is, a tail copula estimation with ρ replaced by $\widehat{\rho}$ does not change its asymptotic behavior. In order to estimate α, we need to solve equation (2.2) as a function of α.

Theorem 2.1. *For any fixed $x, y > 0$ and $|\rho| < 1$, define $\alpha^* := |\ln(x/y)/\ln(\rho \vee 0)|$. Then, $\lambda(\alpha; x, y, \rho)$ is strictly decreasing in α for all $\alpha > \alpha^*$.*

Based on the above theorem, we are able to define an estimator for α as follows. Let $\lambda^{\leftarrow}(\,\cdot\,; x, y, \rho)$ denote the inverse of $\lambda(\alpha; x, y, \rho)$ with respect to α, if it exists. By Theorem 2.1, we know that $\lambda^{\leftarrow}(\,\cdot\,; 1, 1, \rho)$ exists for all $\alpha > 0$. Hence, an obvious estimator for α is $\widetilde{\alpha}(1, 1, k) := \lambda^{\leftarrow}(\widehat{\lambda}^{\text{emp}}(1, 1; k); 1, 1, \widehat{\rho})$ for any estimator $\widehat{\rho}$ of ρ. Since this estimator only employs information at $x = y = 1$, it may not be efficient. Next we define an estimator which takes also $\widehat{\lambda}^{\text{emp}}(x, y; k)$ for other values $(x, y) \in \mathbb{R}^2_+$ into account. Based on Theorem 2.1 we define corresponding ranges for $y/x = \tan\theta$. Since $\lambda(ax, ay) = a\lambda(x, y)$ for any $a > 0$, we only need to consider $(x, y) = (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$ for different angles θ to ensure that $(x, y) = (1, 1)$ is taken into account. Define

$$
\begin{aligned}
\widehat{Q} \;&:=\; \left\{\theta \in \left(0, \frac{\pi}{2}\right) : \widehat{\lambda}^{\text{emp}}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k) < \right. \\
&\qquad\qquad \left. < \lambda\left(\left|\frac{\ln(\tan\theta)}{\ln(\widehat{\rho} \vee 0)}\right|; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \widehat{\rho}\right)\right\}, \\
\widehat{Q}^* \;&:=\; \left\{\theta \in \left(0, \frac{\pi}{2}\right) : |\ln(\tan\theta)| < \widetilde{\alpha}(1, 1; k)\left(1 - k^{-1/4}\right)|\ln(\widehat{\rho} \vee 0)|\right\} \quad \text{and} \\
Q^* \;&:=\; \left\{\theta \in \left(0, \frac{\pi}{2}\right) : |\ln(\tan\theta)| < \alpha\,|\ln(\rho \vee 0)|\right\}.
\end{aligned}
$$

It follows from Theorem 2.1 that there exists a unique $\alpha_1 > |\ln(\tan\theta)/\ln(\widehat{\rho} \vee 0)|$ such that

$$
\lambda(\alpha_1; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \widehat{\rho}) = \widehat{\lambda}^{\text{emp}}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k), \quad \theta \in \widehat{Q}.
$$

Therefore, for $\theta \in \widehat{Q}$ we can define the inverse function of $\lambda(\,\cdot\,; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \widehat{\rho})$ giving

$$
\widetilde{\alpha}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k) \;=\; \lambda^{\leftarrow}\left(\widehat{\lambda}^{\text{emp}}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k); \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \widehat{\rho}\right). \tag{2.7}
$$

Next we have to ensure consistency of this estimator. This can be done by further requiring $\theta \in \widehat{Q}^*$, which implies that the true value of α is larger than $|\ln(\tan\theta)/\ln(\widehat{\rho} \vee 0)|$ with probability tending to one. Thus, our estimator for α is defined as a smoothed version of $\widetilde{\alpha}$. That is, for an arbitrary nonnegative weight function w we define

$$
\widehat{\alpha}(k, w) = \frac{1}{W\left(\widehat{Q} \cap \widehat{Q}^*\right)} \int_{\theta \in \widehat{Q} \cap \widehat{Q}^*} \widetilde{\alpha}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta; k)\, W(d\theta) \tag{2.8}
$$

where W is the measure defined by w. Before we give the asymptotic normality of $\widehat{\alpha}$, we list the following regularity conditions:

(C1) X satisfies relation (2.4), Z has tail dependence function (2.2), $P(G > \cdot\,) \in RV_{-\alpha}$ for some $\alpha > 0$, and $|\rho| < 1$.

(C2) There exists $A(t) \to 0$ such that

$$\lim_{t\to 0} \frac{t^{-1}P(1-F_1(X_1)\le tx, 1-F_2(X_2)\le ty) - \lambda^X(x,y)}{A(t)} \quad = \quad b_{(C2)}(x,y)$$

uniformly on $\mathcal{S}_2$, where $b_{(C2)}(x,y)$ is not a multiple of $\lambda^X(x,y)$.

(C3) $k = k(n) \to \infty$, $k/n \to 0$ and $\sqrt{k}A(k/n) \to b_{(C3)} \in (-\infty, \infty)$ as $n \to \infty$.

Recall that condition (C1) implies that X is asymptotically tail dependent. This is for instance not the case for the multivariate normal distribution, which has representation (2.1) where G is χ^2_d-distributed. This can be interpreted as $\alpha = \infty$, which indeed implies an asymptotically independent tail dependence function; we refer to Abdous, Fougères and Ghoudi (2005) for a treatment of this case. The following theorem gives the asymptotic normality of $\widehat{\alpha}$.

Theorem 2.2. *Suppose that (C1)-(C3) hold, and that w is a positive weight function satisfying $\sup_{\theta\in Q^*} w(\theta) < \infty$. Then, denoting by W the measure defined by w, as $n \to \infty$,*

$$\begin{aligned}&\sqrt{k}\left(\widehat{\alpha}(k,w) - \alpha\right)\\ &\quad \xrightarrow{d} \quad \frac{1}{W(Q^*)}\int_{\theta\in Q^*} \frac{b_{(C3)}b_{(C2)}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta) + \widetilde{B}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)}{\lambda'\left(\alpha; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \rho\right)} W(d\theta),\end{aligned}$$

where $\lambda'(\alpha; x, y, \rho) := \dfrac{\partial}{\partial\alpha}\lambda(\alpha; x, y, \rho)$ *with explicit expression given in the proof,*

$$\widetilde{B}(x,y) \quad = \quad B(x,y) - B(x,0)\left(1 - \frac{\partial}{\partial x}\lambda(x,y)\right) - B(0,y)\left(1 - \frac{\partial}{\partial y}\lambda(x,y)\right)$$

and $B(x,y)$ is a Brownian motion with zero mean and covariance structure

$$\begin{aligned}E\left(B(x_1,y_1)B(x_2,y_2)\right) \;=\; & x_1\wedge x_2 + y_1\wedge y_2 - \lambda(x_1\wedge x_2, y_1) - \lambda(x_1\wedge x_2, y_2)\\ & -\lambda(x_1, y_1\wedge y_2) - \lambda(x_2, y_1\wedge y_2) + \lambda(x_1,y_2) + \lambda(x_2,y_1) + \lambda(x_1\wedge x_2, y_1\wedge y_2).\end{aligned}$$

Remark 2.3. The above limit can be simulated via replacing α and ρ by $\widehat{\alpha}(k,w)$ and $\widehat{\rho}$, respectively, and simulating $\widetilde{B}$ as in Einmahl, de Haan and Li (2006), when the asymptotic bias is negligible. Further studies on estimating the partial derivatives of tail copulas can be found in Peng and Qi (2007).

Next, like in Klüppelberg, Kuhn and Peng (2005), we estimate $\widehat{\rho}$ via the identity $\tau = \frac{2}{\pi}\arcsin\rho$ and the estimator (2.6) and obtain an estimator for $\lambda(x,y)$ by

$$\widehat{\lambda}(x,y;k,w) \quad = \quad \lambda\left(\widehat{\alpha}(k,w); x, y, \widehat{\rho}\right). \tag{2.9}$$

We derive the asymptotic normality of this new estimator $\widehat{\lambda}(x,y;k,w)$ as follows.

Theorem 2.4. *Suppose that the conditions of Theorem 2.2 hold. Then, for* $T > 0$, *we have as* $n \to \infty$,

$$\sup_{0 \le x,y \le T} \left| \sqrt{k} \left(\widehat{\lambda}(x,y;k,w) - \lambda^X(x,y) \right) - \lambda'(\alpha;x,y,\rho) \frac{1}{W(Q^*)} \right.$$
$$\left. \times \int_{\theta \in Q^*} \frac{b_{(C3)} b_{(C2)}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta) + \widetilde{B}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta, t)}{\lambda'(\alpha; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \rho)} W(d\theta) \right| \;=\; o_p(1).$$

3 Theoretical Comparisons

The following corollary gives the optimal choice of the sample fraction k for $\widehat{\alpha}$ in terms of the *asymptotic mean squared error*. As usual, we define the asymptotic mean squared error of $\widehat{\alpha}$ as $k^{-1}\{(E(L))^2 + \mathrm{Var}(L)\}$ if $\sqrt{k}(\widehat{\alpha} - \alpha) \xrightarrow{d} L$. First, denote the *asymptotic bias* of $\widehat{\alpha}$ based on the weight function w by

$$\mathrm{abias}_\alpha(w) \;=\; \frac{1}{W(Q^*)} \int_{\theta \in Q^*} \frac{b_{(C2)}(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)}{\lambda'(\alpha; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \rho)} W(d\theta)$$

and the *asymptotic variance* of $\widehat{\alpha}$ by

$$\mathrm{avar}_\alpha(w) = \frac{1}{(W(Q^*))^2} \times$$
$$\int_{\theta_1 \in Q^*} \int_{\theta_2 \in Q^*} \frac{E\left(\widetilde{B}(\sqrt{2}\cos\theta_1, \sqrt{2}\sin\theta_1) \widetilde{B}(\sqrt{2}\cos\theta_2, \sqrt{2}\sin\theta_2) \right)}{\lambda'(\alpha; \sqrt{2}\cos\theta_1, \sqrt{2}\sin\theta_1, \rho) \lambda'(\alpha; \sqrt{2}\cos\theta_2, \sqrt{2}\sin\theta_2, \rho)} W(d\theta_2) W(d\theta_1).$$

Corollary 3.1. *Assume that (C1)-(C3) hold and* $A(t) \sim ct^\beta$ *as* $t \to 0$ *for some* $c \neq 0$ *and* $\beta > 0$. *Then the asymptotic mean squared error of* $\widehat{\alpha}(k,w)$ *is*

$$\mathrm{amse}_\alpha(k,w) \;=\; c^2 (k/n)^{2\beta} \left(\mathrm{abias}_\alpha(w) \right)^2 + k^{-1} \mathrm{avar}_\alpha(w).$$

By minimizing the above asymptotic mean squared error, we obtain the optimal choice of k *as*

$$k_0(w) \;=\; \left(\frac{\mathrm{avar}_\alpha(w)}{2\beta c^2 (\mathrm{abias}_\alpha(w))^2} \right)^{1/(2\beta+1)} n^{2\beta/(2\beta+1)}.$$

Hence the optimal asymptotic mean squared error of $\widehat{\alpha}$ *is*

$$\mathrm{amse}_\alpha(k_0(w), w) \;=\; \left(\left(\frac{\mathrm{avar}_\alpha(w)}{n} \right)^\beta \mathrm{abias}_\alpha(w) c \sqrt{2\beta} \right)^{2/(2\beta+2)} \left(1 + \frac{1}{2\beta} \right).$$

Firstly, we compare $\widehat{\alpha}(k,w)$ with $\widetilde{\alpha}(1,1;k)$. As a first weight function we choose $w_0(\theta)$ equal to one if $\theta = \pi/4$, and equal to zero otherwise. Since $\widetilde{\alpha}(1,1;k) = \widehat{\alpha}(k,w_0)$, the asymptotic variance and optimal asymptotic mean squared error of $\widetilde{\alpha}(1,1;k)$ are

$$\mathrm{avar}_\alpha(w_0) = k^{-1}\mathrm{avar}_\alpha(w_0) \quad \text{and} \quad \mathrm{amse}_\alpha(w_0) = \mathrm{amse}_\alpha(k_0(w_0), w_0)\,.$$

For simplicity, we only compare $\widehat{\alpha}(k,w_0)$ and $\widehat{\alpha}(k,w_1)$ with the weight function

$$w_1(\theta) \;=\; 1 - \left(\frac{\theta}{\pi/4} - 1\right)^2, \quad 0 \le \theta \le \frac{\pi}{2}. \tag{3.1}$$

In Figure 1, we plot the ratio $\mathrm{ratio}_{\mathrm{var},\alpha} = \mathrm{avar}_\alpha(w_1)/\mathrm{avar}_\alpha(w_0)$ against α for $\rho \in \{0.3, 0.7\}$, which shows that $\widehat{\alpha}(k,w_1)$ has a smaller variance than $\widetilde{\alpha}(1,1;k)$ in many cases, especially when α is large or ρ is small. Hence $\widehat{\alpha}(k,w_1)$ is better than $\widetilde{\alpha}(1,1;k)$ in terms of asymptotic variance. Without doubt, the weight function w_1 is not an optimal one. However, as in kernel smoothing estimation, we believe that the choice of k is more important than the choice of w. In practice, we propose to employ the kernel given in (3.1).

Secondly, we compare $\widehat{\lambda}(x,y;k,w)$ with $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$. It follows from Theorem 2.4 that the asymptotic variance and the asymptotic mean squared error of $\widehat{\lambda}(x,y;k,w)$ are

$$(\lambda'(\alpha;x,y,\rho))^2\,\mathrm{avar}_\alpha(k,w) \quad \text{and} \quad (\lambda'(\alpha;x,y,\rho))^2\,\mathrm{amse}_\alpha(k,w),$$

respectively. As in Corollary 3.1, we obtain the optimal asymptotic mean squared error of $\widehat{\lambda}(x,y;k,w)$ as $(\lambda'(\alpha;x,y,\rho))^2\,\mathrm{amse}_\alpha(k_0(w),w)$. Put

$$\begin{aligned} k_{\mathrm{emp}} &= \left(\frac{E(B^2(x,y))}{2\beta c^2 (b_{(C2)}(x,y))^2}\right)^{1/(2\beta+1)} n^{2\beta/(2\beta+1)} \quad \text{and} \\ \mathrm{amse}_{\mathrm{emp}}(k) &= c^2(k/n)^{2\beta}(b_{(C2)}(x,y))^2 + k^{-1}E(B^2(x,y)). \end{aligned}$$

Then the asymptotic variance and the optimal asymptotic mean squared error of $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$ are

$$\mathrm{avar}_{\lambda^{\mathrm{emp}}}(k,w) = k^{-1}(E\widetilde{B}(x,y))^2 \quad \text{and} \quad \mathrm{amse}_{\lambda^{\mathrm{emp}}}(k,w) = \mathrm{amse}_{\mathrm{emp}}(k_{\mathrm{emp}})\,.$$

In Figure 2, we plot the ratio of the variances of $\widehat{\lambda}(x,y;w_1)$ and $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$ given by

$$\mathrm{ratio}_{\mathrm{var},\lambda} \;=\; \frac{E(B^2(x,y))}{(\lambda'(\alpha;x,y,\rho))^2\,\mathrm{avar}_\alpha(w_1)},$$

for $(x,y) = (\sqrt{2}\cos\phi, \sqrt{2}\sin\phi)$ against $\phi \in (0,\pi/2)$ for different pairs $(\alpha,\rho) \in \{1,5\} \times \{0.3, 0.7\}$, which shows that the new estimator for $\lambda^X(x,y)$ has a smaller variance than the empirical estimator $\widehat{\lambda}^{\mathrm{emp}}(x,y;k)$.

4 Simulation Study

In this section we conduct a simulation study to compare $\widehat{\alpha}(k, w_1)$ with $\widehat{\alpha}(k, w_0) = \widetilde{\alpha}(1, 1, k)$, and to compare $\widehat{\lambda}(x, y; k, w_1)$ with $\widehat{\lambda}^{\mathrm{emp}}(x, y; k)$ by drawing 1 000 random samples with sample size $n = 3\,000$ from an elliptical copula with $P(G > x) = \exp\{-x^{-\alpha}\}$, $x > 0$, for $\alpha = 1$ and $\alpha = 5$, respectively.

For comparison of $\widehat{\alpha}(k, w_1)$ and $\widetilde{\alpha}(1, 1, k)$, we plot the mean squared errors of $\widetilde{\alpha}(1, 1, k)$, $\widehat{\alpha}(k, w_1)$ in Figure 3. We observe that $\widehat{\alpha}(k, w_1)$ has a smaller mean squared error than $\widetilde{\alpha}(1, 1, k)$ in most cases. Further, we plot $\widetilde{\alpha}(1, 1, k)$ and $\widehat{\alpha}(k, w_1)$ based on a particular sample in Figure 5, which shows that $\widehat{\alpha}(k, w_1)$ is much smoother than $\widetilde{\alpha}(1, 1, k)$ with respect to k. This is because $\widehat{\alpha}(k, w_1)$ employs more $\widehat{\lambda}^{\mathrm{emp}}(x, y; k)'s$ and $\widetilde{\alpha}(1, 1, k)$ only uses $\widehat{\lambda}^{\mathrm{emp}}(1, 1; k)$. In summary, one may prefer $\widehat{\alpha}(k, w_1)$ to $\widetilde{\alpha}(1, 1, k)$.

Next we compare the empirical estimator $\widehat{\lambda}^{\mathrm{emp}}(x, y; k)$ with the new $\widehat{\lambda}(x, y; k, w_1)$. We plot the mean squared errors of $\widehat{\lambda}^{\mathrm{emp}}(1, 1; k)$, $\widehat{\lambda}(1, 1, k, w_1)$ in Figure 4. We also plot estimators $\widehat{\lambda}^{\mathrm{emp}}(1, 1; k)$ and $\widehat{\lambda}(1, 1; k, w_1)$ based on a particular sample in Figure 6. Like the comparisons for estimators of α, we observe that $\widehat{\lambda}(1, 1; k, w_1)$ has a slightly smaller mean squared error than $\widehat{\lambda}^{\mathrm{emp}}(1, 1; k)$, but $\widehat{\lambda}(1, 1; k, w_1)$ is much smoother than $\widehat{\lambda}^{\mathrm{emp}}(1, 1; k)$ with respect to k. More improvement of $\widehat{\lambda}(x, y; k, w_1)$ over $\widehat{\lambda}^{\mathrm{emp}}(x, y; k)$ are found when x/y is away from one; see Figure 7.

Finally, we compare $\widehat{\lambda}(x, y; 50, w_1)$ and $\widehat{\lambda}^{\mathrm{emp}}(x, y; 50)$ for different x and y. Again, we plot the mean squared errors of $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ and $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$ for $0 \le \phi \le \pi/2$ in Figure 8. Based on a particular sample, we also plot estimators $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ and $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$ in Figure 9. From these figures, we observe that, when ϕ is away from $\pi/4$, $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ becomes much better than $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$.

In conclusion, with the help of an elliptical copula, we are able to estimate the tail dependence function more efficiently.

5 Elliptical Copula of Arbitrary Dimension

In this section we generalize our results in section 2 to the case, where the dimension $d \geq 2$ is arbitrary.

Theorem 5.1. *Assume that $X = (X_1, \ldots, X_d)^T$ has the same copula as the elliptical vector $Z = (Z_1, \ldots, Z_d)^T$, whose distribution is given in (2.1). W.l.o.g. assume that $AA^T = R$ is the correlation matrix of Z. Let $A_{i\cdot}$ denote the i-th row of A and and let F_U denote the* *uniform distribution on $\mathcal{S}_d$. Then the tail dependence function of X is given by*

$$\lambda^X(x_1,\ldots,x_d) := \lim_{t\to 0} t^{-1}P\left(1-F_1(X_1)<tx_1,\ldots,1-F_d(X_d)<tx_d\right)$$
$$= \int\limits_{u\in\mathcal{S}_d, A_1.\,u>0,\ldots,A_d.\,u>0} \bigwedge_{i=1}^{d} x_i(A_i.\,u)^\alpha\, dF_U(u)\, \Bigg(\int\limits_{u\in\mathcal{S}_d, A_1.\,u>0} (A_1.\,u)^\alpha\, dF_U(u)\Bigg)^{-1}. \tag{5.2}$$

Remark 5.2. (a) For $d=2$ representation (5.2) coincides with (2.2). To see this write $u\in\mathcal{S}_2$ as $u=(\cos\phi,\sin\phi)^T$ for some $\phi\in(-\pi,\pi)$, $A_1.=(1,0)$ and $A_2.=(\rho,\sqrt{1-\rho^2})$. Then, $Au=(\cos\phi,\rho\cos\phi+\sqrt{1-\rho^2}\sin\phi)^T=(\cos\phi,\sin(\phi+\arcsin\rho))^T$, giving the equivalence of (5.2) and (2.2).
(b) For $d\geq 3$ one can also use multivariate polar coordinates and obtain analogous representations. The expression, however, becomes much more complicated.

The estimation procedure in d dimensions is a simple extension of the two-dimensional case. Assume iid observations $X_i=(X_{i1},\ldots,X_{id})^T$, $i=1,\ldots,n$, with an elliptical copula. Then we can estimate ρ_{pq} via Kendall's τ and α_{pq} based on bivariate subvectors (X_{ip},X_{iq}) for $1\leq p,q\leq d$. Denote these estimators by $\widehat{\rho}_{pq}$ and (for any positive weight function w) $\widehat{\alpha}_{pq}(k,w)$, respectively. Then we estimate α and R by

$$\widehat{\alpha}(k,w)=\frac{1}{d(d-1)}\sum_{p\neq q}\widehat{\alpha}_{pq}(k,w) \quad \text{and} \quad \widehat{R}=(\widehat{\rho}_{pq})_{1\leq p,q\leq d}.$$

Note that $\widehat{R}$ is not necessarily positive semi-definite. In that case we can apply algorithm 3.3 in Higham (2002) to project the indefinite correlation matrix to the class of positive semi-definite correlation matrices, say $\widehat{A}\widehat{A}^T=\widehat{R}$; see Klüppelberg and Kuhn (2006) for details. Hence we obtain an estimator for A. This yields an estimator for $\lambda(x_1,\ldots,x_d)$ by replacing α and $A_i.$ in (5.2) by $\widehat{\alpha}(k,w)$ and $\widehat{A}_i.$, respectively. The asymptotic normality of this new estimator can be derived similarly as in Theorems 2.2 and 2.4.

In Figure 10 we give a three-dimensional example. We simulate a sample of length $n=3\,000$ from an elliptical copula with $P(G>x)=\exp\{-x^{-\alpha}\}$, $x>0$, and parameters $\rho_{12}=0.3$, $\rho_{13}=0.5$, $\rho_{23}=0.7$ and $\alpha=5$. In the upper row we plot the true tail dependence function $\lambda^X\left(\sqrt{3}\cos\phi_1,\sqrt{3}\sin\phi_1\cos\phi_2,\sqrt{3}\sin\phi_1\sin\phi_2\right)$, $\phi_1,\phi_2\in(0,\pi/2)$, and each column corresponds to perspective, contour and grey-scale image plot of λ^X, respectively. In the middle and lower row, we plot the corresponding estimators $\widehat{\lambda}(\ldots;100,w_1)$ and $\widehat{\lambda}^{\mathrm{emp}}(\ldots;100)$, respectively. From this figure, we also observe that $\widehat{\lambda}$ becomes much better than $\widehat{\lambda}^{\mathrm{emp}}$ in the three-dimensional case.

Next we apply our estimators to a three-dimensional real data set which consists of $n=4\,903$ daily log returns of currency exchange rates of GBP, USD and CHF with respect to EURO between May 1985 and June 2004. As in Figure 10, we plot the perspective, contour and grey-scale image of $\widehat{\lambda}\left(\sqrt{3}\cos\phi_1, \sqrt{3}\sin\phi_1\cos\phi_2, \sqrt{3}\sin\phi_1\sin\phi_2;\, k, w_1\right)$ and $\widehat{\lambda}^{\mathrm{emp}}(\ldots; k)$; see Figures 11, 12 and 13 for $k = 100$, $k = 150$ and $k = 200$, respectively. Comparing the contour plots (middle columns) of $\widehat{\lambda}$ and $\widehat{\lambda}^{\mathrm{emp}}$, one may conclude that the assumption of having an elliptical copula is not restrictive.

6 Proofs

We shall use the following notation:

$$c_0 = \int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \, d\phi, \qquad c_1 = \int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \ln(\cos\phi)\, d\phi,$$

$$\begin{aligned} D(\alpha, z) &= c_0 \int_z^{\pi/2} (\cos\phi)^\alpha \ln(\cos\phi)\, d\phi - c_1 \int_z^{\pi/2} (\cos\phi)^\alpha \, d\phi \quad \text{and} \\ C(\alpha, z) &= D(\alpha, z) + \left(\rho + \sqrt{1-\rho^2}\tan z\right)^{-\alpha} D(\alpha, \arccos\rho - z). \end{aligned}$$

Recall also from (2.3) that $g(t) = \arctan\left((t-\rho)/\sqrt{1-\rho^2}\right)$ for $t > 0$.

Lemma 6.1. *The tail dependence function $\lambda(\alpha; x, y, \rho)$ has representation* (2.2).

Proof. Klüppelberg, Kuhn and Peng (2007) showed that for an elliptical vector Z

$$\begin{aligned} \lambda^Z(x, y) &= \left(\int_{g\left((x/y)^{1/\alpha}\right)}^{\pi/2} x(\cos\phi)^\alpha \, \mathrm{d}\phi + \int_{-\arcsin\rho}^{g\left((x/y)^{1/\alpha}\right)} y \left(\sin(\phi + \arcsin\rho)\right)^\alpha \, \mathrm{d}\phi \right) \\ &\quad \times \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \, \mathrm{d}\phi \right)^{-1} := \lambda(\alpha; x, y, \rho). \end{aligned}$$

Since $\tan(\arccos\rho) = \left(\tan(\arcsin\rho)\right)^{-1} = \sqrt{1-\rho^2}/\rho$, we have

$$\tan\Big(\arccos\rho - g(t)\Big) = \frac{\tan(\arccos\rho) - \tan(g(t))}{1 + \tan(\arccos\rho)\tan(g(t))} = \frac{t^{-1} - \rho}{\sqrt{1-\rho^2}}, \quad t > 0,$$

which implies that

$$\arccos\rho - g(t) = g(t^{-1}), \quad t > 0. \tag{6.1}$$

Application of a variable transformation to the second summand of (2.2) (setting $\psi = \arccos\rho - \phi$), we obtain (2.2).

Proof of Theorem 2.1. We calculate

$$\cos(g(t)) = \left(1 + \frac{(t-\rho)^2}{1-\rho^2}\right)^{-1/2} = \frac{\sqrt{1-\rho^2}}{\sqrt{1 - 2t\rho + t^2}} = t^{-1}\cos(g(t^{-1})), \quad t > 0. \tag{6.2}$$

It follows from (6.1) and (6.2)

$$x\Big(\cos(g((x/y)^{1/\alpha}))\Big)^{\alpha}\frac{\partial}{\partial\alpha}g((x/y)^{1/\alpha})+y\Big(\cos(g((x/y)^{-1/\alpha}))\Big)^{\alpha}\frac{\partial}{\partial\alpha}g((x/y)^{-1/\alpha}) \;=\; 0. \quad (6.3)$$

It follows from (6.1) and (6.3) that

$$\begin{aligned} la'(\alpha;x,y,\rho) &:= \frac{\partial}{\partial\alpha}\lambda(\alpha;x,y,\rho) \\ &= c_0^{-2}\left[xD\left(\alpha,g\left((x/y)^{1/\alpha}\right)\right)+yD\left(\alpha,g\left((x/y)^{-1/\alpha}\right)\right)\right] \\ &= c_0^{-2}xC\left(\alpha,g\left((x/y)^{1/\alpha}\right)\right). \end{aligned}$$

Since

$$D_{0,1}(\alpha,z) \;:=\; \frac{\partial}{\partial z}D(\alpha,z) \;=\; (\cos z)^{\alpha}\left(c_1-c_0\ln(\cos z)\right),$$

$$\frac{d^2}{dz^2}\Big(c_1-c_0\ln(\cos z)\Big)=c_0\Big(\cos z\Big)^{-2}>0$$

and $c_1-c_0\ln(\cos z)$ is a symmetric function around zero for $z\in(-\pi/2,\pi/2)$, there exists $0<z_0<\pi/2$ such that

$$\begin{cases} c_1-c_0\ln(\cos z)>0, & \text{if } z\in(-\pi/2,-z_0), \\ c_1-c_0\ln(\cos z)=0, & \text{if } z=-z_0, \\ c_1-c_0\ln(\cos z)<0, & \text{if } z\in(-z_0,z_0), \\ c_1-c_0\ln(\cos z)=0, & \text{if } z=z_0, \\ c_1-c_0\ln(\cos z)>0, & \text{if } z\in(z_0,\pi/2) \end{cases}$$

i.e.,

$$\begin{cases} D_{0,1}(\alpha,z)>0, & \text{if } z\in(-\pi/2,-z_0), \\ D_{0,1}(\alpha,z)=0, & \text{if } z=-z_0, \\ D_{0,1}(\alpha,z)<0, & \text{if } z\in(-z_0,z_0), \\ D_{0,1}(\alpha,z)=0, & \text{if } z=z_0, \\ D_{0,1}(\alpha,z)>0, & \text{if } z\in(z_0,\pi/2). \end{cases}$$

Note that z_0 depends on α. Since $D(\alpha,0)=\lim_{z\to\pm\pi/2}D(\alpha,z)=0$, we have

$$\begin{cases} D(\alpha,z)>0, & \text{if } z\in(-\pi/2,0), \\ D(\alpha,z)<0, & \text{if } z\in(0,\pi/2). \end{cases}$$

Hence, if $x/y\in\left[(\rho\vee 0)^{\alpha^*},(\rho\vee 0)^{-\alpha^*}\right]$ for some $\alpha^*\in(0,\infty)$, then $C\left(\alpha,g\left((x/y)^{1/\alpha^*}\right)\right)<0$ for all $\alpha>\alpha^*$. Since also $x/y\in[(\rho\vee 0)^{\alpha},(\rho\vee 0)^{-\alpha}]$ holds for all $\alpha>\alpha^*$, we have $C\left(\alpha,g\left((x/y)^{1/\alpha}\right)\right)<0$ for all $\alpha>\alpha^*$. Hence the theorem follows by choosing $\alpha^*=|\ln(x/y)/\ln(\rho\vee 0)|$.

Proof of Theorem 2.2. Using the same arguments as in Lemma 1 of Huang (1992), p. 30, or Corollary 3.8 of Einmahl (1997), we can show that

$$\sup_{0<x,y<T} \left| \sqrt{k} \left(\widehat{\lambda}^{\mathrm{emp}}(x,y) - \lambda^X(x,y) \right) - b_{(C3)} b_{(C2)}(x,y) - \widetilde{B}(x,y) \right| = o_p(1) \qquad (6.4)$$

as $n \to \infty$. Note that the above equation can also be shown in a way similar to Schmidt and Stadtmüller (2005) by taking the bias term into account. Since $\lambda(\alpha; x, y, \rho)$ in (2.2) is a continuous function of α, by invoking the delta method, the theorem follows from (6.4), $\widehat{\tau} - \tau = o_p(1/\sqrt{k})$ (see Klüppelberg and Kuhn (2006), Theorem 4.6),

$$\sup_{\theta \in Q^*} |\lambda'(\alpha; \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \rho)| < \infty,$$

and a Taylor expansion.

Proof of Theorem 2.4. It easily follows from (2.2) and Theorem 2.2.

Proof of Theorem 5.1. Since copulas are invariant under strictly increasing transformations, we can assume w.l.o.g that $AA^T = R$ is the correlation matrix. Therefore, the $Z_i \stackrel{\mathrm{d}}{=} GA_i\,.\,U$, $1 \le i \le d$, have the same distribution, say F_Z. Hence

$$\begin{aligned} & P\left(1 - F_Z(Z_1) < tx_1, \ldots, 1 - F_Z(Z_d) < tx_d\right) \\ = & \int\limits_{u \in \mathcal{S}_d, A_1\,.\,u>0,\ldots,A_d\,.\,u>0} P\Big(G > \bigvee_{i=1}^{d} \frac{F_Z^{\leftarrow}(1 - tx_i)}{A_i\,.\,u}\Big)\, dF_U(u), \end{aligned} \qquad (6.5)$$

where $F_Z^{\leftarrow}$ denotes the inverse function of F_Z. Since $P(G > \cdot) \in RV_{-\alpha}$ implies that $1 - F_Z \in RV_{-\alpha}$, the inverse function $F_Z^{\leftarrow} \in RV_{-1/\alpha}$ (e.g. Resnick (1987), Proposition 0.8(v)). This implies

$$\lim_{t \to 0} \frac{P\left(G > F_Z^{\leftarrow}(1 - tx_i)/(A_i\,.\,u)\right)}{P\left(G > F_Z^{\leftarrow}(1 - t)\right)} = x_i (A_i\,.\,u)^{\alpha}, \quad i = 1, \ldots, d.$$

Now note that, for all $i = 1, \ldots, d$,

$$\begin{aligned} t &= P(Z_i > F_Z^{\leftarrow}(1 - t)) = P(GA_i\,.\,U > F_Z^{\leftarrow}(1 - t)) \\ &= \int\limits_{u \in \mathcal{S}_d, A_i\,.\,u>0} P\left(G > \frac{F_Z^{\leftarrow}(1 - t)}{A_i\,.\,u}\right) dF_U(u), \end{aligned}$$

giving by means of Potter's bounds (e.g. see (1.20) in Geluk and de Haan (1987)),

$$\begin{aligned} \lim_{t \to 0} \frac{t}{P\left(G > F_Z^{\leftarrow}(1 - t)\right)} &= \lim_{t \to 0} \int\limits_{u \in \mathcal{S}_d, A_i\,.\,u>0} \frac{P(G > F_Z^{\leftarrow}(1 - t)/(A_i\,.\,u))}{P\left(G > F_Z^{\leftarrow}(1 - t)\right)}\, dF_U(u) \\ &= \int\limits_{u \in \mathcal{S}_d, A_i\,.\,u>0} (A_i\,.\,u)^{\alpha}\, dF_U(u) \quad \forall i = 1, \ldots, d. \end{aligned} \qquad (6.6)$$

Applying the same method to (6.5) yields the proof.

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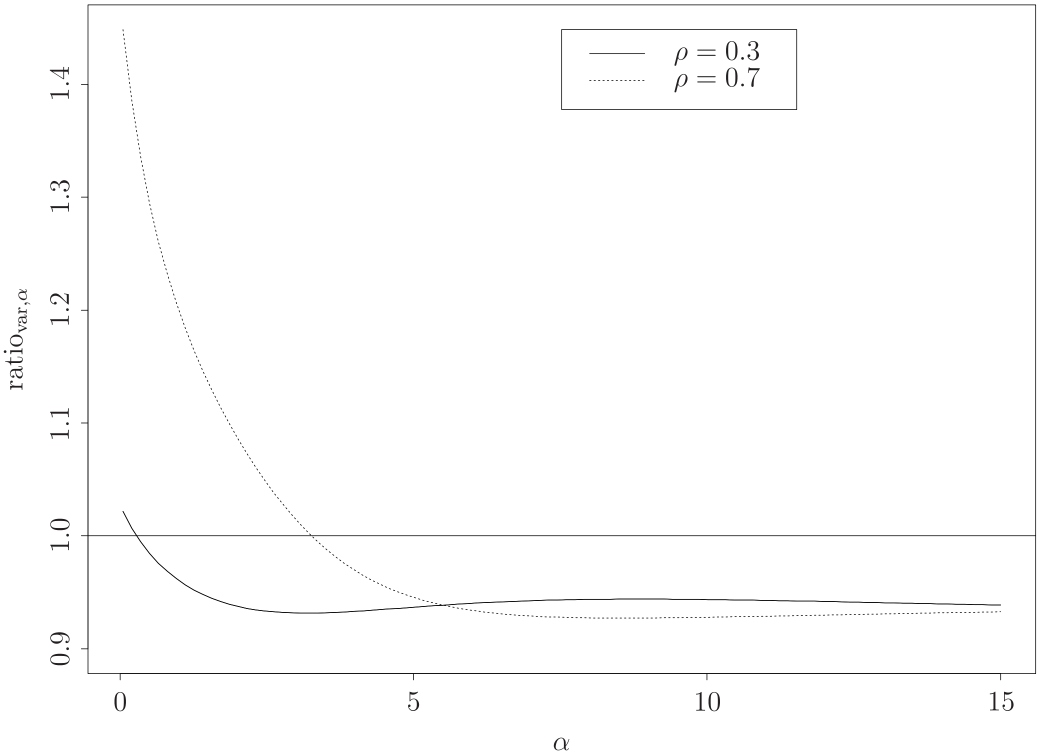

Figure 1: Theoretical ratios, $\mathrm{ratio}_{\mathrm{var},\alpha}$, are plotted against α for $\rho = 0.3$ and 0.7.

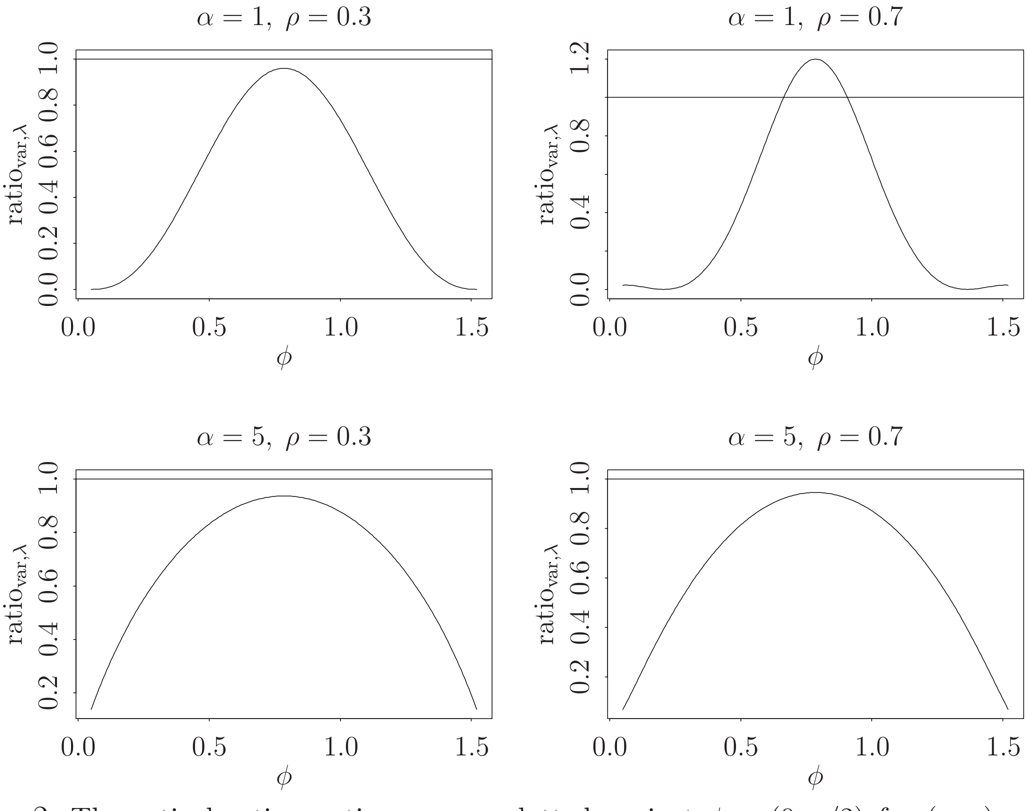

Figure 2: Theoretical ratios, $\mathrm{ratio}_{\mathrm{var},\lambda}$, are plotted against $\phi \in (0, \pi/2)$ for $(\alpha, \rho) \in \{1, 5\} \times \{0.3, 0.7\}$.

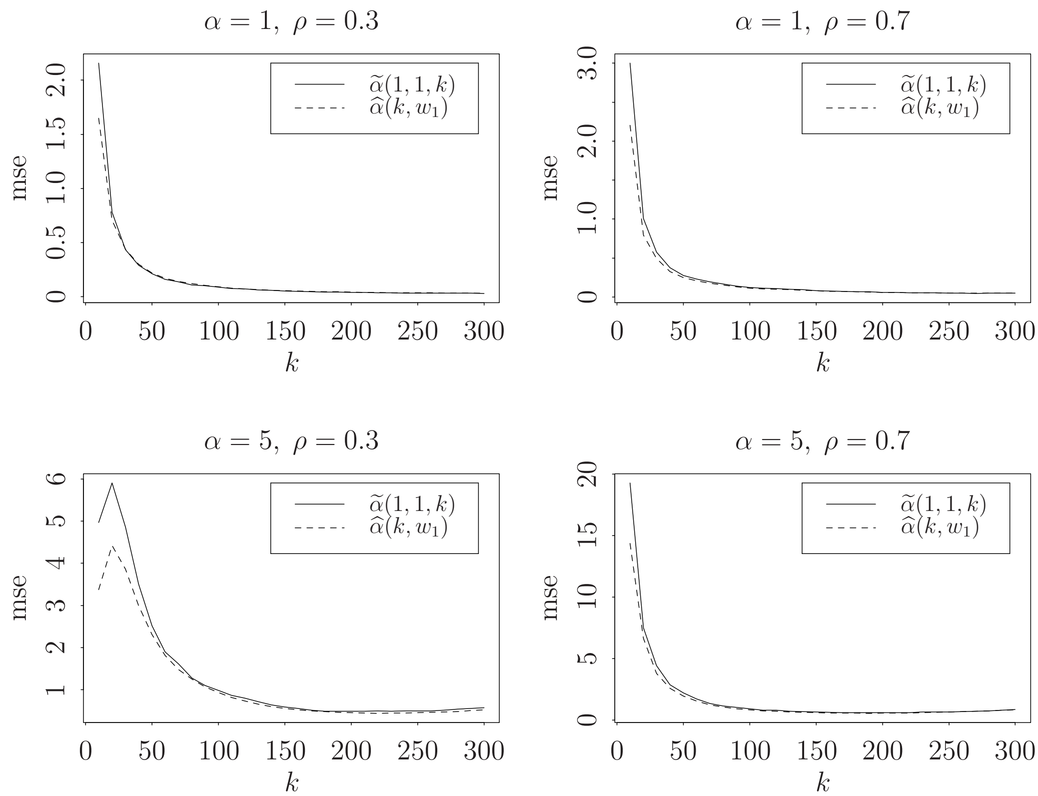

Figure 3: Mean squared errors of $\widetilde{\alpha}(1, 1, k)$ and $\widehat{\alpha}(k, w_1)$ are plotted against $k = 10, 20, \ldots, 300$.

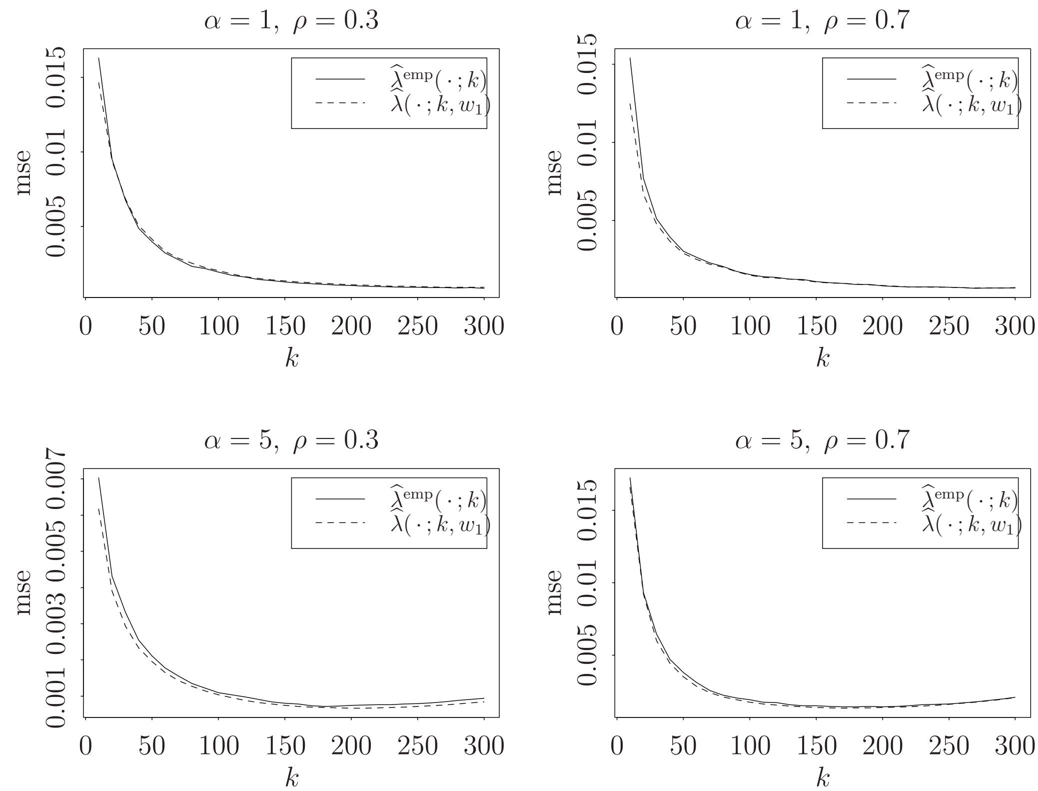

Figure 4: Mean squared errors of $\widehat{\lambda}^{\mathrm{emp}}(1, 1; k)$ and $\widehat{\lambda}(1, 1; k, w_1)$ are plotted against $k = 10, 20, \ldots, 300$.

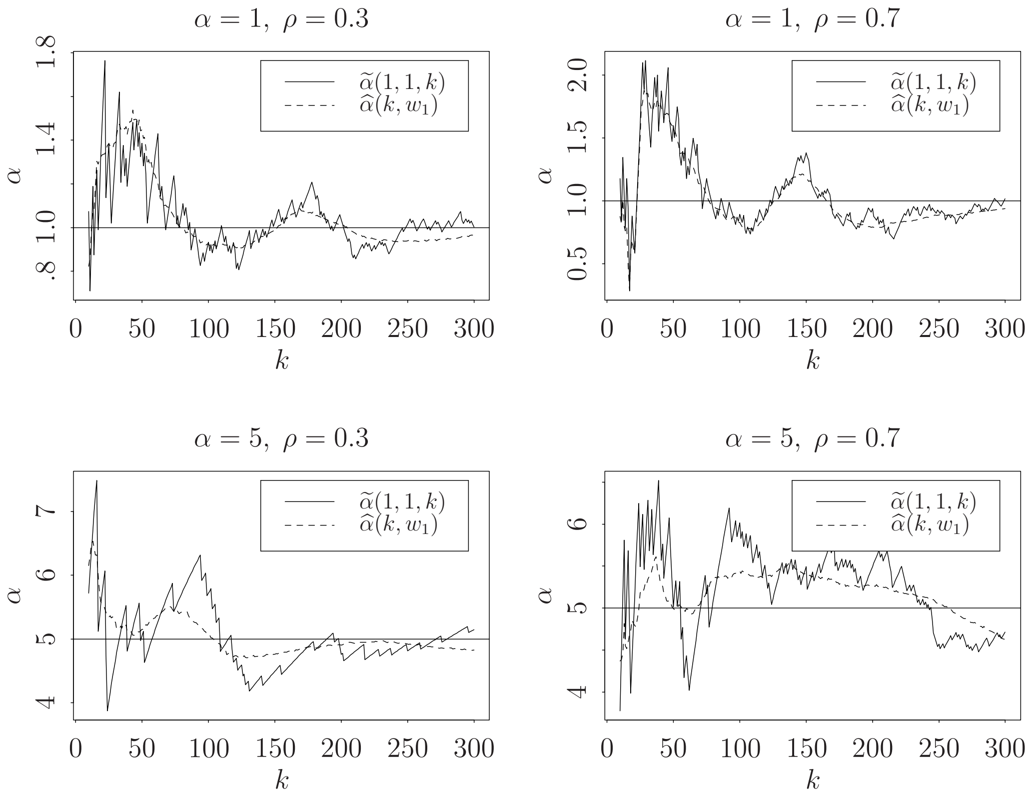

Figure 5: Estimators $\widetilde{\alpha}(1,1,k)$ and $\widehat{\alpha}(k,w_1)$ based on a particular sample are plotted against $k = 10, 11, \ldots, 300$.

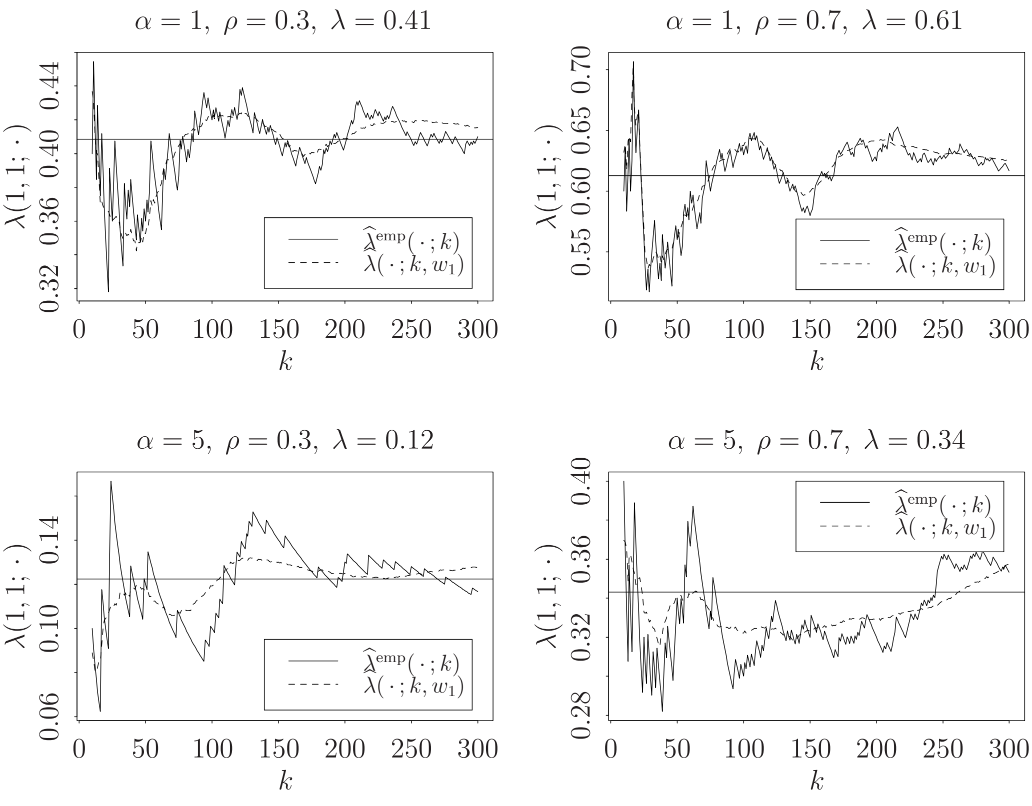

Figure 6: Estimators $\widehat{\lambda}^{\rm emp}(1,1;k)$ and $\widehat{\lambda}(1,1;k,w_1)$ based on a particular sample are plotted against $k = 10, 11, \ldots, 300$.

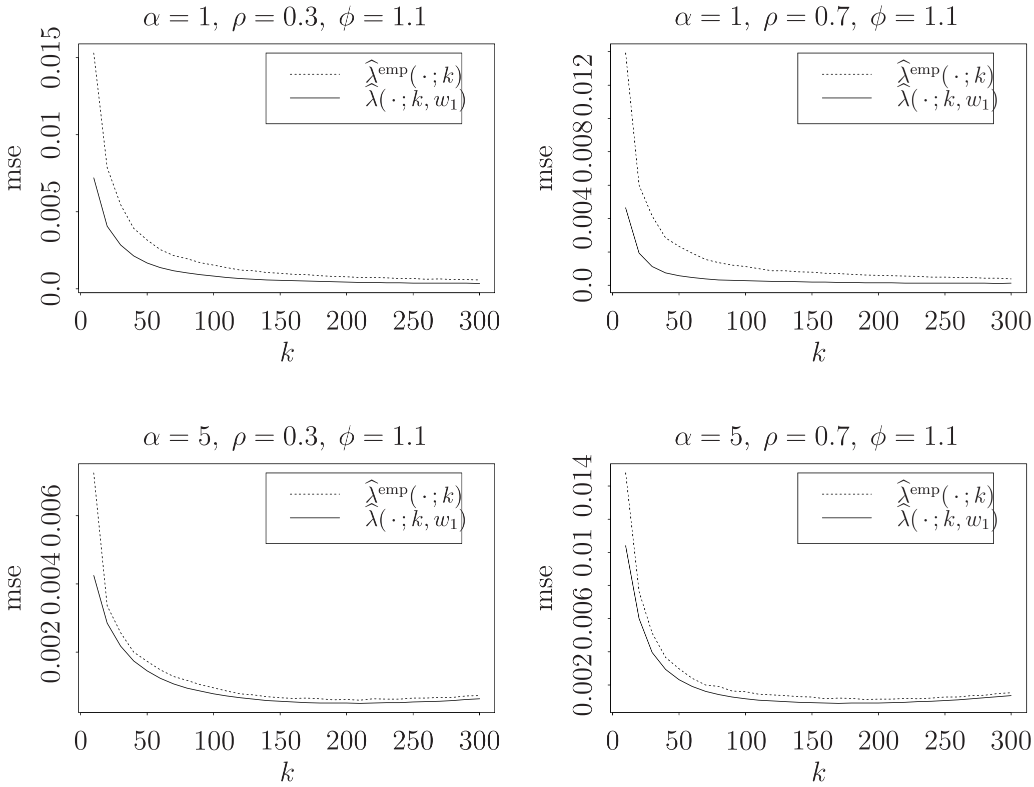

Figure 7: Mean squared errors of $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; k)$ and $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; k, w_1)$ with $\phi = 1.1$ are plotted against $k = 10, 20, \ldots, 300$.

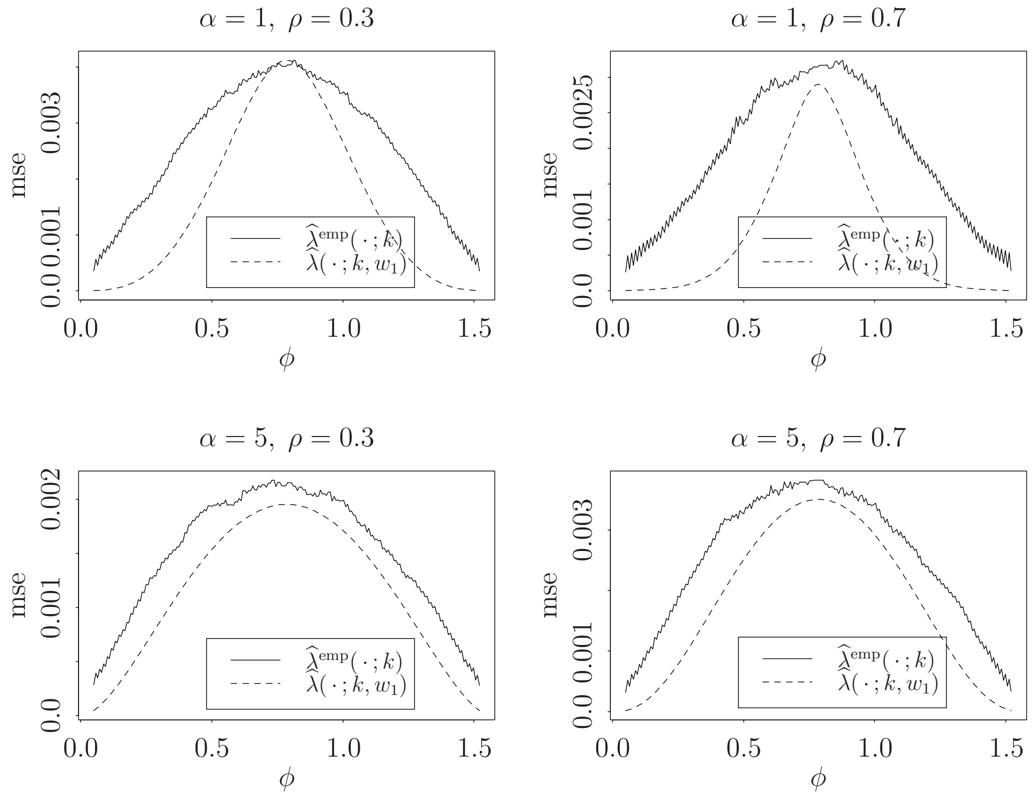

Figure 8: Mean squared errors of $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$ and $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ are plotted against $\phi \in (0, \pi/2)$.

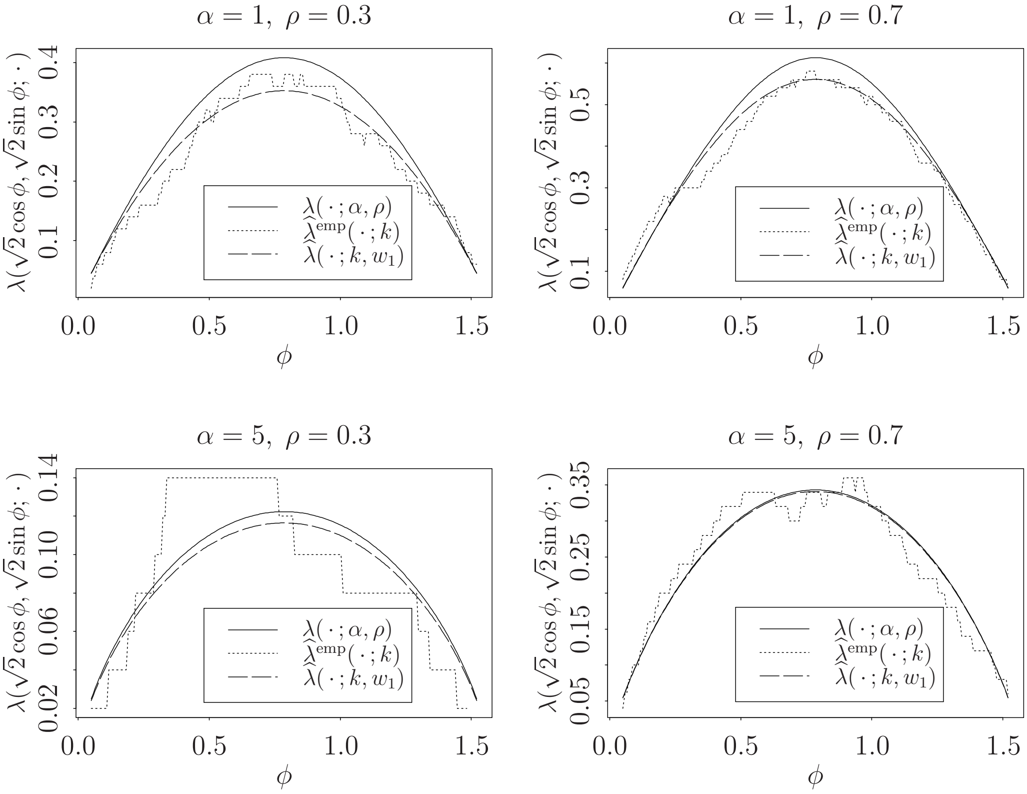

Figure 9: Estimators $\widehat{\lambda}^{\mathrm{emp}}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50)$ and $\widehat{\lambda}(\sqrt{2}\cos\phi, \sqrt{2}\sin\phi; 50, w_1)$ based on a particular sample are plotted against $\phi \in (0, \pi/2)$.

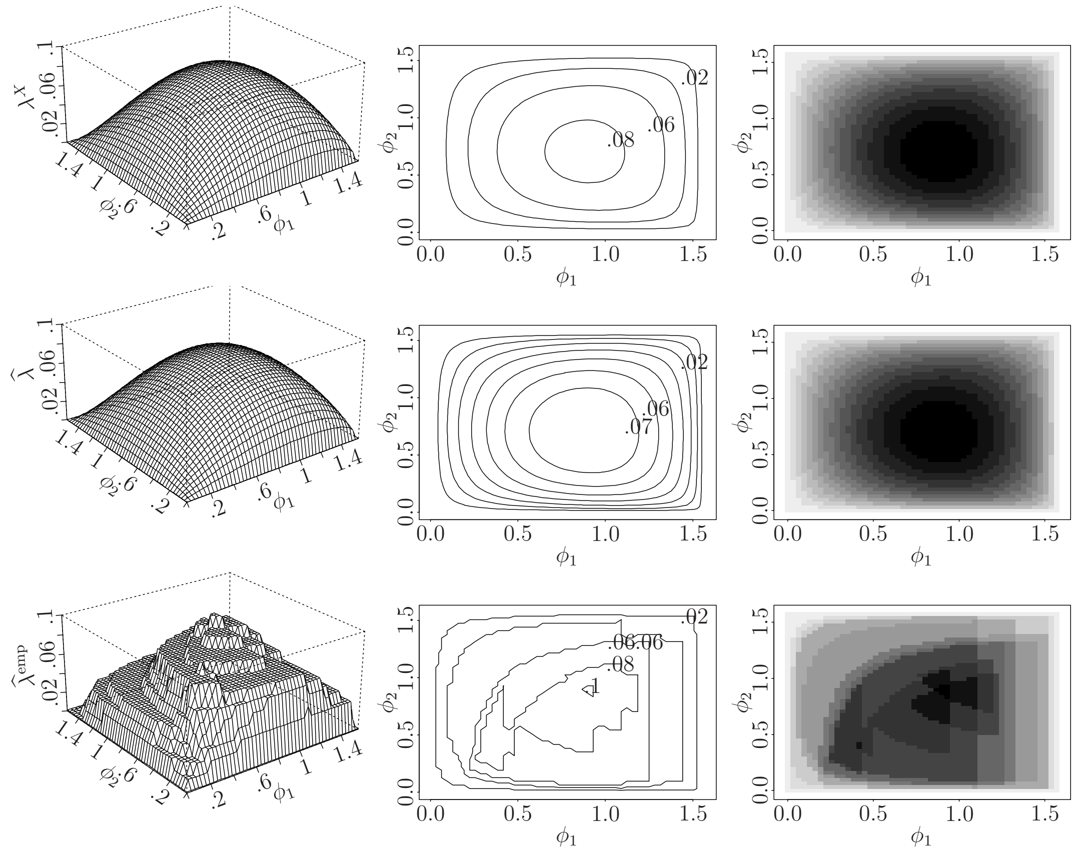

Figure 10: From left to right column: perspective, contour and grey-scale image plot of true $\lambda^X\left(\sqrt{3}\cos\phi_1, \sqrt{3}\sin\phi_1\cos\phi_2, \sqrt{3}\sin\phi_1\sin\phi_2\right)$ with parameters $\rho_{12} = 0.3$, $\rho_{13} = 0.5$, $\rho_{23} = 0.7$ and $\alpha = 5$ (first row) and corresponding estimators based on a particular sample, $\widehat{\lambda}(...; 100, w_1)$ (middle row) and $\widehat{\lambda}^{\mathrm{emp}}(...; 100)$ (lower row).

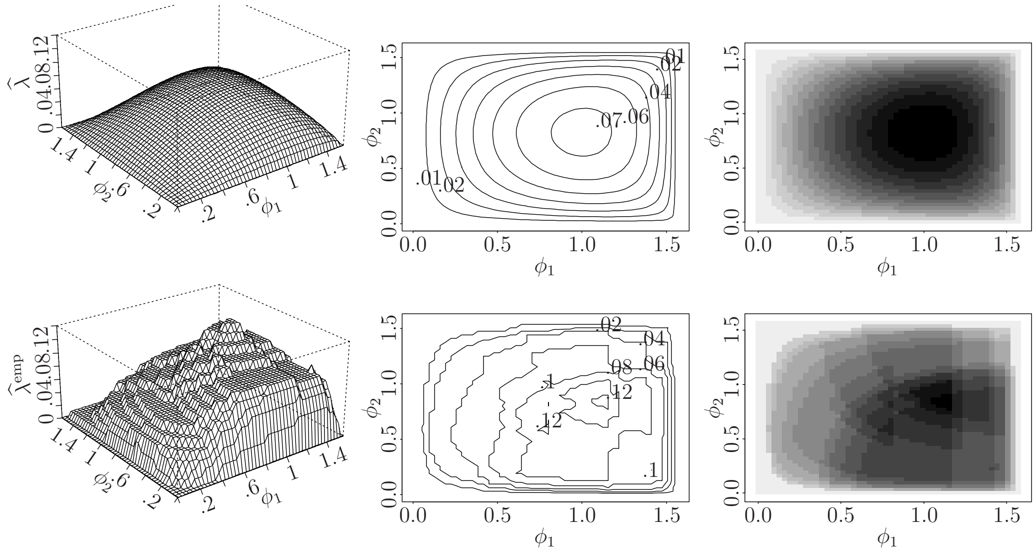

Figure 11: From left to right column: perspective, contour and grey-scale image plot of estimators $\widehat{\lambda}(...; 100, w_1)$ (upper row) and $\widehat{\lambda}^{\mathrm{emp}}(...; 100)$ (lower row) of currencies (GBP, USD, CHF).

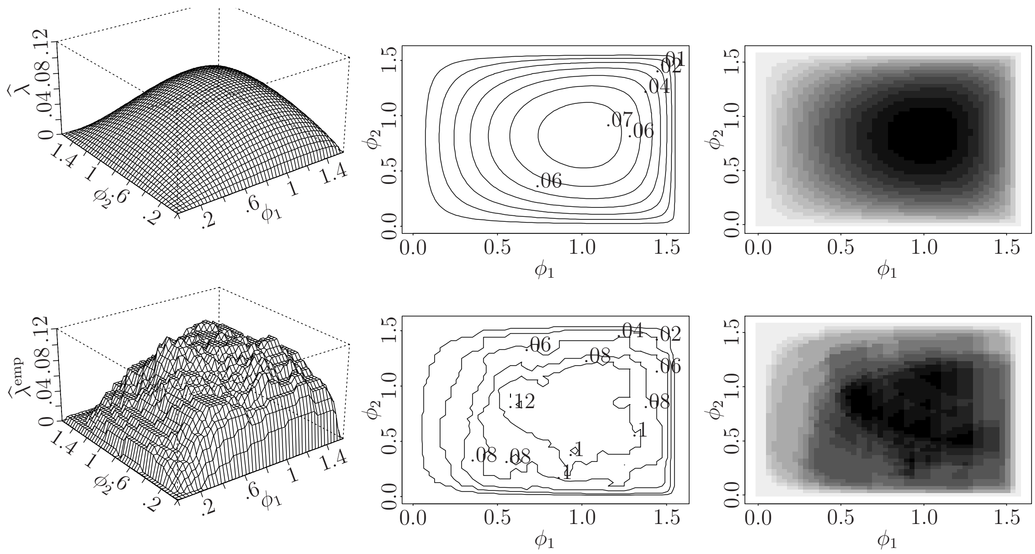

Figure 12: Same as Figure 11 but for $k = 150$.

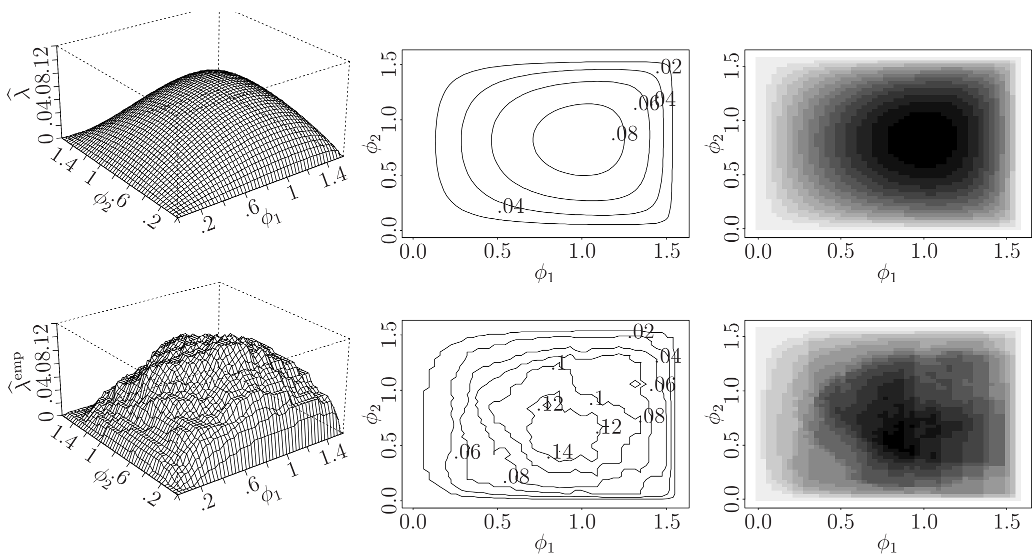

Figure 13: Same as Figure 11 but for $k = 200$.