On the distribution tail of an integrated risk model: a numerical approach

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Abstract

We consider an insurance risk process with the possibility to invest the capital reserve into a portfolio consisting of risky assets and a riskless asset. The stock price is modeled by an exponential Lévy process and the riskless interest rate is assumed to be constant. We aim at the risk assessment of the integrated risk process in terms of a high quantile or the far out distribution tail. We indicate an application to an optimal investment strategy of an insurer.

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1 Introduction

In the paper Klüppelberg and Kostadinova [10] the integrated risk model (1.1) was introduced and its properties were investigated. The main focus of the paper was on asymptotic tail estimation, aiming at a proper risk assessment of the model in terms of such downside risk measures as Value-at-Risk, expected shortfall and others. The asymptotic results of this paper were applied to find an optimal investment strategy in Kostadinova [11, 12]. In the present paper we want to apply numerical methods to find the distribution tail. Given these, one can easily find the optimal portfolio, maximizing the expected wealth subject to a risk bound given in terms of a Value-at-Risk.

We first recall the model under consideration. For $\theta \in [0,1]$ and the investment process $(X_\theta(t))_{t\geq 0}$ we define the *integrated risk process* with the dynamics

$$dU_\theta(t) = c\,dt - dS(t) + U_\theta(t-)\,\frac{dX_\theta(t)}{X_\theta(t-)}\,, \quad t>0\,, \quad U_\theta(0)=u\,,$$

which has the solution (see Lemma 2.2 in [10])

$$U_\theta(t) = e^{L_\theta(t)}\left(u + \int_{(0,t]} e^{-L_\theta(v)}\,(cdv - dS(v))\right)\,, \quad t \geq 0\,, \tag{1.1}$$

where $u>0$ is the initial capital, $c>0$ the premium rate, and $S(t)=\sum_{j=1}^{N(t)} Y_j$, $t\geq 0$, is a compound Poisson process with Poisson intensity $\lambda>0$ and positive claims represented by a generic random variable Y with finite mean μ, modelling the total claim amount at time t. The stochastic process $(e^{L_\theta(t)})_{t\geq 0}$, independent of $(S(t))_{t\geq 0}$, is the stock price process for the mixed investment strategy. More precisely, we assume investment into a Black-Scholes type market consisting of a *riskless bond* and a *risky stock* which follows an exponential Lévy process. Their respective prices $(X_0(t))_{t\geq 0}$ and $(X_1(t))_{t\geq 0}$ follow the equations

$$X_0(t) = e^{\delta t} \quad \text{and} \quad X_1(t) = e^{L(t)}\,, \quad t \geq 0.$$

The constant $\delta>0$ is the *riskless interest rate* and L is a Lévy process with characteristic triplet (γ,σ^2,ν). For $\theta\in[0,1]$ we define $(X_\theta(t))_{t\geq 0}$ to be the *investment process,* controlled by the *constant mix strategy* θ, which follows the dynamics

$$dX_\theta(t) \;=\; X_\theta(t-)\left((1-\theta)\delta\,dt + \theta d\widehat{L}(t)\right)\,, \; t\geq 0\,, \; X_\theta(0)=1. \tag{1.2}$$

Here $\widehat{L}$ is such that $e^{L(t)} = \mathcal{E}(\widehat{L}(t))$ and $\mathcal{E}$ denotes the stochastic exponential of a process. Using Itô's formula, the solution to this SDE is

$$X_\theta(t) = e^{(1-\theta)\delta t}\mathcal{E}(\theta\widehat{L}(t)) = e^{L_\theta(t)}\,, \quad t\geq 0\,,$$

where the Lévy process L_θ has characteristic triplet $(\gamma_\theta, \sigma_\theta^2, \nu_\theta)$. For details on the model we refer to [10] or [3], for background on Lévy processes to Cont and Tankov [1], Sato [16] or Schoutens [17] .

Such models have been considered by various authors for geometric Brownian motion as stock price process. As the ruin probability is a prominent risk measure in insurance, various papers consider the ruin problem for such models; see e.g. Hipp and Plum [9], Gaier [7], Frolova, Kabanov and Pergamenchtchikov [5] and Paulsen [13].

The problem we solve in this paper is triggered by a portfolio optimization problem: determine the optimal investment with respect to the maximal expected wealth subject to a bound on the risk, where we measure risk in terms of a high quantile of an appropriate risk process.

Our paper is organised as follows. We start with a motivating example in Section 2. In Section 3 we derive the partial integro-differential equation (PIDE), which we solve then for the special case of a jump diffusion investment process in Section 4. Finally, we show our method at work by presenting examples for different investment strategies and summarize our findings.

2 A motivating example

We shall be interested in the *net loss process*, which we define as

$$Q_\theta(t) = \int_{(0,t]} e^{L_\theta(t) - L_\theta(v)} \left(dS(v) - c\,dv\right), \quad t \geq 0 . \tag{2.1}$$

Note that Q_θ does not take the risk reserve into account, but simply calculates the balance sheet of the integrated risk model.

We introduce the Value-at-Risk, which is one of the prominent risk measures in finance.

Definition 2.1. *For a fixed time horizon T, an investment strategy θ and $\alpha \in (0,1)$ (typically very small) we define the* Value-at-Risk *by*

$$\mathrm{VaR}(T, \theta, \alpha) = \inf\{x \in \mathbb{R} \,:\, P(Q_\theta(T) > x) \leq \alpha\} .$$

The following optimization problem is typical:

$$\max_{\theta \in [0,1]} E[U_\theta(T)] \quad \text{subject to} \quad \mathrm{VaR}(T, \theta, \alpha) \leq C \tag{2.2}$$

for some given risk bound $C > 0$. Of course, analogous problems can be formulated for other downside risk measures such as the expected shortfall (ES) and the semivariance. Such optimization problems have been considered in a purely financial context by Emmer, Klüppelberg and Korn [4], Gabih, Grecksch and Wunderlich [6], and by

Basak and Shapiro [2] for a geometric Brownian motion price process, and by Emmer and Klüppelberg [3] for a general exponential Lévy market.

The following result is a consequence of Lemma 3.2(a) in [10] and the independence of investment and insurance risk processes.

Proposition 2.2. [Kostadinova [12], Lemma 2.6]
Let $\lambda > 0$ be the intensity of the claim arrival process and $E[Y] = \mu < \infty$. Assume that the net profit condition $c > \lambda\mu$ holds and that the investment process and the parameters satisfy

$$\delta < \ln E[e^{L(1)}] < \infty\,.$$

Then $E[U_\theta(T)]$ is increasing in θ.

With this result we obtain the following optimization problem, equivalent to (2.2):

$$\max_{\theta\in[0,1]} \mathrm{VaR}(T,\theta,\alpha) \le C\,. \tag{2.3}$$

In this paper we prepare the numerical solution of this problem. This means that we provide numerical approximations of $\mathrm{VaR}(T,\theta,\alpha)$ for small $\alpha > 0$. Given these, we can simply read off the optimal value for θ. Our foremost goal in this paper is to derive a partial integro-differential equation (PIDE) for the tail probability $P(Q_\theta(T) > x)$ for positive x, which we can then solve numerically within the relevant region. Such analytic methods have been applied in insurance mathematics to derive explicit impressions for the ruin probability; examples can be found in Rolski et al. [15], e.g. in Section 5.3.

3 Derivation of the PIDE

Denote by Y a typical random claim size. The following is the main result of our paper.

Theorem 3.1. *Define*

$$H(x,t) = P(Q_\theta(t) > x)\,, \quad t \ge 0\,,\, x \in \mathbb{R}, \tag{3.1}$$

and assume that the following conditions hold.

(1) The Lévy measure ν of L satisfies $\int_{|x|>1} e^{2|x|}\nu(dx) < \infty$.

(2) The partial derivative $\partial_t H(x,\cdot)$ exists for each $x \in \mathbb{R}$.

(3) For each $t > 0$, the first and second partial derivatives $\partial_x H(\cdot,t)$ and $\partial_{xx} H(\cdot,t)$ exist, and are continuous and bounded, on $\mathbb{R}$.

Then H is the solution to the PIDE

$$\begin{aligned}
&\partial_t H(x,t) - \lambda\left(EH(x-Y,t) - H(x,t)\right) \\
&= \frac{\sigma_\theta^2}{2}(\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)x) - \gamma_\theta \partial_x H(x,t)x \\
&\quad + \int \left(H(xe^z,t) - H(x,t) - z\partial_x H(x,t)x1_{\{|z|\le 1\}}\right)\nu_\theta(-dz) + c\partial_x H(x,t) \qquad (3.2)
\end{aligned}$$

with boundary condition $H(\cdot,0) = \mathbf{1}_{(-\infty,0)}(\cdot)$.

Proof. For fixed $x \in \mathbb{R}$ and $t > 0$, take $s > 0$ small and consider the probability

$$P\left(Q_\theta(t+s) > x\right) = P\left(\int_{(0,t+s]} e^{L_\theta(t+s)-L_\theta(v)}\left(dS(v) - c\,dv\right) > x\right). \qquad (3.3)$$

We introduce the process $(\overline{L}_\theta(v))_{v\ge 0} := (L_\theta(t+v) - L_\theta(t))_{v\ge 0}$. Due to the independent increments property of Lévy processes, $(\overline{L}_\theta(v))_{v\ge 0}$ is an independent copy of L_θ, independent of $\mathcal{F}_t$, where $(\mathcal{F}_t)_{t\ge 0}$ is the filtration generated by $(X_\theta(t))_{t\ge 0}$. By definition we have

$$\begin{aligned}
Q_\theta(t+s) &= \int_{(0,t+s]} e^{L_\theta(t+s)-L_\theta(v)}\left(dS(v) - c\,dv\right) \\
&= e^{\overline{L}_\theta(s)}\left(\int_{(0,t]} + \int_{(t,t+s]}\right) e^{-(L_\theta(v)-L_\theta(t))}\left(dS(v) - c\,dv\right) \\
&= e^{\overline{L}_\theta(s)}\left(Q_\theta(t) + \int_{(0,s]} e^{-\overline{L}_\theta(u)}\left(d\overline{S}(u) - c\,du\right)\right),
\end{aligned}$$

where in the last line we have set $u = v - t$. Furthermore, we have denoted by $\overline{S}(u) := S(t+u) - S(t)$ the sum of the claims in the interval $(t, t+u]$.

In order to derive a formula for the tail of $Q_\theta(t+s)$ we condition on the number of claims in a small interval $(t, t+s]$. From the total probability formula we have

$$\begin{aligned}
P\left(Q_\theta(t+s) > x\right) &= (1 - \lambda s + o(s))P\left(Q_\theta(t+s) > x \,|\, N(t+s) = N(t)\right) \qquad (3.4) \\
&\quad + \lambda s P\left(Q_\theta(t+s) > x \,|\, N(t+s) = N(t)+1\right) + o(s),
\end{aligned}$$

where $P(N(t+s) = N(t)) = P(\text{no claims in } (t,t+s]) = e^{-\lambda s} = 1 - \lambda s + o(s)$. Consider first the case with no claims in $(t, t+s]$. Conditionally on this, $\int_{(t,t+s]} e^{-\overline{L}_\theta(u)}\,d\overline{S}(u) = 0$, and $\overline{L}_\theta$ is independent of $\mathcal{F}_t$, so we get

$$\begin{aligned}
I_0(s) &:= P\left(Q_\theta(t+s) > x \,|\, N(t+s) = N(t)\right) \\
&= P\left(e^{\overline{L}_\theta(s)}\left(Q_\theta(t) - c\int_{[0,s)} e^{-\overline{L}_\theta(u)}\,du\right) > x\right).
\end{aligned}$$

If there is one claim in the interval $(t, t+s]$, we have that $\int_{(t,t+s]} e^{-\overline{L}_\theta(u)}\,d\overline{S}(u) = Ye^{-\overline{L}_\theta(\overline{T})}$, where $\overline{T}$ is the jump time and Y the jump size of $\overline{S}$ in $(0, s]$. As the Poisson process $\overline{S}$ is independent of $\mathcal{F}_t$, so are $\overline{T}$ and Y. Moreover, due to the order statistics property of the Poisson process, the r.v. $\big(\overline{T}\,|\,\overline{T}\in(t,t+s]\big)\stackrel{d}{=}U_1$ is uniformly distributed in the interval $[0,s]$. Hence

$$\begin{aligned}
I_1(s) \;&:=\; P\left(Q_\theta(t+s)>x\,|\,N(t+s)=N(t)+1\right)\\
&= P\left(e^{\overline{L}_\theta(s)}\left(Q_\theta(t)+Ye^{-\overline{L}_\theta(\overline{T})}-c\int_{[0,s)}e^{-\overline{L}_\theta(u)}\,du\right)>x\;\big|\,\overline{T}\in(0,s]\right)\\
&= P\left(e^{\overline{L}_\theta(s)}\left(Q_\theta(t)+Ye^{-\overline{L}_\theta(U_1)}-c\int_{[0,s)}e^{-\overline{L}_\theta(u)}\,du\right)>x\right).
\end{aligned}$$

Now we want to study equation (3.4) for $s\to0$, which we rewrite as

$$\frac{P\left(Q_\theta(t+s)>x\right)-P\left(Q_\theta(t)>x\right)}{s}=\lambda I_1(s)-\lambda I_0(s)+\frac{I_0(s)-P\left(Q_\theta(t)>x\right)}{s}+\frac{o(s)}{s}\,.$$

We have

(i) $\lim_{s\to0}I_1(s)=P\left(Q_\theta(t)+Y>x\right)$. Indeed, as a Lévy process is cádlág process, we have that $\lim_{s\to0}\overline{L}_\theta(s)=\overline{L}_\theta(0)=0$ a.s. Also $\lim_{s\to0}\int_{[0,s)}e^{-\overline{L}_\theta(v)}\,dv=0$ a.s. Further we have $U_1\to0$ a.s. when $s\to0$, hence also $\lim_{s\to0}\overline{L}_\theta(U_1)=0$ a.s.

(ii) Similarly as for $I_1(s)$, $\lim_{s\to0}I_0(s)=P\left(Q_\theta(t)>x\right)$.

Since $P\left(Q_\theta(t)>x\right)=H(x,t)$, assuming that the limit below exists, from the equation above we obtain the following partial integro-differential equation (PIDE) for H:

$$\partial_tH(x,t)=\lambda\big(EH(x-Y,t)-H(x,t)\big)+\lim_{s\to0}\frac{1}{s}\big(I_0(s)-H(x,t)\big)\,, \tag{3.5}$$

with boundary condition $H(\cdot,0)=\mathbf{1}_{(-\infty,0)}(\cdot)$.

We now calculate the last term in (3.5). For $s>0$ we have

$$\begin{aligned}
&I_0(s)-H(x,t)\\
=\;&\left(P\left(Q_\theta(t)>xe^{-\overline{L}_\theta(s)}\right)-P(Q_\theta(t)>x)\right)\\
&-\left(P\left(Q_\theta(t)>xe^{-\overline{L}_\theta(s)}\right)-P\left(Q_\theta(t)>xe^{-\overline{L}_\theta(s)}+c\int_{[0,s)}e^{-\overline{L}_\theta(v)}\,dv\right)\right)\\
=:\;&J_1(s)-J_2(s)\,,\ \text{say.}
\end{aligned}\tag{3.6}$$

First consider $J_1(s)$, separately for $x>0$ and $x<0$; note that for $x=0$ we have $J_1\equiv0$. For $x>0$, we set $y=\ln x$ and $g(y)=H(e^y,t)$. Then by the independence of the investment process and the insurance risk process we can write

$$J_1(s)\;=\;EH(xe^{-L_\theta(s)},t)-H(x,t)\;=\;Eg(y-L_\theta(s))-g(y)\,.$$

Under Assumption (3) of Theorem 3.1, $g(y)$ is continuous and bounded, and has continuous and bounded first and second derivatives, for $y \in \mathbb{R}$. So we can apply, e.g., Gihman and Skorohod [8], p. 292, to deduce that

$$\lim_{s\to 0} \frac{1}{s} J_1(s) \;=\; \lim_{s\to 0} \frac{1}{s} \left(Eg(y - L_\theta(s)) - g(y) \right) = \mathcal{A} g(y) \,, \quad y \in \mathbb{R} \,, \tag{3.7}$$

where $\mathcal{A}$ is the infinitesimal generator of the Lévy process $-L_\theta$.

Calculating the partial derivatives implicit in the infinitesimal generator, we can check that (3.7) implies, for $x > 0$,

$$\begin{aligned}\lim_{s\to 0} \frac{1}{s} J_1(s) \;&=\; \frac{\sigma_\theta^2}{2} (\partial_{xx} H(x,t) x^2 + \partial_x H(x,t) x) - \gamma_\theta \partial_x H(x,t) x \\ &\quad + \int \left(H(xe^z, t) - H(x,t) - z \partial_x H(x,t) x 1_{\{|z|\le 1\}} \right) \nu_\theta(-dz) \,. \end{aligned} \tag{3.8}$$

Here and in what follows we always take the integral over the support of the corresponding Lévy measure.

For $x < 0$, we set $y = \ln|x|$ and $g(y) = H(-e^y, t)$. Then an analogous calculation gives (3.8) again.

It remains to estimate the second term in (3.6). Define $R(s) = c \int_{[0,s)} e^{-L_\theta(v)}\, dv$. Then we can write, using a Taylor expansion,

$$\begin{aligned} J_2(s) \;&=\; P\left(Q_\theta(t) > xe^{-L_\theta(s)}\right) - P\left(Q_\theta(t) > xe^{-L_\theta(s)} + c\int_{[0,s)} e^{-L_\theta(v)}\, dv\right) \\ &=\; \int\!\!\int \left(P\left(Q_\theta(t) > xe^{-y}\right) - P\left(Q_\theta(t) > xe^{-y} + r\right)\right) dP(L_\theta(s) \le y, R(s) \le r) \\ &=\; \int\!\!\int \left(H(xe^{-y}, t) - H(xe^{-y} + r, t)\right) dP(L_\theta(s) \le y, R(s) \le r) \\ &=\; -E\left[R(s) \partial_x H(\xi(s), t)\right] \,. \end{aligned} \tag{3.9}$$

Here $\xi(s) \in [xe^{-L_\theta(s)}, xe^{-L_\theta(s)} + R(s)]$.

Now let

$$T(s) := -\frac{1}{s} R(s) \partial_x H(\xi(s), t) \,, \quad s > 0 \,.$$

Then $T(s) \ge 0$ a.s. and $J_2(s)/s = ET(s)$. We have $R(s) \overset{\text{a.s.}}{\to} 0$ and also $R(s)/s \overset{\text{a.s.}}{\to} c$, as $s \to 0$, so $\xi(s) \overset{\text{a.s.}}{\to} x$ and consequently,

$$T := \lim_{s\to 0} T(s) = -\lim_{s\to 0} \frac{1}{s} R(s) \partial_x H(\xi(s), t) = -c \partial_x H(x,t) \text{ a.s.}$$

We will have L_1 convergence and deduce that $\lim_{s\to 0} J_2(s)/s = \lim_{s\to 0} ET(s) = T$ if we show that $(T(s))_{s>0}$ is uniformly integrable as $s \to 0$. To see this, recall Assumptions (1) and (3) in Theorem 3.1. Take $\zeta > 0$ and consider

$$E(T(s) 1_{\{T(s) > \zeta\}}) \le \sqrt{E(T^2(s)) P(T(s) > \zeta)}.$$

Let $K_t = \sup_{x\in\mathbb{R}}(-\partial_x H(x,t))$, which is finite by Assumption (3). Now by Theorem 25.3 of Sato [16] and Lemma A.1 of Klüppelberg and Kostadinova [10], $\int_{|x|>1} e^{2|x|}\nu(dx) < \infty$ implies that $Ee^{-2L_\theta(u)} < \infty$ for all $u \in \mathbb{R}$. Using the notation $Ee^{-sL_\theta(t)} = e^{-t\Psi_\theta(s)}$ for $t \geq 0$ and for all $s \in \mathbb{R}$ such that the expectation is finite, we conclude

$$\begin{aligned} E[T^2(s)] &\leq \frac{c^2K_t^2}{s^2} E\left(\int_0^s e^{-L_\theta(v)}dv\right)^2 \\ &= \frac{2c^2K_t^2}{s^2} E\int_0^s\int_u^s e^{-(L_\theta(v)-L_\theta(u))}e^{-2L_\theta(u)}dvdu \\ &= \frac{2c^2K_t^2}{s^2}\int_0^s\int_u^s e^{-(v-u)\Psi_\theta(1)}e^{-u\Psi_\theta(2)}dvdu \\ &\leq \frac{2c^2K_t^2}{s^2}\left(\int_0^s e^{u(\Psi_\theta(1)-\Psi_\theta(2))}du\right)\left(\int_0^s e^{-v\Psi_\theta(1)}dv\right) \\ &= O(1)\,, \quad s \to 0\,. \end{aligned}$$

Since $\lim_{s\to 0} P(T(s) > \zeta) = P(-c\partial_x H(x,t) > \zeta)$ equals 0 for large enough ζ, $(T(s))_{s>0}$ is uniformly integrable, as asserted, and it follows that $T(s) \xrightarrow{L_1} T$ as $s \to 0$. Hence, via (3.9), the second term of (3.6) tends to $c\partial_x H(x,t)$ a.s. as $s \to 0$.

Plugging this into (3.5), we obtain (3.2). □

4 Jump diffusion investment model

In this case

$$L(t) = \gamma\, t + \sigma W(t) + \sum_{j=1}^{M(t)} Z_j\,, \quad t \geq 0\,, \tag{4.1}$$

for $\gamma \in \mathbb{R}$, $\sigma > 0$ and Z_j i.i.d., independent of a Poisson process M with intensity $\eta > 0$ and W a Brownian motion independent of the compound Poisson process. The process L_θ has a similar representation given by

$$L_\theta(t) = \gamma_\theta\, t + \sigma_\theta W(t) + \sum_{j=1}^{M(t)} Z_j^{(\theta)}\,, \quad t \geq 0\,, \tag{4.2}$$

for $\gamma_\theta = \delta + \theta(\gamma - \delta - \sigma^2/2)$, $\sigma_\theta = \theta\sigma$ and $Z_j^{(\theta)} = \ln(1 + \theta(e^{Z_j} - 1))$ i.i.d., independent of the Poisson process M. This means that the Lévy measure $\nu(z) = \eta P(Z \leq z) = \eta F_Z(z)$ of L is transformed into

$$\nu_\theta(z) = \eta P(\ln(1 + \theta(e^Z - 1)) \leq z) = \eta P(Z^{(\theta)} \leq z) = \eta F_Z(\ln(1 + (e^z - 1)/\theta)). \tag{4.3}$$

Recall that L and L_θ jump at the same time and that a jump of size Z of L leads to a jump of size $\ln(1 + \theta(e^Z - 1)) > \ln(1 - \theta)$ of L_θ.

In this case, it is not necessary to compensate the small jumps in the Lévy-Khinchine representation and the PIDE in (3.2) reduces to

$$\begin{aligned} &\partial_t H(x,t) - \lambda\left(EH(x-Y,t) - H(x,t)\right) \\ &= \frac{\sigma_\theta^2}{2}(\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)x) - \gamma_\theta \partial_x H(x,t)x \\ &\quad + \int H(xe^z,t)\nu_\theta(-dz) - \eta H(x,t) + c\partial_x H(x,t)\,. \end{aligned} \tag{4.4}$$

We can rewrite this as

$$\begin{aligned} &\partial_t H(x,t) - \lambda EH(x-Y,t) + (\lambda+\eta)H(x,t) \\ &= \frac{\sigma_\theta^2}{2}\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)((\frac{\sigma_\theta^2}{2} - \gamma_\theta)x + c) + \int H(xe^z,t)\nu_\theta(-dz)\,. \end{aligned}$$

This formula further simplifies, since

$$\int H(xe^z,t)\nu_\theta(-dz) = \eta\int H(xe^{-z},t)F_\theta(dz) = \eta EH(xe^{-Z^{(\theta)}},t)\,.$$

Then

$$\begin{aligned} &\partial_t H(x,t) - \lambda EH(x-Y,t) + (\lambda+\eta)H(x,t) \\ &= \frac{\sigma_\theta^2}{2}\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)\left((\frac{\sigma_\theta^2}{2} - \gamma_\theta)\,x + c\right) + \eta EH(xe^{-Z^{(\theta)}},t)\,. \end{aligned} \tag{4.5}$$

Numerical solution

For the PIDE (4.5) we present a numerical solution using a finite difference (FD) method.

Let us first emphasize that it is not known *a priori* whether a sufficiently smooth (or classical) solution exists; for more details on existence and uniqueness see Seydel [18].

For a numerical solution, we shall assume that the insurance claim Y and the market jump $Z^{(\theta)}$ are absolutely continuous with densities f_Y and f_θ, respectively. By (4.3) we can express f_θ in terms of the density f of a market jump Z of L:

$$f_\theta(z) = \begin{cases} f\left(\ln\left(1 + (e^z-1)/\theta\right)\right)\dfrac{e^z}{e^z-1+\theta}, & z > \ln(1-\theta)\,, \\ 0\,, & z \le \ln(1-\theta)\,. \end{cases}$$

Rewriting (4.5) we have to solve the following initial value problem:

$$\begin{aligned} &\partial_t H(x,t) - \lambda\int_{-\infty}^{x} f_Y(x-y)H(y,t)dy + (\lambda+\eta)H(x,t) \\ &= \frac{\sigma_\theta^2}{2}\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)\left((\frac{\sigma_\theta^2}{2} - \gamma_\theta)\,x + c\right) + \eta\int_{\ln(1-\theta)}^{\infty} f_\theta(z)H(xe^{-z},t)dz\,, \end{aligned} \tag{4.6}$$

with the initial condition $H(\cdot,0) = 1_{(-\infty,0)}$. With a further substitution $u = xe^{-z}$ in the market jump integral, we are able to apply numerical schemes to our problem.

The basic idea is to apply the FD method as for a standard initial value problem (or parabolic PDE). That is, we discretize the derivatives using standard finite differences. For the integrals (they integrate across space for a constant time), we substitute an integration formula, for instance the composite trapezoidal rule (a formula that is of order 2). For stability considerations, we discretize in time such that we obtain an implicit numerical scheme.

The infinite domains of integration require specific numerical treatment. We restrict the computation to the domain $(-R,R) \times (0,T)$ for some $R > 0$. We use boundary conditions $H(-R,t) = 1$ and $H(R,t) = 0$ and approximate those parts of the integrals in (4.6) outside $(-R,R)$ for $x > 0$ by

$$\int_{-\infty}^{-R} f_Y(x-y)H(y,t)dy \approx \overline{F}_Y(x+R)$$

and for $x < 0$ and $-R > x/(1-\theta)$ (where we interpret $x/(1-\theta) = -\infty$ for $\theta = 1$) by

$$\int_{x/(1-\theta)}^{-R} -f_\theta(\ln(x/u))\frac{1}{u}H(u,t)du \approx \int_{\ln(1-\theta)}^{\ln(-x/R)} f_\theta(u)du = F\left(\ln\left(1+(-\frac{x}{R}-1)/\theta\right)\right) .$$

The localization error can be easily derived; see [18] for details.

The result of this discretization is a sequence of linear systems $AH^{(i+1)} = H^{(i)} + b$, $i = 0,\dots,n$ for some $n \in \mathbb{N}$ with $H^{(0)} = 1_{(-\infty,0)}$. In contrast to an ordinary parabolic PDE, A is not a sparse but a dense matrix filled with entries from the two integrals.

Further details and extensions of the method (for instance an improved method of order 2 using a BDF2 discretization in time) can be found in Seydel [18]. We computed the illustrative results of Figure 1 using this improved FD method, comparing it with the results of a Monte Carlo simulation for verification. The model parameters are given in the caption. Depicted are numerical approximations for $P(Q_\theta(T) > x)$ and the corresponding Monte Carlo estimates for different investment parameters θ in the left plots and the corresponding absolute errors in the right plots. As expected the numerical method has its largest errors around 0 due to the initial condition. The errors decrease for positive x much faster than for negative x. Moreover, the method shows higher accuracy, when the investment into risky stock is not too small. On the other hand, the error decreases faster for smaller θ. The approximation becomes for all θ very good in the far out tails. As we are interested in the right tail $P(Q_\theta(T) > x)$ for large $x > 0$, we find the approximation very convincing.

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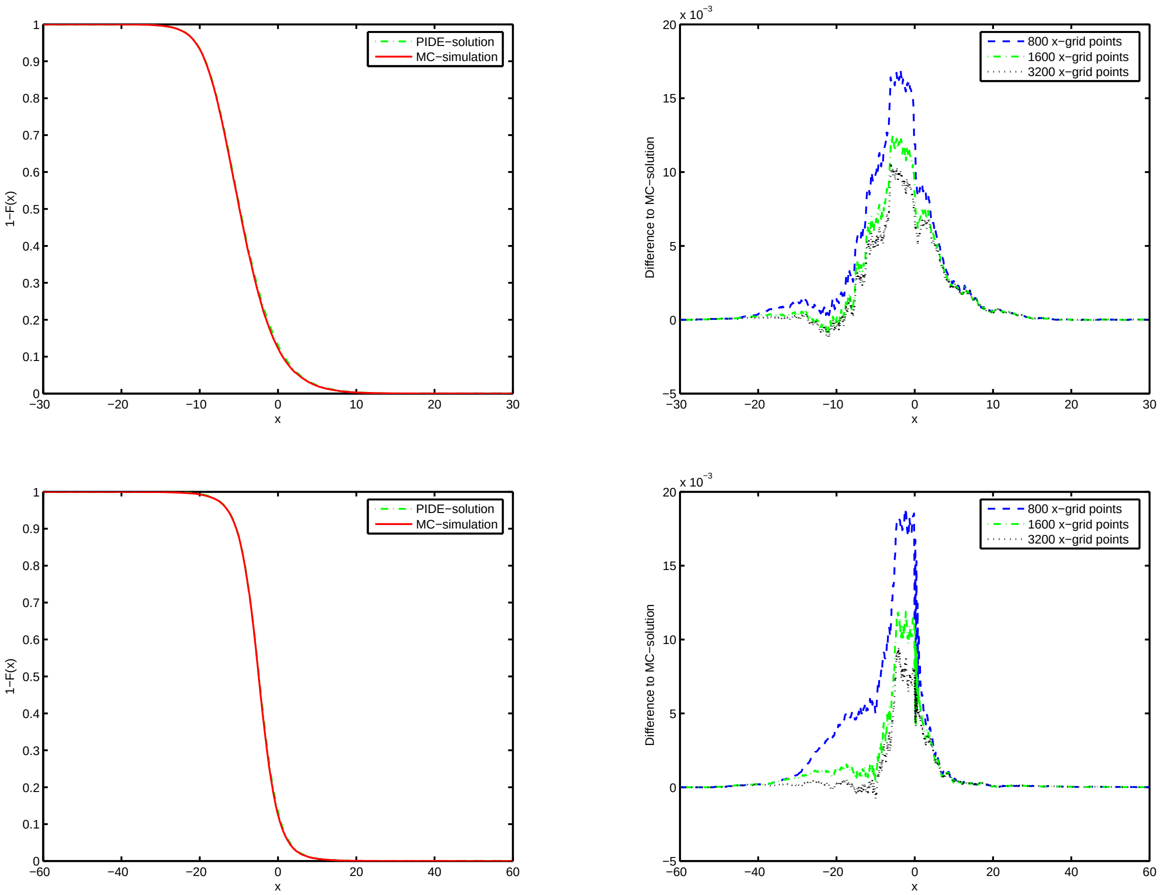

Figure 1: Numerical solution of $H(\cdot,T) = P(Q_\theta(T) > \cdot)$ for $T = 1$ in comparison to a Monte Carlo simulation (left: both solutions plotted, right: difference of both solutions) for three values of θ (first line: $\theta = 0.1$, middle line: $\theta = 0.5$ and last line: $\theta = 0.9$. The following set of parameters has been used. Insurance model: premium rate $c = 10$, standard exponential claim size Y, claims intensity $\lambda = 5$. Investment model: $\gamma = 0.2$, $\sigma = 0.4$, the jump intensity is $\eta = 3$, a jump Z is centered normal with variance 0.09. For the finite difference method we have used 800 x-grid points and 100 t-grid points.