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First Order Quasi-Linear PDEs with BV Boundary Data and Applications to Image Inpainting

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Contents

Chapter 1

Introduction

As *image processing* has become an active field of mathematical research, the task of digital *image inpainting* has also been approached by mathematical methods in the last few years. The questions that we study in this work emerged during the analysis of our image inpainting method *Image Inpainting Based on Coherence Transport* published in [BM07].

1.1 Inpainting

Image inpainting serves the purpose of touching-up damaged or unwanted portions of a picture. In mathematical image processing images are considered as functions of type

$$w : \Omega_0 \to \mathbb{R} ,$$

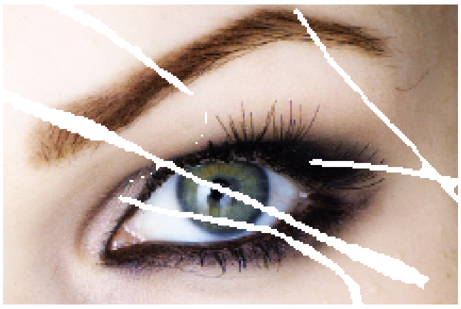
defined on a typically rectangular image domain $\Omega_0 = [a, b] \times [c, d] \subset \mathbb{R}^2$. The value $w(x) \in \mathbb{R}$ often represents an intensity of light which is perceptible as a gray color.
From the mathematician's point of view inpainting is a problem of data interpolation. Apart from Ω_0 we are given a subdomain $\Omega \subset \Omega_0$ which marks the damage or the portion which has to be touched-up. And, the "good" part of the image, which is to be kept, is given as a function

$$u_0 : \Omega_0 \backslash \Omega \to \mathbb{R} ,$$

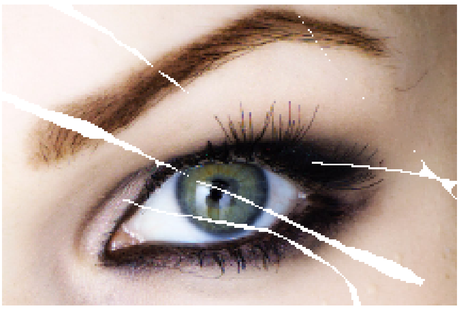
defined on the data domain $\Omega_0 \backslash \Omega$, while u_0 is undefined on Ω. Now, the task is to search for a function $u : \Omega \to \mathbb{R}$, defined on the missing part Ω, which interpolates the data u_0.
Clearly, the interpolation problem, as stated above, might have many solutions, but the very important side condition on an acceptable solution u is that the completed image $\bar{u}$,

$$\bar{u} := u_0 \cdot \mathbb{1}_{\Omega_0 \backslash \Omega} + u \cdot \mathbb{1}_{\Omega} ,$$

(a) vandalized image (courtesy of [Tel04, figure 8.i])



(b) inpainting interstage 1



(c) inpainting interstage 2

(d) inpainted result

Figure 1.1: Scratch removal by inpainting; using the method of [BM07]

defined on the whole image domain Ω_0, should look nice, i.e., u should interpolate the data u_0 in a visually plausible manner.

In order to achieve the latter goal, different approaches to the inpainting problem have been made: for example, [CS02], [CKS02], [MM98], [Mas02], and [Tsc05] have shown that variational principles and PDE methods are fruitful here. But, the resulting PDEs in these works are typically non-linear and of a high order (up to order 4); thus, the numerical algorithms are iterative by nature and computationally expensive.

In the works cited above, the authors started with continuous models and discretized them to obtain algorithms for digital image inpainting. In the article [BM07], we approached the problem from the opposite direction: our point of departure was the discrete inpainting problem. For discrete or digital images the functions $w_h : \Omega_{0,h} \to \mathbb{R}$ are simply matrices and $\Omega_{0,h}$ is the set of matrix coordinates; the subscript h indicates discrete objects.
The simple idea behind the generic algorithm (see [BM07, section 2] and chapter 6) is to fill the inpainting domain Ω_h by traversing its pixels – the points of Ω_h – in a fixed order from the boundary inwards by using weighted means of given or already calculated image values. Thus, the algorithm implements a process with processing order given by the distance-to-boundary map, which is the time of first arrival. See figure 1.1, which

(a) vandalized image (b) inpainted

Figure 1.2: Scratch removal by inpainting; using Telea's method

illustrates some stages of this process. Such a single-pass method was first utilized by Telea (see [Tel04]). Owing to the simple structure of the algorithm, his method performs extremely fast, but produces a blurry fill-in, which, moreover, shows peculiar transport patterns (see figure 1.2).

Analytical results of [BM07]
In order to better understand the geometrical effects of the generic algorithm, we put it in a continuous framework. By the high-resolution vanishing-viscosity limit (see [BM07, section 3]) we have shown that the generic algorithm, for a special class of weights, is consistent with the transport equation

$$\begin{aligned} \langle c(x), \nabla u(x) \rangle &= 0\,, \quad x \in \Omega \backslash \mathcal{S}\,, \\ u|_{\Omega_0 \backslash \Omega} &= u_0\,, \end{aligned} \tag{1.1}$$

a PDE of first order. Hereby, the vector field c depends on the weights used to compute the weighted means. Moreover, the exceptional set $\mathcal{S}$ is the skeleton of the distance-to-boundary map $d(x) = \mathrm{dist}(x, \partial\Omega)$. The skeleton $\mathcal{S}$ comprises the locations of ridges d; these are the points where the time of the first arrival cannot be uniquely associated with a point on the boundary. By equations (1.1), we can give the following rationale: imagine a restorer doing brush strokes in the missing area Ω. Assuming on the one hand that he only uses color given by the data u_0 on $\partial\Omega$ and on the other that brush strokes go along trajectories $x(t)$ – of a vector field c – which constantly carry a single color, we end up with the dynamical system

$$\begin{aligned} x' &= c(x)\,, \quad x(0) = x_0 \in \partial\Omega\,, \\ u' &= 0\,, \qquad\ \ u(0) = u_0(x_0)\,, \end{aligned} \tag{1.2}$$

which describes exactly the characteristics of problem (1.1). Because we paint from every boundary point into Ω, the characteristics, which are the brush strokes, are supposed to meet somewhere; the locations where they meet are contained in the exceptional set $\mathcal{S}$ of equation (1.1).

Eventually, the continuous description was the key to improve the quality (compared to [Tel04]) of the inpainting. Ideally, in order to obtain an aesthetic inpainting, the vector field c would need to reflect the full expertise of our restorer, which, clearly, is impossible. But at least the vector field c should be adapted to and hence depend on the image u. This consideration aims for a quasi-linear model

$$\begin{aligned} \langle c[u](x), \nabla u(x) \rangle &= 0\,, \quad x \in \Omega \backslash \Sigma\,, \\ u|_{\Omega_0 \backslash \Omega} &= u_0\,, \end{aligned} \tag{1.3}$$

within the continuous framework; for the discrete algorithm, that means that the weights need to depend on u. For the improved algorithm, in [BM07], the vector $c[u](x)$ – and thus the weight – includes an estimation of the tangent vector, which is tangent to the level line of u going through x. This is because brush strokes are supposed to continue level lines of u_0 which have been interrupted by Ω_0.
By applying *structure tensor analysis* to the image we estimate *approximate tangent* information or so-called *coherence* information. The structure tensor S is a positive semi-definite 2×2-matrix. Its set-up, basically, consists of the following two steps:

$$\begin{aligned} v(y) &= \int\limits_{B(y)} k_1(y,h)\; u(h)\; dh\,, \\ S(x) &= \int\limits_{B(x)} k_2(x,y)\; \nabla v(y) \cdot \nabla v(y)^T\; dy\,. \end{aligned}$$

The approximate tangent, then, is the eigenvector of S w.r.t. the minimal eigenvalue. The benefit and the robustness of this estimator have been revealed in different works, e.g. [Wei98] and [AMS+06]. A precise description of the structure tensor and its analytical properties is given in chapter 6. Our new weight for the discrete algorithm, then, weights those pixels that lie close to the approximate tangent much stronger; thus, the transport effect is mainly along the approximate tangent.

Practical results of [BM07]
In [BM07], we gave a complete description of the novel algorithm and have compared it to other methods. The inpainting result shown in figure 1.1 has been computed by our improved method. The higher quality, compared to Telea's method, is clearly visible if one compares figures 1.1 and 1.2. For Telea's choice of the weight, the transport field c of the corresponding PDE is exactly ∇d. But, the distance-to-boundary map d depends only on the geometry of Ω and says nothing about the image.
By the structure tensor analysis, our method is computationally more expensive than Telea's, but, compared to the other works cited above, we

have shown that our method is considerably faster while our inpainting results match their level of quality.

Questions left open in [BM07]
As the continuous framework of problems (1.1) and (1.3) has been the key to the improvements, the theory behind these problems is missing. Consequently, interesting questions concerning

- The existence of a solution,
- The uniqueness of the solution,
- The continuous dependence of the solution on the data and the coefficients with respect to a customized topology,

come up. And, the challenges are: the general geometry which inpainting domains can exhibit on the one hand, and the dependence of $c[u]$ on u on the other. The latter is a challenge, because, due to the structure tensor analysis, the vector $c[u](x)$ not only depends on the single value $u(x)$ but on a part $u|_B$ of the image, meaning the dependence on u is of a functional type.

Contribution of this thesis
Looking at equations (1.1) and (1.3), it is clear that these models are not restricted to inpainting or image processing. Thus, we study these equations independent of the particular construction of the coefficients within the inpainting model, but dependent on general features such as continuity, differentiability, etc.
The general framework is developed in chapters 3 and 4. As a consequence of the general theory, we can positively answer the question of the well-posedness of our inpainting model. Coming back to the inpainting model in chapter 6, the corollaries 6.17, 6.18, and 6.19 yield the existence, uniqueness and the continuous dependence.

1.2 Overview of the General Framework

PDEs of the first order, in their general form

$$\begin{aligned} H(x, u(x), \nabla u(x)) = 0\,, \quad & x \in \Omega\,, \\ u|_\Gamma = g\,, \quad & \Gamma \subset \partial\Omega\,, \end{aligned} \tag{1.4}$$

have already been considered in literature. The standard approach to the construction of a solution is the *method of characteristics*, e.g. see [Eva98], [Joh82], [CH62], or [Zau89]. For equation (1.4) the characteristics are given by the system of ODEs

$$
\begin{aligned}
x' &= \nabla_p H(x,u,p)\ , && x(0) = x_0 \in \Gamma\ ,\\
u' &= \langle \nabla_p H(x,u,p), p \rangle\ , && u(0) = g(x_0)\ ,\\
p' &= -\nabla_x H(x,u,p) - \partial_u H(x,u,p) \cdot p\ , && p(0) = p_0\ .
\end{aligned}
$$

And, the assumptions on the function H are such that the existence and uniqueness theory for ODEs applies. For the solution of the PDE (1.4) often only local existence and uniqueness are guaranteed. This is, because the projected characteristics or characteristic base curves – given by the x-component – might cross, or parts of the domain can never be reached by characteristic base curves; the reason often lies within the non-linearity of the function H, w.r.t. the (u,p) component, and the geometry of the domain.

For our linear and quasi-linear cases we want to construct a unique global solution of the problem. We feature the linear case first, because we can concentrate on the domain of the PDE and, in particular, on the exceptional sets.

1.2.1 The Linear Problem

The easiest scenario is a linear problem on the two-dimensional half-space $H^2_+ := \mathbb{R}_+ \times \mathbb{R}$,

$$
\begin{aligned}
a(x,y)\ \partial_x u(x,y) + b(x,y)\ \partial_y u(x,y) &= f(x,y)\ , \qquad \text{in} \quad H^2_+\ ,\\
u(0,y) &= u_0(y)\ .
\end{aligned}
$$

And, from now on, we allow for a source term f on the right hand side. In the linear case, the characteristics are described completely by the reduced system

$$
\begin{aligned}
x' &= a(x,y)\ , && x(0,s) = 0\ ,\\
y' &= b(x,y)\ , && y(0,s) = s\ ,\\
u' &= f(x,y)\ , && u(0,s) = u_0(s)\ .
\end{aligned}
$$

Assuming the coefficients to be C^1, a sufficient condition on the transport field $c = (a,b)^T$, in order to obtain a unique local solution, is

$$
\left\langle \frac{c(0,y)}{|c(0,y)|}, e_1 \right\rangle \geq \beta\ , \quad y \in \mathbb{R} \tag{1.5}
$$

for some $\beta > 0$. The vector $e_1 = (1,0)^T$ is exactly the interior unit normal of the initial manifold ∂H^2_+, and this condition means that every characteristic instantaneously leaves the initial manifold and evolves, at least for short

time, into the interior of H^2_+.
If condition (1.5) is satisfied, the coefficient a is positive; hence, the PDE is equivalent to

$$\partial_x u(x,y) + \frac{b(x,y)}{a(x,y)} \partial_y u(x,y) = \frac{f(x,y)}{a(x,y)} .$$

Applying the method of characteristics to the latter PDE, the variable x can, then, be identified with the time of the characteristics. From this point of view, we can say that the map $T : H^2_+ \to \mathbb{R}$, $T(x,y) = x$ is a *time function*. With respect to T, condition (1.5) rewrites as

$$\left\langle \frac{c(0,y)}{|c(0,y)|}, \frac{\nabla T(0,y)}{|\nabla T(0,y)|} \right\rangle \geq \beta . \tag{1.6}$$

Let us assume that for every s the solution of the IVP

$$y' = \frac{b(x,y)}{a(x,y)} =: \alpha(x,y) , \qquad\qquad y(0,s) = s$$

exists up to the time x_0. Then, in order to extend the solution to $x > x_0$, one restarts the problem on the new initial manifold $\{(x_0, y) : y \in \mathbb{R}\}$, and requires condition (1.6) to hold there as well. To be able to restart everywhere, it is obvious to assume

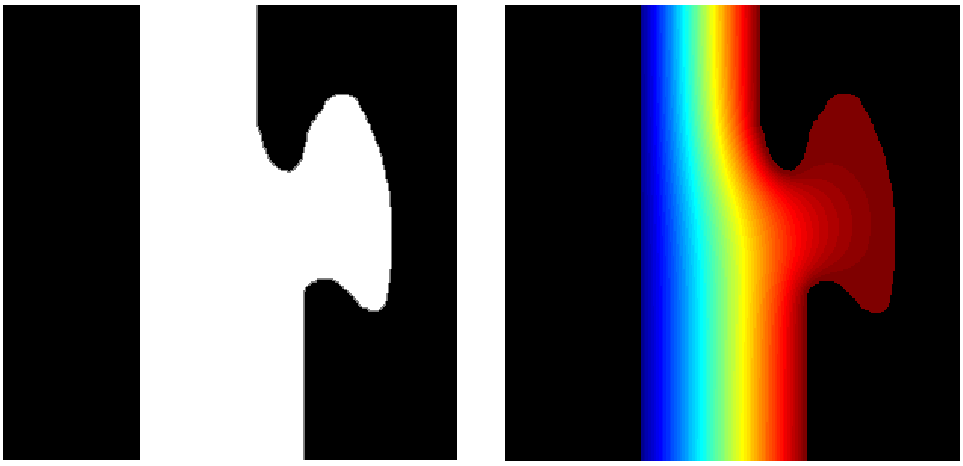
$$\left\langle \frac{c(x,y)}{|c(x,y)|}, \frac{\nabla T(x,y)}{|\nabla T(x,y)|} \right\rangle \geq \beta \qquad \forall\ (x,y) . \tag{1.7}$$

Now, let us attach a *stop set* Σ, by

$$\Sigma := \{(x,y) \in H^2_+ : T(x,y) = \lambda > 0\} .$$

That is, instead of the whole H^2_+, we want to solve on the restricted half-plane $H^2_{0,\lambda} :=]0,\lambda[\times \mathbb{R}$. Let the left boundary of $H^2_{0,\lambda}$ be the initial manifold again. Moreover, we assume that condition (1.7) holds.
Because, on the one hand, condition (1.7) implies that characteristics evolve rightwards and, on the other, that α is bounded on $H^2_{0,\lambda}$, the y-characteristic will exist up to the time $x = \lambda$ and the value $y(\lambda)$ is finite. Hence, every point of Σ will be met by a characteristic, and we obtain a global solution of the PDE on $H^2_{0,\lambda}$.
From the viewpoint of dynamical systems, condition (1.7) says that the time T is a *Lyapunov function* for the flow induced by the transport field, and Σ is an attractor. Moreover, since the Lyapunov condition holds globally on H^2_+, Σ is a global attractor. More on the concept of Lyapunov functions and its utilization can be found, e.g. in [Ama90] and [GJ09].

This interpretation is the key for generalization; we do not necessarily need to identify the x-coordinate with the "standard" time, i.e., $T(x,y) = x$ is not



(a) white: domain of the PDE $\Omega \subset H_+^2$ (b) contours of T, left boundary: $T = 0$, right boundary: $T = \lambda$

Figure 1.3: Example of T

the only possible time function. Let us consider once more the problem on the half-plane. Assume that Σ is a non-selfintersecting curve going through the interior of H_+^2, and assume that the y-component of this curve takes all values of $\mathbb{R}$. Furthermore, assume that we are given a function T, defined on H_+^2, which satisfies

$$T(0, y) = 0 \qquad \text{and} \qquad T|_\Sigma = \lambda \,,$$

and which strictly increases from the y–axis towards Σ. See figure 1.3 for an example. If, now, the transport field c satisfies condition (1.7) with respect to this T, we can argue, as above, for the existence and uniqueness of a global solution of the PDE.

Finally, in the case of a linear problem defined on a simply connected and bounded domain Ω, we will assume that an exceptional curve $\Sigma \subset\subset \Omega$ and a time function $T : \Omega \to \mathbb{R}$ are given. By the way, instead of calling it Lyapunov function, we will always refer to T as time function; in chapter 3, we will see that the characteristics can be transformed such a way that their time is given by the values of T, which motivates the name. Moreover, we assume the set Σ to be the global maximum of T; for this reason we call Σ stop set. By condition (1.7), then, Σ is the global attractor. Within this set-up, it is then possible to construct the unique solution of the linear problem

$$\begin{aligned} \langle c(x), \nabla u \rangle &= f(x) \,, \quad \text{in} \quad \Omega \backslash \Sigma \,, \\ u|_{\partial\Omega} &= u_0 \end{aligned} \tag{1.8}$$

by using the method of characteristics.
Which type of stop sets we can allow for and which conditions make a map T a reasonable time function are discussed in beginning of chapter 3.

In the following, we want to highlight two effects which show up if the stop set Σ is contained in Ω and is not part of the boundary $\partial\Omega$. For this purpose, we consider the simplified example

$$\begin{aligned} \partial_x u(x,y) &= 0\,, \qquad \text{in} \quad \Omega := \,]-1,1[\, \times \mathbb{R}\,, \\ u(-1,y) &= g_l(y)\,, \\ u(1,y) &= g_r(y) \end{aligned} \tag{1.9}$$

with data $g_l, g_r \in C^1(\mathbb{R})$ specified on opposite sides of Ω.

The need for stop sets
Solving problem (1.9) from the left boundary, the solution must be $u(x,y) = g_l(y)$. But, solving from the right boundary, the solution must be $u(x,y) = g_r(y)$. Thus, u can satisfy both boundary conditions if and only if $g_l \equiv g_r$. In the case $g_l \not\equiv g_r$,we must take out a set Σ. Let $\Sigma = \{(h(y),y) : y \in \mathbb{R}\}$ be given by a C^1-function h with $-1 < h < 1$, then $\Omega \backslash \Sigma$ consists of a left and a right part

$$\Omega_l = \{(x,y) \in \mathbb{R}^2 : -1 < x < h(y)\}\,, \quad \Omega_r = \{(x,y) \in \mathbb{R}^2 : h(y) < x < 1\}\,.$$

Analogously, the problem splits up into two sub-problems:

$$\begin{aligned} \partial_x u(x,y) &= 0\,,\ \text{in} \quad \Omega_l \\ u(-1,y) &= g_l(y)\,, \end{aligned} \qquad\qquad \begin{aligned} -\partial_x u(x,y) &= 0\,,\ \text{in} \quad \Omega_r\,, \\ u(1,y) &= g_r(y)\,. \end{aligned}$$

For the sake of completeness,

$$T(x,y) = \begin{cases} \frac{1+x}{1+h(y)}\,, & \text{in} \quad \Omega_l \\ \frac{1-x}{1-h(y)}\,, & \text{in} \quad \Omega_r \end{cases}$$

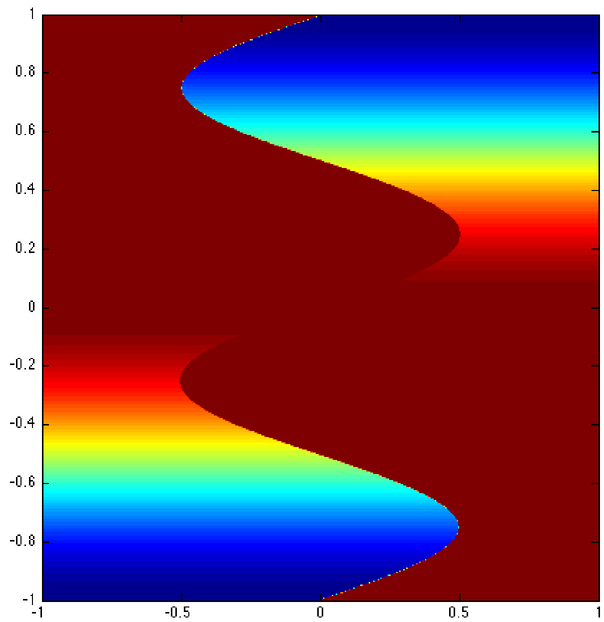
can be used as a time function. Then, condition (1.7) is satisfied with

$$c(x,y) = \begin{cases} e_1\,, & \text{in} \quad \Omega_l \\ -e_1\,, & \text{in} \quad \Omega_r \end{cases}\,,$$

and Σ is globally attractive.
Now, on $\Omega\backslash\Sigma$, the solution $u \in C^1(\Omega\backslash\Sigma)$ is

$$u(x,y) = g_l(y) \cdot \mathbb{1}_{\Omega_l}(x,y) + g_r(y) \cdot \mathbb{1}_{\Omega_r}(x,y)\,, \tag{1.10}$$

and we might view u as a generalized solution of (1.9) in the case $g_l \not\equiv g_r$. We observe that different choices of Σ are possible, and every choice leads to another result. Hence, we must pre-establish Σ in order to make the solution unique.



$$g_l(y) = (y^2-1)^2 \cdot \mathbb{1}_{[-1,0[} + \mathbb{1}_{[0,\infty[} \,,$$
$$g_r(y) = g_l(-y) \,,$$
$$h(y) = \tfrac{1}{2} \sin(2\pi y)$$

Figure 1.4: Contours of solution (1.10)

Properly global solutions

Besides the fact that the PDE is not fulfilled on Σ, we see that, even if the boundary data is as smooth as one desires, the solution will have jump discontinuities at Σ. For illustration: figure 1.4 shows a realization of the solution (1.10).

In order to define a global solution on the whole of Ω, we will have to search for u in a function space whose elements are allowed to have jumps. A suitable space is BV, which consists of the functions of bounded variation. Roughly, these are L^1-functions whose derivatives are finite Radon measures.

In the context of BV, we must rewrite the linear problem (1.8) as

$$\begin{aligned} \langle c(x)\,, Du\rangle &= f(x) \cdot \mathcal{L}^2\,, \qquad \text{in} \quad \Omega\backslash\Sigma\,, \\ u|_{\partial\Omega} &= u_0\,, \end{aligned} \tag{1.11}$$

whereas Du denotes the BV-derivative measure and $\mathcal{L}^2$ the Lebesgue measure. Because the PDE is now measure-valued, the right hand side must also be a measure. For our case, the right hand side is only allowed to be an absolutely continuous measure. Again, we assume the exceptional set Σ to be given in advance together with some reasonable time function T and we require c to satisfy condition (1.7). Working with BV we will see that the solution can be defined globally, that should mean that $u \in BV(\Omega)$ and not only $u \in BV(\Omega \backslash \Sigma)$. Moreover, we can allow for boundary data $u_0 \in BV$.

1.2.2 The Quasi-Linear Problem

The quasi-linear version of problem (1.11) is

$$\begin{aligned} \langle c[u](x)\,, Du\rangle &= f[u](x)\cdot \mathcal{L}^2\,, \qquad \text{in} \quad \Omega\backslash\Sigma\,, \\ u|_{\partial\Omega} &= u_0\,. \end{aligned} \tag{1.12}$$

Here, we allow the dependence of the transport field and the source term on u to be of a functional type. That should mean that they depend not only on the value $u(x)$ but on the whole function, i.e., the coefficients of the PDE are maps

$$\begin{aligned} &f : \mathcal{F} \to \mathcal{G}_1\,,\ u \to f[u]\,, \qquad && f[\,.\,](x) : \mathcal{F} \to \mathbb{R}\ \,,\ u \to f[u](x)\,, \\ &c : \mathcal{F} \to \mathcal{G}_2\,,\ u \to c[u]\,, \qquad && c[\,.\,](x) : \mathcal{F} \to \mathbb{R}^2\,,\ u \to c[u](x)\,, \end{aligned}$$

with $\mathcal{F}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ being subsets of suitable function spaces defined on Ω.

The quasi-linear problem (1.12) will be approached by using the theory of the linear problem. Fixing the functional argument of the coefficients by some $v \in \mathcal{F}$, we obtain the linear PDE

$$\begin{aligned} \langle c[v](x)\,, Du\rangle &= f[v](x)\cdot \mathcal{L}^2\,, \qquad \text{in} \quad \Omega\backslash\Sigma\,, \\ u|_{\partial\Omega} &= u_0\,. \end{aligned}$$

And, if the previous assumptions of the linear case hold true, the linear theory, then, gives us a solution $U[v]$ depending on v. Thus, we view a solution u of the quasi-linear problem to be a fixed point $u = U[u]$ of the operator U.

In the first part of chapter 4, we will utilize the *Schauder fixed point theorem* to establish the existence of a solution. In order to additionally get uniqueness and continuous dependence on the data, we have to restrict the functional dependence to be of *Volterra-type*. The latter should mean that, for fixed x, the values $f[v](x)$, $c[v](x)$ only depend on $v|_{\{T<T(x)\}}$. We will see in the second part of chapter 4 that operator $U[\,.\,]|_{\Omega'}$ is contractive when restricted to suitable subsets $\Omega' \subset \Omega$. The contractiveness will be the key to the uniqueness of the fixed point.

A very similar approach to solving such functional-differential problems on the half space H^n_+ by utilizing fixed point theory, can be found in [Kam99]. This book, in which the author considers classical solutions for time dependent non-linear PDEs with memory effect, was very inspiring.

1.3 Outline

In chapter 2 we collect some results from measure theory and give a short overview of the functions of bounded variation. All the functional analytical and fine properties of BV–functions, which we utilize later on, are summarized there.
In chapter 3 we discuss the linear problem in detail. There, we establish the existence (theorem 3.12) and uniqueness (theorem 3.16) of a global solution in the space $BV(\Omega)$. Moreover, we show (theorem 3.17) that the solution continuously depends on the boundary data as well as on the coefficients of the PDE.
Chapter 4 deals with the quasi-linear problem by using fixed point theory. For general functional dependence, theorem 4.11 establishes the existence of a fixed point as a consequence of the Schauder fixed-point theorem. For the restricted case of Volterra-type dependence, we obtain both uniqueness (theorem 4.22) and continuous dependence (theorem 4.24) by a contraction principle.
In chapter 5 we extend the concept of time functions introduced in chapter 3. As in chapter 3 we have restricted the PDE's domain to a simply connected domain and the stop set to a connected set with tree-like structure, we now can, by the extended concept, allow for non-connected stop sets (with forest-like structure) and n-connected domains.
Chapter 6 is about our inpainting model. In the first part we review the discrete and continuous set-up of the model. In the second part we show that the transport field of the inpainting equation satisfies all assumptions of our quasi-linear theory. The question of well-posedness of our inpainting model will finally be answered by the quasi-linear theory.
Chapter 7 deals with the practical usage of different time functions or orders. The order of pixels used in the algorithm is directly connected to the concept of time functions. In our prior work [BM07] we always ordered the pixels by their euclidean distance to the boundary. But, we have seen that other time functions, which induce orders, are possible. Here, we highlight on a few synthetic inpainting examples how this degree of freedom can be utilized to obtain better inpaintings.

Chapter 2

Basics: Functions of Bounded Variation

2.1 Measure Theory

Because for functions of bounded variation not only positive measures but much more vector-valued measures play the important role, we collect here appropriate definitions from measure theory. The reader can find more on this topic in [AFP00, chapters 1 and 2], [Els05], or [EG92] .

For now let Ω be a non-empty set and $\mathcal{A}$ a σ-Algebra on Ω.

Definition 2.1. *(σ-Additivity)*
A set-function μ of type $\mathcal{A} \to \mathbb{R} \cup \{\pm\infty\}$ or $\mathcal{A} \to \mathbb{R}^d$ ($d \in \mathbb{N}$) is called σ-additive if

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets $A_n \in \mathcal{A}$.

Definition 2.2. *(Positive and vector-valued measures)*

a) (positive measure) A set-function $\mu : \mathcal{A} \to [0, \infty]$ is a positive measure on $(\Omega, \mathcal{A})$ if μ is σ-additive and $\mu(\emptyset) = 0$.

b) (vector-valued measure) A set-function $\mu : \mathcal{A} \to \mathbb{R}^d$ is a vector-valued measure on $(\Omega, \mathcal{A})$ if μ is σ-additive and $\mu(\emptyset) = 0$. In the case $d = 1$, μ is a real-valued measure.

c) *(total variation measure) Let μ a vector-valued measure on $(\Omega, \mathcal{A})$, then*

$$|\mu|(A) := \sup\left\{ \sum_{n=1}^{\infty} |\mu(A_n)| : (A_n)_{n\in\mathbb{N}} \, , \, \bigcup_{n=1}^{\infty} A_n = A \right.$$
$$\left. \text{with pairwise disjoint } A_n \in \mathcal{A} \right\}$$

defines a finite positive measure $|\mu| : \mathcal{A} \to [0,\infty[$ called the total variation measure of μ.

If μ is a real-valued measure, its positive- and its negative part are defined as follows:

$$\mu^+ := \frac{|\mu| + \mu}{2} \, , \qquad \mu^- := \frac{|\mu| - \mu}{2} \, .$$

By now and for the remainder of this section Ω denotes a locally compact separable metric space. $\mathcal{B}(\Omega)$ is the Borel-σ-Algebra on Ω, which is the smallest σ-Algebra containing all open and closed subsets of Ω.

Definition 2.3. *(Borel and Radon measures)*

a) *(Borel measure) A positive measure on $(\Omega, \mathcal{B}(\Omega))$ is called a Borel measure.*

b) *(positive Radon measure) A Borel measure μ on $(\Omega, \mathcal{B}(\Omega))$ with $\mu(K) < \infty$ for every compact set $K \subset \Omega$, is called a positive Radon measure.*

c) *(vector-valued Radon measure) A set-function $\mu : \{A \in \mathcal{B}(\Omega) : A \subset\subset \Omega\} \to \mathbb{R}^d$ is called a vector-valued Radon measure if it is a vector-valued measure on $(K, \mathcal{B}(K))$ for every compact set $K \subset \Omega$.*

   The space of all vector-valued Radon measures is denoted by $[\mathcal{M}_{\mathrm{loc}}(\Omega)]^d$.

d) *(finite Radon measure) If $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^d$ is a vector-valued measure, then μ is called a finite Radon measure.*

   The space of all finite Radon measures is denoted by $[\mathcal{M}(\Omega)]^d$.

In the following, when talking about measures respectively Radon measures, we mean vector-valued measures respectively vector-valued Radon measures, if not explicitly stated otherwise. Furthermore, we denote by $\mathcal{L}^d$ the Lebesgue measure and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure on $\mathbb{R}^d$.

Later on, we will utilize integration w.r.t. vector-valued measures, so we repeat briefly its definition whereas we presume the Lebesgue integral w.r.t. positive measures for functions taking values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Definition 2.4. *(Integrals)*

a) If μ is a real-valued measure on Ω and $u : \Omega \to \overline{\mathbb{R}}$ is $|\mu|$-measurable, then it is $|\mu|$-summable if $\int_\Omega |u(x)|\, d|\mu|(x) < \infty$ and

$$\int\limits_\Omega u(x)\ d\mu(x) = \int\limits_\Omega u(x)\ d\mu^+(x) - \int\limits_\Omega u(x)\ d\mu^-(x)\ .$$

b) If μ is a real-valued measure on Ω and $u : \Omega \to \mathbb{R}^d$, whereas every component u_k is $|\mu|$-summable, then

$$\int\limits_\Omega u(x)\ d\mu(x) = \left(\int\limits_\Omega u_1(x)\ d\mu(x), \ldots, \int\limits_\Omega u_d(x)\ d\mu(x) \right)\ .$$

c) If μ is a vector-valued measure on Ω and $u : \Omega \to \overline{\mathbb{R}}$ is $|\mu|$-summable, then

$$\int\limits_\Omega u(x)\ d\mu(x) = \left(\int\limits_\Omega u(x)\ d\mu_1(x), \ldots, \int\limits_\Omega u(x)\ d\mu_d(x) \right)\ .$$

Moreover, we define two important operations on measures or integrals.

Definition 2.5. *(Restriction)*
Let μ be a positive or real- or vector-valued measure on $(\Omega, \mathcal{A})$ and $R \in \mathcal{A}$. The restriction of μ onto R is defined by

$$\mu \llcorner R(A) = \mu(A \cap R) \quad , \quad \forall\ A \in \mathcal{A}\ .$$

Definition 2.6. *(Push-forward)*
Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measure spaces and let $\xi : \Omega \to \Omega'$ be such that $\xi^{-1}(A') \in \mathcal{A}$ whenever $A' \in \mathcal{A}'$. For every positive or real- or vector-valued measure on $(\Omega, \mathcal{A})$ we define the pushed-forward measure $\xi_\sharp \mu$ on $(\Omega', \mathcal{A}')$ by

$$\xi_\sharp \mu(A') = \mu\left(\ \xi^{-1}(A')\right) \quad , \quad \forall\ A' \in \mathcal{A}'.$$

From this definition we obtain the change of variables rule: If u is a $|\xi_\sharp \mu|$-summable function defined on Ω', then $u \circ \xi$ is $|\mu|$-summable and

$$\int\limits_{\Omega'} u(y)\ d\xi_\sharp \mu(y) = \int\limits_\Omega u \circ \xi(x)\ d\mu(x)\ .$$

Finally, we state a generalization of product measure and product integration.

Definition 2.7.
Let $E_1 \subset \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$ be open sets. Let μ be a positive Radon measure on $(E_1, \mathcal{B}(E_1))$ and let $x \longrightarrow \nu_x$ be a function which assigns to each $x \in E_1$ a $\mathbb{R}^m$-valued finite Radon measure ν_x on $(E_2, \mathcal{B}(E_2))$.

The map $x \longrightarrow \nu_x$ is called μ-measurable if $x \longrightarrow \nu_x(B)$ is μ-measurable for every $B \in \mathcal{B}(E_2)$.

Proposition 2.8. *([AFP00, Proposition 2.26])*
Let E_1, E_2, μ and ν_x as in definition 2.7.

a) If $x \longrightarrow \nu_x(A)$ is μ-measurable for every open $A \subset E_2$ then $x \longrightarrow \nu_x$ is μ-measurable.

b) For every bounded and $\mathcal{B}_\mu(E_1) \times \mathcal{B}(E_2)$-measurable function g the map

$$x \longrightarrow \int_{E_2} g(x,y)\, d\nu_x(y)$$

is μ-measurable.

This proposition suggests the definition of a generalized product.

Definition 2.9. *(Generalized product)*
Let E_1, E_2, μ and ν_x be as in definition 2.7 and assume that

$$\int_{E_1'} |\nu_x|(E_2)\, d\mu(x) < \infty \qquad \forall\, E_1' \subset\subset E_1\, , E_1' \text{ open.}$$

We denote by $\mu \otimes \nu_x$ the $\mathbb{R}^m$-valued Radon measure on $\mathcal{B}(E_1 \times E_2)$ defined by

$$\mu \otimes \nu_x(B) = \int_{E_1} \int_{E_2} \mathbb{1}_B(x,y)\, d\nu_x(y)\, d\mu(x) \quad \forall B \in \mathcal{B}(K \times E_2)\, ,$$

whereas $K \subset E_1$ is any compact set.

Furthermore, approximation by simple functions yields the integration formula

$$\int_{E_1 \times E_2} f(x,y)\, d(\mu \otimes \nu_x)(x,y) = \int_{E_1} \int_{E_2} f(x,y)\, d\nu_x(y)\, d\mu(x)$$

for every bounded Borel function with $\operatorname{supp} f \subset E_1' \times E_2$, $E_1' \subset\subset E_1$.

2.2 Functions of Bounded Variation

Now, we come to that function space which plays the star role. We follow mainly the exposure of [AFP00]; the results can also be found partially in [Zie89]. Throughout this section Ω, the domain of definition for our functions, is required to be an open and bounded subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$) with Lipschitz boundary.

We use the following notation:

- $\langle a, b \rangle$: euclidean scalar product of $a, b \in \mathbb{R}^d$.
- $|a|$: absolute value if $a \in \mathbb{R}$, euclidean norm if $a \in \mathbb{R}^d$, spectral norm if $a \in \mathbb{R}^{d_1 \times d_2}$.
- $C(\Omega)^d = C(\Omega, \mathbb{R}^d)$, with

  $$\|\varphi\|_\infty = \| \, |\varphi| \, \|_\infty = \sup_{x \in \Omega} |\varphi(x)|$$

  if $\varphi \in C(\Omega)^d$ is bounded.
- $L^p(\Omega)^d = L^p(\Omega, \mathbb{R}^d)$, with

  $$\|u\|_{L^p(\Omega)} = \| \, |u| \, \|_{L^p(\Omega)}$$

  if $u \in L^p(\Omega)^d$.

2.2.1 Definition and Characterization

Definition 2.10. *(Functions of bounded variation)*
Let $u \in L^1(\Omega)$. u is said to be a function of bounded variation *if its distributional derivative is representable by a finite Radon measure in Ω, i.e.,*

$$\int_\Omega u(x) \, \partial_k \varphi(x) \, dx = - \int_\Omega \varphi(x) \, dD_k u(x) \qquad \forall \varphi \in C_c^1(\Omega), \quad k = 1, \ldots, d$$

or equivalently

$$\int_\Omega u(x) \; \operatorname{div} \varphi(x) \, dx = - \int_\Omega \langle \varphi(x), dDu(x) \rangle \qquad \forall \varphi \in C_c^1(\Omega)^d$$

for some $\mathbb{R}^d$-valued measure $Du = (D_1 u, \ldots, D_d u)$, $Du \in [\mathcal{M}(\Omega)]^d$. The vector space of all functions of bounded variation is denoted by $BV(\Omega)$.

The notion of variation for functions $u \in L^1_{\text{loc}}(\Omega)$ is given by

Definition 2.11. *(Variation)*
Let $u \in L^1_{\mathrm{loc}}(\Omega)$. The variation $\mathrm{Var}(u,\Omega)$ *of u in Ω is defined by*

$$\mathrm{Var}(u,\Omega) := \sup \left\{ \int_\Omega u(x)\ \mathrm{div}\,\varphi(x)\ dx \ :\ \varphi \in C^1_c(\Omega)^d\ ,\ \|\varphi\|_\infty \le 1 \right\} .$$

And the following proposition characterizes the space $BV(\Omega)$ by the variation, thereby explaining its naming.

Proposition 2.12. *([AFP00, Proposition 3.6])*
Let $u \in L^1(\Omega)$. Then u belongs to $BV(\Omega)$ if and only if $\mathrm{Var}(u,\Omega) < \infty$. For $u \in BV(\Omega)$ the variation $\mathrm{Var}(u,\Omega)$ coincides with the total variation $|Du|(\Omega)$ of Du.

By means of the variation we obtain a generalized notion of perimeter.

Definition 2.13.
For a $\mathcal{L}^d$-measurable set $E \subset \Omega$ the perimeter of E in Ω is defined by

$$P(E,\Omega) := \mathrm{Var}(\mathbb{1}_E,\Omega) .$$

Note: by proposition 2.12 the subset E is a set of finite perimeter in Ω if and only if $\mathbb{1}_E \in BV(\Omega)$.

2.2.2 Topologies on the Space BV

Next we discuss diverse topologies on $BV(\Omega)$. The space $BV(\Omega)$, endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega) ,$$

is a Banach space. But the norm-topology is too strong for most problems stated in $BV(\Omega)$: indeed continuously differentiable functions are not dense in BV. For example consider

$$u : \Omega := \,]-1,1[^2 \to \mathbb{R} \quad , \quad u(x,y) = \begin{cases} 0 & , \ x < 0 \\ 1 & , \ x \ge 0 \end{cases} .$$

It is easy to verify that the derivative measure is $Du = (1,0)^T \cdot \mathcal{H}^1 \llcorner \{x = 0\}$ with variation $|Du|(\Omega) = 2$. Hence, u is of bounded variation but is clearly not a Sobolev-function, i.e., $u \in BV(\Omega) \setminus W^{1,1}(\Omega)$.
For any $v \in W^{1,1}(\Omega) \subset BV(\Omega)$ the BV-derivative measure is $Dv = \nabla v(x) \cdot \mathcal{L}^d$ whereas $\nabla v \in L^1(\Omega)$ is the $W^{1,1}$-derivative of v. Thus, we have

$$\|v\|_{BV(\Omega)} = \|v\|_{W^{1,1}(\Omega)} .$$

By the completeness of $W^{1,1}(\Omega)$ the function u in the example above can never be $\| \, . \, \|_{BV(\Omega)}$-approximated by a sequence of continuously differentiable functions.

Now, we introduce those two topologies that are most commonly used. The first one is the weak* topology.

Definition 2.14. *(Weak* convergence in BV)*
Let $u \in BV(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $BV(\Omega)$. We say that u_n weakly converges in $BV(\Omega)$ to u if:*

1. *u_n tends to u in $L^1(\Omega)$, and*

2. *Du_n weakly* converges to Du in $[\mathcal{M}(\Omega)]^d$, i.e.,*

$$\int_\Omega \varphi(x) \, dDu_n(x) \to \int_\Omega \varphi(x) \, dDu(x) \qquad \forall \, \varphi \in C_0(\Omega) \, .$$

Remark 2.15. *(BV as a dual space)*
It can be proved that $BV(\Omega)$ is the dual of a separable *space, and that the convergence of definition 2.14 corresponds to the weak* convergence in the usual sense. For a construction see [AFP00, pp. 124, 125].*

A simple criterion for weak* convergence is stated in the following proposition.

Proposition 2.16. *(Weak* convergence criterion) ([AFP00, Proposition 3.13])*
Let $u \in BV(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $BV(\Omega)$. Then u_n weakly converges in $BV(\Omega)$ to u if and only if the sequence is $\|.\|_{BV(\Omega)}$-bounded and converges to u w.r.t. $\|.\|_{L^1(\Omega)}$.*

Proposition 2.17. *(Compactness in BV)*
Every $\|.\|_{BV(\Omega)}$-bounded sequence admits a weakly converging subsequence.*

Proof.

See [AFP00, Theorem 3.23]. □

The second topology is called the strict topology and is induced by the metric

$$d(u,v) = \int_\Omega |u(x) - v(x)|\, dx + ||Du|(\Omega) - |Dv|(\Omega)|\ .$$

Definition 2.18. *(Strict convergence)*
Let $u \in BV(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $BV(\Omega)$. We say that u_n strictly converges in $BV(\Omega)$ to u if $d(u_n, u) \to 0$ as $n \to \infty$.

The next theorem characterizes $BV(\Omega)$ by the strict convergence.

Theorem 2.19. *(Approximation by smooth functions) ([AFP00, Theorem 3.9])*
Let $u \in L^1(\Omega)$. Then, $u \in BV(\Omega)$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty(\Omega)$ converging to u in $L^1(\Omega)$ and satisfying

$$L := \lim_{n\to\infty} \int_\Omega |\nabla u_n(x)|\, dx < \infty\ .$$

Moreover, the least constant L is $|Du|(\Omega)$.

In other words $u \in L^1(\Omega)$ is an element of $BV(\Omega)$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty(\Omega)$ which *strictly* converges to u.

Note that by proposition 2.16 strict convergence implies weak* convergence. Moreover, a stronger result holds.

Proposition 2.20.
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $BV(\Omega)$ strictly converging to $u \in BV(\Omega)$ and $g : S^{d-1} \to \mathbb{R}$ continuous. Then for every $\varphi \in C_b(\Omega)$ we have

$$\lim_{n\to\infty} \int_\Omega \varphi(x) \cdot g\left(\frac{Du_n}{|Du_n|}(x)\right) d|Du_n|(x) = \int_\Omega \varphi(x) \cdot g\left(\frac{Du}{|Du|}(x)\right) d|Du|(x)\ ,$$

whereas $\frac{Du}{|Du|}(x)$ denotes the density function of Du w.r.t. $|Du|$.
In particular, the sequence of measures $\mu_n = g\left(\frac{Du_n}{|Du_n|}(x)\right) \cdot |Du_n|$ weakly converges to $\mu = g\left(\frac{Du}{|Du|}(x)\right) \cdot |Du|$ in Ω.*

Proof.

This is a consequence of the Reshetnyak continuity theorem (see [AFP00, Theorem 2.39]). □

2.2.3 Fine Properties of BV-Functions

In the sequel we will expose some fine properties of BV-functions. We begin with regular approximations of $f \in L^p_{\text{loc}}(\Omega)$ through convolution with mollifiers. A family of functions $(\rho_\varepsilon)_{\varepsilon>0}$ is called a family of mollifiers if $\rho_\varepsilon(x) = \varepsilon^{-d} \cdot \rho(x/\varepsilon)$ where $\rho \in C_c^\infty(\mathbb{R}^d)$ with the properties:

$$\rho \geq 0\,, \qquad \rho(-x) = \rho(x)\,, \qquad \int_{\mathbb{R}^d} \rho(x)\, dx = 1\,.$$

For $f \in L^p_{\text{loc}}(\mathbb{R}^d)$, $1 \leq p < \infty$, we have $f * \rho_\varepsilon \in C^\infty(\mathbb{R}^d)$ and

$$\lim_{\varepsilon \to 0_+} \|f * \rho_\varepsilon - f\|_{L^p(A)} = 0 \qquad \forall\ A \subset\subset \mathbb{R}^d\,.$$

Moreover, if f is continuous the convergence is uniform on compact sets. In what follows $(\rho_\varepsilon)_{\varepsilon>0}$ denotes a generic family of mollifiers.

Definition 2.21. *(Approximate limit)*
Let $u \in L^1_{\text{loc}}(\Omega)$. u has an approximate limit z at $x \in \Omega$ if

$$\lim_{r \to 0_+} \frac{1}{\mathcal{L}^d(B_r(x))} \int_{B_r(x)} |u(y) - z| dy = 0\,.$$

The set S_u of points where this property does not hold is called the approximate discontinuity set. For any $x \in \Omega \backslash S_u$ the approximate limit z of u at x is uniquely determined and denoted by $\tilde{u}(x)$.

A representative u is called approximately continuous at x if $x \in \Omega \backslash S_u$ is a Lebesgue point of u, i.e.,

$$u(x) = \tilde{u}(x)\,.$$

For every $u \in L^1_{\text{loc}}(\Omega)$ the set S_u is a Borel set and $\mathcal{L}^d(S_u) = 0$. Furthermore, $\tilde{u} : \Omega \backslash S_u \to \mathbb{R}$ is a Borel-function that coincides $\mathcal{L}^d$-a.e. in $\Omega \backslash S_u$ with u. Moreover, the sequence $u * \rho_\varepsilon$ converges pointwise to $\tilde{u}$ on $\Omega \backslash S_u$ as $\varepsilon \to 0+$.

Definition 2.22. *(Approximate jump points)*
Let $u \in L^1_{\text{loc}}(\Omega)$ and $x \in \Omega$. We say that x is an approximate jump point *of u if there exist $a, b \in \mathbb{R}$, $a \neq b$ and $n \in S^{d-1}$ such that*

$$\lim_{r \to 0_+} \frac{1}{\mathcal{L}^d(B_r^+(x,n))} \int_{B_r^+(x,n)} |u(y) - a|\, dy = 0$$

and

$$\lim_{r \to 0_+} \frac{1}{\mathcal{L}^d(B_r^-(x,n))} \int_{B_r^-(x,n)} |u(y) - b|\, dy = 0\,,$$

where

$$B_r^+(x,n) = \{y \in B_r(x) : \langle y-x, n\rangle > 0\}\ , \quad B_r^-(x,n) = B_r^+(x,-n)\ .$$

The triplet (a,b,n) up to a permutation of (a,b) and a change of sign of n is uniquely determined and denoted by $(u^+(x), u^-(x), n_u(x))$. The set of all approximate jump points is denoted by J_u.

For every $u \in L^1_{\text{loc}}(\Omega)$ the set J_u is a Borel subset of S_u and

$$u^+ : J_u \to \mathbb{R}\ , \quad u^- : J_u \to \mathbb{R}\ , \quad n_u : J_u \to S^{d-1}$$

are Borel functions. Moreover, for $x \in J_u$ the sequence $u * \rho_\varepsilon$ tends to $\frac{u^+(x)+u^-(x)}{2}$ as $\varepsilon \to 0+$.

Analogous to approximate continuity one defines approximate differentiability.

Definition 2.23. *(Approximate differentiability)*
Let $u \in L^1_{\text{loc}}(\Omega)$ and $x \in \Omega\backslash S_u$. We say that u is approximately differentiable *at x if there exist $L \in \mathbb{R}^d$ such that*

$$\lim_{r\to 0_+} \frac{1}{\mathcal{L}^d(B_r(x))} \int\limits_{B_r(x)} \frac{|u(y) - \tilde{u}(x) - \langle L, (y-x)\rangle\,|}{r}\, dy = 0\ .$$

The approximate differential L is uniquely determined and denoted by $\nabla u(x)$. The set of approximate differentiability points is denoted by D_u.

The set D_u is a Borel set and $\nabla u : D_u \to \mathbb{R}^d$ a Borel map.

In order to better understand the approximate jump set J_u we need the notion of $\mathcal{H}^k$-rectifiable sets.

Definition 2.24. *(Rectifiable sets)*
Let $E \subset \mathbb{R}^d$ be a $\mathcal{H}^k$-measurable set. We say that E is countably k-rectifiable if there exist countably many Lipschitz functions $f_j : \mathbb{R}^k \to \mathbb{R}^d$ such that

$$E \subset \bigcup_{j=0}^{\infty} f_j(\mathbb{R}^k)\ .$$

We say that E is countably $\mathcal{H}^k$-rectifiable if furthermore

$$\mathcal{H}^k\left(E\backslash \bigcup_{j=0}^{\infty} f_j(\mathbb{R}^k)\right) = 0\ .$$

Finally, we say that E is $\mathcal{H}^k$-rectifiable if in addition $\mathcal{H}^k(E) < \infty$.

For $k = 0$ countably k-rectifiable sets as well as countably $\mathcal{H}^k$-rectifiable sets correspond to finite or countable sets, while $\mathcal{H}^k$-rectifiable sets correspond to finite sets.

Theorem 2.25. *(Traces on interior rectifiable sets)([AFP00, Theorem 3.77]) Let $u \in BV(\Omega)$ and let $\Gamma \subset \Omega$ be a countably $\mathcal{H}^{d-1}$-rectifiable set oriented by $n : \Gamma \to S^{d-1}$. Then for $\mathcal{H}^{d-1}$-a.e. $x \in \Gamma$ there exist $u_\Gamma^+(x)$ and $u_\Gamma^+(x)$ such that*

$$\lim_{r \to 0_+} \frac{1}{\mathcal{L}^d(B_r^+(x, n(x)))} \int\limits_{B_r^+(x,n(x))} |u(y) - u_\Gamma^+(x)| \, dy = 0$$

and

$$\lim_{r \to 0_+} \frac{1}{\mathcal{L}^d(B_r^-(x, n(x)))} \int\limits_{B_r^-(x,n(x))} |u(y) - u_\Gamma^-(x)| \, dy = 0 \, .$$

Moreover, the restriction of the derivative measure onto Γ is

$$Du \llcorner \Gamma = (u_\Gamma^+(x) - u_\Gamma^-(x)) \cdot n(x) \cdot \mathcal{H}^{d-1} \llcorner \Gamma \, .$$

Now, we give a geometrical description of the derivative measure Du. The Calderon-Zygmund theorem ([AFP00, Theorem 3.83]) tells us that every $u \in BV(\Omega)$ is approximately differentiable at $\mathcal{L}^d$-almost every point and the approximate differential ∇u from definition 2.23 is the density function of the absolutely continuous part of the measure Du, i.e.,

$$Du = Du^a + Du^s \, , \qquad Du^a = \nabla u(x) \cdot \mathcal{L}^d \, ,$$

where the superscripts a, s denote the absolutely continuous and singular part respectively.

The Federer-Vol'pert theorem ([AFP00, Theorem 3.78]) tells us furthermore that the approximate discontinuity set S_u is countably $\mathcal{H}^{d-1}$-rectifiable and that $\mathcal{H}^{d-1}(S_u \backslash J_u) = 0$, i.e., almost every point of discontinuity is an approximate jump point. The singular part Du^s is then decomposed further

$$Du^s = Du^j + Du^c \, , \qquad Du^j = Du^s \llcorner J_u = Du \llcorner J_u \, ,$$

where the superscripts j, c denote the jump part and the so-called Cantor part respectively. Definition 2.22 and theorem 2.25 then yield the geometry of the jump part

$$Du^j = (u^+(x) - u^-(x)) \cdot n_u(x) \cdot \mathcal{H}^{d-1} \llcorner J_u \, .$$

At this point we remark that for $u \in BV(\Omega)$ the implication

$$\mathcal{H}^{d-1}(B) = 0 \quad \Rightarrow \quad |Du|(B) = 0 \qquad \forall \, B \in \mathcal{B}(\Omega)$$

holds true (see [AFP00, Lemma 3.76]). Thus, the Cantor part Du^c measures non-trivially only sets with Hausdorff dimension in between $d - 1$ and d.

Finally, we characterize two function spaces which are contained in $BV(\Omega)$:

a) $u \in BV(\Omega)$ belongs to the Sobolev space $W^{1,1}(\Omega)$ if and only if

$$Du = \nabla u(x) \cdot \mathcal{L}^d \,,$$

i.e., its derivative is absolutely continuous.

b) $u \in BV(\Omega)$ belongs to the space of special BV-functions, denoted by $SBV(\Omega)$, if and only if

$$Du = \nabla u(x) \cdot \mathcal{L}^d + (u^+(x) - u^-(x)) \cdot n_u(x) \cdot \mathcal{H}^{d-1} \llcorner J_u \,,$$

i.e., there is no Cantor part in Du.

For boundary value problems it is interesting to evaluate a function on the boundary.

Theorem 2.26. *(Boundary trace theorem) ([AFP00, Theorem 3.87])*
Let $\Omega \subset \mathbb{R}^d$ be an open set with bounded Lipschitz boundary and $u \in BV(\Omega)$. Then, for $\mathcal{H}^{d-1}$-almost every $x \in \partial\Omega$ there exists the boundary trace value $u|_{\partial\Omega}(x)$ such that

$$\lim_{r \to 0_+} \frac{1}{r^d} \int\limits_{\Omega \cap B_r(x)} |u(y) - u|_{\partial\Omega}(x)| \, dy = 0 \,.$$

Moreover, the trace is integrable:

$$\|u|_{\partial\Omega}\|_{L^1(\partial\Omega, \mathcal{H}^{d-1})} \leq C \cdot \|u\|_{BV(\Omega)}$$

for some constant C depending only on Ω.

Unfortunately, the trace operator is not continuous w.r.t. the weak* topology on $BV(\Omega)$, as the following example shows:

$$u_n :]0,1[\to \mathbb{R} \,, \qquad u_n(x) = \begin{cases} nx & , \ x \in \left]0, \frac{1}{n}\right[\\ 1 & , \ x \in \left[\frac{1}{n}, 1\right[\end{cases} .$$

This sequence of $BV(]0,1[)$-functions weakly* converges to $u \equiv 1$, but $u_n(0) = 0$ does not tend to $u(0) = 1$.

But the trace operator is continuous w.r.t. the strict topology on $BV(\Omega)$.

Theorem 2.27. *(Continuity of the trace operator) ([AFP00, Theorem 3.88])*
Let Ω as in theorem 2.26. Then the trace operator

$$.|_{\partial\Omega} : BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{d-1}) \,,$$

whereas $BV(\Omega)$ is endowed with the strict topology, is continuous.

Finally, we state two properties that are quite useful when working with BV-functions. The first one is a glueing property.

Theorem 2.28. *(Glueing)*
Let $u, v \in BV(\Omega)$. Let $E \subset \Omega$ be an open set with bounded Lipschitz boundary and ∂E be oriented by the outer unit normal $n_{\partial E}$. Let $u^+_{\partial E}$ and $v^-_{\partial E}$ be given $\mathcal{H}^{d-1}$-a.e. on ∂E by theorem 2.25. Then,

$$w := u \cdot \mathbb{1}_{\Omega \setminus E} + v \cdot \mathbb{1}_E \quad \in BV(\Omega) ,$$

and the measure Dw is representable by

$$Dw = Du \llcorner (\Omega \setminus \overline{E}) + (u^+_{\partial E} - v^-_{\partial E}) \cdot n_{\partial E} \cdot \mathcal{H}^{d-1} \llcorner \partial E + Dv \llcorner E .$$

Proof.

This is a consequence of [AFP00, Theorem 3.84], wherein the author considers the more general situation where E is a subset of finite perimeter. □

The second property considers the change of variables.

Theorem 2.29. *([AFP00, Theorem 3.16])*
Let Ω_1, Ω_2 be open subsets of $\mathbb{R}^d$, let $f : \Omega_1 \longrightarrow \Omega_2$ be a bijective and proper Lipschitz map and $u \in BV(\Omega_1)$. Then $f_\sharp u = u \circ f^{-1}$ belongs to $BV(\Omega_2)$ and

$$|D(f_\sharp u)| \leq L^{d-1} f_\sharp |Du| ,$$

where L denotes the least Lipschitz constant of f.

2.2.4 BV-Functions of One Variable and BV-Sections

The problem considered later on is stated on a subset of $\mathbb{R}^2$. So for the boundary data we are concerned with BV-functions of one variable. For BV-functions of one variable much stronger results hold true and in particular there are so-called *good representatives*. Roughly speaking, a representative in the equivalence class of u is called a good representative, if it is maximally continuous.

Theorem 2.30. *(Good representatives) ([AFP00, Theorem 3.28])*
Let $I =]a, b[\subset \mathbb{R}$ be an interval and $u \in BV(I)$. Let A be the set of atoms of Du, i.e., $t \in A$ if and only if $Du(\{t\}) \neq 0$. Then the following statements hold.

a) There exists a unique $c \in \mathbb{R}$ such that

$$u^l(t) = c + Du(]a, t[) \quad , \quad u^r(t) = c + Du(]a, t])$$

are good representatives of u, the left and the right continuous one. Any other $\bar{u} : I \longrightarrow \mathbb{R}$ is a good representative of u if and only if

$$\bar{u}(t) \in \left\{ \lambda \cdot u^l(t) + (1 - \lambda) \cdot u^r(t) : \lambda \in [0, 1] \right\} \quad \forall\, t \in I .$$

b) *Any good representative $\bar{u}$ is continuous in $I \backslash A$ and has a jump discontinuity at every point of A.*

c) *Any good representative $\bar{u}$ is differentiable at $\mathcal{L}^1$-a.e. point of I. The derivative $\bar{u}'$ is the density of Du w.r.t. $\mathcal{L}^1$.*

Note that the set A of atoms is at most countable and that u is essentially bounded.

A simple operation on functions $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ to retrieve a function of one variable is the so-called sectioning or slicing. Let $\Omega \subset \mathbb{R}^d$ as in the beginning of this section and $\nu \in S^{d-1}$. We denote by Ω_ν the projection of Ω onto the hyperplane orthogonal to ν and define the corresponding domain sections

$$\Omega_{y,\nu} := \{t \in \mathbb{R} : y + t\nu \in \Omega\} \quad \text{with} \quad y \in \Omega_\nu \, .$$

Accordingly, for any function $u : \Omega \to \mathbb{R}$ its section is defined by

$$u_{y,\nu} : \Omega_{y,\nu} \to \mathbb{R} \quad , \quad u_{y,\nu}(t) = u(y + t\nu) \, .$$

For a one-dimensional section of a $C^1(\Omega)$-function it is clear that it is a C^1-function of one variable, while for a one-dimensional section of a $BV(\Omega)$-function it is not obvious to obtain a BV-function of one variable. But [AFP00, paragraph 3.11.] tells us:
If $u \in BV(\Omega)$ then for $\mathcal{L}^{d-1}$-a.e. $y \in \Omega_\nu$ the section $u_{y,\nu}$ is a BV-function of one variable, i.e., $u_{y,\nu} \in BV(\Omega_{y,\nu})$.

Furthermore, for the slicing in $C^1(\Omega)$ the chain rule gives a correspondence between ∇u and $u'_{y,\nu}$. In the situation of BV-functions a similar result holds true. We begin with the definition of the directional derivative.

Definition 2.31.
Let $u \in L^1_{\text{loc}}(\Omega)$ and $p \in \mathbb{R}^d$; we say that the distributional derivative of u along p is a measure if there exists a finite Radon measure μ on Ω such that

$$\int_\Omega u(x) \cdot \partial_p \varphi(x) \, dx = - \int_\Omega \varphi(x) \, d\mu(x) \quad , \quad \forall \varphi \in C^1_c(\Omega) \, .$$

The measure μ is uniquely determined and will be denoted by $D_p u$.

The next theorem establishes the relationship between $D_\nu u$ and the derivative $Du_{y,\nu}$.

Theorem 2.32. *([AFP00, Theorem 3.107])*
If $u \in BV(\Omega)$ and $\nu \in S^{d-1}$, then

$$D_\nu u = \mathcal{L}^{d-1} \llcorner \Omega_\nu \otimes Du_{y,\nu}$$

where this is a generalized product measure according to definition 2.9, i.e.,

$$\int_\Omega \varphi(x)\, dD_\nu u(x) = \int_{\Omega_\nu} \int_{\Omega_{y,\nu}} \varphi_{y,\nu}(t)\, dDu_{y,\nu}(t)\, dy\,.$$

For the absolutely continuous part we have

$$D_\nu^a u = \mathcal{L}^{d-1} \llcorner \Omega_\nu \otimes D^a u_{y,\nu}\,.$$

In addition, the precise representative u^*,

$$u^*(x) := \begin{cases} \tilde{u}(x) & , \; x \in \Omega \backslash S_u \\ \frac{u^+(x)+u^-(x)}{2} & , \; x \in J_u \end{cases} ,$$

has a classical directional derivative along ν $\mathcal{L}^d$-a.e. in Ω with

$$\partial_\nu u^*(y+t\nu) = \frac{D^a u_{y,\nu}}{\mathcal{L}^1}(t) = \langle \nabla u(y+t\nu), \nu \rangle$$

for $\mathcal{L}^1$-a.e. $t \in \Omega_{y,\nu}$. Finally, the function $(u^*)_{y,\nu}$ is a good representative in the equivalence class of $u_{y,\nu}$.

Another situation where we are concerned with BV-functions of one variable is when working with BV-functions on a C^1-curve. We begin with the case of an open C^1-curve. The more general case of distributions on manifolds can be found, e.g., in [Hör90].

Definition 2.33. *(Regular parametrization)*
Let $\Gamma \subset \mathbb{R}^d$ be an open C^1-curve. A parametrization $\gamma : I \to \Gamma$ of Γ, whereas $I \subset \mathbb{R}$ is an open interval, is called regular, if $\gamma \in C^1(I, \Gamma)$ is surjective and $\gamma'(s) \neq 0\ \forall s \in I$.

Definition 2.34. *(BV on open rectifiable C^1-curves)*
Let $\Gamma \subset \mathbb{R}^d$ be an open rectifiable C^1-curve. A function $u : \Gamma \to \mathbb{R}$, $u \in L^1(\Gamma, \mathcal{H}^1)$, is a function of bounded variation if for every regular parametrization $\gamma : I \to \Gamma$ of Γ the distributional derivative of $\gamma^* u := u \circ \gamma$ is a finite Radon measure:

$$\int_I \gamma^* u(t)\, \varphi'(t)\, dt = -\int_I \varphi(t)\, dD\gamma^* u(t) \quad \forall\, \varphi \in C_c^1(I)\,.$$

We have to check that $\|u\|_{BV(\Gamma)}$ is independent of the choice of the parametrization. For $\|u\|_{L^1(\Gamma,\mathcal{H}^1)}$ this is clear

$$\|u\|_{L^1(\Gamma,\mathcal{H}^1)} = \int_I |\gamma^* u(t)| \cdot |\gamma'(t)|\, dt\,.$$

Since two regular parametrization differ only by a velocity transformation – that is given $\gamma_1 : I_1 \to \Gamma$ and $\gamma_2 : I_2 \to \Gamma$, there is a bijective $s \in C^1(I_2, I_1)$ such that $\gamma_2(t) = \gamma_1(s(t))$ – we have with $\varphi \in C^1_c(I_1)$

$$\int_{I_1} \gamma_1^* u(h)\varphi'(h)\, dh = \int_{I_2} \gamma_1^* u(s(t))\varphi'(s(t))s'(t)\, dt = \int_{I_2} \gamma_2^* u(t)\psi'(t)\, dt$$

with new test functions $\psi = \varphi \circ s \in C^1_c(I_2)$. Taking the sup over $\|\psi\|_\infty \leq 1$ on the right hand side first, and afterwards taking it over $\|\varphi\|_\infty \leq 1$ on the left hand side shows

$$\mathrm{Var}(\gamma_1^* u, I_1) \leq \mathrm{Var}(\gamma_2^* u, I_2)\ .$$

Reversing the order of taking the suprema shows the inequality in the other direction. Hence, we obtain the variation independent of the choice of the parametrization

$$\mathrm{Var}(u, \Gamma) = |Du|(\Gamma)\ .$$

Note: if $\gamma : I \to \Gamma$ is a regular parametrization with a bounded domain $I \subset \mathbb{R}$ we can say that $\gamma^* u$ is a BV-function of one variable, $\gamma^* u \in BV(I)$.

Likewise, for the next chapter the case where Γ is a simple closed C^1-curve is important. Preparatory, we introduce periodic BV-functions. For that purpose we look at an example first: let $0 < T < \infty$ and consider the T-periodic function u generated by

$$u_0 : \left[-\frac{T}{2}, \frac{T}{2}\right[\to \mathbb{R} \quad , \quad u_0(x) = \begin{cases} 0 & , x \in [-\frac{T}{2}, 0] \\ 1 & , x \in]0, \frac{T}{2}[\end{cases} \quad ,$$

i.e., u is recursively defined by

$$u(x) = \begin{cases} u_0(x) & , x \in [-\frac{T}{2}, \frac{T}{2}[\\ u(x+T) & , x < -\frac{T}{2} \\ u(x-T) & , x \geq \frac{T}{2} \end{cases} \quad .$$

Any restriction of u onto a half-open interval of length T can be used – instead of u_0 – as a generator to produce/reproduce u. When viewing $u|_I = u_0$ on $I =]-\frac{T}{2}, \frac{T}{2}[$ as a $BV(I)$-function, the derivative measure $D(u|_I)$ will detect one jump at $x = 0$ but will never detect the jumps of u at $-\frac{T}{2}$ or $\frac{T}{2}$ (which are the same point because of periodicity). This is because we test $D(u|_I)$ versus $C^1_c(I)$-functions: the support of such a test function does not contain the boundary points of I, thus the test function cannot "see" what is going on there.
If we take instead, say, $I =]-\frac{T}{4}, \frac{3T}{4}[$, then $D(u|_I)$ will detect two jumps, one at $x = 0$ and one at $x = \frac{T}{2}$. For this choice of I nothing spectacular

happens on the boundary of I, because the boundary points are points of continuity of u. More formally, if we take $\varphi \in C_c^1(]-\frac{T}{4}, \frac{3T}{4}[)$, we have

$$\begin{aligned}
\int_{\frac{-T}{4}}^{\frac{3T}{4}} u(t)\ \varphi'(t)\ dt &= \int_{\frac{-T}{4}}^{\frac{T}{2}} u(t)\ \varphi'(t)\ dt + \int_{\frac{T}{2}}^{\frac{3T}{4}} u(t)\ \varphi'(t)\ dt \\
&= \int_{\frac{-T}{4}}^{\frac{T}{2}} u(t)\ \varphi'(t)\ dt + \int_{\frac{-T}{2}}^{\frac{-T}{4}} u(t+T)\ \varphi'(t+T)\ dt \\
&= \int_{\frac{-T}{2}}^{\frac{-T}{4}} u(t)\ \varphi'(t+T)\ dt + \int_{\frac{-T}{4}}^{\frac{T}{2}} u(t)\ \varphi'(t)\ dt = \int_{\frac{-T}{2}}^{\frac{T}{2}} u(t)\ \psi'(t)\ dt
\end{aligned}$$

with a new test function $\psi \in C^1(]-\frac{T}{2}, \frac{T}{2}[)$,

$$\psi(t) = \begin{cases} \varphi(t+T) & , t \in]\frac{-T}{2}, \frac{-T}{4}[\\ \varphi(t) & , t \in]\frac{-T}{4}, \frac{T}{2}[\end{cases} ,$$

but $\psi \notin C_c^1(]-\frac{T}{2}, \frac{T}{2}[)$.

Summarizing, it is not enough to take some interval I of length T and requiring $u|_I$ to belong to $BV(I)$, since this depends on the choice of I. The goal is to adapt the definition of BV in order to retrieve all the features of u from some generator $u|_I$ independent of the choice of I.

Definition 2.35. *(Periodic test functions)*
Let $0 < T < \infty$. We denote by

$$P_T := \{\varphi \in C^1(\mathbb{R}) : \varphi(t+T) = \varphi(t)\}$$

the set of T-periodic test functions.
In the case that $T = \infty$, we set

$$P_\infty := \{\varphi \in C_b^1(\mathbb{R}) : \lim_{t\to-\infty} \varphi(t) = \lim_{t\to+\infty} \varphi(t)\ ,\ \lim_{t\to-\infty} \varphi'(t) = \lim_{t\to+\infty} \varphi'(t)\}\ .$$

Definition 2.36. *(Periodic BV functions)*
Let $0 < T < \infty$. A function $u : \mathbb{R} \to \mathbb{R}$ is a T-periodic BV-function if for every interval I of length T we have $u \in L^1(I)$ and

$$\int_I u(t)\ \varphi'(t)\ dt = -\int_I \varphi(t)\ dDu(t) \quad \forall\ \varphi \in P_T\ .$$

with a finite Radon measure Du on I. The space of T-periodic BV-functions is denoted by BV_T.

Note: if $u \in C^1 \cap BV_T$, then with $\varphi \equiv 1 \in P_T$ and $I =]a, a+T[$ we have

$$0 = \int_I u(t) \cdot 0 \, dt = -\int_I 1 \cdot u'(t) \, dt = u(a) - u(a+T) \, ,$$

that means $u \in P_T$. So the definition of BV_T is a generalization of P_T.

With definition 2.36 in mind we obtain for a T-periodic BV-function u

$$\|u\|_{L^1} = \|u|_I\|_{L^1(I)} \, , \quad |Du| = \sup \left\{ \int_I u(t) \, \varphi'(t) \, dt : \varphi \in P_T, \, \|\varphi\|_\infty \leq 1 \right\} \, ,$$

where I is any interval of length T.

Finally, we define BV on simple closed C^1-curves. Simple closed curves are homeomorphic to a circle, thus, we view them as periodic structures.

Definition 2.37. *(Regular periodic parametrization of simple closed C^1-curves)* *Let $\Gamma \subset \mathbb{R}^d$ be a simple closed C^1-curve. A parametrization $\gamma : \mathbb{R} \to \Gamma$ of Γ is called regular, if*

a) $\gamma \in C^1(\mathbb{R}, \Gamma)$ is surjective and $\gamma'(s) \neq 0 \; \forall s \in \mathbb{R}$,

b) γ is either T-periodic for some $0 < T < \infty$, or – when $T = \infty$ – is injective .

Definition 2.38. *(BV on a simple closed C^1-curve)*
Let $\Gamma \subset \mathbb{R}^d$ be a simple closed C^1-curve. A function $u : \Gamma \to \mathbb{R}$, $u \in L^1(\Gamma, \mathcal{H}^1)$, is a function of bounded variation if for every regular periodic parametrization $\gamma : \mathbb{R} \to \Gamma$ of Γ the distributional derivative of $\gamma^* u := u \circ \gamma$ is a finite Radon measure:

$$\int_I \gamma^* u(t) \, \varphi'(t) \, dt = -\int_I \varphi(t) \, dD\gamma^* u(t) \quad \forall \, \varphi \in P_T \, ,$$

where T is the period of γ and I an interval of length T.

Note: we have $\gamma^* u \in BV_T$ in the case of a finite period T, while T is a feature of γ but not of u. Because of parametrizing Γ periodically, we can retrieve every other parametrization from a given one by a velocity transformation plus a phase shift. Hence, the same argumentation, as already used for BV on open rectifiable C^1-curves, will show that the computation of $\|u\|_{BV(\Gamma)}$ does not depend on the choice of the parametrization.

Chapter 3

The Linear Problem

This chapter is concerned with the existence and uniqueness of global solutions of boundary value problems for linear PDEs of the first order in two-dimensional domains. We start out with a complete description of this problem.

3.1 The Problem and its Requirements

In this section we will first collect all the requirements and then state the full problem afterwards. We begin with the domain.

Requirement 3.1. *(Domains)*
Domains $\Omega \subset \mathbb{R}^2$ are required to satisfy the following conditions:

1. *Ω is open and bounded.*
2. *Ω is simply connected.*
3. *Ω has C^1-boundary.*

Because of 2. and 3. the boundary $\partial\Omega$ is a simple closed C^1-curve. Throughout this chapter we denote by $\gamma : \mathbb{R} \to \partial\Omega$ a generic periodic parametrization of $\partial\Omega$ according to definition 2.37. Furthermore, by $I = [a, b[\subset \mathbb{R}$ we denote an interval such that $\gamma|_I$ is a generator of γ.

In the introduction we pointed out the need for a reasonable substitute for time. Here we consider time functions whose range corresponds to a finite time interval. That means that these time functions will incorporate a stop set, on which they become maximal. Here we state the geometric properties of allowed stop sets.

Requirement 3.2. *(Stop sets)*
Stop sets Σ are required to satisfy the following conditions:

1. *Σ is a closed subset of Ω.*

2. *Σ is either an isolated point, or a connected set with tree-like structure.*

3. *If Σ is not an isolated point, we assume that Σ is made of finitely many rectifiable C^1-arcs Σ_k:*

$$\Sigma = \bigcup_{k=1}^{n} \Sigma_k \ .$$

The collection $\{\Sigma_k\}_{k=1,\dots,n}$ is assumed to be minimal in the number n of arcs, so Σ is decomposed by breaking it up at corners and branching points.

Furthermore, we require for each arc Σ_k that its relative interior $\mathring{\Sigma}_k$ has a given orientation by a continuous unit normal $n_k : \mathring{\Sigma}_k \to S^1$.

Later on, we will need a concept of one-sided limits towards $z \in \mathring{\Sigma}_k$.

Definition 3.3.

a) *Let $P(x)$ be the set of all possible projections of $x \in \Omega\backslash\Sigma$ onto Σ, i.e.,*

$$P(x) = \left\{ p \in \Sigma : |p - x| = \min_{z\in\Sigma} |z - x| \right\} \ .$$

A point $x \in \Omega\backslash\Sigma$ is said to be projectable onto a relatively open arc $\mathring{\Sigma}_k$ of Σ if for every $p \in P(x)$ we have $p \in \mathring{\Sigma}_k$.

b) *Let $x \in \Omega\backslash\Sigma$ be projectable onto $\mathring{\Sigma}_k$. Then, x is on the right hand side or plus-side of $\mathring{\Sigma}_k$ if*

$$\frac{x - p}{|x - p|} = +n_k(p) \qquad \forall \ p \in P(x) \ .$$

Analogously, x is on the left hand side or minus-side of $\mathring{\Sigma}_k$ if

$$\frac{x - p}{|x - p|} = -n_k(p) \qquad \forall \ p \in P(x) \ .$$

c) *A sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in \Omega\backslash\Sigma$ tends to $z \in \mathring{\Sigma}_k$ coming from the plus-side if the sequence converges towards z and almost all elements x_n are on the plus-side according to b); in symbols:*

$$x_n \to z_+ \ .$$

Analogously, we define $x_n \to z_-$, the limit from the minus-side.

Note: not every point is projectable. Moreover, there might be projectable points of which we cannot decide if they belong to the plus-side or to the minus-side. But, locally, in a neighborhood of Σ, every projectable point belongs either to the plus-side or the minus-side.

Next, we collect the properties of admissible time functions. We do this in two steps. In the first collection of requirements we summarize sufficient features which make time functions behave reasonably in between the boundary and the stop set. In the second collection we add requirements considering the behavior close to and on the stop set.

Requirement 3.4. *(Reasonable time functions)*
Time functions $T : \Omega \to \mathbb{R}$ are defined on a domain Ω which satisfies requirement 3.1. We denote by

$$\chi_{T \geq \lambda} := \{x \in \Omega : T(x) \geq \lambda\}$$

upper level-sets of T.

A time function T is required to satisfy the following conditions:

1. *$T \in C(\overline{\Omega})$, i.e., T extends continuously onto $\partial\Omega$.*

2. *The boundary of Ω is the start level: $T|_{\partial\Omega} = 0$.*

3. *T incorporates a stop set Σ in accordance with requirement 3.2, where the following conditions are assumed to hold:*

   a) $T(x) < 1 \Leftrightarrow x \in \overline{\Omega} \backslash \Sigma$.

   b) $T|_\Sigma = 1$, i.e., Σ is the maximal level of T.

4. *T increases strictly from $\partial\Omega$ towards Σ: that means any upper level-set $\chi_{T \geq \lambda}$ is simply connected and*

$$\chi_{T \geq \lambda} = \overline{\chi_{T > \lambda}} \qquad \forall\, \lambda \in [0, 1[\ .$$

5. *For every $\lambda \in [0, 1[$ the set $\chi_{T > \lambda}$ satisfies requirement 3.1 (any proper upper level-set is a future domain).*

   The field of interior unit normals to the λ-levels

$$\chi_{T = \lambda} = \partial\chi_{T > \lambda}\ , \qquad \lambda \in [0, 1[$$

   of T is denoted by $N : \Omega \backslash \Sigma \to S^1$. N is required to be continuously differentiable and extendable onto $\partial\Omega$, i.e., $N \in C^1(\overline{\Omega} \backslash \Sigma)$.

6.* *$T \in C^2(\overline{\Omega})$, with $\nabla T(x) = 0 \Leftrightarrow x \in \Sigma$.*

Ad 6.*: in order to ease things in the passages that follow, we assume that $T \in C^2(\overline{\Omega})$ – i.e., T, ∇T as well as D^2T are continuous with continuous extensions onto $\partial\Omega$ – and $\nabla T(x) = 0 \Leftrightarrow x \in \Sigma$. Because of these assumptions, we obtain a simple description of the field N on $\Omega \backslash \Sigma$:

$$N(x) := \frac{\nabla T(x)}{|\nabla T(x)|} \ .$$

Clearly, N is continuously differentiable and extendable onto $\partial\Omega$.

In the case that Σ is not only an isolated point, we also need a good behavior of the maps T and N at Σ.

Requirement 3.5. *(Behavior of time functions at Σ)*

1. *Requirements on T:*

   Let $y \in \Sigma$ and $h \in S^1$. Let $p = p(y,h)$ be the best possible order for the asymptotic

   $$T(y + rh) = 1 - \mathcal{O}(r^p) \ , \quad r \to 0_+ \ .$$

   We require that there is a bound q such that

   $$\sup_{y \in \Sigma} \sup_{|h|=1} p(y,h) < q \ .$$

2. *Requirements on N:*

   a) *N has one-sided extensions onto the relatively open components $\mathring{\Sigma}_k$ and those extensions are given by $\pm n_k$:*

   $$\begin{aligned} N^+(y) &:= \lim_{x \to y_+} N(x) \ , & N^+(y) &= -n_k(y) \ , \\ N^-(y) &:= \lim_{x \to y_-} N(x) \ , & N^-(y) &= n_k(y) \end{aligned}$$

   for every $y \in \mathring{\Sigma}_k$.

   b) *The derivative DN has one-sided extensions onto the relatively open components $\mathring{\Sigma}_k$, i.e.,*

   $$(DN)^+(y) := \lim_{x \to y_+} DN(x) \ , \quad (DN)^-(y) := \lim_{x \to y_-} DN(x)$$

   exist for every $y \in \mathring{\Sigma}_k$.

   c) *$|DN| \in L^1(\Omega)$, i.e., poles of $|DN|$ at corner-, branching- and terminal nodes of Σ are integrable. This feature is assumed to hold in the case that Σ is an isolated point as well.*

What remains are the assumptions on admissible transport fields. Those are as follows.

Requirement 3.6. *(Transport fields)*
Assuming that a domain Ω and a time function T with stop set Σ according to the requirements stated above are already specified, we require transport fields $c : \Omega\backslash\Sigma \to \mathbb{R}^2$ to satisfy:

1. *$c \in C^1(\Omega\backslash\Sigma)^2$ and c features the following properties:*
   a) *c and Dc are continuously extendable onto $\partial\Omega$.*
   b) *If Σ is not only an isolated point, then c and Dc have one-sided limits on the relatively open C^1-arcs $\mathring{\Sigma}_k$ of Σ:*
   $$c^+(y) = \lim_{x\to y_+} c(x) \quad \textit{and} \quad c^-(y) = \lim_{x\to y_-} c(x)\,,$$
   $$(Dc)^+(y) = \lim_{x\to y_+} Dc(x) \quad \textit{and} \quad (Dc)^-(y) = \lim_{x\to y_-} Dc(x)\,,$$
   for every $y \in \mathring{\Sigma}_k$.
2. *Unit speed and inward-pointing condition:*
   a) *$|c| = 1$.*
   b) *There is a lower bound $\beta > 0$ such that*
   $$\beta \le \langle c(x), N(x)\rangle \le 1 \qquad \forall x \in \overline{\Omega}\backslash\Sigma\,. \tag{3.1}$$
   c) *Conditions a) and b) hold for the one-sided limits as well, i.e.,*
   $$|c^+(y)| = |c^-(y)| = 1$$
   and
   $$\beta \le \left\langle c^+(y), N^+(y)\right\rangle \le 1 \quad , \quad \beta \le \left\langle c^-(y), N^-(y)\right\rangle \le 1\,,$$
   whenever y belongs to some $\mathring{\Sigma}_k$.
3. *Let z_k, $k \in \{1, \dots, m\}$ denote the terminal-, branching- and kink nodes of Σ. For every $\varepsilon > 0$, such that each disk $B_\varepsilon(z_k) \subset\subset \Omega$ is compactly contained in Ω, we define the set*
   $$V_\varepsilon := \Sigma \,\cup\, \bigcup_{k=1}^{m} \overline{B_\varepsilon(z_k)}\,.$$
   a) *For every admissible $\varepsilon > 0$, there is a bound M_ε such that*
   $$|Dc(x)| \le M_\varepsilon \quad , \quad \forall\, x \in \Omega\backslash V_\varepsilon$$
   b) *$|Dc| \in L^1(\Omega)$, poles of $|Dc|$ at z_k, $k \in \{1, \dots, n\}$ are integrable.*

Now that we have collected all assumptions, we finally state the problem.

Problem 3.7. *(Linear problem)*
Let Ω be a domain, $T : \Omega \to \mathbb{R}$ be a time function with a stop set Σ, and $c : \Omega \backslash \Sigma \to \mathbb{R}^2$ be a transport field, all in accordance with the requirements stated above.
Let furthermore $f \in C^1(\overline{\Omega})$ and $u_0 \in BV(\partial\Omega)$.

We search for $u \in BV(\Omega)$, such that

$$\langle c(x), Du \rangle = f(x) \cdot \mathcal{L}^2 \quad \textit{in} \quad \Omega \backslash \Sigma \,, \tag{3.2}$$

$$u|_{\partial\Omega} = u_0 \,. \tag{3.3}$$

Motivation of the requirements on time functions and transport fields
At this point, we want to discuss the motivation of requirement 3.4 which we have claimed to characterize reasonable time functions. In the introduction, we have already considered the problem

$$\begin{aligned} \partial_x u + \alpha(x,y) \cdot \partial_y u &= f(x,y) \qquad \text{in} \qquad H^2_{0,a} \,, \\ u(0,y) &= u_0(y) \,, \end{aligned}$$

for a C^1-function u, where $H^2_{0,a} := \{(x,y) \in \mathbb{R}^2 : 0 < x < a\}$ with start set $\{(x,y) \in \mathbb{R}^2 : x = 0\}$ and stop set $\Sigma = \{(x,y) \in \mathbb{R}^2 : x = a\}$.

Let us use this problem again to motivate the requirements on time functions. The derivation of the characteristic equations results in the following initial value problem (IVP)

$$\begin{aligned} x'(t) &= 1 \,, & x(0) &= 0 \,, \\ y'(t) &= \alpha(\, x(t), y(t) \,) \,, & y(0) &= s \,, \\ u'(t) &= f(\, x(t), y(t) \,) \,, & u(0) &= u_0(s) \,. \end{aligned}$$

For more background on the method of characteristics see e.g. [Eva98, chapter 3].

The solution of the first equation is $x(t) = t$ and so the variable x is typically identified with the characteristic time. In other words, the map $T(x,y) = x$ is the "natural" time function on $H^2_{0,a}$.

Let us first check if this "natural" time function fulfils requirement 3.4:
T is obviously continuous and we observe that the set $\{(x,y) \in \mathbb{R}^2 : x = 0\}$, where data is given, is the 0-level of T while Σ is the maximal level of T. In order to have Σ as the 1-level, one can use $T_2(x,y) = x/a$ instead of T. For $\lambda \in [0, a[$ any upper level-set of T

$$\chi_{T \geq \lambda} = H^2_{\lambda,a}$$

is simply connected and the condition

$$\chi_{T\geq\lambda} = \overline{\chi_{T>\lambda}}$$

is certainly satisfied. Hence, T increases strictly towards Σ in the sense of requirement 3.4 part 4. Since T is in fact a function of one variable given by $h(x) = x$, we can also argue by the monotonicity properties of h. By the way, functions of one variable which are strictly increasing in the usual sense satisfy the conditions in requirement 3.4 part 4. Finally, for part 5, the left boundary $\{(x,y) \in \mathbb{R}^2 : x = \lambda\}$ of any proper upper level-set $\chi_{T>\lambda}$ has the same shape as the left boundary of $\chi_{T>0} = H^2_{0,a}$. And the field of interior unit normals is $N(x,y) = e_1$, which is C^1. This last feature is very important, because it means that, when stopping at some time $\lambda < a$, the restarted problem - on $H^2_{0,a}$ with boundary data $u(\lambda,y)$ on $\{(x,y) \in \mathbb{R}^2 : x = \lambda\}$ - has exactly the same structure as the original one.

For the more general problem 3.7 we want to have virtually the same situation. So, it is quite obvious why we assume parts 1 to 3 of requirement 3.4. Part 4 is more interesting: its first condition,

$$\chi_{T\geq\lambda} = \overline{\chi_{T>\lambda}}\ ,$$

makes sure that T is free of plateaus and that λ-levels of T are closed curves without tentacles.
The second condition, that every upper level-set is required to be simply connected, guarantees that there are no local maxima besides the set Σ and furthermore, that there are no local minima or saddle nodes.
Note, if we allow for a saddle point x with time value $T(x) = \lambda$ then, the upper level-set $\chi_{T>\lambda}$ consists of two disjoint sets and so the problem will split up into two subproblems for time $T > \lambda$ with two parallel time lines so to speak. This scenario will be considered later on in chapter 5.
Part 5 of requirement 3.4 guarantees an analogous self-reproduction feature as the one we have in the half-space example above. In other words, every upper level set of T shall be a ”future” domain.

In section 3.3 we will apply the method of characteristics in order to construct a solution candidate. The characteristic IVP will then look like

$$y' = c(y)\ , \qquad\qquad y(0) = \gamma(s)\ .$$

Requirement 3.6 part 2b) states that the specified time function T is a global Lyapunov function for this dynamical system and the set Σ is an attractor. This might seem unnatural, but it is not: considering the linear problem on the half-plane again, the typical requirement (equation (1.7)) on the coefficients can be rephrased to ”the natural time function $T(x,y) = x$ should be a Lyapunov function".

Remark: in many text books, e.g. [Ama90], a continuous function $L : \Omega \to \mathbb{R}$ is called *Lyapunov function* for the solution $\xi(t,x)$ of an IVP

$$y' = c(y) \,, \qquad\qquad y(0) = x \in \Omega \,,$$

(or more generally a flow ξ) if the orbital derivative

$$L'(x) := \liminf_{t \to 0_+} \frac{L(\xi(t,x)) - L(x)}{t} \leq 0 \qquad \forall x \in \Omega$$

exists for all x and is non-positive. If L is smooth, this requirement is equivalent to

$$\langle \nabla L(x), c(x) \rangle \leq 0 \,.$$

For our case, requirement 3.6 part 2b) means that $L = -T$ is a Lyapunov function.

Requirements 3.4, 3.5, and 3.6 in combination are such that the family of characteristics gives us a customized coordinate system for problem 3.7. This is the matter of next section.

3.2 A Customized Coordinate System

Throughout this section we denote by Ω a domain, by Σ a stop set, by T a time function with the field of normals N and by c a transport field all in accordance with the requirements 3.1, 3.2, 3.4, 3.5, and 3.6.

Lemma 3.8.

a) *Let q be the bound from requirement 3.5 part 1 , let $\varphi(t) := -t^{\frac{1}{q}}$ and let*

$$T_0(x) := 1 + \varphi(1 - T(x)) \,. \tag{3.4}$$

Then, the gradient ∇T_0 of the transformed time function blows up at Σ and is bounded below

$$|\nabla T_0(x)| \geq m_0 > 0 \qquad \forall\, x \in \Omega \backslash \Sigma \,,$$

away from zero.

b) *For every regular C^1-curve $x : [0,a[\,\to \overline{\Omega} \backslash \Sigma$ ($a = \infty$ admissible) that satisfies the following condition*

$$0 < \beta \leq \left\langle N(x(\tau)), \frac{x'(\tau)}{|x'(\tau)|} \right\rangle \,, \qquad \forall \tau \in [0,a[\,, \tag{3.5}$$

the arc-length of x is uniformly bounded by

$$\text{arclength}(x) \leq \frac{1}{\beta \cdot m_0} \,. \tag{3.6}$$

Proof.

a) The function T_0 is well-defined since $0 \leq T \leq 1$ and its derivative is

$$\nabla T_0(x) = \varphi'(1 - T(x)) \cdot (-\nabla T(x)) =: H(x) \cdot \frac{\nabla T(x)}{|\nabla T(x)|} ,$$

with

$$H(x) := \frac{1}{q}(1 - T(x))^{\frac{1-q}{q}} \cdot |\nabla T(x)| > 0 .$$

Let $y \in \Sigma$, $h \in S^1$ and $r > 0$. Then, by requirement 3.5 part 1 we have

$$1 - T(y + rh) = C_1 r^p , \qquad p = p(y, h) , \; C_1 > 0 .$$

Because $T \in C^2(\Omega)$ and $\nabla T|_\Sigma = 0$ we obtain

$$|\nabla T(y + rh)| = C_2 r^{p-1} ,$$

with the same p as before. Putting both results together yields

$$H(y + rh) = C_3 r^{\frac{p(1-q)}{q}} r^{p-1} = C_3 r^{\frac{p-q}{q}} .$$

Since, by requirement 3.5 part 1, $q > p(y, h)$ holds uniformly, we obtain a blow up

$$\lim_{r \to 0_+} |\nabla T_0(y + rh)| = \lim_{r \to 0_+} H(y + rh) = \infty ,$$

for any choice of $y \in \Sigma$, $h \in S^1$.

We will show next that $|\nabla T_0| \geq m_0 > 0$. Assume by contradiction that

$$\inf_{x \in \overline{\Omega} \setminus \Sigma} H(x) = \inf_{x \in \overline{\Omega} \setminus \Sigma} |\nabla T_0(x)| = 0 .$$

and choose an open neighborhood U of Σ such that

$$H|_U \geq M ,$$

for some constant $M > 0$ which is possible because of the blow up. Then the restriction onto the complement $\hat{H} : \overline{\Omega} \setminus U \to \mathbb{R}$, $\hat{H} = H|_{\overline{\Omega} \setminus U}$, being a continuous function, must take the minimum

$$\min_{x \in \overline{\Omega} \setminus U} \hat{H}(x) = 0$$

at some point $\hat{x} \in \overline{\Omega} \setminus U$. But then, the definition of H implies

$$\hat{H}(\hat{x}) = H(\hat{x}) = 0 \quad \Rightarrow \quad |\nabla T(\hat{x})| = 0 ,$$

which is a contradiction, since $\hat{x} \notin \Sigma$. Thus, $H = |\nabla T_0|$ has a minimum greater than zero:

$$m_0 := \min_{x \in \overline{\Omega} \setminus \Sigma} |\nabla T_0(x)| > 0 .$$

b) Using T_0 we estimate the arc-length from above by

$$\begin{aligned} T_0(x(t)) - T_0(x(0)) &= \int_0^t \langle \nabla T_0(x(\tau)), x'(\tau) \rangle \, d\tau \\ &= \int_0^t \left\langle N(x(\tau)), \frac{x'(\tau)}{|x'(\tau)|} \right\rangle \cdot |\nabla T_0(x(\tau))| \cdot |x'(\tau)| \, d\tau \\ &\geq \beta \int_0^t |\nabla T_0(x(\tau))| \cdot |x'(\tau)| \, d\tau \;\geq\; \beta \cdot m_0 \int_0^t |x'(\tau)| \, d\tau \,. \end{aligned}$$

Since $0 \leq T_0 \leq 1$, we end up with

$$\int_0^t |x'(\tau)| \, d\tau \leq \frac{1}{\beta \cdot m_0} \quad , \quad \forall t \in [0, a[\;.$$

The limit $t \longrightarrow a$ finally yields the uniform bound on the arc-length of such curves x

$$\text{arclength}(x) \leq \frac{1}{\beta \cdot m_0} \,,$$

which depends only on β and information from T.

$\square$

Because of its nice properties the transformed version T_0 defined in lemma 3.8 part a) by equation (3.4) will be identified – instead of T – with the time variable of the characteristics. Whenever we speak about T_0 we mean this transformed version of a given time function T.

Lemma 3.9.

a) The initial value problem

$$y' = c(y) \,, \qquad\qquad y(0) = x \in \Omega \backslash \Sigma \,,$$

has a unique maximally continued solution $y :]t_-, t_+[\longrightarrow \mathbb{R}^2$, *with* $-\infty < t_- < 0 < t_+ < \infty$.

Every trajectory y *connects the sets* $\partial\Omega$ *and* Σ, *i.e.,*

$$\lim_{t \to t_-} y(t) \in \partial\Omega \,, \qquad\qquad \lim_{t \to t_+} y(t) \in \Sigma \,.$$

For every point $z \in \mathring{\Sigma}_k$ *in the relative interior of some* C^1*-arc of* Σ, *there are exactly two trajectories which hit* z *in the limit* $t \longrightarrow t_+$, *one for each side of* $\mathring{\Sigma}_k$.

b) The transformed transport field c_0

$$c_0 := \frac{c}{\langle c, \nabla T_0 \rangle} \tag{3.7}$$

features the properties:

- c_0 *is continuously extendable onto* Σ *by* $c_0|_\Sigma \equiv 0$ *.*
- *The solution* y_0 *of*

$$y' = c_0(y) \, , \qquad\qquad y(0) = x \in \Omega \backslash \Sigma \, ,$$

satisfies

$$T_0(y_0(t)) = t + T_0(x) \, .$$

Proof.

a) Consider the system

$$y' = c(y) \, , \qquad\qquad y(0) = x \in \Omega \backslash \Sigma \, , \tag{3.8}$$

Because c is Lipschitz continuous by requirement 3.6 part 3a) there exists a maximally continued, unique solution y with time domain $]t_-, t_+[$ and $0 \in]t_-, t_+[$.

Because of the unit speed condition $|c| = 1$, y never stops inside $\Omega \backslash \Sigma$ and never blows up. The inward-pointing condition (requirement 3.6 part 2b)) implies, by

$$\frac{d}{dt} T_0(y(t)) = \langle \nabla T_0(y(t)), c(y(t)) \rangle \geq m_0 \cdot \beta > 0 \, , \tag{3.9}$$

that $T_0(y(t))$ strictly increases at least by a rate of $m_0 \cdot \beta$.
Thus, y collapses at boundary of $\Omega \backslash \Sigma$ and by (3.9) it follows:

- Going forward $t \longrightarrow t_+$: collapse at Σ after finite time t_+ ,
- Going backward $t \longrightarrow t_-$: collapse at $\partial\Omega$ after finite time t_-.

Because of unit speed the values t_+ , t_- are exactly the arc-lengths, which are finite by lemma 3.8.

Assume now that $z \in \mathring{\Sigma}_k$ and consider the side where $n_k(z)$ points to. According to definition 3.3 b) we call this side the "plus-side" and the opposite side the "minus-side". Since c and Dc both extend from the plus-side onto $\mathring{\Sigma}_k$ by c^+ and $(Dc)^+$, the backward IVP

$$y' = -c(y) \, , \qquad y(0) = z \, , \qquad \text{with } c(z) := c^+(z) \, .$$

has a unique solution that starts at $z \in \mathring{\Sigma}_k$ and evolves away from Σ into the plus-side. Hence, vice versa there is only one solution y of the forward IVP (3.8) that comes from the plus-side, heads for $z \in \mathring{\Sigma}_k$, and hits z in the end. The same argumentation holds true for the minus-side.

b) We consider again the forward IVP (3.8), the initial value x of which satisfies $T_0(x) < 1 = \max_{z\in\Omega} T_0(z)$. For λ with $T_0(x) \leq \lambda < 1$, there is a unique time $\tau(\lambda)$, when the solution y of (3.8) crosses the λ-level of T_0. This is because $T_0(y(t))$ increases strictly by (3.9), so y crosses the λ-level only once .
Then, viewing τ as a function of λ, the implicit function theorem applied to

$$T_0(y(\tau)) = \lambda$$

yields the differentiability of τ w.r.t. λ and the derivative:

$$\tau'(\lambda) = \left.\frac{1}{\langle \nabla T_0(z), c(z)\rangle}\right|_{z=y(\tau(\lambda))} .$$

Using again the inward-pointing condition and recalling that $|\nabla T_0| \geq m_0 > 0$, we infer

$$0 < \tau' \leq \frac{1}{m_0 \cdot \beta} .$$

Let $\lambda_0 := T_0(x)$. Then, the function τ maps $[\lambda_0, 1[$ to $[0, t_+[$ with $\tau(\lambda_0) = 0$ and $\tau(1) = t_+$. Moreover, we have for τ' in the limit:

$$\begin{aligned} \lambda \to 1 \quad &\Rightarrow \quad \tau(\lambda) \to t_+ && \Rightarrow \quad y(\tau(\lambda)) \to z \in \Sigma \\ &\Rightarrow \quad |\nabla T_0(y(\tau(\lambda)))| \to \infty && \Rightarrow \quad \tau'(\lambda) \to 0 . \end{aligned}$$

Using τ, we change now the independent variable

$$y_0(\lambda) := y(\tau(\lambda + \lambda_0)) .$$

Then, y_0 satisfies the initial condition $y_0(0) = x$ and has the derivative

$$\begin{aligned} y_0'(\lambda) &= y'(\tau(\lambda+\lambda_0)) \cdot \tau'(\lambda+\lambda_0) \\ &= \left.\left(c(z) \cdot \frac{1}{\langle \nabla T_0(z), c(z)\rangle}\right)\right|_{z=y(\tau(\lambda+\lambda_0))} = \left.\frac{c(z)}{\langle \nabla T_0(z), c(z)\rangle}\right|_{z=y_0(\lambda)} \\ &= c_0(y_0(\lambda)) . \end{aligned}$$

Consequently, y_0 is the unique solution of

$$y' = c_0(y) \quad , \qquad y(0) = x \in \Omega \backslash \Sigma ,$$

and – by construction – satisfies

$$T_0(y_0(\lambda)) = T_0(y(\tau(\lambda+\lambda_0))) = \lambda + \lambda_0 = \lambda + T_0(x).$$

Because ∇T_0 blows up at Σ while $|c| = 1$, in the limit we have

$$\lim_{y \to z} c_0(y) = 0 \qquad \forall z \in \Sigma \, ,$$

which yields the continuous extension of c_0 onto Σ.

□

By lemma 3.9 part b) , we get the useful property that – when using c_0 from equation (3.7) instead of the original transport field c – the time variable of a characteristic y_0 is given by T_0. We use this feature to introduce new coordinates on $\Omega \backslash \Sigma$ whose conception is similar to polar coordinates on a disk. Whenever we speak about c_0 we mean the transformed version of a given transport field c according to equation (3.7).

Corollary 3.10. *(Polar coordinates)*
Let $\gamma : \mathbb{R} \to \partial\Omega$, a periodic parametrization of $\partial\Omega$ in accordance with definition 2.37. Let $I = [a, b[\subset \mathbb{R}$ be an interval such that $\gamma|_I$ is a generator for γ.

Then, the general solution $\xi(t, s)$ of the forward IVP

$$y' = c_0(y) \, , \qquad\qquad y(0) = \gamma(s) \, ,$$

defines a diffeomorphism $\xi : \,]0, 1[\times]a, b[\to \Omega \backslash (\Sigma \cup S)$ *, where*

$$S = \{\xi(t, a) : t \in \,]0, 1[\} \; .$$

Let $\eta(t, x)$ denote the general solution of the backward IVP

$$y' = -c_0(y) \, , \qquad\qquad y(0) = x \in \Omega \backslash (\Sigma \cup S) \; .$$

Then, the inverse map $\xi^{-1}(x) = (t(x), s(x))^T$ is given by

$$\xi^{-1}(x) = (\; T_0(x) \, , \; \gamma^{-1}(\eta(T_0(x), x)) \;)^T \, , \qquad x \in \Omega \backslash (\Sigma \cup S) \; .$$

The relation between ξ and η is

$$\xi(t, s(x)) = \eta(\; T_0(x) - t \; , \; x \;) \; . \tag{3.10}$$

Proof.

ξ solves the forward IVP and we know from lemma 3.9 that any curve $\xi(\,.\,,s)$ is located in $\Omega\backslash\Sigma$ and connects the sets $\partial\Omega$ and Σ. By this, it is obvious that the set $\Omega\backslash(\Sigma\cup S)$ is simply connected and open in $\mathbb{R}^2$, and that ξ maps $]0,1[\times]a,b[$ onto $\Omega\backslash(\Sigma\cup S)$.
Clearly, ξ is differentiable w.r.t. to time t. Since $\gamma\in C^1(]a,b[,\partial\Omega)$ and $c_0\in C^1(\Omega\backslash\Sigma,\mathbb{R}^2)$, we obtain $\partial_s\xi$ from the variational equation

$$\partial_t(\partial_s\xi)=Dc_0\circ\xi\cdot\partial_s\xi\,,\qquad\qquad \partial_s\xi(0,s)=\gamma'(s)\,.$$

Next we discuss η, the solution of the backward IVP. From lemma 3.9 we know the trajectory $\eta(\,.\,,x)$ hits the boundary after time $t=T_0(x)$. Hence, $\eta(T_0(x),x)$ is the unique point on the boundary that corresponds to x which implies further

$$\eta(T_0(x),x)=\gamma(s(x))\quad\Rightarrow\quad s(x)=\gamma^{-1}(\eta(T_0(x),x))\,.$$

Because $\xi(\,.\,,s(x))$ and $\eta(\,.\,,x)$ both connect the points x and $\gamma(s(x))$ along the same curve in opposite direction and have the same absolute velocity $|c_0|$, it follows that $t(x)=T_0(x)$ and furthermore the relation

$$\xi(t,s(x))=\eta(\;T_0(x)-t\;,\;x\;)\,.$$

With $t(x)$ and $s(x)$ we have the inverse map $\xi^{-1}(x)=(t(x),s(x))^T$. The differentiability properties of γ^{-1},T_0 and c_0 imply the differentiability of $\xi^{-1}(x)$: we obtain $D_x\eta$ from the variational equation for the backward IVP

$$\frac{d}{dt}(D_x\eta)=-Dc_0\circ\eta\cdot D_x\eta\,,\qquad\qquad D_x\eta(0,x)=I\,.$$

$\square$

Remark about the analogy to polar coordinates
Consider the parametrization $\Phi:\,]0,1[\times]-\pi,\pi[\;\rightarrow B_1(0)\backslash(\{0\}\cup S')\,,$

$$\Phi(r,\varphi)=(1-r)\cdot\begin{pmatrix}\cos(\varphi)\\-\sin(\varphi)\end{pmatrix}\quad,\quad S:=\{-r\cdot e_1:r\in\,]0,1[\}\,,$$

of the open unit disk $B_1(0)$ punctured at its center 0 and slitted along S' (negative x-axis). Note that the parametrization of the boundary by $\Phi(0,\varphi)$ is clockwise and thus $\det D\Phi(r,\varphi)=(1-r)>0$.

In comparison, we obtain the following correspondence between Φ and ξ:

	Φ	ξ
Center:	$\{0\}$	Σ
Slit:	S'	S
Radial variable:	r	t
Angular variable:	φ	s

In the following lemma, we collect diverse properties of the polar coordinates introduced just now.

Lemma 3.11.
Let γ be a parametrization of the boundary $\partial\Omega$ and let ξ as in corollary 3.10.

Then,

a) *if γ is clockwise, the jacobian $D\xi = (\partial_t\xi|\partial_s\xi)$ of the diffeomorphism ξ has a positive determinant and the estimate*

$$0 < \det D\xi \leq |\partial_t\xi||\partial_s\xi| \leq \frac{\det D\xi}{\beta} \tag{3.11}$$

holds true.

If γ is counter-clockwise, the assertions hold for $-\det D\xi$.

b) *for each of the relatively open C^1-arcs $\mathring{\Sigma}_k$ of Σ, we can find – w.r.t. the orientation n_k – two subsets $J_{k,+}$ and $J_{k,-}$ of I such that the maps $\xi(1,s)$ with $s \in J_{k,+}$ and $\xi(1,s)$ with $s \in J_{k,-}$ are both regular C^1-parametrizations of $\mathring{\Sigma}_k$.*

c) *$Dc_0 \circ \xi \cdot \partial_s\xi$ is integrable over $[0,1[\times [a,b[$.*

d) *The inverse map ξ^{-1} is one-sided extendable onto the relatively open C^1-arcs $\mathring{\Sigma}_k$ of Σ.*

Proof.

a) By lemma 3.9 part b), we have

$$T_0(\xi(t,s)) = t\,, \qquad\qquad \forall s \in \,]a,b[\;.$$

That means, for a fixed $t \in \,]0,1[$, the function $\xi(t,.) : \,]a,b[\, \to \Omega$ parametrizes the t-level of T_0. This parametrization itself is clockwise if the initial one $\xi(0,s) = \gamma(s)$ is. In this case $\partial_s\xi^\perp$ points into the exterior of $\chi_{T_0\geq t}$, and using the field of normals N, which points into the interior of $\chi_{T_0\geq t}$, we decompose

$$\partial_s\xi^\perp = -N \circ \xi \cdot |\partial_s\xi|$$

into direction and magnitude. For the counter-clockwise case, the same argumentation yields

$$\partial_s \xi^\perp = N \circ \xi \cdot |\partial_s \xi| \; .$$

Since $\partial_t \xi$ points in the same direction as the normed vector $c \circ \xi$, we decompose it similarly $\partial_t \xi = c \circ \xi \cdot |\partial_t \xi|$. Using the identity

$$\det D\xi = \det(\partial_t \xi | \partial_s \xi) = \left\langle \partial_t \xi, -\partial_s \xi^\perp \right\rangle$$

we end up with

$$\begin{aligned} \det D\xi &= |\partial_t \xi||\partial_s \xi| \, \langle c, N \rangle \circ \xi \, , && \text{if } \gamma \text{ clockwise} \\ -\det D\xi &= |\partial_t \xi||\partial_s \xi| \, \langle c, N \rangle \circ \xi \, , && \text{if } \gamma \text{ counter-clockwise} \, . \end{aligned}$$

Because ξ is a diffeomorphism, the determinant is, of course, never zero. The inward-pointing condition (3.1) from requirement 3.6 part 2 on the one hand implies the positiveness of $\pm \det D\xi$ and on the other the relation

$$\pm \det D\xi \leq |\partial_t \xi||\partial_s \xi| \leq \pm \frac{\det D\xi}{\beta} \; .$$

b) Proceeding as in the proof of lemma 3.9 part b), we solve the original forward IVP

$$y' = c(y) \, , \qquad\qquad y(0,s) = \gamma(s) \, ,$$

and obtain its solution $y(\,.\,,s) \; : \; [0, t_+(s)[\; \to \; \Omega$. Then, ξ is given by $\xi(\lambda, s) = y(\; \tau(\lambda, s) \; , s)$, where τ is implicitly defined by

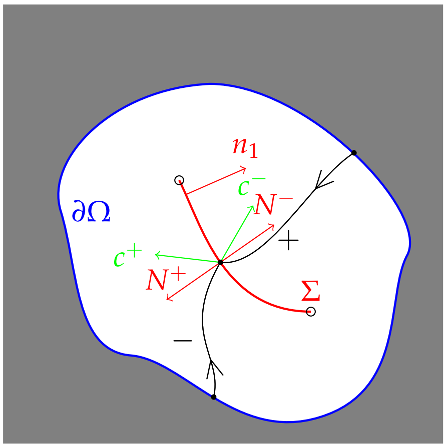
$$T_0(\; y(\tau, s) \;) = \lambda \, , \qquad\qquad 0 \leq \lambda < 1 \, ,$$

Hence, τ depends on $s \in [a, b[$ as well. Clearly, τ is periodic w.r.t. to s and an element of $C^1([0,1[\times [a,b[)$, so the question is what happens in the limit $\lambda \to 1$.

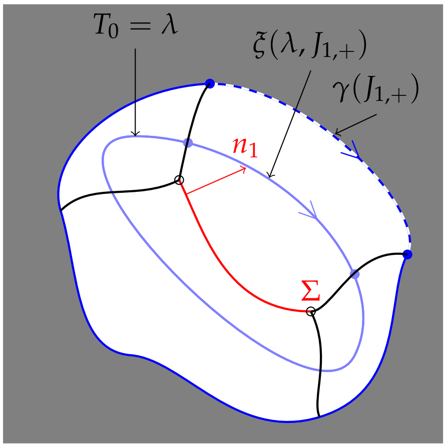
Since $T_0(\xi(1,s)) = 1$ means $\xi(1,s) \in \Sigma$, the function $s \to \xi(1,s)$ must somehow parametrize Σ. From lemma 3.9 part a) we know that if $z \in \mathring{\Sigma}_k$ then there are only two trajectories which meet at z, one for each side of $\mathring{\Sigma}_k$. Let J consist of all $s \in \;]a, b[$, such that $\xi(1,s)$ is a terminal node of one of the arcs $\Sigma_1, \ldots, \Sigma_n$.
Then, – corresponding to a single arc Σ_k – we choose $J_{k,+}$ to consist of all elements of $I \backslash J$ such that the family of curves

$$\xi(\lambda, \,.\,) : J_{k,+} \to \chi_{T_0 = \lambda} \, , \qquad\qquad s \to \xi(\lambda, s)$$

– in the limit $\lambda \to 1$ – reaches $\mathring{\Sigma}_k$ from the plus-side w.r.t. to the orientation n_k. Analogously, considering the minus-side, we choose $J_{k,-}$. For



red: Σ and normals, green: one-sided extensions of c, blue: boundary curve γ, black: two characteristics



red: Σ and a normal, blue: boundary curve γ, dashed blue: part $\gamma(J_{1,+})$ of the boundary, light blue: $\xi(\lambda, .)$, $\xi(\lambda, J_{1,+})$ part between the dots, black: for each terminal node of Σ the first and last characteristic

Figure 3.1: Decomposition of $\xi(\lambda, .)$

an illustration see figure 3.1, where $\Sigma = \Sigma_1$ is oriented by n_1 and γ is clockwise.

In the following, we restrict the discussion to the case of $J_{k,+}$, since for the proof of the other case the same steps are necessary. Depending on where the slit $S = \{\xi(t,a) : t \in]0,1[\}$ is located, the set $J_{k,+}$ is either an open interval or the union of two open intervals.

For $s \in J_{k,+}$ we extend the right hand side of the forward IVP by c^+, then both limits

$$\lim_{t \to t_+(s)} y(t,s) = z(s) \in \mathring{\Sigma}_k \,, \qquad \lim_{t \to t_+(s)} y'(t,s) = c^+(\, z(s) \,) \,,$$

exist and the equation

$$T_0(\, y(\tau, s) \,) = 1$$

has the unique solution $\tau(1,s) = t_+(s)$. Since $|\nabla T_0|$ blows up at Σ (see lemma 3.8 a)), we are not finished with the properties of $\partial_s \tau$. With the estimates

$$\begin{aligned}
&|\tau(1,s) - \tau(1,p)| \\
&\qquad \le |\tau(1,s) - \tau(\lambda,s)| + |\tau(\lambda,s) - \tau(\lambda,p)| + |\tau(\lambda,p) - \tau(1,p)| \\
&\qquad \le |\partial_\lambda \tau(\lambda_1,s)||1-\lambda| + |\tau(\lambda,s) - \tau(\lambda,p)| + |\partial_\lambda \tau(\lambda_2,p)||1-\lambda| \\
&\qquad \le \frac{2}{m_0 \cdot \beta}|1-\lambda| + |\tau(\lambda,s) - \tau(\lambda,p)|
\end{aligned}$$

and

$$|\tau(1,s) - \tau(\lambda,p)| \leq \frac{1}{m_0 \cdot \beta}|1-\lambda| + |\tau(\lambda,s) - \tau(\lambda,p)| \,,$$

whereas $\lambda < \lambda_1, \lambda_2 < 1$ stem from the mean value theorem, we have the continuity of $\tau(1,s) = t_+(s)$ as well as the continuous extension of $\tau(\lambda,s)$ by $\tau(1,s)$. By this result, $\xi(1,s) = y(\tau(1,s),s)$ is continuous.

Next, we study the partial derivative $\partial_s\tau$ and its limit $\lim_{\lambda\to 1}\partial_s\tau(\lambda,s)$. For $0 < \lambda < 1$ we obtain $\partial_s\tau$ by the implicit function theorem applied to

$$T_0(y(\,\tau(\lambda,s)\,,s)) = \lambda \,,$$

the derivative $\partial_s\tau$:

$$0 = \langle \nabla T_0, \partial_t y\rangle\,\partial_s\tau + \langle \nabla T_0, \partial_s y\rangle \qquad \Rightarrow \qquad \partial_s\tau = -\frac{\langle N, \partial_s y\rangle}{\langle N, c\rangle} \,,$$

(see also the proof of lemma 3.9 b)). Because c and N have continuous one-sided extensions, we have

$$\lim_{\lambda\to 1} N(\,y(\tau(\lambda,s),s)\,) = N^+(z(s))\,, \quad \lim_{\lambda\to 1} c(\,y(\tau(\lambda,s),s)\,) = c^+(\,z(s)\,)$$

and thus, it suffices to look at $\partial_s y$. The IVP for $\partial_s y$ gives us

$$\partial_s y(t,s) = \gamma'(s) + \int_0^t Dc\circ y(h,s)\cdot\partial_s y(h,s)\,dh\,.$$

By requirement 3.6 part 1a) Dc continuously extends as well as c, and by 3.6 part 3a) we get $|Dc\circ y(h,s)|$ uniformly bounded when restricting the domain of s to a ε-neighborhood $U_\varepsilon(\sigma)$ of say $\sigma \in J_{k,+}$ with ε small enough:

$$|Dc\circ y(h,s)| \leq M\,, \qquad\qquad (h,s) \in [0,\tau(1,s)] \times U_\varepsilon(\sigma)\,.$$

Gronwall's lemma (see [Wal70]) yields the bound

$$|\partial_s y(t,s)| \leq |\gamma'(s)|e^{Mt} \leq |\gamma'(s)|e^{M\tau(1,s)}\,,$$

which implies the existence of $\lim_{\lambda\to 1}\partial_s y(\,\tau(\lambda,s)\,,s)$. Moreover, the continuity of this limit w.r.t. s and the continuous extension of $(\lambda,p) \to \partial_s y(\,\tau(\lambda,p)\,,p)$ follow from the estimate below:

$$\begin{aligned}
&|\partial_s y(\,\tau(1,s)\,,s) - \partial_s y(\,\tau(\lambda,p)\,,p)| \\
&\leq |\partial_s y(\,\tau(1,s)\,,s) - \partial_s y(\,\tau(\lambda,s)\,,s)| + |\partial_s y(\,\tau(\lambda,s)\,,s) - \partial_s y(\,\tau(\lambda,p)\,,p)| \\
&\leq |\partial_t\partial_s y(\,\tau_*\,,s)||\tau(1,s) - \tau(\lambda,s)| + |\partial_s y(\,\tau(\lambda,s)\,,s) - \partial_s y(\,\tau(\lambda,p)\,,p)| \\
&\leq M|\gamma'(s)|e^{M\tau(1,s)}|\tau(1,s) - \tau(\lambda,s)| + |\partial_s y(\,\tau(\lambda,s)\,,s) - \partial_s y(\,\tau(\lambda,p)\,,p)|\,.
\end{aligned}$$

Consequently, $\partial_s \tau(\lambda, s)$ has exactly the same continuity properties as $\partial_s y$. Summarizing, we have for $s \in J_{k,+}$ that

$$\partial_s \xi(\lambda, s) = \partial_t y(\tau(\lambda, s), s) \cdot \partial_s \tau(\lambda, s) + \partial_s y(\tau(\lambda, s), s) \tag{3.12}$$

extends continuously onto $\lambda = 1$ by $\partial_s \xi(1, s)$.

Finally, for the regularity of $\partial_s \xi(1, s)$, we have to argue that $|\partial_s \xi(1, s)| \neq 0$. In part a), we have already used that $\partial_s \xi = |\partial_s \xi| \cdot N^\perp$ and that

$$\det(c|\partial_s \xi) = |\partial_s \xi| \det(c|N^\perp) = |\partial_s \xi| \, \langle c, N \rangle \geq |\partial_s \xi| \cdot \beta \, .$$

With the representation of $\partial_s \xi$ in equation (3.12) and $\partial_t y = c \circ y$, we obtain

$$|\partial_s \xi(\lambda, s)| \, \langle c, N \rangle \circ \xi(\lambda, s) = \det(c \circ y(\tau(\lambda, s), s) | \partial_s y(\tau(\lambda, s), s)) \, .$$

Hence, $|\partial_s \xi(\lambda, s)|$ becomes zero if and only if $\det(c \circ y | \partial_s y)$ does.

Having the IVPs for $y(t, s)$ and $\partial_s y(t, s)$ in mind, it is easy to check, that the determinant $\det(c \circ y | \partial_s y)$ satisfies the IVP

$$\begin{aligned} \frac{d}{dt} \det(c \circ y | \partial_s y) &= \operatorname{trace}(Dc \circ y) \cdot \det(c \circ y | \partial_s y) \, , \\ \det(c \circ y | \partial_s y)|_{t=0} &= \det(c \circ \gamma(s) | \gamma'(s)) > 0 \, . \end{aligned}$$

Thus, the determinant is given by

$$\begin{aligned} &\det(c \circ y(t, s) | \partial_s y(t, s)) \\ &\qquad = \exp\left(\int_0^t \operatorname{trace}(Dc) \circ y(h, s) \, dh \right) \cdot \det(c \circ \gamma(s) | \gamma'(s)) \, , \end{aligned}$$

and cannot become zero within finite time:

$$\det(c \circ y(t, s) | \partial_s y(t, s)) \neq 0 \quad \text{and} \quad \lim_{t \to t_+(s)} \det(c \circ y(t, s) | \partial_s y(t, s)) \neq 0 \, .$$

Hence, $|\partial_s \xi(\lambda, s)| \neq 0$ for $\lambda \in [0, 1[$ and in the limit $\lambda \to 1$, we have $|\partial_s \xi(1, s)| \neq 0$, because $\det(c \circ y(t, s) | \partial_s y(t, s))$ does not become zero in the limit $t \to t_+(s) = \tau(1, s)$ since $t_+(s)$ is finite.

c) Recalling that ξ is the solution of the forward IVP

$$y' = c_0(y) \, , \qquad y(0) = \gamma(s) \, , \qquad c_0 = \frac{c}{\langle c, \nabla T_0 \rangle} \, ,$$

the norm of $\partial_t \xi$ is

$$|\partial_t \xi| = \frac{1}{\langle c, \nabla T_0 \rangle} \circ \xi \, .$$

And again, we will use the formulas

$$\partial_s \xi = \pm N^\perp \circ \xi \cdot |\partial_s \xi| \qquad \text{and} \qquad N = \frac{\nabla T_0}{|\nabla T_0|}.$$

By the chain rule, the derivative of c_0 is

$$Dc_0 = \frac{1}{\langle c, \nabla T_0 \rangle} \left(Dc - \frac{c}{\langle c, \nabla T_0 \rangle} \left(\nabla T_0^T \cdot Dc + c^T \cdot D^2 T_0 \right) \right) .$$

Using the formulas above, we write the product $Dc_0 \circ \xi \cdot \partial_s \xi$ as

$$\begin{aligned} Dc_0 \circ \xi \cdot \partial_s \xi &= \pm |\partial_s \xi| \cdot Dc_0 \circ \xi \cdot N^\perp \circ \xi = \pm |\partial_t \xi| |\partial_s \xi| \cdot \\ &\quad \left(Dc \cdot N^\perp - \frac{c}{\langle c, N \rangle} \left(N^T \cdot Dc \cdot N^\perp + c^T \cdot \frac{D^2 T_0}{|\nabla T_0|} \cdot N^\perp \right) \right) \circ \xi . \end{aligned}$$

Next, we compute the derivative of $N = \frac{\nabla T_0}{|\nabla T_0|}$

$$\begin{aligned} DN &= \frac{|\nabla T_0| D^2 T_0 - \nabla T_0 \cdot \frac{\nabla T_0^T}{|\nabla T_0|} \cdot D^2 T_0}{|\nabla T_0|^2} \\ &= (I - N \cdot N^T) \cdot \frac{D^2 T_0}{|\nabla T_0|} = N^\perp \cdot N^{\perp T} \cdot \frac{D^2 T_0}{|\nabla T_0|} , \end{aligned}$$

which helps to write the expression

$$c^T \cdot \frac{D^2 T_0}{|\nabla T_0|} \cdot N^\perp = N^{\perp T} \cdot DN \cdot c$$

without $D^2 T_0$.

Finally, we estimate

$$\begin{aligned} &|Dc_0 \circ \xi \cdot \partial_s \xi| \\ &\leq |\partial_t \xi| |\partial_s \xi| \left(|Dc \cdot N^\perp| + \frac{|c|}{\langle c, N \rangle} \left(|N^T \cdot Dc \cdot N^\perp| + |N^{\perp T} \cdot DN \cdot c| \right) \right) \circ \xi \\ &\leq |\partial_t \xi| |\partial_s \xi| \left(|Dc| + \frac{1}{\beta} (|Dc| + |DN|) \right) \circ \xi \\ &\leq \frac{\det D\xi}{\beta} \left(|Dc| + \frac{1}{\beta} (|Dc| + |DN|) \right) \circ \xi , \end{aligned}$$

whereas we have used equation (3.11) in the last step. The right hand side of the last inequality is integrable over $]0,1[\times [a,b[$. This follows from resubstituting ξ and the integrability requirements on Dc and DN over Ω (see requirement 3.6 3.b) and 3.5 2.c)).

Note: if we write $\partial_s\xi(t,s)$ as

$$\partial_s\xi(t,s) = \gamma'(s) + \int_0^t Dc_0 \circ \xi(\tau,s) \cdot \partial_s\xi(\tau,s)\, d\tau$$

and go over to the limit $t \to 1$ for s restricted to $J_{k,+}$, then the result above shows that $\partial_s\xi(1,s)$ is integrable over $J_{k,+}$, and dominated convergence yields the rule

$$\lim_{t\to 1} \int_{J_{k,+}} \partial_s\xi(t,s)\, ds = \int_{J_{k,+}} \partial_s\xi(1,s)\, ds\,. \tag{3.13}$$

d) Let $z \in \mathring{\Sigma}_k$. As argued above, for the plus-side there is a unique $s \in J_{k,+}$ with $\xi(1,s) = z$. By this we define $s^+(z)$, which extends the $s(x)$ component of $\xi^{-1}(x)$. Since $t(x) = T_0(x)$ is defined on Σ, we are done.

□

3.3 Existence of a Solution

The subject of this section is the proof of the following existence theorem.

Theorem 3.12. *(Existence)*
The linear problem 3.7 has a solution.

The proof of the theorem will result from the lemmata which are to follow.

In order to construct a candidate solution we apply the method of characteristics. But first we scale the PDE (3.2) so that we can use the results of the previous section. Let T_0 be the transformed time function from equation (3.4) . In the proof of lemma 3.9 we have seen that

$$0 < \frac{1}{\langle c(x), \nabla T_0(x)\rangle} \le \frac{1}{m_0 \cdot \beta}\,, \qquad \forall\, x \in \Omega\backslash\Sigma\,.$$

Thus, we can scale the original PDE (3.2) by that factor to get the equivalent PDE

$$\langle c_0(x), Du\rangle = f_0(x) \cdot \mathcal{L}^2 \quad \text{in} \quad \Omega\backslash\Sigma\,, \tag{3.14}$$

with

$$f_0(x) = \frac{f(x)}{\langle c(x), \nabla T_0(x)\rangle} \qquad \text{and} \qquad c_0(x) = \frac{c(x)}{\langle c(x), \nabla T_0(x)\rangle}\,.$$

The latter is the transformed transport field as in equation (3.7).

Then, for PDE (3.14) the characteristic equation is exactly the forward IVP from corollary 3.10

$$y' = c_0(y) , \qquad\qquad y(0) = \gamma(s) ,$$

and the family of forward characteristics is ξ (from the same corollary). Clearly, the solution η of the corresponding backward IVP is the family of backward characteristics.

Let now $v(t,s) := u \circ \xi(t,s)$, then – at least formally – the partial derivative of v w.r.t. t is given by

$$\partial_t v(t,s) = \langle \nabla u \circ \xi(t,s), \partial_t \xi(t,s) \rangle = \langle \nabla u, c_0 \rangle \circ \xi(t,s) = f_0 \circ \xi(t,s) ,$$

together with the initial condition

$$v(0,s) = u(\xi(0,s)) = u(\gamma(s)) = \gamma^* u_0(s) .$$

Herein, γ^* denotes the pull-back operation. Thus, by the fundamental theorem of calculus, we obtain

$$v(t,s) = \gamma^* u_0(s) + \int_0^t f_0 \circ \xi(\tau,s)\, d\tau . \tag{3.15}$$

The function v represents our candidate solution u in characteristic variables (t,s). By using the inverse map ξ^{-1} from corollary 3.10 and the relation (3.10) between ξ and η, we push v forward onto $\Omega \backslash \Sigma$ to have $u = v \circ \xi^{-1}$ in original variables x

$$u(x) = u_0(\eta(T_0(x), x)) + \int_0^{T_0(x)} f_0 \circ \eta(\tau, x)\, d\tau . \tag{3.16}$$

For the analysis of the candidate solution, it is useful to decompose it additively:

$$\begin{aligned} v_1(t,s) &= \gamma^* u_0(s) \qquad , \qquad & v_2(t,s) &= \int_0^t f_0 \circ \xi(\tau,s)\, d\tau , \\ u_1(x) &= u_0(\eta(T_0(x),x)) \quad , \qquad & u_2(x) &= \int_0^{T_0(x)} f_0 \circ \eta(\tau,x)\, d\tau . \end{aligned} \tag{3.17}$$

The next lemma shows that the so-constructed candidate belongs to the space of functions which the problem 3.7 is stated for.

Lemma 3.13. *(Element of BV)*
The candidate solution u from (3.16) with its decomposition $u = u_1 + u_2$ from (3.17) has the properties:

a) u is an element of $BV(\Omega\backslash\Sigma) \cap L^\infty(\Omega)$.

Its derivative measure is

$$Du = c_0^\perp(x) \cdot \mu \;+\; \nabla u_2(x) \cdot \mathcal{L}^2 \quad \text{with} \quad \mu := \xi_\sharp \left(\mathcal{L}^1 \otimes D\gamma^* u_0\right) ,$$

and the total variation is bounded by

$$\begin{aligned} |Du|(\Omega\backslash\Sigma) \le M_{\Omega\backslash\Sigma} := \;& \frac{|Du_0|}{\beta \cdot m_0} + \left(\frac{\|f\|_\infty}{\beta} + \frac{\|\nabla f\|_\infty}{\beta^2 \cdot m_0}\right) \cdot \mathcal{L}^2(\Omega) \\ & + \frac{\|f\|_\infty}{\beta^3 \cdot m_0^2} \cdot \left(\|Dc\|_{L^1(\Omega)} + \|DN\|_{L^1(\Omega)}\right) . \end{aligned}$$

The $L^\infty(\Omega)$-norm is bounded by

$$\|u\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\partial\Omega)} + \frac{\|f\|_\infty}{\beta \cdot m_0} .$$

b) u extends onto Σ, i.e., u is an element of $BV(\Omega) \cap L^\infty(\Omega)$.

In comparison to part a) the extension introduces – in the derivative Du – an additional jump part for every C^1-arc Σ_k of Σ :

$$\begin{aligned} Du = \;& c_0^\perp(x) \cdot \mu \;+\; \nabla u_2(x) \cdot \mathcal{L}^2 \\ & + \sum_{k=1}^{n} (u_{\Sigma_k}^+(x) - u_{\Sigma_k}^-(x))\; n_k(x) \cdot \mathcal{H}^1 \llcorner \Sigma_k , \end{aligned}$$

where $u_{\Sigma_k}^-$ and $u_{\Sigma_k}^+$ are the left and right interior BV-traces of u on Σ_k.

The bound on the total variation is added up by

$$|Du|(\Omega) \le M_{\Omega\backslash\Sigma} + 2 \cdot \|u\|_{L^\infty(\Omega)} \cdot \mathcal{H}^1(\Sigma) .$$

Before going into the details of the proof, we summarize some facts concerning the change of variables. If $\varphi \in C_c^1(\Omega)$ or $\varphi \in C_c^1(\Omega\backslash\Sigma)$ is a test function, we will denote by $\psi(t,s) := \varphi \circ \xi(t,s)$ the test function in characteristic variables. By the chain rule we then obtain the derivative with respect to (t,s)

$$\nabla_{t,s}\psi = D\xi^T \cdot \nabla_x \varphi \circ \xi .$$

By inversion we get

$$\det D\xi \cdot \begin{pmatrix} \partial_1\varphi \\ \partial_2\varphi \end{pmatrix} \circ \xi = \begin{pmatrix} \partial_s\xi_2 & -\partial_t\xi_2 \\ -\partial_s\xi_1 & \partial_t\xi_1 \end{pmatrix} \cdot \begin{pmatrix} \partial_t\psi \\ \partial_s\psi \end{pmatrix} ,$$

$$\det D\xi \cdot \nabla_x\varphi \circ \xi = \left(-\partial_s\xi^\perp | \partial_t\xi^\perp\right) \cdot \nabla_{t,s}\psi \,.$$

Let $l = l(k)$ be the non-trivial permutation of $\{1,2\}$, then we write for the k-th component

$$\begin{aligned} \det D\xi \cdot \partial_k\varphi \circ \xi &= (-1)^l \cdot (\partial_s\xi_l \, \partial_t\psi - \partial_t\xi_l \, \partial_s\psi) \\ &= (-1)^l \cdot (\, \partial_t(\partial_s\xi_l \cdot \psi) - \partial_s(\partial_t\xi_l \cdot \psi) \,) \,. \end{aligned} \tag{3.18}$$

The last equality can easily be derived from the product rule.

Finally, we remark that $\psi(t,s)$ is periodic w.r.t. the variable s, since ξ is periodic w.r.t. s, i.e.,

$$\psi(t,a) = \psi(t,b) \,.$$

Proof. (of lemma 3.13)

First, we compute the derivative measure Du, which in both parts is the same process. Let $\varphi \in C^1_c(\Omega)$ be a test function. For the moment we restrict the discussion to subsets Ω_λ of Ω which are lower level-sets of T_0, that is

$$\Omega_\lambda := \Omega \cap \{x \in \Omega : T_0(x) \leq \lambda\} \,,$$

for $0 < \lambda < 1$. Note that $\Omega_1 = \Omega$.

When later on we have $\varphi \in C^1_c(\Omega \backslash \Sigma)$ (respectively $\varphi \in C^1_c(\Omega\backslash\Sigma)^2$), as is the case for part a), we will choose λ big enough such that $\operatorname{supp}\varphi \subset \Omega_\lambda$. For part b) we will pass to the limit $\lambda \to 1$ instead.

We separately compute the derivatives of u_1 and u_2 from the decomposition (3.17). In order to get $D_k u_1$ we have – according to definition 2.10 – to look at the following integral:

$$\begin{aligned} \int\limits_{\Omega_\lambda} u_1(x)\, \partial_k\varphi(x)\, dx &= \int\limits_a^b \int\limits_0^\lambda v_1(t,s)\, \partial_k\varphi \circ \xi(t,s)\ \det D\xi(t,s)\, dt\, ds \\ &= (-1)^l \int\limits_a^b \int\limits_0^\lambda \gamma^* u_0(s)\, (\, \partial_t(\partial_s\xi_l \cdot \psi) - \partial_s(\partial_t\xi_l \cdot \psi)\,)\, dt\, ds \,. \end{aligned}$$

By changing the order of integration and using the integration-by-parts formula for functions of one variable, one obtains

$$= (-1)^l \int_a^b \gamma^* u_0(s) \int_0^\lambda \partial_t(\partial_s \xi_l \cdot \psi)\, dt\, ds - (-1)^l \int_0^\lambda \int_a^b \gamma^* u_0(s)\; \partial_s(\partial_t \xi_l \cdot \psi)\, ds\, dt$$

$$= (-1)^l \int_a^b \gamma^* u_0(s)\;\; [\partial_s \xi_l \cdot \psi]_{t=0}^{\lambda}\, ds \;+\; (-1)^l \int_0^\lambda \int_a^b \partial_t \xi_l \cdot \psi\, dD\gamma^* u_0(s)\, dt\, .$$

In the last step we used the fact that $\gamma^* u_0$ is a periodic BV-function. Because φ has compact support in Ω, we have furthermore

$$\psi(0,s) = \varphi \circ \xi(0,s) = \varphi(\gamma(s)) = 0\, ,$$

so the result reduces to

$$= (-1)^l \int_a^b v_1(\lambda,s)\; \partial_s \xi_l(\lambda,s) \cdot \psi(\lambda,s)\, ds + (-1)^l \int_a^b \int_0^\lambda \partial_t \xi_l \cdot \psi\, dt\, dD\gamma^* u_0(s)\, .$$

For the vector-valued version we test with $\varphi \in C_c^1(\Omega)^2$ and obtain

$$\int_{\Omega_\lambda} u_1(x)\;\operatorname{div}\varphi(x)\; dx = \int_{\Omega_\lambda} u_1(x)\; \partial_1\varphi_1(x)\; dx + \int_{\Omega_\lambda} u_1(x)\; \partial_2\varphi_2(x)\; dx$$

$$\begin{aligned}
&= -\int_a^b -\partial_s \xi_2(\lambda,s) \cdot \psi_1(\lambda,s)\; v_1(\lambda,s)\, ds - \int_a^b \int_0^\lambda -\partial_t \xi_2 \cdot \psi_1\, dt\, dD\gamma^* u_0(s) \\
&\qquad - \int_a^b \partial_s \xi_1(\lambda,s) \cdot \psi_2(\lambda,s)\; v_1(\lambda,s)\, ds - \int_a^b \int_0^\lambda \partial_t \xi_1 \cdot \psi_2\, dt\, dD\gamma^* u_0(s) \\
&= -\int_a^b \left\langle \psi(\lambda,s), \partial_s \xi^\perp(\lambda,s) \right\rangle\; v_1(\lambda,s)\, ds - \int_a^b \int_0^\lambda \left\langle \psi, \partial_t \xi^\perp \right\rangle\, dt\, dD\gamma^* u_0(s)\, .
\end{aligned}$$

By once more using the relations

$$\partial_t \xi^\perp = c_0^\perp \circ \xi \quad , \quad \partial_s \xi^\perp = -N \circ \xi \cdot |\partial_s \xi|\, ,$$

the last result can be written as

$$\begin{aligned}
\int_{\Omega_\lambda} u_1(x)\;\operatorname{div}\varphi(x)\, dx &= -\int_a^b \langle \varphi, -N\rangle \circ \xi(\lambda,s)\; u_1 \circ \xi(\lambda,s)\; |\partial_s \xi(\lambda,s)|\, ds \\
&\qquad - \int_a^b \int_0^\lambda \left\langle \varphi, c_0^\perp \right\rangle \circ \xi\, dt\, dD\gamma^* u_0(s)\, .
\end{aligned}$$

Herein, the first summand integrates w.r.t. the $\mathcal{H}^1$ measure along the λ-level of T_0. For the restriction of measures onto λ-levels of T_0 we will use the abbreviation

$$\mathcal{H}^1 \llcorner \lambda := \mathcal{H}^1 \llcorner \{x \in \Omega : T_0(x) = \lambda\} .$$

In the second integral we rechange variables by pushing-forward the product measure $\mathcal{L}^1 \otimes D\gamma^* u_0$ with the diffeomorphism ξ (see definition 2.6). Let μ denote the pushed-forward measure

$$\mu := \xi_\sharp(\mathcal{L}^1 \otimes D\gamma^* u_0) ,$$

then we finally obtain

$$\begin{aligned}\int\limits_{\Omega_\lambda} u_1(x) \ \operatorname{div} \varphi(x)\, dx = &- \int\limits_{\Omega} \langle \varphi(x), -N(x) \rangle \ u_1(x)\, d\mathcal{H}^1 \llcorner \lambda(x) \\ &- \int\limits_{\Omega_\lambda} \left\langle \varphi(x), c_0^\perp(x) \right\rangle \, d\mu(x) .\end{aligned}$$

For the derivative of u_2 we perform the same steps as above with the integral

$$\int\limits_{\Omega_\lambda} u_2(x)\, \partial_k \varphi(x)\, dx = (-1)^l \int\limits_a^b \int\limits_0^\lambda v_2(t,s)\ (\ \partial_t(\partial_s \xi_l \cdot \psi) - \partial_s(\partial_t \xi_l \cdot \psi)\ )\ dt\, ds .$$

After changing the order of integration we go on with integration by parts:

$$\begin{aligned}&= (-1)^l \int\limits_a^b \int\limits_0^\lambda v_2(t,s)\, \partial_t(\partial_s \xi_l \cdot \psi)\, dt\, ds - (-1)^l \int\limits_0^\lambda \int\limits_a^b v_2(t,s)\, \partial_s(\partial_t \xi_l \cdot \psi)\, ds\, dt \\ &= (-1)^l \int\limits_a^b \left(v_2(\lambda,s)\, \partial_s \xi_l(\lambda,s) \cdot \psi(\lambda,s) - \int\limits_0^\lambda \partial_t v_2\, \partial_s \xi_l \cdot \psi\, dt \right) ds \\ &\quad - \ (-1)^l \int\limits_0^\lambda \int\limits_a^b -\partial_s v_2\, \partial_t \xi_l \cdot \psi\, ds\, dt \\ &= (-1)^l \int\limits_a^b v_2(\lambda,s)\, \partial_s \xi_l(\lambda,s) \cdot \psi(\lambda,s)\, ds \\ &\quad - \int\limits_a^b \int\limits_0^\lambda (-1)^l (\partial_s \xi_l\, \partial_t v_2 \ - \partial_t \xi_l\, \partial_s v_2\ ) \cdot \psi\, dt\, ds .\end{aligned}$$

In the second equality we have used that

$$[v_2\, \partial_t \xi_l \cdot \psi]_{s=a}^{b} = [u_2 \circ \xi\, \partial_t \xi_l \cdot \varphi \circ \xi]_{s=a}^{b} = 0\,,$$

which is a consequence of $\xi(t,a) = \xi(t,b)$. Because of $v_2 = u_2 \circ \xi$ we can – according to equation (3.18) – substitute the last integrand to get

$$= (-1)^l \int\limits_a^b v_2(\lambda,s)\, \partial_s \xi_l(\lambda,s) \cdot \psi(\lambda,s)\, ds \;-\; \int\limits_a^b \int\limits_0^\lambda \det D\xi \cdot \partial_k u_2 \circ \xi \cdot \psi\, dt\, ds\,.$$

By means of the last result and a rechange of variables, we end up with

$$\begin{aligned}\int\limits_{\Omega_\lambda} u_2(x)\ \operatorname{div}\varphi(x)\, dx = &- \int\limits_{\Omega} \langle \varphi(x), -N(x)\rangle\ u_2(x)\, d\mathcal{H}^1 \llcorner \lambda(x) \\ &- \int\limits_{\Omega_\lambda} \langle \varphi(x), \nabla u_2(x)\rangle\ dx\,,\end{aligned}$$

when testing with $\varphi \in C_c^1(\Omega)^2$.

Adding the partial results for u_1 and u_2 gives us

$$\begin{aligned}\int\limits_{\Omega_\lambda} u(x)\ \operatorname{div}\varphi(x)\, dx = &- \int\limits_{\Omega} \langle \varphi(x), -N(x)\rangle\ u(x)\, d\mathcal{H}^1 \llcorner \lambda(x) \\ &- \int\limits_{\Omega_\lambda} \left\langle \varphi(x), c_0^\perp(x)\right\rangle\, d\mu(x) \;-\; \int\limits_{\Omega_\lambda} \langle \varphi(x), \nabla u_2(x)\rangle\ dx\,.\end{aligned} \tag{3.19}$$

Now, we are ready to turn to the proof of the assertions a) and b).

a) In this part we have $\Omega\backslash\Sigma$ as the domain of u. If we test with $\varphi \in C_c^1(\Omega\backslash\Sigma)^2$, we can choose $\lambda < 1$ so big that equation (3.19) reduces to

$$\int\limits_{\Omega} u(x)\ \operatorname{div}\varphi(x)\, dx = - \int\limits_{\Omega} \left\langle \varphi(x)\,,\ c_0^\perp(x)\, d\mu(x) + \nabla u_2(x)\, dx \right\rangle\,.$$

Thus, in this case the derivative measure is given by

$$Du = c_0^\perp(x) \cdot \mu + \nabla u_2(x) \cdot \mathcal{L}^2\,.$$

What remains to show is the boundedness of $\|u\|_{BV(\Omega\backslash\Sigma)}$.

For the total variation $|Du|(\Omega\backslash\Sigma)$ we estimate both summands separately, beginning with

$$\left|\int_\Omega \left\langle \varphi(x), c_0^\perp(x) \right\rangle \, d\mu(x)\right| = \left|\int_a^b \int_0^1 \left\langle \varphi\circ\xi, c_0^\perp\circ\xi \right\rangle \, dt \, dD\gamma^* u_0(s)\right|$$

$$\le \int_a^b \int_0^1 |c_0 \circ \xi| \, dt \, d|D\gamma^* u_0|(s) \cdot \|\varphi\|_\infty$$

$$\le \frac{1}{\beta\cdot m_0} \cdot \int_a^b d|D\gamma^* u_0|(s)\cdot\|\varphi\|_\infty = \frac{|Du_0|}{\beta\cdot m_0}\cdot\|\varphi\|_\infty \,.$$

Clearly, the total variation of this summand is bounded by

$$\left|c_0^\perp(x)\cdot\mu\right|(\Omega\backslash\Sigma) \le \frac{|Du_0|}{\beta\cdot m_0}\,,$$

which is the total variation of the boundary data times the bound on the arc-lengths of characteristics (see lemma 3.8 b)).

The total variation of the second summand is exactly the L^1-norm of ∇u_2:

$$\int_\Omega |\nabla u_2(x)| \, dx = \int_a^b \int_0^1 |\nabla u_2 \circ \xi \cdot \det D\xi| \, dt \, ds$$

$$= \int_a^b \int_0^1 \left| -\partial_s \xi^\perp \, \partial_t v_2 + \partial_t \xi^\perp \, \partial_s v_2 \right| \, dt \, ds$$

$$\le \int_a^b \int_0^1 |\partial_s\xi| \, |\partial_t v_2| \, dt \, ds + \int_a^b \int_0^1 |\partial_t\xi| \, |\partial_s v_2| \, dt \, ds$$

This step is completed if the last two integrals are bounded. The partial derivative of v_2 w.r.t. t is

$$\partial_t v_2 = f_0\circ\xi = \frac{f}{\langle c, \nabla T_0\rangle}\circ\xi = f\circ\xi\cdot|\partial_t\xi|\,.$$

Additionally, by using relation (3.11) from lemma 3.11 a), we obtain the bound

$$\int_a^b \int_0^1 |\partial_s\xi| \, |\partial_t v_2| \, dt \, ds = \int_a^b \int_0^1 |\partial_s\xi| \, |\partial_t\xi| \, |f\circ\xi| \, dt \, ds$$

$$\le \frac{1}{\beta}\int_a^b \int_0^1 \det D\xi \, |f\circ\xi| \, dt \, ds = \frac{1}{\beta}\int_\Omega |f(x)| \, dx \le \frac{1}{\beta}\cdot\|f\|_\infty\cdot\mathcal{L}^2(\Omega)$$

on the first integral.

The second part is a bit lengthier. The partial derivative of v_2 w.r.t. s is

$$\partial_s v_2 = \int_0^t \langle \nabla f_0 \circ \xi(\tau, s), \partial_s \xi(\tau, s) \rangle \; d\tau \; .$$

With this, we estimate the integrand by

$$|\partial_t \xi| \; |\partial_s v_2| \leq \frac{1}{\beta \cdot m_0} \cdot \int_0^1 |\langle \nabla f_0 \circ \xi(\tau, s), \partial_s \xi(\tau, s) \rangle| \; d\tau \; .$$

Because the latter is independent of t, we obtain

$$\int_a^b \int_0^1 |\partial_t \xi| \; |\partial_s v_2| \; dt \; ds \leq \frac{1}{\beta \cdot m_0} \cdot \int_a^b \int_0^1 |\langle \nabla f_0 \circ \xi(\tau, s), \partial_s \xi(\tau, s) \rangle| \; d\tau \; ds \; .$$

Now, the same argumentation that we used in the proof of lemma 3.11 c) to show the integrability of $Dc_0 \circ \xi \cdot \partial_s \xi$ applies here for the integrability of $\langle \nabla f_0 \circ \xi, \partial_s \xi \rangle$: after expanding ∇f_0 we obtain

$$\begin{aligned} h &:= \left(N^T \cdot Dc \cdot N^\perp + N^{\perp T} \cdot DN \cdot c \right) \\ \langle \nabla f_0 \circ \xi, \partial_s \xi \rangle &= |\partial_t \xi| |\partial_s \xi| \left(- \left\langle \nabla f, N^\perp \right\rangle + \frac{f}{\langle c, N \rangle} \cdot h \right) \circ \xi \, . \end{aligned}$$

Hence, there is the integrable upper bound

$$| \langle \nabla f_0 \circ \xi, \partial_s \xi \rangle | \leq \frac{\det D\xi}{\beta} \left(\|\nabla f\|_\infty + \frac{\|f\|_\infty}{\beta \cdot m_0} (|Dc| + |DN|) \right) \circ \xi \, ,$$

which implies

$$\int_a^b \int_0^1 |\partial_t \xi| \; |\partial_s v_2| \, dt \, ds \leq \\ \frac{\|\nabla f\|_\infty}{\beta^2 m_0} \mathcal{L}^2(\Omega) + \frac{\|f\|_\infty}{\beta^3 m_0^2} \left(\|Dc\|_{L^1(\Omega)} + \|DN\|_{L^1(\Omega)} \right) \, .$$

Putting together the partial results we firstly get a bound on the L^1-norm of ∇u_2

$$\begin{aligned} \|\nabla u_2(x)\|_{L^1(\Omega)} \leq \frac{1}{\beta} \cdot \|f\|_\infty \cdot \mathcal{L}^2(\Omega) + \frac{\|\nabla f\|_\infty}{\beta^2 m_0} \mathcal{L}^2(\Omega) \\ + \frac{\|f\|_\infty}{\beta^3 m_0^2} \left(\|Dc\|_{L^1(\Omega)} + \|DN\|_{L^1(\Omega)} \right) \, , \end{aligned}$$

and secondly learn that the total variation $|Du|(\Omega\backslash\Sigma)$ is bounded by $M_{\Omega\backslash\Sigma}$.

By equation (3.16) we have for the L^∞-norm of u

$$\|u\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\partial\Omega)} + \|T_0\|_\infty \|f_0\|_\infty \leq \|u_0\|_{L^\infty(\partial\Omega)} + \frac{\|f\|_\infty}{\beta \cdot m_0} .$$

b) Now, we want to view u as a BV-function on the domain Ω. Thus, test functions stem from $C_c^1(\Omega)^2$, and thus, we have to study the limit $\lambda \longrightarrow 1$ in equation (3.19). In part a) we already have bounds on the total variation of the components concerning $c_0^\perp(x) \cdot \mu$ and $\nabla u_2(x) \cdot \mathcal{L}^2$, which do not depend on λ. Hence, these bounds stay the same, and we can concentrate on the remainder

$$\int_\Omega \langle \varphi(x), -N(x) \rangle \; u(x) \; d\mathcal{H}^1 \llcorner \lambda(x) = \int_a^b \left\langle \psi(\lambda, s), \partial_s \xi^\perp(\lambda, s) \right\rangle \; v(\lambda, s) \; ds .$$

In lemma 3.11 b) we introduced a partition of the interval I,

$$I = \bigcup_{k=1}^n (J_{k,+} \cup J_{k,-}) \;\cup\; J ,$$

such that $\xi(1,.)|_{J_{k,+}}$ and $\xi(1,.)|_{J_{k,-}}$ are both regular parametrizations of $\mathring{\Sigma}_k$ (see figure 3.1), one for the plus-side and one for the minus-side of $\mathring{\Sigma}_k$. For $s \in J$, the characteristic $\xi(\lambda, s)$ hits a singular node of Σ, i.e., terminal-, branching- or kink node, as λ tends to 1.
Let, as before, $z_1, \ldots, z_m$ denote the singular nodes of Σ, then we partition J into a collection $J_{z_1}, \ldots, J_{z_m}$ by:

$$s \in J_{z_l} \quad :\Leftrightarrow \quad \lim_{\lambda \to 1} \xi(\lambda, s) = z_l , \qquad J = \bigcup_{l=1}^m J_{z_l} .$$

By these partitions, we decompose the integral

$$\int_a^b \left\langle \psi(\lambda, s), \partial_s \xi^\perp(\lambda, s) \right\rangle \; v(\lambda, s) \; ds$$

$$= \sum_{l=1}^m \int_{J_{z_l}} \ldots \; ds \; + \sum_{k=1}^n \left(\int_{J_{k,+}} \ldots \; ds + \int_{J_{k,-}} \ldots \; ds \right) .$$

For J_{z_l}-summands, we have the estimate

$$\left| \int_{J_{z_l}} \left\langle \psi(\lambda, s), \partial_s \xi^\perp(\lambda, s) \right\rangle \; v(\lambda, s) \; ds \right| = \left| \int_{\xi(\lambda, J_{z_l})} (\langle \varphi, -N \rangle \cdot u)(x) \; d\mathcal{H}^1(x) \right|$$
$$\leq \|\varphi\|_\infty \cdot \|u\|_{L^\infty(\Omega)} \cdot \mathcal{H}^1(\; \xi(\lambda, J_{z_l}) \;) .$$

Because the curve-arc $\xi(\lambda, J_{z_l})$ degenerates to the single point z_l (see figure 3.1), the right hand side becomes zero in the limit

$$\lim_{\lambda\to 1} \mathcal{H}^1(\ \xi(\lambda, J_{z_l})\ ) = 0\,,$$

and the contribution of those summands vanishes.

For the remaining summands we perform only those which go along $\xi(\lambda, J_{k,+})$, since the same argumentation applies to the others which go along $\xi(\lambda, J_{k,-})$. By lemma 3.11 b) we know that $\xi(1,.) : J_{k,+} \to \mathring{\Sigma}_k$ is a regular parametrization and its tangent is given by the limit

$$\partial_s\xi(1,s) = \lim_{\lambda\to 1} \partial_s\xi(\lambda,s) \qquad s \in J_{k,+}\,.$$

By requirement 3.5 part 2 the field of normals extends to $\mathring{\Sigma}_k$ that means

$$\lim_{\lambda\to 1} -N \circ \xi(\lambda,s) = n_k \circ \xi(1,s) \qquad s \in J_{k,+}\,.$$

Finally, we define the extension u onto $\mathring{\Sigma}_k$ by using the extension of $\xi^{-1}(x)$ – more precisely its second component $s(x)$ – from lemma 3.11 d). Let $z \in \mathring{\Sigma}_k$, with corresponding $s^+(z) \in J_{k,+}$, then we set

$$u_k^+(z) := \lim_{\lambda\to 1} v(\lambda, s^+(z)) = v(1, s^+(z))\,. \tag{3.20}$$

Conversely, the following relation holds true

$$u_k^+ \circ \xi(1,s) = \lim_{\lambda\to 1} u \circ \xi(\lambda,s) = v(1,s)\,, \qquad s \in J_{k,+}\,.$$

Now, we can turn to the limit. For abbreviation let

$$h(\lambda,s) := (\langle\varphi, -N\rangle \cdot u) \circ \xi(\lambda,s)\,,$$

then

$$\begin{aligned}
&\left| \int\limits_{\xi(\lambda,J_{k,+})} (\langle\varphi,-N\rangle\cdot u)(x)\; d\mathcal{H}^1(x) - \int\limits_{\Omega} (\langle\varphi,n_k\rangle\cdot u_k^+)(x)\; d\mathcal{H}^1 \llcorner \Sigma_k(x) \right| \\
&= \left| \int\limits_{J_{k,+}} h(\lambda,s)\cdot|\partial_s\xi(\lambda,s)|\; ds - \int\limits_{J_{k,+}} h(1,s)\cdot|\partial_s\xi(1,s)|\; ds \right| \\
&\le \int\limits_{J_{k,+}} |h(1,s)-h(\lambda,s)|\cdot|\partial_s\xi(1,s)| + \big||\partial_s\xi(1,s)| - |\partial_s\xi(\lambda,s)|\big|\cdot|h(\lambda,s)|\; ds \\
&\le \int\limits_{J_{k,+}} |h(1,s)-h(\lambda,s)|\cdot|\partial_s\xi(1,s)|\; ds \\
&\qquad + \quad \|h\circ\xi^{-1}\|_{L^\infty(\Omega)} \int\limits_{J_{k,+}} \big||\partial_s\xi(1,s)| - |\partial_s\xi(\lambda,s)|\big|\; ds\,.
\end{aligned}$$

By the extensions of N and u the product $|h(1,s) - h(\lambda,s)| \cdot |\partial_s \xi(1,s)|$ tends to zero for every $s \in J_{k,+}$. Furthermore, it has the integrable bound

$$|h(1,s) - h(\lambda,s)| \cdot |\partial_s \xi(1,s)| \le 2\|h \circ \xi^{-1}\|_{L^\infty(\Omega)} \cdot |\partial_s \xi(1,s)| \,,$$

thus, by dominated convergence, the corresponding integral vanishes in the limit. By same argumentation the second integral tends to zero, too.

Summarizing, this means

$$\int_{\xi(\lambda,J_{k,+})} (\langle \varphi, -N\rangle \cdot u)(x)\, d\mathcal{H}^1(x) \to \int_\Omega (\langle \varphi, n_k\rangle \cdot u_k^+)(x)\, d\mathcal{H}^1 \llcorner \Sigma_k(x)\,,$$
$$\int_{\xi(\lambda,J_{k,-})} (\langle \varphi, -N\rangle \cdot u)(x)\, d\mathcal{H}^1(x) \to \int_\Omega (\langle \varphi, -n_k\rangle \cdot u_k^-)(x)\, d\mathcal{H}^1 \llcorner \Sigma_k(x)\,,$$

as $\lambda \to 1$, and together we obtain the jump part for Σ_k

$$\int_{\xi(\lambda,J_{k,+})\cup\xi(\lambda,J_{k,-})} (\langle \varphi, -N\rangle \cdot u)(x)\, d\mathcal{H}^1(x) \;\to$$
$$\int_\Omega (u_k^+(x) - u_k^-(x))\; \langle \varphi, n_k\rangle\,(x)\, d\mathcal{H}^1 \llcorner \Sigma_k(x)\,.$$

The bound on the total variation of the jump part is

$$\left| \int_\Omega (u_k^+(x) - u_k^-(x))\; \langle \varphi, n_k\rangle\,(x)\, d\mathcal{H}^1 \llcorner \Sigma_k(x) \right| \le 2 \cdot \|u\|_{L^\infty(\Omega)} \cdot \mathcal{H}^1(\Sigma_k)\,,$$

whereas $\|\varphi\|_\infty \le 1$. What remains to show, is that the one-sided limits u_k^+, u_k^- defined above are – according to theorem 2.25 – in fact the BV-traces $u_{\Sigma_k}^+$ and $u_{\Sigma_k}^-$ of u on Σ_k. This is true, but we postpone this point to lemma 3.15.

Clearly, the complete additional jump part is given by

$$\sum_{k=1}^n \int_\Omega (u_k^+(x) - u_k^-(x))\; \langle \varphi, n_k\rangle\,(x)\, d\mathcal{H}^1 \llcorner \Sigma_k(x)$$

with the bound

$$\left| \sum_{k=1}^n \int_\Omega (u_k^+(x) - u_k^-(x))\; \langle \varphi, n_k\rangle\,(x)\, d\mathcal{H}^1 \llcorner \Sigma_k(x) \right| \le 2\|u\|_{L^\infty(\Omega)} \cdot \mathcal{H}^1(\Sigma)\,,$$

$\|\varphi\|_\infty \leq 1$, on its total variation. We have shown, that u is in fact an element of $BV(\Omega)$ with $\|u\|_{BV(\Omega)}$ bounded by the given data, and its derivative measure reads

$$Du = \sum_{k=1}^{n} (u_k^+(x) - u_k^-(x))\, n_k(x) \cdot \mathcal{H}^1 \llcorner \Sigma_k \;+\; c_0^\perp(x) \cdot \mu \;+\; \nabla u_2(x) \cdot \mathcal{L}^2 \,.$$

□

Lemma 3.14. *(Solution of the PDE)*
The candidate solution u from (3.16) solves the PDE (3.2) of problem 3.7.

Proof.
By viewing u as an element of $BV(\Omega \backslash \Sigma)$ we have for its derivative

$$Du = c_0^\perp(x) \cdot \mu \;+\; \nabla u_2(x) \cdot \mathcal{L}^2 \,,$$

according to lemma 3.13. Furthermore, the equation

$$\langle c_0(x), Du \rangle = \langle c_0(x), \nabla u_2(x) \rangle \cdot \mathcal{L}^2$$

follows from orthogonality, and the PDE is satisfied if $\langle c_0(x), \nabla u_2(x) \rangle \, \mathcal{L}^2$ equals $f_0(x)\mathcal{L}^2$.
Let us test the measure $\langle c_0(x), \nabla u_2(x) \rangle \cdot \mathcal{L}^2$ with $\varphi \in C_0(\Omega)$, $\|\varphi\|_\infty \leq 1$. By changing variables we obtain – with $\psi = \varphi \circ \xi$ and $v_2 = u_2 \circ \xi$ –

$$\begin{aligned}
\int_\Omega \varphi(x) \, \langle c_0(x), \nabla u_2(x) \rangle \; dx &= \int_a^b \int_0^1 \psi \cdot \langle c_0 \circ \xi, \nabla u_2 \circ \xi \cdot \det D\xi \rangle \; dt \, ds \\
&= \int_a^b \int_0^1 \psi \cdot \left\langle \partial_t \xi, \left(-\partial_s \xi^\perp \partial_t v_2 + \partial_t \xi^\perp \partial_s v_2 \right) \right\rangle \; dt \, ds \\
&= \int_a^b \int_0^1 \psi \cdot \partial_t v_2 \cdot \left\langle \partial_t \xi, -\partial_s \xi^\perp \right\rangle \; dt \, ds \\
&= \int_a^b \int_0^1 \varphi \circ \xi \cdot f_0 \circ \xi \cdot \det D\xi \; dt \, ds = \int_\Omega \varphi(x) f_0(x) \; dx \,.
\end{aligned}$$

Here, we used the fact that $\partial_t v_2 = f_0 \circ \xi$ (see equation (3.17)).
This shows that the total variation of the difference measure is

$$| \, \langle c_0(x), \nabla u_2(x) \rangle \cdot \mathcal{L}^2 - f_0(x) \cdot \mathcal{L}^2 | (\Omega) = 0 \,,$$

and hence the measures are equal. □

In the next lemma we study the traces of the candidate u along level-lines of T_0.

Lemma 3.15. *(Start / restart / stop)*
Let u be the candidate solution from (3.16), then

a) *(start): u satisfies the boundary condition (3.3), i.e.,*

$$\lim_{r\to 0+} \frac{1}{r^2} \int\limits_{\Omega\cap B_r(z)} |u_0(z) - u(x)|\, dx = 0$$

for every Lebesgue point $z \in \partial\Omega$ of u_0.

b) *(restart): for every $z \in \Omega\backslash\Sigma$ that corresponds to a Lebesgue point z' of u_0, that means $z' = \gamma(s(z))$ is a Lebesgue point of u_0, we have*

$$\lim_{r\to 0+} \frac{1}{r^2} \int\limits_{B_r^>(z)} |u(z) - u(x)|\, dx = 0$$

and

$$\lim_{r\to 0+} \frac{1}{r^2} \int\limits_{B_r^<(z)} |u(z) - u(x)|\, dx = 0\,.$$

Here, $B_r^<(z)$ and $B_r^>(z)$ - for r small enough - denote the cut-off disks

$$B_r^<(z) := \{x \in B_r(z) : T_0(x) < T_0(z)\}\,,$$
$$B_r^>(z) := \{x \in B_r(z) : T_0(x) > T_0(z)\}\,.$$

Let $\Gamma := \chi_{T_0=\lambda}$ a λ-level of T_0 for some $0 < \lambda < 1$ and let Γ be oriented by $N|_\Gamma$. If z is restricted to $z \in \Gamma$, the result above means that the traces u_Γ^+ and u_Γ^- are identical. There is no jump across Γ, and if z corresponds to a Lebesgue point of u_0, then it itself is a Lebesgue point of u.

Owing to the identity of both traces the restriction $u|_\Gamma$ is well-defined. Moreover, we have $u|_\Gamma \in BV(\Gamma)$.

c) *(stop): for every $z \in \mathring{\Sigma}_k$ that corresponds to a Lebesgue point z' of u_0 w.r.t. the plus-side of $\mathring{\Sigma}_k$, that means $z' = \gamma(s^+(z))$ is a Lebesgue point of u_0, we have*

$$\lim_{r\to 0+} \frac{1}{r^2} \int\limits_{B_r^+(z)} |u_k^+(z) - u(x)|\, dx = 0\,.$$

Here, $u_k^+(z)$ is defined by equation (3.20) and $B_r^+(z)$ - for r small enough - is the cut-off disk centered at z, restricted to the plus-side.

Hence, the trace $u_{\Sigma_k}^+$ is given by u_k^+. Moreover, we have $u_k^+ \in BV(\mathring{\Sigma}_k)$.

The analogous result holds true w.r.t. the minus-side.

Proof.

a) and b)
Let $z \in \overline{\Omega}\backslash\Sigma$ and let (τ,σ) be its characteristic coordinates, i.e., $z = \xi(\tau,\sigma)$. Choose $r > 0$ sufficiently small, then

$$\frac{1}{r^2}\int\limits_{B_r^>(z)} |u(z) - u(x)|\, dx = \frac{1}{r^2}\int\limits_{s_-(r)}^{s_+(r)}\int\limits_{\tau}^{t_+(r,s)} |v(\tau,\sigma) - v(t,s)| \det D\xi \; dt\; ds\;.$$

Because any level-line of T_0 is a regular C^1-curve, the cut-off disk $B_r^>(z)$ tends to a half disk, oriented along the tangent $\partial_s\xi(\tau,\sigma)$ of this curve, and thus, the functions $s_-(r)$ and $s_+(r)$ are of the form

$$s_-(r) = \sigma - \mathcal{O}(r) \qquad\qquad s_+(r) = \sigma + \mathcal{O}(r)\;.$$

Since $\det D\xi(t,s) \neq 0$, the same argument applies w.r.t. the t-variable, i.e., the second basis vector is $\partial_t\xi(\tau,\sigma)$ and function $t_+(r,s)$ is of the form

$$t_+(r,s) = \tau + \mathcal{O}(r)\;.$$

By an Euler step applied to the variational equation for $\partial_s\xi$ we obtain

$$\partial_s\xi(t,s) = \partial_s\xi(\tau,s) + \mathcal{O}(t-\tau)$$

and therefore

$$\det D\xi(t,s) \leq |\partial_t\xi|\cdot|\partial_s\xi| = \frac{|\partial_s\xi(\tau,s)|}{\beta\cdot m_0} + \mathcal{O}(t-\tau) \leq C + \mathcal{O}(t-\tau)\;,$$

for a suitable constant C. The difference is estimated by

$$|v(t,s) - v(\tau,\sigma)| \leq |\gamma^* u_0(s) - \gamma^* u_0(\sigma)| + |v_2(t,s) - v_2(\tau,\sigma)|\;,$$

with

$$\begin{aligned} |v_2(t,s) - v_2(\tau,\sigma)| &= \left|\int\limits_0^t f_0\circ\xi(h,s)\; dh - \int\limits_0^\tau f_0\circ\xi(h,\sigma)\; dh\right| \\ &\leq \int\limits_0^t |\nabla f_0\circ\xi(h,s_*)|\,|\partial_s\xi(h,s_*)|\,|s-\sigma|\; dh + \left|\int\limits_\tau^t f_0\circ\xi(h,\sigma)\; dh\right|\;. \end{aligned}$$

Because $z \notin \Sigma$, we have for r that the cut-off disk $B_r^>(z)$ is contained in some lower level-set Ω_λ with $\lambda < 1$. Hence, $t \leq \lambda$ for the last two integrals, and following bound exists:

$$M := \|\nabla f_0\|_{L^\infty(\Omega_\lambda)} \cdot \sup_{(h,s_*)\in[0,\lambda]\times[s_-(r),s_+(r)]} |\partial_s\xi(h,s_*)|\;.$$

In summary,

$$\begin{aligned}|v(t,s)-v(\tau,\sigma)| &\le |\gamma^* u_0(s)-\gamma^* u_0(\sigma)| + t\cdot M|s-\sigma| + \|f_0\|_\infty |t-\tau| \\ &\le |\gamma^* u_0(s)-\gamma^* u_0(\sigma)| + \mathcal{O}(s-\sigma)+\mathcal{O}(t-\tau)\ .\end{aligned}$$

Armed with the last result and the estimate of the determinant, we have

$$\begin{aligned}&\frac{1}{r^2}\int\limits_{B_r^>(z)} |u(z)-u(x)|\,dx \\ &\le \frac{1}{r^2}\int\limits_{s_-(r)}^{s_+(r)}\int\limits_{\tau}^{t_+(r)} |\gamma^* u_0(s)-\gamma^* u_0(\sigma)|\cdot C + \mathcal{O}(s-\sigma)+\mathcal{O}(t-\tau)\,dt\,ds \\ &\le \frac{C'}{r}\int\limits_{s_-(r)}^{s_+(r)} |\gamma^* u_0(s)-\gamma^* u_0(\sigma)|\,ds + \mathcal{O}(r)\ .\end{aligned}$$

The latter expression tends, as $r \to 0$, to zero for any Lebesgue-point σ of $\gamma^* u_0$, i.e., for any Lebesgue-point $z' = \gamma(\sigma)$ of u_0.

If $z \in \partial\Omega$, then $\Omega \cap B_r(z) = B_r^>(z)$ and the argumentation above shows that u satisfies the boundary condition as BV-trace.

If $z \in \Omega\backslash\Sigma$, we have got the assertion for the $B_r^>(z)$-case. In the $B_r^<(z)$-case the one and only difference is that, after having changed variables, we have to integrate the t-variable over the interval $[t_-(r,s),\tau]$. For the remainder one must perform the same steps to get the assertion for the $B_r^<(z)$.
Finally, the restriction $u|_\Gamma$ is defined, and when parametrizing Γ regularly by $\xi_\lambda(s) := \xi(\lambda,s)$, we have

$$\xi_\lambda^* u(s) = u\circ\xi(\lambda,s) = v(\lambda,s) = \gamma^* u_0(s) + v_2(\lambda,s) \qquad s \in \mathbb{R}\ .$$

Since $v_2(\lambda,s)$ is a periodic C^1-function, while $\gamma^* u_0(s)$ is a periodic BV-function, the sum $\xi_\lambda^* u$ is a periodic BV-function, and consequently $u|_\Gamma \in BV(\Gamma)$.

c) In order to argue in the same way as in part a) and b) we cannot use diffeomorphism ξ, because $c_0|_\Sigma = 0$ and thus $\det D\xi(1,s) = 0$. But at least $\xi_1(s) := \xi(1,s)$ with $s \in J_{k,+}$ is a regular parametrization of $\mathring{\Sigma}_k$ (see lemma 3.11 b)), so we set up a local diffeomorphism by considering the unscaled backward IVP

$$y' = -c(y)\ , \qquad y(0,s) = \xi(1,s)\ , \qquad s \in J_{k,+}$$

with c extended onto $\mathring{\Sigma}_k$ by c^+.

Let $\tau(s) = \tau(1,s)$, as in the previous section, denote the time when $y(\,.\,,s)$ reaches the boundary $\partial\Omega$. In the proof of lemma 3.11 b) we have seen that $\tau : J_{k,+} \to \mathbb{R}$ is continuously differentiable. Since we parametrize Σ_k by $\xi_1(s)$, and consider the backward characteristics with changed velocity, we get the following correspondence

$$y(\tau(s),s) = \gamma(s)\ .$$

Our candidate solution then, for $t \neq 0$, rewrites as

$$u \circ y(t,s) = \gamma^* u_0(s) + \int_t^{\tau(s)} f \circ y(h,s)\, dh\,, \qquad s\ \in J_{k,+}\ .$$

In the case $t = 0$, we have

$$u_k^+ \circ \xi(1,s) = u_k^+ \circ y(0,s) = \lim_{t \to 0} u \circ y(t,s)\,, \qquad s\ \in J_{k,+}\,,$$

according to equation (3.20).

Let then $z \in \mathring{\Sigma}_k$ with $z = \xi_1(\sigma)$, $\sigma \in J_{k,+}$. For a small enough r, we obtain

$$\frac{1}{r^2} \int_{B_r^+(z)} |u_k^+(z) - u(x)|\, dx$$
$$= \frac{1}{r^2} \int_{s_-(r)}^{s_+(r)} \int_0^{t(r,s)} |u_k^+ \circ y(0,\sigma) - u \circ y(t,s)|\, |\det Dy|\, dt\, ds$$

by changing variables.
Now, for a small enough r, the determinant $|\det Dy(t,s)|$ is approximately $|\det Dy(0,\sigma)|$ with

$$|\det Dy(0,\sigma)| = |\partial_s \xi_1(\sigma)| |c^+(z)| \left| \langle c^+(z), n_k(z) \rangle \right| \neq 0$$

which is non-zero because of the requirement 3.6 part 2c). And, for the functions $s_+(r)$, $s_-(r)$, and $t(r,s)$ we have the same asymptotic, for $r \to 0$, as in the previous part.

Let $\tau_1 = \min(\tau(s), \tau(\sigma))$ and $\tau_2 = \max(\tau(s), \tau(\sigma))$. If $\tau_1 = \tau(s)$, we set

$s_1 = \sigma$, and otherwise we set $s_1 = s$. For the difference we estimate first

$$\begin{aligned}
&\left| \int_0^{\tau(\sigma)} f \circ y(h,\sigma)\; dh - \int_t^{\tau(s)} f \circ y(h,s)\; dh \right| \\
&\leq \left| \int_0^{\tau(\sigma)} f \circ y(h,\sigma)\; dh - \int_0^{\tau(s)} f \circ y(h,s)\; dh \right| + \left| \int_0^{t} f \circ y(h,s)\; dh \right| \\
&\leq \left| \int_{\tau_1}^{\tau_2} f \circ y(h,s_1)\, dh \right| + \left| \int_0^{\tau_1} f \circ y(h,\sigma) - f \circ y(h,s)\, dh \right| + \left| \int_0^{t} f \circ y(h,s)\, dh \right| \\
&\leq \|f\|_\infty \cdot |\tau(\sigma) - \tau(s)| + \|\nabla f\|_\infty \cdot \|\partial_s y(.,s^*)\|_\infty \cdot \tau_1 \cdot |s-\sigma| + \|f\|_\infty t \\
&\leq (\|f\|_\infty \cdot |\partial_s \tau(s^*)| + \|\nabla f\|_\infty \cdot \|\partial_s y(.,s^*)\|_\infty \cdot \tau_1) \cdot |s-\sigma| + \|f\|_\infty t\ .
\end{aligned}$$

Because $\tau(s)$ and $\tau(\sigma)$ are arc-lengths of characteristics, τ_1 is – according to lemma 3.8 – bounded by $1/(\beta \cdot m_0)$. And, as highlighted in the proof of lemma 3.11 b), when s is restricted to small neighborhood around $\sigma \in J_{k,+}$, we also have uniform bounds on $|\partial_s \tau(s^*)|$ and $\|\partial_s y(.,s^*)\|_\infty$. Taking this into account, we end up with

$$\left| \int_0^{\tau(\sigma)} f \circ y(h,\sigma)\; dh - \int_t^{\tau(s)} f \circ y(h,s)\; dh \right| = \mathcal{O}(s-\sigma) + \mathcal{O}(t)\ ,$$

and consequently

$$|u_k^+ \circ y(0,\sigma) - u \circ y(t,s)| = |\gamma^* u_0(\sigma) - \gamma^* u_0(s)| + \mathcal{O}(s-\sigma) + \mathcal{O}(t)\ .$$

As in the previous part, we get then

$$\frac{1}{r^2} \int_{B_r^+(z)} |u_k^+(z) - u(x)|\; dx \quad \leq \quad \frac{C'}{r} \int_{s_-(r)}^{s_+(r)} |\gamma^* u_0(s) - \gamma^* u_0(\sigma)|\; ds + \mathcal{O}(r)\ ,$$

which tends to zero whenever z, by $\gamma(\sigma) = \gamma(s^+(z))$, corresponds to a Lebesgue point of u_0.

By the "old" representation

$$u_k^+ \circ \xi(1,s) = \gamma^* u_0(s) + \int_0^1 f_0 \circ \xi(\tau,s)\; d\tau\ , \qquad s\ \in J_{k,+}\ ,$$

it is obvious that $u_k^+ \circ \xi(1,s)$, on the interval $J_{k,+}$, is a BV-function of one variable and – since $\xi(1,s)$ is regular – that $u_k^+ \in BV(\mathring{\Sigma}_k)$.

□

Remark: part b) of lemma 3.15 is called "restart" because having stopped the characteristics at some intermediate λ-level Γ of T_0, the restarted problem

$$\begin{aligned} \langle c(x), Dw\rangle &= f(x)\cdot \mathcal{L}^2 \quad \text{in} \quad \chi_{T_0>\lambda}\,, \\ w|_\Gamma &= u|_\Gamma\,. \end{aligned}$$

is of the same type as problem 3.7. This is, because lemma 3.15 b) ensures that $u|_\Gamma \in BV(\Gamma)$. Moreover, when applying the same method of construction, then w reproduces u by

$$w = u|_{\chi_{T_0>\lambda}}\,.$$

3.4 Uniqueness and Stability

As seen in the previous section, the linear problem 3.7 has a solution in $BV(\Omega)$. In this section we carry on with the uniqueness of the solution and its continuous dependence on the data.

Theorem 3.16. *(Uniqueness and stability)*
The solution of problem 3.7 is

a) unique and stable w.r.t. perturbations of the boundary data:

Let u and w be solutions of

$$\begin{aligned} \langle c(x), Du\rangle &= f(x)\cdot \mathcal{L}^2 & u|_{\partial\Omega} &= u_0\,, \\ \langle c(x), Dw\rangle &= f(x)\cdot \mathcal{L}^2 & w|_{\partial\Omega} &= w_0\,, \end{aligned}$$

in $\Omega\backslash\Sigma$, then

$$\|u-w\|_{L^\infty(\Omega)} = \|u_0 - w_0\|_{L^\infty(\partial\Omega)}\,.$$

b) stable w.r.t. perturbations of the right hand side of PDE (3.2):

Let u and w be the solutions of

$$\begin{aligned} \langle c(x), Du\rangle &= f(x)\cdot \mathcal{L}^2 & u|_{\partial\Omega} &= u_0\,, \\ \langle c(x), Dw\rangle &= g(x)\cdot \mathcal{L}^2 & w|_{\partial\Omega} &= u_0\,, \end{aligned}$$

in $\Omega\backslash\Sigma$, then

$$\|u-w\|_{L^\infty(\Omega)} \le \frac{\|f-g\|_\infty}{\beta\cdot m_0}\,.$$

Proof.

a) By theorem 3.12 we know that solutions u and w of

$$\langle c(x), Du\rangle = f(x)\cdot\mathcal{L}^2 \qquad u|_{\partial\Omega} = u_0\,,$$
$$\langle c(x), Dw\rangle = f(x)\cdot\mathcal{L}^2 \qquad w|_{\partial\Omega} = w_0$$

exist in $\Omega\backslash\Sigma$. By using the linearity of the PDE the difference $u-w$ clearly satisfies the homogeneous

$$\langle c(x), D(u-w)\rangle = 0 \quad \text{in } \Omega\backslash\Sigma\,, \qquad u|_{\partial\Omega} = (u_0-w_0)\,.$$

We rename the functions $u := u-w$ and $u_0 := u_0 - w_0$ and consider the scaled problem

$$\langle c_0(x), Du\rangle = 0 \quad \text{in } \Omega\backslash\Sigma\,, \qquad u|_{\partial\Omega} = u_0$$

in order to reuse the diffeomorphism ξ.

In the following passage we prove that the latter problem has a unique solution. Let $\varphi\in C_c^1(\Omega\backslash\Sigma)$. We set $\psi = \varphi\circ\xi$ and $v = u\circ\xi$. Moreover, let $\tilde\Omega := \Omega\backslash(\Sigma\cup S)$ as in lemma 3.10.

Next, we want to rewrite the PDE in characteristic variables. If we change the variables first and compute the BV-derivative of v afterwards, we obtain

$$\begin{aligned}
\int_{\tilde\Omega} u(x)\,\partial_k\varphi(x)\,dx &= \int_{\xi^{-1}(\tilde\Omega)} v(t,s)\,\partial_k\varphi\circ\xi(t,s)\,\det D\xi(t,s)\,d(t,s)\\
&= (-1)^l \int_{\xi^{-1}(\tilde\Omega)} v(t,s)\,\big(\,\partial_t(\partial_s\xi_l\cdot\psi) - \partial_s(\partial_t\xi_l\cdot\psi)\,\big)\,d(t,s)\\
&= (-1)^l \left(- \int_{\xi^{-1}(\tilde\Omega)} \partial_s\xi_l\cdot\psi\,dD_t v(t,s) + \int_{\xi^{-1}(\tilde\Omega)} \partial_t\xi_l\cdot\psi\,dD_s v(t,s)\right)\\
&= -\int_{\xi^{-1}(\tilde\Omega)} \psi\cdot(-1)^l\,\big(\partial_s\xi_l\,dD_t v(t,s) - \partial_t\xi_l\,dD_s v(t,s)\big)\,.
\end{aligned}$$

And, if we proceed the other way round, meaning we first compute the BV-derivative of u and then pull back onto characteristic variables, we obtain

$$\begin{aligned}
\int_{\tilde\Omega} u(x)\,\partial_k\varphi(x)\,dx &= -\int_{\tilde\Omega}\varphi(x)\,dD_k u(x) = -\int_{\tilde\Omega}\psi\circ\xi^{-1}(x)\,dD_k u(x)\\
&= -\int_{\xi^{-1}(\tilde\Omega)} \psi(t,s)\,d\xi^{-1}_{\sharp}D_k u(t,s)\,.
\end{aligned}$$

Since we have computed the same thing twice, we have

$$\xi_\sharp^{-1} D_k u = (-1)^l \left(\partial_s \xi_l \; D_t v(t,s) - \partial_t \xi_l \; D_s v(t,s)\right) .$$

By using matrix-vector notation, it reads as

$$\xi_\sharp^{-1} Du = (-\partial_s \xi^\perp | \partial_t \xi^\perp) D_{t,s} v = \det D\xi \cdot D\xi^{-T} D_{t,s} v ,$$

and inversion yields the chain rule

$$D_{t,s} v = \frac{1}{\det D\xi} \cdot D\xi^T \cdot \xi_\sharp^{-1} Du . \tag{3.21}$$

Remark about this chain rule: if $Du = \nabla u(x) \cdot \mathcal{L}^2$ is absolutely continuous then

$$\xi_\sharp^{-1} Du = \nabla u \circ \xi \;\cdot\; \xi_\sharp^{-1}(\mathcal{L}^2 \llcorner \tilde{\Omega}) = \nabla u \circ \xi \;\cdot\; \det D\xi \;\cdot\; \mathcal{L}^2 \llcorner \xi^{-1}(\tilde{\Omega}) .$$

So one gets back the well known chain rule by plugging the last equality into equation (3.21).

From the established chain rule (3.21) we read off

$$\det D\xi \cdot D_t v = \left\langle \partial_t \xi, \xi_\sharp^{-1} Du \right\rangle = \left\langle c_0 \circ \xi, \xi_\sharp^{-1} Du \right\rangle = \xi_\sharp^{-1}(\langle c_0, Du \rangle) = 0 ,$$

and hence the homogeneous PDE in characteristic variables conforms with

$$D_t v = 0 .$$

Since v is a BV-function on $]0,1[\, \times\,]a,b[\, = \xi^{-1}(\tilde{\Omega})$, there exist slices

$$v_s(t) := v(t,s)$$

for almost every $s \in\,]a,b[$ (see section 2.2.4). By theorem 2.32 such a slice v_s is itself a BV-function of one variable on $]0,1[$ and its derivative relates to the partial derivative of v by

$$D_t v = \left(\mathcal{L}^1 \llcorner\,]a,b[\right) \otimes Dv_s .$$

The PDE $D_t v = 0$ clearly implies that the derivative of every slice is zero and therefore – by theorem 2.30 – the slice v_s is equivalent to a constant

$$v(t,s) = \alpha(s)$$

at most depending on s.

Finally, $\alpha(s)$ is fixed by the boundary condition. Let $z \in \partial\Omega$ with $z = \gamma(\sigma)$. Proceeding as in the proof of lemma 3.15 a), we write down the integral for the boundary trace as

$$\frac{1}{r^2} \int\limits_{\Omega \cap B_r(x)} |u(y) - u_0(z)| \, dy = \frac{C'}{r} \int\limits_{s_-(r)}^{s_+(r)} |\alpha(s) - \gamma^* u_0(\sigma)| \, ds + \mathcal{O}(r) \, .$$

Thus, in order to satisfy the boundary condition the only possible choice is $\alpha(\sigma) = \gamma^* u_0(\sigma)$ whenever $\sigma \in \,]a, b[$ is a Lebesgue-point of $\gamma^* u_0$.

In summary, the solution of the homogeneous problem is given by

$$v(t,s) = \gamma^* u_0(s) \qquad \Rightarrow \qquad u(x) = u_0(\eta(T_0(x), x)) \, ,$$

which further implies

$$\|u\|_{L^\infty(\Omega)} = \|u_0\|_{L^\infty(\partial\Omega)} \, .$$

Renaming $u =: u - w$ and $u_0 =: u_0 - w_0$ again, we conclude the stability equation

$$\|u - w\|_{L^\infty(\Omega)} = \|u_0 - w_0\|_{L^\infty(\partial\Omega)} \, ,$$

and obtain the uniqueness of the solution u in the case of $w_0 = u_0$.

b) Here again, we consider the scaled problems

$$\begin{aligned} \langle c_0(x), Du \rangle &= f_0(x) \cdot \mathcal{L}^2 & \qquad u|_{\partial\Omega} &= u_0 \, , \\ \langle c_0(x), Dw \rangle &= g_0(x) \cdot \mathcal{L}^2 & \qquad w|_{\partial\Omega} &= u_0 \, . \end{aligned}$$

By using the uniqueness result from part a), we can write down the solutions as

$$u(x) = u_0(\eta(T_0(x), x)) + \int\limits_0^{T_0(x)} f_0 \circ \eta(\tau, x) \, d\tau \, ,$$

$$w(x) = u_0(\eta(T_0(x), x)) + \int\limits_0^{T_0(x)} g_0 \circ \eta(\tau, x) \, d\tau \, .$$

The difference of the solutions is easily estimated by

$$|u(x) - w(x)| \le \|f_0 - g_0\|_\infty \cdot T_0(x) \le \|f - g\|_\infty \cdot \frac{1}{\beta \cdot m_0} \, ,$$

and hence we obtain

$$\|u - w\|_{L^\infty(\Omega)} \le \frac{\|f - g\|_\infty}{\beta \cdot m_0} \, .$$

□

Having the uniqueness of the solution and the continuous dependence on the right hand side as well as on the boundary data, the last theorem of this chapter is about the continuous dependence of the solution on the transport field.

Theorem 3.17. *(Continuous Dependence)*

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of transport fields and c be a transport field; all according to requirement 3.6. Let $(f_n)_{n\in\mathbb{N}}$, $f_n \in C^1(\overline{\Omega})$ be a sequence of right hand sides and $f \in C^1(\overline{\Omega})$ be a right hand side. Then, consider the family of linear problems

$$\begin{aligned} \langle c_n(x), Du_n\rangle &= f_n(x)\cdot \mathcal{L}^2 \qquad & u_n|_{\partial\Omega} &= u_0\,, \\ \langle c(x), Du\rangle &= f(x)\cdot \mathcal{L}^2 \qquad & u|_{\partial\Omega} &= u_0\,, \end{aligned}$$

on $\Omega\backslash\Sigma$, where the same time function T is used for all problems.

a) *If both sequences $(c_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$ converge uniformly to c, f respectively, i.e.,*

$$\|c_n - c\|_\infty \to 0\,, \qquad\qquad \|f_n - f\|_\infty \to 0\,,$$

and if the lower bound $\beta > 0$ from requirement 3.6 2b) holds uniformly

$$\beta \le \langle c(x), N(x)\rangle \quad \text{and} \qquad \beta \le \langle c_n(x), N(x)\rangle \quad \forall n\in\mathbb{N}\,,$$

and for every $x \in \Omega\backslash\Sigma$, then the sequence of solutions u_n tends to u in $L^1(\Omega)$

$$\|u_n - u\|_{L^1(\Omega)} \to 0 \quad \text{as} \quad n\to\infty\,.$$

b) *If, in addition to the assumptions of a), the derivatives Dc, ∇f, Dc_n, ∇f_n satisfy the following bounds*

$$\begin{aligned} \|Dc\|_{L^1(\Omega)} &\le M_1 \qquad \text{and} \qquad & \|Dc_n\|_{L^1(\Omega)} &\le M_1 \quad \forall n\in\mathbb{N}\,, \\ \|\nabla f\|_\infty &\le M_2 \qquad \text{and} \qquad & \|\nabla f_n\|_\infty &\le M_2 \quad \forall n\in\mathbb{N}\,, \end{aligned}$$

then the sequence of solutions u_n converges weakly to u:*

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in} \quad BV(\Omega)\,, \quad \text{as} \quad n\to\infty\,.$$

Proof.

a) We use the scaled PDEs

$$\begin{aligned} \langle c_{0,n}(x), Du_n\rangle &= f_{0,n}(x)\cdot\mathcal{L}^2 & \qquad u_n|_{\partial\Omega} &= u_0\ ,\\ \langle c_0(x), Du\rangle &= f_0(x)\cdot\mathcal{L}^2 & \qquad u|_{\partial\Omega} &= u_0\ , \end{aligned}$$

again, but now they are scaled differently: the first PDE is scaled with $\frac{1}{\langle c_n, \nabla T_0\rangle}$, while the second one is scaled with $\frac{1}{\langle c, \nabla T_0\rangle}$.

By the uniform convergence of c_n to c and of f_n to f we infer the uniform convergence of $f_{0,n}$ to f_0 and of $c_{0,n}$ to c_0.

In the following, ξ_n and η_n respectively denote the forward and backward characteristics corresponding to the transport fields $c_{0,n}$. Likewise, ξ and η denote the forward and the backward characteristics corresponding to c_0. And, we write the solutions, decomposed additively, according to equation (3.17):

$$\begin{aligned} u_n(x) &= u_0(\eta_n(T_0(x),x)) + \int_0^{T_0(x)} f_{0,n}\circ\eta_n(\tau,x)\,d\tau &&= u_{1,n}(x) + u_{2,n}(x)\ ,\\ u(x) &= u_0(\eta(T_0(x),x)) + \int_0^{T_0(x)} f_0\circ\eta(\tau,x)\,d\tau &&= u_1(x) + u_2(x)\ . \end{aligned}$$

We will show the L^1-convergence of $u_{i,n}$ to u_i , $i\in\{1,2\}$, separately. First, we need to estimate the difference of the backward characteristics. Let $z\in\Omega\backslash\Sigma$ and set $\lambda := T_0(z)$. The backward characteristics are given by the IVPs

$$\begin{aligned} \eta_n' &= -c_{0,n}(\eta_n) & \qquad \eta_n(0,z) &= z\ ,\\ \eta' &= -c_0(\eta) & \qquad \eta(0,z) &= z\ . \end{aligned}$$

The derivative of the difference $\eta_n - \eta$ obviously satisfies

$$(\eta_n-\eta)' = c_0(\eta) - c_{0,n}(\eta_n)\quad ,\quad (\eta_n-\eta)(0,z) = 0\ ,$$

hence, integration yields

$$(\eta_n-\eta)(t,z) = \int_0^t c_0(\eta(\tau,z)) - c_{0,n}(\eta_n(\tau,z))\,d\tau\ .$$

The first estimate is then

$$|\eta_n - \eta|(t,z) \leq \int_0^t |c_0(\eta(\tau,z)) - c_0(\eta_n(\tau,z))| \, d\tau$$
$$+ \int_0^t |c_0(\eta_n(\tau,z)) - c_{0,n}(\eta_n(\tau,z))| \, d\tau \, .$$

By requirement 3.6 3a) we have the bound $|Dc(x)| \leq M_\varepsilon$ on $\Omega \backslash V_\varepsilon$. Let $\Omega_\lambda = \{x \in \Omega : T_0(x) \leq \lambda\}$ denote the lower level-set of T_0 again, then we derive a new bound $|Dc_0(x)| \leq M_\lambda$ which has to hold only on Ω_λ, because all the points $\eta(t,z)$, $\eta_n(t,z)$ are located there when t varies in between zero and $\lambda = T_0(z)$.

With this bound, the next estimate is

$$|\eta_n - \eta|(t,z) \leq \int_0^t M_\lambda \cdot |\eta_n - \eta|(\tau,z) \, d\tau + \lambda \|c_0 - c_{0,n}\|_{L^\infty(\Omega_\lambda)} \, ,$$

and thus Gronwall's lemma yields

$$\begin{aligned} |\eta_n - \eta|(t,z) &\leq \lambda \cdot e^{\lambda \cdot M_\lambda} \cdot \|c_0 - c_{0,n}\|_{L^\infty(\Omega_\lambda)} \\ &=: C_\lambda \cdot \|c_0 - c_{0,n}\|_{L^\infty(\Omega_\lambda)} \end{aligned} \tag{3.22}$$

which holds for all $z \in \Omega_\lambda$ and $t \in [0, \lambda]$.

Now, we head for L^1-convergence of $u_{1,n}$ to u_1. Changing variables yields

$$\int_\Omega |u_1(z) - u_{1,n}(z)| \, dz = \int_a^b \int_0^1 |\gamma^* u_0(s) - u_{1,n} \circ \xi(t,s)| \cdot \det D\xi(t,s) \, dt \, ds$$

with integrable majorant $2 \ \|u_0\|_{L^\infty(\partial\Omega)} \ \det D\xi(t,s)$. So, we will obtain the assertion by dominated convergence if $u_{1,n} \circ \xi(t,s)$ converges pointwise to $\gamma^* u_0(s)$ for almost every (t,s).

By the definition of $u_{1,n}$ we have

$$u_{1,n} \circ \xi(t,s) = u_0(\eta_n(t, \xi(t,s))) = \gamma^* u_0(\, \gamma^{-1}(\eta_n(t, \xi(t,s))) \,) \, .$$

In accordance with equation (3.10) the equality

$$\xi(\tau,s) = \eta(t-\tau, \xi(t,s))$$

holds for $t \geq \tau \geq 0$ and we can transfer the estimate (3.22):

$$\begin{aligned}|\xi(\tau,s) - \eta_n(t-\tau,\xi(t,s))| &= |\eta(t-\tau,\xi(t,s)) - \eta_n(t-\tau,\xi(t,s))| \\ &\leq C_t \cdot \|c_0 - c_{0,n}\|_{L^\infty(\Omega_t)} \,.\end{aligned}$$

In particular, when setting $\tau = 0$, we obtain

$$|\gamma(s) - \eta_n(t,\xi(t,s))| \leq C_t \cdot \|c_0 - c_{0,n}\|_{L^\infty(\Omega_t)} \,.$$

At this stage, we see that, for fixed $(t,s) \in \,]0,1[\, \times \,]a,b[$, the sequence $\eta_n(t,\xi(t,s))$ tends to $\gamma(s)$ as n tends to infinity. Consequently, the sequence $\gamma^{-1}(\eta_n(t,\xi(t,s)))$ will tend to s since γ^{-1} is continuous.

Finally, $\gamma^* u_0$ is a BV-function of one variable on the interval $]a,b[$. By theorem 2.30 there is a good representative and thus we can assume that $\gamma^* u_0$ is piecewise continuous with at most countably many jumps in $]a,b[$, and the left and right limits exist at every jump point. Hence, for almost every $s \in \,]a,b[$ and every $t \in \,]0,1[$ we can say that

$$|\gamma^* u_0(s) - u_{1,n} \circ \xi(t,s)| \to 0 \quad \text{as} \quad n \to \infty \,.$$

So, we conclude by the dominated convergence theorem that

$$\|u_1 - u_{1,n}\|_{L^1(\Omega)} \to 0 \quad \text{as} \quad n \to \infty \,.$$

For the remaining part, we have

$$\begin{aligned}&\int_\Omega |u_2(x) - u_{2,n}(x)| \, dx \\ &\qquad \leq \int_\Omega \int_0^1 |f_0 \circ \eta(\tau,x) - f_{0,n} \circ \eta_n(\tau,x)| \cdot \mathbb{1}_{]0,T_0(x)[}(\tau) \, d\tau \, dx \,.\end{aligned}$$

By the uniform convergence of f_n to f, the family $\{f_n, f\}$ is bounded, let's say, by M. Using furthermore the uniformity of the angle condition we obtain an integrable majorant

$$\begin{aligned}|f_0 \circ \eta - f_{0,n} \circ \eta_n| \cdot \mathbb{1}_{]0,T_0(x)[} &\leq \left| \frac{f}{\langle c, \nabla T_0 \rangle} \circ \eta \right| + \left| \frac{f_n}{\langle c_n, \nabla T_0 \rangle} \circ \eta_n \right| \\ &\leq \frac{2}{\beta \cdot m_0} M \,.\end{aligned}$$

Clearly, the sequence $f_{0,n}$ also converges uniformly to f_0. Hence, for fixed $\tau \in \,]0,1[$ and $x \in \Omega \backslash \Sigma$ we have

$$|f_0 \circ \eta(\tau,x) - f_{0,n} \circ \eta_n(\tau,x)| \cdot \mathbb{1}_{]0,T_0(x)[}(\tau) \to 0 \quad \text{as} \quad n \to \infty \,,$$

and dominated convergence implies

$$\|u_2 - u_{2,n}\|_{L^1(\Omega)} \to 0 \quad \text{as} \quad n \to \infty \,.$$

b) Every u_n is a BV-function and by the additionally required bounds on $\|Dc_n\|_{L^1(\Omega)}$ and $\|\nabla f_n\|_\infty$ the sequence of total variations $|Du_n|(\Omega)$ is bounded (see lemma 3.13). Together with part a), we have a sequence of BV-functions u_n which is $\|.\|_{BV(\Omega)}$-bounded and converges to u w.r.t. $\|.\|_{L^1(\Omega)}$, so proposition 2.16 yields the assertion.

□

In this chapter, we have seen that the linear problem 3.7 has a unique solution in $BV(\Omega)$ and the solution continuously depends on the data. In other words, problem 3.7 is well-posed in $BV(\Omega)$.
Certainly, one might ask what the point of using the space $BV(\Omega)$ is, because, if the boundary data u_0 were C^1, one could solve in $C^1(\Omega\backslash\Sigma)$. In comparison to lemma 3.13 part a) the summand $c_0^\perp \mu$ of the derivative Du for C^1 boundary data

$$\begin{aligned}\int\limits_\Omega \left\langle \varphi(x), c_0^\perp(x)\right\rangle\, d\mu(x) &= \int\limits_a^b \int\limits_0^1 \left\langle \varphi, c_0^\perp \right\rangle \circ \xi(t,s)\, dt\, dD\gamma^* u_0(s) \\ &= \int\limits_a^b \int\limits_0^1 \left\langle \varphi\circ\xi(t,s), c_0^\perp\circ\xi(t,s)\cdot\gamma^* u_0'(s)\right\rangle\, dt\, ds\end{aligned}$$

– by applying equation (3.18) to $u_1\circ\xi(t,s) = \gamma^* u_0(s)$ – will reduce to

$$\begin{aligned}\int\limits_\Omega \left\langle \varphi(x), c_0^\perp(x)\right\rangle\, d\mu(x) &= \int\limits_a^b \int\limits_0^1 \langle \varphi\circ\xi(t,s), \nabla u_1\circ\xi(t,s)\cdot \det D\xi(t,s)\rangle\, dt\, ds \\ &= \int\limits_\Omega \langle \varphi(x), \nabla u_1(x)\rangle\, dx\,.\end{aligned}$$

And, according to lemma 3.13 part a) we end up with

$$Du = \nabla u_1(x)\cdot\mathcal{L}^2 + \nabla u_2(x)\cdot\mathcal{L}^2 = \nabla u(x)\cdot\mathcal{L}^2\,,$$

where the density function ∇u is now the classical derivative, as in the $C^1(\Omega\backslash\Sigma)$-theory.
The advantage of working in $BV(\Omega)$ is that, on the one hand, we have a description of what happens to u on Σ, and on the other hand in $BV(\Omega)$ it is easy to obtain compact subsets (w.r.t. the weak* topology). The latter will be crucial in the next chapter.

Chapter 4

The Quasi-Linear Problem

In this chapter we turn to the quasi-linear problem (see equation (1.12))

$$\begin{aligned} \langle c[u](x), Du \rangle &= f[u](x) \cdot \mathcal{L}^2 \qquad \text{in} \quad \Omega \backslash \Sigma \,, \\ u|_{\partial\Omega} &= u_0 \,. \end{aligned}$$

For the quasi-linear case the transport field as well as the right hand side functionally depend on the function u which we are looking for. That means, for a fixed $x \in \Omega \backslash \Sigma$ the coefficients of the PDE are functionals of the form

$$f[\,.\,](x) : \mathcal{F} \to \mathbb{R} \,, \qquad\qquad c[\,.\,](x) : \mathcal{F} \to \mathbb{R}^2 \,.$$

Herein, $\mathcal{F}$ denotes a suitable subset of a space of functions which are of mapping type $\Omega \to \mathbb{R}$. We will concretize the set $\mathcal{F}$ in the first section of this chapter.

The first goal of this chapter is to prove the existence of a solution of the quasi-linear problem. The plan for doing so is to interpret a solution u as a fixed point of a certain map.
By fixing the functional argument of the coefficients by some $v \in \mathcal{F}$, other than u, we obtain the linear problem

$$\begin{aligned} \langle c[v](x), Du \rangle &= f[v](x) \cdot \mathcal{L}^2 \qquad \text{in} \quad \Omega \backslash \Sigma \,, \\ u|_{\partial\Omega} &= u_0 \,. \end{aligned}$$

If furthermore the transport field $c[v] : \Omega \backslash \Sigma \to \mathbb{R}^2$ and the right hand side $f[v] : \Omega \to \mathbb{R}$ satisfy the requirements of the linear problem 3.7, then the theory discussed in chapter 3 will guarantee the existence of a unique solution. And now, the solution, called $U[v]$, depends on v.

Finally, if the solution $U[v]$ belongs to $\mathcal{F}$ for every choice of $v \in \mathcal{F}$, the solution defines a self-mapping

$$U : \mathcal{F} \to \mathcal{F} \,.$$

Hence, solving the quasi-linear problem is equivalent to searching for a fixed point $u = U[u]$ of the map U.

In order to apply fixed point theory, we have to extend the list of requirements on the coefficients c and f by assumptions concerning the functional argument $v \in \mathcal{F}$. In the first section we will add requirements in order to make the map U continuous. Then, Schauder's fixed point theorem will yield the existence of a fixed point. Unfortunately, Schauder's theorem only guarantees the existence but not the uniqueness. Therefor, in the second section, we add further requirements which suffice to derive the uniqueness, too.

4.1 The Fixed Point Formulation and its Requirements

As in the previous chapter, we start out with the requirements on the coefficients of the PDE and state the problem afterwards.

The assumptions on transport fields then are:

Requirement 4.1. *(Transport fields)*
Let Ω be a domain and T a time function with stop set Σ all in accordance with requirements 3.1, 3.2, 3.4, and 3.5.

Transport fields are maps of the form

$$c : L^1(\Omega) \to C^1(\Omega\backslash\Sigma)^2 \,, \qquad \text{with} \qquad c[\,.\,](x) : L^1(\Omega) \to \mathbb{R}^2 \,,$$

and are required to satisfy:

1. *For fixed $v \in L^1(\Omega)$ the function $c[v] \in C^1(\Omega\backslash\Sigma)^2$ is a transport field according to requirement 3.6.*
2. *Uniformity of the unit speed and angle condition:*
   a) *$|c[v](x)| = 1$ for all $x \in \Omega\backslash\Sigma$ and for all $v \in L^1(\Omega)$.*
   b) *There is a uniform lower bound $\beta > 0$ such that*
   $$\beta \leq \langle c[v](x), N(x)\rangle \leq 1 \qquad \forall\, x \in \overline{\Omega}\backslash\Sigma \quad \text{and} \quad \forall\, v \in L^1(\Omega)\,.$$
   c) *Both conditions hold for the one-sided limits of $c[v]$ on the relatively open C^1-arcs $\mathring{\Sigma}_k$ of Σ.*
3. *Bounds and continuity:*
   a) *The map $D_x c : L^1(\Omega) \to C(\Omega\backslash\Sigma)^{2\times 2}$ – the derivative of $c[v]$ w.r.t. the variable x – is L^1-bounded by*
   $$\|D_x c[v]\|_{L^1(\Omega)} < M_1 \qquad \forall\, v \in L^1(\Omega)\,.$$

b) *c is continuous in the following manner: if $v \in L^1(\Omega)$ and $(v_n)_{n\in\mathbb{N}}$ is a sequence in $L^1(\Omega)$ with $\|v - v_n\|_{L^1(\Omega)} \to 0$, then the sequence of images $c[v_n]$ converges uniformly to $c[v]$,*

$$\|c[v] - c[v_n]\|_\infty \to 0 \,.$$

For the right hand side f, we assume:

Requirement 4.2. *(Right hand sides)*
Right hand sides are maps of the form

$$f : L^1(\Omega) \to C^1(\overline{\Omega}) \,, \qquad \textit{with} \qquad f[\,.\,](x) : L^1(\Omega) \to \mathbb{R} \,,$$

and are required to satisfy:

a) *The map f is bounded by*

$$\|f[v]\|_\infty \le M_2 \qquad \forall\, v \in L^1(\Omega) \,.$$

b) *The map $\nabla_x f : L^1(\Omega) \to C(\overline{\Omega})^2$ – the derivative of $f[v]$ w.r.t. the variable x – is bounded by*

$$\|\nabla_x f[v]\|_\infty \le M_3 \qquad \forall\, v \in L^1(\Omega) \,.$$

c) *f is continuous in the following manner: if $v \in L^1(\Omega)$ and $(v_n)_{n\in\mathbb{N}}$ is a sequence in $L^1(\Omega)$ with $\|v - v_n\|_{L^1(\Omega)} \to 0$, then the sequence of images $f[v_n]$ converges uniformly to $f[v]$,*

$$\|f[v] - f[v_n]\|_\infty \to 0 \,.$$

Finally, we define the subsets of $BV(\partial\Omega)$, $L^1(\Omega)$ and $BV(\Omega)$ with which we will work later on.

Definition 4.3.
Let Ω be a domain and Σ a stop set according to requirements 3.1 and 3.2. Let M_1, M_2, M_3 be the bounds from the requirements stated above.

a) *We denote by*

$$\mathfrak{B} = \mathfrak{B}(\partial\Omega) := \{v \in BV(\partial\Omega) : \|v\|_{L^\infty(\partial\Omega)} \le M_4 \,,\ |Dv| \le M_5\}$$

the set of boundary functions.

b) *Let $M_* \in \mathbb{R}$ be given by*

$$M_* := \left(M_4 + \frac{M_2}{\beta \cdot m_0}\right) \cdot \mathcal{L}^2(\Omega) \,. \tag{4.1}$$

We set

$$\mathfrak{F} = \mathfrak{F}(\Omega) := \{v \in L^1(\Omega) : \|v\|_{L^1(\Omega)} \le M_*\} \,.$$

c) *Let $M_{**} \in \mathbb{R}$ be given by*

$$M_{**} := 2 \cdot \left(M_4 + \frac{M_2}{\beta \cdot m_0} \right) \cdot \mathcal{H}^1(\Sigma) + \frac{M_5}{\beta \cdot m_0} + \left(\frac{M_2}{\beta} + \frac{M_3}{\beta^2 \cdot m_0} \right) \cdot \mathcal{L}^2(\Omega) + \frac{M_2}{\beta^3 \cdot m_0^2} \cdot \left(M_1 + \|DN\|_{L^1(\Omega)} \right) .$$

We set

$$\mathfrak{X} = \mathfrak{X}(\Omega) := \{ v \in BV(\Omega) : \|v\|_{L^1(\Omega)} \leq M_* \ , \ |Dv|(\Omega) \leq M_{**} \} .$$

Now that we have collected all assumptions we state finally the problem.

Problem 4.4. *(Quasi-linear problem)*
Let Ω be a domain and $T : \Omega \to \mathbb{R}$ a time function with stop set Σ according to the requirements 3.1, 3.2, 3.4, and 3.5 . Let furthermore the transport field c and the right hand side f be in accordance with the requirements 4.1 and 4.2. Let finally $u_0 \in \mathfrak{B}$.

We are looking for $u \in BV(\Omega)$, such that

$$\langle c[u](x), Du \rangle = f[u](x) \cdot \mathcal{L}^2 \quad in \quad \Omega \backslash \Sigma \, , \qquad u|_{\partial\Omega} = u_0 \, .$$

As pointed out in the beginning of this chapter, we are interested in a fixed point formulation of problem 4.4. The next corollary will justify the change of viewpoint.

Corollary 4.5.
Let all of the data Ω, Σ, T, c, f and u_0 be as assumed in problem 4.4.

Then,

a) *for fixed $v \in L^1(\Omega)$, the problem*

$$\langle c[v](x), Du \rangle = f[v](x) \cdot \mathcal{L}^2 \quad in \quad \Omega \backslash \Sigma \, , \qquad u|_{\partial\Omega} = u_0 \, ,$$

with transport field $c[v]$ and right hand side $f[v]$, meets all the requirements of the linear problem 3.7.

The unique solution of the latter problem, which we denote by $U[v]$, defines a map / operator

$$U : L^1(\Omega) \to BV(\Omega) \, .$$

b) *The solution operator U, after restriction to $\mathfrak{F}$ or $\mathfrak{X}$, is a self-mapping*

   a) *of type $U : \mathfrak{F} \to \mathfrak{F}$.*
   b) *of type $U : \mathfrak{X} \to \mathfrak{X}$.*

Proof.

a) Let $v \in L^1(\Omega)$ be arbitrary. Then, by requirement 4.1 part 1 the field $c[v]$ is a transport field according to requirement 3.6, while $f[v] \in C^1(\overline{\Omega})$. Thus, the requirements of the linear problem 3.7 are satisfied and the theory in chapter 3 guarantees the existence of a unique solution $U[v]$ belonging to $BV(\Omega)$. In other words, the map $U : L^1(\Omega) \to BV(\Omega)$ is well-defined.

b) Let $v \in L^1(\Omega)$ be arbitrary but fixed. With $U[v]$ being the solution of the linear problem, by lemma 3.13, we get the following estimate on the L^∞-norm

$$\|U[v]\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\partial\Omega)} + \frac{\|f[v]\|_\infty}{\beta \cdot m_0} ,$$

and on the total variation of $U[v]$ we get

$$\begin{aligned} |DU[v]|(\Omega) \leq 2 \cdot \|U[v]\|_{L^\infty(\Omega)} \cdot \mathcal{H}^1(\Sigma) + \frac{|Du_0|}{\beta \cdot m_0} \\ + \left(\frac{\|f[v]\|_\infty}{\beta} + \frac{\|\nabla f[v]\|_\infty}{\beta^2 \cdot m_0} \right) \cdot \mathcal{L}^2(\Omega) \\ + \frac{\|f[v]\|_\infty}{\beta^3 \cdot m_0^2} \cdot \left(\|Dc[v]\|_{L^1(\Omega)} + \|DN\|_{L^1(\Omega)} \right) . \end{aligned}$$

Plugging in the bounds M_1, M_2, M_3 from requirements 4.1 and 4.2, and the bounds M_4, M_5 on $u_0 \in \mathfrak{B}$, it is easy to see that the upper bounds

$$\|U[v]\|_{L^\infty(\Omega)} \leq M_4 + \frac{M_2}{\beta \cdot m_0} ,$$

and

$$\|U[v]\|_{L^1(\Omega)} \leq M_* , \qquad |DU[v]|(\Omega) \leq M_{**}$$

hold independently of v. Summarizing, the operator U is in fact of type

$$U : L^1(\Omega) \to \mathfrak{X} \subset \mathfrak{F} .$$

Because of $\mathfrak{X} \subset \mathfrak{F} \subset L^1(\Omega)$ we can restrict the domain of U to $\mathfrak{F}$ or $\mathfrak{X}$, and thus both maps

$$U : \mathfrak{F} \to \mathfrak{F} , \qquad U : \mathfrak{X} \to \mathfrak{X}$$

are well-defined self-mappings.

□

Now, by corollary 4.5, we can exchange the quasi-linear problem for an equivalent fixed point problem.

Problem 4.6. *(Fixed point problem)*
Let Ω be a domain and $T : \Omega \to \mathbb{R}$ a time function with stop set Σ according to the requirements 3.1, 3.2, 3.4, and 3.5 . Let furthermore the transport field c and the right hand side f be in accordance with requirements 4.1 and 4.2. And let $u_0 \in \mathfrak{B}$.

Let finally

$$U : \mathfrak{X} \to \mathfrak{X}\,, \qquad v \to U[v]\,,$$

be the solution operator of

$$\begin{aligned} \langle c[v](x), Du\rangle &= f[v](x)\cdot\mathcal{L}^2 \quad \text{in} \quad \Omega\backslash\Sigma\,, \\ u|_{\partial\Omega} &= u_0\,. \end{aligned}$$

We are looking for a fixed point $u \in \mathfrak{X}$ of the map U, i.e.,

$$u = U[u]\,.$$

4.2 Existence of a Fixed Point

The goal of this section is to prove the existence of fixed points of the problem 4.6 and thereby the existence of solutions of the quasi-linear problem 4.4. The tool for achieving this objective is the Schauder fixed point theorem.

Theorem 4.7. *(Schauder)([Zei93], [Dei85])*
Let $\mathcal{X}$ be a Banach space and let $\mathcal{M} \subset \mathcal{X}$ be a non-empty, convex, and compact subset. Let the map $F : \mathcal{M} \to \mathcal{M}$ be continuous. Then, F has a fixed point $x \in \mathcal{M}$:

$$x = F(x)\,.$$

The next step is to show that all the assumptions of the Schauder theorem are satisfied in problem 4.6.

Lemma 4.8.
The set $\mathfrak{X}$, defined in Definition 4.3, is non-empty, convex and sequentially compact w.r.t. the BV-weak topology.*

Proof.

The set $\mathfrak{X}$, by its definition, is convex and obviously non-empty. Because $\mathfrak{X}$ is $\|.\|_{BV(\Omega)}$-bounded, the sequential compactness is a consequence of proposition 2.17. $\square$

Lemma 4.9.
The map $U : \mathfrak{X} \to \mathfrak{X}$ from corollary 4.5 b) is sequentially continuous w.r.t. the BV-weak topology.*

Proof.

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{X}$ which tends weakly* to $v \in \mathfrak{X}$ w.r.t. the BV-weak* topology. Then, we have in particular $\|v - v_n\|_{L^1(\Omega)} \to 0$.

We set

$$
\begin{aligned}
c_n &:= c[v_n] \,, & \qquad c &:= c[v] \,, \\
f_n &:= f[v_n] \,, & \qquad f &:= f[v] \,, \\
u_n &:= U[v_n] \,, & \qquad u &:= U[v] \,.
\end{aligned}
$$

By requirements 4.1 and 4.2 we have $\|c_n - c\|_\infty \to 0$ and $\|f_n - f\|_\infty \to 0$, while β in

$$
\beta \le \langle c(x), N(x)\rangle \quad \text{and} \qquad \beta \le \langle c_n(x), N(x)\rangle \quad \forall n \in \mathbb{N}
$$

holds uniformly. Moreover, we have the following bounds

$$
\begin{aligned}
\|Dc\|_{L^1(\Omega)} &\le M_1 \quad \text{and} & \|Dc_n\|_{L^1(\Omega)} &\le M_1 \quad \forall n \in \mathbb{N} \,, \\
\|\nabla f\|_\infty &\le M_3 \quad \text{and} & \|\nabla f_n\|_{\infty)} &\le M_3 \quad \forall n \in \mathbb{N} \,.
\end{aligned}
$$

So, all the assumptions of theorem 3.17 a) and b) are satisfied, which tells us that

$$
U[v_n] = u_n \overset{*}{\rightharpoonup} u = U[v] \,.
$$

$\square$

In order to apply Schauder's theorem we use a result from [Bor02], which characterizes the weak* convergence of sequences in a dual space $\mathcal{X}'$ in the case that $\mathcal{X}$ is separable.

Lemma 4.10.
Let $(\mathcal{X}, \|.\|)$ be a separable normed space. Let $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ be a sequence with $\|\sigma_n\| = 1$ and $\mathcal{X} = \overline{\operatorname{span}\sigma}$. Then,

a) the function

$$\|x'\|_\sigma := \sum_{n=1}^{\infty} 2^{-n}|x'(\sigma_n)| , \qquad x' \in \mathcal{X}' ,$$

defines a norm on the dual space $\mathcal{X}'$.

b) a $\|.\|$-bounded sequence $(x'_k)_{k\in\mathbb{N}}$ in the dual space $\mathcal{X}'$ weakly converges towards $x' \in \mathcal{X}'$ if and only if*

$$\|x'_k - x'\|_\sigma \to 0 \quad as \quad k \to \infty.$$

Proof.

Part a) is lemma 1 in [Bor02], while part b) is theorem 1 in [Bor02]. □

Theorem 4.11. *(Existence)*
The map $U : \mathfrak{X} \to \mathfrak{X}$ from corollary 4.5 b) admits a fixed point $u = U[u]$.

Hence, the quasi-linear problem 4.4 has a solution.

Proof.

In remark 2.15 we mentioned that $BV(\Omega) = \mathcal{X}'$ is the dual of a separable space $\mathcal{X}$. Because $\mathcal{X}$ is separable, we can – according to lemma 4.10 a) – use a new norm $\|.\|_\sigma$ on $BV(\Omega)$.

Keeping lemma 4.10 b) in mind, by using $\|.\|_\sigma$, lemma 4.8 tells us that $\mathfrak{X}$ is non-empty, convex and $\|.\|_\sigma$-compact, while lemma 4.9 tells us that the operator $U : \mathfrak{X} \to \mathfrak{X}$ is $\|.\|_\sigma$-continuous.

Now, we have fulfilled all the assumptions of the Schauder fixed point theorem, which yields the existence of a fixed point $u = U[u]$ of the map U.

Finally, every fixed point $u \in \mathfrak{X}$ of U solves the quasi-linear problem. Since U is the solution operator from corollary 4.5, we have

$$u|_{\partial\Omega} = U[u]|_{\partial\Omega} = u_0 ,$$

and

$$f[u](x) = \langle c[u](x), DU[u]\rangle = \langle c[u](x), Du\rangle \ .$$

□

4.3 Uniqueness and Stability under Volterra-Type Dependence

For the existence result of the previous section we did not make any restrictions on the type of the functional dependence. There the value of the right hand side $f[v](x) \in \mathbb{R}$ at some point $x \in \Omega$ might depend on all values $\{v(z) : z \in \Omega\}$ of the functional argument $v \in L^1(\Omega)$. This type of dependence is often called *dependence of Fredholm-type*. The name stems from the subject of *Fredholm integral equations* where operators of the form

$$A : L^1(]a,b[) \to L^1(]a,b[) \,, \qquad A[v](t) = \int_a^b k(t,\tau,v(\tau))\, d\tau \,,$$

play the important role (see [Hac95]). Here, the value $A[v](t)$ depends on all values of v.

Another interesting case is the *dependence of Volterra-type*. This name stems from the subject of *Volterra integral equations* where operators of the form

$$B : L^1(]a,b[) \to L^1(]a,b[) \,, \qquad B[v](t) = \int_a^t k(t,\tau,v(\tau))\, d\tau \,,$$

are considered (see [Hac95]). The special feature of this case is that the value $B[v](t)$ only depends on the values of v on the interval $]a,t[$. More formally, the operator B has the property

$$B[v](t) = B[\, v \cdot \mathbb{1}_{]a,t[}\,](t) \,,$$

with $\mathbb{1}_{]a,t[}$ denoting the characteristic function of the set $]a,t[$.
If we interpret the variable t as physical time and view v as time-dependent description of some signal, then the operator B is a form of signal processing, which produces a value $B[v](t)$ by employing information about the signal from the time period before t. Since t is often seen as time, another way of saying "the operator B has a functional dependence of Volterra-type" is "the operator B or signal filter B has a memory effect".

In this section we will show the uniqueness of the fixed point of the operator U in the case that the functional dependence of the coefficients c and f – and thus the functional dependence of U– is of Volterra-type. The notion of time, necessary for the dependence of Volterra-type, is again induced by the time function T.

4.3.1 PDE Coefficients with Volterra-Type Dependence and Additional Requirements

Definition 4.12. *(Dependence of Volterra-type)*
Let Ω be a domain and T a time function according to requirements 3.1 and 3.4.

Let $\mathcal{F}(\Omega)$ and $\mathcal{G}(\Omega)$ be function spaces defined on Ω and let f be an operator

$$f : \mathcal{F}(\Omega) \to \mathcal{G}(\Omega) \,, \quad \text{with} \qquad f[\,.\,](x) : \mathcal{F}(\Omega) \to \mathbb{R}^d \,, \quad x \in \Omega \,.$$

Let $T(x)$ be the time of the point $x \in \Omega$. Then, the set $\Omega_{T(x)}$, defined as the following lower level-set of T

$$\Omega_{T(x)} := \{z \in \Omega : T(z) < T(x)\} \,,$$

denotes the "past" w.r.t. $T(x)$.

We say that the functional dependence of f is of Volterra-type (w.r.t. time T) if the equation

$$f[v](x) = f[\, v \cdot \mathbb{1}_{\Omega_{T(x)}} \,](x)$$

is valid.

An analogous definition for problems on the half-space can be found in [Kam99].

Remark: the dependence of Volterra-type incorporates the following domain restriction feature. Let λ be in the range of T. Then, for $x \in \Omega_\lambda$ and $v \in \mathcal{F}(\Omega)$, the inclusion $\Omega_{T(x)} \subset \Omega_\lambda$ implies

$$f[v](x) = f[\, v \cdot \mathbb{1}_{\Omega_{T(x)}} \,](x) = f[\, v \cdot \mathbb{1}_{\Omega_\lambda} \,](x) \,.$$

Hence, the domain restriction (onto Ω_λ)

$$f : \mathcal{F}(\Omega_\lambda) \to \mathcal{G}(\Omega_\lambda)$$

is well-defined.

Let T be a given time function. In what follows, for all dependences on a time function, we mean the same time function T. Then, in this section, we additionally require:

Requirement 4.13. *(Transport fields)*
Let $c : L^1(\Omega) \to C^1(\Omega \backslash \Sigma)^2$ be a transport field according to requirement 4.1.
c is furthermore required to satisfy:

a) the functional dependence of c is of Volterra-type.

b) the map c is Lipschitz in the following manner:

$$\|c[v] - c[w]\|_\infty \le L_1 \cdot \|v - w\|_{L^1(\Omega)} \,.$$

c) the bound on $|D_x c[v](x)|$, from requirement 3.6 3a),

$$|D_x c[v](x)| \le M_\varepsilon \quad , \quad \forall\, x \in \Omega \backslash V_\varepsilon$$

holds uniformly for all $v \in L^1(\Omega)$.

Requirement 4.14. *(Right hand sides)*
Let $f : L^1(\Omega) \to C^1(\overline{\Omega})$ be a right hand side according to requirement 4.2. f is furthermore required to satisfy:

a) the functional dependence of f is of Volterra-type.

b) the map f is Lipschitz in the following manner:

$$\|f[v] - f[w]\|_\infty \leq L_2 \cdot \|v - w\|_{L^1(\Omega)} .$$

Lemma 4.15.
Consider the situation of problem 4.6. If, in addition, the functional dependence of the coefficients c, f is of Volterra-type, then the functional dependence of the solution operator $U : \mathfrak{X} \to \mathfrak{X}$ is of Volterra-type.

Proof.

Let $v \in \mathfrak{X}$ be arbitrary but fixed. Then, according to equation (3.16), $U[v](x)$ is given by

$$U[v](x) = u_0(\eta[v](T_0(x), x)) + \int_0^{T_0(x)} f_0[v] \circ \eta[v](\tau, x)\, d\tau ,$$

where the backward characteristic $\eta[v](., x)$ is the solution of

$$y' = -c_0[v](y) , \qquad\qquad y(0) = x .$$

As in chapter 3, T_0 denotes the transformed version of T according to equation (3.4), while c_0 and f_0 are given by:

$$c_0[v](x) = \frac{c[v](x)}{\langle c[v](x), \nabla T_0(x)\rangle} , \qquad f_0[v](x) = \frac{f[v](x)}{\langle c[v](x), \nabla T_0(x)\rangle} .$$

Clearly, c_0 and f_0 have the same Volterra-type dependence as c and f. Since T_0 has the same level-sets as T – only the names of level lines have changed – we refer to T_0. Let then

$$\Omega_{T_0(x)} := \{z \in \Omega : T_0(z) < T_0(x)\} .$$

For every $t \in]0, T_0(x)[$ we know that $\eta[v](t, x) \in \Omega_{T_0(x)}$. Hence, $\eta[v](\,.\,, x)$ only depends on the restriction $-c_0[v]|_{\Omega_{T_0(x)}}$ which itself only depends on $v \cdot \mathbb{1}_{\Omega_{T_0(x)}}$. So, by the representation of $U[v](x)$ above, it is obvious that

$$U[v](x) = U[\, v \cdot \mathbb{1}_{\Omega_{T_0(x)}}\,](x)$$

holds true. $\square$

Using the domain restriction feature, as discussed above, we define:

Definition 4.16.
Let λ be in the range of T. We denote by

$$\mathfrak{F}_\lambda = \mathfrak{F}(\Omega_\lambda) := \{v|_{\Omega_\lambda} : v \in \mathfrak{F}\} ,$$

and by

$$\mathfrak{X}_\lambda = \mathfrak{X}(\Omega_\lambda) := \{v|_{\Omega_\lambda} : v \in \mathfrak{X}\} ,$$

the domain-restricted versions of $\mathfrak{F}$ and $\mathfrak{X}$ from definition 4.3.

And finally, we consider the domain-restricted problem.

Problem 4.17. *(Domain restricted fixed point problem)*
Assume that all the requirements of problem 4.6 are satisfied and that c and f satisfy the additional requirements stated above. Let furthermore λ be in the range of T.

Let finally

$$U : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda , \qquad v \to U[v] ,$$

be the solution operator of the domain-restricted problem

$$\begin{aligned} \langle c[v](x), Du\rangle &= f[v](x) \cdot \mathcal{L}^2 \quad \text{in} \quad \Omega_\lambda , \\ u|_{\partial\Omega} &= u_0 . \end{aligned}$$

We are looking for a fixed point $u \in \mathfrak{X}_\lambda$ of the map U, i.e.,

$$u = U[u] .$$

The question for existence has already been answered: by lemma 4.15 the operator $U : \mathfrak{X} \to \mathfrak{X}$ is of Volterra-type. Hence, the domain-restricted version $U : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ is well-defined and the argumentation which we have used in the proof of lemma 4.15 shows that it solves the domain-restricted linear problem. Moreover, every fixed point u of the original operator $U : \mathfrak{X} \to \mathfrak{X}$ after restriction $u|_{\Omega_\lambda}$ belongs to $\mathfrak{X}_\lambda$ and is a fixed point of $U : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$.

The next step is to prove the uniqueness of the fixed point of problem 4.17.

4.3.2 Uniqueness of the Fixed Point

In this section we will show that, for any choice of $0 < \lambda < 1$, the operator $U : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ is Lipschitz. Moreover, we will see that $U : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ is in fact contractive for a suitable choice of λ. The latter feature will then imply the uniqueness of the fixed point.

For the estimation of the difference $U[v_1] - U[v_2]$ we prepare by setting up a PDE which is satisfied by the difference. For the purpose of abbreviation, we set

$$c_1 := c[v_1] , \qquad c_2 := c[v_2] ,$$
$$f_1 := f[v_1] , \qquad f_2 := f[v_2] .$$

Let u_1 and u_2 respectively denote the solutions of the two linear problems

$$\begin{aligned} &\langle c_1(x), Du \rangle = f_1(x) \cdot \mathcal{L}^2 \quad \text{in } \Omega_\lambda , && u|_{\partial\Omega} = u_{0,1} , \\ \text{and} \quad &\langle c_2(x), Du \rangle = f_2(x) \cdot \mathcal{L}^2 \quad \text{in } \Omega_\lambda , && u|_{\partial\Omega} = u_{0,2} . \end{aligned}$$

As in the proof of lemma 4.15 we refer to the transformed time T_0 instead of T and denote by Ω_λ the lower level-set of T_0

$$\Omega_\lambda := \{ z \in \Omega : T_0(z) < \lambda \} , \qquad \lambda \in \,]0, 1[\; .$$

For the first considerations we use different boundary data. When setting $u_{0,1} = u_{0,2} = u_0$ later on, we will obtain the relations

$$u_1 = U[v_1] , \quad u_2 = U[v_2] . \tag{4.2}$$

Let w denote the difference $w := u_1 - u_2$. After having subtracted the problems from each other, the difference w must satisfy the linear problem

$$\begin{aligned} \langle c_1(x), Dw \rangle &= (f_1(x) - f_2(x)) \cdot \mathcal{L}^2 - \langle c_1(x) - c_2(x), Du_2 \rangle \quad \text{in } \Omega_\lambda , \\ w|_{\partial\Omega} &= w_0 , \end{aligned}$$

with boundary data $w_0 = u_{0,1} - u_{0,2}$.

By the same argumentation as in the proof of theorem 3.16 we see that w is the unique solution of this PDE. But, in order to solve for w, we cannot directly apply the method of construction from chapter 3, since the right hand side is not an absolutely continuous measure.

Instead we approximate the right hand side by absolutely continuous measures. Since $u_2 \in BV(\Omega)$, by theorem 2.19, there exists a sequence $(u_{2,n})_{n\in\mathbb{N}}$ of $C^\infty(\Omega)$-functions which converges strictly to $u_{2,n}$, i.e.,

$$\|u_2 - u_{2,n}\|_{L^1(\Omega)} \to 0 \quad \text{and} \quad \big| |Du_2|(\Omega) - |Du_{2,n}|(\Omega) \big| \to 0 .$$

Moreover, we have $Du_{2,n} = \nabla u_{2,n}(x) \cdot \mathcal{L}^2$ and $|Du_{2,n}|(\Omega) = \|\nabla u_{2,n}\|_{L^1(\Omega)}$.

Using such a sequence we obtain an approximate problem

$$\begin{aligned} \langle c_1(x), Dw \rangle &= (f_1(x) - f_2(x) - \langle c_1(x) - c_2(x), \nabla u_{2,n}(x) \rangle) \cdot \mathcal{L}^2 \quad \text{in } \Omega_\lambda , \\ w|_{\partial\Omega} &= w_0 , \end{aligned}$$

with a sequence of solutions w_n which we can construct using the same method as in chapter 3. Again, we scale by the factor $\frac{1}{\langle c_1, \nabla T_0\rangle}$ and set

$$c_{1,0} := \frac{c_1}{\langle c_1, \nabla T_0\rangle}\,, \qquad f_{1,0} := \frac{f_1}{\langle c_1, \nabla T_0\rangle}\,,$$
$$c_2^0 := \frac{c_2}{\langle c_1, \nabla T_0\rangle}\,, \qquad f_2^0 := \frac{f_2}{\langle c_1, \nabla T_0\rangle}\,.$$

Note: if we were to be consistent, we would set $f_{2,0} := \frac{f_2}{\langle c_2, \nabla T_0\rangle}$, which differs from f_2^0.

The family of forward characteristics $\xi(\,.\,,s)$ is then given by the IVP

$$y' = c_{1,0}(y)\,, \qquad y(0) = \gamma(s)\,.$$

So, we obtain w_n in characteristic variables by

$$w_n \circ \xi(t,s) = \gamma^* w_0(s) + \int_0^t \big(f_{1,0} - f_2^0 - \langle c_{1,0} - c_2^0, \nabla u_{2,n}\rangle\big) \circ \xi(\tau,s)\, d\tau\,. \tag{4.3}$$

The consideration of the sequence w_n will not be of any use if w_n does not tend to w in an appropriate fashion. We will show the desired convergence in lemma 4.19. But first, we rewrite $w_n \circ \xi$. Because the PDE for u_1 has the same transport field c_1, we have for u_1

$$u_1 \circ \xi(t,s) = \gamma^* u_{0,1} + \int_0^t f_{1,0} \circ \xi(\tau,s)\, d\tau\,,$$

and by the fundamental theorem of calculus, we see

$$u_{2,n} \circ \xi(t,s) - u_{2,n} \circ \gamma(s) = \int_0^t \langle c_{1,0}, \nabla u_{2,n}\rangle \circ \xi(\tau,s)\, d\tau\,.$$

With $w_0 = u_{0,1} - u_{0,2}$ and the last two observations, we have

$$\begin{aligned} w_n \circ \xi(t,s) &= u_1 \circ \xi(t,s) - u_{2,n} \circ \xi(t,s) + u_{2,n} \circ \gamma(s) - \gamma^* u_{0,2}(s) \\ &\quad + \int_0^t \big(\langle c_2^0, \nabla u_{2,n}\rangle - f_2^0\big) \circ \xi(\tau,s)\, d\tau\,. \end{aligned}$$

Subtracting $w = u_1 - u_2$, finally, we end up with

$$\begin{aligned} (w_n - w) \circ \xi(t,s) &= (u_2 - u_{2,n}) \circ \xi(t,s) \; + (\gamma^* u_{2,n}(s) - \gamma^* u_{0,2}(s)) \\ &\quad + \int_0^t \big(\langle c_2^0, \nabla u_{2,n}\rangle - f_2^0\big) \circ \xi(\tau,s)\, d\tau\,. \end{aligned} \tag{4.4}$$

As a second step of preparation, we will show that requirement 4.13 c) implies uniform bounds on the determinant of $D\xi$.

Lemma 4.18.
Let $c : L^1(\Omega) \to C^1(\Omega\backslash\Sigma)^2$ be a transport field according to 4.13. For fixed $v \in L^1(\Omega)$, let $\xi[v] :]0,1[\times]a,b[\to \Omega\backslash(S \cup \Sigma)$ denote the diffeomorphism – according to corollary 3.10 – given by the solution of the IVP

$$y' = c_0[v](y) \qquad\qquad y(0) = \gamma(s)\ .$$

Then, for $0 < \lambda < 1$ there are bounds k_λ and K_λ such that

$$0 < k_\lambda \leq \det D\xi[v](t,s) \leq K_\lambda \qquad\quad \forall\ (t,s) \in]0,\lambda[\times]a,b[\ .$$

The bounds k_λ and K_λ depend only on λ, but not on v. Moreover, k_λ decreases, while K_λ increases monotonically with λ.

Proof.

The right hand side of the IVP is given by

$$c_0[v] = \frac{c[v]}{\langle c[v], \nabla T_0\rangle}\ .$$

Because, by requirement 4.13 part c), there is the uniform bound

$$|D_x c[v](x)| \leq M_\varepsilon \quad , \quad \forall\ x \in \Omega\backslash V_\varepsilon \quad , \quad \forall\ v \in L^1(\Omega)$$

on the derivative $D_x c[v](x)$, a similar bound $M_{0,\varepsilon}$ will hold for $D_x c_0[v](x)$.

The diffeomorphism $\xi[v]$ for every v maps the set $]0,\lambda[\times]a,b[$ onto Ω_λ. We then choose ε so small that

$$\Omega_\lambda \subset \Omega\backslash V_\varepsilon\ ,$$

and obtain a bound that only depends on λ:

$$|D_x c_0[v](x)| \leq M_{0,\varepsilon(\lambda)} \quad , \quad \forall\ x \in \Omega_\lambda \quad , \quad \forall\ v \in L^1(\Omega)\ .$$

By lemma 3.11 a) we have

$$\det D\xi[v] \leq |\partial_t \xi[v]| \cdot |\partial_s \xi[v]| \leq \frac{|\partial_s \xi[v]|}{\beta \cdot m_0}\ .$$

$\partial_s \xi[v]$ is the solution of

$$\partial_s \xi[v]' = D_x c_0[v] \circ \xi[v] \cdot \partial_s \xi[v] \qquad\qquad y(0) = \gamma'(s)\ .$$

Hence, for $t \in [0, \lambda]$, we estimate

$$|\partial_s \xi[v]|(t,s) \le \|\gamma'\|_\infty + \int_0^t M_{0,\varepsilon(\lambda)} |\partial_s \xi[v]|(\tau, s)\, d\tau \,,$$

and an application of Gronwall's lemma yields

$$|\partial_s \xi[v]|(t,s) \le \|\gamma'\|_\infty \exp\left(\lambda \cdot M_{0,\varepsilon(\lambda)}\right) \,.$$

This yields the upper bound K_λ

$$\det D\xi[v](t,s) \le \frac{\|\gamma'\|_\infty \exp\left(\lambda \cdot M_{0,\varepsilon(\lambda)}\right)}{\beta \cdot m_0} =: K_\lambda$$

on $]0, \lambda[\times]a, b[$.

For the lower bound we consider the inverse $\xi[v]^{-1}(x) = (T_0(x), s[v](x))$, with

$$s[v](x) = \gamma^{-1}(\eta[v](T_0(x), x))$$

for $x \in \Omega_\lambda$. Here, $\eta[v](\,.\,, x)$ denotes the backward characteristics given as the solution of

$$y' = -c_0[v](y) \qquad\qquad y(0) = x \in \Omega_\lambda \,.$$

The determinant of $D_x \xi[v]^{-1}$ is bounded by

$$\det D_x \xi[v]^{-1}(x) \le |\nabla T_0(x)| \cdot |\nabla_x s[v](x)|$$

and $\nabla_x s[v](x)^T =$

$$(\gamma^{-1})'(\eta)^T \cdot \left(\partial_t \eta[v](T_0(x), x) \cdot \nabla T_0(x)^T + D_x \eta[v](t, x)|_{t = T_0(x)}\right) \,.$$

We estimate $|D_x \eta[v](t,x)|$ in the same way as $|\partial_s \xi(t,s)|$ and, because $x \in \Omega_\lambda$, obtain

$$|D_x \eta[v](t,x)| \le \exp\left(\lambda \cdot M_{0,\varepsilon(\lambda)}\right) \,.$$

Finally, we see that

$$\det D_x \xi[v]^{-1}(x) \le \frac{\|\nabla T_0\|_{L^\infty(\Omega_\lambda)}}{\min_{s \in [a,b]} |\gamma'(s)|} \cdot \left(\frac{\|\nabla T_0\|_{L^\infty(\Omega_\lambda)}}{\beta \cdot m_0} + \exp\left(\lambda \cdot M_{0,\varepsilon(\lambda)}\right)\right) \,,$$

for $x \in \Omega_\lambda$. We define $1/k_\lambda$ to equal the right hand side of the last inequality. Then, we have

$$\frac{1}{\det D\xi[v](t,s)} = \det\left((D\xi[v](t,s))^{-1}\right) = \det D_x \xi[v]^{-1}(x)|_{x = \xi(t,s)} \le \frac{1}{k_\lambda} \,,$$

for $(t,s) \in]0, \lambda[\times]a, b[$, since in this case $\xi(t,s) \in \Omega_\lambda$.

Both bounds do not depend on the choice of $v \in L^1(\Omega)$ and the monotonicity properties of k_λ and K_λ as functions of λ are obvious. □

In the next lemma we turn to the approximation of w by w_n.

Lemma 4.19.
Let w and w_n be as defined in the preparatory step above. Interpret the $L^1(\Omega_\lambda)$-functions w and w_n as absolutely continuous measures $w(x) \cdot \mathcal{L}^2$ and $w_n(x) \cdot \mathcal{L}^2$ on Ω_λ. Then, the sequence of measures $w_n(x) \cdot \mathcal{L}^2$ weakly converges to $w(x) \cdot \mathcal{L}^2$:*

$$w_n(x) \cdot \mathcal{L}^2 \overset{*}{\rightharpoonup} w(x) \cdot \mathcal{L}^2 \quad , \qquad \textit{as} \quad n \to \infty .$$

Proof.

Let $\varphi \in C_0(\Omega_\lambda)$ be a test function. By changing variables it follows that

$$\int\limits_{\Omega_\lambda} (w_n - w)(x) \cdot \varphi(x)\, dx = \int\limits_a^b \int\limits_0^\lambda ((w_n - w) \cdot \varphi) \circ \xi(t,s) \cdot \det D\xi(t,s)\, dt\, ds .$$

We use the representation of $(w_n - w) \circ \xi$ according to equation (4.4) and study the convergence of the three summands in equation (4.4) separately. The first summand is estimated by

$$\left| \int\limits_a^b \int\limits_0^\lambda ((u_{2,n} - u_2) \cdot \varphi) \circ \xi(t,s) \cdot \det D\xi(t,s)\, dt\, ds \right| \leq \|u_{2,n} - u_2\|_{L^1(\Omega_\lambda)} \|\varphi\|_\infty ,$$

and the right hand side tends to zero, because the sequence $u_{2,n}$ strictly tends to u_2 in $BV(\Omega)$.

For the second summand we write

$$\left| \int\limits_a^b \int\limits_0^\lambda (\gamma^* u_{2,n}(s) - \gamma^* u_{0,2}(s)) \cdot \varphi \circ \xi(t,s) \cdot \det D\xi(t,s)\, dt\, ds \right|$$
$$= \left| \int\limits_a^b (\gamma^* u_{2,n}(s) - \gamma^* u_{0,2}(s)) \cdot |\gamma'(s)| \left(\frac{\int_0^\lambda \varphi \circ \xi(t,s) \cdot \det D\xi(t,s)\, dt}{|\gamma'(s)|} \right) ds \right| .$$

Let k_λ and K_λ be the bounds on the determinant as in lemma 4.18. By the definition of k_λ, we have

$$k_\lambda \leq \det D\xi(0,s) \leq \frac{|\gamma'(s)|}{\beta \cdot m_0} \quad \Leftrightarrow \quad \frac{1}{|\gamma'(s)|} \leq \frac{1}{\beta \cdot m_0 \cdot k_\lambda} .$$

And consequently,

$$\frac{\int\limits_0^\lambda \varphi \circ \xi(t,s) \cdot \det D\xi(t,s)\, dt}{|\gamma'(s)|} \leq \lambda \cdot \frac{K_\lambda}{\beta \cdot m_0 \cdot k_\lambda} \cdot \|\varphi\|_\infty .$$

By the last result, we further estimate:

$$
\begin{aligned}
&\leq \lambda \cdot \frac{K_\lambda}{\beta \cdot m_0 \cdot k_\lambda} \cdot \|\varphi\|_\infty \cdot \int_a^b |\gamma^*(u_{2,n} - u_{0,2})(s)| \cdot |\gamma'(s)| \, ds \\
&\leq \lambda \cdot \frac{K_\lambda}{\beta \cdot m_0 \cdot k_\lambda} \cdot \|\varphi\|_\infty \cdot \int_{\partial\Omega} | \, (u_{2,n} - u_2)|_{\partial\Omega}(x) \, | \, d\mathcal{H}^1(x) \\
&= \lambda \cdot \frac{K_\lambda}{\beta \cdot m_0 \cdot k_\lambda} \cdot \|\varphi\|_\infty \cdot \|(u_{2,n} - u_2)|_{\partial\Omega}\|_{L^1(\partial\Omega, \mathcal{H}^1)} \, .
\end{aligned}
$$

In the last factor we apply the trace operator for BV-functions

$$
\cdot|_{\partial\Omega} : BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^1) \, , \qquad\qquad v \to v|_{\partial\Omega} \, ,
$$

which, by theorem 2.27, is continuous w.r.t. the strict topology on $BV(\Omega)$. Hence, the factor $\|(u_{2,n} - u_2)|_{\partial\Omega}\|_{L^1(\partial\Omega, \mathcal{H}^1)}$ also tends to zero as n tends to infinity.

Let $\psi(t,s) := \varphi \circ \xi(t,s) \cdot \det D\xi(t,s)$. Then, by changing the order of integration, we get for the third summand

$$
\begin{aligned}
&\int_a^b \int_0^\lambda \int_0^t (\langle c_2^0, \nabla u_{2,n} \rangle - f_2^0) \circ \xi(\tau, s) \, d\tau \cdot \psi(t,s) \, dt \, ds \\
&\qquad = \int_a^b \int_0^\lambda (\langle c_2^0, \nabla u_{2,n} \rangle - f_2^0) \circ \xi(\tau, s) \left(\int_\tau^\lambda \psi(t,s) \, dt \right) \, d\tau \, ds \, .
\end{aligned}
$$

By the definition of ψ and since ξ is a diffeomorphism, there is a continuous function $h \in C(\overline{\Omega}_\lambda)$ such that

$$
h \circ \xi(\tau, s) = (\det D\xi(\tau, s))^{-1} \cdot \left(\int_\tau^\lambda \psi(t,s) \, dt \right) \, .
$$

With h, we rewrite the integral

$$
\begin{aligned}
&\int_a^b \int_0^\lambda \int_0^t (\langle c_2^0, \nabla u_{2,n} \rangle - f_2^0) \circ \xi(\tau, s) \, d\tau \cdot \psi(t,s) \, dt \, ds \\
&\qquad = \int_a^b \int_0^\lambda (\langle c_2^0, \nabla u_{2,n} \rangle - f_2^0) \circ \xi(\tau, s) \cdot h \circ \xi(\tau, s) \cdot \det D\xi(\tau, s) \, d\tau \, ds \\
&\qquad = \int_{\Omega_\lambda} (\langle c_2^0, \nabla u_{2,n} \rangle - f_2^0)(x) \cdot h(x) \, dx \, .
\end{aligned}
$$

Next, we use the fact that u_2 solves the PDE

$$\langle c_2^0(x), Du_2 \rangle = f_0^2(x) \cdot \mathcal{L}^2$$

and formulate the last integral as

$$\begin{aligned}
&= \int_{\Omega_\lambda} \langle h(x) \cdot c_2^0(x), \nabla u_{2,n}(x) \rangle \, dx - \int_{\Omega_\lambda} \langle h(x) \cdot c_2^0(x), dDu_2(x) \rangle \\
&= \int_{\Omega_\lambda} \langle \hat{\varphi}(x), \nabla u_{2,n}(x) \rangle \, dx - \int_{\Omega_\lambda} \langle \hat{\varphi}(x), dDu_2(x) \rangle \ .
\end{aligned}$$

In the second equation we have set

$$\hat{\varphi}(x) := h(x) \cdot c_2^0(x)$$

as a new test function which belongs to $C(\overline{\Omega_\lambda})^2$.
Owing again to the strict convergence of $u_{2,n}$ to u_2, we argue by proposition 2.20 that the last integral expression tends to zero as $n \to \infty$. Summarizing the three steps above we obtain

$$\int_{\Omega_\lambda} (w_n - w)(x) \cdot \varphi(x) \, dx \to 0 \qquad \forall \ \varphi \in C_0(\Omega_\lambda) \ ,$$

which means $w_n(x) \cdot \mathcal{L}^2 \overset{*}{\rightharpoonup} w(x) \cdot \mathcal{L}^2$ on Ω_λ. □

Based on the properties of the sequence w_n we will show that the operator U is Lipschitz.

Lemma 4.20.
Let $\lambda, h \geq 0$ be such that $\lambda + h < 1$. We set

$$\Omega_{\lambda+h,\lambda} = \Omega_{\lambda+h} \backslash \Omega_\lambda = \{z \in \Omega : \lambda \leq T_0(z) < \lambda + h\} \ .$$

Then, the difference $w = u_1 - u_2$ satisfies

$$\begin{aligned}
\|w\|_{L^1(\Omega_{\lambda+h})} \leq \ & (\lambda+h) \cdot C_{\lambda+h} \cdot \|u_{0,1} - u_{0,2}\|_{L^1(\partial\Omega, \mathcal{H}^1)} \\
& + C_{\lambda+h} \cdot \mathcal{L}^2(\Omega) \cdot \Big((\lambda+h) \cdot \|f_1 - f_2\|_{L^\infty(\Omega_\lambda)} + h \cdot \|f_1 - f_2\|_{L^\infty(\Omega_{\lambda+h})} \Big) \\
& + C_{\lambda+h} \cdot M_{**} \cdot \Big((\lambda+h) \cdot \|c_1 - c_2\|_{L^\infty(\Omega_\lambda)} + h \cdot \|c_1 - c_2\|_{L^\infty(\Omega_{\lambda+h})} \Big) \ .
\end{aligned}$$

Here, the factor

$$C_\lambda := \frac{K_\lambda}{\beta \cdot m_0 \cdot k_\lambda}$$

is an increasing function of λ.

Proof.

We use the approximation of w by w_n again. Because of the weak* convergence according to lemma 4.19 and because of the lower semi-continuity of the total variation w.r.t. the weak* convergence (for the semi-continuity of norms, e.g., see [AB94]), we have

$$\|w\|_{L^1(\Omega_\lambda)} = \left|w \cdot \mathcal{L}^2\right|(\Omega_\lambda) \leq \liminf_{n\to\infty} \left|w_n \cdot \mathcal{L}^2\right|(\Omega_\lambda) = \liminf_{n\to\infty} \|w_n\|_{L^1(\Omega_\lambda)} ,$$

and thus we can estimate $\|w_n\|_{L^1(\Omega_\lambda)}$ instead. Using the representation of w_n by equation (4.3), we obtain

$$\begin{aligned}
\|w_n\|_{L^1(\Omega_{\lambda+h})} &= \int_a^b \int_0^{\lambda+h} |w_n| \circ \xi(t,s) \cdot \det D\xi(t,s)\; dt\; ds \\
&\leq \int_a^b \int_0^{\lambda+h} |\gamma^*(u_{0,1} - u_{0,2})(s)| \cdot \det D\xi(t,s)\; dt\; ds \\
&\quad + \int_a^b \int_0^{\lambda+h} \int_0^t |f_{1,0} - f_2^0| \circ \xi(\tau,s)\; d\tau \cdot \det D\xi(t,s)\; dt\; ds \\
&\quad + \int_a^b \int_0^{\lambda+h} \int_0^t |\left\langle (c_{1,0} - c_2^0), \nabla u_{2,n} \right\rangle| \circ \xi(\tau,s)\; d\tau \cdot \det D\xi(t,s)\; dt\; ds\; .
\end{aligned}$$

By arguing the same way as in the proof of lemma 4.19 for the first summand we get

$$\begin{aligned}
&\int_a^b \int_0^{\lambda+h} |\gamma^*(u_{0,1} - u_{0,2})(s)| \cdot \det D\xi(t,s)\; dt\; ds \\
&\qquad\qquad \leq\; (\lambda + h) \cdot C_{\lambda+h} \cdot \|u_{0,1} - u_{0,2}\|_{L^1(\partial\Omega,\mathcal{H}^1)}\; .
\end{aligned}$$

For the last summand, let

$$g(\tau,s,t) := |c_{1,0} - c_2^0| \circ \xi(\tau,s) \cdot |\nabla u_{2,n}| \circ \xi(\tau,s) \cdot \det D\xi(t,s)\; .$$

Then, we estimate

$$\begin{aligned}
&\int_a^b \int_0^{\lambda+h} \int_0^t |\left\langle (c_{1,0} - c_2^0), \nabla u_{2,n} \right\rangle| \circ \xi(\tau,s)\; d\tau \cdot \det D\xi(t,s)\; dt\; ds \\
&\quad\leq \int_a^b \int_0^{\lambda+h} \int_0^t g(\tau,s,t)\; d\tau\; dt\; ds = \int_a^b \int_0^{\lambda+h} \left(\int_\tau^{\lambda+h} g(\tau,s,t)\; dt \right)\; d\tau\; ds \\
&\quad\leq \int_a^b \int_0^{\lambda} \left(\int_0^{\lambda+h} g(\tau,s,t)\; dt \right)\; d\tau\; ds + \int_a^b \int_\lambda^{\lambda+h} \left(\int_\lambda^{\lambda+h} g(\tau,s,t)\; dt \right)\; d\tau\; ds\; .
\end{aligned}$$

For the inner integrals, we have

$$\begin{aligned}\int\limits_{\tau}^{\lambda+h} g(\tau,s,t)\;dt &= |c_{1,0}-c_2^0|\circ\xi(\tau,s)\cdot|\nabla u_{2,n}|\circ\xi(\tau,s)\int\limits_{\tau}^{\lambda+h}\det D\xi(t,s)\;dt\\ &\le (|c_{1,0}-c_2^0|\cdot|\nabla u_{2,n}|)\circ\xi(\tau,s)\cdot(\lambda+h-\tau)\cdot K_{\lambda+h}\\ &\le (|c_{1,0}-c_2^0|\cdot|\nabla u_{2,n}|)\circ\xi(\tau,s)\cdot\det D\xi(\tau,s)\cdot(\lambda+h-\tau)\cdot\frac{K_{\lambda+h}}{k_{\lambda+h}}\,.\end{aligned}$$

In the next step we take away the scaling factor, which is in the transport fields and the right hand sides of the PDE, by $1/\langle c_1,\nabla T_0\rangle \le 1/(m_0\cdot\beta)$:

$$\begin{aligned}&\le (|c_1-c_2|\cdot|\nabla u_{2,n}|)\circ\xi(\tau,s)\cdot\det D\xi(\tau,s)\cdot(\lambda+h-\tau)\cdot\frac{K_{\lambda+h}}{\beta\cdot m_0\cdot k_{\lambda+h}}\\ &= (|c_1-c_2|\cdot|\nabla u_{2,n}|)\circ\xi(\tau,s)\cdot\det D\xi(\tau,s)\cdot(\lambda+h-\tau)\cdot C_{\lambda+h}\,.\end{aligned}$$

By the last result we infer on the one hand that

$$\begin{aligned}&\int\limits_a^b\int\limits_0^\lambda\int\limits_0^{\lambda+h} g(\tau,s,t)\;dt\;d\tau\;ds\\ &\qquad\le(\lambda+h)\cdot C_{\lambda+h}\int\limits_{\Omega_\lambda}|c_1-c_2|(x)\cdot|\nabla u_{2,n}|(x)\;dx\\ &\qquad\le(\lambda+h)\cdot C_{\lambda+h}\|c_1-c_2\|_{L^\infty(\Omega_\lambda)}\cdot\|\nabla u_{2,n}\|_{L^1(\Omega_\lambda)}\,,\end{aligned}$$

and on the other hand that

$$\begin{aligned}\int\limits_a^b\int\limits_\lambda^{\lambda+h}\int\limits_\lambda^{\lambda+h} g(\tau,s,t)\;dt\;d\tau\;ds &\le h\cdot C_{\lambda+h}\int\limits_{\Omega_{\lambda+h,\lambda}}|c_1-c_2|(x)\cdot|\nabla u_{2,n}|(x)\;dx\\ &\le h\cdot C_{\lambda+h}\|c_1-c_2\|_{L^\infty(\Omega_{\lambda+h})}\cdot\|\nabla u_{2,n}\|_{L^1(\Omega_{\lambda+h})}\,.\end{aligned}$$

Finally, for the summand

$$\int\limits_a^b\int\limits_0^{\lambda+h}\int\limits_0^t |f_{1,0}-f_2^0|\circ\xi(\tau,s)\;d\tau\cdot\det D\xi(t,s)\;dt\;ds\,,$$

we need to perform the same steps with

$$g(\tau,s,t):=|f_{1,0}-f_2^0|\circ\xi(\tau,s)\cdot\mathbb{1}_{\Omega_{\lambda+h}}\circ\xi(\tau,s)\cdot\det D\xi(t,s)\,,$$

and end up with the same estimates, but $\|c_1-c_2\|_{L^\infty(\Omega_{\lambda+h})}$ has to be replaced by $\|f_1-f_2\|_{L^\infty(\Omega_{\lambda+h})}$ and $\|\nabla u_{2,n}\|_{L^1(\Omega_{\lambda+h})}$ has to be replaced by $\mathcal{L}^2(\Omega_{\lambda+h})$.

Summarizing the last considerations we have an estimate for $\|w_n\|_{L^1(\Omega_{\lambda+h})}$:

$$\begin{aligned}\|w_n\|_{L^1(\Omega_{\lambda+h})} \leq\ & (\lambda+h)\, C_{\lambda+h} \cdot \|u_{0,1}-u_{0,2}\|_{L^1(\partial\Omega,\mathcal{H}^1)} \\ & + C_{\lambda+h}\, \mathcal{L}^2(\Omega_{\lambda+h}) \Big((\lambda+h)\cdot \|f_1-f_2\|_{L^\infty(\Omega_\lambda)} + h\cdot \|f_1-f_2\|_{L^\infty(\Omega_{\lambda+h})} \Big) \\ & + C_{\lambda+h}\, \|\nabla u_{2,n}\|_{L^1(\Omega_{\lambda+h})} \Big((\lambda+h)\cdot \|c_1-c_2\|_{L^\infty(\Omega_\lambda)} + h\cdot \|c_1-c_2\|_{L^\infty(\Omega_{\lambda+h})} \Big)\,.\end{aligned}$$

Because $u_{2,n}$ is chosen according to theorem 2.19, we have

$$\|\nabla u_{2,n}\|_{L^1(\Omega_{\lambda+h})} \to |Du_2|(\Omega_{\lambda+h})\,.$$

Hence, going over to the lim inf and plugging in the estimates

$$|Du_2|(\Omega_{\lambda+h}) \leq M_{**} \qquad \text{and} \qquad \mathcal{L}^2(\Omega_{\lambda+h}) \leq \mathcal{L}^2(\Omega)$$

finally yields the assertion.

Because by lemma 4.18 we know that K_λ increases while k_λ decreases with λ, it is clear that C_λ increases with λ. □

Corollary 4.21.
For any choice of $0<\lambda<1$ the operator $U:\mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ is L^1-Lipschitz

$$\|U[v_1]-U[v_2]\|_{L^1(\Omega_\lambda)} \leq \lambda\cdot\kappa_\lambda\cdot\|v_1-v_2\|_{L^1(\Omega_\lambda)}\,.$$

Here, κ_λ is defined by

$$\kappa_\lambda := C_\lambda\cdot\left(L_2\cdot\mathcal{L}^2(\Omega)+L_1\cdot M_{**}\right),$$

and is an increasing function of λ.

Proof.

Let $v_1, v_2 \in \mathfrak{X}_\lambda$. When we consider the operator U, we always use the same boundary data $u_0 \in \mathfrak{B}$. Hence, as mentioned in equation (4.2) of the preparatory part at the beginning of this section, we have $w = u_1 - u_2 = U[v_1]-U[v_2]$, since $u_{0,1} = u_{0,2} = u_0$. By using lemma 4.20 with $h=0$ we see that

$$\begin{aligned}\|U[v_1]-U[v_2]\|_{L^1(\Omega_\lambda)} \leq\ & \lambda\cdot C_\lambda\cdot\Big(\mathcal{L}^2(\Omega)\cdot\|f_1-f_2\|_{L^\infty(\Omega_\lambda)} \\ & \qquad + M_{**}\cdot\|c_1-c_2\|_{L^\infty(\Omega_\lambda)}\Big)\,.\end{aligned}$$

For the differences $f_1 - f_2$ and $c_1 - c_2$ we use the Volterra-type dependence and the Lipschitz conditions, which we require. That is

$$\begin{aligned}\|f_1 - f_2\|_{L^\infty(\Omega_\lambda)} &= \|f[v_1] - f[v_2]\|_{L^\infty(\Omega_\lambda)} = \|(f[v_1] - f[v_2])|_{\Omega_\lambda}\|_\infty \\ &= \|(f[v_1 \cdot \mathbb{1}_{\Omega_\lambda}] - f[v_2 \cdot \mathbb{1}_{\Omega_\lambda}])|_{\Omega_\lambda}\|_\infty \\ &\leq \|f[v_1 \cdot \mathbb{1}_{\Omega_\lambda}] - f[v_2 \cdot \mathbb{1}_{\Omega_\lambda}]\|_\infty \\ &\leq L_2 \cdot \|v_1 \cdot \mathbb{1}_{\Omega_\lambda} - v_2 \cdot \mathbb{1}_{\Omega_\lambda}\|_{L^1(\Omega)} \\ &= L_2 \cdot \|v_1 - v_2\|_{L^1(\Omega_\lambda)}\end{aligned}$$

and analogously

$$\|c_1 - c_2\|_{L^\infty(\Omega_\lambda)} \leq L_1 \cdot \|v_1 - v_2\|_{L^1(\Omega_\lambda)} .$$

Putting everything together, it follows that

$$\begin{aligned}\|U[v_1] - U[v_2]\|_{L^1(\Omega_\lambda)} &\leq \lambda \cdot C_\lambda \cdot (L_2 \cdot \mathcal{L}^2(\Omega) + L_1 \cdot M_{**}) \cdot \|v_1 - v_2\|_{L^1(\Omega_\lambda)} \\ &\leq \lambda \cdot \kappa_\lambda \cdot \|v_1 - v_2\|_{L^1(\Omega_\lambda)} .\end{aligned}$$

Finally, κ_λ increases with λ, since C_λ does so, too. □

Now that we have brought together all ingredients we are able to show the uniqueness of the fixed point.

Theorem 4.22. *(Uniqueness)*
Consider the solution operator

$$U : \mathfrak{X} \to \mathfrak{X}$$

of the (non-restricted) original problem 4.6, where the transport field $c : L^1(\Omega) \to C^1(\Omega \backslash \Sigma)^2$ and the right hand side $f : L^1(\Omega) \to C^1(\overline{\Omega})$ additionally satisfy the requirements 4.13 and 4.14.

Then, the map U has a unique fixed point $u \in \mathfrak{X}$, $u = U[u]$.

Proof.

First, we show that, for any choice of $0 < \lambda < 1$, the domain-restricted operator $U : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ has a unique fixed point. In order to do so we decompose Ω_λ into finitely many stripes $\Omega_{(l+1)h,lh}$ of "thickness" h.

Let the step size h be such that

$$h < \frac{1}{\kappa_\lambda} ,$$

and let

$$L = \left\lfloor \frac{\lambda}{h} \right\rfloor \in \mathbb{N}$$

be the number of steps. Then,

$$\Omega_\lambda = \bigcup_{l=0}^{L-1} \Omega_{(l+1)h,lh} \cup \Omega_{\lambda,Lh} .$$

For the first step, consider the operator $U : \mathfrak{X}_h \to \mathfrak{X}_h$ on $\Omega_h = \Omega_{h,0}$. By corollary 4.21 and the choice of h we have a contraction

$$\|U[v_1] - U[v_2]\|_{L^1(\Omega_h)} \leq h \cdot \kappa_h \cdot \|v_1 - v_2\|_{L^1(\Omega_h)} \leq h \cdot \kappa_\lambda \cdot \|v_1 - v_2\|_{L^1(\Omega_h)} .$$

If now $u_1 = U[u_1]$ and $u_2 = U[u_2]$ are two fixed points, we have, after domain-restriction onto Ω_h,

$$\|u_1 - u_2\|_{L^1(\Omega_h)} \leq h \cdot \kappa_\lambda \cdot \|u_1 - u_2\|_{L^1(\Omega_h)}$$

and consequently

$$0 \leq (1 - h \cdot \kappa_\lambda)\|u_1 - u_2\|_{L^1(\Omega_h)} \leq 0 .$$

Hence, all fixed points coincide on the stripe Ω_h.

Next, we perform an inductive step. Assume that all fixed points coincide on Ω_{lh}, we show that they must also coincide on $\Omega_{(l+1)h} = \Omega_{lh+h}$. Let $u_1 = U[u_1]$ and $u_2 = U[u_2]$ be two fixed points again. With $w = u_1 - u_2$, by lemma 4.20 we know that

$$\begin{aligned}
&\|u_1 - u_2\|_{L^1(\Omega_{(l+1)h})} \leq C_{(l+1)h} \cdot \\
&\left(\mathcal{L}^2(\Omega) \cdot \left((l+1)h \cdot \|f[u_1] - f[u_2]\|_{L^\infty(\Omega_{lh})} + h \cdot \|f[u_1] - f[u_2]\|_{L^\infty(\Omega_{(l+1)h})}\right)\right. \\
&\left. + M_{**} \cdot \left((l+1)h \cdot \|c[u_1] - c[u_2]\|_{L^\infty(\Omega_{lh})} + h \cdot \|c[u_1] - c[u_2]\|_{L^\infty(\Omega_{(l+1)h})}\right)\right) .
\end{aligned}$$

Because u_1 and u_2 coincide on Ω_{lh}, we have

$$\|f[u_1] - f[u_2]\|_{L^\infty(\Omega_{lh})} = 0 \quad \text{and} \quad \|c[u_1] - c[u_2]\|_{L^\infty(\Omega_{lh})} = 0 ,$$

and thus, the estimate reduces to

$$\begin{aligned}
\|u_1 - u_2\|_{L^1(\Omega_{(l+1)h})} \leq C_{(l+1)h} \cdot h \cdot (\mathcal{L}^2(\Omega) \cdot \|f[u_1] - f[u_2]\|_{L^\infty(\Omega_{(l+1)h})} \\
+ M_{**} \cdot \|c[u_1] - c[u_2]\|_{L^\infty(\Omega_{(l+1)h})}) .
\end{aligned}$$

By using the Lipschitz conditions on c and f again we have

$$\begin{aligned}
\|u_1 - u_2\|_{L^1(\Omega_{(l+1)h})} &\leq h\kappa_{(l+1)k} \cdot \|u_1 - u_2\|_{L^1(\Omega_{(l+1)h})} \\
&\leq h\kappa_\lambda \cdot \|u_1 - u_2\|_{L^1(\Omega_{(l+1)h})} .
\end{aligned}$$

More precisely, because we have assumed that u_1 and u_2 coincide on Ω_{lh}, the latter inequality in fact means

$$\|u_1 - u_2\|_{L^1(\Omega_{(l+1)h,lh})} \le h\kappa_\lambda \cdot \|u_1 - u_2\|_{L^1(\Omega_{(l+1)h,lh})} \;.$$

By the contractiveness, $h\kappa_\lambda < 1$, we see that $\|u_1 - u_2\|_{L^1(\Omega_{(l+1)h,lh})} = 0$ and so the fixed points also coincide on the next stripe $\Omega_{(l+1)h,lh}$.

For the last stripe we have to adapt the step size to

$$\hat{h} = \lambda - Lh \;.$$

But, since $\hat{h} \le h$, the same argumentation applies.

As claimed before, the domain-restricted operator $U : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ has a unique fixed point for any choice of $0 < \lambda < 1$. Now the last step: assume by contradiction that the non-restricted operator $U : \mathfrak{X} \to \mathfrak{X}$ has two different fixed points, u_1 and u_2. Therefor, u_1 and u_2 must differ on a subset $W \subset \Omega$ with $\mathcal{L}^2(W) \neq 0$. Because the stop set Σ has Lebesgue measure zero, $\mathcal{L}^2(\Sigma) = 0$, we can choose $0 < \lambda < 1$ so close to 1 that $\mathcal{L}^2(\Omega_\lambda \cap W) \neq 0$. Thus, we have

$$\|u_1 - u_2\|_{L^1(\Omega_\lambda)} = \|u_1 - u_2\|_{L^1(\Omega_\lambda \cap W)} \neq 0 \;.$$

But, because $u_1|_{\Omega_\lambda}$ and $u_2|_{\Omega_\lambda}$ are fixed points of the domain-restricted operator $U : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$, we also have

$$\|u_1 - u_2\|_{L^1(\Omega_\lambda)} = 0$$

by the previous uniqueness proof. A contradiction. □

4.3.3 Continuous Dependence of the Fixed Point

In this section, we will show that the unique fixed point depends L^1-continuously on the following data: the transport field, the right hand side, and the boundary data. We consider two linear problems:

$$\begin{aligned} \langle c[v](x), Du\rangle &= f[v](x) \cdot \mathcal{L}^2 \quad \text{in} \quad \Omega\backslash\Sigma \;, \\ u|_{\partial\Omega} &= u_0 \;, \end{aligned}$$

and

$$\begin{aligned} \langle \tilde{c}[\tilde{v}](x), D\tilde{u}\rangle &= \tilde{f}[\tilde{v}](x) \cdot \mathcal{L}^2 \quad \text{in} \quad \Omega\backslash\Sigma \;, \\ \tilde{u}|_{\partial\Omega} &= \tilde{u}_0 \;, \end{aligned}$$

where we assume that for both problems the same domain Ω, the same stop set Σ, and the same time function T (with transformed version T_0) are specified.

Moreover, we assume that c and $\tilde{c}$ both satisfy the requirements 4.1 and 4.13 with the same bounds, and that f and $\tilde{f}$ both satisfy the requirements 4.2 and 4.14 with the same bounds. Finally, we assume $u_0 \in \mathfrak{B}$ and $\tilde{u}_0 \in \mathfrak{B}$. By the latter assumptions we are sure that we obtain two solution operators

$$
\begin{aligned}
&U : \mathfrak{X} \to \mathfrak{X} \,, \qquad && v \to U[v] \,, \\
&\tilde{U} : \mathfrak{X} \to \mathfrak{X} \,, \qquad && \tilde{v} \to \tilde{U}[\tilde{v}] \,,
\end{aligned}
$$

which respectively correspond to the two linear problems above and possess the same domain and range $\mathfrak{X}$, which depends on all those bounds.

We view $\tilde{c}$ and $\tilde{f}$ as perturbed versions of c and f. In order to measure the perturbation we introduce the following norm:

Definition 4.23.
For maps of type $g : L^1(\Omega) \to C_b(\Omega \backslash \Sigma)^d$ or of type $g : L^1(\Omega) \to C(\overline{\Omega})^d$, $d \in \mathbb{N}$, we define the norm

$$
\|g\|_0 := \sup_{v \in L^1(\Omega)} \|g[v]\|_\infty \,.
$$

Theorem 4.24. *(Continuous dependence)*
Consider the two solution operators U and $\tilde{U}$ as described above. Let u and $\tilde{u}$ be fixed points of these operators, i.e.,

$$
u = U[u] \,, \qquad \qquad \tilde{u} = \tilde{U}[\tilde{u}] \,.
$$

Then, for every $\varepsilon > 0$, one can find $\delta > 0$ such that

$$
\|u - \tilde{u}\|_{L^1(\Omega)} \leq \varepsilon \,,
$$

whenever

$$
\Big(\|u_0 - \tilde{u}_0\|_{L^1(\partial\Omega, \mathcal{H}^1)} \; + \; \mathcal{L}^2(\Omega) \cdot \|f - \tilde{f}\|_0 \; + \; M_{**} \cdot \|c - \tilde{c}\|_0 \Big) \; \leq \; \delta \,.
$$

Proof.

Let $v, \tilde{v} \in \mathfrak{X}$ be arbitrary but fixed. Let

$$
u_1 := U[v] \qquad \text{and} \qquad u_2 := \tilde{U}[\tilde{v}] \,.
$$

As before, we set up a PDE for the difference $w := u_1 - u_2$ on the restricted domain Ω_λ, $0 < \lambda < 1$:

$$\begin{aligned} \langle c[v](x), Dw \rangle &= (f[v] - \tilde{f}[\tilde{v}])(x) \cdot \mathcal{L}^2 - \langle (c[v] - \tilde{c}[\tilde{v}])(x), Du_2 \rangle \quad \text{in} \quad \Omega_\lambda \, , \\ w|_{\partial\Omega} &= u_0 - \tilde{u}_0 \, . \end{aligned}$$

Again, we choose a sequence $u_{2,n} \in C^\infty(\Omega)$ which strictly converges to u_2 in $BV(\Omega)$. And again, the sequence w_n of solutions to the approximate PDE, which has $\nabla u_{2,n}(x) \cdot \mathcal{L}^2$ instead of Du_2, converges weakly* to w.

In order to proceed as in lemma 4.20 we rewrite the right hand side of the PDE to

$$\begin{aligned} \langle c[v](x), Dw_n \rangle = &\left((f[v] - f[\tilde{v}]) - \langle c[v] - c[\tilde{v}], \nabla u_{2,n} \rangle \right)(x) \cdot \mathcal{L}^2 \\ &+ \left((f[\tilde{v}] - \tilde{f}[\tilde{v}]) - \langle c[\tilde{v}] - \tilde{c}[\tilde{v}], \nabla u_{2,n} \rangle \right)(x) \cdot \mathcal{L}^2 \, . \end{aligned}$$

For the first summand of the new right hand side we will apply the steps from the proof of lemma 4.20. For the second summand we operate in a simpler way. Let $\xi = \xi[v]$ be the characteristics corresponding to the field $c[v]_0$ and let

$$g(x) = \frac{\left| (f[\tilde{v}] - \tilde{f}[\tilde{v}]) - \langle c[\tilde{v}] - \tilde{c}[\tilde{v}], \nabla u_{2,n} \rangle \right|}{\langle c[v], \nabla T_0 \rangle}(x) \, .$$

As in the proof of lemma 4.20 we have to estimate

$$\int_a^b \int_0^{\lambda+h} \int_0^t g \circ \xi(\tau, s) \, d\tau \det D\xi(t, s) \, dt \, ds \leq \ldots \, .$$

After having changed the order of integration and having estimated the determinant, we obtain

$$\begin{aligned} \ldots \leq & (\lambda + h) \cdot \frac{K_{\lambda+h}}{k_{\lambda+h}} \cdot \int_a^b \int_0^{\lambda+h} g \circ \xi(\tau, s) \det D\xi(\tau, s) \, d\tau \, ds \\ \leq & (\lambda + h) \cdot C_{\lambda+h} \cdot \int_{\Omega_{\lambda+h}} \left| (f[\tilde{v}] - \tilde{f}[\tilde{v}]) - \langle c[\tilde{v}] - \tilde{c}[\tilde{v}], \nabla u_{2,n} \rangle \right| (x) \, dx \\ \leq & (\lambda + h) \cdot C_{\lambda+h} \left(\mathcal{L}^2(\Omega) \cdot \| f[\tilde{v}] - \tilde{f}[\tilde{v}] \|_\infty + \| \nabla u_{2,n} \|_{L^1(\Omega)} \cdot \| c[\tilde{v}] - \tilde{c}[\tilde{v}] \|_\infty \right) \\ \leq & (\lambda + h) \cdot C_{\lambda+h} \left(\mathcal{L}^2(\Omega) \cdot \| f - \tilde{f} \|_0 + \| \nabla u_{2,n} \|_{L^1(\Omega)} \cdot \| c - \tilde{c} \|_0 \right) \, . \end{aligned}$$

Putting both estimates together and then going over to the lim inf, we end up with

$$\begin{aligned}
\|w\|_{L^1(\Omega_{\lambda+h})} \leq\ & (\lambda+h)\cdot C_{\lambda+h}\cdot \|u_0-\tilde{u}_0\|_{L^1(\partial\Omega,\mathcal{H}^1)} \\
&+ C_{\lambda+h}\cdot \mathcal{L}^2(\Omega)\left((\lambda+h)\cdot\|f[v]-f[\tilde{v}]\|_{L^\infty(\Omega_\lambda)} + h\cdot\|f[v]-f[\tilde{v}]\|_{L^\infty(\Omega_{\lambda+h})}\right) \\
&+ C_{\lambda+h}\cdot M_{**}\left((\lambda+h)\cdot\|c[v]-c[\tilde{v}]\|_{L^\infty(\Omega_\lambda)} + h\cdot\|c[v]-c[\tilde{v}]\|_{L^\infty(\Omega_{\lambda+h})}\right) \\
&+ (\lambda+h)\cdot C_{\lambda+h}\left(\mathcal{L}^2(\Omega)\cdot\|f-\tilde{f}\|_0 + M_{**}\cdot\|c-\tilde{c}\|_0\right)\ .
\end{aligned}$$

Now we can show the continuous dependence in the domain-restricted situation. Fix $0<\lambda<1$, choose a step size $0<h<1/\kappa_\lambda$ and let

$$L = \left\lfloor \frac{\lambda}{h} \right\rfloor \in \mathbb{N}$$

be the number of steps. Furthermore, let

$$\left(\|u_0-\tilde{u}_0\|_{L^1(\partial\Omega,\mathcal{H}^1)}\ +\ \mathcal{L}^2(\Omega)\cdot\|f-\tilde{f}\|_0\ +\ M_{**}\cdot\|c-\tilde{c}\|_0\right)\ \leq\ \delta\ ,$$

for some $\delta>0$. Let $l\in\mathbb{N}_0$, $l\leq L$. With the result above we estimate on the set $\Omega_{(l+1)h}$:

$$\begin{aligned}
\|w\|_{L^1(\Omega_{(l+1)h})} \leq\ & \lambda\cdot C_\lambda\cdot\delta \\
&+ C_\lambda\cdot\mathcal{L}^2(\Omega)\cdot\left(\lambda\cdot\|f[v]-f[\tilde{v}]\|_{L^\infty(\Omega_{lh})} + h\cdot\|f[v]-f[\tilde{v}]\|_{L^\infty(\Omega_{(l+1)h})}\right) \\
&+ C_\lambda\cdot M_{**}\cdot\left(\lambda\cdot\|c[v]-c[\tilde{v}]\|_{L^\infty(\Omega_{lh})} + h\cdot\|c[v]-c[\tilde{v}]\|_{L^\infty(\Omega_{(l+1)h})}\right)\ .
\end{aligned}$$

By using the Lipschitz condition on c and f and the definition of κ_λ from corollary 4.21 we obtain

$$\|w\|_{L^1(\Omega_{(l+1)h})} \leq\ \lambda C_\lambda\cdot\delta + \lambda\kappa_\lambda\cdot\|v-\tilde{v}\|_{L^1(\Omega_{lh})} + h\kappa_\lambda\cdot\|v-\tilde{v}\|_{L^1(\Omega_{(l+1)h})}\ .$$

Let $\hat{\delta}=\lambda\cdot C_\lambda\cdot\delta$. Now, we plug in the two fixed points u and $\tilde{u}$, i.e., we set $u_1=v=u$ and $u_2=\tilde{v}=\tilde{u}$,

$$\|u-\tilde{u}\|_{L^1(\Omega_{(l+1)h})} \leq \hat{\delta} + \lambda\cdot\kappa_\lambda\cdot\|u-\tilde{u}\|_{L^1(\Omega_{lh})} + h\cdot\kappa_\lambda\cdot\|u-\tilde{u}\|_{L^1(\Omega_{(l+1)h})}\ .$$

We define the error on the set Ω_{lh} to be

$$e_l := \|u-\tilde{u}\|_{L^1(\Omega_{lh})}\ .$$

Then, by our choice of h, the last estimate yields the error recursion

$$0 \leq (1-h\kappa_\lambda)\cdot e_{l+1} \leq \hat{\delta} + \lambda\cdot\kappa_\lambda\cdot e_l\ ,$$

which leads to

$$e_{l+1} \leq \sum_{k=0}^{l} \alpha^k \cdot \frac{\hat{\delta}}{1 - h\kappa_\lambda} \quad \text{with} \quad \alpha := \frac{\lambda \cdot \kappa_\lambda}{1 - h\kappa_\lambda} \,.$$

In summary, we get

$$\|u - \tilde{u}\|_{L^1(\Omega_\lambda)} \leq e_{L+1} \leq \left(\frac{1 - \alpha^{L+1}}{1 - \alpha} \cdot \frac{\lambda \cdot C_\lambda}{1 - h\kappa_\lambda} \right) \cdot \delta$$

and the continuous dependence is obvious.

Let $\varepsilon > 0$. For the full domain Ω we choose λ so close to 1 that

$$\|u - \tilde{u}\|_{L^1(\Omega \setminus \Omega_\lambda)} \leq \frac{\varepsilon}{2} \,.$$

In dependence of this λ we find h and L. What remains to do is to require

$$\delta = \left(\frac{1 - \alpha^{L+1}}{1 - \alpha} \cdot \frac{\lambda \cdot C_\lambda}{1 - h\kappa_\lambda} \right)^{-1} \cdot \frac{\varepsilon}{2} \,,$$

then, we get

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq \varepsilon \,,$$

whenever

$$\Big(\|u_0 - \tilde{u}_0\|_{L^1(\partial\Omega, \mathcal{H}^1)} \;+\; \mathcal{L}^2(\Omega) \cdot \|f - \tilde{f}\|_0 \;+\; M_{**} \cdot \|c - \tilde{c}\|_0 \Big) \;\leq\; \delta \,.$$

$\square$

Chapter 5

Extensions

In chapters 3 and 4 we restricted the discussion to simply connected domains with time functions that have a connected tree-like stop set. In this chapter we will weaken these requirements.

5.1 Extended Concept of Time Functions

We begin by weakening the requirements on time functions, while Ω is assumed to be a domain according to requirement 3.1. Let us consider an example: let $T \in C^2(\overline{\Omega})$ be a function which is zero on the boundary and positive in the interior of Ω. Here, we define Σ to equal the set of stationary points

$$\Sigma := \{x \in \Omega : \nabla T(x) = 0\} \; .$$

For the sake of simplicity, we assume in this example that Σ consists of exactly three points. Two of them are global maxima with the T-value equal to 1, and the remaining one is a saddle point, with its T-value equal to 0.5. The three important levels of T might look as in figure 5.1. Let us now consider the linear problem with Ω, T, and Σ as set up above:

$$\begin{aligned} \langle c(x), Du \rangle &= f(x) \cdot \mathcal{L}^2 \,, \qquad \text{in} \quad \Omega \backslash \Sigma \,, \\ u|_{\partial\Omega} &= u_0 \; . \end{aligned}$$

Here, as in requirement 3.6, c is defined where $N = \nabla T / |\nabla T|$ is defined. Moreover, c is at least as smooth and as extendable as N.

In the situation described above we use the saddle level as an intermediate stop set and solve on the restricted domain $\Omega_{0.5} = \{x \in \Omega : 0 < T(x) < 0.5\}$. Note that on $\Omega_{0.5}$ the map T is a time function in accordance with requirement 3.4. Let u_1 denote the solution on $\Omega_{0.5}$.

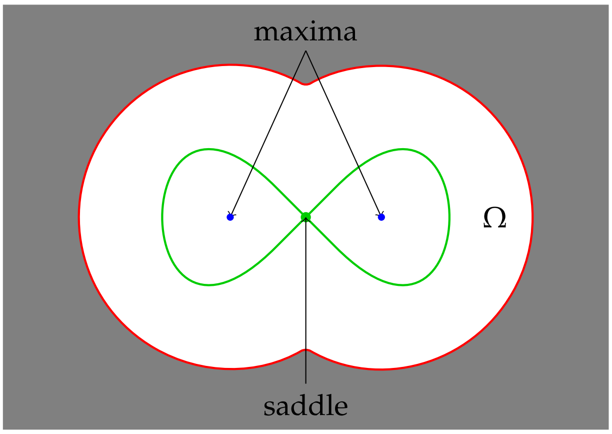


Figure 5.1: white: domain Ω, red: start level $T = 0$, green: saddle level $T = 0.5$, blue: maximal level $T = 1$.

After having reached the saddle level, the remaining part $\Omega_{0.5,1}$ is a disjoint union of two sets, a left one Ω^l and right one Ω^r:

$$\Omega_{0.5,1} = \{x \in \Omega : 0.5 < T(x) < 1\} = \Omega^l \cup \Omega^r \, .$$

According to this domain split we have two new boundaries $\partial\Omega^l$ and $\partial\Omega^r$. We will solve on the remainder by considering two restarted problems

$$\begin{aligned} \left\langle c(x), Du^l \right\rangle &= f(x) \cdot \mathcal{L}^2 \, , \qquad \text{in} \quad \Omega^l \backslash \Sigma \, , \\ u^l|_{\partial\Omega^l} &= u_1|_{\partial\Omega^l} \, , \end{aligned}$$

and

$$\begin{aligned} \langle c(x), Du^r \rangle &= f(x) \cdot \mathcal{L}^2 \, , \qquad \text{in} \quad \Omega^r \backslash \Sigma \, , \\ u^r|_{\partial\Omega^r} &= u_1|_{\partial\Omega^r} \, . \end{aligned}$$

Here, from the perspective of Ω^l and Ω^r, the traces $u_1|_{\partial\Omega^l}$ and $u_1|_{\partial\Omega^r}$ are traces from the outside, since the already computed u_1 is defined on $\Omega_{0.5}$.

If we are able to solve the two restarted problems, we can define a global solution by

$$u = \mathbb{1}_{\Omega_{0.5}} u_1 + \mathbb{1}_{\Omega^l} u^l + \mathbb{1}_{\Omega^r} u^r \, .$$

But some difficulties arise at the saddle level. First let us define the new boundary data.

Consider figure 5.2. Let $s_1 \in [a, b]$ be such that $\xi(\,.\,, s_1) : [0, 0.5[\, \to \Omega$ is the left most characteristic, going bottom-up into the saddle. And, let $s_2 \in [a, b]$

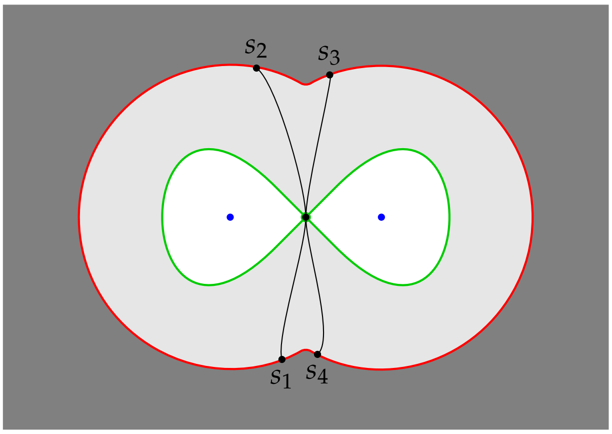


Figure 5.2: light gray: domain of u_1, green: saddle level $T = 0.5$, white: Ω^l and Ω^r, black: left and right most characteristics hitting the saddle point.

be such that $\xi(\,.\,, s_2) : [0, 0.5[\,\to \Omega$ is the left most characteristic, going top-down into the saddle. Then,

$$\xi(0.5, \,.\,) :]s_1, s_2[\,\to \Omega$$

parametrizes $\partial\Omega^l \backslash \{z\}$, whereas $z \in \Omega$ denotes the saddle point. Analogously, we choose s_3 and s_4 such that

$$\xi(0.5, \,.\,) :]s_3, s_4[\,\to \Omega$$

parametrizes $\partial\Omega^r \backslash \{z\}$. According to lemma 3.15 we obtain the new boundary data by

$$\begin{array}{lll} \text{on} \quad \partial\Omega^l \backslash \{z\} & : & u \circ \xi(0.5, s) \,, \quad s \in \,]s_1, s_2[\,, \\ \text{on} \quad \partial\Omega^r \backslash \{z\} & : & u \circ \xi(0.5, s) \,, \quad s \in \,]s_3, s_4[\,. \end{array}$$

Now, that everything is set up for the restart, let us look at the possible difficulties. As we can see in figure 5.2, the new domains might have a corner at the saddle z. This is certainly the case if T looks like

$$T(x, y) = \frac{1 + x^2 - y^2}{2} \,, \quad (x, y) \in B_\varepsilon(z) \,, \quad z = 0$$

in the neighborhood of the saddle z. This means that Ω^l and Ω^r are not domains in accordance with requirements 3.1. Moreover, it is possible that there are infinitely many backward characteristics that, let's say, start in Ω^r and meet the saddle node z, as it was the case for the forward characteristics on $\Omega_{0.5}$. See figure 5.2, where all characteristics $\xi(\,.\,, s)$ for $s \in [s_2, s_3]$ meet z.

As an example, we could use, on $(B_\varepsilon(0)\cap\Omega^r)\backslash\{0\}$, the transport field

$$\begin{aligned} c_*(x,y) &= b\left(\arctan\left(\frac{y}{x}\right)\right)\begin{pmatrix}x\\y\end{pmatrix}+\left(1-b\left(\arctan\left(\frac{y}{x}\right)\right)\right)\nabla T(x,y)\\ &= b\left(\arctan\left(\frac{y}{x}\right)\right)\begin{pmatrix}x\\y\end{pmatrix}+\left(1-b\left(\arctan\left(\frac{y}{x}\right)\right)\right)\begin{pmatrix}x\\-y\end{pmatrix}\\ c(x,y) &= \frac{c_*(x,y)}{|c_*(x,y)|} \end{aligned}$$

with the following C^1-blending function

$$\begin{aligned} b(t) &= b_0(t+\pi/4)-b_0(t-\pi/4+a)\ ,\\ b_0(t) &= \begin{cases} 0 & ,t\le 0\\ 2\frac{t^2}{a^2} & ,0<t\le a/2\\ -\frac{(2t-a)^2-2t^2}{a^2} & ,a/2<t\le a\\ 1 & ,a<t \end{cases}\ ,\\ a &= 0.1\ . \end{aligned}$$

The graph of the blending function is shown in figure 5.3 . And, using polar

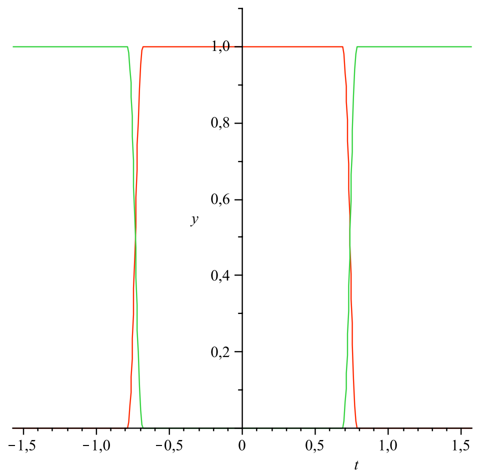


Figure 5.3: Graphs of the blending functions: Red: b, Green: $1-b$.

coordinates $(x,y)=r\cdot(\cos\varphi,\sin\varphi)$ with $(r,\varphi)\in\,]0,\varepsilon[\,\times[-\pi/2,\pi/2]$, it is easy to check that

$$|c_*(x,y)|\le r$$

and that

$$\langle c(x,y),N(x,y)\rangle\ge 1-2b(\varphi)\sin(\varphi)^2>0.15\ .$$

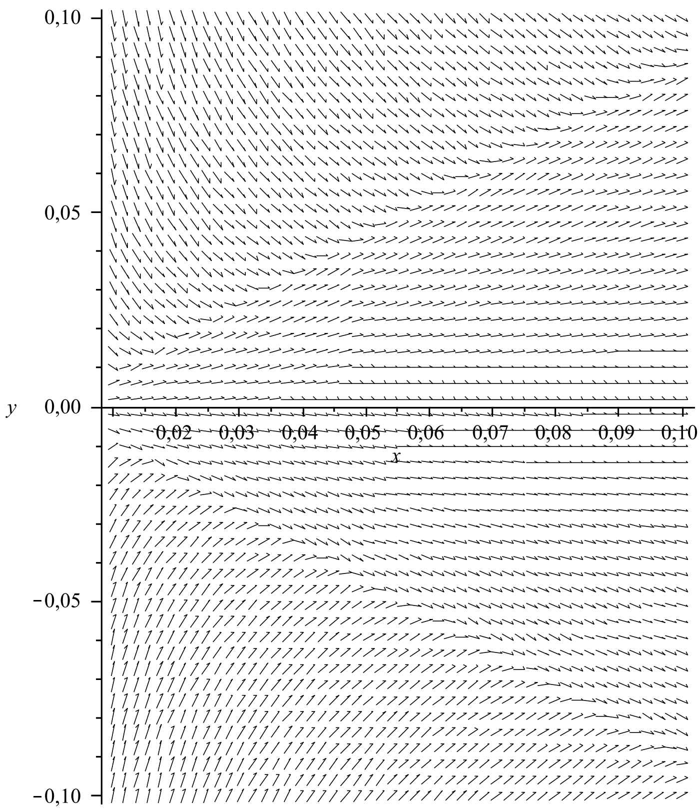


Figure 5.4: Field plot of c.

So, c is an admissible transport field. For illustration the transport field c, on $(B_\varepsilon(0) \cap \Omega^r)\backslash\{0\}$, is plotted in figure 5.4.

By construction, for $\varphi \in [-\pi/4 + a, \pi/4 - a]$, c is given by

$$c(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} .$$

And consequently, the backward characteristics, which solve the IVPs

$$\eta' = -c(\eta) , \qquad \eta(0) \in \Omega^r ,$$

all end in the saddle point $z = 0$ if $\varphi \in [-\pi/4 + a, \pi/4 - a]$. One the other hand, all forward characteristics which start in Ω^r must meet at that maximum of T which is located in Ω^r. Thus, there is a whole area $A \subset \Omega^r$ drawn by characteristics which all connect the saddle and the maximum (see figure 5.5).

The problem, herein, is that the solution u^r might suffer from non-uniqueness. Consider the PDE with $f \equiv 0$. Then the partial solution u_1 is given

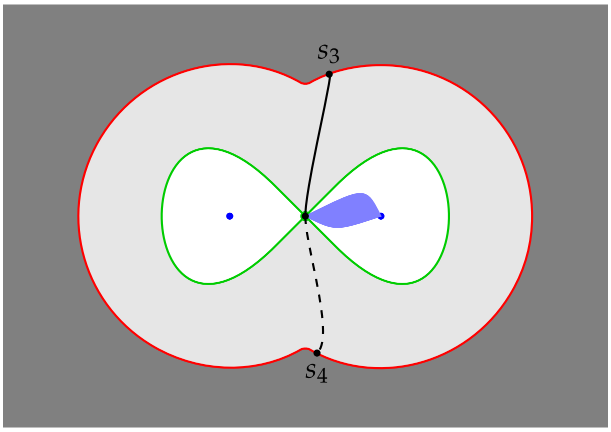


Figure 5.5: black: characteristic which carries the value u^3, dashed black: characteristic which carries the value u^4, light blue: the area drawn by characteristics which connect the saddle and the maximum in Ω^r.

by

$$u_1 \circ \xi(t,s) = \gamma^* u_0(s) ,$$

which implies that at the saddle z, coming from $\Omega_{0.5}$, the two function values

$$u^3 := u_1 \circ \xi(0.5+, s_3+) = \gamma^* u_0(s_3+)$$

and

$$u^4 := u_1 \circ \xi(0.5+, s_4-) = \gamma^* u_0(s_4-)$$

meet. If these two values are equal, we just set $u^r|_A = u^3 = u^4$. But in the case of a jump, i.e.,

$$u^3 \neq u^4 ,$$

the solution u^r on Ω^r cannot be uniquely defined on A, because any characteristic through the set A could carry this jump (see figure 5.6). Another possible way to define u^r on A is to fan out the interval $[u^3, u^4]$ of possible values over all characteristics going through A.

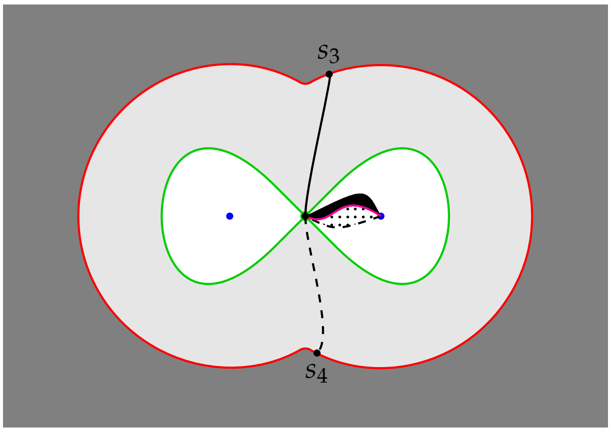


Figure 5.6: black area: area with $u^r = u^3$, dotted area: area with $u^r = u^4$, magenta: an arbitrary characteristic which "carries" the jump from u^3 to u^4.

In order to resolve this difficulty we only allow for saddle points such that, on the one hand, the subsets Ω^l and Ω^r are domains according to requirement 3.1, and on the other hand, the restricted field of normals

$$N : \Omega^l \to S^1 , \qquad\qquad N : \Omega^r \to S^1 ,$$

as well as the restricted transport field

$$c : \Omega^l \to \mathbb{R}^2 , \qquad\qquad c : \Omega^r \to \mathbb{R}^2$$

extend onto the boundaries $\partial\Omega^l$, $\partial\Omega^r$. Hence, the allowed saddle levels look as illustrated in figures 5.7, 5.8, and 5.9 .

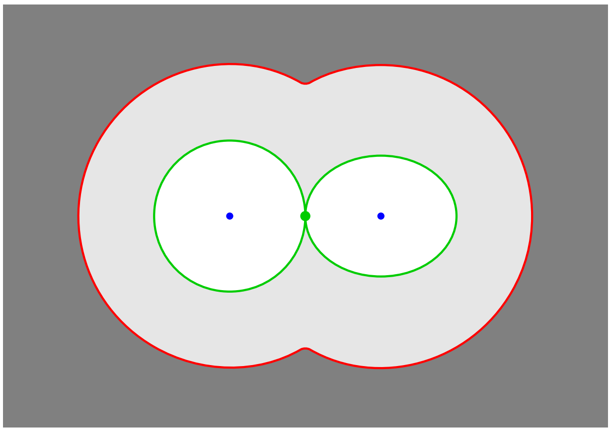


Figure 5.7: white: domain Ω, red: start level $T = 0$, green: saddle level, blue: maximal level.

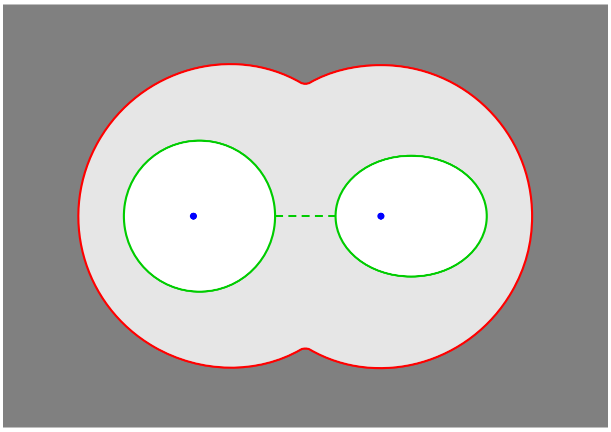


Figure 5.8: white: domain Ω, red: start level $T = 0$, green + dashed green: saddle level, green: restart lines (boundaries of Ω^l and Ω^r), dashed green: stop set for u_1, blue: maximal level.

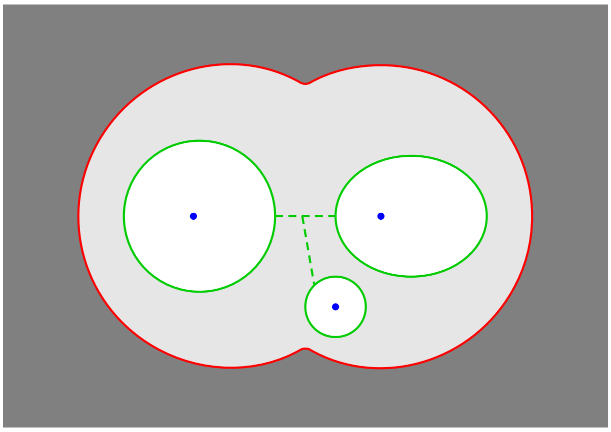


Figure 5.9: white: domain Ω, red: start level $T = 0$, green + dashed green: saddle level, green: restart lines, dashed green: stop set for u_1, blue: maximal level.

Figure 5.8 shows a situation where the saddle level is not completely restarted. Only the completely green parts, which are the boundaries of Ω^l and Ω^r, are to be restarted, while the dashed green line is an effective stop set of the partial solution u_1. Figure 5.9 shows a similar case, which illustrates how we can model a "triple saddle" and stay within the restriction made above.

A situation which we cannot allow for is displayed in figure 5.10. There, the intersection $\partial\Omega^l \cap \partial\Omega^r$ is a line segment and not just a single point. The

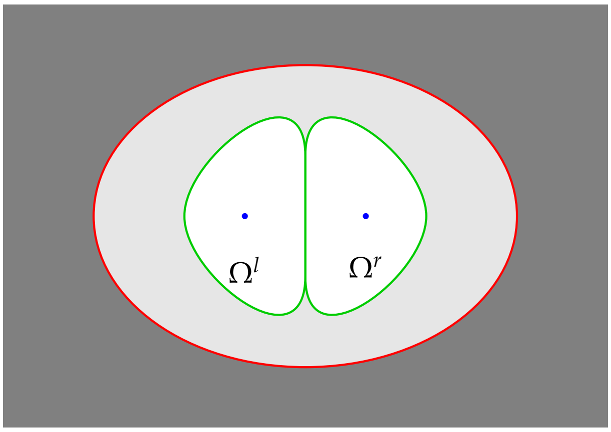


Figure 5.10: light gray + white: domain Ω, red: start level $T = 0$, green: saddle level, blue: maximal level.

difficulty, which arises here, is that the relative interior of this line segment can never be reached by forward characteristics coming from $\Omega \backslash \overline{(\Omega^l \cup \Omega^r)}$. Thus, on this line segment, we do not have data to supply the restarted problems with. A second perspective of the same difficulty: the normal field N does not (one-sided) extend, forward in time, onto $\partial\Omega^l \cap \partial\Omega^r$, i.e.,

$$N : \Omega \backslash \overline{(\Omega^l \cup \Omega^r)} \to S^1$$

does not extend. And thus, considering the linear problem on $\Omega \backslash \overline{(\Omega^l \cup \Omega^r)}$, the set $\partial\Omega^l \cap \partial\Omega^r$ is not a stop set in the sense of chapter 3, since part 2a) of requirement 3.5 does not hold.
However, the extensions backwards in time exist, i.e.,

$$N : \Omega^l \to S^1 \quad \text{and} \quad N : \Omega^r \to S^1$$

have extensions onto $\partial\Omega^l \cap \partial\Omega^r$.

Finally, we summarize how to proceed in the general case. Let Ω be a domain according to 3.1. Then, let $T : \overline{\Omega} \to \mathbb{R}_0^+$ be a continuous function which strictly increases into the interior of Ω and has

- no minima,
- no plateaus,
- no saddle segments as in figure 5.10.

As in the previous chapters we assume $T|_{\partial\Omega} = 0$. We define the set Σ by

$$\Sigma := \{x \in \Omega : x \text{ is a local maximum or a saddle point of } T\} .$$

Σ need not be connected, but consists of $n \in \mathbb{N}$ connected components Σ^k:

$$\Sigma = \bigcup_{k=1}^{n} \Sigma^k .$$

Any component Σ^k plays the role of a stop set and is thus assumed to satisfy requirement 3.2. Moreover, T is constant on each Σ^k:

$$T|_{\Sigma^k} = c_k = \quad \text{const.}$$

Remark: if Σ^k is of the "saddle set" type like the green dashed lines in figures 5.8 and 5.9, then its relative interior $\mathring{\Sigma}^k$ is locally maximal, while only its terminal nodes behave like saddle points. So, such terminal nodes are also called saddle points. Moreover, such saddle sets might degenerate to the case $\mathring{\Sigma}^k = \emptyset$. The latter means that $\Sigma^k = \{z^k\}$ is an isolated saddle point, as shown in the example of figure 5.7.
If Σ^k is a saddle set which contains more than one point and every point of which behaves like a saddle point, then Σ^k will be a saddle segment as in figure 5.10. However, this case has been excluded.

If Σ^k is a saddle set, then

$$\chi_{T=c_k} := \{x \in \Omega : T(x) = c_k\}$$

is the corresponding saddle level of T. It can happen that $\chi_{T=c_k}$ is not connected. If this is the case, we decompose $\chi_{T=c_k}$ into its connected components and discard that components which do not contain any Σ^j. Then, for every $\Sigma^j \subset \chi_{T=c_k}$ we have exactly one connected component L^j of $\chi_{T=c_k}$.
The reason for the non-connectedness is: $\chi_{T=c_k}$ must contain at least Σ^k, but can contain more than one of the Σ^j, since $c_k = c_j$ is admissible. Moreover, $\chi_{T=c_k}$ can have some connected components which do not contain any Σ^j (see figure 5.11). Each one of the defined L^j has a stop part Σ^j and, thus, a necessary restart part $L^j \backslash \mathring{\Sigma}^j$. While all other connected components of $\chi_{T=c_k}$ do not have a stop part, and, thus, represent unnecessary restarts. That is why we discard the latter. Now, we decompose the domain Ω into $m \in \mathbb{N}$ disjoint open components Ω^i by cutting along the restart sets $L^k \backslash \mathring{\Sigma}^k$ (which conform to the fully green lines in figures 5.7, 5.8, and 5.9):

$$\Omega \backslash \bigcup_{k=1}^{n} \left(L^k \backslash \mathring{\Sigma}^k \right) = \bigcup_{i=1}^{m} \Omega^i .$$

For each component Ω^i the start and the stop "times" are given by

$$T_-^i = \min_{x \in \overline{\Omega^i}} T(x) \quad \text{and} \quad T_+^i = \max_{x \in \overline{\Omega^i}} T(x)$$

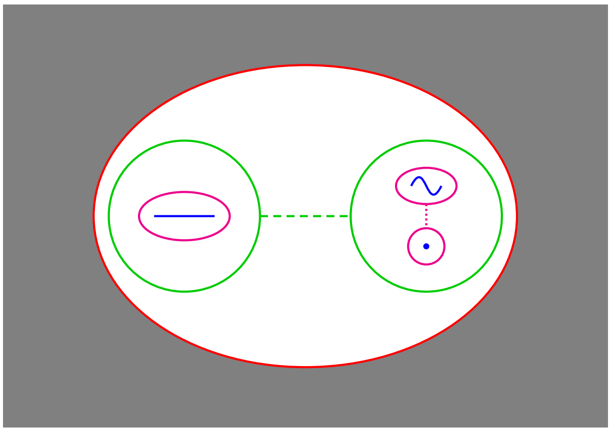


Figure 5.11: white: domain Ω, red: boundary $\partial\Omega$ = level with $T = 0$, green + dashed green: level with $T = c_1 > 0$, magenta + dashed magenta: level with $T = c_2 > c_1$, blue: maximal level with $T = c_3 > c_2$, dashed green: first stop set, green: necessary restart set, dashed magenta: second stop set, magenta right: necessary restart sets, magenta left: unnecessary restart set, blue: last stop sets.

respectively. By these values, we define the start / restart sets

$$\Gamma^i_- = \{x \in \overline{\Omega^i} : T(x) = T^i_-\} ,$$

and the stop / intermediate stop sets

$$\Gamma^i_+ = \{x \in \overline{\Omega^i} : T(x) = T^i_+\} .$$

Any start set Γ^i_- is required to be a simple closed C^1-curve.

For each component Ω^i we assume that $T : \Omega^i \to \mathbb{R}$ satisfies requirement 3.4 parts 1, 2, 3, 5, and 6* with Γ^i_- and Γ^i_+ instead of $\partial\Omega$ and Σ. A reasonable replacement of the growth condition, in part 4 of requirement 3.4, is implicitly satisfied by our decomposition of Ω. Every intermediate stop set Γ^i_+ contains exactly one of the stop sets Σ^k. With respect to Σ^k we assume that requirement 3.5 part 1 holds true. Part 2 of requirement 3.5 is assumed to hold on all of Γ^i_+. Finally, for the transport field c restricted to Ω^i, we assume requirement 3.6 to be satisfied with Γ^i_- and Γ^i_+ instead of $\partial\Omega$ and Σ.

That is all we have to assume so that we can apply the theory discussed in chapter 3. In order to construct the global solution we proceed successively, in ascending order of the intermediate stop times c_k, from one component of Ω to the next by stopping and restarting.

Remark: a restart set Γ_-^i contains finitely many of the saddle points $\{z_j\}_j$. The restart data, then, is a union of traces of previously computed partial solutions. The traces, in turn, are defined on the connected C^1-arcs of $\Gamma_-^i \setminus \bigcup_j \{z_j\}$ and are BV-functions. Since the restart sets Γ_-^i are simple closed C^1-curves, the glued-together traces, by theorem 2.28, define a $BV(\Gamma_-^i)$-function as restart data. Clearly, the restart data in general will have a "new" jump per saddle point contained in Γ_-^i. Here, in particular, the BV-framework proves useful to get well-defined restart data.

5.2 n-Connected Domains

The second extension concerns the connectivity of the domain. Let us start with a simple example. Consider, as domain Ω, a circular ring centered at the origin, $\Omega = B_{R+\rho}(0) \setminus \overline{B_{R-\rho}(0)}$, $R > \rho > 0$, together with the map $T : \Omega \to [0, 1]$,

$$T(x) = 1 - \frac{(|x| - R)^2}{\rho^2} ,$$

as a time function. The level sets of T look as sketched in figure 5.12. In this

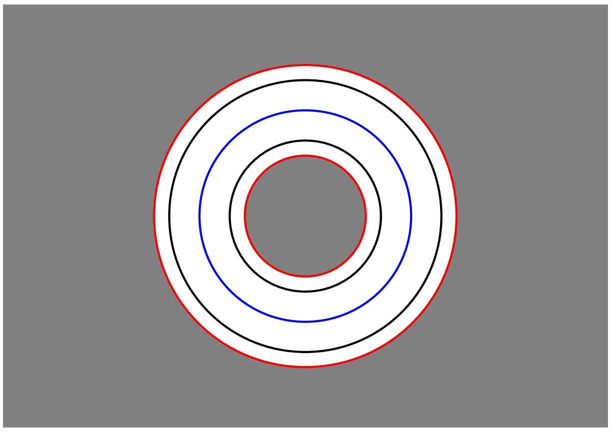


Figure 5.12: white: domain Ω, red: boundary $\partial\Omega$ = level set with $T = 0$, black: level set with $T = 5/9$, blue: maximal level set, $T = 1$.

example our linear problem splits up into two for the two sub-domains Ω^1 and Ω^2

$$\Omega^1 = B_{R+\rho}(0) \setminus \overline{B_R(0)} , \qquad \Gamma_-^1 = \partial B_{R+\rho}(0) , \tag{5.1}$$

$$\Omega^2 = B_R(0) \setminus \overline{B_{R-\rho}(0)} , \qquad \Gamma_-^2 = \partial B_{R-\rho}(0) , \tag{5.2}$$

with corresponding start sets Γ_-^1, Γ_-^2. The common stop set is $\Sigma = \partial B_R(0)$.

Here, the start sets are simple closed C^1-curves while Σ is a stop set in the sense of requirement 3.2. So, the linear theory gives us two partial solutions $u^1 \in BV(\Omega^1)$ and $u^2 \in BV(\Omega^2)$, and finally we obtain the solution $u \in BV(\Omega)$,

$$u = \mathbb{1}_{\Omega^1} u^1 + \mathbb{1}_{\Omega^2} u^2 ,$$

by theorem 2.28, the glueing property of BV-functions. This idea of construction also applies to other types of connectivity (see figure 5.13) and can be combined with the extended concept of time function above (see figure 5.14).

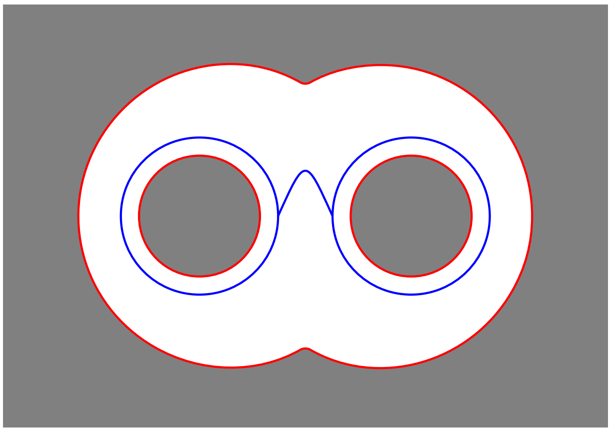


Figure 5.13: white: domain Ω, red: boundary $\partial\Omega$ = level $T = 0$, blue: maximal level set of T.

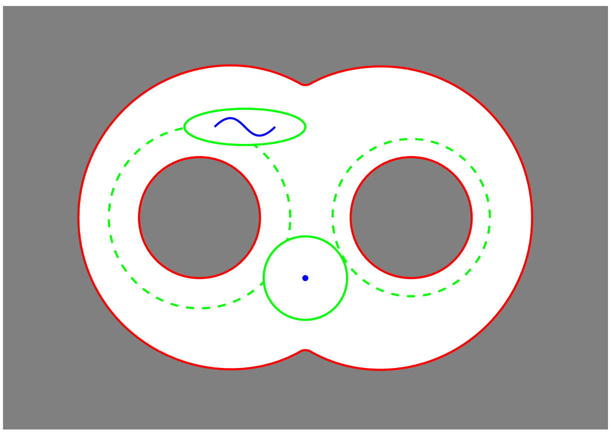


Figure 5.14: white: domain Ω, red: boundary $\partial\Omega$ = level $T = 0$ = start set, green + dashed green: saddle set of T, green: restart set, dashed green: first stop set, blue: maximal level set of T = second stop set.

The only thing we have to assume in the case of an n-connected domain Ω is that each of the n boundary curves, which $\partial\Omega$ is made of, is a simple closed C^1-curve.

So far, we have defined both extensions for the linear theory. In the quasi-linear theory of chapter 4, we assumed that an admissible time function with stop set is fixed in advance. Here, we also assume that a time function of the extended concept is fixed once and for all. So, the decomposition procedure for Ω, as described in section 5.1, can be performed in advance. In order to apply the theory of chapter 4 to a quasi-linear problem now, we just have to replace part 1 of the requirements 4.1 on transport fields by the formulation: "For fixed $v \in L^1(\Omega)$, and for every component Ω^i of Ω, the function $c[v]$, restricted onto Ω^i, satisfies requirement 3.6 with Γ^i_- and Γ^i_+ instead of $\partial\Omega$ and Σ".

Chapter 6

Image Inpainting Based on Coherence Transport

The goal of this chapter is to obtain the well-posedness of the model behind *Inpainting Based on Coherence Transport* (see [BM07]). First, we will review this model and regularize it where necessary. In the second step, we will attain its well-posedness by showing that it fits into the theory which we developed in the previous chapters.

6.1 The Generic Algorithm and its Continuous Formulation

Our starting point is the discrete generic algorithm for gray tone images. We assume that all gray tone images, seen as functions, take values in the real interval $[0, 1]$. Here, we assume that gray tones are mapped onto $[0, 1]$ such that the natural order on the interval reflects the order of the shades of gray by their brightness from black to white. Moreover, we will distinguish between discrete (digital) images defined on finite sets of pixels and continuous (analog) images defined on open subsets of $\mathbb{R}^2$. The latter are thought of as the high-resolution limit of the former. This distinction will be indicated by using the index h for the discrete notions, while omitting it for the corresponding continuous ones. Finally, we identify pixels with their midpoints.

Notation:

a) $\Omega_{0,h}$ is the image domain, the matrix of pixels for the final, restored image $u_h : \Omega_{0,h} \to [0, 1]$.

b) $\Omega_h \subset \Omega_{0,h}$ is the inpainting domain whose values of u_h have to be determined.

c) $\Omega_{0,h} \backslash \Omega_h$ is the data domain whose values of u_h are given as $u_h|_{\Omega_{0,h} \backslash \Omega_h} = u_{0,h}$.

d) $\partial\Omega_h \subset \Omega_h$ is the discrete boundary, i.e., the set of inpainting pixels that have at least one neighbor in the data domain.

Continuous quantities are defined correspondingly. Finally, we define discrete and continuous ε-neighborhoods by

$$B_{\varepsilon,h}(x) := \{y \in \Omega_{0,h} : |y - x| \leq \varepsilon\} , \quad B_\varepsilon(x) := \{y \in \Omega_0 : |y - x| \leq \varepsilon\} .$$

Generic Algorithm: the basic idea is to fill the inpainting domain in a fixed order, from its boundary inwards, by using weighted means of given or already calculated image values.

We number the pixels of the inpainting domain according to the chosen order, $\Omega_h = \{x_1, x_2, \ldots, x_N\}$, and call

$$B^<_{\varepsilon,h}(x_k) := B_{\varepsilon,h}(x_k) \backslash \{x_k, \ldots, x_N\} , \quad k = 1, \ldots, N,$$

the neighborhood of already inpainted pixels. Then, the algorithm reads as follows:

$$u_h|_{\Omega_{0,h} \backslash \Omega_h} = u_{0,h} ,$$

$$u_h(x_k) = \frac{\sum\limits_{y \in B^<_{\varepsilon,h}(x_k)} w(x_k, y) u(y)}{\sum\limits_{y \in B^<_{\varepsilon,h}(x_k)} w(x_k, y)} , \quad k = 1, \ldots, N. \tag{6.1}$$

Here, $w(x, y) \geq 0$ are called the weights of the algorithm and we assume that

$$\sum_{y \in B^<_{\varepsilon,h}(x)} w(x, y) > 0 , \quad x \in \Omega_h .$$

Order: in the generic algorithm, any order which orders the pixels from the boundary inwards can be used. In the article [BM07] this degree of freedom has been fixed by the distance-to-boundary order. That is, the euclidean distance to the boundary d,

$$d(x) = \mathrm{dist}(x, \partial\Omega) , \quad x \in \Omega ,$$

or rather a discrete approximation d_h,

$$d_h(x) = \mathrm{dist}_h(x, \partial\Omega_h) , \quad x \in \Omega_h ,$$

induces the order by

$$d_h(x_j) < d_h(x_k) \quad \Rightarrow \quad j < k\,.$$

In other words, the euclidean distance-to-boundary map d serves as time-like function and this notion of time induces the order. Later on, we will look at the problem from a continuous point of view by the high-resolution limit $h \to 0$. d will then play the role of time, if one uses the distance-to-boundary order. Certainly, the distance-to-boundary map behaves time-like, since it increases strictly into the interior of Ω, but the corresponding field of normals $N = \nabla d$ is not smooth enough to apply the theory of the previous chapters. A discussion of the difficulties, which appear when using d, is postponed to section 6.4.

For the purpose of regularization we take the assumptions from the previous chapters: we require the continuous inpainting domain Ω to be a domain and the order to be induced by a time function $T : \Omega \to \mathbb{R}$ with stop set Σ. Then, for the discrete scenario, we mean by $T_h : \Omega_h \to [0,1]$ the discretized time function, defined on the discretized inpainting domain Ω_h. The order of the pixels is, again, induced by the relation

$$T_h(x_j) < T_h(x_k) \quad \Rightarrow \quad j < k\,.$$

High-Resolution Limit: algorithm (6.1) can be thought of as a forward substitution of the equivalent system of linear equations

$$\sum_{y \in B^<_{\varepsilon,h}(x)} (u(x) - u(y)) w(x,y) = 0, \qquad x \in \Omega_h\,,$$
$$u_h|_{\Omega_{0,h} \setminus \Omega_h} = u_{0,h}\,.$$

Because of the definition of the neighborhoods $B^<_{\varepsilon,h}(x)$ the system is already triangular with respect to the chosen order of pixels. Viewing the system of equations as a discretization of a continuous integral equation we obtain, at least formally, the high-resolution limit

$$\begin{aligned} \frac{1}{\pi\varepsilon^2} \int_{B^<_\varepsilon(x)} (u(x) - u(y)) \cdot w(x,y)\, dy = 0, \qquad x \in \Omega\,, \\ u|_{\Omega_0 \setminus \Omega} = u_0\,, \end{aligned} \tag{6.2}$$

as $h \to 0$. The scale factor in front of the integral will turn out to be convenient later on.

Because the order is induced by a time function, the sets $B^<_{\varepsilon,h}(x)$ are discretizations of the truncated disks

$$B^<_\varepsilon(x) = \{y \in B_\varepsilon(x) : T(y) < T(x)\}\,.$$

Hereby, we extend the time function T onto the data domain by setting

$$T(x) = -\operatorname{dist}(x, \partial\Omega) , \quad x \in \Omega_0 \backslash \Omega .$$

Vanishing Viscosity Limit: the integral equation (6.2) combines directional effects, due to the truncation of the disks and the anisotropic choice of weights, with diffusion caused by the averaging. The amount of viscosity is determined by the radius ε, and in order to distil the directional effects we study the vanishing viscosity limit $\varepsilon \to 0$ for a particular class of weight functions.

Theorem 6.1. *(Vanishing viscosity limit)*
Let $T : \Omega \to \mathbb{R}$ be a time function with stop set Σ. Let $u \in C^1(\Omega\backslash\Sigma)$.

For weights of the form

$$w(x,y) = \frac{1}{|x-y|}\, k\left(x, (x-y)\cdot \varepsilon^{-1}\right)$$

with $k : \Omega\backslash\Sigma \times B_1(0) \to \mathbb{R}_0^+$ uniformly bounded, we have, as $\varepsilon \to 0$,

$$\frac{1}{\pi\varepsilon^2} \int\limits_{B_\varepsilon^<(x)} (u(x) - u(y)) \cdot w(x,y)\, dy = \langle c_*(x), \nabla u(x) \rangle + \mathcal{O}(\varepsilon) ,$$

for every $x \in \Omega\backslash\Sigma$.

For fixed x we express $c_*(x)$ by using polar coordinates with respect to the field of normals $N(x)$. Let the matrix $Q(x) := \left(N(x)|N^\perp(x)\right)$ and let $e(\varphi) := (\cos\varphi, \sin\varphi)^T$. Then,

$$c_*(x) = \frac{1}{\pi}\, Q(x) \cdot \int\limits_{-\pi/2}^{\pi/2} k_*\left(x, Q(x)\cdot e(\varphi)\right)\, e(\varphi)\, d\varphi$$

$$\text{with} \quad k_*(x,\eta) := \int\limits_0^1 k(x, r\,\eta)\, r\, dr .$$

Proof.

Fix $x \in \Omega\backslash\Sigma$ and define the semi-disk

$$S_{\varepsilon,N(x)}(x) := \{ y \in B_\varepsilon(x) : \langle N(x), (x-y) \rangle \geq 0 \} .$$

By construction the inner boundaries of $S_{\varepsilon,N(x)}$ and $B_\varepsilon^<(x)$ touch each other tangentially in x, while $B_\varepsilon^<(x)$ becomes $S_{\varepsilon,N(x)}$ asymptotically as $\varepsilon \to 0$. Hence, the area of the symmetric difference of these two sets is of the order

$\mathcal{O}(\varepsilon^3)$. Since u is assumed to be continuously differentiable in x, we obtain that

$$\frac{u(x)-u(y)}{|x-y|} = \left\langle \frac{x-y}{|x-y|}, \nabla u(x) \right\rangle + \mathcal{O}(\varepsilon)\ , \qquad y \in B_\varepsilon(x)\ ,$$

which implies, moreover, the boundedness of the expression. By using these approximations, we obtain

$$\begin{aligned}
&\frac{1}{\pi\varepsilon^2} \int\limits_{B_\varepsilon^<(x)} (u(x)-u(y)) \cdot w(x,y)\ dy \\
&\qquad = \frac{1}{\pi\varepsilon^2} \int\limits_{S_{\varepsilon,N(x)}(x)} \frac{u(x)-u(y)}{|x-y|}\ k\left(x,(x-y)\cdot\varepsilon^{-1}\right)\ dy + \mathcal{O}(\varepsilon) \\
&\qquad = \frac{1}{\pi\varepsilon^2} \int\limits_{S_{\varepsilon,N(x)}(x)} \left\langle \frac{x-y}{|x-y|}, \nabla u(x) \right\rangle\ k\left(x,(x-y)\cdot\varepsilon^{-1}\right)\ dy + \mathcal{O}(\varepsilon) \\
&\qquad = \frac{1}{\pi} \int\limits_{S_{1,N(x)}(0)} \left\langle \frac{-y}{|-y|}, \nabla u(x) \right\rangle\ k\left(x,-y\right)\ dy + \mathcal{O}(\varepsilon) \\
&\qquad = \left\langle \frac{1}{\pi} \int\limits_{S_{1,N(x)}(0)} k\left(x,-y\right)\ \frac{-y}{|-y|}\ dy, \nabla u(x) \right\rangle + \mathcal{O}(\varepsilon)\ .
\end{aligned}$$

From the last equality, we read

$$c_*(x) = \frac{1}{\pi} \int\limits_{S_{1,N(x)}(0)} k\left(x,-y\right)\ \frac{-y}{|-y|}\ dy\ .$$

Now, we introduce polar coordinates on the semi-disk $S_{1,N(x)}(0)$ by

$$\begin{aligned}
y &= r\cdot Q(x)\cdot e(\varphi) \qquad && (r,\varphi)\ \in [0,1]\times[\pi/2, 3\pi/2]\ , \\
-y &= r\cdot Q(x)\cdot e(\varphi) \qquad && (r,\varphi)\ \in [0,1]\times[-\pi/2, \pi/2]\ ,
\end{aligned}$$

and get

$$\begin{aligned}
c_*(x) &= \frac{1}{\pi} \int\limits_{-\pi/2}^{\pi/2} \int\limits_0^1 k\left(x, r\cdot Q(x)\cdot e(\varphi)\right)\ Q(x)\cdot e(\varphi)\ r\ dr\ d\varphi \\
&= \frac{1}{\pi}\ Q(x)\cdot \int\limits_{-\pi/2}^{\pi/2} \left(\int\limits_0^1 k\left(x, r\cdot Q(x)\cdot e(\varphi)\right)\ r\ dr \right)\ e(\varphi)\ d\varphi \\
&= \frac{1}{\pi}\ Q(x)\cdot \int\limits_{-\pi/2}^{\pi/2} k_*\left(x, Q(x)\cdot e(\varphi)\right)\ e(\varphi)\ d\varphi\ .
\end{aligned}$$

□

By theorem 6.1 the limiting linear inpainting equation then is

$$\begin{aligned} \langle c_*(x), \nabla u(x) \rangle &= 0\,, \qquad \text{in} \quad \Omega\backslash\Sigma \\ u|_{\partial\Omega} &= u_0|_{\partial\Omega}\,. \end{aligned}$$

We have seen in chapter 3, assuming that $c(x) = c_*(x)/|c_*(x)|$ satisfies the requirements 3.6, that we will get jump discontinuities at Σ, even if $u_0|_{\partial\Omega}$ is bounded and smooth. Thus, the second step of regularization is to go over to the BV-formulation

$$\begin{aligned} \langle c(x), Du \rangle &= 0\,, \qquad \text{in} \quad \Omega\backslash\Sigma \\ u|_{\partial\Omega} &= u_0|_{\partial\Omega}\,, \end{aligned} \tag{6.3}$$

which in turn allows for less regular boundary data. Here, in the continuous inpainting scenario of equation (6.3) we assume that the given data u_0 belongs to $BV(\Omega_0\backslash\overline{\Omega})$ and that the BV-trace $u_0|_{\partial\Omega}$ ("trace from outside Ω") belongs to $BV(\partial\Omega)$.

In the following sections we will study weight functions which have the form required in theorem 6.1. Therefor, in the next theorem we summarize sufficient conditions on the kernel k such that the transport field of the inpainting equation meets the assumptions of requirement 3.6.

Theorem 6.2.
Let $T : \Omega \to \mathbb{R}$ be a time function with stop set Σ. Let the kernel $k : \Omega\backslash\Sigma \times B_1(0) \to \mathbb{R}_0^+$ of theorem 6.1 satisfy:

a) uniform bounds:

$$0 < \gamma_1 \le k(x,\eta) \le \gamma_2\,, \qquad \forall\ (x,\eta) \in \Omega\backslash\Sigma \times B_1(0)\,,$$

b) k is continuously differentiable with bounded derivative,

c) for every fixed $\eta \in B_1(0)$ the functions $k(\,.\,,\eta)$, $D_x k(\,.\,,\eta)$, and $D_\eta k(\,.\,,\eta)$ extend onto the boundary $\partial\Omega$ and have one-sided extensions onto the relatively open components $\mathring{\Sigma}_k$ of Σ.

Then,

$$|c_*(x)| \ge \frac{\gamma_1}{\pi}\,, \qquad x \in \Omega\backslash\Sigma\,,$$

and the, hence well-defined, transport field $c(x) = c_*(x)/|c_*(x)|$ satisfies the requirements 3.6.

Proof.

Let $x \in \Omega\backslash\Sigma$. By the representation of $c_*(x)$ we have

$$\langle c_*(x), N(x)\rangle = \left\langle \frac{1}{\pi}\, Q(x) \cdot \int_{-\pi/2}^{\pi/2} k_*\left(x, Q(x)\cdot e(\varphi)\right)\; e(\varphi)\; d\varphi, N(x) \right\rangle$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} k_*\left(x, Q(x)\cdot e(\varphi)\right)\; \cos(\varphi)\; d\varphi \;.$$

The definition of k_*, and the lower bound on k yield the estimate

$$k_*(x, Q(x)\cdot e(\varphi)) := \int_0^1 k(x, r\; Q(x)\cdot e(\varphi))\; r\; dr \geq \frac{\gamma_1}{2} \;.$$

Hence, we obtain

$$|c_*(x)| \geq \langle c_*(x), N(x)\rangle \geq \frac{\gamma_1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos(\varphi)\; d\varphi \geq \frac{\gamma_1}{\pi} \;.$$

By the same argumentation each component w.r.t. the orthonormal basis $N(x)$, $N(x)^\perp$ is bounded above by γ_2/π, which implies

$$|c_*(x)| \leq \frac{\sqrt{2}\gamma_2}{\pi} \;.$$

Thus, we obtain the inward-pointing condition

$$\langle c(x), N(x)\rangle = \frac{\langle c_*(x), N(x)\rangle}{|c_*(x)|} \geq \frac{\gamma_1}{\sqrt{2}\gamma_2} =: \beta > 0 \;.$$

By the differentiability properties of the kernel k we have $c_* \in C^1(\Omega\backslash\Sigma)$. What remains is the extendability of the vector field and its derivative. By definition, k_* has already the same properties a), b), and c) as the kernel k. With $\eta(x,\varphi) = Q(x)\cdot e(\varphi)$ we write

$$c_*(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} k_*(x, \eta(x,\varphi))\eta(x,\varphi)\; d\varphi \;.$$

Then, the derivative of c_* is given by

$$D_x c_*(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \eta D_x k_*(x,\eta) + \eta D_\eta k_*(x,\eta) D_x \eta + k_*(x,\eta) D_x \eta\; d\varphi \;.$$

Moreover, by

$$D_x\eta = (D_xQ)e = (D_xQ)Q^T\eta \ ,$$

we get

$$\begin{aligned} D_x c_*(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} & \eta D_x k_*(x,\eta) + \eta D_\eta k_*(x,\eta)(D_xQ)Q^T\eta \\ & + k_*(x,\eta)(D_xQ)Q^T\eta \, d\varphi \ . \end{aligned} \tag{6.4}$$

Let now $\theta(x) \in [0, 2\pi[$ be the angle of $N(x)$ w.r.t. the standard basis e_1, e_2, i.e., $N(x) = e(\theta(x))$. We then use

$$e(\varphi) \ , \qquad \varphi \in [\theta(x) - \pi/2, \theta(x) + \pi/2]$$

in the integrals instead of

$$\eta(x,\varphi) = Q(x) \cdot e(\varphi) \ , \qquad \varphi \in [-\pi/2, +\pi/2] \ .$$

Hence, on the one hand, we obtain

$$c_*(x) = \frac{1}{\pi} \int_{\theta(x)-\pi/2}^{\theta(x)+\pi/2} k_*(x, e(\varphi)) e(\varphi) \, d\varphi$$

and on the other

$$\begin{aligned} D_x c_*(x) = \frac{1}{\pi} \int_{\theta(x)-\pi/2}^{\theta(x)+\pi/2} & e D_x k_*(x,e) + e D_\eta k_*(x,e)(D_xQ)Q^T e \\ & + k_*(x,e)(D_xQ)Q^T e \, d\varphi \ . \end{aligned}$$

By the extendability of the kernel k and its derivative according to c), together with their upper bounds, the extendability of c_* and Dc_* follows from the above representations by dominated convergence. Now we have shown parts 1 and 2 of requirement 3.6. For part 3 let us consider the representation by equation (6.4) again. The matrix $(D_xQ)Q^T\eta$ writes out as

$$(D_xQ)Q^T\eta = (D_xQ)e(\varphi) = \cos(\varphi) D_x N + \sin(\varphi) D_x N^\perp \ ,$$

and thus, we obtain the following estimate of its norm:

$$|(D_xQ)Q^T\eta| \leq \sqrt{2} |D_x N| \ .$$

Consequently, if M denotes the upper bound on the derivative of k, we get

$$|D_x c_*(x)| \leq M + \sqrt{2} M |D_x N(x)| + \sqrt{2} \gamma_2 |D_x N(x)| \ .$$

Hence, $|D_x c_*|$ belongs to $L^1(\Omega)$, since $|D_x N|$ does. Moreover, there is a uniform bound of $|D_x c_*|$ on $\Omega \backslash V_\varepsilon$ as required in part 3a), since $|D_x N|$ has such a uniform bound on $\Omega \backslash V_\varepsilon$.

Finally, we write the derivative of $c = c_*/|c_*|$ as

$$D_x c(x) = \frac{1}{|c_*(x)|} \left(I - c(x) \cdot c(x)^T \right) \cdot D_x c_*(x) \, .$$

By the features of c_* and $D_x c_*$, shown above, we infer that the transport field c satisfies requirement 3.6. □

In the situation of the last theorem our theory of the linear problem, according to chapter 3, ensures the existence of a unique solution $u \in BV(\Omega)$ of the linear inpainting equation (6.3).
Within the continuous model we always distinguish between the fill-in u, which is defined on the inpainting domain Ω, and the completed image $\bar{u}$

$$\bar{u} = \mathbb{1}_{\Omega_0 \backslash \Omega} \cdot u_0 + \mathbb{1}_\Omega \cdot u \, ,$$

which is defined on the full image domain Ω_0.
By theorem 2.28 the completed image $\bar{u}$ belongs to $BV(\Omega_0)$ with derivative measure

$$D\bar{u} = Du_0 \llcorner (\Omega_0 \backslash \Omega) + Du \llcorner \Omega \, .$$

There are no jumps across $\partial\Omega$.

6.2 Two Linear Models

In this section we present two linear models which we obtain from particular choices of the kernel k of theorem 6.1.

6.2.1 Transport Along Normals

The point of departure of the paper [BM07] was the article of Telea [Tel04], in which the author suggested to perform the generic algorithm with the weight

$$w(x,y) = \frac{|\langle N(x), (x-y) \rangle|}{|x-y|^2} \, . \tag{6.5}$$

In view of theorem 6.1, the kernel k is

$$k(x,\eta) = \frac{|\langle N(x), \eta \rangle|}{|\eta|} \, .$$

And, the vanishing viscosity limit results in a transport field $c = c_*/|c_*|$ equal to the normal,

$$c(x) = N(x)\ ,$$

(see [BM07] part b) of theorem 1). With $c \equiv N$ the theory of chapter 3 applies, even though the kernel does not satisfy all of the assumptions of theorem 6.2.

In practice there are many examples of inpainting domains where the distance-to-boundary map or a distance-to-boundary related time function (to stay within requirement 3.4) induces a reasonable order to perform the generic algorithm. But then, when using the weight (6.5), the realized transport field N only or mainly depends on the geometry of the domain Ω and is not or little adapted to the image.
The next choice of a weight function allows for the practical realization of arbitrary transport fields. This advantage will be used later in section 6.3 in order to adapt the weight to the image.

6.2.2 Guided Transport

Let $g \in C^1(\Omega\backslash\Sigma, S^1)$ be a normed vector field. We assume that g and its derivative ∇g both extend one-sided onto Σ and $\partial\Omega$, and that both $\|g\|_\infty$ and $\|\nabla g\|_\infty$ exist. Then, the one-parameter family of kernels

$$k_\mu(x,\eta) = \sqrt{\frac{\pi}{2}}\mu \exp\left(-\frac{\mu^2}{2}\left\langle g^\perp(x),\eta\right\rangle^2\right)$$

satisfies all assumptions of theorem 6.2 with the uniform bounds

$$\sqrt{\frac{\pi}{2}}\mu \exp\left(-\frac{\mu^2}{2}\right) \le k_\mu(x,\eta) \le \sqrt{\frac{\pi}{2}}\mu\ , \quad \forall\ (x,\eta) \in \Omega\backslash\Sigma \times B_1(0)\ .$$

Hence, by the vanishing-viscosity limit every resulting field c_μ is an admissible transport field with

$$\left\langle c_\mu(x), N(x)\right\rangle \ge \frac{\exp\left(-\frac{\mu^2}{2}\right)}{\sqrt{2}} =: \beta_\mu\ , \quad \forall\ x \in \Omega\backslash\Sigma.$$

In [BM07] (theorem 2) we proved an asymptotic expansion of $c_\mu(x)$, w.r.t. $\mu \to \infty$, which implies the following limit behavior

$$\lim_{\mu\to\infty} c_\mu(x) = \begin{cases} g(x) & ,\langle g(x), N(x)\rangle > 0 \\ -g(x) & ,\langle g(x), N(x)\rangle < 0 \\ N(x) & ,\langle g(x), N(x)\rangle = 0 \end{cases}\ .$$

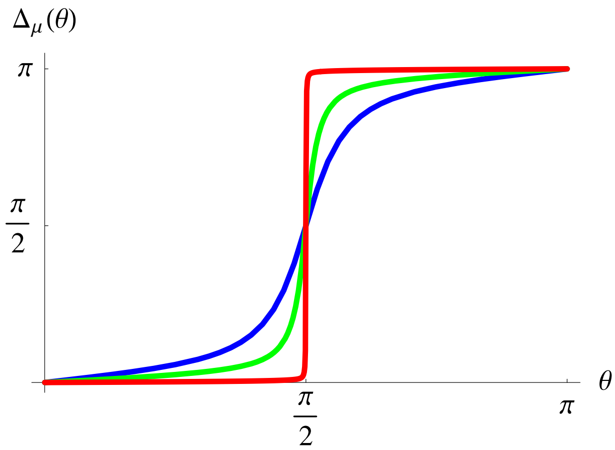


Figure 6.1: Deviation angle Δ_μ: blue: $\mu = 5$, green: $\mu = 10$, red: $\mu = 100$.

The change of sign happens, because c_μ, by theorem 6.2 , always points inwards. So, in the limit $\mu \to \infty$, the transport field equals $\pm g$ wherever $\pm g$ points inwards. Otherwise, it breaks down to N, a vector which naturally points inwards.

For fixed $\mu > 0$ we denote by $\theta(x) = \angle(g(x), N(x)) \in [0, \pi]$, $\cos\theta(x) = \langle g(x), N(x)\rangle$, the angle between $g(x)$ and $N(x)$. Theorem 2 of [BM07] states furthermore that the deviation angle between $g(x)$ and $c_\mu(x)$ is of the form

$$\angle(g(x), c_\mu(x)) = \Delta_\mu\,(\,\theta(x)\,)$$

with a continuous function $\Delta_\mu : [0, \pi] \to [0, \pi]$. And Δ_μ inherits its limit behavior from c_μ:

$$\lim_{\mu\to\infty} \Delta_\mu\,(\,\theta(x)\,) = \begin{cases} 0 & ,\theta(x) < \frac{\pi}{2} \\ \pi & ,\theta(x) > \frac{\pi}{2} \\ \frac{\pi}{2} & ,\theta(x) = \frac{\pi}{2} \end{cases} .$$

Figure 6.1 shows graphs of Δ_μ for different values of μ. Here, we can see that if $\mu > 0$ is set to a large value, then $c_\mu(x)$ approximates $\pm g(x)$ very well ($\Delta_\mu \approx 0$ or $\Delta_\mu \approx \pi$), when $\theta(x)$ is bounded away from $\pi/2$, and continuously fades to $N(x)$, when $\theta(x)$ comes close to $\pi/2$. Moreover, if $\theta(x) = \pi/2$, respectively $\langle g(x), N(x)\rangle = 0$, then $c_\mu(x) = N(x)$ for every $\mu > 0$.
For fixed $\mu > 0$ the transport along c_μ is not, of course, a transport exactly along $\pm g$, but the transport is guided by $\pm g$. For this reason the field g is called guidance field.

Non-normed Guidance Fields: in the case of a non-normed guidance field g we can use $g(x)/|g(x)|$ in the kernel k_μ whenever $|g(x)| \neq 0$. Otherwise, if $|g(x)| = 0$, we are in an undefined situation. But $|g(x)| = 0$ means a special case of $\langle g(x), N(x) \rangle = 0$, and so in this case, the transport vector $c_\mu(x)$ shall equal $N(x)$. For this purpose we introduce a confidence factor $\hat{\alpha} : \Omega \backslash \Sigma \to \mathbb{R}_0^+$,

$$\hat{\alpha}(x) = \alpha(\ q(x)\ ) \ ,$$

as a function of a quality measure $q : \Omega \backslash \Sigma \to \mathbb{R}_0^+$. In general, the quality value $q(x) \geq 0$ shall measure if the vector $g(x)$ defines a direction. Moreover, q shall be as smooth as g. Here, $q(x) = |g(x)|^2$ is a reasonable choice, but others are possible (see the next section).

The function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, which translates the quality- into the confidence measure, is assumed to have the following properties:

- $\alpha \in C^2(\mathbb{R}^+, \mathbb{R}^+)$,
- α is strictly increasing,
- α bounded by $\alpha \leq 1$,
- α' bounded,
- $\lim\limits_{t \to 0_+} \alpha(t) = 0$, $\lim\limits_{t \to 0_+} \alpha'(t) = 0$, and $\lim\limits_{t \to 0_+} \alpha''(t)$ exists.

For a concrete realization of α we will always refer to

$$\alpha(t) = \exp\left(-\frac{1}{t}\right) \ , \tag{6.6}$$

because this choice of α experimentally proved to be good (see [BM07]).

Typically, a quality value q is judged relatively to some reference value $\delta > 0$, i.e., q/δ is the interesting number. Here, we carry over the reference value $\delta > 0$ to α_δ,

$$\alpha_\delta(t) = \alpha\left(\frac{t}{\delta}\right) \ , \quad \delta > 0 \ . \tag{6.7}$$

So, α determines the basic shape of the functions belonging to the family $\{\alpha_\delta\}_{\delta > 0}$.

The family of integral kernels with confidence factor is then given by

$$k_\mu(x, \eta) = \sqrt{\frac{\pi}{2}} \mu \exp\left(-\frac{\mu^2}{2} \cdot \hat{\alpha}(x) \cdot \left\langle \frac{g^\perp(x)}{|g(x)|}, \eta \right\rangle^2\right) \ .$$

If, now, the confidence measure $\hat{\alpha}$ becomes zero at some x, the kernel breaks down to

$$k_\mu(x, \eta) = \sqrt{\frac{\pi}{2}} \mu \ ,$$

and by theorem 6.1 we obtain $c_{\mu,*}(x) = (\mu/\sqrt{2\pi})N(x)$, i.e., $c_\mu(x) = N(x)$ as desired. While, if $0 < \hat{\alpha}(x) \ll 1$, the value $\hat{\alpha}(x)$ damps the effect of the parameter μ. Thus, if $\mu \gg 0$, but $0 < \hat{\alpha}(x) \ll 1$, the guidance effect of $\pm g(x)/|g(x)|$ on $c_\mu(x)$ is not that pronounced.

Guidance Tensor: for the analysis it is more appropriate to wrap the guidance information in a guidance tensor $G(x) \in \mathbb{R}^{2\times 2}$. In this case the kernel k_μ looks like

$$k_\mu(x,\eta) = \sqrt{\frac{\pi}{2}}\mu \exp\left(-\frac{\mu^2}{2}\,\eta^T \cdot G(x)\cdot \eta\right) .$$

For a given guidance vector $g(x)$ the guidance tensor is simply the matrix

$$G(x) = \begin{cases} \alpha(q(x)) \cdot \frac{g^\perp(x)\cdot g^\perp(x)^T}{|g(x)|^2} & ,q(x) > 0 \\ 0 & ,q(x) = 0 \end{cases} .$$

In order to stay within the assumptions of theorem 6.2 α is required to behave such that the singular case, $q(x) = 0$, is a continuously differentiable extension of the regular case, $q(x) > 0$.

6.3 A Quasi-Linear Model

The quasi-linear model of this section is based on the idea of guided transport. So far, for the linear variant of guided transport, the guidance information had to be completely specified in advance, which, in general, is a difficult task. Instead, we now calculate, on the fly, the desired guidance information from the actually known image. This approach will effect the guidance tensor to become a functional of the image function.

6.3.1 Guidance by Coherence Information

We start out by retrieving reasonable guidance information from the image. The basic idea, that almost all geometry based inpainting models have in common, is to try to close broken level lines of the damaged image. In order to be able do so we have to analyze the image structure. More precisely, we have to estimate the current course of a level line. That means, at every point $x \in \Omega$ we estimate an approximate tangent vector to the level line which the point x belongs to. A very robust estimator for coherence (=approximate tangent) information is the so-called structure tensor. In the next step we introduce its concept for C^1-functions, following [AMS+06]. Then, later on, when we will have found the approximate tangent, for guidance, we will plug it into the integral kernel of the guided transport model.

Structure Tensor: let the image $v : \Omega_0 \to \mathbb{R}$ be a C^1-function. Roughly speaking, an approximate tangent is a vector $g(x) \in \mathbb{R}^2$ which satisfies

$$\langle \nabla v(y), g(x) \rangle \approx 0 \,, \qquad |g(x)| = 1,$$

in a neighborhood $U(x)$ of the point x. More precisely, we reformulate this characterization as a weighted least squares problem

$$g(x) = \arg \min_{\substack{h \in \mathbb{R}^2, \\ |h|=1}} \int\limits_{U(x)} K(x-y) \, \langle \nabla v(y), h \rangle^2 \, dy \,,$$

using a non-negative kernel K with $\int_{U(x)} K(x-y)dy = 1$. The so-called structure tensor $S(x) \in \mathbb{R}^{2\times 2}$, then, arises naturally from the equivalent formulation

$$g(x) = \arg \min_{|h|=1} h^T S(x) h \,, \qquad S(x) = \int\limits_{U(x)} K(x-y) \nabla v(y) \cdot \nabla v(y)^T dy \,.$$

By construction the structure tensor $S(x)$ is a symmetric positive semi-definite 2×2-matrix. Hence, an instance of $g(x)$ is a normed eigenvector with respect to the minimal eigenvalue of $S(x)$.

Note that g is not unique. If $S \neq \lambda \cdot I$ the coherence information we obtain is an orientation $\pm g$, which we refer to by its projector $P_0 = g \cdot g^T$ from the unique spectral decomposition

$$\begin{aligned} S &= \lambda_0 \, P_0 + \lambda_1 \, P_1 = \lambda_0 \, g \cdot g^T + \lambda_1 \, g^\perp \cdot g^{\perp T} \,, \\ I &= P_0 + P_1 \,, \end{aligned} \tag{6.8}$$

where

$$0 \leq \lambda_0 < \lambda_1$$

denote the eigenvalues of $S(x)$. In the case of $S = \lambda \cdot I$, the projectors P_0, P_1 are not uniquely defined and hence the orientations $\pm g$, $\pm g^\perp$ are meaningless. A reasonable quality measure should, of course, detect this singular case. So, in order to measure the quality of the projectors or the corresponding orientations we use the so-called coherence measure

$$q = (\lambda_1 - \lambda_0)^2 \,,$$

which becomes zero in the singular case $S = \lambda \cdot I$.

Remarks on the structure tensor concept:

a) The above coherence orientation is exactly the tangent's orientation, for example, if the restriction

$$v|_{U(x)}(y) = f(\langle k, y \rangle) \,, \quad f \in C^1(\mathbb{R}) \,, \quad k \neq 0 \,,$$

of v is a planar wave. On the one hand, with $\nabla v(y) = f'(\langle k, y\rangle)k$, we obtain

$$S(x) = \lambda_1(x) \cdot P_1 = \lambda_1(x) \cdot \frac{k \cdot k^T}{|k|^2} \,,$$
$$\lambda_1(x) = \int\limits_{U(x)} K(x-y)\; f'(\langle k, y\rangle)^2\, dy \cdot |k|^2 \,,$$

and, hence, $g(x) = \pm k^{\perp}/|k|$. On the other hand, the level line through x is orthogonal to the wave vector k, thus its orientation is $\pm k^{\perp}/|k|$.

b) If the integral kernel K, used for the set up of S, is a Dirac-kernel, then

$$S(x) = |\nabla v(x)|^2 \frac{\nabla v(x) \cdot \nabla v(x)^T}{|\nabla v(x)|^2} \,.$$

In this case we have $q(x) = |\nabla v(x)|^4$ and get $g(x) = \pm \nabla v(x)^{\perp}/|\nabla v(x)|$ if $q(x) \neq 0$.

c) If the given function v is not C^1, one typically applies the above concept to a smoothed version of v (see the next section).

Guidance Tensor: based on the coherence information, we set up the guidance tensor by

$$G(x) = \begin{cases} \alpha(\, q(x)\,) \cdot P_1(x) & ,\; q(x) > 0 \\ 0 & ,\; q(x) = 0 \end{cases} \,. \tag{6.9}$$

If $q(x) > 0$, the approximate tangent $\pm g$, which we want to use as guidance vector, is defined. Therefor, the projector which has to be used in the guidance tensor is exactly $P_1(x)$. In addition, the guidance tensor is controlled by the quality of the coherence information.

Finally, solving the spectral decomposition of S (equations (6.8)) for the projector P_1, we can set G up directly, without calculating eigenvectors:

$$G(x) = \begin{cases} \frac{\alpha(\, q(x)\,)}{\sqrt{q(x)}} \cdot (S(x) - \lambda_0 I) & ,\; q(x) > 0 \\ 0 & ,\; q(x) = 0 \end{cases} \,. \tag{6.10}$$

By the construction above, the guidance tensor $G[v]$ is a functional of v, since the structure tensor $S[v]$ is, too. By plugging $G[v]$ into the kernel of guided transport, we obtain

$$k_\mu[v](x, \eta) = \sqrt{\frac{\pi}{2}}\mu \exp\left(-\frac{\mu^2}{2}\, \eta^T \cdot G[v](x) \cdot \eta\right) \,. \tag{6.11}$$

The vanishing viscosity limit of theorem 6.1 plus normalization yields the corresponding transport field $c_\mu[v]$, which now depends on the function / image v.

The quasi-linear model, called *Inpainting Based on Coherence Transport*, is given by

$$\begin{aligned} \langle c_\mu[\bar{u}](x), Du \rangle &= 0 , \quad \text{in} \quad \Omega \backslash \Sigma , \\ u|_{\partial\Omega} &= u_0|_{\partial\Omega} , \\ \bar{u} &= u_0 \cdot \mathbb{1}_{\Omega_0 \backslash \Omega} + u \cdot \mathbb{1}_{\Omega} . \end{aligned} \tag{6.12}$$

The transport, which inpaints the image, is now guided by coherence information.

6.3.2 Structure Tensor with Volterra-Type Dependence

The goal of this section is two-fold. On the one hand, the structure tensor $S[v](x)$ should have the functional dependence on v of Volterra-type w.r.t. the time function T. The reasoning behind this is that we want to retrieve coherence information, at some given point $x \in \Omega$, only from the *actually known* part of the image. On the other hand, we need continuity and/or differentiability properties of S w.r.t. both arguments, x and v. In order to achieve both we have to choose the kernels K, for setting up S, carefully.

Throughout this section we assume that the time function T belongs to $C^1(\overline{\Omega})$, and we extend the time function onto $\Omega_0 \backslash \Omega$ again by

$$T(x) = -\operatorname{dist}(x, \partial\Omega) , \quad x \in \Omega_0 \backslash \Omega .$$

The generic integral kernel $K : \mathbb{R}^2 \to \mathbb{R}$, which will be used later for the set up of the structure tensor, is characterized by the following properties:

a) smoothness: $K \in C^\infty(\mathbb{R}^2)$,

b) non-negativity: $K \geq 0$,

c) unit mass: $\int_{\mathbb{R}^2} K(y)\, dy = 1$,

d) radial symmetry: $\partial_\varphi K(r \cdot e(\varphi)) \equiv 0$,

e) $r \in]0, \infty[\to K(r \cdot e(\varphi))$ decreases strictly on its support.

For a concrete realization of K, we will always refer to

$$K(y) = \frac{1}{2\pi} \exp\left(-\frac{|y|^2}{2} \right) ,$$

a Gaussian kernel.

In order to retrieve coherence information at different scales we will use a family of kernels $\{K_t\}_{t>0}$, generated by K,

$$K_t(y) = \frac{1}{t^2} K\left(\frac{y}{t}\right) , \quad t > 0 .$$

This family inherits all properties of K.
A sound introduction to scale space theory would go beyond the scope of this text. For dealing with the latter subject the reader is referred to [AGLM93]. For the purpose of scales in the context of the structure tensor see [Wei98].

Let the function v of mapping type $v : \Omega_0 \to \mathbb{R}$. In order to get the dependence on v of Volterra-type we restrict the structure tensor's integral to the set $\Omega(x)$, for fixed $x \in \Omega$,

$$\Omega(x) = \{h \in \Omega_0 : T(h) < T(x)\} , \quad \text{with} \quad \mathbb{1}_{\Omega(x)}(h) = H(T(x) - T(h)) .$$

Here $H : \mathbb{R} \to \mathbb{R}$,

$$H(t) := \begin{cases} 0 & , t < 0 \\ 1 & , 0 \le t \end{cases} ,$$

denotes the Heaviside function.

For the fixed choice of the parameter $t = \rho$ the structure tensor is given by

$$S[v](x) = \frac{\int\limits_{\Omega_0} K_\rho(x-y) \cdot \mathbb{1}_{\Omega(x)}(y) \cdot \nabla v(y) \cdot \nabla v(y)^T \, dy}{\int\limits_{\Omega_0} K_\rho(x-y) \cdot \mathbb{1}_{\Omega(x)}(y) \, dy} . \tag{6.13}$$

Two difficulties appear here: for the framework of chapter 4, we have to

1. assume, that S is C^1 w.r.t. the variable x,

2. apply S to functions v belonging to $L^1(\Omega_0)$ or $BV(\Omega_0)$ which, in turn, makes it necessary to smooth v before plugging it into S.

By the way, in practice, the need for smoothing operations is often caused by the given data being noisy.

To fight the first difficulty, we take a C^1-approximation $\mathbb{1}^a_{\Omega(x)}$ of $\mathbb{1}_{\Omega(x)}$,

$$\mathbb{1}^a_{\Omega(h)} = H_a(T(x) - T(h)) ,$$

obtained by some C^1-approximation H_a of the Heaviside function. Let $\tilde{H} \in C^1(\mathbb{R})$, with

$$\tilde{H}(t) = \begin{cases} 0 & , t < 0 \\ \text{strictly increasing} & , 0 \le t < 1 \\ 1 & , 1 \le t \end{cases} .$$

Then, we use the family

$$H_a(t) = \tilde{H}\left(\frac{t}{a}\right) , \qquad a > 0$$

as C^1-approximations H_a of the Heaviside function. For example, one could use

$$\tilde{H}(t) = \begin{cases} 0 & , t < 0 \\ 2t^2 & , 0 \le t < \frac{1}{2} \\ 2t^2 - (2t-1)^2 & , \frac{1}{2} \le t < 1 \\ 1 & , 1 \le t \end{cases} .$$

And, to fight the second difficulty, we define the smoothing operator $\phi_t[v]$ for functions $v \in L^1(\Omega_0)$,

$$\phi_t[v](y) = \int\limits_{\Omega_0} K_t(y-h)\, v(h)\, dh , \quad y \in \Omega_0 .$$

For a fixed choice of $a > 0$ and $\sigma > 0$, we set

$$\hat{v}(y) = \frac{\phi_\sigma[\mathbb{1}^a_{\Omega(x)} \cdot v](y)}{\phi_\sigma[\mathbb{1}^a_{\Omega(x)}](y)} = \frac{\int\limits_{\Omega_0} K_\sigma(y-h) \cdot \mathbb{1}^a_{\Omega(x)}(h) \cdot v(h)\, dh}{\int\limits_{\Omega_0} K_\sigma(y-h) \cdot \mathbb{1}^a_{\Omega(x)}(h)\, dh} . \tag{6.14}$$

By this construction, we make sure that $\hat{v}(y)$ is smooth but depends only on the data $v|_{\Omega(x)}$. Later on, $\hat{v}$ together with $\mathbb{1}^a_{\Omega(x)}$, instead of v and $\mathbb{1}_{\Omega(x)}$, will enter the set up of S according to equation (6.13).

The next lemma collects the features of expression (6.14) as a function of y and x.

Lemma 6.3.

1. *Let $f_1 : \Omega_0 \times \Omega \to \mathbb{R}$ be defined by*

$$f_1(y,x) := \phi_\sigma[\mathbb{1}^a_{\Omega(x)} \cdot v](y) = \int\limits_{\Omega_0} K_\sigma(y-h) \cdot H_a(T(x) - T(h)) \cdot v(h)\, dh .$$

Then, f_1 has the properties:

a) f_1 *is continuous and bounded:* $|f_1(y,x)| \leq \|K_\sigma\|_\infty \cdot \|v\|_{L^1(\Omega_0)}$.

b) f_1 *has a continuous and bounded y-derivative:*

$$\nabla_y f_1(y,x) = \int_{\Omega_0} \nabla K_\sigma(y-h) \cdot H_a(T(x)-T(h)) \cdot v(h)\, dh$$
$$|\nabla_y f_1(y,x)| \leq \|\nabla K_\sigma\|_\infty \cdot \|v\|_{L^1(\Omega_0)} .$$

c) f_1 *has a continuous and bounded x-derivative:*

$$\nabla_x f_1(y,x) = \int_{\Omega_0} K_\sigma(y-h) \cdot H_a'(T(x)-T(h)) \cdot \nabla T(x) \cdot v(h)\, dh$$
$$|\nabla_x f_1(y,x)| \leq \|K_\sigma\|_\infty \cdot \|H_a'\|_\infty \cdot \|\nabla T\|_\infty \cdot \|v\|_{L^1(\Omega_0)} .$$

d) f_1 *has a continuous and bounded mixed derivative:*

$$\nabla_x \nabla_y f_1(y,x) = \int_{\Omega_0} \nabla K_\sigma(y-h) \cdot H_a'(T(x)-T(h)) \cdot \nabla T(x)^T \cdot v(h)\, dh$$
$$|\nabla_x \nabla_y f_1(y,x)| \leq \|\nabla K_\sigma\|_\infty \cdot \|H_a'\|_\infty \cdot \|\nabla T\|_\infty \cdot \|v\|_{L^1(\Omega_0)} .$$

2. *Let* $f_2 : \Omega_0 \times \Omega \to \mathbb{R}$ *be defined by*

$$f_2(y,x) := \phi_\sigma[\mathbb{1}^a_{\Omega(x)}](y) = \int_{\Omega_0} K_\sigma(y-h) \cdot H_a(T(x)-T(h))\, dh .$$

Then, f_2 *has the properties:*

a) f_2 *has the same continuity and differentiability properties as* f_1.

b) f_2 *satisfies bounds analogous to* f_1. $\|v\|_{L^1(\Omega_0)}$ *has just to be substituted by* $\|\mathbb{1}_{\Omega_0}\|_{L^1(\Omega_0)} = \mathcal{L}^2(\Omega_0)$.

c) For $y \in \Omega(x)$, f_2 *is bounded below by* $f_2 > m_\sigma$, *with*

$$m_\sigma = \min_{y \in \Omega} \int_{\Omega_0} K_\sigma(y-h)\, \mathbb{1}^a_{\Omega(y)}(h)\, dh .$$

3. *Let*

$$f_3(y,x) := \frac{\phi_\sigma[\mathbb{1}^a_{\Omega(x)} \cdot v](y)}{\phi_\sigma[\mathbb{1}^a_{\Omega(x)}](y)} = \frac{f_1(y,x)}{f_2(y,x)} .$$

Then, f_3 *has the properties:*

a) f_3 *is well-defined for* $x \in \Omega$ *and* $y \in \Omega(x)$.

b) f_3 is continuous and bounded: $|f_3(y,x)| \leq C_1 \cdot \|v\|_{L^1(\Omega_0)}$.

c) f_3 has a continuous and bounded y-derivative: $|\nabla_y f_3(y,x)| \leq C_2 \cdot \|v\|_{L^1(\Omega_0)}$.

d) f_3 has a continuous and bounded mixed derivative: $|\nabla_x \nabla_y f_3(y,x)| \leq C_3 \cdot \|v\|_{L^1(\Omega_0)}$.

Proof.

1. These statements are true by construction.
2. The statements a) and b) follow from part 1 by setting $v = \mathbb{1}_{\Omega_0}$. Ad c): if $y \in \Omega(x)$, then
$$f_2(x,y) = \int_{\Omega_0} K_\sigma(y-h)\ \mathbb{1}^a_{\Omega(x)}(h)\ dh \geq \min_{y \in \chi_{T=T(x)}} \int_{\Omega_0} K_\sigma(y-h)\ \mathbb{1}^a_{\Omega(x)}(h)\ dh.$$
The last inequality is true, because the biggest cut-offs happen, looking at the shape of the kernel $K_\sigma(y - .)$, when y belongs to the level set $\chi_{T=T(x)}$. Consequently, with $\mathbb{1}^a_{\Omega(x)}(h) = \mathbb{1}^a_{\Omega(y)}(h)$ in the case of $T(y) = T(x)$, we get
$$\begin{aligned} f_2(x,y) &\geq \min_{y \in \chi_{T=T(x)}} \int_{\Omega_0} K_\sigma(y-h)\ \mathbb{1}^a_{\Omega(y)}(h)\ dh \\ &\geq \min_{y \in \Omega} \int_{\Omega_0} K_\sigma(y-h)\ \mathbb{1}^a_{\Omega(y)}(h)\ dh = m_\sigma. \end{aligned}$$
3. The statements are consequences of parts 1 and 2 put together. By 2c), f_3 is well-defined for $x \in \Omega$ and $y \in \Omega(x)$. f_3 is continuously differentiable as stated, since f_1 and f_2 are. After applying the quotient rule, the bounds on f_3 exist and are of the stated form, whereas the constants C_i are combinations of the bounds on f_2 and of the prefactors, regarding the bounds on f_1, in front of $\|v\|_{L^1(\Omega_0)}$.

□

The function $f_3(\,.\,,x)$, from lemma 6.3, equals $\hat{v}$ from equation (6.14). By plugging $f_3(\,.\,,x)$ and $\mathbb{1}^a_{\Omega(x)}$ into equation (6.13) the structure tensor is

$$S(x) = \frac{\int_{\Omega_0} K_\rho(x-y) \cdot \mathbb{1}^a_{\Omega(x)}(y) \cdot \nabla_y f_3(y,x) \cdot \nabla_y f_3(y,x)^T\ dy}{\int_{\Omega_0} K_\rho(x-y) \cdot \mathbb{1}^a_{\Omega(x)}(y)\ dy}.$$

The next lemma collects the features of S as a function of x.

Lemma 6.4.

1. *Let $f_4 : \Omega \to \mathbb{R}^{2\times 2}$ be defined by*

$$f_4(x) := \int_{\Omega_0} K_\rho(x-y) \cdot \mathbb{1}^a_{\Omega(x)}(y) \cdot \nabla_y f_3(y,x) \cdot \nabla_y f_3(y,x)^T \, dy \, .$$

Then, f_4 has the properties:

a) *f_4 is well-defined.*

b) *f_4 is continuous and bounded: $|f_4(x)| \le C_4 \|v\|^2_{L^1(\Omega_0)}$.*

c) *f_4 has a continuous and bounded derivative: $|Df_4(x)| \le C_5 \|v\|^2_{L^1(\Omega_0)}$.*

2. *Let $f_5 : \Omega \to \mathbb{R}^{2\times 2}$ be defined by*

$$f_5(x) := \int_{\Omega_0} K_\rho(x-y) \cdot \mathbb{1}^a_{\Omega(x)}(y) \, dy \, .$$

Then, f_5 has the properties:

a) *f_5 is continuous and bounded: $|f_5(x)| \le 1$.*

b) *f_5 has a continuous and bounded derivative:*

$$|\nabla f_4(x)| \le \|\nabla K_\rho\|_\infty \cdot \mathcal{L}^2(\Omega_0) \, .$$

c) *f_5 is bounded below: $f_5(x) \ge m_\rho > 0$.*

3. *Let $S : \Omega \to \mathbb{R}^{2\times 2}$ be defined by*

$$S(x) := \frac{f_4(x)}{f_5(x)} \, .$$

a) *S is continuous and bounded: $|S(x)| \le C_6 \|v\|^2_{L^1(\Omega_0)}$.*

b) *S has a continuous and bounded derivative:$|DS(x)| \le C_7 \|v\|^2_{L^1(\Omega_0)}$.*

Proof.

1. f_4 is well-defined, because the domain of integration is restricted to the set $\Omega(x)$, where $\nabla_y f_3(y,x)$ is defined according to part 3 of lemma 6.3. By the construction of f_4 and lemma 6.3 f_4 is continuously differentiable. The bound on $|f_4(x)|$, obtained by

$$|f_4(x)| \le \int_{\Omega_0} K_\rho(x-y) \, dy \cdot \|\nabla_y f_3(\,.\,,x)\|^2_{L^\infty(\Omega(x))} \le C_2^2 \cdot \|v\|^2_{L^1(\Omega_0)} \, ,$$

is a consequence of part 3c) of lemma 6.3. So, $C_4 = C_2^2$.

Let $h \in \partial B_1(0)$ be arbitrary but fixed. Then, the product $f_4(x) \cdot h$ belongs to $\in \mathbb{R}^2$ with

$$f_4(x) \cdot h = \int\limits_{\Omega_0} K_\rho(x-y) \cdot \mathbb{1}^a_{\Omega(x)}(y) \cdot \nabla_y f_3(y,x) \cdot \langle \nabla_y f_3(y,x), h \rangle \; dy \, .$$

Consequently, the derivative $D(f_4(x) \cdot h) = Df_4(x) \cdot h \in \mathbb{R}^{2\times 2}$ is given by

$$\begin{aligned} D(f_4(x) \cdot h) = \int\limits_{\Omega_0} & \mathbb{1}^a_{\Omega(x)}(y) \cdot \langle \nabla_y f_3(y,x), h \rangle \cdot \nabla_y f_3(y,x) \cdot \nabla K_\rho(x-y)^T \\ &+ K_\rho(x-y) \cdot H_a'(T(x)-T(y)) \cdot \langle \nabla_y f_3(y,x), h \rangle \cdot \nabla_y f_3(y,x) \cdot \nabla T(x)^T \\ &+ K_\rho(x-y) \cdot \mathbb{1}^a_{\Omega(x)}(y) \cdot \nabla_y f_3(y,x) \cdot h^T \cdot \nabla_x \nabla_y f_3(y,x) \\ &+ K_\rho(x-y) \cdot \mathbb{1}^a_{\Omega(x)}(y) \cdot \langle \nabla_y f_3(y,x), h \rangle \cdot \nabla_x \nabla_y f_3(y,x) \; dy \, . \end{aligned}$$

We then estimate

$$\begin{aligned} |D(f_4(x) \cdot h)| \le \; & \|\nabla K_\rho\|_\infty \cdot \|\nabla_y f_3(.,x)\|^2_{L^\infty(\Omega(x))} \cdot \mathcal{L}^2(\Omega) \cdot |h| \\ &+ \|H_a'\|_\infty \cdot \|\nabla T\|_\infty \cdot \|\nabla_y f_3(.,x)\|^2_{L^\infty(\Omega(x))} \cdot |h| \\ &+ 2 \cdot \|\nabla_y f_3(.,x)\|_{L^\infty(\Omega(x))} \cdot \|\nabla_x \nabla_y f_3(.,x)\|_{L^\infty(\Omega(x))} \cdot |h| \, . \end{aligned}$$

Using the results of part 3 of lemma 6.3 again and the fact that $|h| = 1$, we obtain the stated bound

$$\begin{aligned} |D(f_4(x) \cdot h)| \le \; & \|\nabla K_\rho\|_\infty \cdot \mathcal{L}^2(\Omega) \cdot C_2^2 \cdot \|v\|^2_{L^1(\Omega_0)} \\ &+ \|H_a'\|_\infty \cdot \|\nabla T\|_\infty \cdot C_2^2 \cdot \|v\|^2_{L^1(\Omega_0)} + 2 \cdot C_2 \cdot C_3 \cdot \|v\|^2_{L^1(\Omega_0)} \\ &=: C_5 \|v\|^2_{L^1(\Omega_0)} \, . \end{aligned}$$

The final step is:

$$|Df_4(x)| = \max_{|h|=1} |D(f_4(x) \cdot h)| \le C_5 \|v\|^2_{L^1(\Omega_0)} \, .$$

2. Considering f_2, defined in lemma 6.3, with the parameter σ replaced by ρ, we get

$$f_5(x) = f_2(x,x)|_{\sigma:=\rho} \, .$$

   So, the statements are direct consequences of part 2 of lemma 6.3.

3. By the definition of S the results of parts 1 and 2 put together yield the statements of part 3.

□

In the next step we consider the structure tensor

$$S : L^1(\Omega_0) \to C(\Omega)^{2\times 2}$$

as functional of $v \in L^1(\Omega_0)$ and deduce continuity properties.

Lemma 6.5.

1. *Let f_1 be as in lemma 6.3,*

$$\begin{aligned} &f_1 : L^1(\Omega_0) \to C(\Omega_0 \times \Omega) \ , \\ &f_1[v](y,x) = \int\limits_{\Omega_0} K_\sigma(y-h) \cdot H_a(T(x) - T(h)) \cdot v(h)\, dh \ , \end{aligned}$$

but now regarded as a functional. In the same way we regard the y-derivative $\nabla_y f_1[v]$ as a functional.

$$\begin{aligned} &\nabla_y f_1 : L^1(\Omega_0) \to C(\Omega_0 \times \Omega)^2 \ , \\ &\nabla_y f_1[v](y,x) = \int\limits_{\Omega_0} \nabla K_\sigma(y-h) \cdot H_a(T(x) - T(h)) \cdot v(h)\, dh \ . \end{aligned}$$

Then, we have:

a) f_1 is a bounded linear functional.

b) $\nabla_y f_1$ is a bounded linear functional.

2. *Let f_3 be as in lemma 6.3,*

$$\begin{aligned} &f_3 : L^1(\Omega_0) \to C(\Omega(x) \times \Omega) \ , \\ &f_3[v](y,x) := \frac{f_1[v](y,x)}{f_2(y,x)} \ , \end{aligned}$$

but now regarded as a functional. In the same way we regard the y-derivative $\nabla_y f_3[v]$ as a functional.

$$\begin{aligned} &\nabla_y f_3 : L^1(\Omega_0) \to C(\Omega(x) \times \Omega)^2 \ , \\ &\nabla_y f_3[v](y,x) := \frac{f_2(y,x) \cdot \nabla_y f_1[v](y,x) - f_1[v](y,x) \cdot \nabla_y f_2(y,x)}{f_2(y,x)^2} \ . \end{aligned}$$

Then, we have:

a) f_3 is a bounded linear functional.

b) $\nabla_y f_3$ is a bounded linear functional.

Proof.

The functionals are linear w.r.t. v by construction. The bounds, in the context of linear operators, are exactly the prefactors of the bounds in lemma 6.3 parts 1 and 3. $\square$

Lemma 6.6.

1. *Consider the map* $B : L^1(\Omega_0) \times L^1(\Omega_0) \to C(\Omega)^{2\times 2}$

   $$B[v,w](x) = \frac{1}{f_5(x)} \int_{\Omega_0} K_\rho(x-y) \cdot \mathbb{1}^a_{\Omega(x)}(y) \cdot \nabla_y f_3[v](y,x) \cdot \nabla_y f_3[w](y,x)^T \, dy \, ,$$

   with f_5 *as defined as in lemma 6.4. Then, B is bilinear and continuous w.r.t. v and w, i.e.,*

   $$\|B[v,w]\|_\infty \le C_8 \|v\|_{L^1(\Omega_0)} \cdot \|w\|_{L^1(\Omega_0)} \, .$$

2. *The structure tensor is given by*

   $$S : L^1(\Omega_0) \to C(\Omega)^{2\times 2} \, , \qquad S[v] = B[v,v] \, .$$

   S is Lipschitz-continuous w.r.t. v,

   $$\|S[v] - S[w]\|_\infty \le 2 \cdot C_8 \cdot \max\left\{\|v\|_{L^1(\Omega_0)}, \|w\|_{L^1(\Omega_0)}\right\} \cdot \|v-w\|_{L^1(\Omega_0)} \, .$$

Proof.

1. Since $\nabla_y f_3[v]$ is linear according to lemma 6.5 2.b) the product $\nabla_y f_3[v] \cdot \nabla_y f_3[w]$ is bilinear. Hence B is bilinear. For the continuity we have

   $$\begin{aligned} |B[v,w](x)| &\le \|\nabla_y f_3[v](\,.\,,x)\|_{L^\infty(\Omega(x))} \cdot \|\nabla_y f_3[w](\,.\,,x)\|_{L^\infty(\Omega(x))} \\ &\le C_8 \|v\|_{L^1(\Omega_0)} \cdot \|w\|_{L^1(\Omega_0)} \, . \end{aligned}$$

2. From part 1 follows

   $$\begin{aligned} |S[v](x) - S[w](x)| &\le |B[v, v-w](x)| + |B[v-w, w](x)| \\ &\le 2 \cdot C_8 \cdot \max\left\{\|v\|_{L^1(\Omega_0)}, \|w\|_{L^1(\Omega_0)}\right\} \cdot \|v-w\|_{L^1(\Omega_0)} \, , \end{aligned}$$

   which is the stated assertion.

$\square$

Our set-up of the structure tensor depends on the three positive parameters ρ, σ and a. The last lemma of this section is about the continuity of $S_{\rho,\sigma,a}$ w.r.t. these parameters. Here, we explicitly indicate the dependence on the parameters by the subscript. Equation (6.15) gives an overview where the parameters enter the set-up of $S_{\rho,\sigma,a}$.

$$\left.\begin{array}{r} \left.\begin{array}{r} \left.\begin{array}{l}\sigma,a \longrightarrow f_1^{\sigma,a}\\ \sigma,a\longrightarrow f_2^{\sigma,a}\end{array}\right\} \longrightarrow f_3^{\sigma,a} \\ \rho,a \end{array}\right\} \longrightarrow f_4^{\rho,\sigma,a} \\ \rho,a \longrightarrow f_5^{\rho,a}\end{array}\right\} \longrightarrow S_{\rho,\sigma,a} \tag{6.15}$$

Lemma 6.7.
Let $p = (\rho, \sigma, a) \in (\mathbb{R}^+)^3$ and let $\{p_n\}_{n\in\mathbb{N}}$, $p_n = (\rho_n, \sigma_n, a_n) \in (\mathbb{R}^+)^3$ be a sequence which tends to p. Then, the sequence $S_{p_n}[v]$ of tensor fields tends uniformly to $S_p[v]$, i.e.,

$$\|S_{p_n}[v] - S_p[v]\|_\infty \ \to 0\ , \qquad\qquad as \qquad n \ \to \infty\ .$$

Proof.

Consider $f_1^{\sigma,a}$ as defined in lemma 6.3,

$$f_1^{\sigma,a}(y,x) = \int\limits_{\Omega_0} K_\sigma(y-h) \cdot H_a(T(x) - T(h)) \cdot v(h)\ dh\ .$$

First we show that the weight functions on $\Omega_0 \times \Omega_0$ and $\Omega \times \Omega_0$ converge uniformly. By the assumed form

$$K_\sigma(y) = \frac{1}{\sigma^2} K\left(\frac{y}{\sigma}\right)\ , \qquad\qquad H_a(t) = \tilde{H}\left(\frac{t}{a}\right)\ ,$$

and the assumed differentiability features of K and $\tilde{H}$ using the mean value theorem we obtain the estimates

$$\begin{aligned}
|K_{\sigma_n}(y-h) - K_\sigma(y-h)| &= |\partial_\sigma K_{\sigma_*}(y-h)| \cdot |\sigma_n - \sigma| \\
&\le \left(\frac{2}{\sigma_*^3} K\left(\frac{y-h}{\sigma_*}\right) + \frac{1}{\sigma_*^4}\left|\nabla K\left(\frac{y-h}{\sigma_*}\right)\right| |y-h|\right) \cdot |\sigma_n - \sigma| \\
&\le \left(\frac{2}{\min\{\sigma,\sigma_n\}^3}\|K\|_\infty + \frac{1}{\min\{\sigma,\sigma_n\}^4}\|\nabla K\|_\infty \operatorname{diam}\Omega_0\right) \cdot |\sigma_n - \sigma| \\
&=: C_{\sigma_n} \cdot |\sigma_n - \sigma|\ ,
\end{aligned}$$

and

$$
\begin{aligned}
|H_{a_n}(T(x)-T(h)) - H_a(T(x)-T(h))| &= |\partial_a H_{a_*}(T(x)-T(h))| \cdot |a_n - a| \\
&\leq \left(\frac{1}{a_*} \left| \tilde{H}' \left(\frac{T(x)-T(h)}{a_*} \right) \right| |T(x)-T(h)| \right) \cdot |a_n - a| \\
&\leq \left(\frac{1}{\min\{a, a_n\}} \|\tilde{H}'\|_\infty 2 \|T\|_{L^\infty(\Omega_0)} \right) \cdot |a_n - a| \\
&=: C_{a_n} \cdot |a_n - a| \,,
\end{aligned}
$$

with real-valued bounded sequences C_{σ_n} and C_{a_n}. Consequently, we get

$$
\begin{aligned}
&|f_1^{\sigma_n,a_n}(y,x) - f_1^{\sigma,a}(y,x)| \\
&\leq \int_{\Omega_0} |K_{\sigma_n}(y-h) - K_\sigma(y-h)| \cdot H_{a_n}(T(x)-T(h)) \cdot |v(h)| \, dh \\
&\quad + \int_{\Omega_0} K_\sigma(y-h) \cdot |H_{a_n}(T(x)-T(h)) - H_a(T(x)-T(h))| \cdot |v(h)| \, dh \\
&\leq \|v\|_{L^1(\Omega_0)} \cdot C_{\sigma_n} \cdot |\sigma_n - \sigma| + \|K\|_\infty \cdot \|v\|_{L^1(\Omega_0)} \cdot C_{a_n} \cdot |a_n - a| \,,
\end{aligned}
$$

which shows the uniform convergence of $f_1^{\sigma_n,a_n}$ to $f_1^{\sigma,a}$. The same argumentation applies to $f_2^{\sigma_n,a_n}$, $\nabla_y f_1^{\sigma_n,a_n}$ and $\nabla_y f_2^{\sigma_n,a_n}$.

Let $f_3^{\sigma,a}$ be defined as in lemma 6.3. Then the y-derivative is given by

$$
\nabla_y f_3^{\sigma,a} = \frac{1}{f_2^{\sigma,a}} \nabla_y f_1^{\sigma,a} - \frac{f_1^{\sigma,a}}{(f_2^{\sigma,a})^2} \nabla_y f_2^{\sigma,a} \,.
$$

Clearly, the sequence $\nabla_y f_3^{\sigma_n,a_n}$ converges uniformly to $\nabla_y f_3^{\sigma,a}$ on its domain as a combination of uniformly converging sequences.

Finally, let

$$
S_p = \frac{f_4^{\rho,\sigma,a}}{f_5^{\rho,a}} \quad \text{and} \quad S_{p_n} = \frac{f_4^{\rho_n,\sigma_n,a_n}}{f_5^{\rho_n,a_n}}
$$

according to lemma 6.4. By applying the same argumentation as above we obtain the statement. □

6.3.3 Properties of the Transport Field

In this section we prepare for the existence, uniqueness, and stability results regarding the model of *Inpainting Based on Coherence Transport*. So far, we have discussed the features of the structure tensor. Now, the features of the transport field are the objective. The transport field is obtained by the following set-up chain:

$$S \xrightarrow[\text{Eq. (6.10)}]{} G \xrightarrow[\text{Eq. (6.11)}]{} k \xrightarrow[\text{Theo. 6.1}]{\text{VV-Limit}} c \; .$$

Here, we will show that the analytic properties of S are carried over to c along this chain.

Guidance Tensor: first, we rewrite G in an equivalent form, which is more appropriate for the analysis we are about to do. Consider the symmetric matrix S in its component-by-component description

$$S = \begin{pmatrix} S_0 & S_1 \\ S_1 & S_2 \end{pmatrix} .$$

The characteristic polynomial of S is given by

$$p_S(z) = z^2 - (S_0 + S_2)z + (S_0 S_2 - S_1^2) \; .$$

Now, let q denote the discriminant of p_S

$$q = (S_0 - S_2)^2 + 4S_1^2 \; ,$$

by which we can spell out the eigenvalues of S as

$$\lambda_0 = \frac{1}{2}(S_0 + S_2 - \sqrt{q}) \; , \qquad \lambda_1 = \frac{1}{2}(S_0 + S_2 + \sqrt{q}) \; .$$

Consequently, the discriminant of p_S is exactly the quality (coherence) measure

$$q = (\lambda_1 - \lambda_0)^2 \; .$$

According to equation (6.10), in the regular case $q > 0$, the guidance tensor is given by

$$G = \frac{\alpha(q)}{\sqrt{q}}(S - \lambda_0 I) = \frac{\alpha(q)}{\sqrt{q}} \begin{pmatrix} \frac{S_0 - S_2 + \sqrt{q}}{2} & S_1 \\ S_1 & \frac{S_2 - S_0 + \sqrt{q}}{2} \end{pmatrix} .$$

For the purpose of abbreviation we define the symmetric matrix

$$\tilde{S} = \begin{pmatrix} -S_2 & S_1 \\ S_1 & -S_0 \end{pmatrix} ,$$

which has the same continuity and differentiability features as S. Moreover, we set

$$\alpha_k : \mathbb{R}^+ \to \mathbb{R}^+ \,, \qquad \alpha_k(t) = \frac{\alpha(t)}{2\left(\sqrt{t}\right)^k} \,, \quad k \in \{0,1\} \,. \tag{6.16}$$

Then, G rewrites as

$$G = \frac{\alpha(q)}{2\sqrt{q}}(S + \tilde{S}) + \frac{\alpha(q)}{2} I = \alpha_1(q) \cdot (S + \tilde{S}) + \alpha_0(q) I \,.$$

Lemma 6.8.
The functions α_k, $k \in \{0,1\}$, defined in equation (6.16), satisfy:

a) $\lim\limits_{t \to 0_+} \alpha_k(t) = 0$ *and* $\lim\limits_{t \to 0_+} \alpha_k'(t) = 0$.

b) α_k *and* α_k' *are both bounded.*

Proof.

For α_0 the statements are clearly true, since it inherits these features directly from α. For α_1, we get

$$\alpha_1(t) = \frac{\alpha(t)}{2\sqrt{t}}, \qquad \alpha_1'(t) = \frac{\alpha'(t)}{2\sqrt{t}} - \frac{\alpha(t)}{4(\sqrt{t})^2} \,.$$

By applying L'Hospital's rule the equalities $\lim\limits_{t \to 0_+} \alpha_1(t) = 0$ and $\lim\limits_{t \to 0_+} \alpha_1'(t) = 0$ are a consequence of the limit behavior of α and its first and second derivative. Finally, since α and α' are bounded, in addition we get:

$$\lim_{t \to \infty} \alpha_1(t) = 0 \,, \qquad \lim_{t \to \infty} \alpha_1'(t) = 0 \,.$$

The latter implies the boundedness of α_1 and α_1'. □

The following lemma summarizes the effects on the tensor G as a function of x.

Lemma 6.9.
The guidance tensor, as a function of x, $G : \Omega \to \mathbb{R}^{2\times 2}$

$$G(x) = \begin{cases} \alpha_1(q(x)) \cdot (S(x) + \tilde{S}(x)) + \alpha_0(q(x)) I & , \; q(x) > 0 \\ 0 & , \; q(x) = 0 \end{cases} ,$$

is continuous and bounded with a continuous and bounded derivative

$$|G(x)| \le p_1(\, \|v\|_{L^1(\Omega_0)} \,) \,, \qquad |DG(x)| \le p_2(\, \|v\|_{L^1(\Omega_0)} \,) \,.$$

Here, p_1 and p_2 are polynomials with positive coefficients and with degrees $\deg p_1 = 2$, $\deg p_2 = 6$.

Proof.

Let the sets Ω^1 and Ω^2 be defined by

$$\Omega^1 := \{x \in \Omega : q(x) > 0\} \,, \qquad \Omega^2 := \{x \in \Omega : q(x) = 0\} \,.$$

By the continuity of q, the set Ω^1 is open. Hence, by lemma 6.8 and lemma 6.4 the function

$$G(x) = \begin{cases} \alpha_1(q(x)) \cdot (S(x) + \tilde{S}(x)) + \alpha_0(q(x))I & , \ x \in \Omega^1 \\ 0 & , \ x \in \Omega^2 \end{cases} ,$$

is continuously differentiable on Ω^1 and on the open components of Ω^2 with the derivative

$$DG = \begin{cases} \alpha_1(q) \cdot (DS + D\tilde{S}) + (\alpha_1'(q) \cdot (S + \tilde{S}) + \alpha_0'(q)I)Dq & , \ x \in \Omega^1 \\ 0 & , \ x \in \Omega^2 \end{cases} .$$

G and DG, in addition, both extend continuously onto $\partial\Omega^2$. This is a consequence of lemma 6.8, in particular, of the limit behavior of α_k and α_k', $k \in \{0, 1\}$, as $t \to 0_+$.

By lemma 6.4, G and DG inherit their bounds from S. On the one hand, we get

$$|G(x)| \leq 2C_6\|\alpha_1\|_\infty \|v\|^2_{L^1(\Omega_0)} + \|\alpha_0\|_\infty =: p_1(\ \|v\|_{L^1(\Omega_0)}\ ) \,,$$

and on the other,

$$\begin{aligned} |DG(x)| \leq\ & 32C_6^2C_7\|\alpha_1'\|_\infty \|v\|^6_{L^1(\Omega_0)} + 16C_6C_7\|\alpha_0'\|_\infty \|v\|^4_{L^1(\Omega_0)} \\ & + 2C_7\|\alpha_1\|_\infty \|v\|^2_{L^1(\Omega_0)} \ =: \ p_2(\ \|v\|_{L^1(\Omega_0)}\ ) \,. \end{aligned}$$

The latter inequality uses the fact that

$$Dq = 2(S_0 - S_2)(DS_0 - DS_2) + 8S_1DS_1 \,, \quad |Dq| \leq 16C_6C_7\|v\|^4_{L^1(\Omega_0)} \,.$$

$\square$

By the last two lemmas the distinction of cases in the definition of G is unnecessary: we get

$$G[v](x) = \alpha_1(q[v](x)) \cdot (S[v](x) + \tilde{S}[v](x)) + \alpha_0(q[v](x))I \qquad \forall\ x \in \Omega$$

with α_k as functions of mapping type $\mathbb{R}_0^+ \to \mathbb{R}_0^+$.
Now, we study the properties of the tensor G as a functional of $v \in L^1(\Omega_0)$.

Lemma 6.10.
The guidance tensor, as a functional of v,

$$G : L^1(\Omega_0) \to C(\Omega)^{2\times 2} , \quad v \to G[v] ,$$

is Lipschitz-continuous with

$$\|G[v] - G[w]\|_\infty \le p_3(\ \|v, w\| \ ) \|v - w\|_{L^1(\Omega_0)} .$$

Here, p_3 is a polynomial of degree $\deg p_3 = 5$ with positive coefficients and

$$\|v, w\| := \max \left\{ \|v\|_{L^1(\Omega_0)}, \|w\|_{L^1(\Omega_0)} \right\} .$$

Proof.

Let $D_S q$ denote the derivative of q w.r.t. the components of S,

$$D_S q(S) = (\ 2(S_0 - S_2) \ , \ -2(S_0 - S_2) \ , \ 8S_1) .$$

By the mean value theorem we have for some $t \in]0, 1[$

$$|q[v](x) - q[w](x)| \le |D_S q(h)| \cdot |S[v](x) - S[w](x)|$$

with $h = (1-t)S[v](x) + tS[w](x)$. By the bounds on $S[v]$ and $S[w]$ we obtain the estimate

$$|D_S q(h)| \le 16C_6 \|v, w\|^2 .$$

Writing the difference as

$$\begin{aligned} G[v] - G[w] = & (\alpha_1(q[v]) - \alpha_1(q[w])) \cdot (S + \tilde{S})[v] \\ & + ((S + \tilde{S})[v] - (S + \tilde{S})[w]) \cdot \alpha_1(q[w]) \\ & + (\alpha_0(q[v]) - \alpha_0(q[w])) \cdot I , \end{aligned}$$

we once more apply the mean value theorem to the functions α_k and obtain

$$\begin{aligned} |G[v](x) - G[w](x)| = & \|\alpha_1'\|_\infty \cdot 2C_6 \|v\|^2_{L^1(\Omega_0)} \cdot |q[v](x) - q[w](x)| \\ & + \|\alpha_1\|_\infty \cdot 2|S[v](x) - S[w](x)| \\ & + \|\alpha_0'\|_\infty \cdot |q[v](x) - q[w](x)| \end{aligned}$$

using the bounds on S. Now, we use the estimate on $|q[v](x) - q[w](x)|$ and get

$$\begin{aligned} |G[v](x) - G[w](x)| \le & (\|\alpha_1'\|_\infty \cdot 32C_6^2 \|v, w\|^4 + \|\alpha_1\|_\infty \cdot 2 \\ & + \|\alpha_0'\|_\infty \cdot 16C_6 \|v, w\|^2) \cdot |S[v](x) - S[w](x)| . \end{aligned}$$

And finally, lemma 6.6 yields

$$\begin{aligned} |G[v](x) - G[w](x)| \le & (\|\alpha_1'\|_\infty \cdot 64C_6^2 C_8 \|v, w\|^5 + \|\alpha_1\|_\infty \cdot 4C_8 \|v, w\| \\ & + \|\alpha_0'\|_\infty \cdot 32C_6 C_8 \|v, w\|^3) \cdot \|v - w\|_{L^1(\Omega_0)} \\ = & \ p_3(\|v, w\|) \cdot \|v - w\|_{L^1(\Omega_0)} . \end{aligned}$$

□

The set-up of the guidance tensor depends on four positive parameters: δ, and ρ, σ, a. The latter three are parameters of $S_{\rho,\sigma,a}$, while δ is a parameter of the confidence measure α_δ (see equation (6.7)). With α_δ and definition (6.16) we set

$$\alpha_k^\delta(t) := \frac{\alpha_\delta(t)}{2(\sqrt{t})^k} = \frac{1}{(\sqrt{\delta})^k}\alpha_k\left(\frac{t}{\delta}\right) .$$

With $p = (\delta, \hat{p}) = (\delta, \rho, \sigma, a)$ the guidance tensor is

$$G_p[v](x) = \alpha_1^\delta(q_{\hat{p}}[v](x)) \cdot (S_{\hat{p}}[v](x) + \tilde{S}_{\hat{p}}[v](x)) + \alpha_0^\delta(q_{\hat{p}}[v](x))I .$$

Lemma 6.11.
Let $p = (\delta, \rho, \sigma, a) \in (\mathbb{R}^+)^4$ and let $\{p_n\}_{n\in\mathbb{N}}$, $p_n = (\delta_n, \rho_n, \sigma_n, a_n) \in (\mathbb{R}^+)^4$, be a sequence which tends to p. Then, the sequence $G_{p_n}[v]$ of tensor fields tends uniformly to $G_p[v]$, i.e.,

$$\|G_{p_n}[v] - G_p[v]\|_\infty \to 0 , \qquad \textit{as} \quad n \to \infty .$$

Proof.

By lemma 6.7 we know that $S_{\hat{p}_n}[v]$ and $q_{\hat{p}_n}[v]$ uniformly tend to $S_{\hat{p}}[v]$ and $q_{\hat{p}}[v]$. The remainder follows by an application of the mean value theorem to α_k^δ w.r.t. δ. □

Finally, we discuss the transport field c. Again, we do this within two steps. Firstly, we regard c as a function of x.

Lemma 6.12.
For arbitrary but fixed $v \in L^1(\Omega_0)$ consider the transport kernel

$$k(x,\eta) = \sqrt{\frac{\pi}{2}}\mu \exp\left(-\frac{\mu^2}{2}\eta^T G[v](x)\eta\right) , \qquad x \in \Omega\, , \eta \in B_1(0) .$$

Then, the vanishing viscosity limit yields a transport field c,

$$c : \Omega \to \mathbb{R}^2 \qquad\qquad x \to c(x)$$

that satisfies requirements 3.6.

Proof.

The only thing left to show at this point is that the transport kernel satisfies the assumptions a), b), and c) of theorem 6.2. Since, by equation (6.9), G either equals zero or is a scalar multiple of a projector $G = \hat{\alpha} P_1$, we get

$$\min_{\eta \in B_1(0)} \eta^T G[v](x)\eta = 0 \qquad \max_{\eta \in B_1(0)} \eta^T G[v](x)\eta = \hat{\alpha} \leq 1 .$$

Hence, k is uniformly bounded above and below, away from zero,

$$0 < \sqrt{\frac{\pi}{2}}\mu e^{-\frac{\mu^2}{2}} \leq k \leq \sqrt{\frac{\pi}{2}}\mu ,$$

and assumption a) is satisfied.

k is differentiable with partial derivatives

$$D_x k(x,\eta) = -\sqrt{\frac{\pi}{2}}\frac{\mu^3}{2}\exp\left(-\frac{\mu^2}{2}\eta^T G[v](x)\eta\right)\eta^T DG[v](x)\eta ,$$
$$D_\eta k(x,\eta) = -\sqrt{\frac{\pi}{2}}\mu^3\exp\left(-\frac{\mu^2}{2}\eta^T G[v](x)\eta\right)G[v](x)\eta ,$$

which are uniformly bounded by

$$|D_x k(x,\eta)| \leq \sqrt{\frac{\pi}{2}}\frac{\mu^3}{2}\cdot p_2(\|v\|_{L^1(\Omega_0)}) ,$$
$$|D_\eta k(x,\eta)| \leq \sqrt{\frac{\pi}{2}}\mu^3\cdot p_1(\|v\|_{L^1(\Omega_0)}) ,$$

where p_1 and p_2 stem from lemma 6.9. This is assumption b).

For arbitrary but fixed η, the functions $k(\,.\,,\eta)$, $D_x k(\,.\,,\eta)$, and $D_\eta k(\,.\,,\eta)$ are continuous on Ω, which implies assumption c).

Theorem 6.2 yields the following bounds in particular:

1. inward-pointing condition (see requirement 3.6 part 2):

$$\langle c(x), N(x)\rangle \geq \frac{\exp\left(-\frac{\mu^2}{2}\right)}{\sqrt{2}} =: \beta > 0 , \quad \forall\, x \in \Omega\backslash\Sigma.$$

2. bound on the derivative (see requirement 3.6 part 3):

$$|Dc(x)| \leq \frac{\sqrt{2\pi}}{\mu e^{-\frac{\mu^2}{2}}}\left(\sqrt{\frac{\pi}{2}}\frac{\mu^3}{2}\cdot p_2(\|v\|_{L^1(\Omega_0)}) \right.$$
$$\left. + \sqrt{2}\sqrt{\frac{\pi}{2}}\mu^3\cdot p_1(\|v\|_{L^1(\Omega_0)})|DN(x)| + \sqrt{2}\sqrt{\frac{\pi}{2}}|DN(x)|\right) .$$

For the bound on $\|Dc\|_{L^1(\Omega)}$ we just have to plug in $\|DN\|_{L^1(\Omega)}$ in the last estimate, while for the bound on $\|Dc\|_{L^\infty(\Omega\setminus V_\varepsilon)}$ we just have to plug in $\|DN\|_{L^\infty(\Omega\setminus V_\varepsilon)}$.

□

Secondly, we regard c as functional of v.

Lemma 6.13.
Consider the transport kernel as functional of v:

$$k[v](x,\eta) = \sqrt{\frac{\pi}{2}}\mu \exp\left(-\frac{\mu^2}{2}\eta^T G[v](x)\eta\right) , \qquad x \in \Omega\,, \eta \in B_1(0)\,.$$

Then, the transport field c,

$$c : L^1(\Omega_0) \to C(\Omega\setminus\Sigma)^2\,, \qquad v \to c[v]\,,$$

obtained by the vanishing viscosity, is Lipschitz-continuous.

Proof.

By the mean value theorem we get

$$\begin{aligned}|k[v](x,\eta) - k[w](x,\eta)| &\leq \sqrt{\frac{\pi}{2}}\frac{\mu^3}{2}\exp\left(-\frac{\mu^2}{2}h\right)|\eta^T(G[v](x) - G[w](x))\eta| \\ &\leq \sqrt{\frac{\pi}{2}}\frac{\mu^3}{2}|G[v](x) - G[w](x)|\,,\end{aligned}$$

for some $h \geq 0$. Hence, for k_* and c_* we obtain – both defined in theorem 6.1 –

$$|k_*[v](x,\eta) - k_*[w](x,\eta)| \leq \sqrt{\frac{\pi}{2}}\frac{\mu^3}{4}|G[v](x) - G[w](x)|$$

and

$$\begin{aligned}|c_*[v](x) - c_*[w](x)| &\leq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|(k_*[v](x,\eta) - k_*[w](x,\eta))|_{\eta=Q(x)e(\varphi)}\right| d\varphi \\ &\leq \sqrt{\frac{\pi}{2}}\frac{\mu^3}{4}|G[v](x) - G[w](x)|\,.\end{aligned}$$

Moreover, theorem 6.2 tells us that

$$\frac{1}{\sqrt{2\pi}}\mu e^{-\frac{\mu^2}{2}} \leq |c_*[v](x)| \leq \frac{1}{\sqrt{\pi}}\mu\,.$$

Consequently, the transport field $c[v](x) = c_*[v](x)/|c_*[v](x)|$ satisfies

$$|c[v](x) - c[w](x)| \leq L \cdot |G[v](x) - G[w](x)| \;,$$

for some constant L which only depends on the parameter μ.

Finally, lemma 6.10 yields

$$\|c[v] - c[w]\|_\infty \leq L \cdot p_3(\; \|v, w\| \;) \cdot \|v - w\|_{L^1(\Omega_0)} \;.$$

□

The set-up of the transport field depends on five positive parameters: μ, and δ, ρ, σ, a. The latter four are parameters of $G_{\delta,\rho,\sigma,a}$, while μ is a parameter of the guided transport kernel k_μ. The next lemma is about the continuity of $c_{\mu,\delta,\rho,\sigma,a}$ w.r.t. these parameters.

Lemma 6.14.
Let $p = (\mu, \delta, \rho, \sigma, a) \in (\mathbb{R}^+)^5$ and let $\{p_n\}_{n \in \mathbb{N}}$, $p_n = (\mu_n, \delta_n, \rho_n, \sigma_n, a_n) \in (\mathbb{R}^+)^5$, be a sequence which tends to p. Then, the sequence $c_{p_n}[v]$ of vector fields tends uniformly to $c_p[v]$, i.e.,

$$\|c_{p_n}[v] - c_p[v]\|_\infty \;\rightarrow 0 \;, \qquad\qquad \text{as} \quad n \;\rightarrow \infty \;.$$

Proof.

We set $\hat{p} = (\delta, \rho, \sigma, a)$ and $\hat{p}_n = (\delta_n, \rho_n, \sigma_n, a_n)$. By lemma 6.11 we know that $G_{\hat{p}_n}[v]$ tends uniformly to $G_{\hat{p}}[v]$. The remainder follows by an application of the mean value theorem to k_μ w.r.t. μ and the dominated convergence theorem. □

6.3.4 Existence, Uniqueness and Continuous Dependence

In this section we conclude the existence and uniqueness of the solution to the model of *Inpainting Based on Coherence Transport* and its continuous dependence by using the theory of chapter 4. But a last step of preparation is necessary. So far, we have the transport field c,

$$c : L^1(\Omega_0) \rightarrow C(\Omega \backslash \Sigma)^2 \;, \qquad\qquad v \;\rightarrow c[v] \;,$$

as a functional of v, whereas v is defined on the full domain Ω_0, but inpainting only has to be done in $\Omega \subset \Omega_0$. Now, given some guessed fill-in $v \in L^1(\Omega)$ defined on Ω, we use the corresponding completed image

$$u_0 \cdot \mathbb{1}_{\Omega_0 \backslash \Omega} + v \cdot \mathbb{1}_\Omega \;,$$

which is then defined on the full domain Ω_0, and set

$$c[u_0, v] := c[u_0 \cdot \mathbb{1}_{\Omega_0\setminus\Omega} + v \cdot \mathbb{1}_{\Omega}] \ .$$

Assuming $u_0 \in L^1(\Omega_0\setminus\Omega)$ we obtain a well-defined transport field $c[u_0, .]$ as a functional of $v \in L^1(\Omega)$

$$c[u_0, .] \ : L^1(\Omega) \to C(\Omega\setminus\Sigma)^2 \ , \qquad\qquad v \ \to c[u_0, v] \ ,$$

while the data u_0 becomes a parameter of this map. With this definition, equation (6.12) rewrites as

$$\begin{aligned} \langle c[u_0, u](x), Du \rangle = 0 \ , \quad \text{in} \quad \Omega\setminus\Sigma \ , \\ u|_{\partial\Omega} = u_0|_{\partial\Omega} \ . \end{aligned} \tag{6.17}$$

In order to apply the fixed point concept of chapter 4 we need to concretize the subsets of function spaces for our solution operator according to definition 4.3. We adapt the latter to:

Definition 6.15.

a) The set of boundary functions / data is

$$\mathfrak{B}(\partial\Omega) := \{v \in BV(\partial\Omega) : \|v\|_{L^\infty(\partial\Omega)} \leq M_4 \ , \ |Dv| \leq M_5\} \ .$$

b) Let $M_* \in \mathbb{R}$ be given by

$$M_* := M_4 \cdot \mathcal{L}^2(\Omega) \ .$$

The set of L^1-fill-in-functions on Ω is

$$\mathfrak{F} = \mathfrak{F}(\Omega) := \{v \in L^1(\Omega) : \|v\|_{L^1(\Omega)} \leq M_*\} \ .$$

c) Let $M_{**} \in \mathbb{R}$ be given by

$$M_{**} := \ 2 \cdot M_4 \cdot \mathcal{H}^1(\Sigma) + \frac{M_5}{\beta \cdot m_0} \ .$$

The set of BV-fill-in-functions on Ω is

$$\mathfrak{X} = \mathfrak{X}(\Omega) := \{v \in BV(\Omega) : \|v\|_{L^1(\Omega)} \leq M_* \ , \ |Dv|(\Omega) \leq M_{**}\} \ .$$

d) The set of data functions on the data domain $\Omega_0\setminus\Omega$ is

$$\mathfrak{B}(\Omega_0\setminus\Omega) := \{v \in BV(\Omega_0\setminus\Omega) : \|v\|_{L^1(\Omega_0\setminus\Omega)} \leq M_6 \ , \ v|_{\partial\Omega} \in \mathfrak{B}(\partial\Omega)\} \ .$$

The right hand side of the inpainting model is identical with zero, hence the constants M_2 and M_3, assumed in requirements 4.2, equal zero. In turn, the constants M_* and M_{**} of definition 4.3 reduce to those of definition 6.15.

Lemma 6.16.
Let $u_0 \in \mathfrak{B}(\Omega_0\backslash\Omega)$ and let $p = (a, \sigma, \rho, \delta, \mu) \in (\mathbb{R}^+)^5$ be a fixed choice of the parameters concerning the set-up of the transport field. Consider now the transport field with functional domain $\mathfrak{F}(\Omega)$

$$c_p[u_0, .] \ : \mathfrak{F}(\Omega) \to C(\Omega\backslash\Sigma)^2 \,, \qquad\qquad v \ \to c[u_0, v] \,.$$

Then, $c_p[u_0, .]$ satisfies requirements 4.1 and 4.13.

Moreover, the solution operator according to corollary 4.5 is a well-defined self-mapping

a) of type $U : \mathfrak{F} \to \mathfrak{F}$.

b) of type $U : \mathfrak{X} \to \mathfrak{X}$.

Proof.

The largest part is already proven by lemmas 6.12 and 6.13. What remains to show is the uniformity of the constants w.r.t. v. For arbitrary $v \in \mathfrak{F}(\Omega)$ the completed image
$$\bar{v} = u_0 \cdot \mathbb{1}_{\Omega_0\backslash\Omega} + v \cdot \mathbb{1}_{\Omega}$$
belongs to $L^1(\Omega_0)$ with $\|\bar{v}\|_{L^1(\Omega_0)} \leq M_6 + M_*$, while $c_p[u_0, v] = c_p[\bar{v}]$. Hence, the bounds on $D_x c_p[u_0, v](x)$, according to lemma 6.12, hold uniformly w.r.t. v, and the local Lipschitz-constant of $c_p[u_0, .]$, according to lemma 6.13, is now a global one.

In chapter 4 we assumed, for the sake of simplicity, that the functional domain of the transport field and that of the right hand side both equal $L^1(\Omega)$. But, in fact, it is enough if the intersection of these two functional domain contains the interesting subset $\mathfrak{F}(\Omega)$. So, the solution operator is a well-defined self-mapping of the stated types and the argumentation is exactly that of corollary 4.5. □

Corollary 6.17. *(Existence and uniqueness)*
Let $\Omega \subset \Omega_0$ be a domain and $T \in C^2(\Omega)$ be a time function with stop set Σ in accordance with chapter 5. Let $p = (a, \sigma, \rho, \delta, \mu) \in (\mathbb{R}^+)^5$ be a fixed choice of the parameters concerning the set-up of the transport field. Let $u_0 \in \mathfrak{B}(\Omega_0\backslash\Omega)$. Then, the model of Inpainting Based on Coherence Transport, *i.e., equations (6.17), has a unique solution.*

Proof.

By lemma 6.16 we are in the framework of chapter 4. The statement here is a consequence of theorem 4.11 and theorem 4.22. □

Corollary 6.18. *(Continuous dependence on the data image)*
Let $\Omega \subset \Omega_0$ be a domain and $T \in C^2(\Omega)$ be a time function with stop set Σ in accordance with chapter 5. Let $p = (a, \sigma, \rho, \delta, \mu) \in (\mathbb{R}^+)^5$ be a fixed choice of the parameters concerning the set-up of the transport field. Let $u_0 \in \mathfrak{B}(\Omega_0 \backslash \Omega)$. Let $\{u_{0,n}\}_{n\in\mathbb{N}}$ be a sequence with $u_{0,n} \in \mathfrak{B}(\Omega_0 \backslash \Omega)$ that tends to u_0 w.r.t. the strict topology on $BV(\Omega_0 \backslash \Omega)$. Moreover, let u and u_n be the unique solutions of

$$\begin{aligned} \big\langle c_p[u_0, u](x), Du \big\rangle &= 0\,, \quad in \quad \Omega \backslash \Sigma\,, \\ u|_{\partial\Omega} &= u_0|_{\partial\Omega}\,, \end{aligned}$$

and

$$\begin{aligned} \big\langle c_p[u_{0,n}, u_n](x), Du_n \big\rangle &= 0\,, \quad in \quad \Omega \backslash \Sigma\,, \\ u_n|_{\partial\Omega} &= u_{0,n}|_{\partial\Omega}\,, \end{aligned}$$

respectively.

Then,

$$\|u_n - u\|_{L^1(\Omega)} \;\rightarrow 0\,, \qquad\qquad as \quad n \rightarrow \infty\,.$$

Proof.

This continuity result is a consequence of theorem 4.24. The transport fields $c_p[u_0, .]$ and $c_p[u_{0,n}, .]$, $n \in \mathbb{N}$, are all of the same class, i.e., they satisfy the same inward-pointing condition, the same bounds on their x-derivative and the same Lipschitz-constant (see lemmas 6.12 and 6.13). These common constants are the basic ingredients for theorem 4.24. Let $\varepsilon > 0$. By the latter theorem, we can find $\delta > 0$ such that

$$\|u_n - u\|_{L^1(\Omega)} \leq \varepsilon\,.$$

whenever

$$\Big(\|(u_0 - u_{0,n})|_{\partial\Omega}\|_{L^1(\partial\Omega, \mathcal{H}^1)} \;+\; M_{**} \cdot \|c_p[u_0, .] - c_p[u_{0,n}, .]\|_0 \Big) \;\leq\; \delta\,.$$

And, reviewing the proof of theorem 4.24, δ only depends on ε and the common constants.

Because of the assumed strict convergence of $u_{0,n}$ to u_0, on the one hand, we have

$$\|u_{0,n} - u_0\|_{L^1(\Omega_0 \backslash \Omega)} \;\rightarrow 0\,, \qquad\qquad \text{as} \quad n \rightarrow \infty\,,$$

because this is part of the strict topology (see definition 2.18). On the other hand, we have

$$\|(u_0 - u_{0,n})|_{\partial\Omega}\|_{L^1(\partial\Omega,\mathcal{H}^1)} \to 0\,, \qquad \text{as} \quad n \to \infty\,,$$

because of the continuity of the trace operator w.r.t. the strict topology (see theorem 2.27).

Let L_1 denote the common Lipschitz constant of the transport fields. For $v \in \mathfrak{F}(\Omega)$ consider the completed images

$$\begin{aligned} \bar{v} &= u_0 \cdot \mathbb{1}_{\Omega_0\setminus\Omega} + v \cdot \mathbb{1}_\Omega\,, \\ \bar{v}_n &= u_{0,n} \cdot \mathbb{1}_{\Omega_0\setminus\Omega} + v \cdot \mathbb{1}_\Omega\,. \end{aligned}$$

Then, we get

$$\begin{aligned} \|c_p[u_{0,n}, v] - c_p[u_0, v]\|_\infty &= \|c_p[\bar{v}_n] - c_p[\bar{v}]\|_\infty \\ &\le L_1\|\bar{v}_n - \bar{v}\|_{L^1(\Omega)} = L_1\|u_{0,n} - u_0\|_{L^1(\Omega_0\setminus\Omega)}\,, \end{aligned}$$

which yields

$$\|c_p[u_0, \,.\,] - c_p[u_{0,n}, \,.\,]\|_0 \le L_1\|u_0 - u_{0,n}\|_{L^1(\Omega_0\setminus\Omega)} \to 0\,, \quad \text{as} \quad n \to \infty\,.$$

If $m \in \mathbb{N}$ is chosen big enough, then for all $n \ge m$ the ε-δ-condition above is certainly satisfied. □

Corollary 6.19. *(Continuous dependence on the parameters)*
Let $\Omega \subset \Omega_0$ be a domain and $T \in C^2(\Omega)$ be a time function with stop set Σ in accordance with chapter 5. Let $u_0 \in \mathfrak{B}(\Omega_0\setminus\Omega)$ be a fixed choice of the data function. Let $p = (a, \sigma, \rho, \delta, \mu) \in (\mathbb{R}^+)^5$. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence with $p_n \in (\mathbb{R}^+)^5$ that tends to p. Moreover, let u and u_n be the unique solutions of

$$\begin{aligned} \langle c_p[u_0, u](x), Du\rangle &= 0\,, \quad \text{in} \quad \Omega\setminus\Sigma\,, \\ u|_{\partial\Omega} &= u_0|_{\partial\Omega}\,, \end{aligned}$$

and

$$\begin{aligned} \langle c_{p_n}[u_0, u_n](x), Du_n\rangle &= 0\,, \quad \text{in} \quad \Omega\setminus\Sigma\,, \\ u_n|_{\partial\Omega} &= u_0|_{\partial\Omega}\,, \end{aligned}$$

respectively.

Then,

$$\|u_n - u\|_{L^1(\Omega)} \to 0\,, \qquad \text{as} \quad n \to \infty\,.$$

Proof.

Let $v \in \mathfrak{F}(\Omega)$ with the completed image

$$\bar{v} = u_0 \cdot \mathbb{1}_{\Omega_0 \setminus \Omega} + v \cdot \mathbb{1}_{\Omega} \,.$$

Reviewing lemma 6.7 (the parameter-continuity of the structure tensor), one can see that the pre-factors, which depend on $\|\bar{v}\|_{L^1(\Omega)}$, are now uniform. This feature transfers over to the guidance tensor (lemma 6.11) and to the transport field (lemma 6.14). Hence, we have the uniform convergence

$$\|c_{p_n}[u_0, .] - c_p[u_0, .]\|_0 \;\to 0 \,, \qquad \text{as} \quad n \to \infty \,.$$

For fixed $n \in \mathbb{N}$ the bounds (that are the inward-pointing condition, the bounds on the x-derivative, and the Lipschitz-constant – according to the requirements 4.1 and 4.13) of a single $c_{p_n}[u_0, .]$ all depend on the set of parameters p_n. Thus, we have sequences, made of those bounds, which correspond to the sequence $c_{p_n}[u_0, .]$. The former sequences are itself bounded. From this feature we obtain the required common bounds, i.e., if $m \in \mathbb{N}$ is big enough, the transport fields $c_{p_n}[u_0, .]$, for $n \geq m$ are all of the same class.

Now, the assertion follows from theorem 4.24. The remaining argumentation is the same as in the previous corollary. □

Remark: in corollaries 6.18 and 6.19 the sequence u_n, in fact, converges BV-weakly* to u due to proposition 2.16.

6.4 Distance-To-Boundary Map as Time

At this point, we want discuss the difficulties of our theory, which arise, when the euclidean distance-to-boundary map induces the order, or rather when it is used as time.

In the situation of theorem 6.1 the vanishing viscosity limit applies at every point $x \in \Omega$ where $N(x)$, the normal to the time-level of x, is uniquely determined. Consequently, the resulting transport field c, and thus the PDE, is defined on $\Omega \setminus \Sigma$. Here, the exceptional set of the PDE-domain is exactly the stop set Σ.

Now, let us see what the exceptional set is in the case of $d(x) = \operatorname{dist}(x, \partial\Omega)$. This function is the unique viscosity solution of the Dirichlet problem for the eikonal equation

$$\begin{aligned} |\nabla d| &= 1 \quad \text{in} \quad \Omega \,, \\ d|_{\partial\Omega} &= 0 \,. \end{aligned}$$

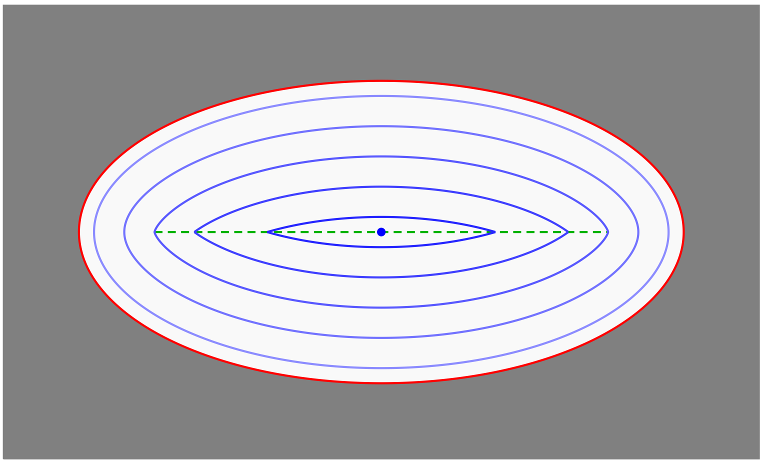


Figure 6.2: Distance-to-boundary map of an ellipse, red: boundary of Ω, blue: different level lines, dashed green: skeleton.

The corresponding field of normals is given by

$$N(x) = \nabla d(x) \; .$$

The subset, where N does not exist, is the *skeleton* $\mathcal{S}$ of the domain Ω. There are at least four different equivalent definitions of the skeleton (a.k.a. medial axis) (see [Kim04]). Here, we choose $\mathcal{S}$ as the set of singularities (locations of ridges) of the map d. That is, $\mathcal{S}$ is the smallest closed set such that $d \in C^1(\Omega \backslash \mathcal{S})$. For illustration: figures 6.2 and 6.3 show different level lines and the skeleton of the distance-to-boundary map in the case of an ellipse and in the case of a rectangle.
Now, when using d, the vanishing viscosity limit applies at every point $x \in \Omega \backslash \mathcal{S}$, and the exceptional set of the PDE-domain is the skeleton $\mathcal{S}$.

The set $\mathcal{S}$ has the desired shape: if the boundary $\partial\Omega$ is a sufficiently regular curve, it can be shown that $d \in C^2(\Omega \backslash \mathcal{S})$ and that $\mathcal{S}$ is a connected set with tree-like structure consisting of finitely many C^1-arcs (see [CCM97]).
But even if the domain Ω is of a simple shape, the skeleton $\mathcal{S}$ is not a stop set in the sense of requirement 3.4 part 3, since

$$d|_{\mathcal{S}} \neq \text{const.}$$

The fact that $\mathcal{S}$ is not a stop set causes some difficulties. Stop sets Σ, as in the previous chapters, feature the property that there is no transport across Σ. Here, when using d, we define Σ to be that subset of $\mathcal{S}$ where no transport across takes place.

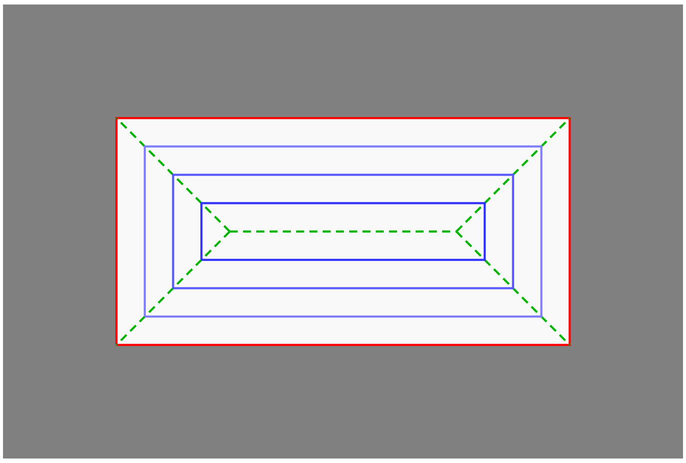


Figure 6.3: Distance-to-boundary map of a rectangle, red: boundary of Ω, blue: different level lines, dashed green: skeleton.

But now Σ depends on the choice of the transport field. In the case of transport along normals, the transport field is given by

$$c = N = \nabla d \; .$$

The skeleton $\mathcal{S}$ then, is exactly the set where the characteristics of $c = N$ intersect for the first time. Hence, transport across $\mathcal{S}$ is impossible here, and we have $\Sigma = \mathcal{S}$. But, in the case of guided transport, together with d as time, we have shown in [BM07] theorem 3 that parts of the skeleton become transparent. This means, that there are parts of the skeleton, where transport across happens, so $\Sigma \neq \mathcal{S}$.
The only part of $\mathcal{S}$ which always belongs to the stop set Σ, independent of c, is the subset $\mathcal{S}_{\max}$ which consists of the local maxima of d. Summarizing the observations we obtain

$$\mathcal{S}_{\max} \subseteq \Sigma(c) \subseteq \mathcal{S} \; ,$$

while $\Sigma(c)$ varies with the transport field c.

The basic difficulties are

- If $x \in \mathcal{S}$, the level set

  $$\{z \in \Omega : d(z) = d(x)\} \; ,$$

  typically has a kink at x. If now c is such that the point x is transparent, the boundary of the remaining

  $$\{z \in \Omega : d(z) > d(x)\} \; ,$$

  inpainting domain is not C^1.

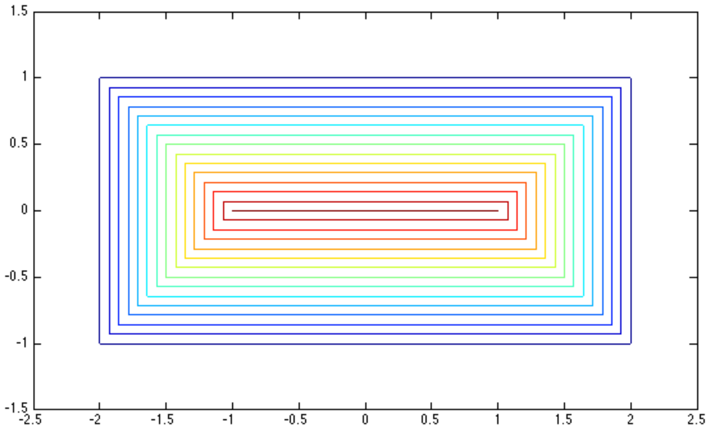
- Typically, c is as smooth as $N = \nabla d$. Hence, c is not differentiable at any point of $\mathcal{S}$.
- If $x \in \mathcal{S}$ is a transparent triple (or n-fold) point of the skeleton, i.e., three (or n) arcs of the skeleton meet there, then x often happens to be a saddle node of d, and the saddle level has a kink at x. Here, additionally, the saddle point difficulties discussed in chapter 5 appear.
- For the quasi-linear theory of chapter 4 it is crucial that the stop set Σ is independent of all admissible c. There, $c[v]$ varies with v, but we have several uniformity requirements, in particular, T and Σ both have to be fixed.
  But, here, $\Sigma(c) \subset \mathcal{S}$ moves with c and thus with v.

Work-around
If Ω has a simple shape, for example if Ω is a rectangle, we suggest the following work-around:

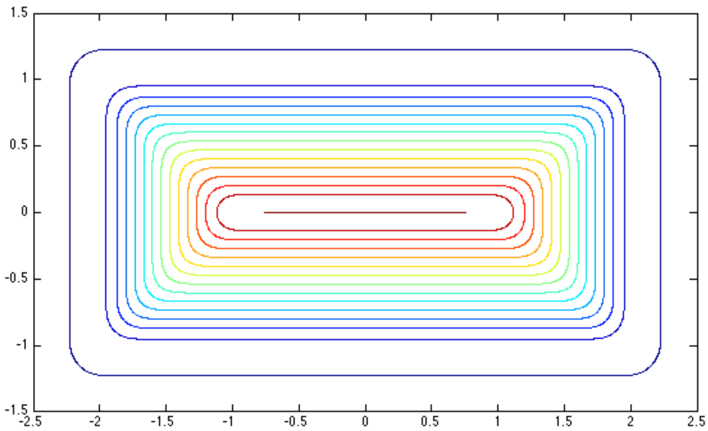
1. Choose the inpainting domain a bit bigger, i.e., take $\Omega^\delta := \Omega + B_\delta(0)$, $0 < \delta \ll 1$.
2. Extend the distance-to-boundary map by zero, i.e.,
$$\hat{d} : \Omega_0 \to \mathbb{R}\,, \quad \hat{d}(x) = \mathbb{1}_\Omega(x) \cdot d(x)\,.$$
3. Take a smoothed version of $\hat{d}$ as time T. Let K be a smoothing kernel as in section 6.3.2, but with $\operatorname{supp} K = B_1(0)$, and set
$$T : \Omega^\delta \to \mathbb{R}\,, \quad T(x) = K_\delta * \hat{d}(x)\,.$$

Figure 6.4 shows the effect for a rectangular Ω. In this example the part $\mathcal{S}_{\max}$, which belongs to the stop set independently of the transport field c, is the central line segment of $\mathcal{S}$ (see also figure 6.3). Comparing figure 6.4 (a) and (b) we can see that the stop set of T (the maxima of T) is a slightly shortened version of $\mathcal{S}_{\max}$, while the problematic parts of $\mathcal{S}$, because they have been be smoothed, have vanished. The latter effect can be understood in the way that those parts of $\mathcal{S}$, which might become transparent depending on the concrete c, are now made transparent for every choice of c.
Finally, in this example, Ω^δ and T, together with Σ, satisfy the requirements of chapter 3, and we can solve the problem with the changed data $u_0^\delta = u_0|_{\Omega_0 \setminus \Omega^\delta}$.

Remark: this work-around is not general enough; in the case of $\hat{d}$ having a saddle node the corresponding saddle node of T might not be admissible in view chapter 5. If it works, the question of what happens in the limit $\delta \to 0$ arises. Unfortunately, we must leave this question open.



(a) contours of $\hat{d}$



(b) contours of T

Figure 6.4: Work-around for a rectangular Ω

Chapter 7

Experiments on Different Orders

In this chapter, we will report on a few computational experiments concerning the utilization of the ”new” parameter *time* respectively *order*. The generic algorithm of section 6.1 depends on a prescribed order, which orders the pixels from the boundary inwards. In all previous experiments (see [BM07]) the pixels were ordered by their euclidean distance to boundary. For all types of domains the approximate distance-to-boundary map is easy to generate by the *fast marching method* (see [Set99] and [Kim04]), but it is not always the best choice if one wants to get a nice looking inpainting result. Here, we present three other ways of setting up a discrete time-like map. We show a few examples where they yield better inpaintings than the distance-to-boundary order.

The generic algorithm is performed in its *coherence transport* version. That is, the weight function w has the form as assumed in theorem 6.1 with the kernel k given by equation (6.11). The execution of the coherence transport algorithm, then, depends on the choice of four parameters:

- ε, the averaging radius,
- μ, the guidance strength,
- σ and ρ, the scale parameters of the smoothing operations in the structure tensor.

The remaining two parameters of the structure and guidance tensor are fixed to $a = 1$ and $\delta = 1$. Finally, the algorithm is supplied with the data image u_0 and a sorted list of the pixels which are to be inpainted. Any item of this list has the form

$$[i \quad j \quad T_h(i,j)]\ ,$$

(a) damaged image, white Ω_h (b) inpainted image (c) inpainted image with contours of d_h

Figure 7.1: Broken diagonal

(a) damaged image, white Ω_h (b) inpainted image (c) inpainted image with contours of d_h

Figure 7.2: Two broken diagonals

whereas (i, j) are the pixel coordinates and $T_h(i, j)$ is the time value of the pixel. The list is sorted in ascending order of the $T_h(i, j)$-values.

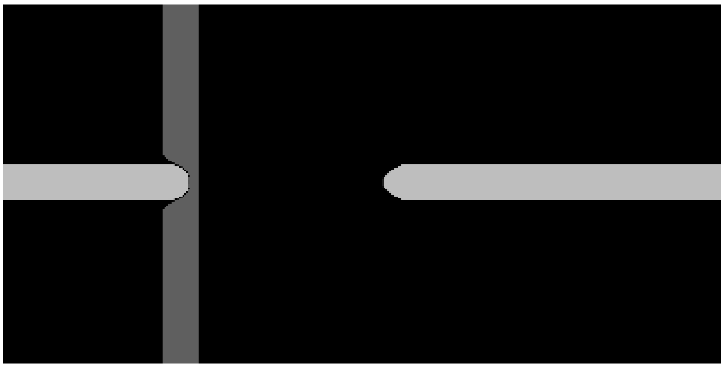
We consider four examples, where distance-to-boundary ordering is not favorable:

1. Example: The broken diagonal.
Figure 7.1 (a) shows a damaged image with the damaged area painted white. The desired inpainting result would be the restored diagonal. But the algorithm performed with distance-to-boundary ordering and the set of parameters

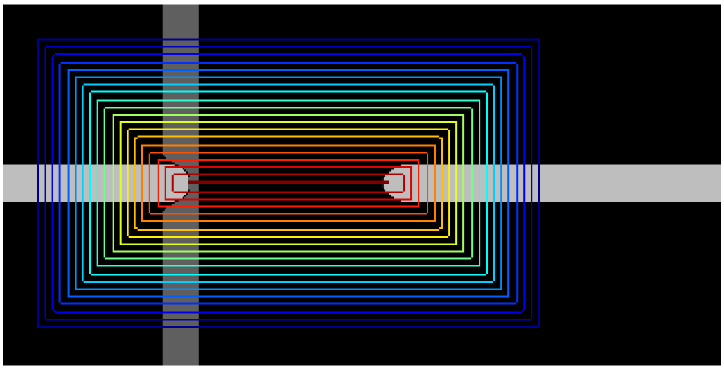
$$[\varepsilon, \mu, \sigma, \rho] = [3, 50, 0.5, 5]$$

yields the result shown in 7.1 (b), where the diagonal is not restored. The diagonal is only partly continued correctly. In figure 7.1 (c), the result is overlayed with the contours of the distance-to-boundary map d_h, and it shows that the undesired effect is due to the ”wrong” location of the stop set.

(a) damaged image, white Ω_h



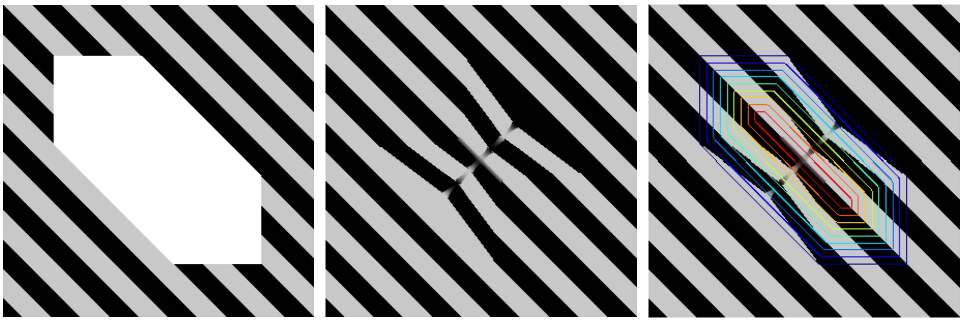
(b) inpainted image



(c) inpainted image with contours of d_h

Figure 7.3: Broken junction

2. *Example:* Two broken diagonals.
The second example, figure 7.2, has the same Ω_h and is performed with the same parameters as the first example. We emphasize here that the appearance of an undesired effect depends on how the edge that needs to be continued is located in relation to the inpainting domain. Here the bottom-



(a) damaged image (b) inpainted image (c) inpainted image with contours of d_h

Figure 7.4: Stripe pattern

left-to-top-right diagonal is continued as desired, while the continuation of the top-left-to-bottom-right diagonal suffers from a badly located stop set.

3. Example: The broken junction.
Figure 7.3 (a) shows a damaged cross junction. A cross junction would, in any case, geometrically be the simplest object for completion. The algorithm performed with the parameters

$$[\varepsilon, \mu, \sigma, \rho] = [5, 100, 0.5, 10]$$

yields the result shown in figure 7.3 (b) while in figure 7.3 (c) the result is overlayed with the contours of d_h. Here, the stop set, which is the central arc of the skeleton, has the wrong location again. The bar coming from the right hand side can never reach its counterpart.

4. Example: The stripe pattern.
Figure 7.4 (a) shows a damaged stripe pattern. The algorithm performed with the set of parameters

$$[\varepsilon, \mu, \sigma, \rho] = [5, 100, 0.5, 10]$$

yields the result shown in figure 7.4 (b). In figure 7.4 (c) the result is overlayed with the contours of the distance-to-boundary map. The difficulty, here, is that the tangent of the edges is orthogonal to the lower left and the upper right segment of $\partial\Omega_h$. And thus, as explained in section 6.2.2, the transport vector c switches to the normal N.

Let us see if we can do better.

7.1 Order by Harmonic Interpolation

In examples 1, 2, and 3, the wrong location of the stop set caused problems. Now, we describe the construction of a discrete time function T_h for which we can prescribe the location of the stop set arcs and the exact time when these arcs are reached.

Because the boundary is the start set, the discrete time function T_h must equal zero on $\partial\Omega_h$. In addition, we take at least one or more discrete curves Γ_h^k, $k \in \{1, \ldots, n\}$, which are contained in Ω_h and belong to the stop set Σ_h. Moreover, for every Γ_h^k we specify a time value $t_k > 0$ when this curve has to be reached. The remainder of T_h then, is calculated by harmonic interpolation. That is, we solve the discrete Laplace equation

$$\begin{aligned} \Delta_h T_h &= 0 && \text{in} \quad \Omega_h \setminus \bigcup_{k=1}^{n} \Gamma_h^k \,, \\ T_h &= 0 && \text{on} \quad \partial\Omega_h \,, \\ T_h &= t_k && \text{on} \quad \Gamma_h^k \,, \quad k = 1, \ldots, n \,. \end{aligned}$$

Hereby, the discretization Δ_h of the Laplacian is due to the five-point-stencil

$$\Delta_h = \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} ,$$

with

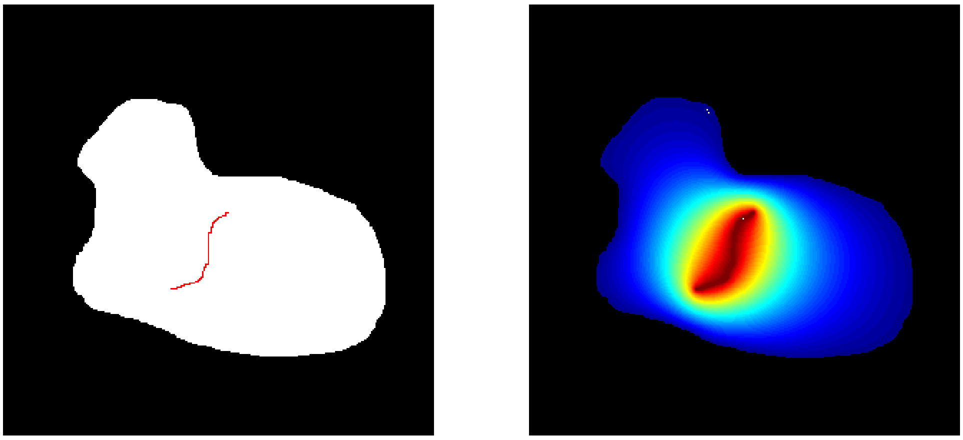
$$\Delta_h T_h(i,j) = T_h(i-1,j) + T_h(i,j-1) - 4T_h(i,j) + T_h(i,j+1) + T_h(i+1,j) \,.$$

Since harmonic interpolation provides a minimum and maximum principle, this construction of T_h can be imagined as the setting up a tent roof over the domain Ω_h where Γ_h^k are the locations of the tent poles, and every tent pole of Γ_h^k has the length t_k.
Unfortunately, not every choice of curves Γ_h^k, with time values t_k, results in a valid time-like function. In the case of a single curve Γ_h the resulting T_h must be time-like because of the minimum and maximum principle. Figure 7.5 shows an example. If there are two or more curves Γ_h^k, the question whether the resulting T_h is time-like or not depends on the location of the curves in relation to each other and the differences of their prescribed values t_k . Figure 7.6 shows an example with three curves Γ_h^1, Γ_h^2 and Γ_h^3 with

$$t_1 = t_2 = 250 \quad > \quad t_3 = 50 \,.$$

Here, all points of Γ_h^3 are local minima of T_h.
But, if we keep the geometry of Ω_h, Γ_h^1, Γ_h^2 and Γ_h^3, and change the prescribed times to

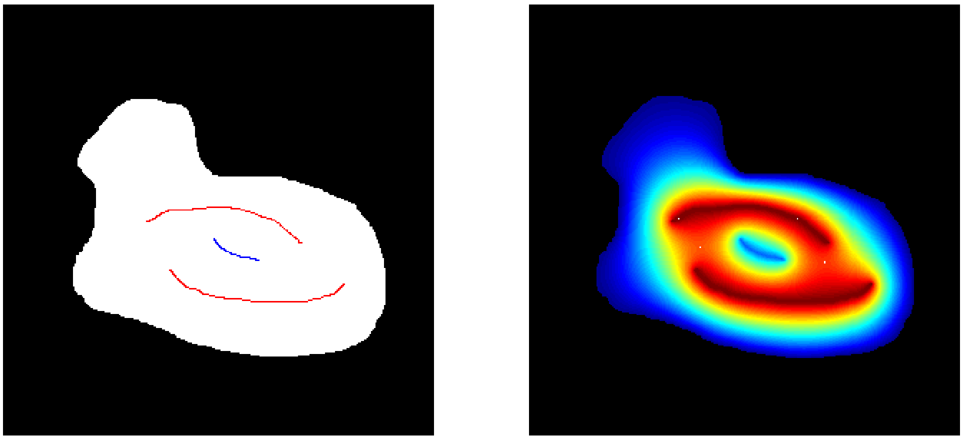
$$t_1 = t_2 = 250 \quad > \quad t_3 = 249 \,,$$



(a) white: Ω_h, red: Γ_h^1 with $t_1 = 127$

(b) contour plot of T_h

Figure 7.5: Single Γ_h yields a valid T_h



(a) white: Ω_h, red: Γ_h^1 and Γ_h^2 with $t_1 = t_2 = 250$,

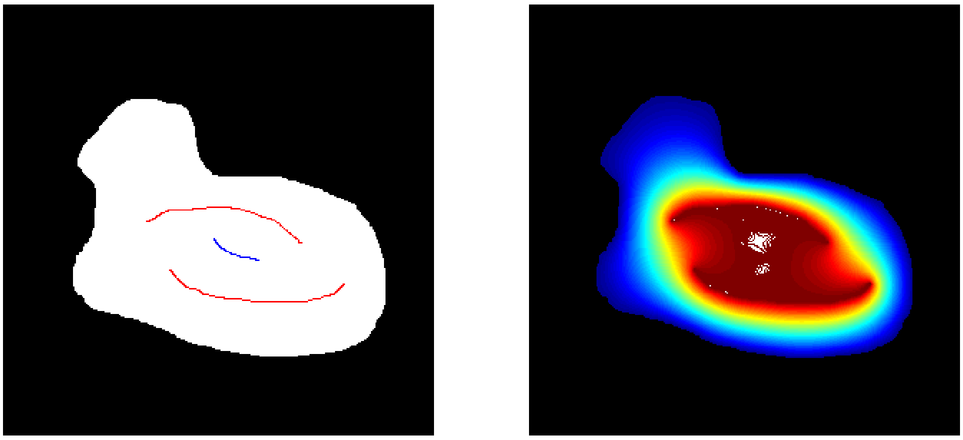
blue: Γ_h^3 with $t_3 = 50$

(b) contour plot of T_h

Figure 7.6: Non-valid T_h

then T_h does not have any minima (see figure 7.7). So, the resulting T_h is admissible. Generally speaking, if we have two or more curves Γ_h^k, with different prescribed time values t_k, and if the values t_k are chosen unfavorably, then the resulting T_h might possess local minima on some of the Γ_h^k. Remark: the suggested construction only works for the discrete case, since the corresponding high-resolution limit, as $h \to 0$, results in an ill-posed problem.

Let us review example 1. Figure 7.8 shows the broken diagonal again. In Figure 7.8 (a) we have the damaged image with the single curve Γ_h^1, $t_1 = 127$ shown in red. Figure 7.8 (b) and Figure 7.8 (c) show the inpainted result,



(a) white: Ω_h, red: Γ_h^1 and Γ_h^2 with $t_1 = t_2 = 250$, blue: Γ_h^3 with $t_3 = 249$

(b) contour plot of T_h

Figure 7.7: Valid T_h

(a) damaged image, white Ω_h, red Γ_h^1

(b) inpainted image

(c) inpainted image with contours of T_h

Figure 7.8: Broken diagonal

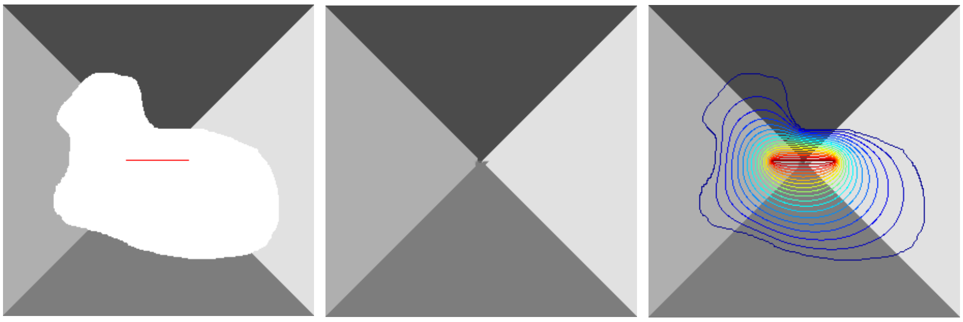
the latter is overlayed with the contours of T_h. The set of parameters is the same as before. Now, the inpainting method is able to close the broken diagonal because the stop set Σ_h has a good location.
If we think of Σ_h as the initial scratch, which has been dilated to Ω_h over the time T_h, then the backward filling-in process, if Σ_h is well located, makes the matching opposite sides come together. Clearly, if we deliberately place $\Gamma_h^1 = \Sigma_h$ badly, then the method must fail (see figure 7.9).

Ad example 2: in Figure 7.10 (a) we have the damaged image with the single curve Γ_h^1, $t_1 = 127$ shown in red. Figure 7.10 (b) and Figure 7.10 (c) show the inpainted result, the latter is overlayed with the contours of T_h. The set of parameters is the same as before. Again, the good location of Σ_h makes for a good result.

In the same way we are able to restore the cross junction of example 3 (see

(a) damaged image, white Ω_h, red Γ_h^1 (b) inpainted image (c) inpainted image with contours of T_h

Figure 7.9: Broken diagonal



(a) damaged image, white Ω_h, red Γ_h^1 (b) inpainted image (c) inpainted image with contours of T_h

Figure 7.10: Two broken diagonals

figure 7.11). Here, Γ_h^1 is a single point placed at the center of the cross junction. The set of parameters is the same as before. Which of the bars is closed in the end depends on the coherence strength. The brighter bar has the higher contrast w.r.t. the black background and is thus of stronger coherence. This is the reason why this bar is closed.

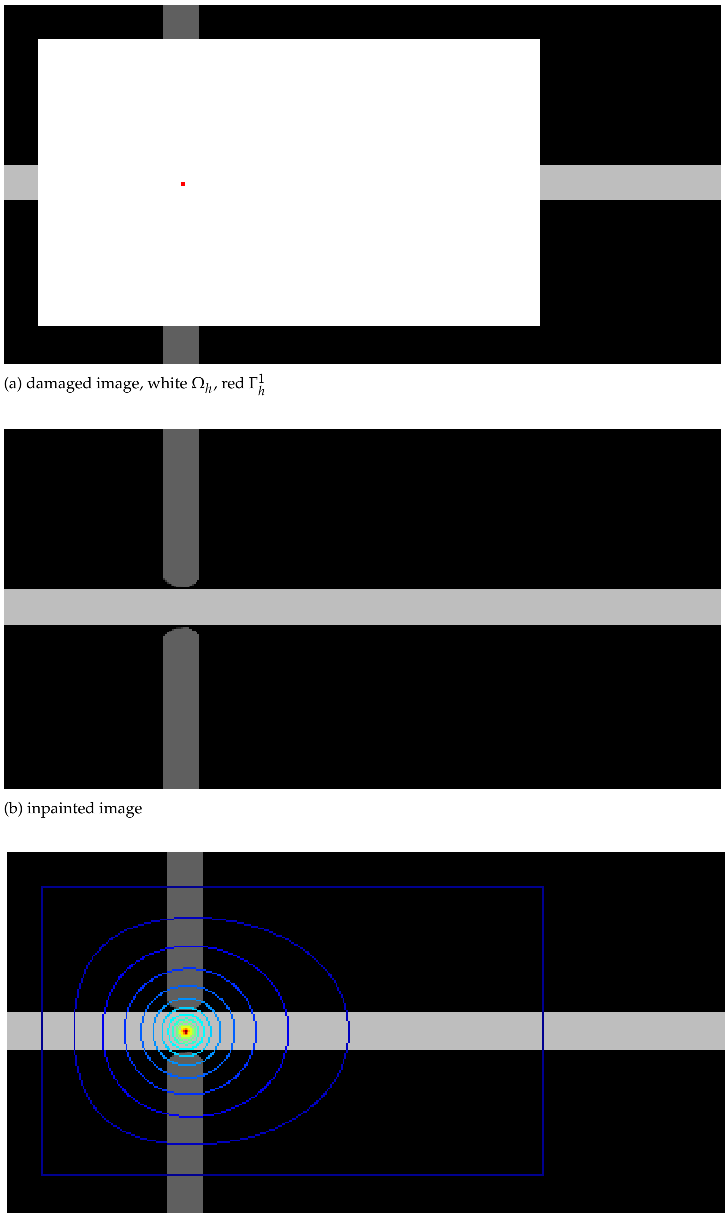
7.2 Order by Modified Distance to Boundary

The special difficulty of example 4 is that the guidance vector, i.e., the desired transport vector, does not point inwards on parts of the boundary. To combat this we suggest a modification of the distance-map set-up.

The euclidean distance-to-boundary map d is the viscosity solution of

$$|\nabla d| = 1 \quad \text{in} \quad \Omega\,, \qquad\qquad d|_{\partial\Omega} = 0\,.$$

We modify the distance-map set-up by searching for the euclidean distance



(a) damaged image, white Ω_h, red Γ_h^1

(b) inpainted image

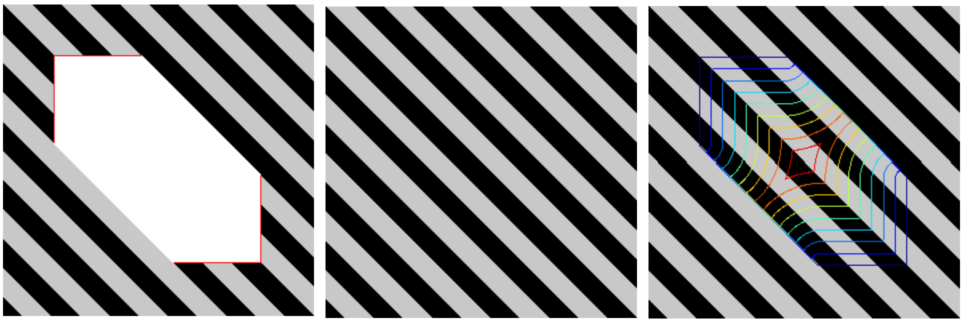
(c) inpainted image with contours of T_h

Figure 7.11: Broken junction

d_* to a subset Γ of the boundary $\partial\Omega$, i.e.,

$$|\nabla d_*| = 1 \quad \text{in} \quad \Omega\,, \qquad\qquad\qquad d_*|_\Gamma = 0\,.$$

We classify the points which shall not belong to Γ. Assume $x \in \partial\Omega$ satisfies $d_*(x) = 0$, then the boundary normal is given by $N(x) = \nabla d_*(x)$. Now, if



(a) damaged image, white Ω_h, red active boundary Γ (b) inpainted image (c) inpainted image with contours of $d_{*,h}$

Figure 7.12: Stripe pattern

we have at x

$$N^{\perp T} \cdot G \cdot N^{\perp} = \alpha \langle g, N \rangle^2 = 0 \,,$$

whereas G is the guidance tensor, α the confidence measure and g the guidance vector, then either there is no guidance ($\alpha = 0$) or the guidance vector does not point inwards. Such a boundary point shall not belong to Γ. In fact, we use the stronger condition

$$0 \le N^{\perp T} \cdot G \cdot N^{\perp} \le \gamma$$

with a threshold parameter $0 < \gamma \le 1$. Complementarily, the set of active boundary points Γ is given by

$$\Gamma = \{x \in \partial\Omega : N^{\perp T} \cdot G \cdot N^{\perp} > \gamma\} \,.$$

Clearly, the new parameter γ must be chosen such that Γ is not empty.

We have applied this idea to the stripe pattern of example 4. The result is shown in figure 7.12. Our standard parameters $[\varepsilon, \mu, \sigma, \rho]$ have the same values as before, while the additional parameter is set to $\gamma = 0.1$. The discrete approximation $d_{*,h}$ was computed using the fast marching method. The overlayed contour plot of $d_{*,h}$ in figure 7.12 (c) shows that the inward-pointing condition holds everywhere on the domain Ω_h.

7.3 Order by Distance to Skeleton

The third approach to obtaining an order is to use the distance to a prescribed stop part of the skeleton. Let $\mathcal{S}^k$, $k \in \{1, \ldots, n\}$ be curves in the image domain Ω_0. The curves $\mathcal{S}^k$ will later belong to the skeleton $\mathcal{S}$. Let, then, T_* be the viscosity solution of

$$\begin{aligned} |\nabla T_*| &= 1 \quad \text{in} \quad \Omega_0 \,, \\ T_* &= 0 \quad \text{on} \quad \mathcal{S}^k \,, \quad k \in \{1, \ldots, n\} \,, \end{aligned}$$

(a) damaged image, white Ω_h, red $\mathcal{S}_h^1$ (b) inpainted image (c) inpainted image with contours of T_h

Figure 7.13: Broken diagonal

and let

$$T_{*,\max} = \max_{x\in\Omega} T_*(x) \ .$$

The desired time-like function is defined by

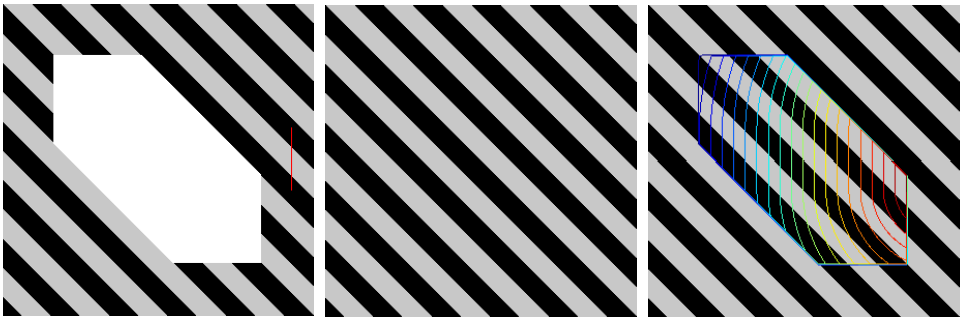
$$T(x) = T_{*,\max} - T_*(x) \ , \quad x \in \Omega \ .$$

Warning: as in the case of harmonic interpolation (see section 7.1) one must check if T is admissible, i.e., if T is free of local minima.

Figure 7.13 shows the result for example 1. The red curve in figure 7.13 (a) defines $\mathcal{S}_h^1$ (discrete). The set of parameters is the same as before. The discrete approximation $T_{*,h}$ was computed using the fast marching method. The inpainted result here is the same as in figure 7.8 (b), but the order has changed in comparison to 7.8 (c). If there is only one curve $\mathcal{S}_h^1$ which is completely contained in Ω, then order-by-harmonic-interpolation (with $\Gamma_h^1 = \mathcal{S}_h^1$) and order-by-distance-to-skeleton will yield very similar results. If there are two or more curves, then order-by-harmonic-interpolation allows for different stop times, while order-by-distance-to-skeleton has exactly one stop time on all of those curves. In contrast to order-by-distance-to-skeleton the order-by-harmonic-interpolation method requires $T_h|_{\partial\Omega_h} = 0$.

Moreover, since T_* is defined on Ω_0, we can place $\mathcal{S}^k$ outside of the inpainting domain Ω. We use this possibility to restore the stripe pattern of example 4. The result is shown in figure 7.14. The parameters are the same as before. It is obvious from the level lines of T_h (see 7.14 (c)) that the guidance vector always points inwards. Thus, we get the desired result again here.

Remark: The construction here, as well as that of section 7.2 produces time-like functions whereas not the whole boundary belongs to the start set. Moreover, it is possible that parts of the boundary belong to the stop set.



(a) damaged image, white Ω_h, red $\mathcal{S}_h^1$ (b) inpainted image (c) inpainted image with contours of T_h

Figure 7.14: Stripe pattern

Thus, the functions are not time functions in the sense of chapter 3. In practice both approaches work fine.

7.4 Order by Distance to Boundary and Natural Images

In the previous sections we considered synthetic images, because their image geometry is easy to understand. Thus, we were able to prescribe orders which are adapted to the image or rather to an expected result. When we face natural inpainting problems, as shown in figures 7.15 (a) and 7.16 (a), it is not as easy to prescribe an adapted order. This is because

- the geometry of the image is harder to understand,
- the damaged region is complicated.

Moreover, if the damaged region Ω consists of many connected components, we have to prescribe orders or time functions for every single component. This can be time consuming.

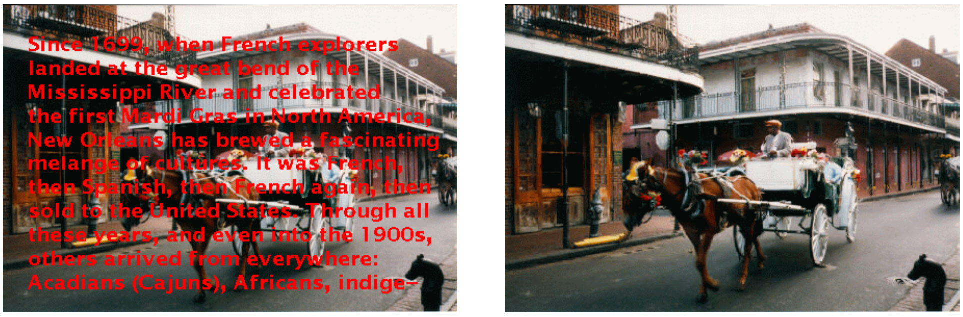
In contrast, the distance-to-boundary map can be computed fast and easily for every type of inpainting domain. And, inpainting with distance-to-boundary order often produces results of high quality when applied to natural inpainting problems (see figures 7.15 (b) and 7.16 (b)).

Bearing in mind the shape of the damage and its location in relation to the image geometry, generally, one will obtain good results, if the damage is such that level lines have been broken by scratches (By scratches we mean rather thin and lengthy damages). This is because, if the damage is of this type, the skeleton of Ω, being a simplified version of the scratch, is well placed.

(a) vandalized image

(b) inpainted: $[\varepsilon, \mu, \sigma, \rho] = [5, 25, 1.4, 4]$

Figure 7.15: Scratch removal



(a) original image (courtesy of [BSCB00, figure 6])

(b) inpainted: $[\varepsilon, \mu, \sigma, \rho] = [4, 25, 2, 3]$

Figure 7.16: Removal of superimposed text

The images shown in figures 7.15 (a) and 7.16 (a), like many other natural inpainting problems, have this type of damage. Thus, our inpainting method using distance-to-boundary order is able to produce results pleasing to the eye.

Miscellaneous Symbols and Notations

Sets

$\mathbb{N}$	natural numbers	
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$	
$\mathbb{R}$	real numbers	
$\overline{\mathbb{R}}$	extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$	
$\mathcal{B}(\Omega)$	Borel-σ-Algebra on Ω	p. 14
S_u	approximate discontinuity set of u	p. 21
J_u	approximate jump set of u	p. 21

$\mathbb{R}^d$ and $\mathbb{R}^{d_1 \times d_2}$

$\lvert a \rvert$	euclidean norm if $a \in \mathbb{R}^d$, or spectral norm if $a \in \mathbb{R}^{d_1 \times d_2}$
$\langle x, y \rangle$	euclidean scalar product of $x, y \in \mathbb{R}^d$
$a^\perp$	$a = (a_1, a_2) \in \mathbb{R}^2$, $a^\perp := (-a_2, a_1)$
$B_\varepsilon(x)$	$B_\varepsilon(x) := \{y \in \mathbb{R}^d : \lvert y - x \rvert < \varepsilon\}$
S^{d-1}	$S^{d-1} := \{y \in \mathbb{R}^d : \lvert y \rvert = 1\} = \partial B_1(0)$

Measures and measure spaces

$\mathcal{L}^d$	Lebesgue measure on $\mathbb{R}^d$	p. 14
$\mathcal{H}^k$	k-dimensional Hausdorff measure (on $\mathbb{R}^d$)	p. 14
$[\mathcal{M}_{\mathrm{loc}}(\Omega)]^m$	$\mathbb{R}^m$-valued Radon measures on Ω	p. 14
$[\mathcal{M}(\Omega)]^m$	finite $\mathbb{R}^m$-valued Radon measures on Ω	p. 14

$\mu \llcorner A$	measure restriction : $(\mu \llcorner A)(B) := \mu(A \cap B)$	p. 15
$\xi_\sharp \mu$	push-forward of μ by ξ	p. 15
$\mu \otimes \nu_x$	generalized product measure	p. 16
$\frac{\nu}{\mu}$	Randon-Nikodym density of ν w.r.t. μ	

Functions and function spaces

$\mathbb{1}_E$	characteristic function of the set E	
$C(\Omega)$	real continuous functions	
$C_c(\Omega)$	functions of $C(\Omega)$ with compact support	
$C_0(\Omega)$	closure of $C_c(\Omega)$ w.r.t. the sup-norm	
$C_b(\Omega)$	bounded functions of $C(\Omega)$	
$C^k(\Omega)$	k-times continuously differentiable functions	
$C^k(\Omega)^d$	$C^k(\Omega)^d := C^k(\Omega, \mathbb{R}^d)$	
$L^p(\Omega)$	p-integrable functions	
$L^p_{\mathrm{loc}}(\Omega)$	locally p-integrable functions	
$W^{k,p}(\Omega)$	k-times weakly differentiable Sobolev functions with derivatives in $L^p(\Omega)$	
$BV(\Omega)$	functions of bounded variation	p. 17
$SBV(\Omega)$	special BV-functions	
BV_T	periodic BV-functions	p. 29
P_T	periodic test functions	p. 29
$\mathfrak{F}$	$\mathfrak{F} \subset L^1$, domain of solution operator U	p. 81
$\mathfrak{X}$	$\mathfrak{X} \subset BV$, domain of solution operator U	p. 81
$\mathfrak{B}$	boundary data	p. 81

Functions of bounded variation: $u \in BV(\Omega)$

$\mathrm{Var}(u, \Omega)$	variation of u in Ω	p. 18
$P(E, \Omega)$	perimeter of E in Ω	p. 18
Du	derivative measure	
Du^a	absolutely continuous part of Du	

Du^s	singular part of Du	
Du^j	jump part of Du	
Du^c	Cantor part of Du	
$u_\Gamma^\pm$	interior traces of u on both sides of Γ	p. 23
$u\|_{\partial\Omega}$	boundary trace of u	p. 24

Special identifiers throughout chapters 3, 4, 5, and 6

Ω	$\Omega \subset \mathbb{R}^2$, domain	p. 31
Σ	stop set	p. 31
T	time function	p. 33
T_0	transformed time function	p. 38
N	field of normals	p. 33
c	transport field	p. 35
c_0	transformed transport field	p. 41
f	right hand side	p. 36
f_0	transformed right hand side	p. 51
u_0	(boundary) data	p. 36
U	solution operator	p. 82

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