Extremes of Subexponential Lévy Driven Moving Average Processes

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Abstract

In this paper we study the extremal behavior of a stationary continuous-time moving average process $Y(t) = \int_{-\infty}^{\infty} f(t-s)\, dL(s)$ for $t \in \mathbb{R}$, where f is a deterministic function and L is a Lévy process whose increments, represented by $L(1)$, are subexponential and in the maximum domain of attraction of the Gumbel distribution. We give necessary and sufficient conditions for Y to be a stationary, infinitely divisible process, whose stationary distribution is subexponential, and in this case we calculate its tail behavior. We show that large jumps of the Lévy process in combination with extremes of f cause excesses of Y and thus properly chosen discrete-time points are sufficient to specify the extremal behavior of the continuous-time process Y. We describe the extremal behavior of Y completely by a weak limit of marked point processes. A complementary result guarantees the convergence of running maxima of Y to the Gumbel distribution.

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1 Introduction

We investigate the extremal behavior of a stationary *continuous-time moving average* (MA) process

$$Y(t) = \int_{-\infty}^{\infty} f(t-s)\, dL(s) \quad \text{for } t \in \mathbb{R}\,, \tag{1.1}$$

where the *kernel function* $f : \mathbb{R} \to \mathbb{R}$ is measurable, and the driving process $L = \{L(t)\}_{t\in\mathbb{R}}$ is a *Lévy process*. Recall that a Lévy process L has independent and stationary increments, $L(0) = 0$, and L is stochastically continuous. Moreover, L is characterized by the *Lévy-Khinchine representation* $\mathbb{E}(\exp(iuL(t))) = \exp(t\psi(u))$ for $t \geq 0$, $u \in \mathbb{R}$ with

$$\psi(u) = ium - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{iux} - 1 - iu\kappa(x)\right) \nu(dx), \tag{1.2}$$

and $\kappa(x) = x\,\mathbf{1}_{[-1,1]}(x)$. The quantities (m, σ^2, ν) are called the *generating triplet* of the Lévy process L. Here $m \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}$, called *Lévy measure*, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\,\nu(dx) < \infty$; we refer to the monographs of Applebaum [1] and Sato [23] for background on Lévy processes. Prominent examples of MA processes are CARMA processes (cf. Brockwell [4]) and stochastic differential delay equations (cf. Gushchin and Küchler [14]). Both families include Ornstein-Uhlenbeck-processes.

We concentrate in this paper on increments of the Lévy process L in the *maximum domain of attraction of the Gumbel distribution* (MDA(Λ)): a distribution function $F \in \text{MDA}(G)$, where G is a non-degenerate distribution function (d. f.), if there exist constants $a_T > 0$, $b_T \in \mathbb{R}$ for $T > 0$ such that $\lim_{T\to\infty} T(1 - F(a_T x + b_T)) = -\log G(x)$ for $x \in \mathbb{R}$. The symbol Λ stands for Gumbel distribution. Without precise referencing we use results from classical extreme value theory; we refer to Embrechts et al. [9], Chapter 3 for more details.

Complementary results for MA processes in the maximum domain of attraction of the Fréchet distribution have been investigated in the early work of Rootzén [21] for stable processes and for regularly varying mixed MA processes in Fasen [11].

Throughout the paper we assume the following condition, which is sufficient for the existence and the infinitely divisibility of Y. Firstly, we define $\mathbb{L}^\delta := \{f : \mathbb{R} \to \mathbb{R} \text{ measurable}, \int_{-\infty}^{\infty} |f(s)|^\delta\, \lambda(ds) < \infty\}$, $\delta > 0$, where λ denotes the Lebesgue measure on $\mathbb{R}$.

Condition (M). *Let Y be a MA process as given in (1.1). The Lévy measure ν of L satisfies $\nu\,(1, \cdot \vee 1]\,/\nu(1,\infty) \in \text{MDA}(\Lambda)$ with infinite right endpoint, and tail balance condition*

$$\lim_{x\to\infty} \frac{\nu\,(-\infty, -x)}{\nu\,(x,\infty)} = \frac{1-p}{p} \tag{1.3}$$

for some $p \in (0,1]$. The kernel function $f : \mathbb{R} \to \mathbb{R}$ is bounded, and one of the following conditions hold:

(M1) $f \in \mathbb{L}^1$.

(M2) $f \in \mathbb{L}^2$ *and* $\mathbb{E}L(1) = 0$.

If the support of ν is bounded below, we assume w. l. o. g. $\nu(-\infty, -1) = 0$.

We can also replace $p \in (0,1]$ and $\nu\,(1, \cdot \vee 1]\,/\nu(1,\infty) \in \mathrm{MDA}(\Lambda)$ with infinite right endpoint by $p \in [0,1)$ and $\nu\,[-(\cdot \vee 1), -1)\,/\nu(-\infty,-1) \in \mathrm{MDA}(\Lambda)$ with infinite right endpoint. Embrechts et al. [9], Corollary 3.3.32 and Sato [23], Corollary 25.8, imply that $L(1)$ has moments of all order and the tails of ν decrease faster than polynomial. Furthermore, the right tail of ν is rapidly varying, i. e. $\lim_{x\to\infty} \nu(xt,\infty)/\nu(x,\infty) = 0$ for $t > 1$. Notice, if f is bounded, then $f \in \mathbb{L}^1$ implies $f \in \mathbb{L}^2$.

This paper is on the extremal behavior of *subexponential Lévy driven MA processes*. Subexponentiality is a property of the right tail of a distribution. Consequently, it has been defined originally for positive r. v. s. In the context of this paper $L(1)$ has a distribution on the whole of $\mathbb{R}$, which has a subexponential right tail. The definition of a subexponential r. v. has been extended from a positive r. v. to a r. v. on $\mathbb{R}$ by Willekens [24] and we start with the definition.

Throughout the paper we use the following standard notation: we write $\overline{F} = 1 - F$ for the right tail of the d. f. F, F^{2*} for the convolution $F * F$ and $\overline{F^{2*}} = 1 - F^{2*}$. $X \overset{d}{=} Y$, if the distributions of the random variables (r. v. s) X and Y coincide. The abbreviation i. d. stands for infinitely divisible. For real functions g and h we write $g(t) \sim h(t)$ for $t \to \infty$, if $g(t)/h(t) \to 1$ as $t \to \infty$, and we denote $g^+(t) = \max\{0, g(t)\}$, $g^-(t) = \max\{0, -g(t)\}$, $g^+ = \sup_{t\in\mathbb{R}} g^+(t)$, $g^- = \sup_{t\in\mathbb{R}} g^-(t)$ and $\int_{-\infty}^{\infty} \nu(x/g(s),\infty)\,\lambda(ds) = \int_{g(s)\neq 0} \nu(x/g(s),\infty)\,\lambda(ds)$. The symbol $\overset{T\to\infty}{\Longrightarrow}$ stands for weak convergence for $T \to \infty$.

Definition 1.1 *Let F be a d. f. on $\mathbb{R}$ with $F(x) < 1$ for every $x \in \mathbb{R}$. Then F belongs to the class of subexponential distributions, denoted by $\mathcal{S}$, if the following conditions hold:*

(i) $F \in \mathcal{L}$, *which means for all $y \in \mathbb{R}$ locally uniformly* $\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = 1$.

(ii) $\lim_{x\to\infty} \overline{F^{2*}}(x)/\overline{F}(x)$ *exists and is finite.*

If $F \in \mathcal{S}$ and Z is a r. v. with d. f. F, then we write $Z \in \mathcal{S}$. The class $\mathcal{S}$ is closed under *tail-equivalence*, i. e. if $F \in \mathcal{S}$ and G is a d. f. with $\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = q \in (0,\infty)$, then also $G \in \mathcal{S}$. A survey of the class of subexponential distributions with support on $\mathbb{R}_+$ is provided by Goldie and Klüppelberg [12], see also Embrechts et al. [9], Section A3. The following result summarizes mostly known properties of subexponentials on $\mathbb{R}$ needed for this paper, which can be found in Cline [5], Cline and Samorodnitsky [6] and Pakes [18]. Only *(vi)* is a new and easy consequence of the other results.

Proposition 1.2

(i) *If* $F \in \mathcal{L}$, *then* $\overline{F}(x/2)^2 = o(\overline{F}(x))$ *for* $x \to \infty$ *and* $\lim_{x\to\infty} e^{\epsilon x}\overline{F}(x) = \infty$ *for* $\epsilon > 0$.

(ii) *If* $F \in \mathcal{S}$, *then* $\lim_{x\to\infty} \overline{F^{2*}}(x)/\overline{F}(x) = 2$.

(iii) *Suppose* $F \in \mathcal{S}$, F_i *are d. f. s with* $\lim_{x\to\infty} \overline{F_i}(x)/\overline{F}(x) = q_i \geq 0$ *for* $i = 1, 2$ *and* $G = F_1 * F_2$. *Then,* $\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x) = q_1 + q_2$. *If* $q_i > 0$ *for some* $i \in \{1, 2\}$, *then also* F_i, $G \in \mathcal{S}$. *Moreover, for* $q_1 > 0$,

$$\lim_{x\to\infty} \int_{-\infty}^{x/2} \frac{\overline{F}_2(x-u)}{\overline{F}_1(x)}\, F_1(du) = \frac{q_2}{q_1}, \quad \lim_{x\to\infty} \int_{-\infty}^{x/2} \frac{\overline{F}_1(x-u)}{\overline{F}_1(x)}\, F_2(du) = 1.$$

(iv) *Let* F *be an i. d. distribution function with Lévy measure* ν. *Then,*

$$F \in \mathcal{S} \quad \Longleftrightarrow \quad \frac{\nu\,(1, \cdot \vee 1]}{\nu\,(1, \infty)} \in \mathcal{S} \quad \Longleftrightarrow \quad \overline{F}(x) \sim \nu(x, \infty) \text{ for } x \to \infty.$$

(v) *If* $X \in \mathcal{S}$ *has only support on* $\mathbb{R}_+$ *and* Y *is a bounded r. v., then* $XY \in \mathcal{S}$.

(vi) *If* X, Y *are i. d.,* $X \in \mathcal{S}$ *and* $\nu_Y(x, \infty)/\nu_X(x, \infty) \to 0$ *as* $x \to \infty$. *Then,*

$$\mathbb{P}(Y > x) = o(\mathbb{P}(X > x)) \quad \text{for } x \to \infty.$$

The class of subexponential distributions includes all distributions with regularly varying tails, the loggamma distribution and the heavy-tailed Weibull distribution. A prominent example in the context of this paper is the following:

Example 1.3 (Extended heavy-tailed Weibull model) Let the right tail of the d. f. F behave like $\overline{F}(x) \sim \exp(-u(x))$ for $x \to \infty$, where there exists a $v > 1$ such that $u(tx) \leq x^\alpha u(t)$ for all $t \geq v$, $x > 1$ and some $\alpha \in (0, 1)$, then $F \in \mathcal{S}$ (cf. Baltrunas et al. [2], Proposition 3.7, Lemma 3.8). If u is twice differentiable with $0 < -u''(x)/u'(x)^2 \overset{x\to\infty}{\longrightarrow} 0$, then $F \in \mathrm{MDA}(\Lambda)$ (cf. Embrechts et al. [9], Example 3.3.23). Thus, the heavy-tailed Weibull distribution $\overline{F}(x) = K \exp(-x^\alpha)$, $x > 0$, $\alpha \in (0, 1), K > 0$, belongs to $\mathcal{S} \cap \mathrm{MDA}(\Lambda)$. □

For the main results of this paper, presented in Section 4, about extremes of subexponential Lévy driven MA processes, we are imposing the following more restrictive condition.

Condition (G). *Let Y be a measurable and separable version of the MA process as given in (1.1) satisfying Condition (M) and $\mathbb{P}(|Y(t)| < \infty$ for all $t \in \mathbb{R}) = 1$. Let $L(1) \in \mathcal{S} \cap \mathrm{MDA}(\Lambda)$ with $a_T > 0$, $b_T \in \mathbb{R}$, $u_T = a_T x + b_T$ for $x \in \mathbb{R}$ such that*

$$\lim_{T\to\infty} T\mathbb{P}(f^+ L(1) > u_T) = \exp(-x).$$

We suppose $f \in \mathbb{L}^1$, $f^+ \geq f^-$, and for $i = 1, 2$, $P^{(i)} := \mathrm{card}\, O_i < \infty$, where

$$O_i := \left\{\alpha \in \mathbb{R} \,:\, f(\alpha) = (-1)^{(i+1)} f^+\right\} = \{\alpha_1^{(i)}, \ldots, \alpha_{P^{(i)}}^{(i)}\},$$

$O_1 \neq \emptyset$ and, if $f^- = f^+$ and $p < 1$, then also $O_2 \neq \emptyset$. If $p = 1$, then $O_2 := \emptyset$.

Note, (1.3), $L(1) \in \mathcal{S}$ and Proposition 1.2 imply $\mathbb{P}(|L(1)| > x) \sim p^{-1}\mathbb{P}(L(1) > x)$ for $x \to \infty$. Condition (G) excludes kernel functions, which are piecewise constant in their extremes.

The paper is organized as follows. In Section 2 we give conditions for the stationarity of Y and calculate the tail behavior of the Lévy measure of Y under Condition (M). If $L(1) \in \mathcal{S}$ and if $-L(1)$ satisfies weak conditions, we can transfer the results to the tail behavior of Y. Furthermore, we present the most important example, namely Poisson shot noise processes. Poisson shot noise processes form the basic structure for our results.

In Section 3 we derive results on weak convergence of point processes of subexponential sequences in a general setup. These are fundamental results for our continuous-time process as its extreme behavior is governed by a discrete-time skeleton. Furthermore, we derive path properties if a high level exceedance occurs. Such results apply also immediately to discrete-time MA processes.

Such results of Sections 2 and 3 are applied in Section 4 to subexponential Lévy driven MA processes in $\mathrm{MDA}(\Lambda)$, which means that (G) is satisfied. As can be seen from (1.1) if $\triangle L(t^*) = L(t^*) - L(t^*-)$ for some $t^* \in \mathbb{R}$ is extremely large then $Y(t)$ behaves roughly like $f(t - t^*)\triangle L(t^*)$ for any $t \in \mathbb{R}$. Thus, our investigation on the extremal behavior of Y is based on a discrete-time skeleton $\{Y(t_n)\}_{n\in\mathbb{N}}$, where the discrete-time random sequence $\{t_n\}_{n\in\mathbb{N}}$ is chosen as to incorporate those times, where big jumps of the Lévy process and extremes of the kernel function occur. We embed the process $\{Y(t_n)\}_{n\in\mathbb{N}}$ in a sequence of point processes and derive the weak limit of this sequence. Not surprisingly, we find a strong analogy to the point process behavior of discrete-time MA processes and corresponding results of Davis and Resnick [8] and Rootzén [22]. We model the path behavior of the continuous-time process near high level excursions by a mark on the point process. Obviously marks are influenced by the kernel function and its local extremes. High level excursions of Y are, in contrast to regularly varying models, no longer persistent; in the limit they collapse into singular time points, where also extremes of the kernel function occur. Choosing another normalization we show, that the marks behave asymptotically like the deterministic functions $f(\cdot)/f^+$ or $-f(\cdot)/f^+$. Our findings point out that our discrete-time skeleton reflects local extremes of Y. Finally, we derive the limit distribution of running maxima. We conclude with the proofs of our results in Section 5.

2 Stationarity and tail behavior

Under certain conditions the integral given in (1.1) is well-defined as a limit in probability of integrals of step functions approximating f. This has been shown by Rajput and Rosinski [19], Theorem 2.7. They give necessary and sufficient conditions, which are formulated in terms of the kernel function f and the generating triplet of $L(1)$. Under these assumptions Y is i. d., and by the structure of a MA process Y is stationary. The following Proposition gives sufficient conditions to ensure that these assumptions are satisfied. For the proof of Proposition 2.1 we refer to Proposition 1.1.7 of Fasen [10] and of Proposition 2.2 to Section 5.

Proposition 2.1 (Existence) *Let Y be a MA process as given in (1.1) satisfying Condition (M). Then Y is well-defined, i. d. and stationary. The generating triplet of the marginal distribution of Y is (m_Y, σ_Y^2, ν_Y), where*

$$\begin{aligned} m_Y &= \int_{-\infty}^{\infty} m f(s) + \int_{-\infty}^{\infty} (\kappa(xf(s)) - f(s)\kappa(x))\, \nu(dx)\, \lambda(ds), \\ \sigma_Y^2 &= \sigma^2 \int_{-\infty}^{\infty} f^2(s)\, \lambda(ds), \\ \nu_Y(x,\infty) &= \int_{-\infty}^{\infty} \nu\left(\frac{x}{f^+(s)}, \infty\right) \lambda(ds) + \int_{-\infty}^{\infty} \nu\left(-\infty, \frac{-x}{f^-(s)}\right) \lambda(ds) \text{ for } x > 0. \end{aligned} \tag{2.1}$$

Proposition 2.2 *Let Y be a MA process as given in (1.1) satisfying Condition (M). Suppose $Z^{(1)}$ is a r.v. having d. f. $\nu\left(1, \cdot \vee 1\right] / \nu(1,\infty)$, and $Z^{(2)}$ is a r.v. having d. f. $\nu\left[-(\cdot \vee 1), -1\right) / \nu(-\infty,-1)$. Let A be a Borel set on $\mathbb{R}$ such that there exist a Borel set $B_y = \{t \in \mathbb{R} : |f(t)| \geq y\} \subseteq A$, where B_y has a finite positive Lebesgue measure and $B_y \subseteq B_{y-\delta} \subseteq A$ for some $\delta > 0$. Moreover, we assume $f^- \leq f^+$ and U_A is a uniform r. v. on A independent of $Z^{(1)}$ and $Z^{(2)}$.*

(a) Then for $x \to \infty$,

$$\nu_Y(x,\infty) \sim \lambda(A)\nu(1,\infty)\mathbb{P}(f^+(U_A)Z^{(1)} > x) + \lambda(A)\nu(-\infty,-1)\mathbb{P}(f^-(U_A)Z^{(2)} > x).$$

(b) Let $L(1) \in \mathcal{S}$, and if $f^- = f^+$ and $L(1)$ has an infinite left endpoint, we suppose $-L(1) \in \mathcal{S}$. Then $f(U_A)L(1) \in \mathcal{S}$ if and only if $Y(t) \in \mathcal{S}$ for $t \in \mathbb{R}$. In this

case

$$\mathbb{P}(Y(t) > x) \sim \lambda(A)\mathbb{P}(f(U_A)L(1) > x) \quad \text{for } x \to \infty.$$

If $|f(t)| \to 0$ for $|t| \to \infty$, then there exists a $t_0 > 0$ such that we can choose $A = (-s, s)$ for any $s \geq t_0$. The interval $(-t_0, t_0)$ contains all time points, where f achieves its maxima and minima. In this case $U_A = sU$, where U is a uniform r. v. on $(-1, 1)$. If the kernel function f is positive, then $f(sU)L(1) \in \mathcal{S}$ by Proposition 1.2 *(vi)*; further conditions can be found in Fasen [10], Remark 1.3.5. The next Lemma is the basis for the results in Section 4.

Lemma 2.3 *Let Y be a MA process as given in (1.1) satisfying Condition (M) with $L(1) \in \mathcal{S}$ and $f^- \leq f^+$. Suppose for every $\epsilon > 0$ there exists a Borel set $B_y = \{t \in \mathbb{R} : |f(t)| \geq y\}$ with $0 < \lambda(B_y) \leq \epsilon$. Then*

$$\mathbb{P}(|Y(t)| > x) = o(\mathbb{P}(f^+|L(1)| > x)) \quad \text{for } x \to \infty.$$

Lemma 2.3 does not hold if f is piecewise constant in a local extreme. For this reason, we need $\operatorname{card} O_i < \infty$, $i = 1, 2$, in Condition (G).

Example 2.4 (Poisson shot noise process) Consider in (1.1) as driving process a compound Poisson process L with

$$L(t) = \sum_{j=1}^{N(t)} Z_j \quad \text{and} \quad L(-t) = \sum_{j=1}^{-N(-t-)} Z_{-j} \quad \text{for } t \geq 0, \tag{2.2}$$

where $\{N(t)\}_{t\in\mathbb{R}}$ is a Poisson process on $\mathbb{R}$ with intensity $\mu > 0$ and jump times $\{\Gamma_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, $\cdots < \Gamma_{-1} < 0 < \Gamma_1 < \cdots$, which is independent of the i. i. d. sequence $\{Z_k\}_{k\in\mathbb{Z}}$. Favorably for such a Y under Condition (M), is the representation

$$Y(t) = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} f(t - \Gamma_j)Z_j \quad \text{for } t \in \mathbb{R}. \tag{2.3}$$

We call Y given in (2.3) a *Poisson shot noise process*. If additionally f is positive and Z_1 has only support on $\mathbb{R}_+$, we call Y a *positive Poisson shot noise process*. In particular, for a non-increasing $f : [0, \infty) \to [0, \infty)$ the positive Poisson shot noise process is non-increasing between successive jumps of L, and thus Y has a local maximum in t if and only if $t \in \{\Gamma_k\}_{k\in\mathbb{N}}$. This means $\{Y(\Gamma_k)\}_{k\in\mathbb{N}}$ are the local extremes of Y on $\mathbb{R}_+$ and characterize the extremal behavior of Y.

The Lévy measure ν of L is $\nu(x, \infty) = \mu\mathbb{P}(Z_1 > x)$ for $x \in \mathbb{R}$ (cf. Sato [23], Theorem 4.3). Proposition 1.2 *(iv)* gives $L(1) \in \mathcal{S}$ if and only if $Z_1 \in \mathcal{S}$, and in that case,

$$\mathbb{P}(L(1) > x) \sim \mu\mathbb{P}(Z_1 > x) \quad \text{for } x \to \infty. \tag{2.4}$$

If Y is a positive Poisson shot noise process and $|f(t)| \to 0$ for $|t| \to \infty$, then by Proposition 2.2 and Lemma 2.3 there exists a $t_0 > 0$ such that for $s \geq t_0$ and $x \to \infty$,

$$\mathbb{P}(Y(t) > x) \sim 2s\mathbb{P}(f(sU)L(1) > x) = o(\mathbb{P}(f^+|L(1)| > x)),$$

where U is a uniform r. v. on $(-1, 1)$ independent of $L(1)$. □

Example 2.5 (Discrete-time MA process) Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be an i. i. d. sequence of r. v. s and $\{c_k\}_{k\in\mathbb{Z}}$ be a sequence of real constants with $c^- \leq c^+$. Then

$$Y_n = \sum_{k=-\infty}^{\infty} c_{n-k}\xi_k \quad \text{for } n \in \mathbb{Z} \tag{2.5}$$

is called a *discrete-time MA process*. Let $\xi_1 \in \mathcal{S} \cap \mathrm{MDA}(\Lambda)$ be i. d. with infinite right endpoint. Let additionally the tail balance condition $\lim_{x\to\infty} \mathbb{P}(\xi_1 < -x)/\mathbb{P}(\xi_1 > x) = p^{-1}(1-p)$ for $p \in (0, 1]$ holds. This model can be considered as a special case of Y in (1.1): choose $f(t) = \sum_{k=-\infty}^{\infty} c_k \mathbf{1}_{[k-1,k)}(t)$ for $t \in \mathbb{R}$. The continuous-time MA process Y viewed at discrete-time points $Y(n) = \sum_{k=-\infty}^{\infty} c_{n-k}[L(k+1) - L(k)]$ for $n \in \mathbb{Z}$, is a discrete-time MA process with $\xi_k = L(k+1) - L(k)$. By Proposition 2.2 the process Y and, hence also the discrete-time MA process $\{Y_n\}_{n\in\mathbb{Z}}$ is well-defined and stationary, if either $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, or $\mathbb{E}\xi_k = 0$ and $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$. In particular, MA processes with the long memory property $\sum_{k=-\infty}^{\infty} \gamma(k) = \infty$, where γ denotes the covariance function, are included. Write $P^{(1)} = \#\{k : c_k = c^+\}$ and $P^{(2)} = \#\{k : c_k = -c^+\}$. Then we have by Proposition 2.2,

$$\mathbb{P}(Y_n > x) \sim \left(P^{(1)} + p^{-1}(1-p)P^{(2)}\right) \mathbb{P}(c^+\xi_1 > x) \quad \text{for } x \to \infty.$$

3 Extremal behavior of subexponential sequences

In this section we investigate the extremal behavior of processes, not necessarily stationary with marginals in $\mathcal{S} \cap \mathrm{MDA}(\Lambda)$. Throughout this section, we continue the example of a discrete-time MA process as it provides a good intuition.

We follow Resnick [20] and introduce point processes to describe the extremal behavior precisely. Let S denote the locally compact and separable Hausdorff space $[0, \infty) \times \mathbb{R}$ with the Borel σ-field $\mathcal{B}(S)$, and $M_P(S)$ denotes the class of point measures on S with metric ρ that generates the topology of vague convergence. A measure of the form $\sum_{k\in I} \varepsilon_{x_k}$, where $x_k \in S$, I is at most countable and ε_{x_k} denotes the Dirac measure in x_k, is a point measure. The space $(M_P(S), \rho)$ is a complete and separable metric space provided with the Borel σ-field $\mathcal{M}_P(S)$. A *point process* in S is a random element in $(M_P(S), \mathcal{M}_P(S))$, i. e. a measurable map from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into $(M_P(S), \mathcal{M}_P(S))$. Given a Radon measure ϑ on $\mathcal{B}(S)$, a point process κ is called *Poisson random measure* with *intensity measure* ϑ, denoted by PRM(ϑ), if $\kappa(A)$ is Poisson distributed with intensity $\vartheta(A)$ for every $A \in \mathcal{B}(S)$ and if for mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{B}(S)$, $n \in \mathbb{N}$, the r. v. s $\kappa(A_1), \ldots, \kappa(A_n)$ are independent. More about point processes can be found in Daley and Vere-Jones [7] and Kallenberg [16].

First we study the extremal behavior of discrete-time processes via point processes. This result will be used in Section 4 to derive the point process behavior of the discrete-time sequence $\{Y(t_n)\}_{n\in\mathbb{N}}$, where Y is the MA process as given in (1.1) and $\{t_n\}_{n\in\mathbb{N}}$ is a properly chosen discrete-time random sequence.

Proposition 3.1 *Let $\{Z_k\}_{k\in\mathbb{N}}$ be identically distributed r. v. s in $\mathcal{S} \cap \mathrm{MDA}(\Lambda)$ and $\{\theta_k\}_{k\in\mathbb{N}}$ be a sequence of r. v. s. Suppose $a_T > 0$, $b_T \in \mathbb{R}$ and $u_T = a_T x + b_T$ are constants such that*

$$\lim_{T\to\infty} T\mathbb{P}(Z_1 > u_T) = \exp(-x) \quad \text{for } x \in \mathbb{R}$$

holds. Furthermore, assume there exists a sequence $\{\Theta_k\}_{k\in\mathbb{N}}$ with $\Theta_k \stackrel{d}{=} \Theta_1$ for $k \in \mathbb{N}$ such that $\theta_k \le \Theta_k$ a. s., Θ_k is independent of Z_k for every $k \in \mathbb{N}$, and

$$\mathbb{P}(\Theta_1 > x) = o(\mathbb{P}(Z_1 > x)) \text{ for } x \to \infty.$$

Let $\{\Gamma_k\}_{k\in\mathbb{N}}$ be the points of a Poisson process with intensity $\mu > 0$, and for $\alpha \in \mathbb{R}$ arbitrary let $s_k \in [\Gamma_{k-1} + \alpha, \Gamma_{k+1} + \alpha)$ for $k \in \mathbb{N}$, setting $\Gamma_0 := 0$. Denote by

$$\widetilde{\kappa}_T = \sum_{k=1}^{\infty} \varepsilon_{\left(k/T, a_T^{-1}(Z_k - b_T)\right)} \quad \text{and} \quad \kappa_T = \sum_{k=1}^{\infty} \varepsilon_{\left(s_k\mu/T, a_T^{-1}(Z_k + \theta_k - b_T)\right)} \quad \text{for } T > 0$$

point processes in $M_P(S)$. Suppose there exists a point process κ in $M_P(S)$ with $\kappa([s,t) \times \{x\}) = 0$ a. s. for $s, t \ge 0$ such that $\widetilde{\kappa}_T \stackrel{T\to\infty}{\Longrightarrow} \kappa$. Let $I = [s,t) \times (x, \infty) \subseteq S$. Then

$$\lim_{T\to\infty} \mathbb{P}(\kappa_T(I) \neq \widetilde{\kappa}_T(I)) = 0 \quad \text{and} \quad \kappa_T \stackrel{T\to\infty}{\Longrightarrow} \kappa.$$

In particular, if $\{Z_k\}_{k\in\mathbb{N}}$ is an i. i. d. sequence, then κ is a PRM(ϑ) with intensity measure $\vartheta(dt \times dx) = dt \times \exp(-x)\, dx$.

Example 3.2 (Continuation of Example 2.5) Let $c_{i_1} = \cdots = c_{i_{P^{(1)}}} = c^+$, $c_{j_1} = \cdots = c_{j_{P^{(2)}}} = -c^+$, and else $|c_k| < c^+$. In the case $p = 1$ set $P^{(2)} := 0$. Furthermore, let $a_T > 0$, $b_T \in \mathbb{R}$ and $u_T = a_T x + b_T$ for $x \in \mathbb{R}$ such that $\lim_{T\to\infty} T\mathbb{P}(c^+\xi_1 > u_T) = \exp(-x)$. For $k \in \mathbb{Z}$ define the stationary processes

$$\overline{\xi}_k = -\xi_{k-j_1} - \ldots - \xi_{k-j_{P^{(2)}}} + \xi_{k-i_1} + \ldots + \xi_{k-i_{P^{(1)}}} \quad \text{and} \quad \theta_k = Y_k - c^+\overline{\xi}_k.$$

Let $\sum_{k=1}^{\infty} \varepsilon_{(s_{ki},P_{ki})}$ be independent PRM(ϑ_i), $i = 1, 2$, with $\vartheta_1(dt \times dx) = dt \times \exp(-x)\,dx$ and $\vartheta_2(dt \times dx) = dt \times p^{-1}(1-p)\exp(-x)\,dx$ respectively. On the one hand

$$\sum_{k=1}^{\infty} \varepsilon_{\left(k/T, a_T^{-1}(c^+\overline{\xi}_k - b_T)\right)} \stackrel{T\to\infty}{\Longrightarrow} P^{(1)} \sum_{k=1}^{\infty} \varepsilon_{(s_{k1},P_{k1})} + P^{(2)} \sum_{k=1}^{\infty} \varepsilon_{(s_{k2},P_{k2})}. \tag{3.1}$$

On the other hand $\mathbb{P}(|\theta_k| > x) \sim K\mathbb{P}(\widetilde{c}\xi_1 > x) = o(\mathbb{P}(c^+|\overline{\xi}_1| > x))$ for $x \to \infty$ and some $K > 0$ by Example 2.5 and the rapidly varying tails of ξ_1, where $\widetilde{c}$ is the second largest value of $\{|c_k|\}_{k\in\mathbb{Z}}$. Hence, by Proposition 3.1, for $I = [s,t) \times (x,\infty) \subseteq S$ we have

$$\mathbb{P}\left(\sum_{k=1}^{\infty} \varepsilon_{\left(k/T, a_T^{-1}(Y_k - b_T)\right)}(I) \neq \sum_{k=1}^{\infty} \varepsilon_{\left(k/T, a_T^{-1}(c^+\overline{\xi}_k - b_T)\right)}(I)\right) \stackrel{T\to\infty}{\longrightarrow} 0, \tag{3.2}$$

and by (3.1) we obtain

$$\sum_{k=1}^{\infty} \varepsilon_{\left(k/T, a_T^{-1}(Y_k - b_T)\right)} \stackrel{T\to\infty}{\Longrightarrow} P^{(1)} \sum_{k=1}^{\infty} \varepsilon_{(s_{k1},P_{k1})} + P^{(2)} \sum_{k=1}^{\infty} \varepsilon_{(s_{k2},P_{k2})}.$$

This result extends Theorem 3.3 of Davis and Resnick [8], who proved it under the condition $\sum_{k=-\infty}^{\infty} |c_k|^\delta < \infty$ for some $\delta \in (0,1)$. □

Proposition 3.1 gives a criterion for point process convergence of a discrete-time subexponential sequence with marginals in MDA(Λ). In the continuous-time setting of a MA process as given in (1.1), we apply the results to a properly chosen discrete-time skeleton $\{Y(t_n)\}_{n\in\mathbb{N}}$. But then also the behavior of the continuous-time process between the discrete-time points matters. The question arises how long the sample paths of Y stays on a high level, and how it reverts to its mean after exceeding a high threshold. The following Lemma is essential for describing the sample path of Y after a high level exceedance.

Lemma 3.3 *Let $Y = \{Y(t)\}_{t\in\mathbb{R}}$ be a stochastic process in $\mathbb{R}$ with decomposition*

$$Y(t) = \widetilde{f}(t)Z + \widetilde{Y}(t) \quad \text{for } t \in \mathbb{R},$$

where $Z \in \mathcal{S} \cap \mathrm{MDA}(\Lambda)$ is a r. v. independent of $\widetilde{Y} = \{\widetilde{Y}(t)\}_{t\in\mathbb{R}}$, and $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ is a deterministic function with $\widetilde{f}^- \leq \widetilde{f}^+ < \infty$. Furthermore, assume there exist constants $a_T > 0$, $b_T \in \mathbb{R}$ and $u_T = a_T x + b_T$ for $x \in \mathbb{R}$ such that

$$\lim_{T\to\infty} T\mathbb{P}(\widetilde{f}^+ Z > u_T) = \exp(-x).$$

Define $\tau = \widetilde{f}^+ Z + \theta$, where θ is independent of Z and satisfies

$$\mathbb{P}(\theta > x) = o(\mathbb{P}(\widetilde{f}^+ Z > x)) \quad \text{for } x \to \infty. \tag{3.3}$$

Then the following assertions hold:

(a) Let $J \subseteq \mathbb{R}$ *and* $\mathbb{P}(\sup_{t\in J} |\widetilde{Y}(t)| < \infty) = 1$. *Then, we have*

$$\lim_{T\to\infty} \mathbb{P}\left(\sup_{t\in J} \left| \frac{Y(t)}{b_T} - \frac{\widetilde{f}(t)}{\widetilde{f}^+} \right| > \epsilon \middle| \tau > u_T \right) = 0.$$

(b) Let $O = \{\alpha_1, \ldots, \alpha_P\}$ *be a finite set in* $\mathbb{R}$ *such that* $\widetilde{f}(t) = \widetilde{f}^+$ *for* $t \in O$. *For* $y_1, \ldots, y_P \in \mathbb{R}$, *and* $y = \max\{0, y_1, \ldots, y_P\}$ *we have*

$$\lim_{T\to\infty} \mathbb{P}\left(Y(\alpha_1) > u_T + a_T y_1, \ldots, Y(\alpha_P) > u_T + a_T y_P \middle| \tau > u_T \right) = \exp(-y).$$

(c) Let $t \in \mathbb{R}$ *with* $\widetilde{f}(t) < \widetilde{f}^+$ *and* $\mathbb{P}(\widetilde{Y}(t) > x) = o(\mathbb{P}(\widetilde{f}^+ Z > x))$ *for* $x \to \infty$. *Then,*

$$\lim_{T\to\infty} \mathbb{P}\left(Y(t) > u_T + a_T y \middle| \tau > u_T \right) = 0 \quad \text{for } y \in \mathbb{R}.$$

Remark 3.4 *Let* $\alpha \in \mathbb{R}$ *with* $\widetilde{f}(\alpha) = \widetilde{f}^+$, $\mathbb{P}(\widetilde{Y}(\alpha) > x) = o(\mathbb{P}(\widetilde{f}^+ Z > x))$ *for* $x \to \infty$ *and* $\tau = Y(\alpha)$, *where we suppose* $\widetilde{Y}$ *is a. s. bounded on every compact set on* $\mathbb{R}$. *Then, Lemma 3.3 (a) describes the sample paths behavior of* Y, *if it has an exceedance over the threshold* u_T *at time point* α. *More precisely, let* X_T *for* $T > 0$ *be processes in some measurable metric space* $(\widetilde{\mathbb{D}}, \widetilde{\mathcal{D}})$, *where uniform convergence on compacta is sufficient for convergence. The process* X_T *is defined to have the distribution*

$$\mathbb{P}(X_T \in D) = \mathbb{P}(Y \in D | Y(\alpha) > u_T) \quad \text{for } D \in \widetilde{\mathcal{D}}.$$

Then Lemma 3.3 (a) states that X_T/b_T *converges weakly to the deterministic function* $\widetilde{f}(\cdot)/\widetilde{f}^+$. *Thus, the sample path of* Y/b_T *after an exceedance of* $Y(\alpha)$ *above* u_T *is asymptotically* $\widetilde{f}(\cdot)/\widetilde{f}^+$. *For* $P = 1$, *the exponential limit in (b) corresponds to the limiting generalized Pareto distribution for scaled excesses in* $\mathrm{MDA}(\Lambda)$. □

Example 3.5 (Continuation of Example 3.2) Suppose $P^{(1)} = 1$, $P^{(2)} = 0$ and $c_0 = c^+$. Let $k \in \mathbb{Z}$ be fixed. Define the discrete-time process $Y(n) = Y_n$, $\widetilde{f}(n) = c_{n-k}$, $\widetilde{Y}(n) = Y_n - \widetilde{f}(n)\xi_k$ for $n \in \mathbb{Z}$ and $Z = \xi_k$. Let X_T be a stochastic process with $\mathbb{P}(X_T \in D) = \mathbb{P}(Y \in D | Y_k > u_T)$ for $D \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}})$. Then Lemma 3.3 *(a)* implies $X_T/b_T \overset{T\to\infty}{\Longrightarrow} \{c_{n-k}/c^+\}_{n\in\mathbb{Z}}$. Applying Rootzén [22], Lemma 3.4, yields

$$\sum_{k=1}^{\infty} \varepsilon_{(k/T, a_T^{-1}(Y_k - b_T), (Y_n/b_T)_{n\in\mathbb{Z}})} \overset{T\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \varepsilon_{(s_{k1}, P_{k1}, (c_{k-n}/c^+)_{n\in\mathbb{Z}})}.$$

For a subclass of the extended heavy tailed Weibull distribution with the specific tail $\mathbb{P}(\xi_1 > x) \sim K x^{\beta} \exp(-x^{\alpha})$ for $x \to \infty$, $K > 0$, $\beta \in \mathbb{R}$, $\alpha \in (0, 1)$, this result can be found in Rootzén [22], Theorem 8.6. □

4 Extremal behavior of a Lévy driven MA process

In this section we study the extremal behavior of a subexponential Lévy driven MA process Y as given in (1.1) satisfying Condition (G). To this end we use a discrete-time skeleton. This means we investigate the extremal behavior of a discrete-time sequence $\{Y(t_n)\}_{n\in\mathbb{N}}$, where the discrete-time random sequence $\{t_n\}_{n\in\mathbb{N}}$ is chosen properly by the jump times of the driving Lévy process and the extremes of the kernel function. We shall show that the extremes of $\{Y(t_n)\}_{n\in\mathbb{N}}$ coincide with the extremes of Y on high levels.

Therefore, we decompose L in two independent Lévy processes according to its jump sizes: $L = L_1 + L_2$ with Lévy measure

$$\nu_1(A) = \nu(A \cap (1,\infty)) + \nu(A \cap (-\infty,-1)) \quad \text{for } A \in \mathcal{B}(\mathbb{R})$$

and generating triplet $(0,0,\nu_1)$ of L_1. The Lévy process L_2 has generating triplet $(m,\sigma^2,\nu-\nu_1)$. Then L_1 is a compound Poisson process whose jumps have modulus larger than 1, and L_2 has jumps with modulus only smaller than 1. Hence, $L_1(t) = \sum_{j=1}^{N(t)} Z_j$ for $t \geq 0$, where $N = \{N(t)\}_{t\in\mathbb{R}}$ is a Poisson process with intensity $\mu = \nu_1(\mathbb{R})$, and jump times $\Gamma = \{\Gamma_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, $\cdots < \Gamma_{-1} < 0 < \Gamma_1 < \cdots$. The sequence $Z = \{Z_k\}_{k\in\mathbb{Z}}$ consists of i. i. d. random variables with d. f. s $\nu_1(-\infty,\cdot]/\mu$. Furthermore, N and Z are independent. This decomposition of L induces a decomposition of Y giving $Y = Y_1 + Y_2$, where for $i = 1, 2$,

$$Y_i(t) = \int_{-\infty}^{\infty} f(t-s)\, dL_i(s) \quad \text{for } t \in \mathbb{R} \tag{4.1}$$

are independent MA processes. Then Y_1 has the modification

$$Y_1(t) = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} f(t-\Gamma_j) Z_j \quad \text{for } t \in \mathbb{R}, \tag{4.2}$$

where the right hand side is defined pathwise. First we give a short motivation for the choice of the discrete-time random sequence $\{t_n\}_{n\in\mathbb{N}}$. Consider the Poisson shot noise process Y_1 given in (4.2), then

$$Y_1(\Gamma_k + t) = f(t)Z_k + \sum_{\substack{j=-\infty \\ j\neq k,0}}^{\infty} f(t+\Gamma_k-\Gamma_j)Z_j \quad \text{for } k \in \mathbb{N},\, t \in \mathbb{R}.$$

For subexponential $\{Z_k\}_{k\in\mathbb{Z}}$ some Z_k is likely to be large in comparison to other terms of the sequence. Then $Y_1(\Gamma_k + t)$ behaves roughly like $f(t)Z_k$. The process $\{f(t)Z_k\}_{t\in\mathbb{R}}$ achieves a maximum only for some $t \in O_1$. Similar results hold for large negative jumps and a minimum $t \in O_2$ of the kernel function. This suggests that $Y_1(t_n)$ with

$$t_n \in \{\Gamma_k + \alpha_l^{(1)} : k \in \mathbb{N},\, l = 1, \ldots, P^{(1)}\} \,\cup \{\Gamma_k + \alpha_l^{(2)} : k \in \mathbb{N},\, l = 1, \ldots, P^{(2)}\}$$

is a local extreme value of Y_1, if the absolute value of the jump of L is large.

Theorem 4.1 *Let Y be a MA process as given by (1.1) satisfying Condition (G), where Y has the decomposition (4.1) with (4.2). For $i = 1, 2$, $l = 1, \ldots, P^{(i)}$, define point processes in $M_P(S)$ by*

$$\kappa_T^{(i,l)} = \sum_{k=1}^{\infty} \varepsilon_{\left((\Gamma_k + \alpha_l^{(i)})/T,\, a_T^{-1}(Y(\Gamma_k + \alpha_l^{(i)}) - b_T)\right)}, \qquad \widetilde{\kappa}_T^{(i)} = \sum_{k=1}^{\infty} \varepsilon_{\left(k/(T\mu),\, a_T^{-1}((-1)^{(i+1)} f + Z_k - b_T)\right)}.$$

Let $\sum_{k=1}^{\infty} \varepsilon_{(s_{ki}, P_{ki})} = \kappa^{(i)}$ be a PRM(ϑ_i), $i = 1, 2$, with intensity measure $\vartheta_1(dt \times dx) = dt \times \exp(-x)\, dx$ and $\vartheta_2(dt \times dx) = dt \times p^{-1}(1-p)\, \exp(-x)\, dx$ respectively. Suppose $\kappa^{(1)}$ and $\kappa^{(2)}$ are independent. Furthermore, define the point processes

$$\kappa_T = \sum_{l=1}^{P^{(1)}} \kappa_T^{(1,l)} + \sum_{l=1}^{P^{(2)}} \kappa_T^{(2,l)}, \qquad \widetilde{\kappa}_T = P^{(1)} \widetilde{\kappa}_T^{(1)} + P^{(2)} \widetilde{\kappa}_T^{(2)}, \qquad \kappa = P^{(1)} \kappa^{(1)} + P^{(2)} \kappa^{(2)}.$$

Let $I = [s, t) \times (x, \infty) \subseteq S$. Then for $i = 1, 2$, $l = 1, \ldots, P^{(i)}$, we have

$$\lim_{T \to \infty} \mathbb{P}(\kappa_T^{(i,l)}(I) \neq \widetilde{\kappa}_T^{(i)}(I)) = 0 \qquad \text{and} \qquad \kappa_T \overset{T \to \infty}{\Longrightarrow} \kappa.$$

The limit process of the *point process of exceedances* $\kappa_T(\cdot \times (x, \infty))$ for $x > 0$ fixed, is the sum of two independent compound Poisson random measures with constant cluster sizes $P^{(1)}$ and $P^{(2)}$ respectively. If f has at most one maximum and at most one minimum the limit process κ is a Poisson random measure. This case reflects no clusters on high levels.

The sample paths behavior near high level excursions is modelled by *marked point processes*. For our model a marked point process is a point process in $S \times [-\infty, \infty]^m$ for $m \in \mathbb{N}$. The coordinates higher than three describe the behavior of the continuous-time process in the neighborhood of an exceedance over the threshold u_T in the second coordinate. More about the concept of marked point processes can be found in Daley and Vere-Jones [7], Section 6.4. The following corollary describes the behavior of marked point processes.

Corollary 4.2 *Let the assumptions of Theorem 4.1 hold. Suppose $t_1, \ldots, t_m \in \mathbb{R}$, $i \in \{1, 2\}$ is fixed and $\alpha^{(i)} \in O_i$. Then the following statements hold.*

(a) *Let*
$$K_T = \sum_{k=1}^{\infty} \varepsilon_{\left((\Gamma_k+\alpha^{(i)})/T, a_T^{-1}(Y(\Gamma_k+\alpha^{(i)})-b_T), \{a_T^{-1}(Y(\Gamma_k+t_j)-b_T)\}_{j=1,\ldots,m}\right)},$$
$$K = \sum_{k=1}^{\infty} \varepsilon_{(s_{ki}, P_{ki}, \{P_{ki}\,\mathbf{1}_{\{f(t_j)=(-1)^{(i+1)}f^+\}} + \varepsilon_{-\infty}\,\mathbf{1}_{\{f(t_j)\neq(-1)^{(i+1)}f^+\}}\}_{j=1,\ldots,m})}$$
be point processes in $M_P(S\times[-\infty,\infty]^m)$. *Then* $K_T \stackrel{T\to\infty}{\Longrightarrow} K$.

(b) *Let*
$$K_T = \sum_{k=1}^{\infty} \varepsilon_{\left((\Gamma_k+\alpha^{(i)})/T, a_T^{-1}(Y(\Gamma_k+\alpha^{(i)})-b_T), \{Y(\Gamma_k+t_j)/b_T\}_{j=1,\ldots,m}\right)},$$
$$K = \sum_{k=1}^{\infty} \varepsilon_{(s_{ki}, P_{ki}, \{(-1)^{(i+1)}f(t_j)/f^+\}_{j=1,\ldots,m})}$$
be point processes in $M_P(S\times\mathbb{R}^m)$. *Then* $K_T \stackrel{T\to\infty}{\Longrightarrow} K$.

(c) *Define* $P = P^{(i)}$, $\alpha_l = \Gamma_k + \alpha_l^{(i)}$, $l = 1,\ldots,P$, *and* $\alpha = \Gamma_k + \alpha^{(i)}$ *for some* $k\in\mathbb{N}$. *For* $y_1,\ldots,y_P\in\mathbb{R}$, $y=\max\{0,y_1,\ldots,y_P\}$, $\theta_1 = 1$ *and* $\theta_2 = p^{-1}(1-p)$ *we have*

$$\lim_{T\to\infty} \mathbb{P}\left(Y(\alpha_1) > u_T + a_T y_1, \ldots, Y(\alpha_P) > u_T + a_T y_P \,|\, Y(\alpha) > u_T\right) = \exp(-\theta_i y).$$

(d) *Let* $t\notin O_i$ *and* $y\in\mathbb{R}$. *Then*

$$\lim_{T\to\infty} \mathbb{P}(Y(\Gamma_k+t) > u_T + a_T y | Y(\Gamma_k+\alpha^{(i)}) > u_T) = 0.$$

Remark 4.3

(i) Theorem 4.1 states that exceedances of $\{Y(\Gamma_k+\alpha_l^{(i)})\}_{k\in\mathbb{N}}$ *above the threshold* u_T *behave like the exceedances of* $\{(-1)^{(i+1)}f^+Z_k\}_{k\in\mathbb{Z}}$ *above* u_T *for* $T\to\infty$. *Hence, the influence of small jumps of the Lévy process, represented in* Y_2, *are negligible for the extremal behavior of* $\{Y(\Gamma_k+\alpha_l^{(i)})\}_{k\in\mathbb{N}}$, *since* Z_k *represents the jumps of* L *with modulus larger than* 1. *Furthermore, this result means that extremely large jumps of the Lévy process cause extremely large jumps of the MA process. Fasen [10], Theorem 1.4.5 shows the converse that under more restrictive assumptions on the kernel function extreme large jumps of the MA process can only be caused by extreme large jumps of the Lévy process.*

(ii) The discrete-time skeleton $\{Y(t_n)\}_{n\in\mathbb{N}}$ *reflects the local maxima of the process on high levels; see Corollary 4.2 (b). Notice that in the last coordinate of* K_T *in (b) the normalization* b_T *represents the behavior of* $Y(\Gamma_k+\alpha_l^{(i)})$.

(iii) The extremal behavior of a continuous-time MA process is similar to the extremal behavior of a discrete-time MA processes, cf. Examples 3.2 and 3.5. In both cases the cluster behavior depends on the number of extremes of the kernel function. □

In the following theorem we calculate the normalizing constants of running maxima of Y.

Theorem 4.4 *Let Y be a MA process as given in (1.1) satisfying Condition (G), where Y has the decomposition (4.1) with (4.2). Assume the kernel function f satisfies $\int_{-\infty}^{\infty} \sup_{0\le t\le 1} |f(t+s)|\,\lambda(ds) < \infty$. Write $M(T) = \sup_{0\le t\le T} Y(t)$ for $T > 0$. Then,*

$$\lim_{T\to\infty} \mathbb{P}\left(a_T^{-1}(M(T) - b_T) \le x\right) = \exp(-[1 + p^{-1}(1-p)\,\mathbf{1}\{f^- = f^+\}]\, e^{-x}) \text{ for } x \in \mathbb{R}.$$

We impose a stronger condition on the kernel f than in (G), because we compute an upper bound for Y, which only exists under this additional assumption. For a Poisson shot noise process with non-negative, non-increasing kernel function, the normalizing constants of running maxima have already been calculated by Lebedev [17].

Remark 4.5 *If f is flat in its maximum and either $f^- < f^+$ or f is also flat in its minimum with value $-f^+$, the convergence of running maxima of Y is also ensured. Following the proof of Theorem 4.4 line by line and replacing the suprema in $X_n^{(i)}$ by the infima, a lower bound for $\sup_{t\in[n-1,n)} Y(t)$ can be found, without using Theorem 4.1.* □

5 Proofs

Proof of Proposition 2.2. *(a).* Using Davis and Resnick [8], Proposition 1.1, there exist $x_0, K > 0$, $\omega : (x_0/f^+, \infty) \to \mathbb{R}_+$ absolutely continuous with density ω', $\lim_{x\to\infty} \omega'(x) = 0$ and $\lim_{x\to\infty} \omega(x) = \infty$ such that for $x \ge x_0$:

$$\begin{aligned}&\frac{\int_{A^c} \nu(x/f^+(s), \infty)\,\lambda(ds) + \int_{A^c} \nu(-\infty, -x/f^-(s))\,\lambda(ds)}{\nu(x/y, \infty)}\\ &\qquad \le \frac{K}{\delta^2}\left(\frac{\omega(x/y)}{x/y}\right)^2 \int_{A^c} |f(s)|^2\,\lambda(ds). \qquad (5.1)\end{aligned}$$

By the rule of L'Hospital $\lim\limits_{x\to\infty} \omega(x)/x = \lim\limits_{x\to\infty} \omega'(x) = 0$. Hence, by (5.1) and $f \in \mathbb{L}^2$, we have

$$\begin{aligned}0 &\le \frac{\int_{A^c} \nu(x/f^+(s), \infty)\,\lambda(ds) + \int_{A^c} \nu(-\infty, -x/f^-(s))\,\lambda(ds)}{\int_A \nu(x/f^+(s), \infty) + \nu(-x/f^-(s), \infty)\,\lambda(ds)}\\ &\le C\frac{\int_{A^c} \nu(x/f^+(s), \infty)\,\lambda(ds) + \int_{A^c} \nu(-\infty, -x/f^-(s))\,\lambda(ds)}{\lambda(B_y)\nu(x/y, \infty)} \overset{x\to\infty}{\longrightarrow} 0.\end{aligned}$$

This means with $\mu_1 = \nu(1, \infty)$ and $\mu_2 = \nu(-\infty, -1)$ for $x \to \infty$,

$$\begin{aligned}&\int_{-\infty}^{\infty} \nu(x/f^+(s), \infty)\,\lambda(ds) + \int_{-\infty}^{\infty} \nu(-\infty, -x/f^-(s))\,\lambda(ds)\\ &\quad \sim \int_A \nu(x/f^+(s), \infty)\,\lambda(ds) + \int_A \nu(-\infty, -x/f^-(s))\,\lambda(ds)\\ &\quad = \lambda(A)\mu_1\mathbb{P}(f^+(U_A)Z^{(1)} > x) + \lambda(A)\mu_2\mathbb{P}(f^-(U_A)Z^{(2)} > x). \qquad (5.2)\end{aligned}$$

(b). If $L(1) \in \mathcal{S}$ and $-L(1) \in \mathcal{S}$ respectively, we obtain by Proposition 1.2 *(iv)* for $x \to \infty$,

$$\mathbb{P}(L^+(1) > x) \sim \mu_1 \mathbb{P}(Z^{(1)} > x) \quad \text{and} \quad \mathbb{P}(L^-(1) > x) \sim \mu_2 \mathbb{P}(Z^{(2)} > x)$$

respectively. Thus, standard arguments (for details see Fasen [10], Lemma 1.3.4) and *(a)* yields for $x \to \infty$,

$$\begin{aligned} \nu_Y(x,\infty) &\sim \lambda(A)\mathbb{P}(f^+(U_A)L^+(1) > x) + \lambda(A)\mathbb{P}(f^-(U_A)L^-(1) > x) \\ &= \lambda(A)\mathbb{P}(f(U_A)L(1) > x). \end{aligned} \tag{5.3}$$

If ν has a finite left endpoint, also the support of the Lévy measure of $f^-(U_A)L^-(1)$ is bounded below. Moreover $f^+(U_A)L^+(1) \in \mathcal{S}$ by Proposition 1.2 *(v)*. We have by Theorem 26.1 of Sato [23], and Proposition 1.2 *(i)*,

$$\mathbb{P}(f^-(U_A)L^-(1) > x) = o(\mathbb{P}(f^+(U_A)L^+(1) > x)) \quad \text{for } x \to \infty.$$

Then (5.3) follows again with $\mathbb{P}(f^-(U_A)Z^{(2)} > x) = 0$ for large x and *(a)*.

Thus, by (5.3) and Proposition 1.2 *(iv)* we obtain the r. v. $Y(t) \in \mathcal{S}$ if and only if $f(U_A)L(1) \in \mathcal{S}$. In this case $\mathbb{P}(Y(t) > x) \sim \nu_Y(x,\infty) \sim \lambda(A)\mathbb{P}(f(U_A)L(1) > x)$ for $x \to \infty$. □

Proof of Lemma 2.3. Let $\epsilon > 0$ and $0 < \lambda(B_y) \le \epsilon$. Similarly to (5.1), there exist $K_y, x_0 > 0, \omega : (x_0/f^+,\infty) \to \mathbb{R}_+$ absolutely continuous with $\lim_{x\to\infty} \omega(x)/x = 0$ such that for $x > x_0$,

$$\begin{aligned} \frac{\nu_Y(x,\infty)}{\nu(x/f^+,\infty)} &= \int_{-\infty}^{\infty} \frac{\nu\left(x/f^+(s),\infty\right)}{\nu(x/f^+,\infty)}\,\lambda(ds) + \int_{-\infty}^{\infty} \frac{\nu\left(-\infty,-x/f^-(s)\right)}{\nu(x/f^+,\infty)}\,\lambda(ds) \\ &\le K_y \left(\frac{\omega(x/y)}{x/y}\right)^2 \int_{B_y^c} |f(s)|^2\,\lambda(ds) + \epsilon, \end{aligned}$$

which tends to 0 as $x \to \infty$ and $\epsilon \downarrow 0$. On a similar way we obtain

$$\lim_{x\to\infty} \frac{\nu_Y(-\infty,-x)}{\nu(x/f^+,\infty)} = 0.$$

The statement follows by Proposition 1.2 *(vi)*. □

The main step of proving Proposition 3.1 is the following Lemma.

Lemma 5.1 *Let $Z \in \mathcal{S} \cap \mathrm{MDA}(\Lambda)$ be independent of the r. v. s θ and X. Suppose there exist constants $a_T > 0$, $b_T \in \mathbb{R}$, such that for $u_T = a_T x + b_T$ with $x \in \mathbb{R}$,*

$$\lim_{T\to\infty} T\mathbb{P}(Z > u_T) = \exp(-x). \tag{5.4}$$

For $\epsilon > 0$ *define* $v_T = a_T\epsilon$.

(a) Suppose $\mathbb{P}(\theta > x) = o(\mathbb{P}(Z > x))$ *for* $x \to \infty$. *Then,*

$$\lim_{T\to\infty} T\mathbb{P}(\theta + Z > u_T, Z \le u_T - v_T) = 0, \tag{5.5}$$

$$\lim_{A\uparrow\infty}\lim_{T\to\infty} T\mathbb{P}\left(\theta + Z > u_T, |Z - u_T| > a_T A\right) = 0. \tag{5.6}$$

(b) Then, $\lim_{T\to\infty} T\mathbb{P}(\theta + Z \le u_T, Z > u_T + v_T) = 0$.

(c) Suppose $\mathbb{P}(X > x) \sim q\,\mathbb{P}(Z > x)$ *for* $x \to \infty$ *and* $q > 0$. *Then,*

$$\lim_{T\to\infty} T\mathbb{P}(X + Z > u_T, X \le u_T - v_T, Z \le u_T - v_T) = 0.$$

Proof of Lemma 5.1. Let F_Z, F_θ and F_X be the d. f. s of Z, θ and X respectively.

(a) Note that $u_T \to \infty$, $v_T \to \infty$, $a_T/b_T \to 0$ and also $u_T/2 - v_T = (x/2-\epsilon)a_T + b_T/2 \to \infty$ for $T \to \infty$. Hence, we can assume that $u_T/2 < u_T - v_T$. Now, suppose for the moment that for $T \to \infty$,

$$\int_{u_T/2}^{u_T-v_T} \overline{F}_\theta(u_T - y)F_Z(dy) = o(\overline{F}_Z(u_T)), \tag{5.7}$$

$$\int_{-\infty}^{u_T/2} \overline{F}_\theta(u_T - y)F_Z(dy) = o(\overline{F}_Z(u_T)), \tag{5.8}$$

$$\overline{F}_Z\left(u_T/2\right)\overline{F}_\theta\left(u_T/2\right) = o(\overline{F}_Z(u_T)). \tag{5.9}$$

Then we obtain for $T \to \infty$,

$$\mathbb{P}\left(\theta + Z > u_T, Z \le u_T - v_T, \theta \le u_T/2\right) \le \int_{u_T/2}^{u_T-v_T} \overline{F}_\theta(u_T - y)F_Z(dy) = o(\overline{F}_Z(u_T)),$$

$$\mathbb{P}\left(\theta + Z > u_T, Z \le u_T/2\right) = \int_{-\infty}^{u_T/2} \overline{F}_\theta(u_T - y)F_Z(dy) = o(\overline{F}_Z(u_T)).$$

Hence, the last two inequalities and (5.9) give

$$\begin{aligned}
&\mathbb{P}(\theta + Z > u_T, Z \le u_T - v_T)\\
&\quad\le \mathbb{P}\left(\theta + Z > u_T, Z \le u_T - v_T, \theta \le u_T/2\right) + \ldots\\
&\qquad \ldots + \mathbb{P}\left(\theta + Z > u_T, Z \le u_T/2\right) + \mathbb{P}\left(Z > u_T/2, \theta > u_T/2\right)\\
&\quad= o(\overline{F}_Z(u_T)) \quad \text{for } T \to \infty.
\end{aligned}$$

Applying (5.4) yields (5.5). On the other hand, we estimate

$$\begin{aligned}
&\frac{\mathbb{P}(\theta + Z > u_T, |Z - u_T| > a_T A)}{\mathbb{P}(Z > u_T)}\\
&\quad= \int_{-\infty}^{u_T - a_T A} \frac{\overline{F}_\theta(u_T - y)}{\overline{F}_Z(u_T)} F_Z(dy) + \int_{u_T + a_T A}^{\infty} \frac{\overline{F}_\theta(u_T - y)}{\overline{F}_Z(u_T)} F_Z(dy)\\
&\quad\le \sup_{z > a_T A} \frac{\overline{F}_\theta(z)}{\overline{F}_Z(z)} \frac{\overline{F_Z^{2*}}(u_T)}{\overline{F}_Z(u_T)} + \frac{\overline{F}_Z(u_T + a_T A)}{\overline{F}_Z(u_T)}.
\end{aligned} \tag{5.10}$$

For the first summand in (5.10) the assumption $\mathbb{P}(\theta > x) = o(\mathbb{P}(Z > x))$ for $x \to \infty$, Proposition 1.2 *(ii)* and the fact that $u_T, a_T \to \infty$ for $T \to \infty$ gives

$$\lim_{T\to\infty} \sup_{z > a_T A} \frac{\overline{F}_\theta(z)}{\overline{F}_Z(z)} \frac{\overline{F_Z^{2*}}(u_T)}{\overline{F}_Z(u_T)} = 0. \tag{5.11}$$

Applying (5.4) again gives for the second summand in (5.10),

$$\lim_{T\to\infty} \frac{\overline{F}_Z(u_T + a_T A)}{\overline{F}_Z(u_T)} = \lim_{T\to\infty} \frac{\overline{F}_Z\left(a_T(x+A) + b_T\right)}{\overline{F}_Z\left(a_T x + b_T\right)} = \frac{\exp(-x-A)}{\exp(-x)} \overset{A\to\infty}{\longrightarrow} 0. \tag{5.12}$$

The result (5.6) follows then by (5.10)-(5.12).

Next we prove (5.7)-(5.9). By the same argument as used for (5.11) and the fact u_T, $v_T \to \infty$ for $T \to \infty$ we obtain (5.7):

$$\int_{u_T/2}^{u_T - v_T} \frac{\overline{F}_\theta(u_T - y)}{\overline{F}_Z(u_T)} F_Z(dy) \le \sup_{z \ge v_T} \frac{\overline{F}_\theta(z)}{\overline{F}_Z(z)} \frac{\overline{F_Z^{2*}}(u_T)}{\overline{F}_Z(u_T)} \overset{T\to\infty}{\longrightarrow} 0.$$

Moreover, we obtain (5.8) by Proposition 1.2 *(iii)*. Finally, (5.9) follows from Proposition 1.2 *(i)*, which gives

$$0 \le \lim_{T\to\infty} \frac{\overline{F}_\theta\left(u_T/2\right) \overline{F}_Z\left(u_T/2\right)}{\overline{F}_Z(u_T)} = \lim_{T\to\infty} \frac{\overline{F}_\theta\left(u_T/2\right)}{\overline{F}_Z\left(u_T/2\right)} \lim_{T\to\infty} \frac{\overline{F}_Z\left(u_T/2\right) \overline{F}_Z\left(u_T/2\right)}{\overline{F}_Z(u_T)} = 0.$$

Statement (5.9) also holds, if θ and Z are tail-equivalent.

(b) We have again by (5.4) and $v_T \to \infty$ as $T \to \infty$,

$$\lim_{T\to\infty} T\mathbb{P}(Z > u_T + v_T, \theta + Z \le u_T) \le \lim_{T\to\infty} T\mathbb{P}(Z > u_T + v_T)\mathbb{P}(\theta \le -v_T) = 0.$$

(c) Since $F_X \in \mathcal{S}$, we know that $\overline{F}_X(u_T - y)/\overline{F}_Z(u_T) \to q$ for $T \to \infty$ locally uniformly in y. Moreover, by Proposition 1.2 *(iii)*,

$$\lim_{T\to\infty} \int_{-\infty}^{u_T/2} \frac{\overline{F}_X(u_T - y)}{\overline{F}_Z(u_T)} F_Z(dy) = q.$$

Thus by Pratt's Lemma (Resnick [20], Exercise 5.4.2.4),

$$\begin{aligned}
\lim_{T\to\infty} \frac{\mathbb{P}(X + Z > u_T, X \le u_T - v_T, Z \le u_T/2)}{\mathbb{P}(Z > u_T)} &= \lim_{T\to\infty} \int_{v_T}^{u_T/2} \frac{\overline{F}_X(u_T - y)}{\overline{F}_Z(u_T)} F_Z(dy) \\
&= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{\overline{F}_X(u_T - y)}{\overline{F}_Z(u_T)} 1_{[v_T, u_T/2]}(y) F_Z(dy) = 0.
\end{aligned}$$

By symmetry also $\mathbb{P}(X+Z > u_T, X \leq u_T/2, Z \leq u_T - v_T) = o(\overline{F}_Z(u_T))$ for $T \to \infty$. Hence, by (5.9),

$$\begin{aligned}
&\mathbb{P}\left(X + Z > u_T, X \leq u_T - v_T, Z \leq u_T - v_T\right) \\
&\quad \leq \mathbb{P}\left(X + Z > u_T, X \leq u_T - v_T, Z \leq u_T/2\right) + \dots \\
&\qquad \dots + \mathbb{P}\left(X + Z > u_T, X \leq u_T/2, Z \leq u_T - v_T\right) + \mathbb{P}\left(X > u_T/2\right)\mathbb{P}\left(Z > u_T/2\right) \\
&\quad = o(\overline{F}_Z(u_T)) \quad \text{for } T \to \infty.
\end{aligned}$$

□

Proof of Proposition 3.1. Denote by $\zeta_T = \sum_{k=1}^{\infty} \varepsilon_{\left(k/T, a_T^{-1}(Z_k + \theta_k - b_T)\right)}$ for $T > 0$ point processes in $M_P(S)$. Let $\epsilon > 0$ be arbitrary. Write $I_\epsilon = [s,t) \times (x - \epsilon, x + \epsilon]$. Then,

$$\begin{aligned}
\{\zeta_T(I) \neq \widetilde{\kappa}_T(I)\} \subseteq & \bigcup_{k \in [Ts, Tt)} \{\theta_k + Z_k > u_T, Z_k \leq u_T - v_T\} \cup \dots \\
& \dots \bigcup_{k \in [Ts, Tt)} \{\theta_k + Z_k \leq u_T, Z_k > u_T + v_T\} \cup \{\widetilde{\kappa}_T(I_\epsilon) > 0\}.
\end{aligned}$$

Hence, by Lemma 5.1 *(a,b)* and the independence of Θ_1 and Z_1 we obtain

$$\begin{aligned}
\mathbb{P}\left(\zeta_T(I) \neq \widetilde{\kappa}_T(I)\right) \leq\; & T(t-s)\mathbb{P}(\Theta_1 + Z_1 > u_T, Z_1 \leq u_T - v_T) + \mathbb{P}(\widetilde{\kappa}_T(I_\epsilon) > 0) + \dots \\
& \dots + T(t-s)\mathbb{P}(\Theta_1 + Z_1 \leq u_T, Z_1 > u_T + v_T) \\
& \xrightarrow{T \to \infty} \mathbb{P}(\kappa(I_\epsilon) > 0) \xrightarrow{\epsilon \downarrow 0} 0.
\end{aligned}$$

By a modification of an argument of Hsing and Teugels [15] (see the proofs of their Theorem 4.2, Lemma 2.1 and for more details Fasen [10], Corollary 1.2.2) we have $\lim_{T\to\infty} \mathbb{P}(\kappa_T(I) \neq \zeta_T(I)) = 0$. Thus the assertion

$$\lim_{T\to\infty} \mathbb{P}(\kappa_T(I) \neq \widetilde{\kappa}_T(I)) \leq \lim_{T\to\infty} \mathbb{P}(\kappa_T(I) \neq \zeta_T(I)) + \lim_{T\to\infty} \mathbb{P}(\zeta_T(I) \neq \widetilde{\kappa}_T(I)) = 0$$

follows. We conclude $\kappa_T \overset{T\to\infty}{\Longrightarrow} \kappa$ by Rootzén [22], Lemma 3.3. □

Proof of Lemma 3.3. Let $\epsilon > 0$ be arbitrary.

(a) We decompose the probability:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in J}\left|\frac{Y(t)}{b_T} - \frac{\widetilde{f}(t)}{\widetilde{f}^+}\right| > \epsilon \,\middle|\, \tau > u_T\right) \qquad (5.13)\\
&\quad = \mathbb{P}\left(\sup_{t\in J}\left|\frac{Y(t)}{b_T} - \frac{\widetilde{f}(t)}{\widetilde{f}^+}\right| > \epsilon, |\widetilde{f}^+ Z - u_T| > a_T A \,\middle|\, \tau > u_T\right) + \dots \\
&\qquad \dots + \mathbb{P}\left(\sup_{t\in J}\left|\frac{Y(t)}{b_T} - \frac{\widetilde{f}(t)}{\widetilde{f}^+}\right| > \epsilon, |\widetilde{f}^+ Z - u_T| \leq a_T A \,\middle|\, \tau > u_T\right).
\end{aligned}$$

The first term in (5.13) satisfies the inequality

$$\mathbb{P}\left(\sup_{t\in J}\left|\frac{Y(t)}{b_T}-\frac{\widetilde{f}(t)}{\widetilde{f}^+}\right|>\epsilon,|\widetilde{f}^+Z-u_T|>a_TA\,\middle|\,\tau>u_T\right)$$
$$\leq\frac{\mathbb{P}\left(|\widetilde{f}^+Z-u_T|>a_TA,\tau>u_T\right)}{\mathbb{P}(\tau>u_T)}. \tag{5.14}$$

Furthermore, by (3.3) and Proposition 1.2 *(iii)*,

$$\lim_{T\to\infty}T\mathbb{P}(\tau>u_T)=\lim_{T\to\infty}T\mathbb{P}(\widetilde{f}^+Z+\theta>u_T)=\exp(-x). \tag{5.15}$$

Then, by using Lemma 5.1 *(a)* we conclude

$$\lim_{A\uparrow\infty}\lim_{T\to\infty}\frac{\mathbb{P}\left(|\widetilde{f}^+Z-u_T|>a_TA,\tau>u_T\right)}{\mathbb{P}(\tau>u_T)}=0. \tag{5.16}$$

For the second term in (5.13) we have

$$\mathbb{P}\left(\sup_{t\in J}\left|\frac{Y(t)}{b_T}-\frac{\widetilde{f}(t)}{\widetilde{f}^+}\right|>\epsilon,|\widetilde{f}^+Z-u_T|\leq a_TA\right)$$
$$\leq\mathbb{P}\left(\sup_{t\in J}|\widetilde{Y}(t)|>b_T\epsilon-a_T(A+x)\right)\mathbb{P}\left(\left|\widetilde{f}^+Z-u_T\right|\leq a_TA\right), \tag{5.17}$$

where we used the independence of $\widetilde{Y}$ and Z in the last step. Furthermore,

$$T\mathbb{P}\left(\left|\widetilde{f}^+Z-u_T\right|\leq a_TA\right)\leq T\mathbb{P}\left(\widetilde{f}^+Z>u_T-a_TA\right)\overset{T\to\infty}{\longrightarrow}\mathrm{e}^{-x+A}\text{ for }T\to\infty \tag{5.18}$$

holds. Thus, by (5.15), (5.17), (5.18) and $b_T\epsilon-a_T(A+x)\to\infty$ for $T\to\infty$ (cf. Embrechts et al. [9], p. 149) we obtain

$$\mathbb{P}\left(\sup_{t\in J}\left|\frac{Y(t)}{b_T}-\frac{\widetilde{f}(t)}{\widetilde{f}^+}\right|>\epsilon,|\widetilde{f}^+Z-u_T|\leq a_TA\,\middle|\,\tau>u_T\right)\overset{T\to\infty}{\longrightarrow}0. \tag{5.19}$$

Combining (5.13), (5.14), (5.16) and (5.19) yields the assertion.

(b) First we show

$$\lim_{T\to\infty}\mathbb{P}\left(\sup_{t\in O}|Y(t)-\tau|>a_T\epsilon\,\middle|\,\tau>u_T\right)=0. \tag{5.20}$$

Define $v_T=a_T\epsilon$. We proceed as in *(a)* and decompose the probability

$$\mathbb{P}\left(\sup_{t\in O}|Y(t)-\tau|>v_T\,\middle|\,\tau>u_T\right) \tag{5.21}$$
$$=\mathbb{P}\left(\sup_{t\in O}|\widetilde{Y}(t)-\theta|>v_T,\widetilde{f}^+Z>u_T-v_T\,\middle|\,\tau>u_T\right)+\dots$$
$$\dots+\mathbb{P}\left(\sup_{t\in O}|\widetilde{Y}(t)-\theta|>v_T,\widetilde{f}^+Z\leq u_T-v_T\,\middle|\,\tau>u_T\right).$$

For the first summand of (5.21) we get by the independence of $\widetilde{Y} - \theta$ and Z

$$\mathbb{P}\left(\sup_{t\in O}|\widetilde{Y}(t)-\theta| > v_T, \widetilde{f}^+Z > u_T - v_T \middle| \tau > u_T\right) \tag{5.22}$$
$$\leq \frac{\mathbb{P}\left(\sup_{t\in O}|\widetilde{Y}(t)-\theta| > v_T\right)\mathbb{P}\left(\widetilde{f}^+Z > u_T - v_T\right)}{\mathbb{P}(\tau > u_T)} \overset{T\to\infty}{\longrightarrow} 0.$$

The last term tends to zero, since $a_T \to \infty$, $T\mathbb{P}(\widetilde{f}^+Z > u_T - v_T) \to \exp(-x+\epsilon)$ for $T \to \infty$ and (5.15) holds.

Using Lemma 5.1 *(a)* and (5.15) we get for the second summand of (5.21)

$$\mathbb{P}\left(\sup_{t\in O}\left|\widetilde{Y}(t)-\theta\right| > v_T, \widetilde{f}^+Z \leq u_T - v_T \middle| \tau > u_T\right)$$
$$\leq \frac{\mathbb{P}\left(\tau > u_T, \widetilde{f}^+Z \leq u_T - v_T\right)}{\mathbb{P}\left(\tau > u_T\right)} \overset{T\to\infty}{\longrightarrow} 0. \tag{5.23}$$

Therefore (5.20) is proven by (5.21)-(5.23). Invoking again (5.15) we see that

$$\mathbb{P}\left(\tau > u_T + a_T y_i \middle| \tau > u_T\right) \overset{T\to\infty}{\longrightarrow} \exp(-\max\{y_i, 0\}). \tag{5.24}$$

Taking (5.20) into account we obtain the second statement of *(b)*.

(c) By considering Proposition 1.2 *(iii)* we have for $|\widetilde{f}(t)| < \widetilde{f}^+$,

$$\mathbb{P}\left(Y(t) > a_T(x+y) + b_T\right) = o\left(\mathbb{P}\left(\widetilde{f}^+Z > a_T(x+y)+b_T\right)\right) \quad \text{for } T\to\infty.$$

With (5.15) we conclude

$$\mathbb{P}\left(Y(t) > u_T + a_T y \middle| \tau > u_T\right) \leq \frac{\mathbb{P}\left(Y(t) > a_T(x+y)+b_T\right)}{\mathbb{P}\left(\tau > a_T x + b_T\right)} \overset{T\to\infty}{\longrightarrow} 0.$$

If $\widetilde{f}(t) = -\widetilde{f}^+$, then with Lemma 5.1 *(a)* we have

$$\begin{aligned}
&\lim_{T\to\infty} \mathbb{P}(Y(t) > u_T + a_T y | \tau > u_T) \\
&\quad \leq \lim_{T\to\infty} \frac{\mathbb{P}(-\widetilde{f}^+Z + \widetilde{Y}(t) > u_T + a_T y, \widetilde{f}^+Z > u_T - v_T)}{\mathbb{P}(\tau > u_T)} \\
&\quad \leq \lim_{T\to\infty} \frac{\mathbb{P}(\widetilde{Y}(t) > u_T + a_T y)}{\mathbb{P}(\tau > u_T)} = 0.
\end{aligned}$$

□

For the proofs of Theorem 4.1 and Corollary 4.2 we first show the following Lemma.

Lemma 5.2 *Suppose the assumptions of Theorem 4.1 hold. Then for* $t \in \mathbb{R}, k \in \mathbb{N}$, *there exists a sequence* $\{\Theta_k(t)\}_{k\in\mathbb{N}}$ *with* $\Theta_k(t)$ *independent of* Z_k *and*

$$|Y(\Gamma_k + t) - f(t)Z_k| \leq \Theta_k(t) \quad \text{a. s.} \tag{5.25}$$

Furthermore there exists a r. v. Θ *with* $\Theta_k(t) \stackrel{d}{=} \Theta$ *for* $k \in \mathbb{N}, t \in \mathbb{R}$, *and*

$$\mathbb{P}(\Theta > x) = o(\mathbb{P}(f^+|Z_1| > x)) \quad \textit{for } x \to \infty.$$

Proof of Lemma 5.2. Choose $k > 0$ fixed, and define the shifted compound Poisson process $\{\widetilde{L}(t)\}_{t\in\mathbb{R}}$ with jump times $\{-\widetilde{\Gamma}_{-j}\}_{j\in\mathbb{Z}\backslash\{0\}}$, where

$$\widetilde{\Gamma}_j = \begin{cases} \Gamma_k & \text{for } j = k, \\ \Gamma_k - \Gamma_{k-j} & \text{for } j \in \mathbb{Z}\backslash\{k\}, \end{cases}$$

with corresponding jump sizes $|Z_{k+j}|$ at time $-\widetilde{\Gamma}_{-j}$ and intensity μ. Then,

$$Y_1(\Gamma_k + t) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} f(t + \widetilde{\Gamma}_{k-m})Z_m = \sum_{\substack{j=-\infty \\ j\neq -k}}^{\infty} f(t + \widetilde{\Gamma}_{-j})Z_{k+j} \quad \text{for } t \in \mathbb{R},$$

and we obtain

$$|Y_1(\Gamma_k + t) - f(t)Z_k| \leq \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} |f(t + \widetilde{\Gamma}_{-j})||Z_{k+j}| =: \widetilde{Y}_1(t), \tag{5.26}$$

where $\widetilde{Y}_1(t)$ is a modification of the MA process $\int_{-\infty}^{\infty} |f(t-s)|\, d\widetilde{L}(s)$. Thus,

$$|Y(\Gamma_k + t) - f(t)Z_k| \leq \widetilde{Y}_1(t) + |Y_2(\Gamma_k + t)| =: \Theta_k(t). \tag{5.27}$$

Note, that $\Theta_k(t)$ is independent of Z_k. Choose $\Theta \stackrel{d}{=} \widetilde{Y}_1(0) + |Y_2(0)|$. By the independence of Γ, Z and Y_2 as well as the stationarity of $\widetilde{Y}_1$ and Y_2 we obtain

$$\mathbb{P}(|Y(\Gamma_k + t) - f(t)Z_k| > x) \leq \mathbb{P}(\Theta_k(t) > x) = \mathbb{P}(\Theta > x) \quad \text{for } k \in \mathbb{N}, x > 0. \tag{5.28}$$

Using Proposition 2.2 *(b)* and Proposition 1.2 *(v)* we have $|f(U_A)|\widetilde{L}(1), \widetilde{Y}_1(0) \in \mathcal{S}$, and by Lemma 2.3 also $\mathbb{P}(\widetilde{Y}_1(0) > x) = o(\mathbb{P}(f^+|Z_1| > x))$ for $x \to \infty$. Since ν_{Y_2} has a bounded support, applying Theorem 26.1 of Sato [23] and Proposition 1.2 *(i)* yields

$$\mathbb{P}(|Y_2(0)| > x) = o(\mathbb{P}(\widetilde{Y}_1(0) > x)) \quad \text{for } x \to \infty.$$

Hence, with Proposition 1.2 *(iii)* we conclude

$$\mathbb{P}(\Theta > x) = \mathbb{P}(\widetilde{Y}_1(0) + |Y_2(0)| > x) = o(\mathbb{P}(f^+|Z_1| > x)) \quad \text{for } x \to \infty. \tag{5.29}$$

□

Proof of Theorem 4.1. Recall that by (2.4) the normalizing constants of Z_k are $a_{T/\mu}$, $b_{T/\mu}$ and, if $p<1$, the normalizing constants of $-Z_k$ are $a_{(1-p)T/(p\mu)}$, $b_{(1-p)T/(p\mu)}$. Considering Lemma 5.2 the assumptions of Proposition 3.1 are satisfied, and thus,

$$\lim_{T\to\infty}\mathbb{P}(\kappa^{(1,l)}_{T/\mu}(I)\neq\widetilde{\kappa}^{(1)}_{T/\mu}(I))=\lim_{T\to\infty}\mathbb{P}(\kappa^{(2,l)}_{T/\mu}(I)\neq\widetilde{\kappa}^{(2)}_{T/\mu}(I))=0.$$

Hence,

$$\mathbb{P}(\kappa_T(I)\neq\widetilde{\kappa}_T(I))\leq\sum_{l=1}^{P^{(1)}}\mathbb{P}(\kappa^{(1,l)}_{T}(I)\neq\widetilde{\kappa}^{(1)}_{T}(I))+\sum_{l=1}^{P^{(2)}}\mathbb{P}(\kappa^{(2,l)}_{T}(I)\neq\widetilde{\kappa}^{(2)}_{T}(I))\overset{T\to\infty}{\longrightarrow}0.$$

By Proposition 3.1 of Davis and Resnick [8] also $\widetilde{\kappa}_T\overset{T\to\infty}{\Longrightarrow}\kappa$. A conclusion of Rootzén [22], Lemma 3.3, is $\kappa_T\overset{T\to\infty}{\Longrightarrow}\kappa$. □

Proof of Corollary 4.2. For statement *(a)* we consider w.l.o.g. the case $m=2$, $t_1=\alpha^{(i)}_{\widetilde{l}}$ with $f(t_1)=(-1)^{(i+1)}f^+$ and $f(t_2)\neq(-1)^{(i+1)}f^+$. Let $I=I_0\times I_1\times I_2\times I_3=(s,t]\times(x_1,y_1]\times[x_2,y_2]\times[x_3,y_3]$ be relatively compact sets on $S\times[-\infty,\infty]^2$. Define the point processes

$$\begin{aligned}K^{(1)}_T&=\sum_{k=1}^{\infty}\varepsilon_{\left((\Gamma_k+\alpha^{(i)})/T,a_T^{-1}(Y(\Gamma_k+\alpha^{(i)})-b_T),a_T^{-1}(Y(\Gamma_k+\alpha^{(i)}_{\widetilde{l}})-b_T),\varepsilon_{-\infty}\right)},\\ K^{(2)}_T&=\sum_{k=1}^{\infty}\varepsilon_{\left(k/(T\mu),a_T^{-1}((-1)^{(i+1)}f^+Z_k-b_T),a_T^{-1}((-1)^{(i+1)}f^+Z_k-b_T),\varepsilon_{-\infty}\right)}\end{aligned}$$

in $M_P(S\times[-\infty,\infty]^2)$. Thus,

$$\mathbb{P}(K_T(I)\neq K^{(2)}_T(I))\leq\mathbb{P}(K_T(I)\neq K^{(1)}_T(I))+\mathbb{P}(K^{(1)}_T(I)\neq K^{(2)}_T(I)).\tag{5.30}$$

On the one hand we have by (5.27), (5.29), Lemma 3.3 *(c)* for some $\epsilon>0$, and $\widetilde{x}=x_3$ if $x_3>-\infty$, and $\widetilde{x}=y_3$ if $x_3=-\infty$,

$$\begin{aligned}&\mathbb{P}(K_T(I)\neq K^{(1)}_T(I))\\&\leq T(\epsilon+t-s)\mathbb{P}(f(t_2)Z_k+\Theta_k(t_2)>a_T\widetilde{x}+b_T,(-1)^{(i+1)}f^+Z_k+\Theta_k(\alpha^{(i)})>a_Tx_1+b_T)\\&\quad\overset{T\to\infty}{\longrightarrow}0,\end{aligned}\tag{5.31}$$

and on the other hand by Theorem 4.1 we have for $x_2>-\infty$,

$$\begin{aligned}&\mathbb{P}(K^{(1)}_T(I)\neq K^{(2)}_T(I))\\&\quad\leq\mathbb{P}(\kappa^{(i,l)}_T(I_0\times I_1)\neq\widetilde{\kappa}^{(i)}_T(I_0\times I_1))+\mathbb{P}(\kappa^{(i,\widetilde{l})}_T(I_0\times I_2)\neq\widetilde{\kappa}^{(i)}_T(I_0\times I_2))\overset{T\to\infty}{\longrightarrow}0.\end{aligned}\tag{5.32}$$

If $x_2=-\infty$ similar arguments as in (5.31) show $\lim_{T\to\infty}\mathbb{P}(K^{(1)}_T(I)\neq K^{(2)}_T(I))=0$. By Proposition 3.1 of Davis and Resnick [8] also $K^{(2)}_T\overset{T\to\infty}{\Longrightarrow}K$. Again reasoning as in Rootzén [22], Lemma 3.3, and (5.30)-(5.32) we obtain $K_T\overset{T\to\infty}{\Longrightarrow}K$.

Statement *(b)* is a conclusion of Lemma 3.3 *(a)*, Lemma 5.2 and similar arguments as in *(a)* (cf. Rootzén [22], Lemma 3.4). The statements *(c)* and *(d)* follow by Lemma 3.3 *(b, c)*, and Lemma 5.2. □

Proof of Theorem 4.4. Let $c_n = \sup_{t\in[n-1,n+1)} f^+(t)$ and $d_n = \sup_{t\in[n-1,n+1)} f^-(t)$ for $n \in \mathbb{Z}$. Since $\int_{-\infty}^{\infty} \sup_{0\le t\le 1} |f(t+s)|\,\lambda(ds) < \infty$ we conclude $\sum_{n=-\infty}^{\infty} c_n < \infty$ and $\sum_{n=-\infty}^{\infty} d_n < \infty$. Now, we use the decomposition $L = \widetilde{L}_1 - \widetilde{L}_2 + \widetilde{L}_3$, where $\widetilde{L}_1$ and $\widetilde{L}_2$ respectively are positive compound Poisson processes with characteristic triplet $(0, 0, \widetilde{\nu}_1)$ and $(0, 0, \widetilde{\nu}_2)$ respectively, where $\widetilde{\nu}_1(A) = \nu(A \cap (1,\infty))$ and $\widetilde{\nu}_2(A) = \nu(A \cap (-\infty,-1))$ for $A \in \mathcal{B}(\mathbb{R})$, and $\widetilde{L}_3 = L_2$ has characteristic triplet $(m, \sigma^2, \nu - \widetilde{\nu}_1 - \widetilde{\nu}_2)$. Define

$$X_n^{(1)} = \sum_{k=-\infty}^{\infty} c_{n-k}\xi_k^{(1)}, \quad X_n^{(2)} = \sum_{k=-\infty}^{\infty} d_{n-k}\xi_k^{(2)}, \quad X_n^{(3)} = \sup_{t\in[n-1,n)} \int_{-\infty}^{\infty} f(t-s)\, d\widetilde{L}_3(s)$$

for $n \in \mathbb{N}$, where $\xi_k^{(i)} = \widetilde{L}_i(k) - \widetilde{L}_i(k-1)$ for $k \in \mathbb{Z}$, $i = 1, 2$. Both $X_n^{(1)}$ and $X_n^{(2)}$ are finite a. s. by Example 2.5, and since $\mathbb{P}(|Y(t)| < \infty$ for every $t \in \mathbb{R}) = 1$ also $|X_n^{(3)}| < \infty$ a. s. As $\widetilde{L}_1$ and $\widetilde{L}_2$ respectively, are increasing, we have

$$\sup_{t\in[n-1,n)} Y(t) \le X_n^{(1)} + X_n^{(2)} + X_n^{(3)} =: X_n. \tag{5.33}$$

Since $\{\xi_k^{(i)}\}_{k\in\mathbb{N}}$ is an i. i. d. sequence with $\xi_k^{(i)} \stackrel{d}{=} \widetilde{L}_i(1)$ and $X^{(i)} = \{X_n^{(i)}\}_{n\in\mathbb{N}}$ is a discrete-time MA process, which satisfies the assumptions of Example 3.2, we obtain for $i = 1, 2$, (only $i = 1$ in the case $f^- < f^+$ or $p = 1$)

$$\kappa_T^{(i)} = \sum_{k=1}^{\infty} \varepsilon_{\left(k/T,\, a_T^{-1}(X_k^{(i)} - b_T)\right)} \stackrel{T\to\infty}{\Longrightarrow} \widetilde{P}^{(i)}\kappa^{(i)}$$

with $\kappa^{(i)}$ as given in Theorem 4.1. Furthermore, $\widetilde{P}^{(1)} = \operatorname{card}\{k : c_k = f^+\}$ and $\widetilde{P}^{(2)} = \operatorname{card}\{k : d_k = f^+\}$. The processes $X^{(1)}$ and $X^{(2)}$ are independent. By the use of Example 2.5 we have $\mathbb{P}(X_n^{(i)} > x) \sim \widetilde{P}^{(i)}\mathbb{P}(f^+\widetilde{L}_i(1) > x)$ for $x \to \infty$, $i = 1, 2$ (only $i = 1$ in the case $f^- < f^+$ or $p = 1$). Regarding Goldie and Resnick [13], Theorem 2.3, (if $f^- = f^+$ and $p < 1$) and Proposition 3.1 (if $f^- < f^+$ or $p = 1$) we conclude

$$\sum_{k=1}^{\infty} \varepsilon_{\left(k/T,\, a_T^{-1}(X_k^{(1)} + X_k^{(2)} - b_T)\right)} \stackrel{T\to\infty}{\Longrightarrow} \widetilde{P}^{(1)}\kappa^{(1)} + \widetilde{P}^{(2)}\kappa^{(2)}.$$

The sample path of Y_1 are a.s. càdlàg so that Y_2 is also separable. Using Braverman and Samorodnitsky [3], Lemma 2.1, and the Markov-inequality we obtain $\mathbb{P}(|X_k^{(3)}| >$

$x) = O(\exp(-x))$ for $x \to \infty$ such that by Proposition 1.2 *(i)* we also have $\mathbb{P}(|X_k^{(3)}| > x) = o(\mathbb{P}(f^+|L(1)| > x))$ for $x \to \infty$. Applying Proposition 3.1 yields

$$\sum_{k=1}^{\infty} \varepsilon_{\left(k/T, a_T^{-1}(X_k^{(1)}+X_k^{(2)}+X_k^{(3)}-b_T)\right)} \overset{T\to\infty}{\Longrightarrow} \widetilde{P}^{(1)}\kappa^{(1)} + \widetilde{P}^{(2)}\kappa^{(2)}.$$

Thus, for $I = (0,1] \times (x,\infty)$ we have on the one hand with (5.33)

$$\begin{aligned} \lim_{T\to\infty} \mathbb{P}(a_T^{-1}(M(T) - b_T) \le x) &\ge \lim_{T\to\infty} \mathbb{P}\left(\sum_{k=1}^{\infty} \varepsilon_{\left(k/T, a_T^{-1}(X_k^{(1)}+X_k^{(2)}+X_k^{(3)}-b_T)\right)}(I) = 0\right) \\ &= \mathbb{P}(\kappa^{(1)}(I) = 0)[\mathbf{1}_{\{f^- < f^+\}} + \mathbf{1}_{\{f^-=f^+\}}\,\mathbb{P}(\kappa^{(2)}(I) = 0)]. \qquad (5.34) \end{aligned}$$

On the other hand, Theorem 4.1 gives

$$\begin{aligned} \lim_{T\to\infty} \mathbb{P}(a_T^{-1}(M(T) - b_T) \le x) &\le \mathbb{P}(P^{(1)}\kappa^{(1)}(I) + P^{(2)}\kappa^{(2)}(I) = 0) \qquad (5.35) \\ &= \mathbb{P}(\kappa^{(1)}(I) = 0)[\mathbf{1}_{\{f^- < f^+\}} + \mathbf{1}_{\{f^-=f^+\}}\,\mathbb{P}(\kappa^{(2)}(I) = 0)]. \end{aligned}$$

Taking $\mathbb{P}(\kappa^{(1)}(I) = 0) = \exp(-\mathrm{e}^{-x})$ and $\mathbb{P}(\kappa^{(2)}(I) = 0) = \exp(-p^{-1}(1-p)\mathrm{e}^{-x})$ into account we obtain by (5.34) and (5.35) the result. □

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