Estimating high quantiles for electricity prices by stable linear models

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Abstract

We estimate conditional and unconditional high quantiles for electricity spot prices based on a linear model with stable innovations. This approach captures the impressive peaks in such data and, as a four-parametric family captures also the assymmetry in the innovations. Moreover, it allows for explicit formulas of quantiles, which can then be calculated recursively from day to day. We also prove that conditional quantiles of step $h \in \mathbb{N}$ converge for $h \to \infty$ to the corresponding unconditional quantiles. The paper is motivated by the daily spot prices from the Singapore New Electricity Market, which serves an example to show our method at work.

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1 Introduction

The liberation of electricity markets requires new models and methods for risk management in these highly volatile markets. The data exhibit certain features of commodity data as well as financial data. We summarize the following stylized features of daily electricity prices (see for example [8] or [13] and references therein).

(a) Seasonal behaviour in yearly, weekly and daily cycles.

(b) Stationary behaviour.

(c) Non-Gaussianity manifested for instance by high kurtosis.

(d) Extreme spikes.

In this paper we shall present statistical estimation of high quantiles, unconditional and conditional ones. Whereas in [7] we discussed the fit of a continuous-time model aiming at pricing of electricity spot price derviatives, in this paper we fit a simple discrete-time ARMA model enriched by a trend and seasonality component. We propose an ARMA-model with stable innovations, which is able to capture in particular the impressive spikes. Such a model seems to be appropriate for the estimation of high quantiles aiming at risk management. In this model it is also straightforward how to estimate high quantiles, conditional as well as unconditional ones. For the conditional quantiles we present not only the one-step, but also multi-step predicted quantiles. We also show that when the number of steps increases, the conditional quantiles converge to the unconditional ones.

Our analysis is motivated by daily spot prices from the Singapore New Electricity Market available at *www.ema.gov.sg* as depicted in Figure 1. The data are measured in Singapore-Dollar per MegaWatthour (SGD/MWh) and run from January 1, 2005 to April 11, 2007, resulting in $n = 831$ data points.

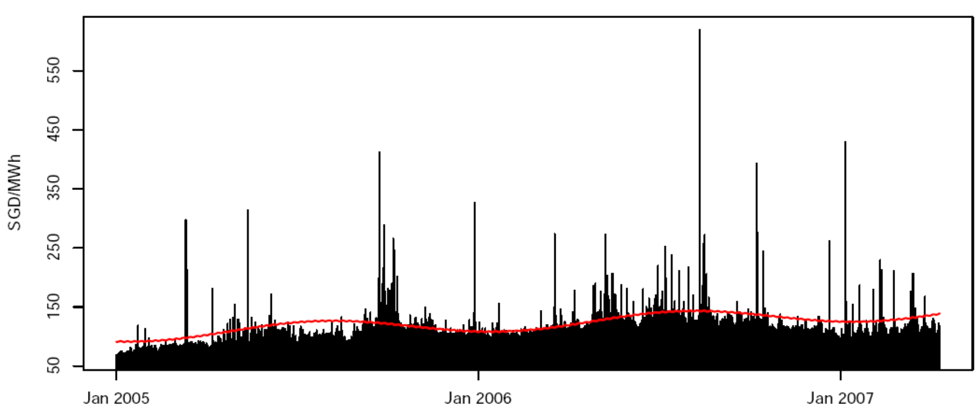

Figure 1: Time series plot of the Singapore data (January 2005 - April 2007) $(X_t)_{t=1,\ldots,n}$ for $n = 831$ data points with estimated seasonality function $\widehat{\Lambda}_t$ (in red) as modeled in Section 2.

Our paper is organised as follows. In Section 2 we present the data, which we model by a deterministic component (capturing trend and seasonality) and a stationary time series. Invoking classical model selection procedures for linear models we find that a pure AR model does not capture the features of the data, but an ARMA(1,2) model is suggested. As expected, the estimated residuals are assymmetric with high positive peaks, and we use a four-parametric stable model for the innovations. Section 3 is devoted to the quantile estimation, where we use that the estimated parameters correspond to a causal and invertible model. In Section 3.1 we apply well-known results from stable linear models to estimate the unconditional quantiles, whereas in Section 3.2 we derive formulas for conditional quantiles, the recursion formulas and the almost sure convergence result of conditional quantiles to the unconditional ones. Throughout we use the spot price data for illustration.

2 Data and Model

We model our data as

$$X_t = \Lambda_t + Y_t\,, \quad t \in \mathbb{Z}\,, \tag{2.1}$$

where Λ_t combines trend and saisonality (yearly and weekly) in the following deterministic model

$$\Lambda_t = \beta_0 + \beta_1 \cos\left(\frac{\tau_1 + 2\pi t}{365}\right) + \beta_2 \cos\left(\frac{\tau_2 + 2\pi t}{7}\right) + \beta_3 t\,, \quad t \in \mathbb{Z}\,, \tag{2.2}$$

and $(Y_t)_{t\in\mathbb{Z}}$ is a stationary ARMA model, which has to be further specified later. We estimate Λ by a robust least squares method (see [6], Ch. 1) and obtain the following estimated parameters.

$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\tau}_1$	$\widehat{\beta}_2$	$\widehat{\tau}_2$	$\widehat{\beta}_3$
103.66	-13.13	2136.16	-0.7686	32.04	0.04597

Table 2.1: Estimates of the parameters of (2.2)

Figures 1 and 2 show the original time series with estimated seasonality function and the deseasonalised series, respectively.

Define now $\widehat{Y}_t = X_t - \widehat{\Lambda}_t$ for $t = 1, \ldots, n$. Then we test $(\widehat{Y}_t)_{t=1,\ldots,n}$ for stationarity by an augmented Dickey-Fuller test and a Phillips-Perron test (cf. [1], Ch. 4, for details). Both tests result in a p-value less than 0.01. We want, however, remark that our data are heavy-tailed, in particular, as we shall see below, they have infinite variance. Consequently, variance and autocorrelation function do not exist. It has, however, been shown in [4] that empirical variance and autocorrelations converge to a stable vector. This means that the empirical quantities make still sense, but the confidence bands are much wider. See also [5], Section 7.3. Having clarified this, we assume stationarity for $(Y_t)_{t\in\mathbb{Z}}$ in (2.1).

In the same spirit we fit an ARMA model of appropriate order based on the empirical autocorrelation and partial autocorrelation function in Figure 3; cf. [2], Ch. 3.

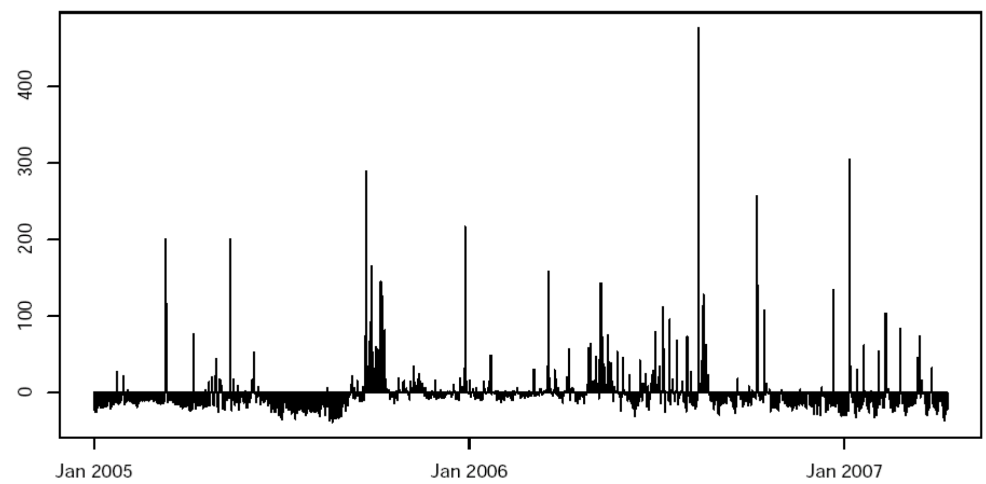

Figure 2: Deseasonalised series $(\widehat{Y}_t)_{t=1,\ldots,n}$.

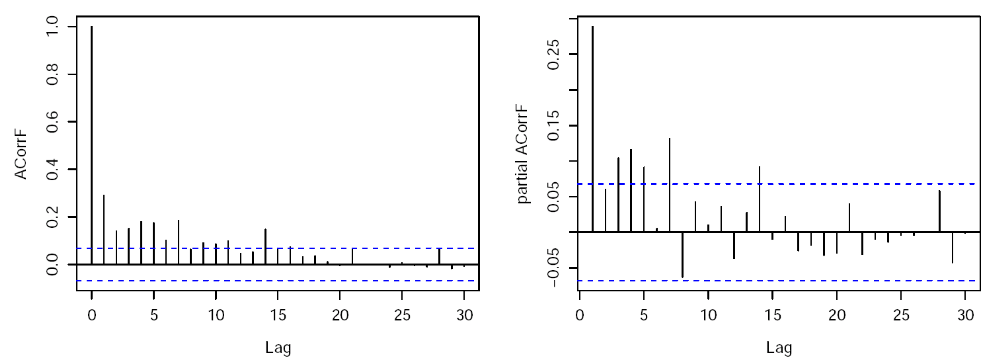

Figure 3: Empirical autocorrelation function and empirical partial autocorrelation function of the desaisonalised time series $(\widehat{Y}_t)_{t=1,\ldots,n}$.

Moreover, we use the AICC and BIC criterions for all ARMA(p, q) models with $p + q \leq 3$; the result is depicted in Figure 4; cf. [3], Section 5.5 for such criterions. From this it is obvious that a simple AR model is not sufficient to model the data properly. The best model with respect to both criterions was an ARMA$(1, 2)$ for $(Y_t)_{t\in\mathbb{Z}}$; i.e. with $Y_0 = 0$ and iid $(Z_t)_{t\in\mathbb{Z}}$ whose distribution we will specify later,

$$Y_t = \phi_1 Y_{t-1} + Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}\,, \quad t \in \mathbb{Z}\,. \tag{2.3}$$

An explorative analysis of the residuals $(\widehat{Z}_t)_{t=1,\ldots,n}$ clarifies two points.

(1) The empirical autocorrelation function of the residuals and of the squared residuals shows no significant dependence; cf. Figure 5. This implies that a linear model is appropriate for these data and more sophisticated models as suggested in Swider and Weber [12] are not necessary.

(2) Figure 6 shows that the distribution of the residuals is highly non-symmetric and has a heavy right tail. The empirical distribution function of $(\widehat{Z}_t)_{t=1,\ldots,n}$ is shown in the left plot

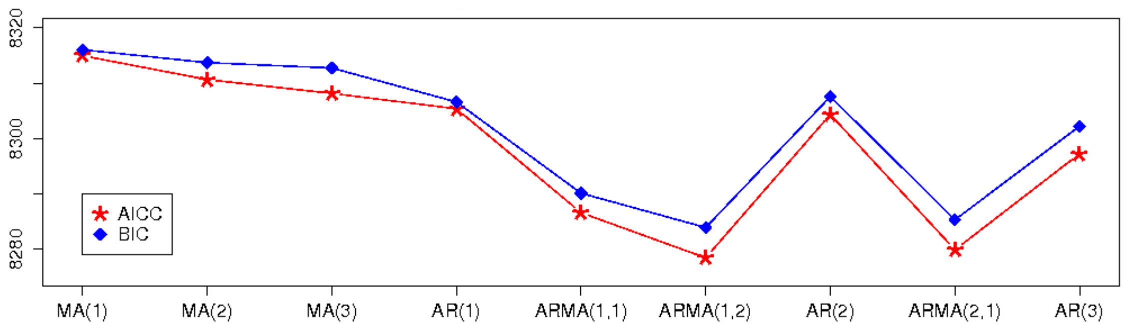

Figure 4: Values of the AICC and BIC criterions for the ARMA(p, q) model with $p+q \leq 3$ for the deseasonalised time series $(\widehat{Y}_t)_{t=1,\dots,n}$

	$\widehat{\phi_1}$	$\widehat{\theta}_1$	$\widehat{\theta}_2$
ARMA$(1,2)$	0.930	-0.689	-0.123
	(0.879 , 0.980)	(-0.776 , -0.602)	(-0.197 , -0.0489)

Table 2.2: MLEs of the ARMA$(1,2)$ coefficients with 95%-confidence intervals for the deseasonalised time series $(\widehat{Y}_t)_{t=1,\dots,n}$

of Figure 7. The right plot shows the mean excess plot; i.e. the empirical counterpart of

$$e(u) = E(X - u \mid X > u)\,, \quad u \geq 0\,.$$

The increase of the estimated mean excess function shows clearly that the right tail is Pareto like. As is confirmed by a Hill plot, the corresponding tail index is clearly smaller than 2.

For definitions and notation we refer to Embrechts et al. [5], Ch. 6 and Resnick [11], Section 4.

This tail pattern, in combination with the asymmetry, indicates that a four-parameter stable distribution is an appropriate model for the residuals $(Z_t)_{t\in\mathbb{Z}}$. It is defined as follows.

Definition 2.1 (Stable random variable, stable distribution)**.** *A random variable Z has an* α-stable distribution *with index of stability $\alpha \in (0, 2]$, skewness parameter $\beta \in [-1, 1]$, scale*

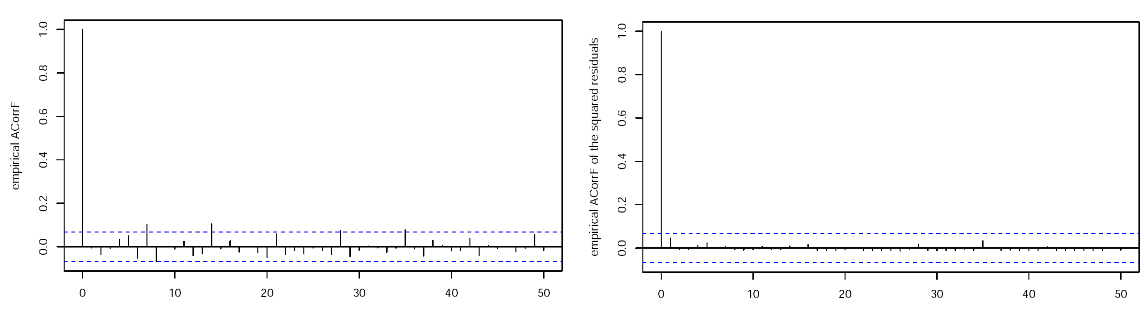

Figure 5: Plot of the autocorrelation functions of the residuals $(\widehat{Z}_t)_{t=1,\dots,n}$ and the squared residuals $(\widehat{Z}_t^2)_{t=1,\dots,n}$.

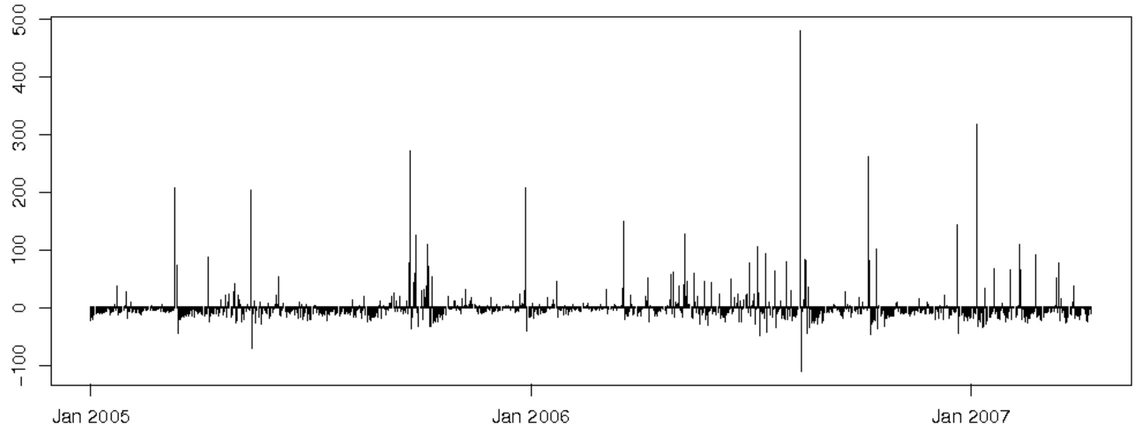

Figure 6: Plot of the residuals $(\widehat{Z}_t)_{t=1,\ldots,n}$.

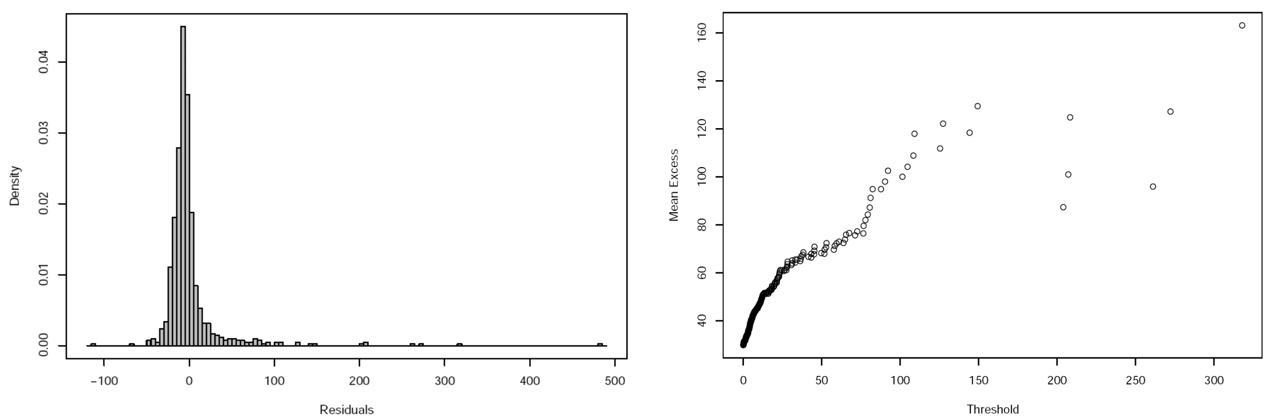

Figure 7: Left: Plot of the histogram of the residuals $(\widehat{Z}_t)_{t=1,\ldots,n}$. Right: Mean excess plot of the positive residuals.

parameter $\gamma > 0$*, and location parameter* $\delta \in \mathbb{R}$*, if*

$$Z \stackrel{d}{=} \begin{cases} \gamma\left(V - \beta \tan\left(\frac{\pi\alpha}{2}\right)\right) + \delta, & \alpha \neq 1, \\ \gamma V + \delta, & \alpha = 1, \end{cases}$$

where V *is a random variable with characteristic function*

$$\varphi_V(t) = Ee^{itV} = \begin{cases} \exp\left\{-|t|^{\alpha}\left(1 - i\beta \tan\left(\frac{\pi\alpha}{2}\left(\text{sign } t\right)\right)\right)\right\}, & \alpha \neq 1, \\ \exp\left\{-|t|\left(1 + i\beta\frac{2}{\pi}\left(\text{sign } t\right)\right)\log\left(|t|\right)\right\}, & \alpha = 1. \end{cases}$$

The sign function sign t *is defined as usual:* sign $t = -1, 0, 1$ *according as* $t < 0, t = 0, t > 0$*, respectively.*
We denote the class of stable distributions by $\mathcal{S}(\alpha, \beta, \gamma, \delta)$*. A random variable* $Z \sim \mathcal{S}(\alpha, \beta, \gamma, \delta)$ *is also referred to as* α-stable*.*
If $\delta = 0$ *and* $\gamma = 1$ *the distribution is called standardized* α*-stable.*

For more information on stable distributions we refer to [10] or any other monograph on stable distributions. The advantage of choosing a stable model lies in the fact that stable distributions are closed with respect to linear transformations. The following result can be found in [10], Section 1.6.

Theorem 2.2. *Let $(X_j)_{j\in\mathbb{N}_0}$ be iid $\mathcal{S}(\alpha,\beta,\gamma,\delta)$ distributed random variables and $(\psi_j)_{j\in\mathbb{N}_0}$ a real-valued sequence with $\sum_j |\psi_j|^\alpha < \infty$. Then*

$$\sum_{j=0}^{\infty} \psi_j X_j \;\sim\; \mathcal{S}\left(\alpha, \overline{\beta}, \overline{\gamma}, \overline{\delta}\right),$$

with

$$\begin{aligned}
\overline{\beta} &= \beta \sum_{j=0}^{\infty} |\psi_j|^\alpha \operatorname{sign}(\psi_j) \Big(\sum_{j=0}^{\infty} |\psi_j|^\alpha \Big)^{-1}, \\
\overline{\gamma} &= \gamma \Big(\sum_{j=0}^{\infty} |\psi_j|^\alpha \Big)^{\frac{1}{\alpha}}, \\
\overline{\delta} &= \begin{cases} \delta \sum_{j=0}^{\infty} \psi_j + \tan\left(\frac{\pi\alpha}{2}\right) \Big(\overline{\beta}\,\overline{\gamma} - \beta\gamma \sum_{j=0}^{\infty} \psi_j \Big), & \alpha \neq 1, \\ \delta \sum_{j=0}^{\infty} \psi_j + \dfrac{2}{\pi} \Big(\overline{\beta}\,\overline{\gamma} \log(\overline{\gamma}) - \beta\gamma \sum_{j=0}^{\infty} \psi_j \log|\psi_j \gamma| \Big), & \alpha = 1. \end{cases}
\end{aligned}$$

Various estimation procedures have been proposed in the literature. In this paper we use MLE, which gives the following parameter estimates for the residuals $(\widehat{Z}_t)_{t=1,\dots,n}$.

	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\delta}$
ML-estimators	1.282650	0.442722	7.012304	-7.610320

Table 2.3: MLEs of the parameters α, β, γ and δ of the stable distribution for the residuals $(\widehat{Z}_t)_{t=1,\dots,n}$.

Again one may ask for asymptotic properties of these estimators as the usual regularity conditions on the model to guarantee asymptotic normality are not satisfied. It has, however, been shown in [9] that MLEs for heavy-tailed models have even better rate of convergence than models with finite variance.

3 Estimating high quantiles

After having fitted an appropriate model to our data we now turn to the estimation of high quantiles, where we present unconditional as well as conditional ones. Recall the model from (2.3)

$$Y_t = \phi_1 Y_{t-1} + Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, \quad t \in \mathbb{Z}, \tag{3.1}$$

where $(Z_t)_{t\in\mathbb{Z}}$ are iid $\mathcal{S}(\alpha,\beta,\gamma,\delta)$ with estimators $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ and $\widehat{\delta}$ given in Table 2.3.

We shall invoke the following result.

Proposition 3.1 ([2], Proposition 12.5.2). *Let $(Z_t)_{t\in\mathbb{Z}}$ be iid $\mathcal{S}(\alpha, \beta, \gamma, \delta)$ distributed. Denote by $\theta(z) = 1 + \theta_1 z + \cdots + \theta_1 z^q$ and by $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, and assume that $\phi(z) \neq 0$ for $|z| \leq 1$. Then the difference equations $\phi(B)\, Y_t = \theta(B)\, Z_t$ for $t \in \mathbb{Z}$ (B denotes the backshift operator) have the unique strictly stationary and causal solution*

$$Y_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}\,, \quad t \in \mathbb{Z}\,, \tag{3.2}$$

where the coefficients $(\psi_j)_{j\in\mathbb{Z}}$ are chosen so that $\psi_j = 0$ for $j < 0$ and

$$\sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}\,, \quad |z| \leq 1\,. \tag{3.3}$$

If in addition $\phi(z)$ and $\phi(z)$ have no common zeroes, then the process $(Y_t)_{t\in\mathbb{Z}}$ is invertible if and only if $\theta(z) \neq 0$ for $|z| \leq 1$. In that case

$$Z_t = \sum_{j=0}^{\infty} \pi_j Y_{t-j}\,, \quad t \in \mathbb{Z}\,, \tag{3.4}$$

where the coefficients $(\pi_j)_{j\in\mathbb{Z}}$ are chosen so that $\psi_j = 0$ for $j < 0$ and

$$\sum_{j=0}^{\infty} \pi_j z^j = \frac{\phi(z)}{\theta(z)}\,, \quad |z| \leq 1\,. \tag{3.5}$$

Note that with the estimated coefficients of Table 2.2 the model (3.1) is causal and invertible. Invoking (3.3) and (3.5) gives the following parameters ψ and π for an ARMA(1,2) model as functions of the model parameters $\phi_1, \theta_1, \theta_2$

$$\psi_j = \begin{cases} 1 & \text{for } j = 0, \\ \theta_1 + \phi_1 & \text{for } j = 1, \\ \theta_2 \phi_1^{j-2} + \theta_1 \phi_1^{j-1} + \phi_1^j & \text{for } j \geq 2\,, \end{cases} \tag{3.6}$$

$$\pi_j = \begin{cases} 1 & \text{for } j = 0, \\ -\theta_1 - \phi_1 & \text{for } j = 1, \\ -\theta_2 \pi_{j-2} - \theta_1 \pi_{j-1} & \text{for } j \geq 2\,. \end{cases} \tag{3.7}$$

In our situation, the parameters ψ are given by (3.6). In order to compute the unconditional quantiles, we then plug in the MLEs for $\phi_1, \theta_1, \theta_2$, which results in $\widehat{\psi_j}$ as presented in Table 3.4.

3.1 Unconditional quantiles

For $q \in (0, 1)$ and $t \in \mathbb{Z}$ the *unconditional q-quantile* of X_t is defined as the generalised inverse of its distribution function; i.e.

$$x_q(t) := \inf\{x \in \mathbb{R} \,:\, P(X_t \leq x) \geq q\}\,.$$

$\widehat{\psi}_0$	$\widehat{\psi}_1$	$\widehat{\psi}_2$	$\widehat{\psi}_3$	$\widehat{\psi}_4$	$\widehat{\psi}_5$	$\widehat{\psi}_6$
1	0.241	0.101	0.094	0.087	0.081	0.081×0.930

Table 3.4: Estimates for ψ_j for $0 \le j \le 5$; their rate of decrease is given by $\widehat{\phi}_1 = 0.930$ (cf. Table 2.2).

The unconditional quantiles of Y_t and Z_t are defined analogously, and denoted by $y_q(t)$ and $z_q(t)$. First note that, by definition (2.1) of the model, the quantiles of X_t and Y_t are linked by

$$x_q(t) = \Lambda_t + y_q(t)\,, \quad \text{for } q \in (0,1)\,, t \in \mathbb{Z}\,.$$

In order to estimate the quantile of X_t, it suffices thus to estimate the quantile of Y_t. By stationarity and causality of Y, the unconditional quantile $y_q(t) = y_q$ becomes independent of t and can be expressed as

$$y_q := \inf\Big\{y \in \mathbb{R} \,:\, P\Big(\sum_{j=0}^{\infty} \psi_j Z_{t-j} \le y\Big) \ge q\Big\}\,, \quad q \in (0,1)\,.$$

Theorem 3.2. *Assume model (3.2) with $(Z_j)_{j\in\mathbb{N}_0}$ iid $\mathcal{S}(\alpha,\beta,\gamma,\delta)$ distributed. For $q \in (0,1)$ denote by s_q the q-quantile of an $\mathcal{S}\left(\alpha,\overline{\beta},1,0\right)$ distributed random variable. Then the q-quantile y_q is given by*

$$y_q = \overline{\gamma}\; s_q + \overline{\delta}\,, \qquad q \in (0,1)\,. \tag{3.8}$$

The parameters $\overline{\beta}$, $\overline{\gamma}$ und $\overline{\delta}$ are given by

$$\begin{aligned}
\overline{\beta} &= \beta \sum_{j=0}^{\infty} \psi_j^{\alpha}\operatorname{sign}(\psi_j)\Big(\sum_{j=0}^{\infty}|\psi_j|^{\alpha}\Big)^{-1},\\
\overline{\gamma} &= \gamma\,\Big(\sum_{j=0}^{\infty}|\psi_j|^{\alpha}\Big)^{1/\alpha},\\
\overline{\delta} &= \begin{cases} \delta\sum_{j=0}^{\infty}\psi_j + \tan\left(\frac{\pi\alpha}{2}\right)\Big(\overline{\beta}\,\overline{\gamma} - \beta\gamma\sum\limits_{j=0}^{\infty}\psi_j\Big), & \alpha \neq 1\,,\\ \delta\sum\limits_{j=0}^{\infty}\psi_j + \dfrac{2}{\pi}\Big(\overline{\beta}\,\overline{\gamma}\log\left(\overline{\gamma}\right) - \beta\gamma\sum\limits_{j=0}^{\infty}\psi_j\log|\psi_j\gamma|\Big), & \alpha = 1\,. \end{cases}
\end{aligned}$$

$\widehat{\alpha}$	$\widehat{\overline{\beta}}$	$\widehat{\overline{\gamma}}$	$\widehat{\overline{\delta}}$
1.282650	0.442722	13.20421	-15.19818

Table 3.5: Estimates for α, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ as in Theorem 3.2 (for the residuals $(\widehat{Z}_t)_{t=1,\dots,n}$.)

The quantiles of the corresponding stable distribution for appropriate high levels are presented in Table 3.6.

From this and (3.8) we obtain the corresponding estimated unconditional quantiles $\widehat{y}_q$ of the stationary random variable $\widehat{Y}_t$ as given in Table 3.7.

q	0.95	0.99	0.999
$\widehat{s}_q$	5.309276	17.50723	102.0260

Table 3.6: 95%-, 99%- and 99.9% quantiles of a $\mathcal{S}(\widehat{\alpha}, \widehat{\widehat{\beta}}, 1, 0)$ distributed random variable.

q	0.95	0.99	0.999
$\widehat{y}_q$	54.9066	215.9709	1331.974
number of exceedances	44	5	0

Table 3.7: 95%-, 99%- and 99.9% quantiles of the stationary time series $(\widehat{Y}_t)_{t=1,...,n}$ with simple backtesting procedure, counting the number of exceedances of the estimated quantiles (44=5.3%, 5=0.6% of data).

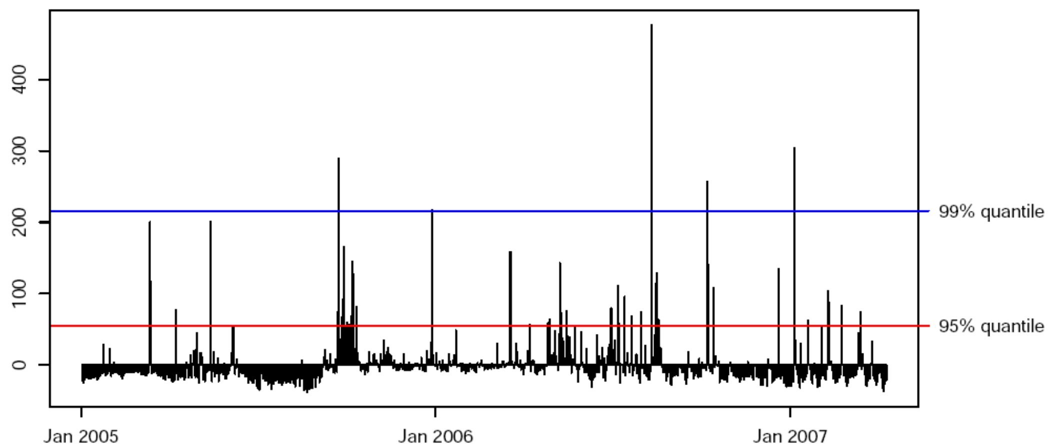

Figure 8: Time series $(\widehat{Y}_t)_{t=1,...,n}$ with 95%- and 99% quantile (the 99.9% quantile is outside of the figure).

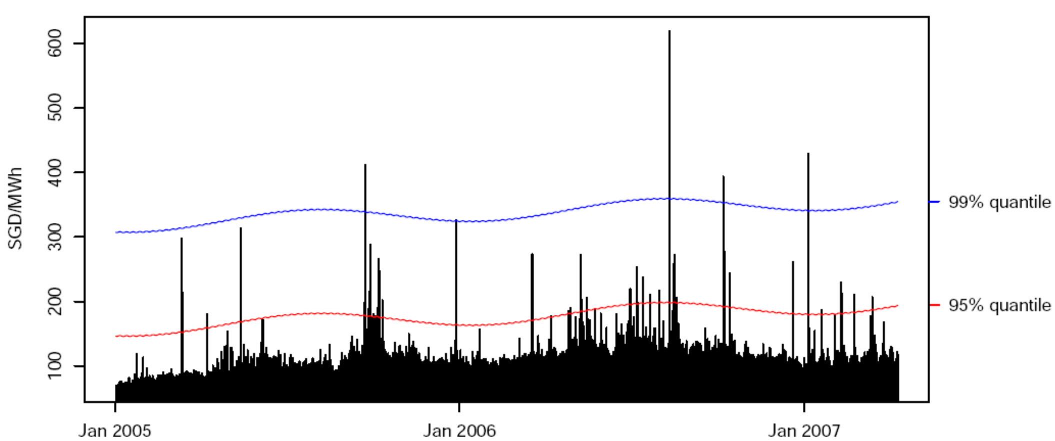

Figure 9: Data $(X_t)_{t=1,...,n}$ with 95%- and 99% quantile (the 99.9% quantile is outside of the figure).

3.2 Conditional quantiles

For $q \in (0,1)$ the *conditional q-quantile* of X_t of *step h*, where $t \in \mathbb{Z}$ and $h \in \mathbb{N}$, is defined as the generalised inverse of its conditional distribution function, conditioned on the observation of the time series $(X_t)_{t\in\mathbb{Z}}$ up to time $t-h$:

$$x_q^h(t) := \inf\Big\{x \in \mathbb{R} \,:\, P\Big(X_t \le x \mid X_s\,,\ s \le t-h\Big) \ge q\Big\}\,.$$

Note that by definition (2.1), the observation sigma algebra generated by $\{X_s\,,\ s \le t-h\}$ is the same as the one generated by $\{Y_s\,,\ s \le t-h\}$. Therefore, provided we know Λ_t we can express the conditional q-quantile of X_t as

$$x_q^h(t) = \Lambda_t + y_q^h(t)\,,$$

where

$$y_q^h(t) := \inf\Big\{y \in \mathbb{R} \,:\, P\Big(Y_t \le y \mid Y_s\,,\ s \le t-h\Big) \ge q\Big\}$$

is the conditional q-quantile of Y_t of step h, conditioned on the evolution of the time series $(Y_t)_{t\in\mathbb{Z}}$ up to time $t-h$.

Theorem 3.3. *Assume a causal and invertible model of the form* (3.2) *with* $(Z_j)_{j\in\mathbb{N}_0}$ *iid* $\mathcal{S}(\alpha,\beta,\gamma,\delta)$ *distributed. For* $q \in (0,1)$ *and* $h \in \mathbb{N}$ *denote by* s_q^h *the q-quantile of an* $\mathcal{S}\left(\alpha, \overline{\beta^h}, 1, 0\right)$ *distributed random variable. Then the conditional q-quantile* $y_q^h(t)$ *of step h is given by*

$$y_q^h(t) = \overline{\gamma^h}\, s_q^h + \overline{\delta^h} + \sum_{j=h}^{\infty} \psi_j Z_{t-j}\,, \qquad q \in (0,1)\,,\ h \in \mathbb{N}\,. \tag{3.9}$$

The parameters $\overline{\beta^h}$, $\overline{\gamma^h}$ *und* $\overline{\delta^h}$ *are given by*

$$\begin{aligned}
\overline{\beta^h} &= \beta \sum_{j=0}^{h-1} \psi_j^{\alpha} \operatorname{sign}(\psi_j) \Big(\sum_{j=0}^{h-1} |\psi_j|^{\alpha}\Big)^{-1},\\
\overline{\gamma^h} &= \gamma \,\Big(\sum_{j=0}^{h-1} |\psi_j|^{\alpha}\Big)^{1/\alpha},\\
\overline{\delta^h} &= \begin{cases} \delta \sum_{j=0}^{h-1} \psi_j + \tan\left(\frac{\pi\alpha}{2}\right)\Big(\overline{\beta}\,\overline{\gamma} - \beta\gamma \sum_{j=0}^{h-1} \psi_j\Big), & \alpha \ne 1\,,\\ \delta \sum_{j=0}^{h-1} \psi_j + \dfrac{2}{\pi}\Big(\overline{\beta}\,\overline{\gamma} \log\left(\overline{\gamma}\right) - \beta\gamma \sum_{j=0}^{h-1} \psi_j \log|\psi_j \gamma|\Big), & \alpha = 1\,. \end{cases}
\end{aligned}$$

Note that by the invertibility of the model, the term $\sum_{j=h}^{\infty} \psi_j Z_{t-j}$ in (3.9) is known in the sense that it is measurable with respect to the sigma algebra generated by $\{Y_s\,,\ s \le t-h\}$.

Proof. Using the causality and invertibility of the model, we get

$$\begin{aligned} P(Y_t \le y \,|\, Y_s\,,\, s \le t-h) &= P(\sum_{j=0}^{\infty} \psi_j Z_{t-j} \le y \,|\, Y_s\,,\, s \le t-h) \\ &= P(\sum_{j=0}^{h-1} \psi_j Z_{t-j} \le y - \sum_{j=h}^{\infty} \psi_j Z_{t-j} \,|\, Z_s\,,\, s \le t-h) \\ &= P(\sum_{j=0}^{h-1} \psi_j Z_{t-j} \le y - u)\Big|_{u=\sum_{j=h}^{\infty} \psi_j Z_{t-j}}\,, \end{aligned}$$

where we have used the independence of the innovations Z_t in the last equality. We thus have to compute the unconditional q-quantile of $\sum_{j=0}^{h-1} \psi_j Z_{t-j}$, which as in Theorem 3.2 with $\psi_j = 0$ for all $j \ge h$ is given by $\overline{\gamma^h}\ s_q^h + \overline{\delta^h}$. The conditional quantile (3.9) then follows and the theorem is proved. □

The following result is a contribution to the discussion about conditional versus unconditional quantiles.

Corollary 3.4. *For every $t \in \mathbb{Z}$, the following almost sure convergence holds*

$$y_q^h(t) \overset{h\to\infty}{\longrightarrow} y_q \quad a.s..$$

Proof. Comparing Theorems 3.2 and 3.3, from the definition of the parameters we have for $h \to \infty$ the following convergence: $\overline{\gamma^h} \to \overline{\gamma}$, $\overline{\delta^h} \to \overline{\delta}$, and $\overline{\beta^h} \to \overline{\beta}$. From the characteristic function (2.4) one then easily infers that $s_q^h \to s_q$ and, consequently, $(\overline{\gamma^h}\ s_q^h + \overline{\delta^h}) \to (\overline{\gamma}\ s_q + \overline{\delta})$ for $h \to \infty$. Further, by assumption of causality of the model as in (3.2), the sum $\sum_{j=h}^{\infty} \psi_j Z_{t-j} \overset{h\to\infty}{\longrightarrow} 0$ almost surely. This proves the corollary. □

For computational purposes, in the relevant situation of our ARMA(1,2) model, it is useful to note the following formulation of the conditional quantile.

Corollary 3.5. *Let $(Y_t)_{t\in\mathbb{Z}}$ be the solution of a causal and invertible $ARMA(1,2)$ model as given in (3.1), whose innovations $(Z_t)_{t\in\mathbb{Z}}$ are iid $\mathcal{S}(\alpha,\beta,\gamma,\delta)$ distributed. Then $y_q^h(t)$, the conditional q-quantile of step h at time t, is given by*

$$\begin{cases} y_q^1(t) = \overline{\gamma^h}\ s_q^1 + \overline{\delta^h} + \phi_1 Y_{t-1} + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} & \text{for } q \in (0,1)\,,\ h = 1\,, \\ y_q^h(t) = \overline{\gamma^h}\ s_q^h + \overline{\delta^h} + \phi_1^h Y_{t-h} + (\phi_1^{h-1}\theta_1 + \phi_1^{h-2}\theta_2) Z_{t-h} + \phi_1^{h-1}\theta_2 Z_{t-h-1} & \text{for } q \in (0,1)\,,\ h \ge 2\,. \end{cases} \tag{3.10}$$

Here, the quantile s_q^h and the parameters $\overline{\gamma^h}$ and $\overline{\delta^h}$ are as in Theorem 3.3.

Proof. From (3.6) it follows that

$$\psi_{h+1} = \begin{cases} \phi_1 \psi_h + \theta_1 & \text{for } h = 0\,, \\ \phi_1 \psi_h + \theta_2 & \text{for } h = 1\,, \\ \phi_1 \psi_h & \text{for } h \ge 2\,, \end{cases} \tag{3.11}$$

and we get

$$\sum_{j=h}^{\infty} \psi_j Z_{t-j} = \begin{cases} \phi_1^h \sum\limits_{j=0}^{\infty} \psi_j Z_{t-h-j} + \theta_1 Z_{t-h} + \theta_2 Z_{t-h-1} & \text{for } h = 1\,, \\ \phi_1^h \sum\limits_{j=0}^{\infty} \psi_j Z_{t-h-j} + (\phi_1^{h-1}\theta_1 + \phi_2^{h-2}\theta_2) Z_{t-h} + \phi_1^{h-1}\theta_2 Z_{t-h-1} & \text{for } h \geq 2\,. \end{cases}$$

By (3.2) we have $\sum\limits_{j=0}^{\infty} \psi_j Z_{t-h-j} = Y_{t-h}$, which proves the Corollary. □

Once we have computed the first conditional quantile, we can use the following recursive relation for the computation of further quantiles.

Lemma 3.6. *The conditional q-quantile $y_q^h(t+1)$ at time $t+1$ can be expressed in terms of the conditional q-quantile $y_q^h(t)$ at time t as follows:*

$$\begin{cases} y_q^1(t+1) = (1-\phi_1)(\overline{\gamma^h}\, s_q^1 + \overline{\delta^h}) + \phi_1\, y_q^1(t) + \psi_1\, Z_t + \theta_2\, Z_{t-1} & \textit{for } h = 1\,, \\ y_q^h(t+1) = (1-\phi_1)(\overline{\gamma^h}\, s_q^h + \overline{\delta^h}) + \phi_1\, y_q^h(t) + \psi_h\, Z_{t+1-h} & \textit{for } h \geq 2\,, \end{cases}$$

where by the model specification (3.1), Z_{t+1-h} is obtained recursively in terms of the observed Y_{t+1-h}, Y_{t-h} and previous innovations Z_{t-h}, Z_{t-1-h} as

$$Z_{t+1-h} = Y_{t+1-h} - \phi_1 Y_{t-h} - \theta_1 Z_{t-h} - \theta_2 Z_{t-1-h}\,.$$

Proof. We have, using the form (3.9) of the conditional quantile, that

$$\begin{aligned} y_q^h(t+1) =& \overline{\gamma^h}\, s_q^h + \overline{\delta^h} + \sum_{j=h}^{\infty} \psi_j Z_{t+1-j} \\ =& \overline{\gamma^h}\, s_q^h + \overline{\delta^h} + \sum_{j=h}^{\infty} \psi_j Z_{t-j} + \sum_{j=h}^{\infty} \psi_j (Z_{t+1-j} - Z_{t-j}) \\ =& y_q^h(t) + \psi_h Z_{t+1-h} + \sum_{j=h}^{\infty} (\psi_{j+1} - \psi_j) Z_{t-j}\,. \end{aligned} \tag{3.12}$$

Invoking (3.11) we get for $h = 1$

$$\begin{aligned} \sum_{j=1}^{\infty} (\psi_{j+1} - \psi_j) Z_{t-j} =& (\phi_1 - 1) \sum_{j=1}^{\infty} \psi_j Z_{t-j} + \theta_2 Z_{t-1} \\ =& (\phi_1 - 1)(y_q^h(t) - \overline{\gamma^h}\, s_q^1 - \overline{\delta^h}) + \theta_2 Z_{t-1}\,, \end{aligned}$$

and for $h \geq 2$

$$\begin{aligned} \sum_{j=h}^{\infty} (\psi_{j+1} - \psi_j) Z_{t-j} =& (\phi_1 - 1) \sum_{j=h}^{\infty} \psi_j Z_{t-j} \\ =& (\phi_1 - 1)(y_q^h(t) - \overline{\gamma^h}\, s_q^h - \overline{\delta^h})\,. \end{aligned}$$

Substituting in (3.12) yields the result. □

We now consider the estimated conditional quantiles $\widehat{y}_q^1(t)$ of step $h = 1$ of our data, assuming that $\Lambda(t)$ is known for all $t \geq 0$. To this end we plug in the estimated parameters $\widehat{\phi}_1, \widehat{\theta}_1, \widehat{\theta}_2$ and residuals $\widehat{Z}_t$ in (3.10) and obtain

$$\widehat{y}_q^1(t) = s_q^1 + \widehat{\phi}_1 Y_{t-1} + \widehat{\theta}_1 \widehat{Z}_{t-1} + \widehat{\theta}_2 \widehat{Z}_{t-2}\,, \tag{3.13}$$

where s_q^1 is the q-quantile of a $\mathcal{S}(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta})$ distributed random variable. The estimates $\widehat{Z}_t$ are obtained via the invertibility of the model (cf. Proposition 3.1) and are given by

$$\widehat{Z}_s := \sum_{j=0}^{\infty} \widehat{\pi}_j Y_{s-j}\,, \quad s \in \mathbb{Z}\,.$$

Here the estimates $\widehat{\pi}_j$ are constructed by plugging in $\widehat{\phi}_1, \widehat{\theta}_1, \widehat{\theta}_2$ in (3.7). Note that after computing (3.13) once, we can use Lemma 3.6 to determine the conditional quantiles recursively for the next time steps.

The conditional quantiles for $q = 0.95$ and $q = 0.99$ can be seen in figure 10.

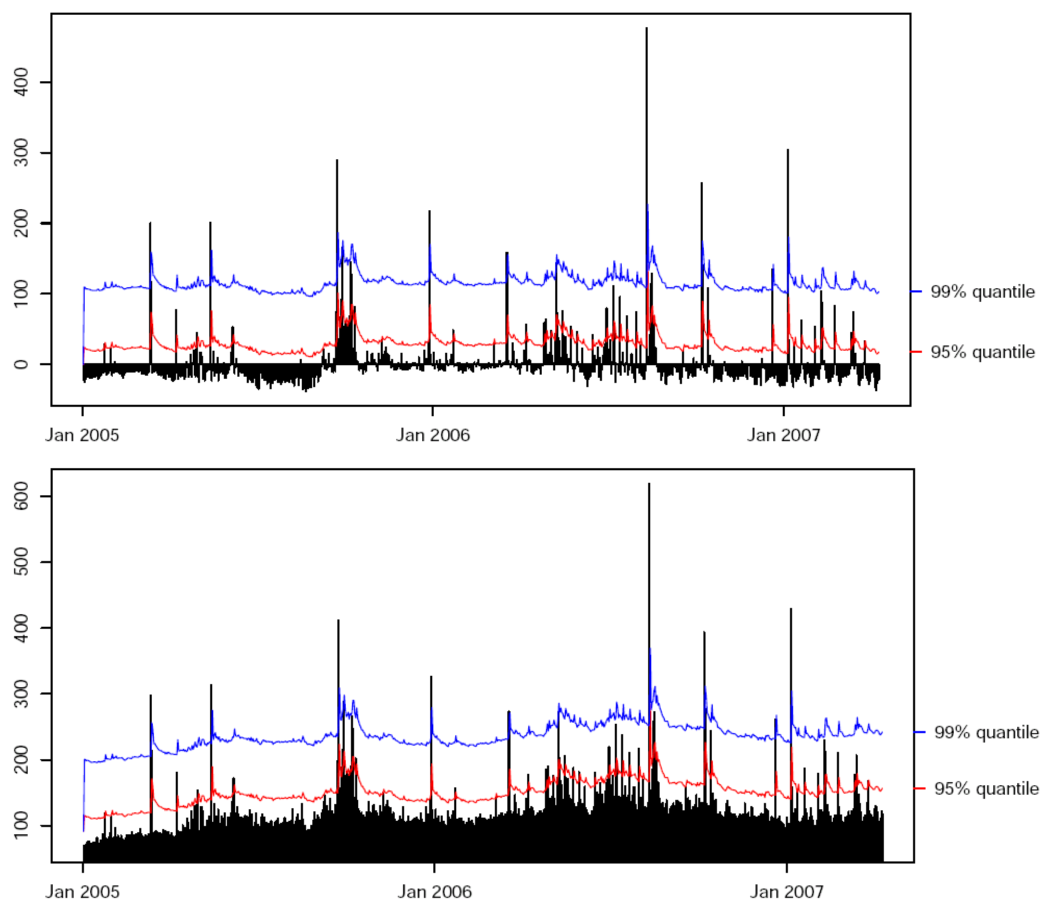

Figure 10: Conditional quantiles for $q = 0.95$ and $q = 0.99$. Top: stationary time series, bottom: original data. For a simple backtesting procedure we count the number of exceedances of the estimated quantiles. There are 57 exceedances for the 95%-quantile and 11 for the 99%-quantile. Thus both quantiles are underestimated.

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