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A Sequential Reduction Scheme in Reduced Order Modelling of Large Scale Systems

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Abstract

An approximate solution for Lyapunov equations is constructed using Krylov subspaces for the purpose of order reduction. It is proved that order reduction by low rank gramians calculated in this way and used for balancing and truncation is equivalent to applying moment matching by Krylov subspace methods to the original large scale system. It is also shown that some of the Hankel singular values of the large scale system are approximated by the Hankel singular values of the reduced model. A two step order reduction is then proposed where a Krylov subspace method and truncated balanced realization (TBR) are applied, sequentially. This improves the result of Krylov subspace method while being computationally much cheaper than TBR.

Keywords: Model Order Reduction; Balancing and Truncation; Lyapunov Equations; Krylov Subspaces

1 Introduction

In recent years, in circuit simulation, Micro-Electro-Mechanical systems and other fields, the computer supported modelling leads to accurate but complicated systems, mostly described by high dimensional differential equations. Simulation and analysis of such large scale models are complicated, the computational time is stretched and numerical errors may lead to incorrect results. Therefore, approximating large scale systems with a smaller dimension model is a demanding problem today.

A well-accepted method in order reduction of Linear Time Invariant (LTI) systems is truncated balanced realization (TBR) [8, 18, 21]. TBR is based on finding a similarity transformation which transforms a system to a balanced form in which the controllability and observability gramians are equal and diagonal. To find gramians of a system, two corresponding Lyapunov equations are to be solved. Another option to balance a system is using a cross gramian approach where solving a Sylvester equation is involved [26]. While TBR is theoretically attractive and yields to excellent approximate models in practice, its use for model reduction of large scale systems is costly by growing computational complexity in solving two Lyapunov equations or a Sylvester equation.

Besides TBR, *moment matching using Krylov subspaces* is one of the best methods in order reduction of large scale systems [5, 6, 7, 9, 22, 27]. This approach is based in matching some of

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the first coefficients of the Taylor series expansion of the original and reduced transfer functions around a suitable point. Such a reduced model can be calculated using Krylov subspaces by means of Arnoldi, Lanczos [6] or two-sided Arnoldi [4, 25] algorithms in a relatively short time compared to TBR. The most important advantages of this method are its simplicity and numerical accuracy, while there is no general error bound and no general guaranty to preserve stability. In practice, a Krylov subspace method does not lead to reduced systems with better accuracy than TBR.

In this context, combining the preceding approaches to take advantages of all nice properties of them is a challenging problem. The aim would be finding a numerically reliable method which can be applied for the reduction of large scale systems with almost the same accuracy as TBR.

One option to achieve this goal is to implement some algorithms that can solve large Lyapunov or Sylvester equations, finding the so called low rank gramians. Low rank gramians are used for approximate balancing and truncation. A lot of research has been done to solve large Lyapunov [1, 10, 12, 16, 17, 19, 20, 23] and Sylvester equations [2, 11, 26] and Krylov subspaces have been extensively used. The result of the reduction procedure by approximate TBR, depending on the solving iterations, can be close to the exact TBR.

In this paper, we propose another approach to find a reduced order system close to exact TBR. It is based on applying the so called sequential reduction scheme. The excellent performance of the sequential approach was also observed in [13], where it was investigated empirically. This approach finds a more accurate reduced system compared to the Krylov subspace methods and is much faster than TBR. In practice, the accuracy of the proposed method is very close to TBR with much less numerical effort and the method can easily be applied to reduce large scale systems.

In the following, first we show a connection between approximate TBR and Krylov subspace methods. By means of Galerkin conditions and Krylov subspaces, two Lyapunov equations are solved to find approximate gramians of a large scale system. *We show that using these approximate solutions for order reduction by TBR leads to exactly the same reduced model as applying moment matching using Krylov subspace methods.*

One result of this equivalence is that the Hankel singular values of the reduced system by a Krylov subspace method are approximations of the Hankel singular values of the original model. It is observed in practice that this approximation is improved by going to higher orders and larger Hankel singular values are approximated better.

Another result of the equivalence, is a two-step procedure to find a reduced order model. In the first step, a Krylov subspace method is applied to reduce the order of the original system to the values that TBR works well (a few hundreds) and then by TBR, the result of the previous step is further reduced to go to lower orders.

Similar to the idea in TBR, in sequential reduction approach, it is tried to keep the larger and well-approximated Hankel singular values. Depending on the approximation error of the Hankel singular values, the resulting reduced model can be close to the result of pure TBR which are also confirmed in practice. This gives us a stopping criterion for suitable choice of the reduced orders in both sequential steps.

The paper is organized as follows: In sections 2 and 3, we review known reduction methods using Krylov subspaces and TBR, respectively. Section 4 presents low rank solutions of Lyapunov equations used in balancing and we suggest to use particular Krylov subspaces for the projection

matrices $\mathbf{V}$ and $\mathbf{W}$. In section 5.1, the approximate solution of Lyapunov equations are used for approximate balancing and truncation and the sequential reduction scheme is proposed, which is the main result of the paper. Section 6 presents some remarks on Sylvester equation and invariance properties of the two-step reduction scheme.

2 Reduction by moment matching

We consider the dynamical multi-input multi-output (MIMO) system of the form,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{1}$$

with m inputs, p outputs and n states.

The *Krylov subspace* is defined as,

$$\mathcal{K}_q(\mathbf{A}_1, \mathbf{B}_1) = span\{\mathbf{B}_1, \mathbf{A}_1\mathbf{B}_1, \cdots, \mathbf{A}_1^{q-1}\mathbf{B}_1\}, \tag{2}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n\times n}$, and the columns of the matrix $\mathbf{B}_1 \in \mathbb{R}^{n\times m}$ are called starting vectors. Considering the state space representation (1), two Krylov subspaces $\mathcal{K}_{q_1}(\mathbf{A}^{-1}, \mathbf{A}^{-1}\mathbf{B})$ and $\mathcal{K}_{q_2}(\mathbf{A}^{-T}, \mathbf{A}^{-T}\mathbf{C}^T)$, both with the same rank are called *input* and *output Krylov subspaces*, respectively. The values of q_1 and q_2 must be chosen such that the rank of the subspaces are the same. For the case that the basic vectors are linearly independent, the relationship $mq_1 = pq_2$ must be satisfied.

For the system (1), the so-called *moments* are defined as,

$$\mathbf{m}_i = \mathbf{C}\mathbf{A}^{-i-1}\mathbf{B} \quad , \quad i = 0, 1, \cdots . \tag{3}$$

The $p \times m$ matrices $\mathbf{m}_i$ are the negative coefficients of the Taylor series expansion (around zero) of the system's transfer function [7, 27].

If the mapping

$$\mathbf{x} = \mathbf{V}\mathbf{x}_k, \mathbf{V} \in \mathbb{R}^{n\times q}, \mathbf{x} \in \mathbb{R}^n, \mathbf{x}_k \in \mathbb{R}^q, \tag{4}$$

where $q < n$, is applied to the system (1) and then the state equation is multiplied by the transpose of a matrix $\mathbf{W} \in \mathbb{R}^{n\times q}$, a model of reduced order q is found,

$$\begin{cases} \dot{\mathbf{x}}_k(t) = \mathbf{A}_r\mathbf{x}_k(t) + \mathbf{B}_r\mathbf{u}(t), \\ \mathbf{y}_k = \mathbf{C}_r\mathbf{x}_k(t), \end{cases} \tag{5}$$

where,

$$\mathbf{A}_r = \left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{B}_r = \left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{B}, \quad \mathbf{C}_r = \mathbf{C}\mathbf{V}. \tag{6}$$

In this paper, we consider that $\mathbf{W}^T\mathbf{V}$ is invertible. It can be proved that if $\mathbf{V}$ is a basis of the input Krylov subspace *or* $\mathbf{W}$ is a basis of the output Krylov subspace then the first $\frac{q}{m}$ or $\frac{q}{p}$ moments match, and the method is called a *one-sided Krylov method*. When both, $\mathbf{V}$ *and* $\mathbf{W}$, are bases of the input and output Krylov subspaces respectively, then the first $\frac{q}{m} + \frac{q}{p}$ moments match and the method is called a *two-sided Krylov method* [24, 27]. In one-sided methods, typically $\mathbf{W} = \mathbf{V}$ is chosen.

In Krylov subspace methods, with moment matching around zero, the reduced order model has a good performance at low frequencies. Therefore,

$$\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} \simeq \mathbf{C}\mathbf{V}\left(s\mathbf{I} - \left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\right)^{-1}\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{B}, \tag{7}$$

and the approximation is good for small s.

To compute the matrices $\mathbf{V}$ and $\mathbf{W}$, some suitable algorithms can be applied which are the *Arnoldi algorithm* [7] in the one sided methods and the *Lanczos algorithm* [7] and the *two-sided Arnoldi algorithm* [25].

3 Truncated balanced realization

Suppose that the system (1) is controllable, observable and stable, then there exist unique and positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$ which are solutions of the following Lyapunov equations,

$$\mathbf{AP} + \mathbf{PA}^T + \mathbf{BB}^T = \mathbf{0}, \tag{8}$$
$$\mathbf{A}^T\mathbf{Q} + \mathbf{QA} + \mathbf{C}^T\mathbf{C} = \mathbf{0}. \tag{9}$$

The matrices $\mathbf{P}$ and $\mathbf{Q}$ are called controllability and observability gramians, respectively. It is well known that the solution of these equations can be written as,

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}dt \ , \ \mathbf{Q} = \int_0^\infty e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}t}dt. \tag{10}$$

An important set of parameters for model reduction are *Hankel singular values* [18]. These are defined as the square roots of eigenvalues of the product $\mathbf{PQ}$ and are invariant to change of state space transformation and only depend on the input-output behaviour of the system.

By solving the corresponding Lyapunov equations, it is possible to balance the system i.e. a similarity transformation is applied to the system such that the gramians for the new realization are equal and diagonal. The idea of reduction is to remove some blocks related to the small Hankel singular values in the balanced system in order to keep large and important Hankel singular values. As proposed in [15], this can be done by applying a projection to the original non-balanced system (1) in the following steps:

1. Find the Cholesky factorization of the gramian matrices $\mathbf{P} = \mathbf{L}_c\mathbf{L}_c^T$ and $\mathbf{Q} = \mathbf{L}_o\mathbf{L}_o^T$.
2. Calculate the singular value decomposition of the matrix $\mathbf{L}_o^T\mathbf{L}_c = \mathbf{U}\mathbf{\Sigma}\mathbf{K}^T$, where $\mathbf{\Sigma}$ is a diagonal matrix containing the Hankel singular values of the system in descending order.
3. Find the matrices $\mathbf{U}_r$ and $\mathbf{K}_r$ as the first r columns of the matrices $\mathbf{U}$ and $\mathbf{K}$ and $\mathbf{\Sigma}_r = diag(\sigma_1, \cdots, \sigma_r)$ to form the projection matrices $\mathbf{T} = \mathbf{L}_c\mathbf{K}_r\mathbf{\Sigma}_r^{-\frac{1}{2}}$ and $\mathbf{S} = \mathbf{L}_o\mathbf{U}_r\mathbf{\Sigma}_r^{-\frac{1}{2}}$.
4. Apply the projection to system (1) to find the reduced order model of order r,

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{S}^T\mathbf{AT}\hat{\mathbf{x}}(t) + \mathbf{S}^T\mathbf{Bu}(t), \\ \hat{\mathbf{y}} = \mathbf{CT}\hat{\mathbf{x}}(t). \end{cases} \tag{11}$$

4 Approximate solution to Lyapunov equations

In this section, we establish a low rank solution to the Lyapunov equations to be used in order reduction. First, the Lyapunov equations are solved based on Galerkin type conditions by arbitrary projection matrices $\mathbf{V}$ and $\mathbf{W}$. Then some hints are given to choose $\mathbf{V}$ and $\mathbf{W}$ with respect to order reduction.

4.1 Galerkin projection

Here, we use Galerkin type conditions, directly applied to the Lyapunov equations (8), (9); see also [12, 23]. The goal is to find approximate solutions $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ such that, the $n \times n$ error matrices,

$$\mathbf{R}_c = \mathbf{A}\hat{\mathbf{P}} + \hat{\mathbf{P}}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T \ , \mathbf{R}_o = \mathbf{A}^T\hat{\mathbf{Q}} + \hat{\mathbf{Q}}\mathbf{A} + \mathbf{C}^T\mathbf{C},$$

satisfy the Galerkin conditions,

$$\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{R}_c\mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1} = \mathbf{0} \ , \mathbf{V}^T\mathbf{R}_o\mathbf{V} = \mathbf{0},$$

where $\mathbf{V}$ and $\mathbf{W}$ can be any full rank matrices with q columns such that $\mathbf{W}^T\mathbf{V}$ is nonsingular.

Now, consider the small Lyapunov equations,

$$\mathbf{A}_r\mathbf{P}_r + \mathbf{P}_r\mathbf{A}_r^T + \mathbf{B}_r\mathbf{B}_r^T = \mathbf{0} \ , \mathbf{A}_r^T\mathbf{Q}_r + \mathbf{Q}_r\mathbf{A}_r + \mathbf{C}_r^T\mathbf{C}_r = \mathbf{0}, \tag{12}$$

where, the matrices $\mathbf{A}_r$, $\mathbf{B}_r$ and $\mathbf{C}_r$ are defined in (6) and assuming that $\mathbf{A}_r$ is stable. The solution of equations (12) are the gramians of the reduced system (5). If the solutions of the original equations (10) are approximated by,

$$\hat{\mathbf{P}} = \mathbf{V}\mathbf{P}_r\mathbf{V}^T, \tag{13}$$

$$\hat{\mathbf{Q}} = \mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1}\mathbf{Q}_r\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T, \tag{14}$$

then the corresponding Galerkin conditions are satisfied,

$$\begin{aligned}
&\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{R}_c\mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1} \\
&\quad = \left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\left[\mathbf{A}\mathbf{V}\mathbf{P}_r\mathbf{V}^T + \mathbf{V}\mathbf{P}_r\mathbf{V}^T\mathbf{A}^T + \mathbf{B}\mathbf{B}^T\right]\mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1} \\
&\quad = \left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{P}_r + \mathbf{P}_r\mathbf{V}^T\mathbf{A}^T\mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1} + \left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{B}\mathbf{B}^T\mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1} \\
&\quad = \mathbf{A}_r\mathbf{P}_r + \mathbf{P}_r\mathbf{A}_r^T + \mathbf{B}_r\mathbf{B}^T = \mathbf{0}.
\end{aligned}$$

For the observability gramian, the proof is similar.

For the case $q = n$, the matrices $\mathbf{V}$ and $\mathbf{W}$ are invertible square matrices, the error matrices are zero and exact solutions are found. Furthermore, by choosing a positive definite approximate solution, the error matrices are also positive semi-definite, and if the Cholesky factors of the error matrices are spanned by the projection matrices $\mathbf{V}$ and $\mathbf{W}$, then the error matrix is zero.

4.2 Approximating the explicit solution

In the previous subsection, approximate solutions of Lyapunov equations with arbitrary $\mathbf{V}$ and $\mathbf{W}$ are found such that the errors are constrained by some Galerkin type conditions. The question arises of how to choose $\mathbf{V}$ and $\mathbf{W}$ in practice so that the resulting approximate gramians $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ are useful in order reduction.

In approaches similar to the section 4.1, some authors, e.g. [11, 23], have used $\mathbf{V}$ and $\mathbf{W}$ as bases of Krylov subspaces $\mathcal{K}_{q_1}(\mathbf{A}, \mathbf{B})$ and $\mathcal{K}_{q_2}(\mathbf{A}^T, \mathbf{C}^T)$ with *positive* powers of $\mathbf{A}$ in which the large eigenvalues of $\mathbf{A}$ are dominant. This method leads to good approximation of the time function in the explicit integral solutions (10) for *small* values of t. In order reduction, however the eigenvalues that are far from the imaginary axis (corresponding to fast dynamics) are *less* important. Therefore, to find an approximate solution to Lyapunov equations for the purpose

of order reduction of non-stiff systems, it is more useful to use the subspaces $\mathcal{K}_{q_1}(\mathbf{A}^{-1}, \mathbf{A}^{-1}\mathbf{B})$ and $\mathcal{K}_{q_2}(\mathbf{A}^{-T}, \mathbf{A}^{-T}\mathbf{C}^T)$ with *negative* powers of $\mathbf{A}$. This leads to a good approximation of the time functions in the explicit solutions (10) for *large* values of t. The difficulty of the inverse of the large matrix $\mathbf{A}$ in the Krylov subspace is solved by using the LU-factorization in the Arnoldi and Lanczos algorithms to find the associated bases. In fact, there is no need to calculate the inverse and only some large linear equations should be solved.

Suppose that the outputs of the original model (7) are all states, $\mathbf{C} = \mathbf{I}$. Based on moment matching we have,

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \simeq \mathbf{V}\left(s\mathbf{I} - \left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\right)^{-1}\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{B} \tag{15}$$

where $\mathbf{V}$ and $\mathbf{W}$ are bases of the input and output Krylov subspaces and $\frac{q}{m}$ moments match. By duality and with the choice $\mathbf{B} = \mathbf{I}$, we have,

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \simeq \mathbf{C}\mathbf{V}\left(s\mathbf{I} - \left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\right)^{-1}\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T \tag{16}$$

and $\frac{q}{p}$ moments match. Because of matching the moments around zero, the transfer function is well approximated for small values of s. Now, assume that all eigenvalues of the matrix $\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}$ are stable; i.e. located in the left half complex plane. By using the final value theorem, the equivalent time domain functions are good approximation of each other for large t,

$$e^{\mathbf{A}t}\mathbf{B} \simeq \mathbf{V}e^{\left(\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\right)t}\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{B}, \tag{17}$$

$$\mathbf{C}e^{\mathbf{A}t} \simeq \mathbf{C}\mathbf{V}e^{\left(\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\right)t}\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T. \tag{18}$$

Using the approximations (17) and (18) into (10) results in approximate gramians,

$$\hat{\mathbf{P}} = \mathbf{V}\mathbf{P}_r\mathbf{V}^T\ ,\ \hat{\mathbf{Q}} = \mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1}\mathbf{Q}_r\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T,$$

where,

$$\mathbf{P}_r = \int_0^\infty e^{\mathbf{A}_r t}\mathbf{B}_r\mathbf{B}_r^T e^{\mathbf{A}_r^T t}dt\ ,\ \mathbf{Q}_r = \int_0^\infty e^{\mathbf{A}_r^T t}\mathbf{C}_r^T\mathbf{C}_r e^{\mathbf{A}_r t}dt.$$

The matrices, $\mathbf{P}_r$ and $\mathbf{Q}_r$ are the solutions of the small Lyapunov equations (12). This choice satisfies the Galerkin conditions with a meaningful choice of $\mathbf{V}$ and $\mathbf{W}$.

4.3 Approximating the Hankel singular values

By using the approximate gramians, it is possible to approximate some of the Hankel singular values of a large scale system that are the eigenvalues of the matrix,

$$\begin{aligned}\sigma_i^2 = \lambda_i(\mathbf{P}\mathbf{Q}) \approx \lambda_i(\hat{\mathbf{P}}\hat{\mathbf{Q}}) &= \lambda_i(\mathbf{V}\mathbf{P}_r\mathbf{V}^T\mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1}\mathbf{Q}_r\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T)\\ &= \lambda_i(\mathbf{V}\mathbf{P}_r\mathbf{Q}_r\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T).\end{aligned}$$

It is known that changing the sequence of multiplication does not change the nonzero eigenvalues [14]. Therefore, the nonzero eigenvalues of the matrices $\mathbf{V}\mathbf{P}_r\mathbf{Q}_r\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T$ and $\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{V}\mathbf{P}_r\mathbf{Q}_r = \mathbf{P}_r\mathbf{Q}_r$ are equal, which are square of the Hankel singular values of

the system (5). In this way, q Hankel singular values of the original large scale system (1) are approximated by the Hankel singular values of the small projected system (5).

In practice, because of the properties of the input and output Krylov subspaces, this method approximates the *largest* Hankel singular values of the original model. The higher number of iterations to calculate the matrices $\mathbf{V}$ and $\mathbf{W}$ (i.e. larger q), the better the large Hankel singular values are approximated. Roughly speaking, by increasing q, the largest Hankel singular values tend to be approximated better; i.e. for $q \geq i$, approximation of σ_i is improved by increasing the value of q.

5 Sequential order reduction

In this section, the approximate solutions $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ of the Lyapunov equations (8), (9) are used for balancing and truncation. We show that a reduced model using a two-sided Krylov subspace method is an approximate truncated balanced realization of the original system but the reduced order model is not balanced.

5.1 Reduction steps

Suppose that the first r significant Hankel singular values of the system (1) must be preserved in the reduced order model. As discussed in section 4.3, to have a good approximation of the first r Hankel singular values, q must be larger than r and the approximate solutions $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ have rank q. These approximate gramians are symmetric and their Cholesky factorization can be found by using the Cholesky factorization of the matrices $\hat{\mathbf{P}}_r$ and $\hat{\mathbf{Q}}_r$,

$$\begin{aligned}\hat{\mathbf{P}} &= \mathbf{V}\mathbf{L}_{rc}\mathbf{L}_{rc}^T\mathbf{V}^T = \mathbf{V}\mathbf{L}_{rc}\left(\mathbf{V}\mathbf{L}_{rc}\right)^T \\ \hat{\mathbf{Q}} &= \mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1}\mathbf{L}_{ro}\mathbf{L}_{ro}^T\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T = \mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1}\mathbf{L}_{ro}\left(\mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1}\mathbf{L}_{ro}\right)^T.\end{aligned}$$

Then SVD of a small $q \times q$ matrix is to be calculated as,

$$\mathbf{L}_{ro}^T\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{V}\mathbf{L}_{rc} = \mathbf{L}_{ro}^T\mathbf{L}_{rc} = \hat{\mathbf{U}}\hat{\mathbf{\Sigma}}\hat{\mathbf{K}}^T.$$

where,

$$\hat{\mathbf{\Sigma}} = diag(\hat{\sigma}_1, \cdots, \hat{\sigma}_r, \hat{\sigma}_{r+1}, \cdots, \hat{\sigma}_q) = \begin{bmatrix} \hat{\mathbf{\Sigma}}_r & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{\Sigma}}_1 \end{bmatrix}.$$

containing q nonzero approximated Hankel Singular values of the original system (1) sorting in descending order. To find a reduced order model of order q the projection,

$$\mathbf{T} = \mathbf{V}\underbrace{\mathbf{L}_{rc}\hat{\mathbf{K}}\hat{\mathbf{\Sigma}}^{-\frac{1}{2}}}_{\mathbf{T}_r}, \mathbf{S} = \mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1}\underbrace{\mathbf{L}_{ro}\hat{\mathbf{U}}\hat{\mathbf{\Sigma}}^{-\frac{1}{2}}}_{\mathbf{S}_r}, \tag{19}$$

must be applied to the original system (1) to find the reduced model. The matrices $\mathbf{S}_r$ and $\mathbf{T}_r$ are in fact the similarity transformation which balances the system (5). Therefore, to find the reduced model of order q, applying a two-sided Krylov subspace method to the original high order system is enough and the Lyapunov equations are implicitly but approximately solved.

Now, consider that q is chosen such that $\sigma_1, \cdots, \sigma_r$ are well approximated. To reduce the original system to order r, the projection matrices,

$$\bar{\mathbf{T}} = \mathbf{V}\bar{\mathbf{T}}_r, \bar{\mathbf{S}} = \mathbf{W}\left(\mathbf{V}^T\mathbf{W}\right)^{-1}\bar{\mathbf{S}}_r,$$

should be applied to the original system where $\bar{\mathbf{S}}_r$ and $\bar{\mathbf{T}}_r$ are the first r columns of the matrices $\mathbf{S}_r$ and $\mathbf{T}_r$ in (19), respectively. Then, the reduced system of order r can be calculated as,

$$\begin{cases} \dot{\mathbf{x}}_r(t) = \bar{\mathbf{S}}_r^T \left(\mathbf{W}^T\mathbf{V}\right)^{-1} \mathbf{W}^T\mathbf{A}\mathbf{V}\bar{\mathbf{T}}_r\mathbf{x}_r(t) + \bar{\mathbf{S}}_r^T \left(\mathbf{W}^T\mathbf{V}\right)^{-1} \mathbf{W}^T\mathbf{B}\mathbf{U}(t), \\ \mathbf{y}_r = \mathbf{C}\mathbf{V}\bar{\mathbf{T}}_r\mathbf{x}_r(t). \end{cases} \tag{20}$$

The system (20) is exactly the same as truncated balanced realization of system (5) to order r.

The following steps are proposed for order reduction of a large scale model:

0. Choose a suitable order r and the number of iterations $q > r$ (q is the order of the reduced system by Krylov methods).
1. (a) Calculate a pair of bases for input and output Krylov subspaces using Two-sided Arnoldi or Lanczos algorithms.
   (b) Calculate the reduced order system (5) of order q.
2. Apply balancing and truncation to the model from the last step to go to order r.

The two main steps applied for order reduction are first Krylov subspace method, which is very fast and efficient and second TBR, which is applied to a small system of order q for further reduction and go to a smaller order. The norm of the error system by this method is very close to the one from an exact TBR in (11). In practice, this closeness mostly depends on the approximation error of the largest Hankel Singular values, which is improved by increasing the difference $q - r$.

Furthermore, the reduced system of order r is stable if the reduced system of order q is stable. In other words, stability preserving by the sequential method depends on stability preserving by Krylov methods applied in the first step.

5.2 Stopping criterion

In balancing and truncation, Hankel singular values are used to find a suitable order for the reduced system leading to keep large Hankel singular values and delete others by truncating the balanced system. In order to find a reduced model which keeps the approximated Hankel singular values with small error, we propose a criterion to find convenient orders q and r for the first reduction using Krylov subspaces and the second reduction using TBR, respectively.

If i is the iteration counter, in every iteration of a Krylov subspace method, there exists a reduced model of order i. The Hankel singular values of the reduced model are calculated in every iteration and then sorted in decreasing order. In each iteration, a new column is conceived and added to the previous set. These values are sorted and saved in the columns of a $q \times q$ matrix as follows,

$$\mathbf{H} = \begin{bmatrix} \sigma_1^1 & \sigma_1^2 & \cdots & \sigma_1^q \\ 0 & \sigma_2^2 & \cdots & \sigma_2^q \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_q^q \end{bmatrix}_{q\times q}, \quad \begin{bmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \vdots \\ \hat{\sigma}_q \end{bmatrix} = \begin{bmatrix} \sigma_1^i \\ \sigma_2^i \\ \vdots \\ \sigma_q^i \end{bmatrix}, \tag{21}$$

where σ_j^i is the approximation of the j-th largest Hankel singular value of the original model calculated by an i-th order model. The iterations must be continued to be sure that all large

Hankel singular values are approximated with small errors. In practical applications, this can be done by checking the convergence of σ_r^q where $|\sigma_r^q - \sigma_r^{q+1}|$ is smaller than a threshold and σ_{r+1}^q is much smaller than $\sigma_1^q, \cdots, \sigma_r^q$.

6 Further remarks

In the following subsections, the same method of reduction is extended to Sylvester equations and some properties of the sequential reduction is discussed.

6.1 Solving Sylvester equation

In section 4, the two Lyapunov equations are solved to find approximate low order controllability and observability gramians. It is also possible to balance a system by only solving one Sylvester equation and find a matrix called cross gramian [26]. The cross gramian $\mathbf{X}$ for a square and symmetric system (1) ($m = p$ and all Markov parameters are symmetric) is defined as the solution of the Sylvester equation,

$$\mathbf{AX} + \mathbf{XA} + \mathbf{BC}^T = 0. \tag{22}$$

If $\mathbf{A}$ is stable, the explicit solution of this equation is written as,

$$\mathbf{X} = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{C}^T e^{\mathbf{A}t} dt. \tag{23}$$

The most important properties of the cross gramian are as follows:

1. Multiplication of controllability and observability gramians of a square symmetric system is equal to the square of the cross gramian, $\mathbf{X}^2 = \mathbf{PQ}$.
2. The Hankel singular values are equal to the absolute value of the eigenvalues of the cross gramian.
3. There exists a nonsingular matrix $\mathbf{Z}$ such that the system with similarity transformation $\mathbf{Z}$ is balanced and $\mathbf{Z}$ diagonalizes the matrix $\mathbf{X}$, i.e. the system,

$$\begin{cases} \dot{\mathbf{x}}_b(t) = \mathbf{Z}^{-1}\mathbf{A}\mathbf{Z}\mathbf{x}_b(t) + \mathbf{Z}^{-1}\mathbf{B}\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}\mathbf{Z}\mathbf{x}_b(t). \end{cases}$$

   is balanced and $\mathbf{Z}^{-1}\mathbf{X}\mathbf{Z} = diag\{\pm\sigma_1 \cdots \pm \sigma_n\}$ is diagonal where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are Hankel singular values of the system.

With the same idea used for solving Lyapunov equations, here, the Sylvester equation (22) is solved. Consider $\mathbf{X}_r$ is the solution of the Sylvester equation,

$$\mathbf{A}_r\mathbf{X}_r + \mathbf{X}_r\mathbf{A}_r + \mathbf{B}_r\mathbf{C}_r = \mathbf{0}, \tag{24}$$

which finds the cross gramian of the reduced system (5). An approximate solution of equation (22) can be written as follows,

$$\hat{\mathbf{X}} = \mathbf{V}\mathbf{X}_r \left(\mathbf{W}^T\mathbf{V}\right)^{-1} \mathbf{W}^T, \tag{25}$$

and the Galerkin condition,

$$\left(\mathbf{W}^T\mathbf{V}\right)^{-1} \mathbf{W}^T\mathbf{R}_x\mathbf{V} = \mathbf{0},$$

is satisfied where,

$$\mathbf{A}\hat{\mathbf{X}} + \hat{\mathbf{X}}\mathbf{A} + \mathbf{B}\mathbf{C}^T = \mathbf{R}_x.$$

In the case that $\mathbf{V}$ and $\mathbf{W}$ are bases of the input and output Krylov subspaces, the approximate solution (25) approximates the explicit integral solution (23) for *large* values of t. By using the approximation from (17) and (18) we have,

$$\begin{aligned}\mathbf{X} &= \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{C}e^{\mathbf{A}t}dt \\ &\simeq \int_0^\infty \mathbf{V}e^{\left(\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\right)t}\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{B}\mathbf{C}\mathbf{V}e^{\left(\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\right)t}\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T dt = \hat{\mathbf{X}}.\end{aligned}$$

Therefore,

$$\hat{\mathbf{X}} = \mathbf{V}\mathbf{X}_r\left(\mathbf{W}^T\mathbf{V}\right)^{-1}\mathbf{W}^T \quad \text{and} \quad \mathbf{X}_r = \int_0^\infty e^{\mathbf{A}_r t}\mathbf{B}_r\mathbf{C}_r e^{\mathbf{A}_r t}dt.$$

As it can be seen, $\mathbf{X}_r$ is the solution of the small Sylvester equation (24).

6.2 Invariance properties

The goal of this section is describing the invariance properties of the approximate gramians found in the previous sections to changing the bases of Krylov subspaces. These properties are important in the sense that applying two different algorithms to find the bases for the Krylov subspaces should lead to the same approximate gramians.

Theorem 1. *The approximated gramians* (13) *and* (14) *of the system (1), based on the gramians of the reduced model (5), are independent of the particular choice of the bases* $\mathbf{V}$ *and* $\mathbf{W}$ *of the input and output Krylov subspaces.*

Proof: Consider two pairs of bases $\mathbf{V}_1, \mathbf{W}_1$ and $\mathbf{V}_2, \mathbf{W}_2$ to be used in Lyapunov equation (12) resulting in $\mathbf{P}_{r1}, \mathbf{Q}_{r1}$ and $\mathbf{P}_{r2}, \mathbf{Q}_{r2}$, respectively. Because the columns of $\mathbf{V}_1$ and $\mathbf{V}_2$ form two bases for the input Krylov subspace and the columns of $\mathbf{W}_1$ and $\mathbf{W}_2$ form two bases for the output Krylov subspace then,

$$\exists \mathbf{H}_v \in \mathbb{R}^{q\cdot m \times q\cdot m}, \mathbf{H}_w \in \mathbb{R}^{q\cdot p \times q\cdot p} : \mathbf{V}_2 = \mathbf{V}_1\mathbf{H}_v, \mathbf{W}_2 = \mathbf{W}_1\mathbf{H}_w. \tag{26}$$

where $\mathbf{H}_v$ and $\mathbf{H}_w$ are invertible. By putting (26) into the Lyapunov equations in (12), it can be shown that,

$$\mathbf{P}_{r2} = \mathbf{H}_v^{-1}\mathbf{P}_{r1}\mathbf{H}_v^{-T} \quad \text{and} \quad \mathbf{Q}_{r2} = \mathbf{H}_v^T\mathbf{Q}_{r1}\mathbf{H}_v, \tag{27}$$

Now, we put equation (27) into approximate solutions (13) and (14),

$$\begin{aligned}\hat{\mathbf{P}}_2 &= \mathbf{V}_2\mathbf{P}_{2r}\mathbf{V}_2^T = \mathbf{V}_1\mathbf{H}_v\mathbf{H}_v^{-1}\mathbf{P}_{r1}\mathbf{H}_v^{-T}\mathbf{H}_v^T\mathbf{V}_1 = \mathbf{V}_1\mathbf{P}_{r1}\mathbf{V}_1 = \hat{\mathbf{P}}_{r1}, \\ \hat{\mathbf{Q}}_{r2} &= \mathbf{W}_2\left(\mathbf{V}_2^T\mathbf{W}_2\right)^{-1}\mathbf{Q}_{r2}\left(\mathbf{W}_2^T\mathbf{V}_2\right)^{-1}\mathbf{W}_2^T \\ &= \mathbf{W}_1\left(\mathbf{V}_1^T\mathbf{W}_1\right)^{-1}\mathbf{H}_v^{-T}\mathbf{H}_v^T\mathbf{Q}_{r1}\mathbf{H}_v\mathbf{H}_v^{-1}\left(\mathbf{W}_1^T\mathbf{V}_1\right)^{-1}\mathbf{W}_1^T \\ &= \mathbf{W}_1\left(\mathbf{V}_1^T\mathbf{W}_1\right)^{-1}\mathbf{Q}_{r1}\left(\mathbf{W}_1^T\mathbf{V}_1\right)^{-1}\mathbf{W}_1^T = \hat{\mathbf{Q}}_{r1},\end{aligned}$$

Therefore, by changing the bases, the approximate solutions do not change.□

The Hankel singular values of the system (1) are approximated by the Hankel singular values of the system (5). Therefore, the approximate Hankel singular values of the system (1) is also independent of the particular choice of bases of the Krylov subspaces.

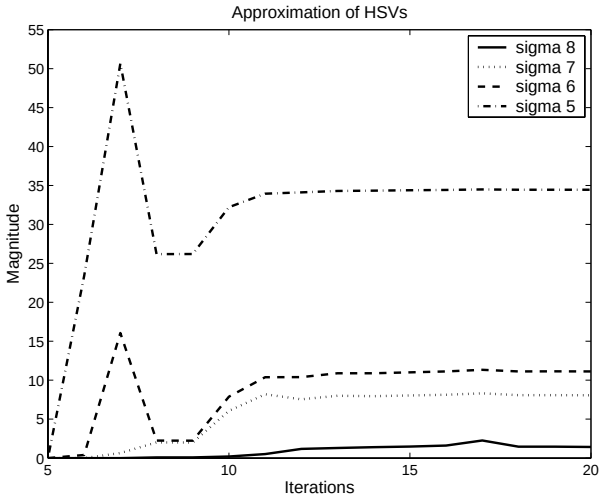

Figure 1: Approximation of the some of the Hankel singular values of example 1 in terms of iterations.

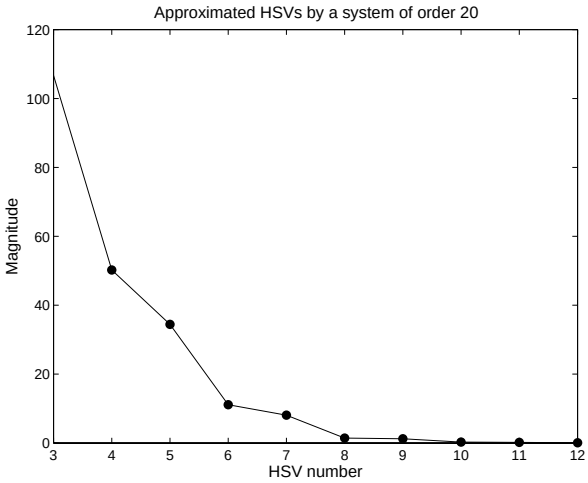

Figure 2: Original and Approximated Hankel singular values (3rd to 12th) of example 1 using reduced model of order 20.

7 Technical examples

We consider two different examples from [3][1]. We apply the sequential reduction approach to reduce the order of these systems and compare the results with the existing methods of TBR and Krylov subspace methods.

The first example is the model of an atmospheric storm track. The order of the system is 598 with one input and one output. The two-sided Arnoldi algorithm has been applied to this system to find the approximated Hankel singular values. In figure 1, the approximation of $\hat{\sigma}_5, \cdots, \hat{\sigma}_8$ in terms of iteration of the Krylov method are shown. It can be seen that the approximations tend to constant values by going to higher dimensions and they converge faster for larger Hankel singular values. A comparison of the original Hankel singular values to the ones in figure 1 shows that all approximations converge to the exact values. A system of order 20 seems to be a good choice for the first step of the sequential reduction scheme. In figure 2, the Approximated Hankel singular values using a system of order 20 is illustrated. One option for the second step of reduction using TBR is going to order 7.

For comparison, the original system has also been reduced by applying only TBR and only Krylov subspace methods directly to the original system. By using TBR a reduced system of order 7 is found. Because the reduced order model of order 7 by using a Krylov subspace method is unstable, we reduce the model to order 8. The result of reduction methods are compared in table 1. The errors are defined as $\|\frac{g(s)-g_r(s)}{g(s)}\|_\infty$ where $g(s)$ and $g_r(s)$ are the original and reduced order models and the elapsed time is the time MATLAB needs on a Pentium IV system to find the reduced order model including the stopping criterion for the sequential method.

As it can be expected, the Krylov subspace method has the highest error but is the fastest method to find the reduced order system. In comparison to TBR, the sequential approach is much faster and leads to the same value of error. Although for the stopping criterion, a couple of Lyapunov equations or a Sylvester equation should be solved, the sequential approach is a good choice to find a reduced system close to direct TBR in a relatively much shorter time.

[1]The models are available online at http://www.win.tue.nl/niconet/NIC2/benchmodred.html.

Table 1: Comparison of the different reduced order models for the Example 1.

Method	Order	H_∞ error norm	Elapsed time
Krylov Method	8	0.0134	0.3s
Sequential	7	0.0018	2.7s
TBR	7	0.0018	176s

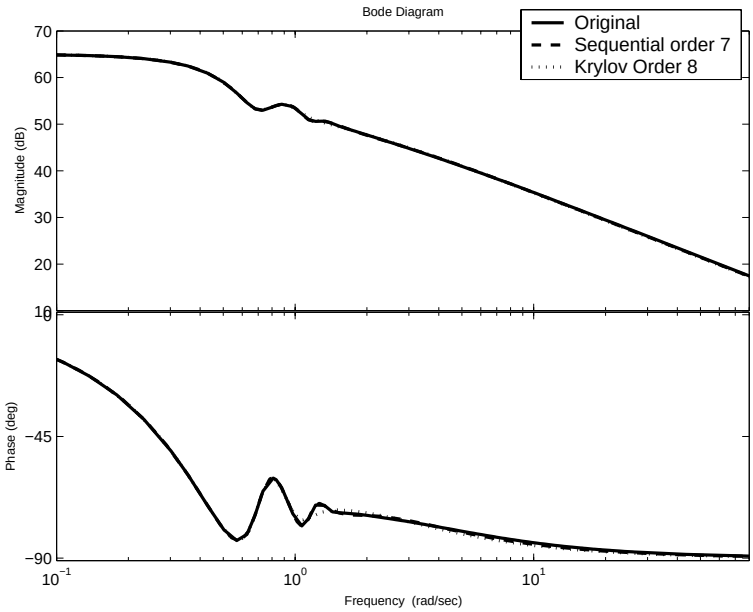

Figure 3: Bode diagram of the original and reduced order models.

In figures 3, the Bode diagram of the original and reduced systems are shown. The reduced system by the sequential approach has a good accuracy and is better than the direct Krylov method specially in phase plot, although the order of two reduced systems are different.

The second example is a clamped beam model. The order of the system is 348 with one input and one output. The model is obtained by spatial discretization of an appropriate partial differential equation. The input represents the force applied to the structure at the free end, and the output is the resulting displacement. The same procedure as in the previous example is applied to reduce the order of this system. In figure 4, the approximation of $\hat{\sigma}_{15}, \hat{\sigma}_{17}, \hat{\sigma}_{19}, \hat{\sigma}_{21}$ in terms of iteration of the Krylov method are shown. The same as before, the approximations converge to constant values which are the original Hankel Singular values and the convergence is faster for larger Hankel singular values. A system of order 60 can be a good choice for the first step of the sequential reduction scheme. In figure 5, the approximated Hankel singular values using a system of order 60 is illustrated. The Hankel singular values decrease with a slow rating. We choose order 20 for the second step of reduction using TBR.

The original system has also been reduced by applying TBR and Krylov subspace methods directly to the original system to go to order 20. The result of reduction methods are compared in table 2.

The same results as in example 1 can be concluded: the Krylov subspace method has the highest error and shortest elapsed time, the sequential approach leads to an error very close to direct

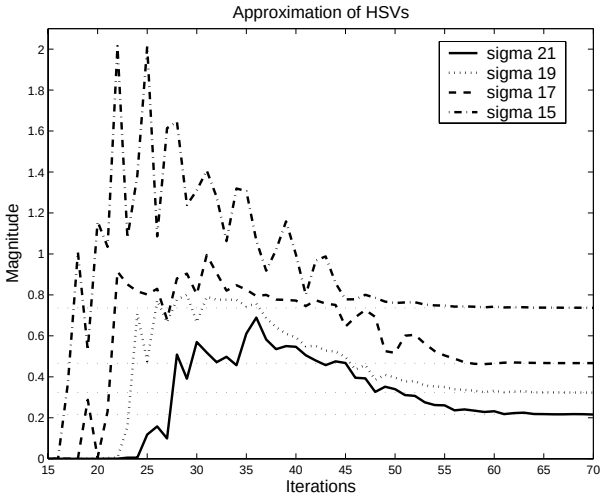

Figure 4: Approximation of the some of the Hankel singular values of example 2 in terms of iterations.

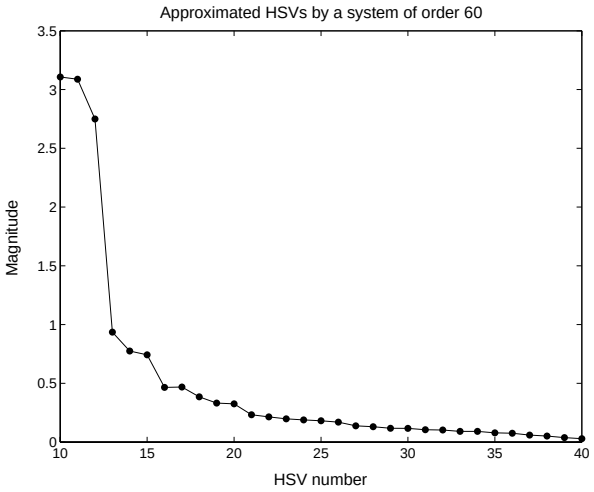

Figure 5: Original and Approximated Hankel singular values (10th to 40th) of example 2 using reduced model of order 60.

TBR but achieved in a shorter time.

Table 2: Comparison of the different reduced order models for the Example 2.

Method	Order	H_∞ error norm	Elapsed time
Krylov Method	20	0.0017	0.1s
Sequential	20	9.4044e-005	5.2s
TBR	20	8.7899e-005	35s

In figures 6, the Bode diagram of the original and reduced systems are shown. The reduced system by the sequential approach has a much better accuracy than the direct Krylov method.

8 Conclusions

In this paper, based on approximately solving the Lyapunov equations by Krylov subspace methods, the equivalence of Krylov subspace reduction methods and approximate TBR has been shown. The equivalence property results in a sequential order reduction scheme to reduce the order of a large scale system yielding a result close to TBR. The proposed approach uses a two-sided Krylov method and balancing and truncation, sequentially. This method is fast, numerically robust and leads to better results than Krylov subspace methods with an error close to the one of direct balancing and truncation of the original system.

It was also shown that the Hankel singular values of a reduced system calculated by a two-sided Krylov subspace method approximate the Hankel singular values of the original large scale system. Based on this fact, a stopping criterion to find suitable orders for the reduced systems in each step of the sequential reduction is proposed to preserve the large approximated Hankel singular values.

The effectiveness of the sequential method is shown using different examples and the results are compared to TBR and Krylov subspace method which confirm the theoretical results.

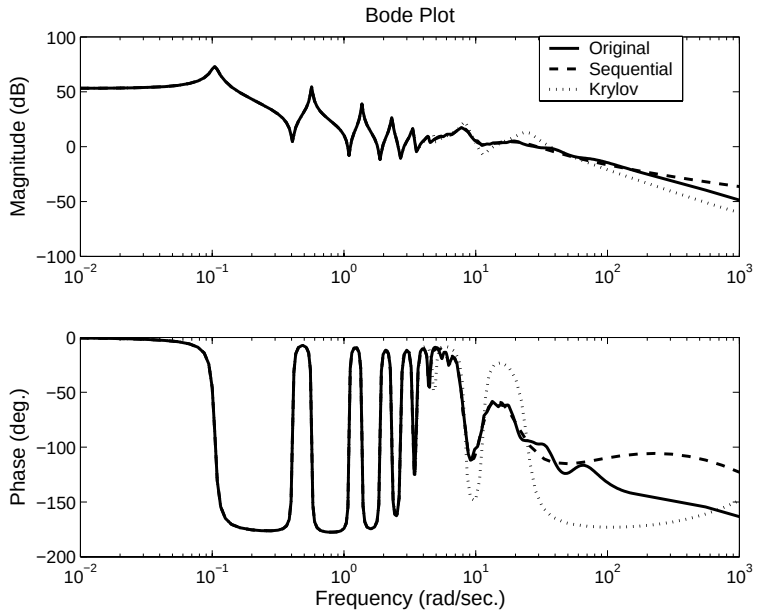

Figure 6: Bode diagram of the original and reduced order models of order 20 of example 2

Similar results have been shown for large Sylvester equations where the original system is SISO or symmetric. The invariance of the approximate gramians has also been investigated.

More discussions on related topics like a detailed comparison with the existing approximate TBR methods, stability preserving, convergence of the stopping criterion are left for future works.

References

[1] P. Benner, J.-R. Li, and T. Penzl. Numerical Solution of Large Lyapunov Equations, Riccati equations, and Linear-Quadratic Control Problems. *Numerical Linear Algebra with Applications*, 15(9):755–777, 2008.

[2] P. Benner, J.-R. Li, and N. Truhar. On the ADI Method for Sylvester Equations. Preprint, Fakultät für Mathematik, TU Chemnitz, Germany, August 2008.

[3] Y. Chahlaoui and P. Van Dooren. A collection of Benchmark Examples for Model Reduction of Linear Time Invariant Dynamical Systems. SLICOT Working Note 2002-2, Feb. 2002. Available by anonymous ftp from ftp://wgs.esat.kuleuven.ac.be/pub/WGS/REPORTS/SLWN2002-2.ps.Z.

[4] J. Cullum and T. Zhang. Two-sided Arnoldi and Nonsymmetric Lanczos Algorithms. *SIAM Journal on Matrix Analysis and Applications*, 24(2):303–319, 2002.

[5] P. Feldmann and R.W. Freund. Efficient Linear Circuit Analysis by Padé Approximation via the Lanczos Process. *IEEE trans. Coputer-Aided Design of ICs and Systems*, 14(5):639649, 1995.

[6] R. W. Freund. Passive Reduced-Order Modeling via Krylov Subspace Methods. *Numerical Analysis Manuscript*, (00-3-02), March 2000. Available online at http://cm.bell-labs.com/cs/doc/00.

[7] R. W. Freund. Model Reduction Methods Based on Krylov Subspaces. *Numerical Analysis Manuscript*, (03-4-01), 2003.

[8] K. Glover. All optimal Hankel-norm Approximations of Linear Multivariable Systems and their Error Bounds. *International Journal of Control*, 39:1115–1193, 1984.

[9] E. J. Grimme. *Krylov Projection Methods for Model Reduction.* PhD thesis, Dep. of Electrical Eng., Uni. of Illinois at Urbana Champaign, 1997.

[10] S. Gugercin, D. C. Sorensen, and A. C. Antoulas. A Modified Low-Rank Smith Method for Large-Scale Lyapunov Equations. *Numerical Algorithms*, 32(1):27–55, 2003.

[11] D. Y. Hu and L. Reichel. Krylov Subspace methods for Sylvester equation. *Linear Algebra and its Applications*, 172:283–313, 1992.

[12] I. M. Jaimoukha and E. M. Kasenally. Krylov Subspace Methods for Solving Large Lyapunov Equations. *SIAM J. Numer. Anal.*, 31(1):227–251, 1994.

[13] M. Kamon, F. Wang, and J. White. Generating Nearly Optimally Compact Models from Krylov-Subspace Based Reduced-Order Models. *IEEE Trans. Circuit and Systems*, 47(4):239–248, 2000.

[14] P. Lancaster and M. Tismenetsky. *The Theory of Matrices.* Academic Press, second edition, 1985.

[15] A. J. Laub, M. T. Heath, C. C. Paige, and R. C. Ward. Computation of System Balancing Transformations and Other Applications of Simultaneous Diagonalization Algorithms. *IEEE Trans. on Automatic Control*, AC-32(2):115–122, 1987.

[16] R. J. Li and J. White. Efficient Model Reduction of Interconnect via Approximate System Grammians. In *Int. Conf. Coputer Aided design*, San Jose, CA, 1999.

[17] R. J. Li and J. White. Reduction of Large Circuit Models via Low Rank Approximate Gramians. *International Journal of Applied Mathematics and Computer Science*, 11(5):1151–1171, 2001.

[18] B. C. Moore. Principal Component Analysis in Linear Systems: Controllability, Observability and Model Reduction. *IEEE Trans. Aut. Control*, AC-26:17–32, 1981.

[19] T. Penzl. A Cyclic Low-Rank Smith Method for Large Sparse Lyapunov Equations. *SIAM J. Sci. Comp.*, 21(4):1401–1418, 2000.

[20] T. Penzl. Lyapack, User's Guide (Version 1). Preprint SFB 393/00-33, TU Chemnitz, 2000.

[21] J. R. Phillips, L. Daniel, and L. M. Silveira. Guaranteed Passive Balancing Transformations for Model Order Reduction. *IEEE trans. Coputer-Aided Design of ICs and systems*, 22(8):1027–1041, 2003.

[22] D. Ramaswamy and J. White. Automatic Generation of Small-Signal Dynamic Macromodel from 3-D Simulation. In *Int. Conf. Modeling and Simulation of Microsystems*, pages 27–30, South Carolina, U.S.A., 2001.

[23] Y. Saad. Numerical Solution of Large Lyapunov Equations. In M. Kaashoek, J. V. Schuppen, and A. Ran, editors, *Signal Processing, Scattering, Operator Theory and Numerical Methods*, pages 503–511, Boston, MA, 1990. Birkhauser.

[24] B. Salimbahrami and B. Lohmann. Krylov Subspace Methods in Linear Model Order Reduction: Introduction and Invariance Properties. Scientific report, Institute of Automation, University of Bremen, 2002. Available at http://www.rt.mw.tum.de/salimbahrami/Invariance.pdf.

[25] B. Salimbahrami, B. Lohmann, T. Bechtold, and J. G. Korvink. A Two-Sided Arnoldi-Algorithm with Stopping Criterion and an application in Order Reduction of MEMS. *Mathematical and Computer Modelling of Dynamical Systems (MCMDS)*, 11(1):79–93, 2005.

[26] D. Sorensen and A. C. Antoulas. The Sylvester Equation and approximate Balanced Reduction. *Linear Algebra and its Applications*, 351-352(15):671–700, 2002.

[27] C. Villemagne and R. E. Skelton. Model Reductions using a Projection Formulation. *Internat. J. Control*, 46:2141–2469, 1987.