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Homogenization, linearization and dimension reduction in elasticity with variational methods

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Contents

1. Introduction

The main objective of this thesis is the derivation of effective theories for thin elastic bodies featuring periodic microstructures, starting from nonlinear three-dimensional elasticity. Our approach is based on the variational point of view and the derivation is expressed in the language of Γ-convergence. A peculiarity of thin elastic objects is their capability to undergo large deformations at low energy. In this thesis we are particularly interested in regimes leading to limiting theories featuring this phenomenon[1]. Mathematically, this corresponds to a scaling of the energy that leads to a *linearization*[2] effect in the limiting process.

Our main result is the rigorous, ansatz free **derivation of a homogenized Cosserat theory for inextensible rods**[3] as a Γ-limit of nonlinear three-dimensional elasticity. The starting point of our derivation is an energy functional that describes an elastic body with a periodic material microstructure with small period, say ε. We suppose that the elastic body is slender and occupies a thin cylindrical domain in $\mathbb{R}^3$ with small diameter h. A special feature of this setting is the presence of the two small length scales ε and h. We prove that the associated energy sequence converges to a homogenized Cosserat rod theory as both fine-scales h and ε *simultaneously* converge to zero. The limiting energy is finite only for rod configurations. Generally speaking, a rod configuration is a pair consisting of a one-dimensional deformation (i.e. a map from the mid line of the cylindrical domain to $\mathbb{R}^3$) and an associated frame. In particular, it has the capability to capture the curvature and torsion associated to a deformed (infinitesimally thin) rod. For such configurations the energy is quadratic in the associated curvature and torsion. Interestingly, it turns out that the precise form of the limiting energy not only depends on the assumed material law, but also on the limit of the ratio $\frac{h}{\varepsilon}$ as both fine-scales converge to zero. In particular, we show that the effective coefficients appearing in the limiting energy are determined by a linear variational problem that is different for each of the three fine-scale coupling regimes

$$h \gg \varepsilon, \qquad h \sim \varepsilon \quad \text{ and } \quad h \ll \varepsilon.$$

To our knowledge this is the first rigorous result in this direction.

We would like to emphasize that in this problem effects due to *homogenization* as well as *dimension reduction and linearization* are present. The development of appropriate mathematical methods for multiscale problems (mainly in the context of elasticity)

[1] In the literature these regimes are usually called the bending regime (in the case of elastic plates, see [FJM02]) and the bending-torsion regime (in the case of elastic rods, see [MM03]).

[2] In the sense of an expansion of the energy.

[3] See page 172 for a very brief survey of the theory of elastic rods.

that simultaneously involve homogenization, dimension reduction and linearization is a further focus and discussed in Part II of this thesis.

Over the last years, in engineering and physics there has been a tendency to produce smaller and smaller devices and a demand to create new materials with designed properties. The physical behavior of such materials is often determined by complex patterns spanning several length scales, and therefore a proper understanding of the interplay of microscopic and macroscopic properties can have a great impact on the development of these materials (cf. [CDD+03]). Although the content of this thesis is theoretical, we believe that the developed methods are also interesting for applications; for instance in the context of optimal design problems involving periodic elastic plates and rods.

Before we provide a more complete and detailed outline of the results derived in this thesis, we briefly comment on the fields of homogenization and dimension reduction which both are popular research areas of their own importance.

Classically, the theory of **homogenization** studies the behavior of a model (typically a partial differential equation or an energy functional) with heterogeneous coefficients that periodically oscillate on a small scale, say ε. The central idea behind homogenization is based on the observation that in many cases it is possible to use the smallness of the fine-scale parameter ε to derive a *reduced model*[4] that still captures the behavior of the initial situation in a sufficiently precise manner — at least from the macroscopic perspective. The theory of homogenization renders a rigorous way to derive such a limiting model by analyzing the behavior as the fine-scale ε converges to zero. Various methods have been developed in this context, for instance asymptotic expansion methods (e.g. see A. Bensoussan, J.L. Lions and G. Papanicolaou [BLP78], E. Sanchez-Palencia [SP80]) or the H-convergence methods due to F. Murat and L. Tartar [Tar77, Tar09, FMT09]. The latter are also suitable for the more general setting of monotone operators and non-periodic microstructures. In this thesis we use the method of *two-scale convergence* [Ngu89, All92], which can be interpreted as an intermediate convergence between weak and strong convergence in L^p and has the capability to capture rapid oscillations on a prescribed fine-scale. Recently, under the name periodic unfolding (see [CDG02, Dam05, Vis06, Vis07]) two-scale convergence has been reinvestigated and related to the dilation technique (see [AJDH90, BLM96]).

In variational problems (as considered in this thesis) one is interested in the minimizers of energy functionals. In this case homogenization results can be proved and stated in a natural way in the language of De Giorgi's Γ-convergence (see [DGF75, DGDM83, DM93]). In elasticity the first homogenization results in this direction are due to P. Marcellini [Mar78] for convex energies and A. Braides [Bra85] and S. Müller [Mül87] for non-convex energies.

Another area of research in elasticity with a longstanding history is the **derivation of lower dimensional theories** — such as membrane, plate, string and rod models — from three-dimensional elasticity. The classical approaches are mostly ansatz based and

[4]In this context reduced means that the limiting model only involves macroscopic quantities.

can be viewed as the attempt to regard the lower-dimensional theories as constrained versions of three-dimensional elasticity in the situation where the three-dimensional body is slender and subject to additional constitutive restrictions (see the classical work of L. Euler, D. Bernoulli, A. Cauchy, G. R. Kirchhoff and of many modern authors). In contrast, the intention of *variational dimension reduction* is to derive a lower dimensional elasticity theory by proving Γ-convergence (of an appropriately scaled version) of the pure three-dimensional elastic energy as the geometry of the slender body becomes singular. In particular, no additional constitutive restrictions (as in ansatz based approaches) are allowed. For this reason, in the literature such results are often called *rigorous*.

The first result in this direction is due to E. Acerbi et al. [ABP91]. They derived an elastic string theory as Γ-limit from three-dimensional elasticity in the so called membrane regime[5]. Shortly after, H. Le Dret and A. Raoult derived a similar result for the two-dimensional limiting case, namely a nonlinear membrane theory from three-dimensional elasticity (see [LDR95]). As typical for the membrane regime, both limiting theories are not resistant to compression and bending. In contrast, G. Friesecke, R.D. James and S. Müller derived in their seminal work [FJM02] the nonlinear plate theory as Γ-limit from three-dimensional elasticity in the bending regime[6]. At the core of this and (a huge number of) related results is the *geometric rigidity estimate* (see [FJM02]) that allows to control the L^2-distance of a deformation gradient to an appropriate constant rotation by the L^2-distance of the gradient to the entire group of rotations. Based on this estimate, a whole hierarchy of plate models has been rigorously derived (see [FJM06]) and — particularly interesting for the situation considered in the last part of this thesis — the nonlinear bending-torsion theory for inextensible rods has been established as Γ-limit from three-dimensional elasticity by M.G. Mora and S. Müller (see [MM03]). The geometric rigidity estimate also plays a central role in many parts of this thesis.

Although the amount of research in the field of homogenization and dimension reduction respectively, is quite large, only a small number of rigorous results exist for the combination of homogenization and variational dimension reduction in nonlinear elasticity and — as far as we know — *only settings related to the membrane regime* have been considered (see A. Braides, I. Fonseca and G. Francfort [BFF00], Y.C. Shu [Shu00], J.-F. Babadijan and M. Baía [BB06]). While in the membrane regime quasiconvexification and relaxation methods are dominant and in most cases abstract representation theorems of the theory of Γ-convergence are needed, the analysis in the bending regime (as considered here) is very different: In virtue of the energy scaling, the rigidity properties of the problem dominate the behavior and as a consequence, linearization effects come into play. We are going to see that this allows us to derive the limiting theory not only in a more explicit way, but also enables us to gain insight in the physics of the fine-scale behavior of the initial models by retracing the explicit construction.

[5]The terminus membrane regime stems from 3d to 2d dimension reduction problems and refers to the energy scaling which corresponds to energy per volume.

[6]For a slender domain $\Omega_h \subset \mathbb{R}^3$ with a volume that scales like h^d with $d = 1$ (for plates) and $d = 2$ (for rods), the bending regime corresponds to the energy scaled by $h^{-(2+d)}$.

In the following we give a more detailed and complete outline of this thesis. We mainly focus on the main result of Part III and its relation to the analysis of Part II.

As already mentioned, our primary result is the derivation of an elastic rod theory from three-dimensional elasticity, that is presented in Chapter 8 of this thesis. In Chapter 7 we study a simplification of this problem already showing most of the interesting behavior. For simplicity we stick to this setting in the remainder of this introduction: Namely, we study the functional

$$(1.1)\qquad W^{1,2}(\Omega_h;\mathbb{R}^2)\ni v\mapsto \frac{1}{h^3}\int_{\Omega_h} W(x_1/\varepsilon,\nabla v(x))\,\mathrm{d}x,\qquad \Omega_h:=(0,L)\times(-h/2,h/2),$$

which is the stored energy of an elastic body, deformed by the map $v:\Omega_h\to\mathbb{R}^2$ and occupying the thin, two-dimensional domain Ω_h. The potential $W(y,F)$ is assumed to be a frame indifferent, non-negative integrand that is zero for $F\in SO(2)$ and non-degenerate in the sense that

$$\operatorname*{ess\,inf}_y W(y,F)\geq c'\operatorname{dist}^2(F,SO(2))\qquad \text{for all } F\in\mathbb{M}(2).$$

We assume that W is $[0,1)$-periodic in its first component and suppose that it admits a quadratic Taylor expansion at the identity, i.e.

$$W(y,Id+F)=Q(y,F)+o(|F|^2)$$

where $Q(y,F)$ is a suitable integrand, quadratic in F. These quite generic assumptions correspond to a laterally (i.e. in the "length"-direction x_1) periodic, hyperelastic material with period ε and a stress free reference state. The non-degeneracy condition combined with the quadratic expansion can be interpreted as a generalization of Hooke's law to the geometrically nonlinear setting — in the sense that for infinitesimal small strains a linear stress-strain relation holds. In Chapter 7 we show that as h and ε converge to zero, the elastic energy in (1.1) Γ-converges to a limiting functional that is finite only for bending deformations $u\in W^{2,2}_{\mathrm{iso}}((0,L);\mathbb{R}^2)$ and in this case takes the form

$$\frac{q_\gamma}{12}\int_0^L \boldsymbol{\kappa}^2_{(u)}(x_1)\,\mathrm{d}x_1$$

where $\boldsymbol{\kappa}_{(u)}$ is the curvature of u and q_γ an effective stiffness coefficient that is derived from the quadratic form Q by a subtle relaxation procedure depending on the limiting ratio $\gamma\in[0,\infty]$ with $\frac{h}{\varepsilon}\to\gamma$. The derived energy can be interpreted as a planar theory for inextensible rods, since on the one hand deformations that stretch or compress the infinitesimal thin rod are penalized by infinite energy, and on the other hand for bending deformations the energy is quadratic in curvature.

For the analysis it is convenient to study the *scaled* but equivalent formulation

$$\mathcal{I}^{\varepsilon,h}(u):=\frac{1}{h^2}\int_\Omega W(x_1/\varepsilon,\nabla_h u(x))\,\mathrm{d}x$$

where Ω denotes the *fixed domain* $(0, L)\times(-1/2, 1/2)$ and $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ is related to the initial deformation via the scaling $u(x_1, x_2) = v(x_1, hx_2)$ and has the meaning of a *scaled deformation*. Thereby, we have

$$(\nabla v)(x_1, hx_2) = (\nabla_h u)(x_1, x_2)$$

where $\nabla_h u := (\partial_1 u|\frac{1}{h}\partial_2 u)$ is a *scaled deformation gradient*.

We are going to see that the frame indifference of the elastic potential allows us to express the overall behavior of the energy $\mathcal{I}^{\varepsilon,h}(u)$ by means of the nonlinear strain $E_h := h^{-1}\left(\sqrt{\nabla_h u^{\mathrm{T}}\nabla_h u} - Id\right)$ via the integral

$$\frac{1}{h^2}\int_\Omega W(x_1/\varepsilon, Id + hE_h(x))\,\mathrm{d}x.$$

The property, that W admits a quadratic expansion at Id, suggests that we can linearize the expression above and (at least formally and modulus terms of higher order) we can replace the previous integral by $\int_\Omega Q(x_1/\varepsilon, E_h(x))\,\mathrm{d}x$. Indeed, in Section 5.2 we present a result concerning the **simultaneous homogenization and linearization** of integral functionals (also covering more general settings) that makes this observation rigorous. More specifically, we prove that whenever (E_h) two-scale converges to a map E the inequality

$$\liminf_{h\to 0}\frac{1}{h^2}\int_\Omega W(x_1/\varepsilon, Id + hE_h(x))\,\mathrm{d}x \geq \iint\limits_{\Omega\times(0,1)} Q(y, E(x,y))\,\mathrm{d}x\,\mathrm{d}y$$

is valid, and the stronger statement

$$\lim_{h\to 0}\frac{1}{h^2}\int_\Omega W(x_1/\varepsilon, Id + hE_h(x))\,\mathrm{d}x = \iint\limits_{\Omega\times(0,1)} Q(y, E(x,y))\,\mathrm{d}x\,\mathrm{d}y$$

holds, whenever E is the strong two-scale limit of (E_h), provided the sequence's L^∞-norm grows with a sufficiently slow rate. It is important to note that the expression on the right hand sides captures oscillations on scale ε along the sequence (E_h).

In view of this preliminary analysis the general strategy of the Γ-convergence proof is quite natural:

In a first part we provide a **compactness result** (adapted from [FJM02]) which considers sequences (u_h) with equibounded energy and guarantees that for suitable subsequences (u_h) and (E_h) converge (two-scale converge, resp.) to a bending deformation $u \in W^{2,2}_{\text{iso}}(\Omega;\mathbb{R}^2)$ and a limiting strain E, respectively.

Now the main challenge in the proof of the Γ-convergence result is to establish a precise link between the limiting strain E and the limiting deformation u. This is done in Section 7.4.2, where we prove a **two-scale characterization of the nonlinear limiting strain** for arbitrary sequences of deformations with finite bending energy. In particular, the characterization is sensitive to the limit of the ratio $\frac{h}{\varepsilon}$ and **sharp** in the sense that any limit that obeys the characterization can be recovered by an appropriate sequence of deformations. In Section 7.7.1 we elaborate on this property and explicitly construct such sequences. The proof of this characterization result is based on two insights: First, a careful approximation of $\nabla_h u_h$ in $W^{1,2}$ by maps $R_h : (0, L) \to SO(2)$ shows that "curvature oscillations" only play a role if $\varepsilon \gg h$. Secondly, we establish a decomposition of the deformation that has the overall form $u_h = v_h + hw_h$ where v_h is a deformation obtained by extending a one-dimensional bending deformation via a standard Cosserat ansatz and w_h is a corrector of higher order. Since the construction of v_h is quite explicit, we can easily characterize its contribution the limiting strain E. On the other hand, we can identify the contribution of the corrector term w_h by means of a **two-scale characterization of scaled gradients** which we establish in Chapter 6. In view of this, the Γ-convergence statement mainly follows by combining the simultaneous homogenization and linearization result with the sharp two-scale characterization of the limiting strain.

We complete the result in Chapter 7 by taking one-sided boundary conditions and forces into account. Moreover, in Section 7.5 we prove that for low energy sequences the associated nonlinear strain strongly two-scale converges. In Section 7.6 we justify the claim that the effective coefficient in the fine-scale coupling regimes $\varepsilon \gg h$ and $\varepsilon \ll h$ can equivalently be computed by firstly reducing the dimension and secondly homogenizing the reduced energy and vice versa. We proof this by applying the result that **homogenization and linearization commute in finite elasticity**, which is the main content of Chapter 5. There, we review and extend related results from joint work with S. Müller (see [MN10]). In the last part of Chapter 7 we demonstrate that the developed strategy can be applied to more **advanced settings**, including layered and prestressed rods. This extends results in [Sch07] to rapidly oscillating materials.

In Chapter 8 we show that a **homogenized Cosserat theory for elastic rods** emerges as a Γ-limit from three-dimensional elasticity. This is the analogon to the main result of Chapter 7; for brevity, we only prove the pure Γ-convergence statement. Eventually, in Chapter 9 we present partial results for the derivation of a **homogenized bending theory for elastic plates** from three-dimensional elasticity.

The first and second part of the thesis are structured as follows: In Part I we mainly introduce the notions of two-scale convergence and Γ-convergence and present some known lower semicontinuity results for integral functionals. The aim of this part is to permit easy reference throughout the thesis. Except for the content of Section 2.2 and 2.3, where we provide some new results related to two-scale convergence, the content of this part might be considered to be standard.

Part II elaborates on the interplay between homogenization, linearization and dimension reduction in general settings. As already mentioned, in Chapter 5 we prove that **linearization and homogenization commute** in the sense of Γ-convergence for a large class of elastic potentials. Moreover, we study the asymptotic behavior of elastic energies for simultaneous linearization and homogenization. As a by-product we prove that homogenized, linearized elasticity can be obtained as Γ-limit from nonlinear three-dimensional elasticity with cellular periodic materials. This combines recent results from G. Dal Maso et al. [DMNP02] with homogenization methods.

In Chapter 6 we develop new two-scale methods suited for dimension reduction problems. As a main result we prove a sharp **characterization of two-scale limits** of sequences of **scaled gradients** which naturally emerge in the context of gradient integral functionals on thin domains with a small thickness, say h. It turns out that the general structure of such a limit is sensitive to the ratio between the fine-scale associated to two-scale convergence and the fine-scale h associated to the scaling of the gradient. As mentioned before, this plays a key role in the two-scale characterization of the limiting strain of Part III.

Because of the inconvenient length of this thesis we would like to conclude this introduction by suggesting a *quick tour* leading the hounded reader to the main results of this thesis in Part III:

1. For readers unfamiliar with periodic unfolding we recommend to start with the brief motivation of two-scale convergence and its link to periodic unfolding (see page 13).

2. In Section 3.3 we consider convex integral functionals and demonstrate the general strategy for the homogenization of variational problems with two-scale convergence methods. We believe that this is also instructive for the understanding of the more elaborated results in Part III.

3. In Section 5.2 we prove the simultaneous homogenization and linearization result which is an important ingredient for the analysis in Part III.

4. We recommend to register Definition 6.2.3 which entails a slight variant of two-scale convergence suited for in-plane oscillations. Section 6.3 contains the two-scale characterization result for scaled gradients and explains the dependency of the limiting theories derived in Part III on the ratio $\frac{h}{\varepsilon}$.

5. The main Γ-convergence results in Part III are contained in Section 7.1, 7.2, 7.4 and Chapter 8.

Part I.

Mathematical preliminaries

2. Two-scale convergence

The notion of two-scale convergence was introduced by G. Nguetseng in [Ngu89] and employed in the theory of homogenization by many researcher, in particular G. Allaire e.g. in [All92]. Loosely speaking, it can be interpreted as an intermediate convergence between weak and strong convergence in L^p and has the capability to capture fine oscillation properties of sequences. Recently, a reinvestigation of this notion, motivated by the dilation technique (see [AJDH90, BLM96]), led to the periodic unfolding method (cf. [CDG02, Vis06, Vis07]) and revealed that two-scale convergence can be equivalently defined as weak convergence in an appropriate space.

In the first part of this chapter we motivate and recall the basic notion of two-scale convergence from the point of view of the dilation technique following [Dam05, Vis07, MT07]. In Section 2.2 we consider piecewise constant functions that are coherent to a fine lattice. In particular we develop some criteria when a two-scale limit of such a sequence is equal to the weak limit — which means that the sequence "does not carry oscillations" on the tested scale. Furthermore, we present an analog result for the associated piecewise affine interpolations. Eventually, in Section 2.3 we study the interplay between two-scale convergence and linearization. The analytical tools developed in the subsequent sections are frequently used throughout this contribution.

Motivation. A basic problem in homogenization is the identification of limits that emerge from weakly converging sequences where the loss of mass (and therefore the loss of strong convergence) is caused by fine oscillations. As a prototypical example, we consider the product

$$\int_{\mathbb{R}^n} u_\varepsilon(x)\psi(x, x/\varepsilon)\,\mathrm{d}x \tag{2.1}$$

where $(u_\varepsilon)_\varepsilon$ is a weakly convergent sequence in $L^2(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n; C_{\mathrm{per}}(Y))$ with $Y := [0,1)^n$. For instance, one may think of u_ε as the solution of a variational problem or partial differential equation with oscillating coefficients given by $\psi(x, x/\varepsilon)$. The understanding of the limiting behavior of (2.1) as $\varepsilon \to 0$ is essential in the context of homogenization.

For small ε the function $\psi_\varepsilon(x) := \psi(x, x/\varepsilon)$ is rapidly oscillating. As a consequence, the sequence $(\psi_\varepsilon)_\varepsilon$ is not strongly convergent in general, but converges only weakly in $L^2(\mathbb{R}^n)$ (to the map $x \mapsto \int_Y \psi(x,y)\,\mathrm{d}y$). Thereby, (2.1) is a product of weakly convergent sequences, and therefore we cannot pass to the limit by elementary methods. The heart of two-scale convergence is the following observation: There exist a map $u \in L^2(\mathbb{R}^n \times Y)$ and a subsequence (that we do not relabel) such that

$$\int_{\mathbb{R}^n} u_\varepsilon(x)\psi(x, x/\varepsilon)\,\mathrm{d}x \to \iint_{\mathbb{R}^n \times Y} u(x,y)\psi(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad \text{as } \varepsilon \to 0 \text{ for all } \psi \in L^2(\mathbb{R}^n; C_{\mathrm{per}}(Y)). \tag{2.2}$$

This result (refered to as two-scale compactness) was first proved by G. Nguetseng in [Ngu89]. Shortly afterwards G. Allaire in [All92] followed this idea and developed the theory of "two-scale convergence", which revealed itself to be a powerful, but simple method in the homogenization of periodic problems.

In the following we give a brief proof of (2.2) with methods related to periodic unfolding. We follow ideas in [Dam05, Vis06] with the aim to motivate the main idea behind two-scale convergence and to illustrate its relation to periodic unfolding. We start with the observation that for each positive ε the union

$$\mathbb{R}^n = \bigcup_{\xi \in \mathbb{Z}^n} \varepsilon(\xi + Y)$$

is a tessellation of $\mathbb{R}^n$. Since this is particularly true for $\varepsilon = 1$, we can assign to any point $x \in \mathbb{R}^d$ a unique translation point $\lfloor x \rfloor \in \mathbb{Z}^n$ such that x belongs to the translated cell $\lfloor x \rfloor + Y$. Because of $Y = [0,1)^n$, the translation point $\lfloor x \rfloor$ is obviously the (vectorial) integer part of x, i.e.

$$\lfloor x \rfloor = \max\{\, \xi \in \mathbb{Z}^n \,;\, \xi \le x \ \text{(componentwise)} \,\}.$$

For each $\varphi \in L^1(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}x = \sum_{\xi \in \mathbb{Z}^n} \int_{\varepsilon(\xi+Y)} \varphi(x)\,\mathrm{d}x = \sum_{\xi \in \mathbb{Z}^n} \varepsilon^n \int_Y \varphi(\varepsilon\xi + \varepsilon y)\,\mathrm{d}y = \iint_{\mathbb{R}^n \times Y} \varphi(\varepsilon\lfloor x/\varepsilon \rfloor + \varepsilon y)\,\mathrm{d}y\,\mathrm{d}x \tag{2.3}$$

where the second equality is derived by the change of coordinates $y = x/\varepsilon$. Moreover, the last identity is valid, because $\lfloor x/\varepsilon \rfloor = \xi$ for all $\xi \in \mathbb{Z}^d$ and $x \in \varepsilon(\xi + Y)$.

Although being elementary, equation (2.3) already comprises the central idea of the periodic unfolding method. In order to carve out the implications of (2.3), let us introduce the operator

$$\mathcal{T}_\varepsilon : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n \times Y), \qquad (\mathcal{T}_\varepsilon \varphi)(x,y) := \varphi(\varepsilon\lfloor x/\varepsilon \rfloor + \varepsilon y).$$

The idea of the periodic unfolding method is to study the convergence properties of the sequence (u_ε) by analyzing the "unfolded" sequence $(\mathcal{T}_\varepsilon u_\varepsilon)$. To this end, we set $\varphi(x) =$

$u_\varepsilon(x)\psi_\varepsilon(x)$ in equation (2.3). Because of $\mathcal{T}_\varepsilon(u_\varepsilon\psi_\varepsilon) = (\mathcal{T}_\varepsilon u_\varepsilon)(\mathcal{T}_\varepsilon\psi_\varepsilon)$, equation (2.3) yields

$$\int_{\mathbb{R}^n} u_\varepsilon(x)\psi_\varepsilon(x)\,\mathrm{d}x = \iint_{\mathbb{R}^n\times Y} (\mathcal{T}_\varepsilon u_\varepsilon)(x,y)(\mathcal{T}_\varepsilon\psi_\varepsilon)(x,y)\,\mathrm{d}y\,\mathrm{d}x. \tag{2.4}$$

Using the periodicity of $\psi(\cdot,\cdot)$ in its second component, we deduce that

$$(\mathcal{T}_\varepsilon\psi_\varepsilon)(x,y) = \psi(\varepsilon\lfloor x/\varepsilon\rfloor + \varepsilon y, y).$$

Because

$$\varepsilon\lfloor x/\varepsilon\rfloor + \varepsilon y \to x \qquad \text{uniformly as} \qquad \varepsilon \to 0$$

and due to the continuity of the translation-operator in $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$, we observe that

$$\mathcal{T}_\varepsilon\psi_\varepsilon \to \psi \qquad \text{strongly in } L^2(\mathbb{R}^n\times Y).$$

Hence, whenever

$$\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \qquad \text{weakly in } L^2(\mathbb{R}^n\times Y), \tag{2.5}$$

we can pass to the limit on the right hand side of (2.4) and arrive at

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} u_\varepsilon(x)\psi(x, x/\varepsilon)\,\mathrm{d}x = \iint_{\mathbb{R}^n\times Y} u(x,y)\psi(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad \text{for all} \quad \psi \in L^2(\mathbb{R}^n; C_{\mathrm{per}}(Y)).$$

It remains to show that (2.5) is valid at least for a subsequence. Therefore, we set $\varphi(x) := |u_\varepsilon(x)|^2$ in (2.3). We obtain

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \iint_{\mathbb{R}^n\times Y} \mathcal{T}_\varepsilon(|u|^2)\,\mathrm{d}y\,\mathrm{d}x = \iint_{\mathbb{R}^n\times Y} |\mathcal{T}_\varepsilon u|^2\,\mathrm{d}y\,\mathrm{d}x = \|\mathcal{T}_\varepsilon u\|^2_{L^2(\mathbb{R}^n\times Y)}$$

and deduce that $\mathcal{T}_\varepsilon$ is a linear isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n\times Y)$. Because by assumption (u_ε) is bounded in $L^2(\mathbb{R}^d)$, the unfolded sequence $(\mathcal{T}_\varepsilon u_\varepsilon)$ is bounded in $L^2(\mathbb{R}^n\times Y)$. Because bounded sequences in $L^2(\mathbb{R}^n\times Y)$ are relatively compact with respect to weak convergence, (2.5) follows for a subsequence and the proof is complete. □

As already mentioned, the first proofs of this two-scale compactness result are due to G. Nguetseng [Ngu89] and G. Allaire [All92]. Roughly speaking, in their proof they gain compactness by showing that the sequence of functionals associated to (u_ε) by

$$L^2(\mathbb{R}^n; C_{\mathrm{per}}(Y)) \ni \psi \mapsto \int_{\mathbb{R}^n} u_\varepsilon(x)\psi(x, x/\varepsilon)\,\mathrm{d}x$$

is compact with respect to the weak star topology in the dual space of $L^2(\mathbb{R}^n; C_{\mathrm{per}}(Y))$. In contrast, the proof presented above is more elementary. Here, the two-scale compactness immediately follows from the relative compactness of the unfolded sequence $(\mathcal{T}_\varepsilon u_\varepsilon)$ and the observation that $\mathcal{T}_\varepsilon$ is a linear isometry.

We would like to remark that an operator similar to $\mathcal{T}_\varepsilon$ (called dilation operator) was introduced in the context of homogenization of porous media by T. Arbogast, J. Douglas and U. Hornung [AJDH90] for the first time. In [BLM96] A. Bourgeat, S. Luckhaus and A. Mikelić showed that weak convergence of the unfolded (or dilated) sequence and two-scale convergence of the initial sequence are equivalent. More recently, D. Cioranescu, A. Damlamian and G. Griso (see [CDG02, CDG08]) and A. Visintin [Vis06, Vis07] reinvestigated two-scale convergence from the point of view of the dilation method and established a general approach to periodic homogenization which nowadays is often called "periodic unfolding".

2.1. Definition and basic properties

Throughout this section $\mathbb{E}$ denotes a d-dimensional Euclidean space with inner product $\langle\cdot,\cdot\rangle$, norm $|\cdot|$ and orthonormal basis $\{e_1,\dots,e_d\}$. Unless stated otherwise we suppose that $p,q\in[1,\infty]$ with $1/p+1/q=1$ where we use the convention $\frac{1}{\infty}=0$.

In this chapter we set $Y:=[0,1)^n$ and suppose that Ω is a (possibly unbounded) subset of $\mathbb{R}^n$. In order to avoid technical difficulties regarding the boundary of $\partial\Omega$ we suppose that $\mathcal{H}^n(\partial\Omega)=0$ (which for instance is satisfied for domains with Lipschitz boundary). We assign to each function $v:\Omega\to\mathbb{E}$ its extension to $\mathbb{R}^n$ by zero according to

$$v^{\mathrm{Ext}}:\mathbb{R}^n\to\mathbb{E},\qquad v^{\mathrm{Ext}}(x):=\begin{cases}v(x) & \text{if } x\in\Omega\\ 0 & \text{else.}\end{cases}$$

Unless stated otherwise, $(\varepsilon_k)_{k\in\mathbb{N}}$ denotes an arbitrary (but fixed) sequence of positive real numbers that converges to zero as $k\to\infty$. For brevity, we represent this sequence by (ε) and write ε to denote a generic element of the sequence. Moreover, we write (u_ε) to refer to a sequence that is indexed by (ε).

Let us remark that the notion of two-scale convergence and the results in this chapter can be extended in a straightforward way to the case where (u_ε) and (ε) are rather families than mere sequences.

Definition of two-scale convergence. The "classical" definition of two-scale convergence in $L^2(\Omega)$ (see [All92, Ngu89, LNW02]) says that a sequence (u_ε) in $L^2(\Omega)$ is two-scale convergent to a function $u\in L^2(\Omega\times Y)$ if

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\psi(x,\tfrac{x}{\varepsilon})\,\mathrm{d}x\to\iint_{\Omega\times Y}u(x,y)\psi(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for all $\psi\in L^2(\Omega;C_{\mathrm{per}}(Y))$.

In the following we give a different (but in the situations considered here) equivalent definition based on the periodic unfolding operator.

Definition 2.1.1 (Two-scale convergence)**.**

(a) For all positive ε we define the operator $\mathcal{T}_\varepsilon$ from the space of measurable functions on Ω with values in $\mathbb{E}$ to the space of measurable functions on $\mathbb{R}^n\times Y$ with values in $\mathbb{E}$ by

$$(\mathcal{T}_\varepsilon u)(x,y) := u^{\mathrm{Ext}}(\varepsilon\lfloor x/\varepsilon\rfloor + \varepsilon y).$$

(b) Let (u_ε) be a sequence of measurable functions from Ω to $\mathbb{E}$ and u a measurable function from $\mathbb{R}^n\times Y$ to $\mathbb{E}$. We say that (u_ε) *strongly two-scale converges* to u in $L^p(\Omega\times Y;\mathbb{E})$ (for $\varepsilon\to 0$) and write

$$u_\varepsilon \xrightarrow{2} u \qquad \text{strongly two-scale in } L^p(\Omega\times Y;\mathbb{E}),$$

whenever

$$\mathcal{T}_\varepsilon u_\varepsilon \to u \qquad \text{strongly in } L^p(\mathbb{R}^n\times Y;\mathbb{E}).$$

We similarly define weak (and for $p=\infty$ weak star) two-scale convergence and denote them by

$$u_\varepsilon \xrightharpoonup{2} u \qquad \text{weakly two-scale in } L^p(\Omega\times Y;\mathbb{E}),$$
$$u_\varepsilon \xrightharpoonup{2\star} u \qquad \text{weakly star two-scale in } L^\infty(\Omega\times Y;\mathbb{E}).$$

Lemma 2.1.2 (e.g. see [Vis06])**.** *Let $u\in L^1(\Omega;\mathbb{E})$ and $\varepsilon>0$. Then*

$$\int_\Omega u(x)\,\mathrm{d}x = \iint_{\mathbb{R}^n\times Y} (\mathcal{T}_\varepsilon u)(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Proposition 2.1.3 (see [AJDH90, CDG02, Vis06])**.** *Let $p\in[1,\infty]$. The restriction of $\mathcal{T}_\varepsilon$ to $L^p(\Omega;\mathbb{E})$ is a (nonsurjective) linear isometry from $L^p(\Omega;\mathbb{E})$ to $L^p(\mathbb{R}^n\times Y;\mathbb{E})$.*

As an immediate consequence we obtain the following two-scale compactness result:

Proposition 2.1.4 (see Proposition 3.1 in [Vis06])**.** *Let $p\in(1,\infty)$ and (u_ε) a sequence in $L^p(\Omega;\mathbb{E})$. If the sequence (u_ε) is bounded w.r.t. the norm in $L^p(\Omega;\mathbb{E})$, then (u_ε) is weakly two-scale relatively compact in $L^p(\Omega\times Y;\mathbb{E})$, i.e. we can extract from any subsequence a further subsequence that weakly two-scale converges in $L^p(\Omega\times Y;\mathbb{E})$.*

Lemma 2.1.5 (see Proposition 2.7 in [Vis06])**.** *Let $p\in[1,\infty)$, (u_ε) a sequence in $L^p(\Omega;\mathbb{E})$ and $u\in L^p(\Omega\times Y;\mathbb{E})$.*

(1) If (u_ε) weakly two-scale converges to u, then

$$\liminf_{\varepsilon\to 0}\|u_\varepsilon\|_{L^p(\Omega;\mathbb{E})} \geq \|u\|_{L^p(\Omega\times Y;\mathbb{E})}\,.$$

(2) If (u_ε) strongly two-scale converges to u, then

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^p(\Omega;\mathbb{E})} = \|u\|_{L^p(\Omega\times Y;\mathbb{E})}\,.$$

(3) *If $p \in (1,\infty)$, then a sequence (u_ε) strongly two-scale converges to u if and only if (u_ε) weakly two-scale converges to u and* $\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^p(\Omega;\mathbb{E})} = \|u\|_{L^p(\Omega\times Y;\mathbb{E})}$.

Remark 2.1.6. It is important to note that in general the support of the map $\mathcal{T}_\varepsilon u$ with $u \in L^p(\Omega;\mathbb{E})$ is (slightly) larger then $\Omega\times Y$, namely we have

$$\operatorname{supp}(\mathcal{T}_\varepsilon u) \subset \{\, (x,y) \in \mathbb{R}^n\times Y \,:\, \operatorname{dist}(x,\Omega) \le \sqrt{n}\varepsilon \,\}. \tag{2.6}$$

As a consequence, for bounded domains Ω the operator $\mathcal{T}_\varepsilon : L^p(\Omega;\mathbb{E}) \to L^p(\Omega\times Y;\mathbb{E})$ is **not** an isometry (as it is sometimes wrongly stated in the literature). For the same reason weak convergence of the unfolded sequence $(\mathcal{T}_\varepsilon u_\varepsilon)$ in $L^p(\Omega\times Y;\mathbb{E})$ is in general not sufficient to guarantee weak two-scale convergence of (u_ε) in $L^p(\Omega\times Y;\mathbb{E})$. An example for an unbounded sequence (u_ε) in $L^2((0,1))$ with $\|\mathcal{T}_\varepsilon u_\varepsilon\|_{L^2((0,1)\times Y)} \to 0$ can be found in [MT07].

Remark 2.1.7. For any sequence $(u_\varepsilon) \subset L^p(\Omega;\mathbb{E})$ and map $u \in L^p(\mathbb{R}^n\times Y;\mathbb{E})$, $p \in [1,\infty)$ with

$$\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \qquad \text{weakly in } L^p(\mathbb{R}^n\times Y;\mathbb{E})$$

(2.6) implies that the support of u is contained in $\overline{\Omega}\times Y$. The assumption $\mathcal{H}^n(\partial\Omega) = 0$ implies that $L^p(\Omega\times Y;\mathbb{E}) = L^p(\overline{\Omega}\times Y;\mathbb{E})$. For this reason we can identify u with a map in $L^p(\Omega\times Y;\mathbb{E})$.

Remark 2.1.8. In Section 6.2 we present a variant of two-scale convergence which is suited to situations where it is sufficient to capture oscillations of a sequence only in "some" directions (as it is the case for elastic thin films featuring laterally periodic microstructures).

In the following we gather some known properties of two-scale convergence. For an extensive introduction and as a source for proofs that we left out, we refer to [Vis06, LNW02, MT07].

The classical definition of two-scale convergence. The next results reveal that Definition 2.1.1 is equivalent to the classical definition of two-scale convergence.

Lemma 2.1.9 (cf. Lemma 2.1 in [Vis06]). *Let $q \in [1,\infty)$ and consider a function ψ belonging to one of the following spaces*

$$L^q(\Omega; C_{\text{per}}(Y;\mathbb{E})), \qquad L^q_{\text{per}}(Y; C(\overline{\Omega};\mathbb{E})), \qquad C^\infty_c(\Omega; C^\infty_{\text{per}}(Y;\mathbb{E})).$$

Then the sequence $(\psi^\varepsilon) \subset L^q(\Omega;\mathbb{E})$ given by

$$\psi^\varepsilon(x) := \psi(x, x/\varepsilon)$$

converges strongly two-scale to ψ in $L^q(\Omega\times Y;\mathbb{E})$.

Proposition 2.1.10 (cf. Proposition 2.5 [Vis06] and Theorem 10 in [LNW02]). *Let $p \in [1,\infty]$. For a sequence $(u_\varepsilon) \subset L^p(\Omega;\mathbb{E})$ and $u \in L^p(\Omega\times Y;\mathbb{E})$ the following conditions are equivalent:*

(1) (u_ε) *is weakly two-scale convergent to* u *in* $L^p(\Omega{\times}Y;\mathbb{E})$

(2) (u_ε) *is bounded in* $L^p(\Omega;\mathbb{E})$ *and*

$$\left\{\begin{aligned} &\lim_{\varepsilon\to 0}\int_\Omega \langle u_\varepsilon(x),\, \psi(x,\tfrac{x}{\varepsilon})\rangle\,\mathrm{d}x = \iint\limits_{\Omega\times Y}\langle u(x,y),\,\psi(x,y)\rangle\,\mathrm{d}y\,\mathrm{d}x\\ &\qquad\qquad \text{for all } \psi\in C_c^\infty(\Omega;C^\infty_{\mathrm{per}}(Y;\mathbb{E})).\end{aligned}\right. \tag{2.7}$$

Moreover, if $p\in(1,\infty)$ *and* Ω *is bounded, then (1) and (2) are equivalent to*

(3) (2.7) *holds for all* $\psi\in L^q(\Omega;C_{\mathrm{per}}(Y;\mathbb{E}))$.

Two-scale convergence as an intermediate convergence.

Lemma 2.1.11 (cf. Theorem 1.3 [Vis06])**.** *Let* $p\in[1,\infty)$, (u_ε) *a sequence in* $L^p(\Omega;\mathbb{E})$, $u\in L^p(\Omega;\mathbb{E})$ *and* $u_0\in L^p(\Omega{\times}Y;\mathbb{E})$.

(1) If (u_ε) *strongly converges to* u, *then* (u_ε) *strongly two-scale converges to* u.

(2) If (u_ε) *strongly two-scale converges to* u_0, *then* (u_ε) *weakly two-scale converges to* u_0.

(3) If (u_ε) *weakly two-scale converges to* u_0, *then* (u_ε) *weakly converges to* $\int_Y u_0(\cdot,y)\,\mathrm{d}y$.

Product rules.

Proposition 2.1.12. *Let* $p\in(1,\infty)$, $q\in(1,\infty]$ *and* $1/p+1/q=1/r$, *let* $(u_\varepsilon)\subset L^p(\Omega;\mathbb{E})$ *weakly two-scale converge to* u *in* $L^p(\Omega{\times}Y;\mathbb{E})$ *and* $(w_\varepsilon)\subset L^q(\Omega;\mathbb{E})$ *strongly two-scale converge to* w *in* $L^q(\Omega{\times}Y;\mathbb{E})$, *then*

$$\langle u_\varepsilon,\, w_\varepsilon\rangle \xrightarrow{2} \langle u,\, w\rangle \qquad \text{weakly two-scale in } L^r(\Omega{\times}Y;\mathbb{R}).$$

Proof. By definition, we immediately have

$$\mathcal{T}_\varepsilon(\langle u_\varepsilon,\, v_\varepsilon\rangle) = \langle \mathcal{T}_\varepsilon u_\varepsilon,\, \mathcal{T}_\varepsilon v_\varepsilon\rangle\,.$$

Now the statement follows from the corresponding result in $L^r(\mathbb{R}^n{\times}Y;\mathbb{E})$. □

Proposition 2.1.13. *Let* $p\in(1,\infty)$, *let* $(u_\varepsilon)\subset L^p(\Omega;\mathbb{E})$ *weakly two-scale converge to* u *in* $L^p(\Omega{\times}Y;\mathbb{E})$ *and let* (χ_ε) *denote a bounded sequence in* $L^\infty(\Omega)$. *If* (χ_ε) *converges to* $\chi\in L^\infty(\Omega)$ *in measure, then*

$$\chi_\varepsilon u_\varepsilon \xrightarrow{2} \chi u \qquad \text{weakly two-scale in } L^p(\Omega{\times}Y;\mathbb{E}).$$

Proof. Let $q \in (1,\infty)$ satisfy $1/p + 1/q = 1$. By assumption, the sequence $(\chi_\varepsilon u_\varepsilon)$ is bounded in $L^p(\Omega;\mathbb{E})$, and therefore weakly two-scale relatively compact (see Proposition 2.1.4). Thus, there exist a subsequence (not relabeled) and a map $w \in L^p(\Omega\times Y;\mathbb{E})$ such that

$$\int_\Omega \langle \chi_\varepsilon(x) u_\varepsilon(x),\, \psi(x, x/\varepsilon)\rangle \,\mathrm{d}x \to \iint_{\Omega\times Y} \langle w(x,y),\, \psi(x,y)\rangle \,\mathrm{d}y\,\mathrm{d}x.$$

for all $\psi \in L^q(\Omega; C_{\mathrm{per}}(Y;\mathbb{E}))$. It is sufficient to prove that $w(x,y) = \chi(x)u(x,y)$ for a.e. $(x,y) \in \Omega\times Y$. Let ψ be an arbitrary two-scale test function in $L^q(\Omega; C_{\mathrm{per}}(Y;\mathbb{E}))$ and let U be a compact subset of Ω that contains $\operatorname{supp}\psi$. We first show that

$$\chi_\varepsilon u_\varepsilon \xrightarrow{2} \chi u \qquad \text{weakly two-scale in } L^1(U\times Y;\mathbb{E}). \tag{2.8}$$

Since (χ_ε) is bounded in $L^\infty(\Omega)$ and because U is compact, the map

$$U \ni x \mapsto \sup_{\varepsilon>0} \operatorname*{ess\,sup}_{x\in U} |\chi_\varepsilon(x)|$$

is a function in $L^q(U)$ that dominates each $\chi_\varepsilon|_U$. Thus, because $\chi_\varepsilon|_U$ converges to $\chi|_U$ in measure, the dominated convergence theorem implies that

$$\chi_\varepsilon|_U \to \chi|_U \qquad \text{strongly in } L^q(U;\mathbb{E}).$$

Now Lemma 2.1.11 teaches that the map $U\times Y \ni (x,y) \mapsto \chi(x)$ is the strong two-scale limit of $(\chi_\varepsilon|_U)$ in $L^q(U\times Y)$ and by applying the product rule (see Proposition 2.1.12) convergence (2.8) follows.

As a consequence, the uniqueness of two-scale limits implies that

$$w(x,y) = \chi(x)u(x,y) \qquad \text{for a.e. } (x,y) \in U\times Y.$$

Because the previous reasoning can be repeated for arbitrary $U \subset\subset \Omega$, the proof is complete. $\square$

Two-scale convergence of gradients. In the following we consider bounded sequences in $W^{1,p}(\Omega;\mathbb{E})$. Define

$$W^{1,p}_{\mathrm{per},0}(Y;\mathbb{E}) := \left\{ \psi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{E}) \,:\, \psi \in L^p_{\mathrm{per}}(Y;\mathbb{E}),\ \int_Y \psi \,\mathrm{d}y = 0 \right\}.$$

Proposition 2.1.14 (cf. Theorem 20 [LNW02], Proposition 4.2 [Vis06])**.** *Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^n$, $p \in (1,\infty)$ and (u_ε) a bounded sequence in $W^{1,p}(\Omega;\mathbb{E})$. If*

$$u_\varepsilon \xrightarrow{2} u(x,y) \qquad \textit{weakly two-scale in } L^p(\Omega\times Y;\mathbb{E}),$$

then u is independent of $y \in Y$ and (u_ε) converges to u weakly in $W^{1,p}(\Omega;\mathbb{E})$. Moreover, there exists a subsequence (not relabeled) and $u_1 \in L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y;\mathbb{E}))$ such that

$$\nabla u_\varepsilon \xrightarrow{2} \nabla u(x) + \nabla_y u_1(x,y) \qquad \textit{weakly two-scale in } L^p(\Omega\times Y;\mathbb{E}^n).$$

Remark 2.1.15. In Section 6.3 we present an extension of this result comprising a new characterization of two-scale cluster points that emerge from sequences of scaled gradients, in particular for sequences of vector fields in the form

$$\left(\ \partial_1 u_\varepsilon \ \middle|\ \partial_2 u_\varepsilon \ \middle|\ \tfrac{1}{h(\varepsilon)}\partial_3 u_\varepsilon\ \right)$$

where (u_ε) is a bounded sequence in $W^{1,2}(\Omega)$ with $\Omega \subset \mathbb{R}^3$ and $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$.

Proposition 2.1.16. *Let* $\psi \in C_c^\infty(\mathbb{R}^n; C_{\mathrm{per}}^\infty(Y;\mathbb{E}))$ *and define* $\psi_\varepsilon(x) := \psi_\varepsilon(x, x/\varepsilon)$. *Then*

$$\varepsilon\,\nabla\psi_\varepsilon \xrightarrow{2} \nabla_y\,\psi \qquad \textit{strongly two-scale in } L^p(\mathbb{R}^n;\mathbb{E}^n)$$

for all $p \in [1,\infty]$.

Proof. By the chain rule we have

$$\varepsilon\,\nabla\psi_\varepsilon(x) = \varepsilon(\nabla\psi)(x, {}^x\!/\!_\varepsilon) + (\nabla_y\,\psi)(x, {}^x\!/\!_\varepsilon).$$

Now the first term on the right hand side strongly converges to 0 in $L^p(\Omega;\mathbb{E}^n)$ while the second term strongly two-scale converges to $\nabla_y\,\psi(x,y)$. □

2.2. Two-scale properties of piecewise constant approximations

Coherent maps. Let $\mathcal{L}_{\delta,c} := \delta\mathbb{Z}^n + c$ denote the n-dimensional standard lattice dilated by $\delta > 0$ and translated by $c \in \delta Y$. To any discrete map $\mathfrak{u} : \mathcal{L}_{\delta,c} \to \mathbb{E}$ we can assign a map $u : \mathbb{R}^n \to \mathbb{E}$ according to

$$u(x) := \sum_{\xi\in\mathcal{L}_{\delta,c}} 1_{\delta Y}(x-\xi)\mathfrak{u}(\xi).$$

Then u is piecewise constant; more precisely, it is constant on each of the cells $\delta Y + \xi$ with $\xi \in \mathcal{L}_{\delta,c}$. We say that such a map is *coherent to the* (δ, c)*-lattice* (see Definition 2.2.1 below).

In this section we study the two-scale convergence behavior of coherent piecewise constant maps. Roughly speaking, we are going to show that whenever we have a two-scale convergent sequence (u_ε) where each u_ε is coherent to a δ_ε-lattice with $\delta_\varepsilon = \varepsilon$ or $\delta_\varepsilon \gg \varepsilon$, then the sequence's two-scale limit is independent of the fast variable.

Definition 2.2.1. Let $\delta > 0$ and Ω be an open (possibly unbounded) subset of $\mathbb{R}^n$.

(a) We say that a measurable map $u : \mathbb{R}^n \to \mathbb{E}$ is *coherent to a* (δ, c)*-lattice* if

$$\int_Y u\big(\,\delta\lfloor \tfrac{x-c}{\delta}\rfloor + \delta y + c\,\big)\,\mathrm{d}y = u(x) \qquad \text{for almost every } x \in \mathbb{R}^n.$$

(b) We say that a measurable map $u : \Omega \to \mathbb{E}$ is *δ-coherent* if there exists a measurable map $\tilde{u} : \mathbb{R}^n \to \mathbb{E}$ such that $u = \tilde{u}|_\Omega$ and $\tilde{u}$ is coherent to a (δ, c)-lattice for a translation $c \in \delta Y$.

(c) We say that a measurable map $u : \Omega \to \mathbb{E}$ is *δ-coherent in the interior of* Ω, if there exists a translation $c \in \delta Y$ such that

$$\int_Y u\big(\,\delta\lfloor\tfrac{x-c}{\delta}\rfloor + \delta y + c\,\big)\,\mathrm{d}y = u(x)$$

$$\text{for almost every } x \in \Omega \text{ with } \operatorname{dist}(x, \partial\Omega) > 3\sqrt{n}\delta.$$

Remark 2.2.2. Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\delta > 0$ and $c \in \delta Y$. The set

$$\Omega_\delta := \bigcup \Big\{\, \xi + \delta Y \;:\; \xi \in \mathcal{L}_{\delta,c} \text{ such that } \xi + \delta Y \subset \Omega \,\Big\}$$

is the largest union of cells of the (δ, c)-lattice covered by Ω. Note that every $x \in \Omega$ that satisfies

$$\operatorname{dist}(x, \partial\Omega) > 3\sqrt{n}\delta.$$

belongs to Ω_δ.

Lemma 2.2.3. *Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain. Let $(u_\varepsilon) \subset L^p(\Omega; \mathbb{E})$ be a weakly two-scale convergent sequence with limit u in $L^2(\Omega \times Y; \mathbb{E})$ and suppose that there exists a subsequence (ε') such that each $u_{\varepsilon'}$ is ε'-coherent in the interior of Ω. Then*

$$u(x, y) = \int_Y u(x, \bar{y})\,\mathrm{d}\bar{y} \qquad \text{for a.e. } x \in \Omega \text{ and } y \in Y.$$

Proof. Extend each map u_ε to $\mathbb{R}^n$ by zero. Moreover, we extend the limit $u(x, y)$ to $\mathbb{R}^n \times \mathbb{R}^n$ by zero in its first variable and by Y-periodicity in its second variable. Then we have

$$u_\varepsilon \xrightarrow{2} u \qquad \text{weakly two-scale in } L^p(\mathbb{R}^n \times Y; \mathbb{E}).$$

We pass to a subsequence (which we do not relabel) such that u_ε is ε-coherent. Hence, there exists a sequence $c_\varepsilon \in \mathbb{R}^n$ with $|c_\varepsilon| \in \varepsilon Y$ such that

$$\int_Y u_\varepsilon\big(\,\varepsilon\lfloor\tfrac{x-c_\varepsilon}{\varepsilon}\rfloor + \varepsilon y + c_\varepsilon\,\big)\,\mathrm{d}y = u_\varepsilon(x)$$

for almost every

$$x \in \Omega_\varepsilon := \{\, x \in \Omega \;:\; \operatorname{dist}(x, \partial\Omega) > 3\sqrt{n}\varepsilon\,\}.$$

<u>*Step 1.*</u> We first consider the case where $c_\varepsilon = 0$ for all ε and set

$$\mathring{u}_\varepsilon(x) := \int_Y u_\varepsilon(\varepsilon\lfloor x/\varepsilon\rfloor + \varepsilon y)\,\mathrm{d}y.$$

Then $\mathring{u}_\varepsilon$ weakly two-scale converges in $L^2(\Omega{\times}Y;\mathbb{E})$ to the map

$$\int_Y u(\cdot,y)\,\mathrm{d}y \tag{2.9}$$

as it is easy to see, because $(\mathcal{T}_\varepsilon\mathring{u}_\varepsilon)(x,y)=\mathring{u}_\varepsilon(x)$ (cf. [Vis07]). On the other side, we claim that

$$\mathring{u}_\varepsilon \xrightarrow{2} u \qquad \text{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{E}). \tag{2.10}$$

To this end, let $\varphi\in C_c^\infty(\Omega;C^\infty_{\mathrm{per}}(Y;\mathbb{E}))$ and set $\varphi^\varepsilon(x):=\varphi(x,{}^x\!/\!_\varepsilon)$. Because Ω_ε covers Ω except for a thin, tubular neighborhood of the Lipschitz boundary $\partial\Omega$ with diameter $3\sqrt{n}\varepsilon$, the support of φ^ε is contained in Ω_ε provided ε is sufficiently small. Moreover, the ε-coherence implies that

$$\mathring{u}_\varepsilon(x)=u_\varepsilon(x) \qquad \text{for a.e. } x\in\Omega_\varepsilon$$

and consequently

$$\int_\Omega \langle\mathring{u}_\varepsilon(x),\,\varphi^\varepsilon(x)\rangle\,\mathrm{d}x=\int_{\Omega_\varepsilon}\langle u_\varepsilon(x),\,\varphi^\varepsilon(x)\rangle\to\iint_{\Omega\times Y}\langle u(x,y),\,\varphi(x,y)\rangle\,\mathrm{d}y\,\mathrm{d}x$$

for all smooth two-scale test functions φ. Since $(\mathring{u}_\varepsilon)$ is bounded in $L^2(\Omega;\mathbb{E})$, this already implies (2.10). Because of the uniqueness of two-scale limits, we deduce that $u(x,y)$ is equal to the map in (2.9), and therefore independent of the fast variable y.

Step 2. We consider the general case where $c_\varepsilon\in\varepsilon Y$. We pass to a subsequence (not relabeled) such that ${}^{c_\varepsilon}\!/\!_\varepsilon\to c_0$ and define the maps

$$\tilde{u}_\varepsilon(x):=u_\varepsilon(x+c_\varepsilon).$$

By construction $(\tilde{u}_\varepsilon)$ is a bounded sequence in $L^2(\mathbb{R}^n;\mathbb{E})$. We claim that

$$\tilde{u}_\varepsilon(x)\xrightarrow{2}u(x,y+c_0)\qquad\text{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{E}).$$

In order to prove this, let $\varphi\in C_c^\infty(\Omega;C^\infty_{\mathrm{per}}(Y;\mathbb{E}))$ and set $\varphi^\varepsilon(x):=\varphi(x,{}^x\!/\!_\varepsilon)$. Because $c_\varepsilon\to0$, the support of the test function φ^ε translated by $-c_\varepsilon$ is still contained in Ω, if ε is sufficiently small. This justifies the computation

$$\begin{aligned}\int_\Omega\langle\tilde{u}_\varepsilon(x),\,\varphi(x,{}^x\!/\!_\varepsilon)\rangle\,\mathrm{d}x&=\int_\Omega\langle u_\varepsilon(x+c_\varepsilon),\,\varphi(x,{}^x\!/\!_\varepsilon)\rangle\,\mathrm{d}x\\&=\int_\Omega\Big\langle u_\varepsilon(x),\,\varphi\big(x-c_\varepsilon,\tfrac{x-c_\varepsilon}{\varepsilon}\big)\Big\rangle\,\mathrm{d}x\end{aligned}$$

Because of ${}^{c_\varepsilon}\!/\!_\varepsilon\to c_0$, the map $x\mapsto\varphi(x{-}c_\varepsilon,\frac{x-c_\varepsilon}{\varepsilon})$ strongly two-scale converges to $\varphi(x,y{-}c_0)$ in $L^{p/p-1}(\Omega;\mathbb{E})$ (cf. [NS10]), and consequently

$$\tilde{u}_\varepsilon\xrightarrow{2}u(x,y+c_0)\qquad\text{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{E}).$$

On the other hand, each map $\tilde{u}_\varepsilon$ is ε-coherent in the sense of Step 1 and we deduce that

$$u(x,y+c_0)=\int_Y u(x,\bar{y}+c_0)\,\mathrm{d}\bar{y}$$

for all $x\in\Omega$ and $y\in Y$. In view of the Y-periodicity of $u(x,y)$ in its second variable, the proof is complete. □

Lemma 2.2.4. *Let $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $(u_\varepsilon) \subset L^p(\Omega;\mathbb{E})$ be a weakly two-scale convergent sequence with limit u in $L^2(\Omega\times Y;\mathbb{E})$ and suppose that each u_ε is h_ε-coherent in the interior of Ω with*

$$\lim_{\varepsilon\to 0} h_\varepsilon = 0 \qquad \textit{and} \qquad \liminf_{\varepsilon\to 0} \frac{\varepsilon}{h_\varepsilon} = 0.$$

Then

$$u(x,y) = \int_Y u(x,\bar{y})\,\mathrm{d}\bar{y} \qquad \textit{for a.e. } x \in \Omega \textit{ and } y \in Y.$$

Proof. We extend each map u_ε to $\mathbb{R}^n$ by zero. Moreover, we extend the limit $u(x,y)$ to $\mathbb{R}^n\times\mathbb{R}^n$ by zero in its first variable and by Y-periodicity in its second variable. By assumption, there exist a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and a sequence $(c_k) \subset \mathbb{R}^n$ such that

$$\frac{\varepsilon_k}{h_{\varepsilon_k}} \le \frac{1}{2k}$$

and

$$\int_Y u_{\varepsilon_k}(h_{\varepsilon_k}\lfloor \tfrac{x-c_k}{h_{\varepsilon_k}}\rfloor + h_{\varepsilon_k}y + c_k)\,\mathrm{d}y = u_{\varepsilon_k}(x)$$

for all

$$x \in \Omega_k := \{\, x \in \Omega \,:\, \operatorname{dist}(x,\partial\Omega) > 3\sqrt{n}\varepsilon_k \,\}.$$

We only consider the case $c_k = 0$. The proof for $c_k \neq 0$ can be reduced to the case $c_k = 0$ by a reasoning similar to Step 2 of Lemma 2.2.3. Define

$$v_k(x) := u_{\varepsilon_k}(x) - \int_Y u_{\varepsilon_k}(\varepsilon_k\lfloor x/\varepsilon_k\rfloor + \varepsilon_k y)\,\mathrm{d}y$$

and set $h_k := h_{\varepsilon_k}$ for brevity. The sequence (v_k) weakly two-scale converges in $L^p(\Omega;\mathbb{E})$ to the map

$$u(x,y) - \int_Y u(x,\bar{y})\,\mathrm{d}\bar{y}.$$

Hence, it is sufficient to prove that (v_k) weakly two-scale converges to 0.

This can be seen as follows: For all $k \in \mathbb{N}$ define the sets

$$Z_k := \Big\{\, p \in h_k\mathbb{Z}^n \,:\, p + h_k(0,1)^n \subset \{\, x \in \Omega \,:\, \operatorname{dist}(x,\partial\Omega) > 5\sqrt{n}h_k \,\} \Big\}$$
$$A_k := \bigcup_{p\in Z_k} (p + [\varepsilon_k, h_k - \varepsilon_k]^n)\,.$$

The set Z_k contains all lattice points in Ω that are sufficiently far away from $\partial\Omega$. Since $\mathcal{H}^{n-1}(\partial\Omega)$ is bounded and $\partial\Omega$ Lipschitz, there exists a positive constant c' such that

$$\#Z_k \ge \frac{\mathcal{H}^n(\Omega)}{h_k^n} - c' h_k^{n-1}$$

and consequently

$$\mathcal{H}^n(A_k) = \#Z_k\, \mathcal{H}^n(p + [\varepsilon_k, h_k - \varepsilon_k]^n) = \#Z_k\, |h_k - 2\varepsilon_k|^n \\ \geq (1 - \tfrac{1}{k})^n\, h_k^n \#Z_k \geq (1 - \tfrac{1}{k})^n \left(\mathcal{H}^n(\Omega) - c' h_k \right).$$

Thus, $\mathcal{H}^n(A_k) \to \mathcal{H}^n(\Omega)$ and because A_k is a subset of Ω, the previous estimate implies that $1_{\Omega \setminus A_k}$ converges to 0 boundedly in measure. In view of Proposition 2.1.13, this implies that

$$1_{\Omega \setminus A_k}\, v_k \xrightarrow{2} 0 \qquad \text{weakly two-scale in } L^p(\Omega; \mathbb{E}). \tag{2.11}$$

Now let $p \in Z_k$. Because u_{ε_k} is h_k-coherent (with $c_k = 0$), we have

$$u_{\varepsilon_k}(x) = \int_Y u_{\varepsilon_k}(h_k \lfloor x/h_k \rfloor + h_k y)\, \mathrm{d}y \qquad \text{for a.e. } x \in p + (0, h_k)^n.$$

In particular, this implies that u_{ε_k} restricted to the cube $p + (0, h_k)^n$ is constant. On the other hand, we observe that

$$x \in p + (\varepsilon_k, h_k - \varepsilon_k)^n \;\Rightarrow\; \forall y \in Y \,:\, (\, \varepsilon_k \lfloor x/\varepsilon_k \rfloor + \varepsilon_k y\,) \in p + (0, h_k)^n.$$

In combination with the constancy of u_{ε_k} on the cube $p + (0, h_k)^n$, this implies that

$$\int_Y u_{\varepsilon_k}(\varepsilon_k \lfloor x/\varepsilon_k \rfloor + \varepsilon_k y)\, \mathrm{d}y = u_{\varepsilon_k}(x) \qquad \text{for a.e. } x \in p + (\varepsilon_k, h_k - \varepsilon_k)^n.$$

Since A_k is a union of cubes of the form $p + (\varepsilon_k, h_k - \varepsilon_k)^n$ with $p \in Z_k$, the previous identity implies that

$$1_{A_k}\, v_k = 0 \qquad \text{for a.e. } x \in \Omega$$

and together with (2.11) we see that (v_k) indeed weakly two-scale converges to 0. □

Regularization δ-coherent maps in one dimension. In this paragraph we suppose that $\omega \subset \mathbb{R}$ is an open interval and $Y := [0, 1)$. For $c \in \mathbb{R}$, we set

$$\mathcal{L}_{\delta,c}(\omega) := \{\, \xi \in \delta\mathbb{Z}^n + c \,:\, [\xi, \xi + \delta) \cap \omega \neq \emptyset \,\}$$

which is the "smallest" (δ, c)-lattice that covers ω. Obviously, if a map $u : \omega \to \mathbb{E}$ is δ-coherent, then there exist a translation $c \in [0, \delta)$ and a unique discrete map

$$\mathfrak{u} \,:\, \mathcal{L}_{\delta,c}(\omega) \to \mathbb{E} \qquad \text{such that} \qquad u(x) = \sum_{\xi \in \mathcal{L}_{\delta,c}(\omega)} 1_{[0,\delta)}(x - \xi)\mathfrak{u}(\xi)$$

for almost every $x \in \omega$. For brevity, we write $\mathcal{L}_{(u)}$ instead of $\mathcal{L}_{\delta,c}(\omega)$. For $p \in [1, \infty)$ we define the *p-variation* of a δ-coherent map u according to

$$\operatorname{Var}_p u := \sum_{\xi \in \mathcal{L}_{(u)} \setminus \max \mathcal{L}_{(u)}} |\mathfrak{u}(\xi + \delta) - \mathfrak{u}(\xi)|^p\,.$$

Remark 2.2.5. If $u : \omega \to \mathbb{E}$ is δ-coherent, then u is a function of bounded p-variation and $\operatorname{Var}_p u$ coincides with the usual p-variation seminorm for functions in $BV_p(\omega)$.

Lemma 2.2.6. *Let $p \in [1, \infty)$, Suppose that $u : \omega \to \mathbb{E}$ is δ-coherent. Then there exist a piecewise affine map $v \in W^{1,\infty}(\omega; \mathbb{E})$ with*

$$\partial_1 v \in L^\infty(\omega; \mathbb{E}) \text{ is } \delta\text{-coherent}$$

and

$$\int_\omega |u - v|^p \, \mathrm{d}x_1 \leq \frac{\delta}{p+1} \operatorname{Var}_p(u), \qquad \int_\omega |\nabla v|^p \, \mathrm{d}x_1 \leq \delta^{1-p} \operatorname{Var}_p(u).$$

Proof. For brevity we set $\mathcal{L}_\star := \mathcal{L}_{(u)} \setminus \max \mathcal{L}_{(u)}$. Let $\mathfrak{u} : \mathcal{L}_{(u)} \to \mathbb{E}$ denote the discrete map satisfying

$$u(x_1) = \sum_{\xi \in \mathcal{L}_{(u)}} 1_{[0,\delta)}(x_1 - \xi)\mathfrak{u}(\xi) \qquad \text{for almost every } x_1 \in \omega$$

and define $W := \cup_{\xi \in \mathcal{L}_{(u)}} [\xi, \xi + \delta]$. Let $v : W \to \mathbb{E}$ be the linear interpolation of $\mathfrak{u}$, i.e.

$$v(x) := \begin{cases} \mathfrak{u}(\xi) + \dfrac{x - \xi}{\delta}(\mathfrak{u}(\xi + \delta) - \mathfrak{u}(\xi)) & \text{if } \exists \xi \in \mathcal{L}_\star \text{ such that } x \in [\xi, \xi + \delta) \\ \mathfrak{u}(\max \mathcal{L}_{(u)}) & \text{if } x \in [\max \mathcal{L}_{(u)}, \max \mathcal{L}_{(u)} + \delta] \end{cases}$$

It is easy to show that $v|_\omega$ fulfills the claimed properties. Therefore, we only show the estimate for $\|u - v\|^p_{L^p(\omega;\mathbb{E})}$. Because u is δ-coherent, we have

$$\begin{aligned} \int_\omega |u - v|^p \, \mathrm{d}x_1 &\leq \sum_{\xi \in \mathcal{L}_{(u)}} \int_{(\xi, \xi+\delta)} |\mathfrak{u}(\xi) - v(x_1)|^p \, \mathrm{d}x_1 \\ &\leq \sum_{\xi \in \mathcal{L}_\star} |\mathfrak{u}(\xi + \delta) - \mathfrak{u}(\xi)|^p \int_{(0,\delta)} \frac{x_1}{\delta} \, \mathrm{d}x_1 = \frac{\delta}{p+1} \operatorname{Var}_p(u). \end{aligned}$$

□

Proposition 2.2.7. *Let (δ_ε) be a family of positive numbers such that* $\lim_{\varepsilon \to 0} \delta_\varepsilon = 0$. *Let (u_ε) be a bounded family in $L^2(\omega; \mathbb{E})$. Suppose that*

(a) u_ε is δ_ε-coherent for each ε.

(b) $\limsup_{\varepsilon \to 0} \delta_\varepsilon^{-1} \operatorname{Var}_2(u_\varepsilon) < +\infty$.

Then there exists a sequence $(v_\varepsilon) \subset W^{1,2}(\omega; \mathbb{E})$ such that

(1) (v_ε) is bounded in $W^{1,2}(\omega; \mathbb{E})$.

(2) The sequence $(u_\varepsilon - v_\varepsilon)$ strongly converges to zero in $L^2(\omega; \mathbb{E})$.

(3) If (u_ε) weakly converges to u in $L^2(\omega;\mathbb{E})$, then u belongs to $W^{1,2}(\omega;\mathbb{E})$ and

$$\begin{aligned} v_\varepsilon \to u &\qquad \textit{strongly in } L^2(\omega;\mathbb{E}) \\ u_\varepsilon \to u &\qquad \textit{strongly in } L^2(\omega;\mathbb{E}). \end{aligned}$$

Moreover, there exists a map $u_0 \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y;\mathbb{E}))$ such that

$$\partial_1 v_\varepsilon \stackrel{2}{\rightharpoonup} \partial_1 u(x_1) + \partial_y u_0(x_1, y) \qquad \textit{weakly two-scale in } L^2(\omega{\times}Y;\mathbb{E})$$

for a suitable subsequence (not relabeled). If additionally either

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon}{\delta_\varepsilon} = 0 \qquad \textit{or} \qquad \delta_\varepsilon = \varepsilon,$$

then $u_0 = 0$.

Proof. Let v_ε denote the approximation constructed in Lemma 2.2.6. Then we have

$$\frac{3}{\delta_\varepsilon^2}\|u_\varepsilon - v_\varepsilon\|^2_{L^2(\omega;\mathbb{E})} + \|\partial_1 v_\varepsilon\|^2_{L^2(\omega;\mathbb{E}^n)} \le \frac{\mathrm{Var}_2(u_\varepsilon)}{\delta_\varepsilon}$$

for each ε. Now assumption (b) and the boundedness of (u_ε) immediately imply statement (1) and (2). We prove (3). To this end, we suppose that (u_ε) weakly converges to $u \in L^2(\omega;\mathbb{E})$. As a consequence of (1) and (2), also (v_ε) weakly converges to u in $L^2(\omega;\mathbb{E})$. In view of Proposition 2.1.14 we deduce that there exist a subsequence (not relabeled) and a map $u_0 \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y;\mathbb{E}))$ such that

$$\begin{aligned} v_\varepsilon &\rightharpoonup u && \text{weakly in } W^{1,2}(\omega;\mathbb{E}) \\ \partial_1 v_\varepsilon &\stackrel{2}{\rightharpoonup} \partial_1 u(x) + \partial_y u_0(x,y) && \text{weakly two-scale in } L^2(\omega{\times}Y;\mathbb{E}). \end{aligned}$$

Due to the compactness of the embedding $W^{1,2}(\omega;\mathbb{E}) \subset L^2(\omega;\mathbb{E})$ and the uniqueness of the limit u, the convergence $v_\varepsilon \to u$ also holds strongly in $L^2(\omega;\mathbb{E})$ for the entire sequence. Now (2) implies that also u_ε strongly converges to u in $L^2(\omega;\mathbb{E})$. In order to prove the last part of (3), suppose that

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon}{\delta_\varepsilon} = 0 \qquad \text{or} \qquad \delta_\varepsilon = \varepsilon. \tag{2.12}$$

By construction $\partial_1 v_\varepsilon$ is δ_ε-coherent; thus, assumption (2.12) allows us to apply either Lemma 2.2.3 or Lemma 2.2.4 and we deduce that

$$\partial_1 u(x_1) + \partial_1 u_0(x_1, y) = \int_Y \partial u(x_1) + \partial_y u_0(x_1, y)\,\mathrm{d}y$$

for almost every $(x_1, y) \in \omega{\times}Y$, and consequently $u_0 = 0$. □

2.3. Two-scale convergence and linearization

In this section we suppose that $\Omega \subset \mathbb{R}^n$ is an open and bounded domain with Lipschitz boundary and assume that (h_ε) is a sequence of positive real numbers that converges to 0 as $\varepsilon \to 0$.

Let $\Phi \in C^1(\mathbb{E};\mathbb{E})$ and consider a weakly two-scale convergent sequence (u_ε) in $L^p(\Omega;\mathbb{E})$. In the sequel we present some methods to study the two-scale convergence behavior of sequences of the form

$$w_{h,\varepsilon}(x) := \frac{\Phi(hu_\varepsilon(x)) - \Phi(0)}{h} \tag{2.13}$$

as h and ε simultaneously converge to zero. In particular, the subsequent analysis covers the case where

$$w_{h,\varepsilon}(x) := \frac{\sqrt{(Id + h\,\nabla u_\varepsilon(x))^{\mathrm{T}}(Id + h\,\nabla u_\varepsilon(x))} - Id}{h}$$

and (u_ε) is a sequence of maps in $W^{1,2}(\Omega;\mathbb{R}^n)$. This situation is related to elasticity where u_ε has the meaning of a scaled displacement and $w_{h,\varepsilon}$ can be interpreted as the scaled nonlinear strain. In Corollary 2.3.4 we study this situation explicitly.

As an introductory example, let us consider (2.13) and assume that

$$u_\varepsilon \xrightarrow{2} u \qquad \text{weakly two-scale in } L^p(\Omega{\times}Y)$$

with $p \in [1,\infty)$. Moreover, we suppose that there exists a constant M such that

$$\operatorname*{ess\,sup}_{x\in\Omega} |u_\varepsilon(x)| \leq M \qquad \text{for all } \varepsilon. \tag{2.14}$$

For $\Phi \in C^1(\mathbb{R})$, it is natural to expect that

$$w_\varepsilon := \frac{\Phi(h_\varepsilon u_\varepsilon) - \Phi(0)}{h_\varepsilon} \xrightarrow{2} \Phi'(0)\,u(x,y) \qquad \text{weakly two-scale in } L^p(\Omega{\times}Y). \tag{2.15}$$

Indeed, this is the case: Since Φ is of class C^1, we can rewrite w_ε by means of the fundamental theorem of calculus, and for almost every $x \in \Omega$ we obtain

$$w_\varepsilon(x) = \frac{\Phi(h_\varepsilon u_\varepsilon(x)) - \Phi(0)}{h_\varepsilon} = \left(\int_0^1 \Phi'(sh_\varepsilon u_\varepsilon(x))\,\mathrm{d}s\right) u_\varepsilon(x)$$

where Φ' denotes the derivative of Φ. Because of the uniform bound (2.14), we have

$$\operatorname*{ess\,sup}_{(x,y)\in\Omega\times Y} \left|\int_0^1 \Phi'(sh_\varepsilon(\mathcal{T}_\varepsilon u_\varepsilon)(x,y))\,\mathrm{d}s - \Phi'(0)\right| \leq \sup_{|a|\leq h_\varepsilon M} \left|\Phi'(a) - \Phi'(0)\right|.$$

Now the continuity of $a \mapsto \Phi'(a)$ and the assumption that $h_\varepsilon \to 0$ imply that

$$\int_0^1 \Phi'(sh_\varepsilon u_\varepsilon(\cdot))\,\mathrm{d}s \xrightarrow{2} \Phi'(0) \qquad \text{strongly two-scale in } L^\infty(\Omega\times Y) \tag{2.16}$$

and in virtue of the product rule (Proposition 2.1.12) convergence (2.15) follows.

The situation gets more interesting if we weaken the regularity of Φ and drop the assumption that (u_ε) is uniformly bounded:

Proposition 2.3.1. *Let $p \in (1,\infty)$ and let $\Phi \in C(\mathbb{E};\mathbb{E})$ satisfy the following properties:*

(a) There exists a linear map $\Lambda : \mathbb{E} \to \mathbb{E}$ such that

$$\limsup_{\substack{a\to 0\\ a\neq 0}} \frac{|\Phi(a)-\Phi(0)-\Lambda(a)|}{|a|} = 0,$$

(b) Φ is locally p-Lipschitz continuous, i.e. there exists a constant L such that

$$|\Phi(a)-\Phi(b)| \le L(1+|a|^{p-1}+|b|^{p-1})\,|a-b| \qquad \text{for all } a,b\in\mathbb{E}.$$

If (u_ε) is a bounded sequence in $L^p(\Omega;\mathbb{E})$, then the sequence

$$w_\varepsilon := \frac{\Phi(h_\varepsilon u_\varepsilon)-\Phi(0)}{h_\varepsilon}$$

is weakly two-scale relatively compact in $L^1(\Omega\times Y;\mathbb{E})$ and we have

$$w_\varepsilon \xrightarrow{2} \Lambda(u) \qquad \text{weakly two-scale in } L^1(\Omega\times Y;\mathbb{E})$$

whenever

$$u_\varepsilon \xrightarrow{2} u \qquad \text{weakly two-scale in } L^p(\Omega\times Y;\mathbb{E}).$$

The proof relies on the following observation:

Lemma 2.3.2. *Let $p\in[1,\infty)$, $\alpha\in(0,1)$ and (u_ε) a bounded sequence in $L^p(\Omega;\mathbb{E})$. For each ε define*

$$\chi_\varepsilon^\alpha(x) := \begin{cases} 1 & \text{if } |u_\varepsilon(x)| \le h_\varepsilon^{\alpha-1}\\ 0 & \text{else.}\end{cases}$$

Then $(\chi_\varepsilon^\alpha)$ is a bounded sequence in $L^\infty(\Omega)$ and satisfies

$$\chi_\varepsilon^\alpha \to 1 \qquad \text{strongly in } L^r(\Omega)$$

for all $r\in[1,\infty)$. Moreover, if (u_ε) weakly two-scale converges to u in $L^p(\Omega\times Y;\mathbb{E})$ with $p\in(1,\infty)$, then

$$\chi_\varepsilon^\alpha u_\varepsilon \xrightarrow{2} u \qquad \text{weakly two-scale in } L^p(\Omega\times Y;\mathbb{E}). \tag{2.17}$$

Proof. By definition we have

$$\int_\Omega |u_\varepsilon|^p \,\mathrm{d}x \geq h_\varepsilon^{p(\alpha-1)} \int_\Omega (1-\chi_\varepsilon^\alpha)\,\mathrm{d}x.$$

Because $|1-\chi_\varepsilon^\alpha|^r = (1-\chi_\varepsilon^\alpha)$ for all $r \in [1,\infty)$ and $h_\varepsilon^{-p(\alpha-1)} \to 0$, we deduce that $(1-\chi_\varepsilon^\alpha)$ strongly converges to 0 in $L^r(\Omega)$, and consequently $\chi_\varepsilon^\alpha = 1-(1-\chi_\varepsilon^\alpha)$ converges to 1. By applying Proposition 2.1.13 to the product $\chi_\varepsilon^\alpha u_\varepsilon$, we see that (2.17) holds, whenever (u_ε) weakly two-scale converges to u. □

Proof of Proposition 2.3.1. *Step 1.* For all $a \in \mathbb{E}$ define the remainder

$$\operatorname{rest}(a) := \Phi(a) - \Phi(0) - \Lambda(a)$$

and set $\rho(\alpha) := \sup\left\{ |a|^{-1} |\operatorname{rest}(a)| \,:\, a \in \mathbb{E},\ 0 < |a| \leq \alpha \right\}$ for $\alpha > 0$ and $\rho(0) := 0$.

Then $\rho : [0,\infty) \to [0,\infty]$ is a monotonically increasing map with $\rho(\alpha) \to 0$ as $\alpha \downarrow 0$ and

$$|\operatorname{rest}(a)| \leq \rho(|a|)\,|a| \qquad \text{for all } a \in \mathbb{E}.$$

Step 2. Because (u_ε) is (as a bounded sequence) relatively compact with respect to weak two-scale convergence (see Proposition 2.1.4), it is sufficient to assume that

$$u_\varepsilon \xrightarrow{2} u \qquad \text{weakly two-scale in } L^p(\Omega\times Y;\mathbb{E}).$$

Thereby, we only have to show that

$$w_\varepsilon \xrightarrow{2} \Lambda(u) \qquad \text{weakly two-scale in } L^1(\Omega\times Y;\mathbb{E}).$$

Set

$$\chi_\varepsilon(x) := \begin{cases} 1 & \text{if } |u_\varepsilon(x)| \leq h^{-1/2} \\ 0 & \text{else.} \end{cases}$$

and define $U_\varepsilon := \mathcal{T}_\varepsilon(\chi_\varepsilon u_\varepsilon)$. In virtue of Lemma 2.3.2, we have

$$U_\varepsilon \rightharpoonup u \qquad \text{weakly in } L^p(\mathbb{R}^n\times Y;\mathbb{E})$$

and by construction U_ε is uniformly bounded in the following sense

$$\operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^n\times Y} |h_\varepsilon U_\varepsilon(x,y)| \leq h_\varepsilon^{1/2}. \tag{2.18}$$

We compute

$$\mathcal{T}_\varepsilon(\chi_\varepsilon w_\varepsilon) = \frac{\Phi(h_\varepsilon U_\varepsilon) - \Phi(0)}{h_\varepsilon} = \Lambda(U_\varepsilon) + \frac{\operatorname{rest}(h_\varepsilon U_\varepsilon)}{h_\varepsilon}.$$

In view of Step 1 and due to (2.18), we can estimate the remainder according to

$$h_\varepsilon^{-1}\,|\operatorname{rest}(h_\varepsilon U_\varepsilon)| \leq \rho(h_\varepsilon^{1/2})\,|U_\varepsilon|\,.$$

Because $\rho(h_\varepsilon^{1/2}) \to 0$ as $\varepsilon \to 0$ and because (U_ε) is a bounded sequence in $L^p(\mathbb{R}^n \times Y;\mathbb{E})$, we find that

$$h_\varepsilon^{-1}\,\mathrm{rest}(h_\varepsilon U_\varepsilon) \to 0 \qquad \text{strongly in } L^p(\mathbb{R}^n \times Y;\mathbb{E}).$$

On the other side, we have

$$\Lambda(U_\varepsilon) \rightharpoonup \Lambda(U) \qquad \text{weakly in } L^p(\mathbb{R}^n \times Y;\mathbb{E}),$$

because Λ is (as a linear map) continuous with respect to weak convergence. So far, we have shown that

$$\chi_\varepsilon w_\varepsilon \overset{2}{\rightharpoonup} u \qquad \text{weakly two-scale in } L^p(\Omega \times Y;\mathbb{E}). \tag{2.19}$$

Because $p > 1$, convergence (2.19) also holds with respect to weak two-scale convergence in $L^1(\Omega \times Y;\mathbb{E})$. Hence, to complete the proof, it is sufficient to show that

$$(1-\chi_\varepsilon)w_\varepsilon \to 0 \qquad \text{strongly in } L^1(\Omega;\mathbb{E}). \tag{2.20}$$

In order to justify this, we utilize the p-Lipschitz continuity of Φ which yields the estimate

$$\begin{aligned}\int_\Omega |(1-\chi_\varepsilon)w_\varepsilon|\,\mathrm{d}x &\le L\int_\Omega (1-\chi_\varepsilon)(1+|h_\varepsilon u_\varepsilon|^{p-1})\,|u_\varepsilon|\,\mathrm{d}x\\ &\le L\int_\Omega (1-\chi_\varepsilon)\,|u_\varepsilon|\,\mathrm{d}x + h_\varepsilon^{p-1}\,\|u_\varepsilon\|^p_{L^p(\Omega;\mathbb{E})}\end{aligned}$$

The sequence (u_ε) is bounded in $L^p(\Omega;\mathbb{E})$ with $p>1$; thus, the first term on the right hand side vanishes, because $(1-\chi_\varepsilon)$ strongly converges to 0 in $L^{p/(p-1)}(\Omega)$ (see Lemma 2.3.2). The second term on the right hand side vanishes due to $h_\varepsilon^{p-1} \to 0$. Thus, (2.20) follows and the proof is complete. □

Corollary 2.3.3. *Let $p \in (1,\infty)$ and $\Phi \in C(\mathbb{E},\mathbb{E})$ as in Proposition 2.3.1 and suppose that $(u_\varepsilon) \subset L^p(\Omega;\mathbb{E})$ is a weakly two-scale convergent sequence with limit $u \in L^p(\Omega \times Y;\mathbb{E})$. If Φ is globally Lipschitz continuous, i.e.*

$$|\Phi(a)-\Phi(b)| \le L\,|a-b| \qquad \text{for all } a,b \in \mathbb{E},$$

then

$$w_\varepsilon := \frac{\Phi(h_\varepsilon u_\varepsilon)-\Phi(0)}{h_\varepsilon} \overset{2}{\rightharpoonup} \Lambda u \qquad \text{weakly two-scale in } L^p(\Omega \times Y;\mathbb{E}).$$

If (u_ε) strongly two-scale converges to u in $L^p(\Omega \times Y;\mathbb{E})$ and additionally satisfies

$$(\star) \qquad \limsup_{\varepsilon\to 0}\,\operatorname*{ess\,sup}_{x\in\Omega} |h_\varepsilon u_\varepsilon(x)| = 0$$

then

$$w_\varepsilon := \frac{\Phi(h_\varepsilon u_\varepsilon)-\Phi(0)}{h_\varepsilon} \overset{2}{\to} \Lambda u \qquad \text{strongly two-scale in } L^p(\Omega \times Y;\mathbb{E}).$$

Proof. Because of the global Lipschitz continuity, the sequence (w_ε) is bounded in $L^p(\Omega;\mathbb{E})$. Hence, due to Proposition 2.1.4 the sequence (w_ε) is weakly two-scale relatively compact in $L^p(\Omega{\times}Y;\mathbb{E})$. On the other side, any weak two-scale cluster point of (w_ε) must be equal to $\Lambda(u)$ due to the previous proposition. As a consequence, the entire sequence weakly two-scale converges to $\Lambda(u)$ in $L^p(\Omega{\times}Y;\mathbb{E})$.

Now we suppose that (u_ε) is strongly two-scale convergent and satisfies $(\star)$. From the first part of the proof we know that w_ε weakly two-scale converges to Λu. In virtue of Lemma 2.1.5 it sufficient to prove that

$$\|\mathcal{T}_\varepsilon w_\varepsilon - \Lambda(\mathcal{T}_\varepsilon u_\varepsilon)\|_{L^p(\mathbb{R}^n\times Y;\mathbb{E})}\,.$$

But this follows due to $(\star)$. □

Eventually, we apply the previous results to an explicit situation that is related elasticity.

Corollary 2.3.4. *Let (F_ε) be a sequence in $L^2(\Omega;\mathbb{M}(d))$. If*

$$F_\varepsilon \xrightarrow{2} F \qquad \textit{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{M}(d))$$

then

$$\frac{(Id+h_\varepsilon F_\varepsilon)^T(Id+h_\varepsilon F_\varepsilon)-Id}{h_\varepsilon} \xrightarrow{2} 2\,\mathrm{sym}\,F \qquad \textit{weakly two-scale in } L^1(\Omega{\times}Y;\mathbb{M}(d))$$
$$\frac{\sqrt{(Id+h_\varepsilon F_\varepsilon)^T(Id+h_\varepsilon F_\varepsilon)}-Id}{h_\varepsilon} \xrightarrow{2} \mathrm{sym}\,F \qquad \textit{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{M}(d))$$

Proof. For $A\in\mathbb{M}(d)$ set $\Phi_1(A) := (Id+A)^{\mathrm{T}}(Id+A)$ and $\Phi_2(A) := \sqrt{\Phi_1(A)}$. Then Φ_1,Φ_2 satisfy the requirements from Proposition 2.3.1 for $p=2$ and

$$\Lambda_1(A) := 2\,\mathrm{sym}\,A, \qquad \Lambda_2(A) := \mathrm{sym}\,A.$$

Moreover, Φ_2 is globally Lipschitz continuous. Thus, the statements follow immediately from Proposition 2.3.1 and Corollary 2.3.3. □

Example 2.3.5. In the following we illustrate how the previous corollary might be used in the context of finite elasticity. Let $(u_\varepsilon)\in W^{1,2}(\Omega;\mathbb{R}^n)$ be a sequence of deformations satisfying

$$\det\nabla u_\varepsilon(x)\geq 0 \qquad \text{for almost every } x\in\Omega \qquad \text{and} \qquad u_\varepsilon|_{\partial\Omega}(x)=x.$$

The Cauchy strain tensor associated to u_ε is defined as the map

$$C_\varepsilon : \Omega\to\mathbb{M}_{\mathrm{sym}}(n), \qquad C_\varepsilon(x) := \nabla u_\varepsilon(x)^{\mathrm{T}}\,\nabla u_\varepsilon(x).$$

Note that in physically relevant situations the elastic energy associated to a deformation can be written as a function of the Cauchy strain. We suppose that

$$\limsup_{\varepsilon\to 0}\frac{1}{h_\varepsilon}\int_\Omega \mathrm{dist}^2(\nabla u_\varepsilon(x),SO(n))^2\,\mathrm{d}x<\infty. \tag{2.21}$$

Because $\operatorname{dist}^2(F, SO(n)) \geq \left|\sqrt{F^{\mathrm{T}}F} - Id\right|^2$ for all $F \in \mathbb{M}(n)$, condition (2.21) means that (u_ε) is a sequence of deformations with infinitesimal small strain (in the limit). For this reason it is convenient to introduce the scaled nonlinear strain

$$E_\varepsilon : \Omega \to \mathbb{M}_{\mathrm{sym}}(n) \qquad E_\varepsilon := \frac{\sqrt{C_\varepsilon} - Id}{h_\varepsilon}.$$

Moreover, we are going to see in the following chapters that (2.21) and the Dirichlet boundary condition imposed on u_ε imply that

$$g_\varepsilon(x) := \frac{u_\varepsilon(x) - x}{h_\varepsilon}$$

defines a bounded sequence in $W^{1,2}(\Omega; \mathbb{R}^n)$. The map g_ε can be interpreted as a scaled displacement. Now the nonlinear strain can be rewritten as follows:

$$E_h = \frac{\sqrt{(Id + h_\varepsilon \nabla g_\varepsilon)^{\mathrm{T}}(Id + h_\varepsilon \nabla g_\varepsilon)} - Id}{h_\varepsilon}.$$

Hence, whenever (g_ε) converges weakly to a map g in $W^{1,2}(\Omega; \mathbb{R}^n)$ and

$$\nabla g_\varepsilon \xrightarrow{2} \nabla g(x) + \nabla_y g_0(x, y) \qquad \text{weakly two-scale in } L^2(\Omega{\times}Y; \mathbb{M}(n)),$$

then the previous corollary implies that

$$E_h \xrightarrow{2} \operatorname{sym} \nabla g(x) + \operatorname{sym} \nabla_y g_0(x, y) \qquad \text{weakly two-scale in } L^2(\Omega{\times}Y; \mathbb{M}(n)).$$

We see that — although the nonlinear strain E_ε is related to the gradient ∇g_ε in a nonlinear way, and therefore a priori a "nice" interplay with weak (two-scale) convergence cannot be expected — the application of the linearization methods developed in this section yield an explicit link between the two-scale limits of (E_ε) and (∇g_ε).

3. Integral functionals

In this chapter we recall some known lower semicontinuity results for integral functionals and present their generalization to oscillating integral functionals where lower bound inequalities can be stated by means of two-scale convergence. Eventually, in Section 3.3 we briefly discuss the homogenization of periodic, convex integral functionals with the aim to demonstrate the general homogenization scheme based on periodic unfolding by means of a simple, but instructive example.

Throughout this chapter Ω denotes an open, bounded subset of $\mathbb{R}^n$, U an open, bounded subset of $\mathbb{R}^m$ and $\mathbb{E}$ an d-dimensional Euclidean space. Furthermore, we denote by $\mathcal{L}(A)$ and $\mathcal{B}(A)$ the σ-algebra of Lebesgue- and Borel-measurable subsets of A, respectively. We use the notation $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_k$ to refer to the product σ-algebra generated by a finite set $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of σ-algebras.

3.1. Basic properties and lower semicontinuity

Let us consider integral functionals of the type

$$E \mapsto \iint_{\Omega\times U} f(x, y, E(x, y))\, \mathrm{d}x\, \mathrm{d}y \tag{3.1}$$

where f is a map from $\Omega\times\mathbb{R}^m\times\mathbb{E}$ to the extended reals $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and E a measurable function from $\Omega\times U$ to $\mathbb{E}$. A necessary prerequisite for the functional in (3.1) to be well-defined is the *sup-measurability* of f, which means that the superposition map

$$f_E : \Omega\times U \mapsto \bar{\mathbb{R}}, \qquad (x, y) \mapsto f(x, y, E(x, y))$$

is measurable for all measurable functions $E : \Omega\times U \to \mathbb{E}$. Another requirement for the expression in (3.1) is the integrability of the superposition map f_E.

For our purpose it is convenient to suppose that f satisfies the following conditions:

i. (Measurability). The map $f : \Omega\times\mathbb{R}^m\times\mathbb{E} \to \bar{\mathbb{R}}$ is measurable either with respect to $\mathcal{L}(\Omega)\otimes\mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{E})$ or with respect to $\mathcal{B}(\Omega) \otimes \mathcal{L}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{E})$.

ii. (Integrability). There exists a constant $c_0 \in \mathbb{R}$ such that

$$f(x, y, E) \geq c_0 \qquad \text{for all } E \in \mathbb{E} \text{ and a.e. } (x, y) \in \Omega\times\mathbb{R}^m.$$

Definition 3.1.1. We call a map $f : \Omega\times\mathbb{R}^m\times\mathbb{E} \to \bar{\mathbb{R}}$ that satisfies i. and ii. a *measurable integrand*.

Remark 3.1.2. If f is a measurable integrand, then the superposition map f_E is $\mathcal{L}(\Omega\times U)$-measurable for all Lebesgue-measurable maps $E : \Omega\times U \to \mathbb{E}$ (see e.g. [Vis07]) and the integral in (3.1) is well-defined (and possibly takes the value $+\infty$). Moreover, if

$$\pi : \Omega \to U \qquad \text{is a } \mathcal{L}(\Omega){-}\mathcal{L}(U)\text{-measurable map}$$

then also the superposition

$$\Omega \ni x \mapsto f(x, \pi(x), E(x))$$

is Lebesgue-measurable for all Lebesgue-measurable maps $E : \Omega \to \mathbb{E}$.

As a consequence, (in the case where $m = n$ and $Y := [0,1)^n$) the functionals

$$L^p(\Omega;\mathbb{E}) \ni u \mapsto \int_\Omega f(x, x/\varepsilon, u(x))\,\mathrm{d}x$$
$$L^p(\Omega\times Y;\mathbb{E}) \ni u \mapsto \iint_{\Omega\times Y} f(x, y, u(x,y))\,\mathrm{d}y\,\mathrm{d}x$$

are well-defined for $p \in [1,\infty]$ and $\varepsilon > 0$.

Remark 3.1.3. If a measurable integrand $f : \Omega\times\mathbb{R}^m\times\mathbb{E} \to \bar{\mathbb{R}}$ is additionally lower-semi-continuous in its third component, then f is a normal integrand (see e.g. [Dac08]). Moreover, any function of Carathéodory-type is included in our notion of measurable integrands. Nevertheless, Definition 3.1.1 renders not the most general class of integrands that can be considered in this context. For more details, we refer to [Vis07, BD98, App88].

Next, we introduce some properties of integrands which we will frequently encounter throughout this contribution.

Definition 3.1.4. Let $f : \Omega\times\mathbb{R}^m\times\mathbb{E} \to \mathbb{E}$ be a measurable integrand. We say

(a) f is *finite*, if $f(x,y,E) \in \mathbb{R}$ for all $(x,y,E) \in \Omega\times\mathbb{R}^m\times\mathbb{E}$.

(b) f is *convex*, if $f(x,y,\cdot)$ is convex for a.e. $(x,y) \in \Omega\times\mathbb{R}^m$.

(c) f is *continuous* (*lower semicontinuous*), if $(x,E) \mapsto f(x,y,E)$ is continuous (lower semicontinuous) for a.e. $y \in \mathbb{R}^m$ respectively.

(d) f is *Y-periodic* with $Y := [0,1)^m$, if

$$f(x, y+k, E) = f(x,y,E) \qquad \text{for all } E \in \mathbb{E},\ k \in \mathbb{Z}^m \text{ and a.e. } (x,y) \in \Omega\times\mathbb{R}^m.$$

(e) f satisfies the *p-summability condition*, if there exists a positive constant c_1 such that

$$|f(x,y,E)| \le c_1(1 + |E|^p) \qquad \text{for all } E \in \mathbb{E} \text{ and almost every } (x,y) \in \Omega\times\mathbb{R}^m.$$

In the following lemma we state some known continuity results for integral functionals (cf. e.g. [Dac08]).

Lemma 3.1.5. *Let $f : \Omega\times\mathbb{R}^m\times\mathbb{E} \to \bar{\mathbb{R}}$ be a measurable integrand, $p \in [1,\infty)$ and U an open and bounded subset of $\mathbb{R}^m$. Define*

$$\mathcal{G} : L^p(\Omega\times U;\mathbb{E}) \to \mathbb{R}\cup\{+\infty\}, \qquad \mathcal{G}(u) := \iint_{\Omega} f(x,y,u(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

Then the functional $\mathcal{G}$ is well defined and:

(1) If f is lower semicontinuous (i.e. f is a normal integrand), then $\mathcal{G}$ is lower semicontinuous w.r.t. to strong convergence in $L^p(\Omega\times U;\mathbb{E})$.

(2) If f is lower semicontinuous and convex, then $\mathcal{G}$ is convex and lower semicontinuous w.r.t. to weak convergence in $L^p(\Omega\times U;\mathbb{E})$.

(3) If f is continuous and satisfies the p-summability condition, then $\mathcal{G}$ is continuous w.r.t. to strong convergence in $L^p(\Omega\times U;\mathbb{E})$.

We briefly sketch the proof, which can be found in [Dac08, Vis07] for instance.

Proof. Without loss of generality assume $f \geq 0$. Let (u_k) be a sequence in $L^p(\Omega\times U;\mathbb{E})$. If $u_k \to u$ strongly, then we can extract a subsequence (not relabeled) with $u_k(x,y) \to u(x,y)$ for a.e. $(x,y) \in \Omega\times U$. The lower semicontinuity of f implies that

$$\liminf_{k\to\infty} f(x,y,u_k(x,y)) \geq f(x,y,u(x,y))$$

and by Fatou's Lemma we obtain (1).

Now assume that f is convex and (u_k) is weakly convergent to u. By Mazur's Theorem we can construct a sequence (w_k) that strongly converges to u in $L^p(\Omega\times U;\mathbb{E})$ such that w_k is a convex combination of the functions $\{\, u_i \,:\, i \leq k \,\}$. The convexity of f and the previous reasoning yield (2).

Statement (3) follows by applying (1) to f and $-f$. □

3.2. Periodic integral functionals and two-scale lower semicontinuity

Lemma 3.2.1. *Let $f : \Omega\times\mathbb{R}^n\times\mathbb{E} \to \bar{\mathbb{R}}$ be a measurable integrand, $p \in [1,\infty)$. Suppose that f is $[0,1)^n =: Y$-periodic in its second variable. For $\varepsilon > 0$ we define*

$$\begin{aligned} &\mathcal{G}^\varepsilon : L^p(\Omega;\mathbb{E}) \to \bar{\mathbb{R}}, && \mathcal{G}^\varepsilon(u) := \int_\Omega f(x, {}^x\!/\!_\varepsilon, u(x))\,\mathrm{d}x, \\ &\mathcal{G}^0 : L^p(\Omega\times Y;\mathbb{E}) \to \bar{\mathbb{R}}, && \mathcal{G}^0(u) := \iint_{\Omega\times Y} f(x,y,u(x,y))\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

Then the functionals $\mathcal{G}^\varepsilon$ and $\mathcal{G}^0$ are well-defined and:

(1) If f is lower semicontinuous (i.e. f is a normal integrand), then $\mathcal{G}^\varepsilon$ and $\mathcal{G}^0$ are lower semicontinuous w.r.t. to strong convergence.

(2) If f is lower semicontinuous and convex, then $\mathcal{G}^\varepsilon$ and $\mathcal{G}^0$ are convex and lower semicontinuous w.r.t. to weak convergence.

(3) If f is continuous and satisfies the p-summability condition, then $\mathcal{G}^\varepsilon$ and $\mathcal{G}^0$ are continuous w.r.t. to strong convergence.

The proof is similar to the one of Lemma 3.1.5 and omitted here. The following proposition entails a continuity and lower-semicontinuity result with respect to strong and weak two-scale convergence respectively.

Proposition 3.2.2 (see e.g. Proposition 1.3 in [Vis07])**.** *In the situation of Lemma 3.2.1 we have:*

(1) If f is lower semicontinuous, then

$$\liminf_{\varepsilon\to 0} \mathcal{G}^\varepsilon(u_\varepsilon) \geq \mathcal{G}^0(u),$$
$$\textit{provided}\qquad u_\varepsilon \xrightarrow{2} u \qquad \textit{strongly two-scale in } L^p(\Omega{\times}Y;\mathbb{E}).$$

(2) If f is continuous and satisfies the p-summability condition, then

$$\lim_{\varepsilon\to 0} \mathcal{G}^\varepsilon(u_\varepsilon) = \mathcal{G}^0(u),$$
$$\textit{provided}\qquad u_\varepsilon \xrightarrow{2} u \qquad \textit{strongly two-scale in } L^p(\Omega{\times}Y;\mathbb{E}).$$

(3) If f is lower semicontinuous and convex, then

$$\liminf_{\varepsilon\to 0} \mathcal{G}^\varepsilon(u_\varepsilon) \geq \mathcal{G}^0(u),$$
$$\textit{provided}\qquad u_\varepsilon \xrightharpoonup{2} u \qquad \textit{weakly two-scale in } L^p(\Omega{\times}Y;\mathbb{E}).$$

Proof. For convenience we extend f to $\mathbb{R}^n{\times}\mathbb{R}^n{\times}\mathbb{E}$ by zero. Moreover, without loss of generality we assume that $f \geq 0$. It is easy to check that for each map $u_\varepsilon \in L^p(\Omega;\mathbb{E})$ and $\varepsilon > 0$ we have

$$\mathcal{G}^\varepsilon(u) = \iint\limits_{\mathbb{R}^n\times Y} f(\varepsilon\lfloor x/\varepsilon\rfloor + \varepsilon y, y, \mathcal{T}_\varepsilon u_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

If (u_ε) strongly converges to u, then

$$(\varepsilon\lfloor x/\varepsilon\rfloor + \varepsilon y, \mathcal{T}_\varepsilon u_\varepsilon(x,y)) \to (x, u(x,y)) \qquad \text{as } \varepsilon\to 0$$

almost everywhere for a subsequence (not relabeled). Thus, if the integrand f is lower-semicontinuous, we obtain

$$\liminf_{\varepsilon\to 0} f(\varepsilon\lfloor x/\varepsilon\rfloor + \varepsilon y, y, \mathcal{T}_\varepsilon u_\varepsilon(x,y)) \geq f(x,y,u(x,y)).$$

Now (1) follows by Fatou's lemma (first for the subsequence and then for the entire sequence, because the choice of the subsequence was sufficiently arbitrary).

Statement (2) and (3) can be proved by a reasoning similar to the one used in the proof of Lemma 3.1.5. □

3.3. Convex homogenization

In this section we present a short proof of the classical homogenization problem of convex, periodic integral functionals based on the two-scale (lower-semi-) continuity result depicted in Proposition 3.2.2. This problem is already well understood and we refer in this context to [All92] where the problem was treated with two-scale methods for the first time. Our aim is to illustrate the general strategy of the two-scale method for homogenization problems in a simple setting.

We suppose that Ω is an open and bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $Y := [0,1)^n$ and $f : \Omega\times\mathbb{R}^n\times\mathbb{R}^n \to \bar{\mathbb{R}}$ a measurable integrand. We furthermore suppose that

(a) f is Y-periodic, continuous and convex in the sense of Definition 3.1.4, and

(b) f satisfies the standard growth- and coercivity condition of order p

$$\frac{1}{c}\,|F|^p - c \le f(x,y,F) \le c(1+|F|^p)$$

for almost every x, y and a positive constant c.

For $\varepsilon > 0$ we define the functionals

$$\mathcal{G}^\varepsilon : W^{1,p}(\Omega) \to \mathbb{R} \qquad \mathcal{G}^\varepsilon(u) := \int\limits_\Omega f(x, {}^x\!/\!_\varepsilon, \nabla u(x))\,\mathrm{d}x$$

and

$$\mathcal{G}^0 : W^{1,p}(\Omega)\times L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y)) \to \mathbb{R}$$
$$\mathcal{G}^0(u,u_0) := \iint\limits_{\Omega\times Y} f(x,y,\nabla u(x) + \nabla_y u_0(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

Theorem 3.3.1. *Let (ε) denote an arbitrary vanishing sequence of positive numbers and let $p \in (1,\infty)$.*

(1) Let (u_ε) be an arbitrary sequence in $W^{1,p}(\Omega)$ such that

$$\limsup_{\varepsilon\to 0}\left\{\left|\int_\Omega u_\varepsilon\,\mathrm{d}x\right| + \mathcal{G}^\varepsilon(u_\varepsilon)\right\} < \infty. \tag{3.3}$$

Then there exist a subsequence (not relabeled) and a pair

$$(u,u_0) \in W^{1,p}(\Omega)\times L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y))$$

such that

$$(\star) \qquad \begin{cases} u_\varepsilon \rightharpoonup u & \textit{weakly in } W^{1,p}(\Omega) \\ \nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \nabla_y u_0 & \textit{weakly two-scale in } L^p(\Omega\times Y;\mathbb{R}^n). \end{cases}$$

(2) Suppose that $(u_\varepsilon) \subset W^{1,p}(\Omega)$ *converges to a pair*

$$(u, u_0) \in W^{1,p}(\Omega)\times L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y))$$

in the sense of $(\star)$*. Then*

$$\liminf_{\varepsilon\to 0} \mathcal{G}^\varepsilon(u_\varepsilon) \geq \mathcal{G}^0(u, u_0).$$

(3) For any pair $(u, u_0) \in W^{1,p}(\Omega)\times L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y))$ *there exists a sequence* (u_ε) *in* $W^{1,p}(\Omega)$ *converging to* (u, u_0) *in the sense of* $(\star)$ *such that*

$$\lim_{\varepsilon\to 0} \mathcal{G}^\varepsilon(u_\varepsilon) = \mathcal{G}^0(u, u_0).$$

Proof. Because f has standard p-growth, condition (3.3) implies that for a subsequence (not relabeled) (∇u_ε) as well as $(\int_\Omega u_\varepsilon \,\mathrm{d}x)$ are bounded sequences in $L^p(\Omega;\mathbb{R}^n)$ and $\mathbb{R}$, respectively. In virtue of Poincaré's inequality we see that (u_ε) is bounded in $W^{1,p}(\Omega)$, and therefore weakly convergent in $W^{1,p}(\Omega)$ up to a subsequence. Now $(\star)$ follows due to Proposition 2.1.14 for a further subsequence and (1) is proved.

Statement (2) directly follows by applying the two-scale lower semicontinuity result (see Proposition 3.2.2) in connection with (1).

It remains to prove (3). It is well known that the inclusions

$$C^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega), \qquad C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Y)) \subset L^p(\Omega; W^{1,2}_{\mathrm{per}}(Y))$$

are dense with respect to the strong topology. Hence, for each $\delta > 0$ we can find maps $u_\delta \in C^\infty(\overline{\Omega})$, $v_\delta \in C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Y))$ such that

$$\|u_\delta - u\|_{W^{1,p}(\Omega)} + \|v_\delta - u_0\|_{L^p(\Omega;W^{1,2}_{\mathrm{per}}(Y))} \leq \delta.$$

For each ε define

$$u_{\delta,\varepsilon}(x) := u_\delta(x) + \varepsilon v_\delta(x, x/\varepsilon).$$

Then

$$\nabla u_{\delta,\varepsilon}(x) = \nabla u_\delta(x) + (\nabla_y v_\delta)(x, x/\varepsilon) + \varepsilon(\nabla v_\delta)(x, x/\varepsilon)$$

and it is easy to check (see Lemma 2.1.9) that

$$\begin{aligned} &u_{\delta,\varepsilon} \to u_\delta && \text{strongly in } L^p(\Omega) \\ &\nabla u_{\delta,\varepsilon} \overset{2}{\longrightarrow} \nabla u_\delta(x) + \nabla_y v_\delta(x,y) && \text{strongly two-scale in } L^p(\Omega\times Y;\mathbb{R}^n) \end{aligned}$$

as $\varepsilon \to 0$. Define

$$c_{\delta,\varepsilon} := \|u_{\delta,\varepsilon} - u\|_{W^{1,p}(\Omega)} + \|\mathcal{T}_\varepsilon \nabla u_{\delta,\varepsilon} - (\nabla u + \nabla_y u_0)\|_{L^p(\mathbb{R}^n \times Y;\mathbb{R}^n)}.$$

The previous reasoning shows that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} c_{\delta,\varepsilon} = 0.$$

This allows us to apply a diagonalization argument that is due to H. Attouch (see Lemma A.2.1); thus, there exists a diagonal sequence $\delta(\varepsilon)$ such that $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$ and $c_{\delta(\varepsilon),\varepsilon} \to 0$ as $\varepsilon \to 0$. Now define

$$u_\varepsilon := u_{\delta(\varepsilon),\varepsilon}.$$

By construction (u_ε) is a sequence in $W^{1,p}(\Omega)$ and fulfills

$$\begin{aligned} u_\varepsilon &\to u && \text{strongly in } L^p(\Omega)\\ \nabla u_\varepsilon &\xrightarrow{2} \nabla u(x) + \nabla_y u_0(x,y) && \text{strongly two-scale in } L^p(\Omega\times Y;\mathbb{R}^n). \end{aligned}$$

Because the latter implies that $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^p(\Omega;\mathbb{R}^n)$ (cf. Lemma 2.1.11), we infer that (u_ε) weakly converges to u in $W^{1,p}(\Omega)$.

By assumption the integrand f is continuous and satisfies the p-summability condition in the sense of Definition 3.1.4. Hence, we can apply Proposition 3.2.2 (2) and deduce that

$$\lim_{\varepsilon\to 0}\mathcal{G}^\varepsilon(u_\varepsilon) = \mathcal{G}^0(u,u_0).$$

□

Corollary 3.3.2. *Set*

$$\mathcal{G}_{\hom} : W^{1,p}(\Omega) \to \mathbb{R}, \qquad \mathcal{G}_{\hom}(u) := \inf\left\{ \mathcal{G}^0(u,u_0) \,:\, u_0 \in L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y)) \right\}.$$

(1) Suppose that (u_ε) is a weakly converging sequence in $W^{1,p}(\Omega)$ with limit u. Then

$$\liminf_{\varepsilon\to 0}\mathcal{G}^\varepsilon(u_\varepsilon) \geq \mathcal{G}_{\hom}(u).$$

(2) Let $u \in W^{1,p}(\Omega$. Then there exists a sequence (u_ε) in $W^{1,p}(\Omega)$ weakly converging to u such that

$$\lim_{\varepsilon\to 0}\mathcal{G}^\varepsilon(u_\varepsilon) = \mathcal{G}_{\hom}(u).$$

Proof. This is an immediate consequence of the previous theorem and the fact that the problem

$$\text{minimize} \qquad u_0 \mapsto \mathcal{G}^0(u,u_0) \qquad \text{subject to } u_0 \in L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y))$$

attains its minimum. □

Lemma 3.3.3. *Set*

$$f_{\hom}: \Omega{\times}\mathbb{R}^n \to \mathbb{R}, \qquad f_{\hom}(x,F) := \inf_{\varphi\in W^{1,p}_{\mathrm{per}}(Y)} \int_Y f(x,y,F+\nabla_y\,\varphi(y))\,\mathrm{d}y.$$

Then for each $u \in W^{1,p}(\Omega)$ we have

$$\mathcal{G}_{\hom}(u) = \int_\Omega f_{\hom}(x,\nabla u(x))\,\mathrm{d}x.$$

The proof of this result (which we omit here) is non-trivial and we refer to [Dam05, CDDA06, BF07]. A possible strategy is to show that the multifunction

$$\Phi: \Omega{\times}\mathbb{R}^n \to P^{W^{1,p}_{\mathrm{per},0}(Y)}, \qquad \Phi(x,F) \in \operatorname*{argmin}_{\varphi\in W^{1,p}_{\mathrm{per},0}(Y)} \int_Y f(x,y,F+\nabla_y\,\varphi(y))\,\mathrm{d}y$$

admits a measurable selection. We like to remark that in the case when $f(x,y,F)$ is quadratic in F, this difficulty can be avoided by studying the Euler-Lagrange equations of the minimization problem in the definition of $f_{\hom}$.

Discussion of the general strategy. In the language of Γ-convergence (see Section 4.2) the previous results prove that the sequence $(\mathcal{G}^\varepsilon)$ Γ-converges to $\mathcal{G}_{\hom}$. To make this precise we extend $\mathcal{G}^\varepsilon$ and $\mathcal{G}_{\hom}$ to $L^p(\Omega)$ by setting

$$\mathcal{G}^\varepsilon(u) := +\infty \quad \text{and} \quad \mathcal{G}_{\hom}(u) := +\infty \qquad \text{for all } u \in L^p(\Omega)\setminus W^{1,p}(\Omega).$$

Then the previous results imply that

$$\mathcal{G}^\varepsilon \xrightarrow{\Gamma} \mathcal{G}_{\hom} \qquad \text{in } L^p(\Omega)$$

with respect to strong convergence in $L^p(\Omega)$. Moreover, Theorem 3.3.1 (1) yields equi-coercivity of the sequence $(\mathcal{G}^\varepsilon)$ in $L^p(\Omega)$. (2) and (3) in combination with Corollary 3.3.2 and Lemma 2.3.2 prove that the lower bound and recovery sequence condition of the sequential characterization of Γ-convergence are satisfied.

Although being an elementary example, it is instructive — in particular for the analysis of the more involved problems that we address in this thesis — to carve out the steps that led to the previous Γ-convergence result:

(1) **Compactness**. We prove that sequences with an equibounded energy are relatively compact, i.e. we can extract subsequences that converge in a certain two-scale sense. In the example above, convergence meant strong convergence in L^p and weak two-scale convergence of the gradient.

(2) **Lower bound** (also called liminf-inequality). We study the convergence behavior of the energy along sequences that converge in the sense of (1). In particular, in a first step, we establish an *intermediate liminf-inequality* where the limit inferior of the energy is bounded from below by an intermediate functional that

additionally depends on the two-scale limiting behavior of the sequence. In the example above, this step is contained in Theorem 3.3.1 (2), where we proved that

$$\liminf_{\varepsilon\to 0} \mathcal{G}^{\varepsilon}(u_{\varepsilon}) \geq \mathcal{G}^{0}(u, u_0)$$

where u is the strong limit of (u_ε) in $L^p(\Omega)$ while u_0 captures the oscillation properties of the sequence (∇u_ε) in the two-scale sense. The seeked Γ-limit should only depend on the "one-scale limit" u and not on the two-scale auxiliary map u_0, which may depend on the choice of the subsequence. For this reason we *identify* in a second step the *two-scale behavior* of sequences with equibounded energy, in the sense that we characterize the auxiliary maps that are associated to cluster points of such sequences. In the example above, this meant to characterize the structure of u_0 and has been done in Theorem 3.3.1 (1) where we showed that

$$u_0 \in L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y)).$$

Based on this **identification** we obtain the sought-after liminf-inequality by *relaxing the intermediate functional* with respect to all admissible auxiliary maps. In the previous example, this has been done in Corollary 3.3.2 (1) and led to the definition

$$\mathcal{G}_{\mathrm{hom}}(u) := \inf \left\{ \mathcal{G}^{0}(u, u_0) \,:\, u_0 \in L^p(\Omega; W^{1,p}_{\mathrm{per},0}(Y)) \right\}.$$

(3) **Recovery sequence** (also called upper bound). In this step we show that the energy of the functional derived in step (2) is optimal, in the sense that there exists a convergent sequence such that the associated energies converge to the energy of the limit. Such a sequence is called recovery sequence. As in the lower bound step, we construct this sequence in two stages. In the example above, these stages constitute as follows: First, we consider the intermediate functional $\mathcal{G}_0$ and construct a recovery sequence for any pair (u, u_0). In the second stage we analyze the minimization problem associated to the intermediate functional and prove that

$$\mathcal{G}_{\mathrm{hom}}(u) = \mathcal{G}^{0}(u, u_0^{\star})$$

for a suitable auxiliary map $u_0^{\star}$. As a consequence the recovery sequence can be constructed by applying the previous construction to the pair $(u, u_0^{\star})$.

(4) **Analysis of the limiting functional.** So far (1) - (3) yield Γ-convergence to a limiting functional that is defined implicitly, namely by the relaxation construction in the last part of (2). Usually, by analyzing the limiting functional one can find a simplified and streamlined presentation of the Γ-limit. In the example above, this step was established by means of Lemma 3.3.3 and revealed that the Γ-limit $\mathcal{G}_{\mathrm{hom}}$ is a integral functional with an homogenized integrand f_{hom}. In homogenization the minimization problem

$$\inf_{\varphi \in W^{1,p}_{\mathrm{per}}(Y)} \int_Y f(x, y, F + \nabla_y \varphi(y))\,\mathrm{d}y.$$

appearing in the definition of the homogenized integrand is called *cell-problem*.

Remark 3.3.4. The functional $\mathcal{G}^0$ already appeared in the seminal work by G. Allaire in [All92]. A. Mielke and A. Timofte call $\mathcal{G}^0$ the two-scale Γ-limit of the sequence $(\mathcal{G}^\varepsilon)$ (see [MT07]).

4. Gamma-convergence and the direct method of the calculus of variations

In this section we recall the basic notion of Γ-convergence introduced by De Giorgi in the 1970's. Γ-convergence is a variational notion of convergence that is naturally suited to study the asymptotic behavior of families of minimization problems that are parametrized by a small parameter.

As a classical example, such a situation emerges in the homogenization of variational problems where the physically interesting states are minimizers of an energy functional. There the small parameter, say ε, has the meaning of a length scale that describes the typical size of the microstructure of an oscillating material. A natural believe is that one can use the smallness of ε to derive a reduced model that still captures the behavior of the situation in a sufficiently precise manner — at least from a macroscopic perspective. The notion of Γ-convergence establishes a rigorous mathematical language that allows to implement this idea by studying the convergence behavior of the family of energy functionals as ε tends to zero. This procedure leads to a reduced functional, called the Γ-limit, that roughly speaking captures the behavior of the situation for all small, but finite ε optimally. Furthermore, the notion of Γ-convergence is tailor-made to guarantee (under suitable compactness conditions) the convergence of minimizers and minima — and thus, "convergence of the physically interesting information".

Γ-convergence is closely related to Tonelli's direct method of the calculus of variations, which is a classical way to prove the existence of minimizers for variational problems. In the next section we describe this method and recall the notions of lower semicontinuity and coercivity, which are the main ingredients of the direct method. In Section 4.2 we briefly recall the notion of Γ-convergence and gather basic properties.

Our main aim is to fix the notation and to simplify the referencing in the subsequent chapters of this contribution. For an extensive introduction and for the proofs of the results in this chapter, we refer to the monographs of G. Dal Maso [DM93] and I. Fonseca and G. Leoni [FL07].

4.1. The direct method of the calculus of variations

In the following, we always suppose that X is a topological space.

Definition 4.1.1 (compactness)**.** We say that a subset $K \subset X$ is

(a) *sequentially compact*, if every sequence in K has a subsequence which converges to a point in K,

(b) *relatively compact*, if the closure of K is compact,

(c) *relatively sequentially compact*, if the closure of K is sequentially compact.

Definition 4.1.2 (lower semicontinuity)**.** We say that a function $F : X \to \bar{\mathbb{R}}$ is

(a) *lower semicontinuous*, if for all $\alpha \in \mathbb{R}$ the set $\{ x \in X : F(x) < \alpha \}$ is open,

(b) *sequentially lower semicontinuous*, if

$$F(x) \leq \liminf_{k \to \infty} F(x_k)$$

for all $x \in X$ and all sequences (x_k) that converge to x in X.

Definition 4.1.3 (coercivity)**.** We say that a function $F : X \to \bar{\mathbb{R}}$ is *coercive* (resp. *sequentially coercive*), if for all $\alpha \in \mathbb{R}$ the set $\{ x \in X : F(x) \leq \alpha \}$ is compact (resp. sequentially compact).

Remark 4.1.4. It is well known that in general the sequential notions are weaker than the topological ones: In particular, lower semicontinuity implies sequential lower semicontinuity, compactness implies sequential compactness and coercivity implies sequential coercivity. It is important to note that the **converse is true when X is a metric space** (or more generally: a topological space that satisfies the first axiom of countability).

Theorem 4.1.5 (The direct method of the calculus of variations)**.** *Let $F : X \to \bar{\mathbb{R}}$ be sequentially coercive and sequentially lower semicontinuous. Then F attains its minimum in X. Moreover, if F is not identically $+\infty$, then every minimizing sequence of has a convergent subsequence.*

Proof. Without loss of generality we assume that F is not identically $+\infty$. In this case we have $\inf_X F < \infty$. Let $(x_n)_{n \in \mathbb{N}}$ be a minimizing sequence, i.e.

$$\lim_{n \to \infty} F(x_n) = \inf_X F.$$

Because F is sequentially coercive, the sequence (or at least a suitable tail of the sequence) lies in a sequentially compact set of X and we can pass to a subsequence (x_{n_k}) that converges to some x in X. In virtue of the sequential lower semicontinuity of f we see that

$$\inf_X F \leq F(x) \leq \liminf_{k \to \infty} F(x_{n_k}) = \lim_{n \to \infty} F(x_n) = \inf_X F.$$

□

The property of a point x to be a minimizer of a function $F : X \to \bar{\mathbb{R}}$ is completely independent of the topological structure of X. In contrast to this, the necessary conditions for applying the direct method in order to prove existence of a minimizer are of topological nature. Thus, the direct method permits a certain freedom of choosing the topology of X. In this context it is important to note that the conditions of F being coercive and lower semicontinuous, respectively, are antagonistic: The weaker the topology, the easier it is for a function F to be coercive, but the harder it is for F to be lower semicontinuous.

Integral functionals. In this thesis we frequently encounter situations where F is a functional of the type

$$F : \mathcal{A} \to \bar{\mathbb{R}}, \qquad F(u) := \int_\Omega g(x, \nabla u(x))\,\mathrm{d}x$$

where $\mathcal{A}$ is a convex and compact subset of $W^{1,p}(\Omega;\mathbb{R}^n)$, $p \in (1,\infty)$ and $g : \Omega \times \mathbb{M}(n) \to \mathbb{R} \cup \{+\infty\}$ is a measurable integrand.

In view of Lemma 3.1.5, we already know necessary conditions for F being sequentially lower semicontinuous:

- If $g(x,\cdot)$ is lower semicontinuous, then F is lower semicontinuous with respect to strong convergence in $W^{1,p}(\Omega;\mathbb{R}^n)$.
- If $g(x,\cdot)$ is lower semicontinuous and convex, then F is lower semicontinuous with respect to weak convergence in $W^{1,p}(\Omega;\mathbb{R}^n)$.

On the other hand, if g satisfies certain growth conditions, we can ensure that F is coercive: For instance, assume that g satisfies

$$c_0\,|A|^p - c_1 \le g(x,A) \qquad \text{for all } A \in \mathbb{M}(n) \text{ and almost every } x \in \Omega \tag{4.1}$$

with some positive constants c_0, c_1. Then for each $\alpha \in \mathbb{R}$ the sublevel sets $U_\alpha := \{\, u \in \mathcal{A} \,:\, F(u) \le \alpha \,\}$ satisfy

$$\|\nabla u\|^p_{L^p(\Omega;\mathbb{M}(n))} \le \frac{\alpha + c_1}{c_0} \qquad \text{for all } u \in U_\alpha.$$

Now suppose that $\mathcal{A}$ has a certain structure, in the sense that

($\star$) the norm in $W^{1,p}(\Omega;\mathbb{R}^n)$ and the semi-norm $u \mapsto \|\nabla u\|_{L^p(\Omega;\mathbb{M}(n))}$ induce the same topology on $\mathcal{A}$.

Then the set $U_\alpha \subset W^{1,p}(\Omega;\mathbb{R}^n)$ is bounded with respect to the norm of $W^{1,p}(\Omega;\mathbb{R}^n)$ and since $W^{1,p}(\Omega;\mathbb{R}^n)$ is reflexive, we deduce that U_α is sequentially compact with respect to the weak topology. Moreover, since $\mathcal{A}$ is convex and compact, it is also weakly compact and we see that U_α is a sequentially compact subset of $\mathcal{A}$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$. Consequently, F is coercive with respect to the weak topology whenever ($\star$) is satisfied.

For this reason, we can apply the direct method and obtain the following existence result:

Corollary 4.1.6. *Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $\mathcal{A}$ be a convex and compact subset of $W^{1,p}(\Omega;\mathbb{R}^n)$, $p \in (1,\infty)$ that satisfies $(\star)$ and let $g : \Omega\times\mathbb{M}(n) \to \bar{\mathbb{R}}$ be a measurable, convex and lower semicontinuous integrand (in the sense of Definition 3.1.4) that satisfies the p-growth condition (4.1). Consider the functional*

$$F : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}, \qquad F(u) := \int_\Omega g(x, \nabla u(x))\,\mathrm{d}x.$$

Then F has a minimizer in $\mathcal{A}$ and every minimizing sequence admits a subsequence that converges to a minimizer of F in $\mathcal{A}$ with respect to weak convergence in $W^{1,p}(\Omega;\mathbb{R}^n)$.

If $\mathcal{A}$ is additionally contained in one of the sets

$$\Big\{ u \in L^2(\Omega;\mathbb{R}^n) \,:\, \int_\Omega u\,\mathrm{d}x = m \Big\} \qquad \text{with } m \in \mathbb{R}$$

$$\Big\{ u \in L^2(\Omega;\mathbb{R}^n) \,:\, \|u\|_{L^2(\Omega;\mathbb{R}^n)} \le c \Big\} \qquad \text{with } c > 0$$

$$\Big\{ u \in W^{1,p}(\Omega;\mathbb{R}^n) \,:\, (u-g) \in W^{1,p}_{\Gamma,0}(\Omega;\mathbb{R}^n) \Big\} \qquad \text{with } g \in W^{1,p}(\Omega;\mathbb{R}^n) \text{ and } \Gamma \subset \partial\Omega \text{ with positive measure,}$$

then $\mathcal{A}$ satisfies $(\star)$. This can be easily shown by means of the Poincaré and Poincaré-Friedrichs inequality, respectively. In elasticity (in particular in linear elasticity) the integrand g does not satisfy the standard p-growth condition, but rather a growth condition of Korn-type, i.e.

$$c_0\,|\mathrm{sym}\,A|^p - c_1 \le g(x,A) \qquad \text{for all } A \in \mathbb{M}(n) \text{ and almost every } x \in \Omega.$$

In this case the sequential coercivity of the associated integral functional can be shown for a wide class of domains $\mathcal{A}$ in a similar way by using Korn's inequality. Moreover, we like to remark that for non-convex integrands the notion of quasi-convexity is (under suitable growth-conditions) a sufficient as well as necessary condition for an integral functional to be weakly sequentially lower semicontinuous.

4.2. Gamma-convergence

Let X be a topological space. We denote the set of all open neighborhoods of x in X by $\mathcal{N}(x)$ and consider a sequence of functions (F_h) from X to the extended reals $\bar{\mathbb{R}}$, where (h) denotes a vanishing sequence of positive numbers.

Definition 4.2.1 (Γ-convergence (see DalMaso [DM93]))**.** The *lower* Γ*-limit* and the *upper* Γ*-limit* of the sequence (F_h) are the functions from X to $\bar{\mathbb{R}}$ defined by

$$(\Gamma\text{-}\liminf_{h\to 0} F_h)(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{h\to 0} \inf_{y\in U} F_h(y)$$

$$(\Gamma\text{-}\limsup_{h\to 0} F_h)(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{h\to 0} \inf_{y\in U} F_h(y)$$

If there exists a function F from X to $\bar{\mathbb{R}}$ such that

$$\Gamma\text{-}\liminf_{h\to 0} F_h = \Gamma\text{-}\limsup_{h\to 0} F_h = F,$$

then we write $F = \Gamma\text{-}\lim_{h\to 0} F_h$ or $F_h \xrightarrow{\Gamma} F$ and we say that the sequence (F_h) Γ-*converges* to F (in X) and call F the Γ-*limit* of (F_h) (in X).

Lemma 4.2.2. *The lower (upper)* Γ*-limit of a sequence of functions from* X *to* $\bar{\mathbb{R}}$ *is lower semicontinuous.*

Definition 4.2.3 (equi-coercivity)**.** We say (F_h) is equi-coercive (on X), if there exists a lower semicontinuous coercive function $\Psi : X \to \bar{\mathbb{R}}$ such that $F_h \geq \Psi$ on X for each h.

Proposition 4.2.4 (Convergence of minima and minimizers)**.** *Let* (F_h) *be equi-coercive sequence of functions from* X *to* $\bar{\mathbb{R}}$ *and suppose that* (F_h) Γ*-converges to* F *in* X*. Then*

(1) F *is coercive and*

$$\min_{x\in X} F(x) = \lim_{h\to 0} \inf_{x\in X} F_h(x).$$

(2) Suppose that $\min_{x\in X} F(x) \in \mathbb{R}$*. Let* (x_h) *be a sequence of almost minimizers, i.e.*

$$\limsup_{h\to 0} \left(F_h(x_h) - \inf_{x\in X} F_h(x) \right) = 0,$$

then (x_h) *is sequentially compact and any cluster point of the sequence is a minimizer of* F*.*

Proposition 4.2.5 (Stability with respect to continuous perturbations)**.** *Let* (F_h) *be a sequence of functions from* X *to* $\bar{\mathbb{R}}$ *and suppose that* (F_h) Γ*-converges to* F *in* X*. If* $G : X \to \mathbb{R}$ *is a continuous function, then* $(F_h + G)$ Γ*-converges to* $F + G$ *in* X*.*

Γ-convergence in metric spaces. The definition of Γ-convergence in a general topological space is quite cumbersome. Nevertheless, in the case where X is a metric space, Γ-convergence can be characterized sequentially. Moreover, we are going to see that in most cases we can stick to this sequential characterization.

Definition 4.2.6. The *sequential lower* Γ*-limit* and the *sequential upper* Γ*-limit* of the sequence (F_h) are the functions from X to $\bar{\mathbb{R}}$ defined by

$$\text{seq}\text{-}\Gamma\text{-}\liminf_{h\to 0} F_h(x) = \inf \Big\{ \liminf_{h\to 0} F_h(x_h) \ : \ (x_h) \subset X,\, x_h \to x \Big\},$$
$$\text{seq}\text{-}\Gamma\text{-}\limsup_{h\to 0} F_h(x) = \inf \Big\{ \limsup_{h\to 0} F_h(x_h) \ : \ (x_h) \subset X,\, x_h \to x \Big\}.$$

If there exists a function F from X to $\bar{\mathbb{R}}$ such that

(a) for every $x \in X$ and for every sequence (x_h) converging to x in X we have

$$F(x) \leq \liminf_{h \to 0} F_h(x_h)$$

(b) for every $x \in X$ there exists a sequence (x_h) converging to x in X such that

$$F(x) = \lim_{h \to 0} F_h(x_h),$$

then we say that the sequence (F_h) *sequentially* Γ*-converges* to F (in X) and set $\operatorname{seq-}\Gamma\text{-}\lim_{h \to 0} F_h := F$.

Proposition 4.2.7 ([DM93])**.** *Assume that (X, d) is a metric space (or more general: a first countable topological space). Then the sequential Γ-lower (upper) limit is equal to the Γ-lower (upper) limit. In particular $F_h \xrightarrow{\Gamma} F$ in X, if and only if (F_h) sequentially Γ-converges to F.*

In most applications in this thesis we study functionals defined on a subspace of $X := W^{1,p}(\Omega; \mathbb{R}^n)$ endowed with the weak topology. For $p \in (1, \infty)$ the space X is a reflexive Banach space, and therefore the topology of a norm bounded subsets of X is metrizable. As a consequence of this observation we obtain the following:

Proposition 4.2.8 (Γ-convergence w.r.t. weak convergence)**.** *Assume that X is a reflexive Banach space endowed with its weak topology and that the sequence (F_h) is equi-coercive in the weak topology of X. Then the lower Γ-limit and the sequential lower Γ-limit are equal and the sequence Γ-converges if and only if it sequentially Γ-converges.*

Part II.

Variational multiscale methods for integral functionals

5. Linearization and homogenization commute in finite elasticity

5.1. Introduction and main result

In this chapter we consider integral functionals of the type

$$u \mapsto \int_\Omega W(x/\varepsilon, \nabla u(x))\,\mathrm{d}x, \qquad u \in W^{1,p}(\Omega;\mathbb{R}^n) \tag{5.1}$$

where ε is a small positive scale parameter, Ω an open, bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ and $W : \mathbb{R}^n \times \mathbb{M}(n) \to \mathbb{R} \cup \{+\infty\}$ is a measurable integrand, $Y := [0,1)^n$-periodic in its first variable.

Functionals of this type model various situations in physics and engineering. We are particularly interested in applications to elasticity. In this context, the integral in (5.1) is the elastic energy of a periodic composite material with period ε that is deformed by the map $u : \Omega \to \mathbb{R}^n$. The matrix

$$\nabla u(x) := \left(\partial_1 u(x) \,\middle|\, \cdots \,\middle|\, \partial_n u(x) \right) \in \mathbb{M}(n)$$

is called deformation gradient. We are interested in situations where ε is small, which means that the scale of the composite's microstructure and the macroscopic dimension of the body are clearly separated.

In situations where the deformation is close to a rigid deformation, say $|\nabla u - Id| \sim h$, it is convenient to consider the energy

$$\mathcal{I}^{h,\varepsilon}(g) := \frac{1}{h^2}\int_\Omega W(x/\varepsilon,\, Id + h\,\nabla g(x))\,\mathrm{d}x, \qquad g \in W^{1,p}(\Omega;\mathbb{R}^n) \tag{5.2}$$

which is a scaled, but equivalent formulation of (5.1) by means of the (scaled) displacement

$$g(x) := \frac{u(x) - x}{h}.$$

For small parameters ε and h it is natural to hope that we can replace the functional (5.2) by an *effective* model, which is simpler than the initial one, but nevertheless catches the essential behavior of the original model from a macroscopic perspective. In this context, the limit $h \to 0$ corresponds to *linearization*, while the theory of *homogenization* renders a rigorous way from (5.1) to a simplified model by analyzing the asymptotic behavior of (5.1) as $\varepsilon \to 0$.

It is well known that in the case where W is convex and of polynomial growth with respect to its second component, the *homogenization* of (5.1) is the integral functional $\int_\Omega W^{(1)}_{\text{hom}}(\nabla u(x))\,\mathrm{d}x$, where the homogenized integrand is given by the *one-cell homogenization formula*

$$W^{(1)}_{\text{hom}}(F) := \inf\left\{ \int_Y W(y, F + \nabla\varphi(y))\,\mathrm{d}y \;:\; \varphi \in W^{1,p}_{\text{per}}(Y;\mathbb{R}^n) \right\}.$$

This result goes back to P. Marcellini [Mar78] and was extensively studied with various methods (cf. e.g. [Tar77], [Tar09], [DM93], [All92]).

In contrast, for non-convex potentials typically instabilities may arise in the homogenization procedure — even when the one-cell homogenization formula predicts no loss of rank-one convexity (see [AT84, TM85, GMT93]). For this reason, it turns out that the relaxation of W over one periodicity cell Y is not sufficient for homogenization in the non-convex setting. Nevertheless, A. Braides [Bra85] and S. Müller [Mül87] showed in the 1980s that in the case where W satisfies a growth-, coercivity- and Lipschitz condition of order p, i.e.

$$\begin{cases} \frac{1}{C}\,|F|^p - C \le W(y,F) \le C(1+|F|^p) \quad \text{and} \\ |W(y,F) - W(y,G)| \le C(1+|F|^{p-1}+|G|^{p-1})\,|F-G| \end{cases} \tag{5.3}$$

for a positive constant C, the functional (5.1) can be homogenized in the sense of Γ-convergence and the Γ-limit is an integral functional of type (5.1) with an homogenized integrand given by the *multi-cell homogenization formula*

$$W^{(\text{mc})}_{\text{hom}}(F) := \inf_{k\in\mathbb{N}} \inf\left\{ \frac{1}{k^n}\int_{kY} W(y, F+\nabla\varphi(y))\,\mathrm{d}y \;:\; \varphi \in W^{1,p}_{\text{per}}(kY;\mathbb{R}^n) \right\}.$$

With regard to linearization, G. Dal Maso, M. Negri and D. Percivale treated in [Per99] the limit $h \to 0$ of the functional (5.2) (for fixed ε and subject to Dirichlet boundary data) and derived linear elasticity as a Γ-limit of finite elasticity. They assumed, as it is common in elasticity, that the reference configuration is a natural state, i.e.

$$W(y, Id) = 0 \quad \text{and} \quad W(y,F) \ge 0 \tag{W2}$$

and considered frame indifferent stored energy functions that are of class C^2 in a neighborhood of $SO(n)$ and that satisfy the non-degeneracy condition

$$W(y,F) \ge C\,\operatorname{dist}^2(F, SO(n)) \tag{W3}$$

where C is a positive constant. In [MN10] we proved a variant of their argument (see Theorem 5.3.10 below) which is adapted to the (slightly weaker) assumption that W has a quadratic Taylor expansion at the identity, i.e.

$$\exists\, Q \in \mathfrak{Q}(Y;n) \;:\; \limsup_{\substack{G\to 0\\ G\neq 0}} \operatorname*{ess\,sup}_{y\in Y} \frac{|W(y, Id+G) - \langle \mathbb{L}(y)G,\, G\rangle|}{|G|^2} = 0. \tag{W4}$$

Here $\mathfrak{Q}(Y;n)$ denotes the set of all measurable integrands $Q : Y\times\mathbb{M}(n) \to \mathbb{R}$ that are Y-periodic in the first, quadratic in the second variable and bounded in the sense that $\operatorname{ess\,sup}_{y\in\mathbb{R}^n} \sup_{|G|=1} Q(y,G) < \infty$.

A natural approach to derive an effective model for the situation where both fine-scales ε and h are small, is to consecutively pass to the limits corresponding to homogenization and linearization. Obviously, there are two different orderings to do so, namely linearization after homogenization and vice versa. It is stringent to ask whether both ways lead to the same result. In other words:

Do linearization and homogenization commute in finite elasticity?

For W satisfying (W2), (W3), (W4) and (5.3) the author proved in joint work with S. Müller (see [MN10]) that homogenization and linearization commute indeed. We stated this result on the level of the integrands, as well as in the language of Γ-convergence on the level of the corresponding functionals. A key insight in [MN10] is the observation that the homogenized integrand $W^{(\mathrm{mc})}_{\mathrm{hom}}$ admits a quadratic Taylor expansion at Id, the quadratic term of which is given by the homogenization of the quadratic term in the expansion of $W({}^{x}\!/\!{}_{\varepsilon},\cdot)$, i.e.

$$\limsup_{\substack{G\to 0\\ G\neq 0}} \frac{\left|W^{(\mathrm{mc})}_{\mathrm{hom}}(Id+G) - Q_{\mathrm{hom}}(G)\right|}{|G|^2} = 0. \tag{5.4}$$

We like to remark that assumption (5.3) guarantees that the homogenization of (5.2) can be expressed by means of the homogenized integrand $W^{(\mathrm{mc})}_{\mathrm{hom}}$. However, the very same assumption (particularly the p-growth condition) excludes stored energy functions with the physical behavior

$$W(y,F) = +\infty \quad \text{if } \det F \le 0 \qquad \text{and} \qquad W(y,F)\to+\infty \quad \text{as } \det F\to 0. \tag{5.5}$$

In this chapter we extend the results in [MN10] to stored energy functions W that only need to satisfy (W2), (W3), (W4). In particular, we can take elastic potentials fulfilling (5.5) into account. More precisely, we consider the functionals

$$\mathcal{I}^{h,\varepsilon}(g) := \begin{cases} \dfrac{1}{h^2}\displaystyle\int_\Omega W({}^{x}\!/\!{}_{\varepsilon}, Id + h\,\nabla g(x))\,\mathrm{d}x & \text{if } g\in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)\\ +\infty & \text{else,}\end{cases}$$

$$\mathcal{I}^{\varepsilon}_{\mathrm{lin}}(g) := \begin{cases} \displaystyle\int_\Omega Q({}^{x}\!/\!{}_{\varepsilon}, \nabla g(x))\,\mathrm{d}x & \text{if } g\in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)\\ +\infty & \text{else,}\end{cases}$$

$$\mathcal{I}(g) := \begin{cases} \displaystyle\int_\Omega Q_{\mathrm{hom}}(\nabla g(x))\,\mathrm{d}x & \text{if } g\in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)\\ +\infty & \text{else.}\end{cases}$$

Here $Q_{\text{hom}}(\cdot)$ denotes the homogenization of the quadratic integrand Q from the expansion (W4), i.e.

$$(5.6) \qquad Q_{\text{hom}}(F) := \min \left\{ \int_Y Q(y, F + \nabla\varphi)\,\mathrm{d}y \,:\, \varphi \in W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n) \right\}.$$

Moreover, $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ denotes the space of functions $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ with $u = 0$ on $\Gamma \subset \partial\Omega$. We suppose that Γ is a measurable subset of $\partial\Omega$ with positive $n-1$-dimensional Hausdorff measure and satisfies the (regularity) property that

$$W^{1,\infty}(\Omega;\mathbb{R}^n) \cap W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$$

is strongly dense in $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$.

Remark 5.1.1. Conditions (W3) and (W4) imply that the quadratic form Q is an integrand in $L^\infty_{\text{per}}(Y;C(\mathbb{M}(n)))$ and a positive semidefinite quadratic form with respect to its second component. Moreover, by the non-degeneracy condition (W3) we see that $Q(y,\cdot)$ restricted to the subspace of symmetric $n\times n$ matrices is positive definite and in particular we have

$$Q(y,F) \geq C\,|\text{sym}\,F|^2$$

where C is the constant from condition (W3) (see Lemma 5.2.4 below). As a consequence, it is not difficult to see that for fixed $F \in \mathbb{M}(n)$ the right hand side of (5.6) is a minimization problem that has a unique minimizer, say $\varphi_F \in W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n)$, which is characterized by the Euler-Lagrange equation

$$(5.7) \qquad \int_Y \langle \mathbb{L}(y)(F + \nabla\varphi_F(y)),\, \nabla\psi(y)\rangle\,\mathrm{d}y = 0 \qquad \text{for all } \psi \in W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n)$$

where $\mathbb{L} \in L^\infty_{\text{per}}(Y;\mathbb{T}_{\text{sym}}(n))$ is the fourth order tensor field defined by

$$\langle \mathbb{L}(y)A,\, B\rangle = \frac{1}{2}\left(Q(y, A+B) - Q(y,A) - Q(y,B) \right)$$

for $A, B \in \mathbb{M}(n)$ and a.e. $y \in \mathbb{R}^n$. This observation relies on the fact that $W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n)$ is a Banach space and that

$$\varphi \mapsto \int_Y Q(y, \nabla\varphi(y))\,\mathrm{d}y$$

is a norm on $W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n)$ that is equivalent to the standard norm, as can be seen in virtue of Korn's inequality for periodic maps (see Proposition A.1.3), the ellipticity estimate (5.24) and the constraint $\int_Y \varphi\,\mathrm{d}y = 0$. If W is additionally frame indifferent, then $Q(y,F)$ vanishes for skew symmetric F and we can replace ∇g by $\text{sym}\,\nabla g$ in the definition of $\mathcal{I}^\varepsilon_{\text{lin}}$ and $\mathcal{I}$; thus, both energies indeed apply to linearized elasticity.

Our main results are the following:

Theorem 5.1.2. *Suppose that W satisfies (W2), (W3), (W4) and consider the functional*

$$\mathcal{I}^h_{\text{hom}}(g) := \inf\Big\{ \liminf_{\varepsilon\to 0} \mathcal{I}^{\varepsilon,h}(g_\varepsilon) \,:\, (g_\varepsilon) \subset L^2(\Omega;\mathbb{R}^n) \\ \text{with } g_\varepsilon \to g \text{ strongly in } L^2(\Omega;\mathbb{R}^n) \Big\}.$$

Then

$$\Gamma\text{-}\lim_{h\to 0} \mathcal{I}^h_{\text{hom}} = \mathcal{I}$$

with respect to strong convergence in $L^2(\Omega;\mathbb{R}^n)$.

(For the proof see page 69).

Theorem 5.1.3. *Suppose that W satisfies (W2), (W3), (W4). Then the following diagram commutes*

$$\begin{array}{ccc} \mathcal{I}^{\varepsilon,h} & \xrightarrow{(1)} & \mathcal{I}^{\varepsilon}_{\text{lin}} \\ {\scriptstyle (2)}\big\downarrow & & \big\downarrow{\scriptstyle (3)} \\ \mathcal{I}^h_{\text{hom}} & \xrightarrow[(4)]{} & \mathcal{I}^0 \end{array} \tag{5.8}$$

Here $(1),(4)$ and (3) mean Γ-convergence w.r.t. to strong convergence in $L^2(\Omega;\mathbb{R}^n)$ as $h\to 0$ and $\varepsilon \to 0$, respectively; while (2) means that $\mathcal{I}^h_{\text{hom}} = \Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{I}^{\varepsilon,h}$ with respect to strong convergence in $L^2(\Omega;\mathbb{R}^n)$.

(For the proof see page 72).

The functional $\mathcal{I}^h_{\text{hom}}$ defined in Theorem 5.1.2 is exactly the lower Γ-limit of the sequence $(\mathcal{I}^{\varepsilon,h})_\varepsilon$ with respect to strong convergence in $L^2(\Omega;\mathbb{R}^n)$ (see Definition 4.2.1). In contrast to the Γ-limit, the lower Γ-limit of a sequence always exists. Moreover, if a sequence is Γ-convergent, then by definition the lower Γ-limit and the Γ-limit itself are equal. As a consequence, if in the situation of the previous two theorems W additionally satisfies (5.3), then (as already mentioned) $(\mathcal{I}^{\varepsilon,h})_\varepsilon$ is Γ-convergent and we particularly have

$$\mathcal{I}^h_{\text{hom}}(g) = \begin{cases} \dfrac{1}{h^2}\displaystyle\int_\Omega W^{(\text{mc})}_{\text{hom}}(Id + h\,\nabla g(x))\,\mathrm{d}x & \text{if } g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n) \\ +\infty & \text{else.} \end{cases}$$

Thus, Theorem 5.1.2 and Theorem 5.1.3 comprise the corresponding results in [MN10] as a special case.

The theorems are accompanied by the following equi-coercivity results.

Proposition 5.1.4. *Suppose that W satisfies (W2), (W3), (W4) and set*

$$\Psi(g) := \begin{cases} \|g\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} & \text{if } g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^2), \\ +\infty & \text{else.} \end{cases}$$

(1) The map $\Psi : L^2(\Omega;\mathbb{R}^n) \to [0,\infty]$ is lower semicontinuous and coercive in the strong topology.

(2) There exists a positive constant C such that

$$\min\left\{ \mathcal{I}^{\varepsilon,h}(g),\, \mathcal{I}^h_{\hom}(g) \right\} \geq C\,\Psi(g)$$

for all $g \in L^2(\Omega;\mathbb{R}^n)$ and all $h,\varepsilon > 0$.

(3) There exists a positive constant C such that

$$\min\{ \mathcal{I}^{\varepsilon}_{\mathrm{lin}}(g),\, \mathcal{I}(g),\, \} \geq C\,\Psi(g)$$

for all $g \in L^2(\Omega;\mathbb{R}^n)$ and all $\varepsilon > 0$.

(For the proof see page 66).

In contrast to the situation considered in [MN10] where $\mathcal{I}^h_{\hom}$ matches the Γ-limit of $\mathcal{I}^{\varepsilon,h}$ (corresponding to homogenization) and is given by the multi-cell homogenization formula, the energy $\mathcal{I}^h_{\hom}$ is a lower Γ-limit and cannot be represented by an explicit formula in the setting considered here. As a consequence, it turns out to be imperative to understand the behavior of $(\mathcal{I}^{\varepsilon,h})$ as ε, h simultaneously tend to zero. This is done by means of a novel two-scale linearization result which we present in the next section. Moreover, we prove — as a by-product — that $\mathcal{I}$ is also the Γ-limit of $\mathcal{I}^{\varepsilon,h}$ as ε and h simultaneously converge to zero:

Theorem 5.1.5. *Suppose that W satisfy (W2), (W3), (W4) and let $\varepsilon : (0,\infty) \to (0,\infty)$ be a map with $\lim_{h\to 0}\varepsilon(h) = 0$. Then*

$$\Gamma\text{-}\lim_{h\to 0} \mathcal{I}^{\varepsilon(h),h} = \mathcal{I}$$

with respect to strong convergence in $L^2(\Omega;\mathbb{R}^n)$.

(For the proof see page 74).

5.2. Simultaneous linearization and homogenization of elastic energies

In this section we prove a two-scale linearization result with the capability to capture the two-scale limiting behavior of functionals of the type

$$L^2(\Omega;\mathbb{M}(n)) \ni G \mapsto \frac{1}{h^2}\int_\Omega W(x, x/\varepsilon, Id + hG(x))\,\mathrm{d}x$$

as h and ε simultaneously converge to 0. The result is a useful tool — not only for the analysis in this part of the thesis, but also for the subsequent chapters. For this reason we state the result in a setting that is more general as it is necessary for the specific problem considered in Theorem 5.1.2.

Theorem 5.2.1. *Let $W : \Omega\times\mathbb{R}^n\times\mathbb{M}(n) \to \overline{\mathbb{R}}$ be a measurable integrand satisfying*

$$W(x,y,Id) = 0 \quad \text{and} \quad W \geq 0.$$

Suppose that there exists

$$\mathbb{L} \in C(\overline{\Omega}; L^\infty_{\text{per}}(Y;\mathbb{T}_{\text{sym}}(n)))$$

such that

$$\limsup_{\substack{G\to 0\\ G\neq 0}} \operatorname*{ess\,sup}_{(x,y)\in\Omega\times\mathbb{R}^n} \frac{|W(x,y,Id+G) - \langle \mathbb{L}(x,y)G,\, G\rangle|}{|G|^2} = 0. \tag{5.9}$$

Let $\varepsilon : (0,\infty) \to (0,\infty)$ be a map with $\lim_{h\to 0}\varepsilon(h) = 0$.

(1) Let $(G_h) \subset L^2(\Omega;\mathbb{M}(n))$ converge weakly two-scale to G in $L^2(\Omega\times Y;\mathbb{M}(n))$. Then

$$\liminf_{h\to 0} \frac{1}{h^2}\int_\Omega W(x, x/\varepsilon(h), Id + hG_h)\,\mathrm{d}x \geq \iint\limits_{\Omega\times Y} \langle \mathbb{L}(x,y)G,\, G\rangle\,\mathrm{d}y\,\mathrm{d}x.$$

(2) Let $(G_h) \subset L^2(\Omega;\mathbb{M}(n))$ converge strongly two-scale to G in $L^2(\Omega\times Y;\mathbb{M}(n))$ and suppose that

$$\limsup_{h\to 0} \operatorname*{ess\,sup}_{x\in\Omega} |hG_h(x)| = 0.$$

Then

$$\lim_{h\to 0} \frac{1}{h^2}\int_\Omega W(x, x/\varepsilon(h), Id + hG_h)\,\mathrm{d}x = \iint\limits_{\Omega\times Y} \langle \mathbb{L}(x,y)G,\, G\rangle\,\mathrm{d}y\,\mathrm{d}x.$$

Here, two-scale convergence is understood with respect to the fine-scale $\varepsilon{=}\varepsilon(h)$.

Remark 5.2.2. The measurability of W has to be understood in the sense Definition 3.1.1.

Proof of Theorem 5.2.1. For convenience we set

$$Q(x,y,F) := \langle \mathbb{L}(x,y)F,\, F\rangle\,.$$

We like to remark that the assumptions on W straightforwardly imply that there exist a closed ball $K \subset \mathbb{M}(n)$ with center 0 and a monotone map $\rho : [0,\infty) \to [0,\infty)$ with $\lim_{r\to 0}\rho(r) = \rho(0) = 0$ such that

$$|W(x,y,Id+F) - Q(x,y,F)| \leq \rho(|F|)\,|F|^2 \tag{5.10}$$

for all $F \in K \subset \mathbb{M}(n)$ and a.e. $(x,y) \in \Omega\times\mathbb{R}^n$.

<u>*Step 1.*</u> We prove the lower bound statement (1). The proof relies on a “careful Taylor expansion” of W used in [FJM02]. We extend their idea with regard to two-scale

convergence. Roughly speaking, the strategy is the following: Because of the quadratic expansion of W (condition (W4)), we can approximate $W(x,y,Id+F)$ by means of the quadratic form Q provided $|F|$ is small enough. This suggests to split Ω into a good set Ω_h, where $|hG_h(x)|$ is sufficiently small, and a bad set $\Omega_h^c = \Omega \setminus \Omega_h$. On the good set we approximate W by the quadratic expansion and then obtain a lower bound by means of the lower semicontinuity of convex integral functionals. On the bad set the non-negativity of W allows us to ignore the energy portion generated in Ω_h^c without increasing the energy. Since the measure of the bad set becomes negligible as $h \to 0$, it turns out that the decomposition above is sufficiently precise.

For the rigorous proof define

$$\Omega_h := \{ \, x \in \Omega \, : \, |G_h(x)| \geq h^{-1/2} \, \}$$

and let 1_{Ω_h} denote the indicator function associated to Ω_h. We consider the sequence $(\widetilde{G}_h)$ defined by

$$\widetilde{G}_h := 1_{\Omega_h} G_h.$$

Because (G_h) weakly two-scale converges to G, we can apply Lemma 2.3.2 and deduce that $\widetilde{G}_h$ weakly two-scale converges to G in $L^2(\Omega{\times}Y;\mathbb{M}(n))$ as well.

Due to the non-negativity of W we trivially have

$$\frac{1}{h^2}\int_\Omega W(x, {}^x\!/\!_{\varepsilon(h)}, Id + hG_h(x))\,\mathrm{d}x \geq \frac{1}{h^2}\int_\Omega W(x, {}^x\!/\!_{\varepsilon(h)}, Id + h\widetilde{G}_h(x))\,\mathrm{d}x.$$

Because of the estimate $\operatorname{ess\,sup}_{x\in\Omega}\left|h\widetilde{G}_h(x)\right| \leq h^{1/2}$, we deduce that for almost every $x \in \Omega$ the matrix $h\widetilde{G}(x)$ belongs to the closed ball $K \subset \mathbb{M}(n)$ — provided h is small enough. Hence, for small h we can approximate the right hand side in the previous estimate by means of the quadratic form Q (cf. (5.10)):

$$\frac{1}{h^2}\int_\Omega W(x, {}^x\!/\!_{\varepsilon(h)}, Id{+}h\widetilde{G}_h(x))\,\mathrm{d}x \geq \int_\Omega Q(x, {}^x\!/\!_{\varepsilon(h)}, \widetilde{G}_h(x))\,\mathrm{d}x - \rho(h^{1/2})\left\|\widetilde{G}_h\right\|^2_{L^2(\Omega;\mathbb{M}(n))}.$$

The quadratic functional on the right hand side is lower semicontinuous with respect to weak two-scale convergence in $L^2(\Omega{\times}Y;\mathbb{M}(n))$ (see Proposition 3.2.2), while the remainder vanishes as $h \to 0$ due to the boundedness of $(\widetilde{G}_h)$ and $\lim_{h\to 0}\rho(h) = 0$. Since $(\widetilde{G}_h)$ two-scale converges to G, we infer that

$$\liminf_{h\to 0}\frac{1}{h^2}\int_\Omega W(x, {}^x\!/\!_{\varepsilon(h)}, Id + hG_h(x))\,\mathrm{d}x \geq \iint\limits_{\Omega\times Y} Q(x,y,G(x,y))\,\mathrm{d}x\,\mathrm{d}y.$$

Step 2. We prove statement (2). By assumption the matrix $hG_h(x)$ belongs to K for a.e. $x \in \Omega$ provided h is sufficiently small. Hence, (5.10) implies that

$$\begin{aligned}&\limsup_{h\to 0}\frac{1}{h^2}\int_\Omega W(x, {}^x\!/\!_{\varepsilon(h)}, Id + hG_h(x))\,\mathrm{d}x\\ &\leq \limsup_{h\to 0}\frac{1}{h^2}\int_\Omega Q(x, {}^x\!/\!_{\varepsilon(h)}, Id + hG_h(x))\,\mathrm{d}x + \limsup_{h\to 0}\int_\Omega \rho(|hG_h(x)|\,|G_h(x)|^2\,\mathrm{d}x.\end{aligned}$$

The second integral vanishes since $\rho(h\,|G_h(x)|) \to 0$ uniformly. The first integral is a quadratic functional that is continuous with respect to strong two-scale convergence (see Proposition 3.2.2). Since $G_h \xrightarrow{2} G$ strongly two-scale, we can pass to the limit and obtain

$$\limsup_{h\to 0} \frac{1}{h^2} \int_\Omega W(x, x/\varepsilon(h), Id + hG_h(x))\,\mathrm{d}x \le \iint_{\Omega\times Y} Q(x,y,G(x,y))\,\mathrm{d}x\,\mathrm{d}y.$$

On the other side we can apply statement (1) of the current theorem and deduce that

$$\iint_{\Omega\times Y} Q(x,y,G(x,y))\,\mathrm{d}x\,\mathrm{d}y \le \liminf_{h\to 0} \frac{1}{h^2} \int_\Omega W(x, x/\varepsilon(h), Id + hG_h(x))\,\mathrm{d}x.$$

Now the combination of both estimates imply the claimed convergence and the proof is complete. □

Below we give a sufficient condition for the requirements in Theorem 5.2.1 by means of the regularity of W in a neighborhood of Id.

Lemma 5.2.3. *Let $W : \Omega\times\mathbb{R}^n\times\mathbb{M}(n) \to \overline{\mathbb{R}}$ be a measurable integrand. Assume that W is Y-periodic in its second variable and satisfies*

$$W(x,y,Id) = 0 \quad \text{and} \quad W \ge 0.$$

If there exists a closed set $U \subset \mathbb{M}(n)$ containing Id such that W restricted to $\Omega\times Y\times U$ belongs to the space

$$C(\overline{\Omega}; L^\infty_{\mathrm{per}}(Y; C^{2,1}(U))),$$

then W satisfies the assumptions of the previous theorem and $\mathbb{L}$ is the unique map in $C(\overline{\Omega}; L^\infty_{\mathrm{per}}(Y; \mathbb{T}_{\mathrm{sym}}(n)))$ satisfying

$$\langle \mathbb{L}(x,y)G,\, G\rangle = \frac{1}{2}\frac{\partial^2 W}{\partial F^2}(x,y,Id)[G,G] \tag{5.11}$$

for all $G \in \mathbb{M}(n)$, $x \in \Omega$ and a.e. $y \in Y$.

Proof. Set

$$Q(x,y,G) := \frac{1}{2}\frac{\partial^2 W}{\partial F^2}(x,y,Id)[G,G]$$

and let $\mathbb{L}(x,y) \in \mathbb{T}_{\mathrm{sym}}(n)$ denote the unique symmetric tensor with

$$\langle \mathbb{L}(x,y)G,\, G\rangle = Q(x,y,G) \qquad \text{for all } G \in \mathbb{M}(n).$$

Since $W \in C(\overline{\Omega}; L^\infty_{\mathrm{per}}(Y; C^{2,1}(U)))$ with $Id \in U$ and due to the periodicity of W, the map $(x,y) \mapsto \mathbb{L}(x,y)$ belongs to $C(\overline{\Omega}; L^\infty_{\mathrm{per}}(Y; \mathbb{T}_{\mathrm{sym}}(n))$.

We prove (5.9). To this end let K denote a closed ball in $\mathbb{M}(n)$ with center 0 and $K + Id \subset U$. Let $G \in K$. Then the line segment $\{\, Id + sG \,:\, s \in [0,1] \,\}$ belongs to K and a quadratic Taylor expansion of W with center Id yields

$$W(x,y,Id+G) = W(x,y,Id) + \frac{\partial W}{\partial F}(x,y,Id)[G] + \int_0^1 (1-s)\frac{\partial^2 W}{\partial F^2}(x,y,Id+sG)[G,G]\,\mathrm{d}s. \tag{5.12}$$

for a.e. (x,y). Now the first two terms on the right hand side are zero, since $W(x,y,Id)$ is the minimum of W. By assumption, the map $G \mapsto \frac{\partial^2 W}{\partial F^2}(x,y,Id+G)$ is Lipschitz continuous with a constant $L > 0$ uniform in (x,y). This implies

$$\left|\frac{\partial^2 W}{\partial F^2}(x,y,Id+F)[H,H] - Q(x,y,H)\right| \leq L\,|F|\,|H|^2$$

for all $F \in K$, $H \in \mathbb{M}(n)$ and a.e. (x,y). We apply this estimate to (5.12) end deduce that

$$\begin{aligned}|W(x,y,Id+G) - Q(x,y,G)| &= \left|\int_0^1 (1-s)\frac{\partial^2 W}{\partial F^2}(x,y,Id+sG)[G,G]\,\mathrm{d}s - Q(x,y,G)\right|\\ &\leq L\,|G| \int_0^1 (1-s)s\,\mathrm{d}s\ \ |G|^2\end{aligned}$$

for all $G \in K$ and a.e. (x,y). Thus,

$$\limsup_{\substack{G\to 0\\ G\neq 0}}\ \operatorname*{ess\,sup}_{(x,y)\in\Omega\times\mathbb{R}^n} \frac{|W(x,y,Id+G) - \langle \mathbb{L}(x,y)G,\,G\rangle|}{|G|^2} = 0.$$

□

In the next lemma we gather some properties of $\mathbb{L}(x,y)$ for the situation where W is related to finite elasticity.

Lemma 5.2.4. *Let $W : \Omega\times\mathbb{R}^n\times\mathbb{M}(n) \to \overline{\mathbb{R}}$ be a measurable integrand. Suppose that Id is a natural state, i.e. $W(x,y,Id)=0$ for a.e. $(x,y) \in \Omega\times\mathbb{R}^n$, and that W satisfies the non-degeneracy condition*

$$\exists C > 0\ :\ \operatorname*{ess\,inf}_{(x,y)\in\Omega\times\mathbb{R}^n} W(x,y,F) \geq C\,\mathrm{dist}^2(F, SO(n)) \qquad \text{for all } F \in \mathbb{M}(n).$$

Suppose that there exists a symmetric second order tensor field

$$\mathbb{L} \in L^\infty(\Omega\times\mathbb{R}^n; \mathbb{T}_{\mathrm{sym}}(n))$$

such that

$$\limsup_{\substack{G\to 0\\ G\neq 0}}\ \operatorname*{ess\,sup}_{(x,y)\in\Omega\times\mathbb{R}^n} \frac{|W(x,y,Id+G) - \langle \mathbb{L}(x,y)G,\,G\rangle|}{|G|^2} = 0. \tag{5.13}$$

Then:

(1) *There exists a constant* $C > 0$ *such that*

$$\operatorname*{ess\,inf}_{(x,y)\in\Omega\times\mathbb{R}^n} \langle \mathbb{L}(x,y)G,\, G\rangle \geq c\,|\mathrm{sym}\,G|^2$$

for all $G \in \mathbb{M}(n)$.

(2) *If* W *is additionally frame indifferent, i.e.*

$$W(x,y,F) = W(x,y,RF) \qquad \textit{for all } F \in \mathbb{M}(n),\; R \in SO(n)$$

and a.e. $(x,y) \in \Omega\times\mathbb{R}^n$, *then*

$$\begin{aligned}&\langle \mathbb{L}(x,y)\,\mathrm{skew}\,G,\, \mathrm{skew}\,G\rangle = 0\\ &\langle \mathbb{L}(x,y)F,\, G\rangle = \langle \mathbb{L}(x,y)G,\, F\rangle = \langle \mathbb{L}(x,y)\,\mathrm{sym}\,G,\, F\rangle = \langle \mathbb{L}(x,y)\,\mathrm{sym}\,G,\, \mathrm{sym}\,F\rangle\,.\end{aligned}$$

for a.e. (x,y) *and all matrices* $G, F \in \mathbb{M}(n)$.

Proof. Set

$$Q(x,y,G) := \langle \mathbb{L}(x,y)G,\, G\rangle\,.$$

Step 1. Let $G \in \mathbb{M}(n)$. By assumption we have for all sufficiently small $h > 0$

$$\left|\frac{1}{h^2}W(x,y,Id+hG) - Q(x,y,G)\right| \leq \rho(h\,|G|)\,|G|^2 \tag{5.14}$$

with $\rho(r) \to 0$ as r tends to zero. This implies

$$\lim_{h\to 0}\frac{1}{h^2}W(x,y,Id+hG) = Q(x,y,G). \tag{5.15}$$

Moreover, in view of the non-degeneracy condition we compute

$$Q(x,y,G) \geq C\liminf_{h\to 0}\left\{\frac{1}{h^2}\,\mathrm{dist}^2(Id+hG, SO(n)) - \rho(h\,|G|)\,|G|^2\right\}$$

for some uniform constant C. If h is small enough, the matrix $Id+hG$ has positive determinant. Hence, we can factorize it by means of the polar factorization as

$$Id + hG = R_h\sqrt{(Id+hG)^{\mathrm{T}}(Id+hG)}$$

where R_h is a suitable rotation. We can rewrite the distance to $SO(n)$ in terms of this factorization and obtain

$$\mathrm{dist}^2(Id+hG, SO(n)) = \left|\sqrt{(Id+hG)^{\mathrm{T}}(Id+hG)} - Id\right|^2.$$

Since

$$\lim_{h\to 0}\frac{\sqrt{(Id+hG)^{\mathrm{T}}(Id+hG)} - Id}{h} = \mathrm{sym}\,G,$$

(5.14) yields statement (1).

Step 2. Let $G \in \mathbb{M}(n)$ and $h > 0$ sufficiently small, so that $\det(Id + hG) > 0$. Then

$$W(x, y, Id + hG) = W\left(x, y, Id + h\frac{\sqrt{(Id + hG)^{\mathrm{T}}(Id + hG)} - Id}{h}\right)$$

as can be seen by a polar factorization and frame indifference. Since

$$\frac{\sqrt{(Id + hG)^{\mathrm{T}}(Id + hG)} - Id}{h} \to \operatorname{sym} G,$$

we deduce from (5.14) that $\frac{1}{h^2}W(x, y, Id + hG) \to Q(x, y, \operatorname{sym} G)$ and a comparison with (5.15) yields the identity

$$Q(x, y, G) = Q(x, y, \operatorname{sym} G).$$

Now the claimed identities follow from the previous identity and the formula

$$2\,\langle \mathbb{L}(x, y)G,\, F\rangle = Q(x, y, F + G) - Q(x, y, F) - Q(x, y, G).$$

□

5.3. Proof of the main results

5.3.1. Equi-coercivity

In this section, we prove that the non-degeneracy condition (W3) and the Dirichlet boundary condition yield equi-coercivity of the functionals $\mathcal{I}^{\varepsilon,h}$, $\mathcal{I}^h_{\hom}$, $\mathcal{I}^{\varepsilon}_{\mathrm{lin}}$ and $\mathcal{I}$ with respect to strong convergence in $W^{1,2}(\Omega; \mathbb{R}^n)$ — as it is stated in Proposition 5.1.4. The proof for $\mathcal{I}^{\varepsilon,h}$, $\mathcal{I}^h_{\hom}$ relies on the following estimate:

Proposition 5.3.1. *There exists a positive constant C such that*

$$\int_\Omega \operatorname{dist}^2(Id + h\,\nabla g(x), SO(n))\,\mathrm{d}x \geq C\,h^2\,\|g\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} \tag{5.16}$$

for all $g \in W^{1,2}_{\Gamma,0}(\Omega; \mathbb{R}^n)$.

A variant of the previous proposition can be found along the lines in [Per99]. For the sake of completeness we briefly sketch the proof, which roughly speaking relies on two observations. The first one is a consequence of the geometric rigidity estimate (see Theorem 5.3.2 below) and says that we can approximate the map $x \mapsto Id + h\,\nabla g(x)$ by a constant rotation, say $R_h \in SO(n)$, in such a way that the L^2-distance is controlled by the left hand side of inequality (5.16).

Theorem 5.3.2 (Geometric rigidity [FJM02]). *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. There exists a constant $C(U)$ with the following property: For each $v \in W^{1,2}(U;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\int_U |\nabla v(x) - R|^2 \,\mathrm{d}x \leq C(U) \int_U \mathrm{dist}^2(\nabla v(x), SO(n))\,\mathrm{d}x.$$

Moreover, the constant $C(U)$ is invariant under uniform scaling of U.

The second insight is that the Dirichlet boundary condition imposed on $\Gamma \subset \partial\Omega$ implies that the rotation R_h is close to Id; namely, we are going to show that $\left|h^{-1}(R_h - Id)\right|^2$ can be controlled by the left hand side of (5.16) as well. In this regard we need the following result:

Lemma 5.3.3. *Let $\Gamma \subset \partial\Omega$ be a bounded $\mathcal{H}^{n-1}$-measurable set with $0 < \mathcal{H}^{n-1}(\Gamma) < +\infty$. Define for $F \in \mathbb{M}(n)$*

$$|F|_\Gamma^2 := \min_{c \in \mathbb{R}^n} \int_\Gamma |Fx - c|^2 \,\mathrm{d}\mathcal{H}^{n-1}(x).$$

Then there exists a positive constant C such that

$$|F|^2 \leq C\,|F|_\Gamma^2 \qquad \textit{for all } F \in \mathbb{M}_C(n)$$

where $\mathbb{M}_C(n)$ denotes the union of the cone generated by $Id - SO(n)$ and the set of skew symmetric matrices in $\mathbb{M}(n)$.

(For the proof see Lemma 3.3 in [Per99].)

Proof of Proposition 5.3.1. Let $h > 0$ and $g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$. Due to Theorem 5.3.2 there exists a rotation $R \in SO(n)$ with

$$\int_\Omega |Id + h\,\nabla g(x) - R|^2 \,\mathrm{d}x \leq c' \int_\Omega \mathrm{dist}^2(Id + h\,\nabla g(x), SO(n))\,\mathrm{d}x. \tag{5.17}$$

Here and below, c' denotes a positive constant that may change from line to line, but can be chosen independent of h and g. Since g vanishes on $\Gamma \subset \partial\Omega$, we have

$$\|g\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} \leq c' \int_\Omega |\nabla g(x)|^2 \,\mathrm{d}x$$

due to the Poincaré-Friedrichs inequality (see Proposition A.1.1). By means of the decomposition

$$h\,\nabla g(x) = (Id + h\,\nabla g(x) - R) - (Id - R)$$

inequality (5.17) implies

$$h^2\,\|g\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} \leq c' \left(\int_\Omega \mathrm{dist}^2(Id + h\,\nabla g(x), SO(n))\,\mathrm{d}x + |Id - R|^2 \right).$$

Hence, it remains to prove that the distance $|Id - R|^2$ can be controlled by the right hand side of (5.17) as well. To this end, we define

$$u(x) := (Id - R)x + hg(x) - u_\Omega \quad \text{with} \quad u_\Omega := \frac{1}{\mathcal{H}^n(\Omega)} \int_\Omega (Id - R)x + hg(x)\,\mathrm{d}x.$$

The map u belongs to $W^{1,2}(\Omega;\mathbb{R}^n)$ and has vanishing mean value; thus, the Poincaré inequality and the continuity of the trace operator imply

$$\int_\Gamma |u(x)|^2 \,\mathrm{d}\mathcal{H}^{n-1}(x) \leq c' \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x.$$

Since $g(x) = 0$ for $x \in \Gamma$, we have $u(x) = (Id - R)x + u_\Omega$ for all $x \in \Gamma$ and Lemma 5.3.3 yields

$$\begin{aligned} |Id - R|^2 \leq C\,|Id - R|_\Gamma^2 &\leq c' \int_\Gamma |u(x)|^2 \,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &\leq c' \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x. \end{aligned}$$

On the other hand, we have $\nabla u(x) = Id + h\,\nabla g(x) - R$ and in view of (5.17) we obtain the estimate

$$|Id - R|^2 \leq c' \int_\Omega \mathrm{dist}^2(Id + h\,\nabla g(x), SO(n))\,\mathrm{d}x, \tag{5.18}$$

which completes the proof. □

Remark 5.3.4. The presence of a boundary condition is crucial in Proposition 5.3.1. For instance, let R denote an arbitrary rotation that is different from Id. Consider the displacement

$$g_h(x) := \frac{R - Id}{h} x.$$

Then the gradient of the deformation $x \mapsto x + hg_h(x)$ belongs to $SO(n)$ for all x and the left hand side in (5.16) vanishes. But we have

$$\liminf_{h\to 0} h^2 \int_\Omega |\nabla g_h|^2 \,\mathrm{d}x > 0.$$

Remark 5.3.5. The estimate in Proposition 5.3.1 is also valid for periodic functions. In particular we showed in [MN10] the following: There exists a positive constant C such that

$$\frac{1}{h^2} \int_{kY} \mathrm{dist}^2(Id + h\,\nabla\psi(x), SO(n))\,\mathrm{d}x \geq C \int_{kY} |\nabla\psi(x)|^2 \,\mathrm{d}x$$

for all $h > 0$, $k \in \mathbb{N}$ and maps $\psi \in W^{1,2}_{\mathrm{per}}(kY;\mathbb{R}^n)$.

Proof of Proposition 5.1.4, statement (1) and (2). *Step 1.* We prove (1). Let (g_j) be a strongly convergent sequence in $L^2(\Omega;\mathbb{R}^n)$ with limit g. We have to show that

$$\liminf_{j\to\infty} \Psi(g_j) \geq \Psi(g). \tag{5.19}$$

For the proof we only have to consider the situation where the left hand side is finite. In this case we can pass to a subsequence (not relabeled) with $\sup_j \Psi(g_j) < \infty$ such that $\lim_{j\to\infty} \Psi(g_j)$ exists and equals the left hand side of (5.19). Obviously, this implies that (g_j) is a bounded sequence in $W^{1,2}(\Omega;\mathbb{R}^n)$ and satisfies the vanishing Dirichlet boundary condition on Γ. We conclude that (g_j) weakly converges to g in $W^{1,2}(\Omega;\mathbb{R}^n)$ and that g satisfies the boundary condition as well. Now (5.19) directly follows from the lower semicontinuity of the norm with respect to weak convergence.

Step 2. Due to the non-degeneracy condition and the fact that $\mathcal{I}^{\varepsilon,h}$ can only be finite for maps in $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$, Proposition 5.3.1 implies that

$$\mathcal{I}^{\varepsilon,h}(g) \geq c_0\, \Psi(g) \qquad \text{for all } g \in L^2(\Omega;\mathbb{R}^n) \tag{5.20}$$

for a positive constant c_0. Let $g \in L^2(\Omega;\mathbb{R}^n)$. We claim that

$$\mathcal{I}^h_{\hom}(g) \geq c_0\, \Psi(g). \tag{5.21}$$

This can be seen as follows: Since $\mathcal{I}^h_{\hom}$ is the lower Γ-limit of $(\mathcal{I}^{\varepsilon,h})_\varepsilon$, there exists a sequence (g_ε) converging to g in $L^2(\Omega;\mathbb{R}^n)$ such that

$$\mathcal{I}^h_{\hom}(g) = \liminf_{\varepsilon\to 0} \mathcal{I}^{\varepsilon,h}(g_\varepsilon).$$

Now the lower bound (5.20) and the lower semicontinuity of the map Ψ proves (5.21). □

The equi-coercivity of the linearized functionals $\mathcal{I}^\varepsilon_{\mathrm{lin}}$ and $\mathcal{I}$ relies on the following Korn inequality of Friedrichs type:

Proposition 5.3.6. *There exists a positive constant C such that*

$$\int_\Omega |\operatorname{sym}\nabla g(x)|^2 \,\mathrm{d}x \geq C\, \|g\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} \qquad \textit{for all} \qquad g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n).$$

Proof. We recall the Korn inequality (see Proposition A.1.2): For every $g \in W^{1,2}(\Omega;\mathbb{R}^n)$ there exists a skew symmetric matrix $A \in \mathbb{M}(n)$ such that

$$\int_\Omega |\nabla g(x) + A|^2 \,\mathrm{d}x \leq C_\Omega \int_\Omega |\operatorname{sym}\nabla g(x)|^2 \,\mathrm{d}x \tag{5.22}$$

where C_Ω is a positive constant that only depends on Ω.

Let $g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ and choose A as above. Then

$$u(x) := g(x) + Ax - u_\Omega \qquad \text{with } u_\Omega := \frac{1}{\mathcal{H}^n(\Omega)} \int_\Omega g(x) + Ax \,\mathrm{d}x$$

defines a map in $W^{1,2}(\Omega;\mathbb{R}^n)$ with vanishing mean value. We estimate the modulus of the skew symmetric matrix A by means of Lemma 5.3.3:

$$|A|^2 \leq C\,|A|^2_\Gamma \leq c' \int_\Gamma |Ax - u_\Omega|^2 \,\mathrm{d}\mathcal{H}^{n-1}(x).$$

Here and below, c' denotes a constant that may change from line to line, but only depends on Ω and Γ. Since $u(x) = Ax - u_\Omega$ on Γ, the right hand side is controlled by $\|u\|^2_{L^2(\partial\Omega;\mathbb{R}^n)}$. Now the continuity of the trace operator and Poincaré's inequality yield

$$|A|^2 \leq c' \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x$$

and because of $\nabla u = \nabla g + A$, Korn's inequality (5.22) implies that

$$|A|^2 \leq c' \int_\Omega |\mathrm{sym}\, \nabla g(x)|^2 \, \mathrm{d}x. \tag{5.23}$$

On the other hand, we have

$$\|g\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \leq c' \, \|\nabla g\|_{L^2(\Omega;\mathbb{M}(n)} \leq c' \left(\|\nabla u\|_{L^2(\Omega;\mathbb{M}(n)} + |A| \, \mathcal{H}^n(\Omega) \right)$$

by the Poincaré-Friedrichs inequality and because of $\nabla g(x) = \nabla u(x) - A$; thus, the right hand side can be estimated by means of (5.22) and (5.23) and the proof is complete. □

Proof of Proposition 5.1.4, statement (3). In virtue of Lemma 5.2.4, the non-degeneracy condition (W3) and assumption (W4), we have

$$\operatorname*{ess\,inf}_{y \in Y} Q(y, F) \geq C \, |\mathrm{sym}\, F|^2 \qquad \text{for all } F \in \mathbb{M}(n) \tag{5.24}$$

and a positive constant C. We claim that the same estimate holds for Q_{hom} as well. Since the map $F \mapsto Q_{\mathrm{hom}}(F)$ is continuous and quadratic, we only have to prove that there exists a constant $C > 0$ such that

$$Q_{\mathrm{hom}}(\mathrm{sym}\, F) \geq C \tag{5.25}$$

for all symmetric $F \in \mathbb{M}(n)$ with $|F| = 1$. Let F be such a matrix. In view of Remark 5.1.1 there exists a map $\varphi_F \in W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^n)$ such that

$$Q_{\mathrm{hom}}(F) = \int_Y Q(y, F + \nabla \varphi_F) \, \mathrm{d}y.$$

We apply estimate (5.24) and compute

$$\begin{aligned} \frac{1}{C} Q_{\mathrm{hom}}(F) &\geq \int_Y |\mathrm{sym}(F + \nabla\varphi_F(y))|^2 \, \mathrm{d}y \\ &= |\mathrm{sym}\, F|^2 + 2 \int_Y \langle \mathrm{sym}\, F, \, \mathrm{sym}\, \nabla\varphi_F(y) \rangle \, \mathrm{d}y + \int_Y |\mathrm{sym}\, \nabla\varphi_F(y)|^2 \, \mathrm{d}y. \end{aligned}$$

Now the first term on the right hand side equals 1 and the third term is positive. The second term vanishes, because of

$$\int_Y \langle \mathrm{sym}\, F, \, \mathrm{sym}\, \nabla\varphi_F(y) \rangle \, \mathrm{d}y = \int_Y \langle \mathrm{sym}\, F, \, \nabla\varphi_F(y) \rangle \, \mathrm{d}y = 0$$

where we used the fact that gradients of periodic functions and constant matrices are orthogonal in $L^2(Y;\mathbb{M}(n))$. We conclude that $Q_{\text{hom}}(F)$ is strictly positive and because the set of all symmetric matrices with modulus equal to 1 is compact, (5.25) follows.

Now the validity of statement (3) is a direct consequence of the periodic Korn inequality (see Proposition A.1.3). □

5.3.2. Proof of Theorem 5.1.2

We have to show the following:

(1) (Lower bound). For each sequence $(g_h) \subset L^2(\Omega;\mathbb{R}^n)$ with limit g there holds

$$\liminf_{h\to 0} \mathcal{I}^h_{\text{hom}}(g_h) \geq \mathcal{I}(g). \tag{5.26}$$

(2) (Upper bound). For each map $g \in L^2(\Omega;\mathbb{R}^n)$ there exists a sequence $(g_h) \subset L^2(\Omega;\mathbb{R}^n)$ converging to g such that

$$\lim_{h\to 0} \mathcal{I}^h_{\text{hom}}(g) = \mathcal{I}(g). \tag{5.27}$$

Lower bound

In contrast to the situation discussed in [MN10], the lower Γ-limit $\mathcal{I}^h_{\text{hom}}$ (which is associated to homogenization) cannot be expressed by an explicit formula in general. Nevertheless, we can approximate $\mathcal{I}^h_{\text{hom}}(g)$ by appropriately evaluating the initial energy $\mathcal{I}^{\varepsilon,h}$ (see Lemma 5.3.7 below). Moreover, the equi-coercivity of $\mathcal{I}^{\varepsilon,h}$ enables us to establish a lim inf-inequality in the situation where ε and h simultaneously tend to zero (see Proposition 5.3.8). The combination of both observations eventually leads to the lower bound estimate.

Lemma 5.3.7. *Let $h > 0$, $\eta > 0$ and $g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$. Then there exists $\tilde\varepsilon \in (0,h)$ and a map $\tilde g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ such that*

$$\begin{cases} \mathcal{I}^h_{\text{hom}}(g) + \eta \geq \mathcal{I}^{\tilde\varepsilon,h}(\tilde g) \\ \|\tilde g - g\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq h \end{cases}$$

Proof. We only have to consider the case where $\mathcal{I}^h_{\text{hom}}(g)$ is finite. By definition, there exists a sequence $(g_\varepsilon) \subset L^2(\Omega;\mathbb{R}^n)$ converging to g such that

$$\mathcal{I}^h_{\text{hom}}(g) + \eta/2 \geq \liminf_{\varepsilon\to 0} \mathcal{I}^{\varepsilon,h}(g_\varepsilon). \tag{5.28}$$

We pass to a subsequence (not relabeled) such that $\lim_{\varepsilon\to 0} \mathcal{I}^{\varepsilon,h}(g_\varepsilon)$ exists and equals the right hand side of (5.28). Consequently, for all ε sufficiently small we have

$$\varepsilon < h, \qquad \mathcal{I}^h_{\text{hom}}(g) + \eta \geq \mathcal{I}^{\varepsilon,h}(g_\varepsilon) \quad \text{and} \quad \|g - g_\varepsilon\|_{L^2(\Omega;\mathbb{R}^n)} < h.$$

Let $\tilde\varepsilon$ denote such a sufficiently small positive number and set $\tilde g := g_{\tilde\varepsilon}$. Since $\mathcal{I}^{\tilde\varepsilon,h}(\tilde g)$ is finite, we deduce that $\tilde g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ and the proof is complete. □

Proposition 5.3.8. *Suppose that W satisfies the conditions (W2), (W3) and (W4), and let $\varepsilon : (0,\infty) \to (0,\infty)$ be a map with $\lim_{h\to 0}\varepsilon(h) = 0$. Then for any sequence (g_h) in $L^2(\Omega;\mathbb{R}^n)$ with limit $g \in L^2(\Omega;\mathbb{R}^2)$ we have*

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(g_h) \geq \mathcal{I}(g). \tag{5.29}$$

Proof. *Step 1.* We only have to consider the case where the left hand side of (5.29) is finite. In this case we can pass to a subsequence of (h) — that we do not relabel — such that $\lim_{h\to 0}\mathcal{I}^{\varepsilon(h),h}(g_h)$ exists and equals the left hand side of (5.29). Moreover, we assume without loss of generality that $\mathcal{I}^{\varepsilon(h),h}(g_h) \leq C$ for all h and a uniform positive constant C. Due to the equi-coercivity (see Proposition 5.1.4), we find that (g_h) is a bounded sequence in $W^{1,2}(\Omega;\mathbb{R}^n)$ and satisfies the Dirichlet boundary condition. This implies that (g_h) weakly converges to g in $W^{1,2}(\Omega;\mathbb{R}^n)$, because, by assumption, $g_h \to g$ in $L^2(\Omega;\mathbb{R}^n)$. Thus, $g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$.

Step 2. In view of Proposition 2.1.14, we can pass to a further subsequence (not relabeled) such that

$$\nabla g_h \xrightharpoonup{2} \nabla g + \nabla_y \varphi \qquad \text{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{M}(n))$$

for a suitable map $\varphi \in L^2(\Omega; W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n))$. Note that in the limit, only the periodic profile φ depends on the chosen subsequence. Now assumption (W4) allows us to pass to the limit by means of Theorem 5.2.1; thus,

$$\begin{aligned}\liminf_{h\to 0}\mathcal{I}^{\varepsilon(h),h}(g_h) &= \liminf_{h\to 0}\frac{1}{h^2}\int_\Omega W\left(x/\varepsilon(h), Id + h\,\nabla g_h(x)\right)\mathrm{d}x\\ &\geq \iint_{\Omega\times Y} Q(y, \nabla g(x) + \nabla_y\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

Since $\varphi(x,\cdot) \in W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n)$ for a.e. $x \in \Omega$, the right hand side is bounded from below by

$$\int_\Omega Q_{\text{hom}}(\nabla g(x))\,\mathrm{d}x = \mathcal{I}(g).$$

This expression is independent of the subsequence, and therefore (5.29) follows. □

Proof of Theorem 5.1.2, lower bound. Let (g_h) be an arbitrary sequence in $L^2(\Omega;\mathbb{R}^n)$ with limit g. We only have to consider the case where

$$\liminf_{h\to 0}\mathcal{I}^h_{\text{hom}}(g_h) \tag{5.30}$$

is finite. Since $(\mathcal{I}^h_{\text{hom}})_h$ is equi-coercive (see Proposition 5.1.4), we can assume without loss of generality (similarly to Step 1 in the proof of Proposition 5.3.8) that (g_h) is a weakly convergent sequence in $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ with limit g such that $\lim_{h\to 0}\mathcal{I}^h_{\text{hom}}(g_h)$ exists and equals (5.30).

Due to Lemma 5.3.7 we can assign to each $\eta > 0$ and $h > 0$ a map $\tilde{g}_h \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ and a number $\varepsilon(h) \in (0,h)$ such that

$$\mathcal{I}^h_{\text{hom}}(g_h) + \eta \geq \mathcal{I}^{\varepsilon(h),h}(\tilde{g}_h) \quad \text{and} \quad \|g_h - \tilde{g}_h\|_{L^2(\Omega;\mathbb{R}^n)} \leq h. \tag{5.31}$$

The latter property implies that $(\tilde{g}_h)$ converges to g strongly in $L^2(\Omega;\mathbb{R}^n)$ and in view of Proposition 5.3.8 we obtain

$$\liminf_{h\to 0} \mathcal{I}^h_{\text{hom}}(g_h) \geq \liminf_{h\to 0}(\mathcal{I}^{\varepsilon(h),h}(\tilde{g}_h) - \eta) \geq \mathcal{I}(g) - \eta.$$

Because this is valid for all $\eta > 0$, the proof is complete. □

Upper bound

Lemma 5.3.9. *Let W satisfy (W4). Then there exist a closed ball $K \subset \mathbb{M}(n)$ with center 0 and a monotone map $\rho : [0,\infty) \to [0,\infty)$ with $\lim_{r\to 0}\rho(r) = \rho(0) = 0$ such that*

$$|W(y, Id + F) - Q(y,F)| \leq \rho(|F|)\,|F|^2$$

for all $F \in K$ and a.e. $y \in Y$.

The proof is obvious and omitted here.

For the sequel it is convenient to introduce the two-scale limiting functional

$$\mathcal{I}^0 : W^{1,2}(\Omega;\mathbb{R}^n)\times L^2(\Omega;W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n)) \to [0,\infty)$$
$$\mathcal{I}^0(g,\varphi) := \iint_{\Omega\times Y} Q(y, \nabla g(x) + \nabla_y\,\varphi(x,y))\,dy\,dx.$$

Proof of Theorem 5.1.2, upper bound. Let $g \in L^2(\Omega;\mathbb{R}^n)$. We only have to consider the case where $\mathcal{I}(g)$ is finite. In this case g belongs to $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ and we have

$$\mathcal{I}(g) = \mathcal{I}^0(g,\varphi)$$

for a suitable map $\varphi \in L^2(\Omega;W^{1,2}_{\text{per},0}(Y;\mathbb{R}^n))$. Because $\mathcal{I}^0$ is continuous with respect to strong convergence and due to density arguments, there exist maps

$$g_\eta \in W^{1,\infty}(\Omega;\mathbb{R}^n) \cap W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n) \quad \text{and} \quad \varphi_\eta \in C^\infty_c(\Omega;C^\infty_{\text{per}}(Y;\mathbb{R}^n))$$

such that

$$\left|\mathcal{I}^0(g_\eta,\varphi_\eta) - \mathcal{I}(g)\right| + \|g_\eta - g\|_{W^{1,2}(\Omega;\mathbb{R}^n)} + \|\varphi_\eta - \varphi\|_{L^2(\Omega;W^{1,2}(Y;\mathbb{R}^n))} < \eta. \tag{5.32}$$

Set $g_{\eta,\varepsilon}(x) := g_\eta(x) + \varepsilon\varphi_\eta(x, {}^x\!/\!_\varepsilon)$. Then $g_{\eta,\varepsilon}$ belongs to $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ and satisfies

$$\nabla g_{\eta,\varepsilon} \xrightarrow{2} \nabla g_\eta + \nabla_y\,\varphi_\eta \qquad \text{strongly two-scale in } L^2(\Omega\times Y;\mathbb{M}(n)).$$

Note that this implies

$$\lim_{\varepsilon\to 0}\int_\Omega Q\left(x/\varepsilon, \nabla g_{\eta,\varepsilon}(x)\right)\,\mathrm{d}x \to \mathcal{I}^0(g_\eta,\varphi_\eta)$$

for all η. Moreover, the sequence $(g_{\eta,\varepsilon})_\varepsilon$ uniformly converges to g. For this reason we have

$$\mathcal{I}^h_{\mathrm{hom}}(g_\eta) \le \limsup_{\varepsilon\to 0}\mathcal{I}^{\varepsilon,h}(g_{\eta,\varepsilon}) = \limsup_{\varepsilon\to 0}\frac{1}{h^2}\int_\Omega W(x/\varepsilon, Id + h\,\nabla g_{\eta,\varepsilon}(x))\,\mathrm{d}x.$$

Since $g_{\eta,\varepsilon}$ and its gradient is uniformly bounded in ε, we have

$$\sup_{x\in\Omega}|h\,\nabla g_{\eta,\varepsilon}(x)| \le hC_\eta$$

for some constant C_η independent of ε and h. This means that for h sufficiently small, $\nabla g_{\eta,\varepsilon}(x)$ belongs to the closed ball K from Lemma 5.3.9 for all $x\in\Omega$ and all ε. Thus, we can utilize the quadratic expansion of W and deduce with Lemma 5.3.9 that

$$\begin{aligned}\mathcal{I}^h_{\mathrm{hom}}(g_\eta) &\le \limsup_{\varepsilon\to 0}\frac{1}{h^2}\int_\Omega W(x/\varepsilon, Id + h\,\nabla g_{\eta,\varepsilon}(x))\,\mathrm{d}x\\ &\le \limsup_{\varepsilon\to 0}\int_\Omega Q\left(x/\varepsilon, \nabla g_{\eta,\varepsilon}\right)\,\mathrm{d}x + \rho(hC_\eta)C_\eta \le \mathcal{I}^0(g_\eta,\varphi_\eta) + \rho(hC_\eta)C_\eta^2\\ &\le \mathcal{I}(g) + \rho(hC_\eta)C_\eta^2 + \eta.\end{aligned}$$

Consequently, we arrive at

$$\limsup_{\eta\to 0}\limsup_{h\to 0}\mathcal{I}^h_{\mathrm{hom}}(g_\eta) \le \mathcal{I}(g).$$

In view of Attouch's diagonalization argument (see Lemma A.2.1), there exists a diagonal sequence $\eta(h)$ with $\lim_{h\to 0}\eta(h) = 0$ and

$$\limsup_{h\to 0}\mathcal{I}^h_{\mathrm{hom}}(g_{\eta(h)}) \le \mathcal{I}(g).$$

Set $g_h := g_{\eta(h)}$. Then g_h converges to g strongly in $W^{1,2}(\Omega;\mathbb{R}^n)$ due to (5.32) and we have

$$\mathcal{I}(g) \ge \limsup_{h\to 0}\mathcal{I}^h_{\mathrm{hom}}(g_h) \ge \liminf_{h\to 0}\mathcal{I}^h_{\mathrm{hom}}(g_h) \ge \mathcal{I}(g)$$

where the last inequality holds, because of the lower bound estimate in statement (1). □

5.3.3. Proof of Theorem 5.1.3

In this section we prove that the diagram (5.8) in Theorem 7.1.1 commutes. Therefore, it remains to show that

$$(1)\qquad \Gamma\text{-}\lim_{h\to 0}\mathcal{I}^{\varepsilon,h} = \mathcal{I}^\varepsilon_{\mathrm{lin}} \qquad\qquad (3)\qquad \Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{I}^\varepsilon_{\mathrm{lin}} = \mathcal{I}$$

hold w.r.t. strong convergence in $L^2(\Omega;\mathbb{R}^n)$.

Convergence (3) follows by standard results from convex homogenization (see e.g. Section 3.3, [FM86], [OSI84]), while convergence (1), which corresponds to linearization, follows by the theorem below.

Theorem 5.3.10. *Suppose that W satisfies (W2) and*

$$(\mathrm{W4}_\Omega)\qquad \exists\, \mathbb{L}\in L^\infty(\Omega;\mathbb{T}_{\mathrm{sym}}(n))\,:\, \limsup_{\substack{G\to 0\\ G\neq 0}}\ \operatorname*{ess\,sup}_{x\in\Omega} \frac{|W(y,Id+G)-\langle \mathbb{L}(x)G,\,G\rangle|}{|G|^2}=0,$$

and set $Q(x,F):=\langle \mathbb{L}(x)F,\,F\rangle$. Consider the functional

$$\mathcal{I}^h(g):=\begin{cases}\dfrac{1}{h^2}\displaystyle\int_\Omega W(x,Id+h\,\nabla g(x))\,\mathrm{d}x & \textit{if } g\in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)\\ +\infty & \textit{else.}\end{cases}$$

Then the family $(\mathcal{I}_h)$ Γ-converges w.r.t. strong convergence in $L^2(\Omega;\mathbb{R}^n)$ to the functional

$$\mathcal{I}_{\mathrm{lin}}(g):=\begin{cases}\dfrac{1}{h^2}\displaystyle\int_\Omega Q(x,\nabla g(x)) & \textit{if } g\in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)\\ +\infty & \textit{else.}\end{cases}$$

This result can be found in our paper [MN10] and is a slight variant of an argument used in [Per99] where linearized elasticity is derived as a Γ-limit of finite elasticity. For the sake of completeness we briefly recall the proof from [MN10]:

Proof. *Step 1.* (Upper bound). We have to prove that for any $g\in L^2(\Omega;\mathbb{R}^n)$ there exists a sequence (g_h) strongly converging to g in $L^2(\Omega;\mathbb{R}^n)$ such that

$$\lim_{h\to 0}\mathcal{I}^h(g_h)=\mathcal{I}_{\mathrm{lin}}(g).$$

In view of Proposition 5.1.4, we only have to consider the case $g\in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$. The continuity of $\mathcal{I}_{\mathrm{lin}}\big|_{W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)}$ with respect to strong convergence in $W^{1,2}(\Omega;\mathbb{R}^n)$ and the definition of the space $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ allows us to restrict the subsequent analysis to the case where $g\in W^{1,\infty}(\Omega;\mathbb{R}^n)\cap W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$. In this situation we utilize the quadratic expansion in assumption (W4) by applying Lemma 5.3.9 and obtain the estimate

$$\left|\frac{1}{h^2}\int_\Omega W(x,Id+h\,\nabla g(x))\,\mathrm{d}x-\int_\Omega Q(x,\nabla g(x))\,\mathrm{d}x\right|\le \operatorname*{ess\,sup}_{x\in\Omega}\rho(h\,|\nabla g(x)|)\int_\Omega|\nabla g|^2\,\mathrm{d}x$$

which holds whenever h is small enough. Consequently, we deduce that

$$\lim_{h\to 0}\frac{1}{h^2}\int_\Omega W(x,Id+h\,\nabla g(x))\,\mathrm{d}x=\int_\Omega Q(x,\nabla g(x))\,\mathrm{d}x.$$

Step 2. (Lower bound). Let (g_h) be a sequence in $L^2(\Omega;\mathbb{R}^n)$ with (strong) limit $g \in L^2(\Omega;\mathbb{R}^n)$. We show that

$$\liminf_{h\to 0} \mathcal{I}^h(g_h) \geq \mathcal{I}_{\text{lin}}(g).$$

As usual, we only have to consider the case where the left hand side is finite. In this case the equi-coercivity (cf. Proposition 5.1.4) implies that $g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ and enables us to pass to a subsequence (not relabeled) such that

$$g_h \rightharpoonup g \qquad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^n) \qquad \text{and} \qquad \limsup_{h\to 0} \mathcal{I}^h(g_h) = \liminf_{h\to 0} \mathcal{I}^h(g_h).$$

We define the set

$$\Omega_h := \{\, x \in \Omega \,:\, |\nabla g_h(x)| \geq h^{-1/2} \,\}$$

and can see that the sequence (H_h) with $H_h := 1_{\Omega_h} \nabla g_h$ weakly converges to ∇g (see Proposition 2.3.1). As in the proof of the lower bound part of Theorem 5.2.1, we obtain

$$\begin{aligned}\liminf_{h\to 0} \frac{1}{h^2} \int_\Omega W(x, Id + h\,\nabla g_h(x))\,\mathrm{d}x &\geq \liminf_{h\to 0} \frac{1}{h^2} \int_\Omega W(x, Id + hH_h(x))\,\mathrm{d}x \\ &\geq \int_\Omega Q(x, \nabla g(x))\,\mathrm{d}x = \mathcal{I}_{\text{lin}}(g).\end{aligned}$$

□

5.4. Proof of Theorem 5.1.5

We have to prove the following:

(1) (Lower bound). For each sequence $(g_h) \subset L^2(\Omega;\mathbb{R}^n)$ with limit g there holds

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(g_h) \geq \mathcal{I}(g). \tag{5.33}$$

(2) (Upper bound). For each map $g \in L^2(\Omega;\mathbb{R}^n)$ there exists a sequence $(g_h) \subset L^2(\Omega;\mathbb{R}^n)$ converging to g such that

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(g) = \mathcal{I}(g). \tag{5.34}$$

The lower bound part is covered by Proposition 5.3.8. For the upper bound part we proceed as we did in the proof of the upper bound part of Theorem 5.1.2. As usual it is sufficient to assume that $g \in W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$. Recalling the definition of the functional $\mathcal{I}^0$ we choose maps $g_\eta \in W^{1,\infty}(\Omega;\mathbb{R}^n) \cap W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ and $\varphi \in C^\infty_c(\Omega; C^\infty_{\text{per}}(Y;\mathbb{R}^n))$ such that

$$\mathcal{I}^0(g_\eta, \varphi_\eta) \leq \mathcal{I}(g) + \eta$$

and define

$$g_{\eta,h}(x) := g_\eta(x) + \varepsilon(h)\varphi_\eta(x, x/\varepsilon(h)).$$

Obviously, $(\nabla g_{\eta,h})$ strongly two-scale converges to $\nabla g_\eta + \nabla_y \varphi_\eta$ and satisfies

$$\lim_{h\to 0} \sup_{x\in\Omega} |h \nabla g_{\eta,h}(x)| = 0.$$

Hence, we can apply Theorem 5.2.1 and deduce that

$$\limsup_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(g_{\eta,h}) = \mathcal{I}^0(g_\eta, \varphi_\eta) \le \mathcal{I}(g) + \eta.$$

As in the proof of Theorem 5.1.2, we obtain the recovery sequence by choosing a suitable diagonal sequence. □

6. Two-scale convergence methods for slender domains

6.1. Introduction and motivation

In this chapter we introduce some two-scale convergence methods that are suited for homogenization problems in the context of thin films. In particular, we present a new characterization of two-scale cluster points that emerge from sequences of scaled gradients, i.e. sequences of vector fields in the form

$$\left(\ \partial_1 u_h \ \middle|\ \partial_2 u_h \ \middle|\ \tfrac{1}{h}\partial_3 u_h\ \right) \tag{6.1}$$

where (u_h) is a bounded sequence in $W^{1,2}(\Omega)$ with $\Omega \subset \mathbb{R}^3$. The characterization has the capability to capture lateral oscillations with respect to a given fine-scale ε and is sensitive to the limiting behavior of the ratio ${}^{h}\!/\!_{\varepsilon}$ as both fine-scales tend to zero.

Scaled gradients naturally appear in the context of thin films — as we illustrate in the following introductory example: Let $\Omega_h := \omega{\times}(hS)$ be a thin, cylindrical domain in $\mathbb{R}^3$ where $\omega \subset \mathbb{R}^2$ is open and bounded, S a bounded, one-dimensional interval and h a small positive number. We decompose each point $x \in \mathbb{R}^3$ according to

$$x = (\hat{x}, \bar{x}) \quad \text{with} \quad \hat{x} \in \mathbb{R}^2,\ \bar{x} \in \mathbb{R}$$

and consider the functional

$$\mathcal{E}^{\varepsilon,h}(v) := \frac{1}{h}\int_{\Omega_h} g({}^{\hat{x}}\!/\!_{\varepsilon}, \nabla v(x))\,\mathrm{d}x, \qquad v \in W^{1,2}(\Omega_h;\mathbb{R}^3) \tag{6.2}$$

where $g : \mathbb{R}^2{\times}\mathbb{M}(3) \to \mathbb{R}$ is a measurable integrand, $[0,1)^2 =: Y$-periodic in its first variable and ε is a small positive number. Functionals of this type are related to elastic thin-films with laterally periodic inhomogeneities. In this context $\mathcal{E}^{\varepsilon,h}$ is the elastic energy "per thickness" of a film with thickness h and a material microstructure on length scale ε. It is convenient to change coordinates according to

$$x = (\hat{x}, \bar{x}) \mapsto \left(\hat{x}, \frac{1}{h}\bar{x}\right), \qquad \Omega_h \to \Omega := \omega{\times}S.$$

This allows us to work on the fixed domain Ω and to consider the scaled, but equivalent energy

$$\mathcal{I}^{\varepsilon,h}(u) := \int_{\Omega} g({}^{\hat{x}}\!/\!_{\varepsilon}, \nabla_{2,h}\, u(x))\,\mathrm{d}x, \qquad u \in W^{1,2}(\Omega;\mathbb{R}^3)$$

where $\nabla_{2,h} u$ denotes the scaled gradient of u and is defined according to (6.1). The scaled energy is related to the initial one as follows:

$$\mathcal{E}^{\varepsilon,h}(v) = \mathcal{I}^{\varepsilon,h}(u) \quad \text{and} \quad (\nabla_{2,h} u)(x) = (\nabla v)(\hat{x}, h\bar{x}) \quad \text{for} \quad u(x) = v(\hat{x}, h\bar{x}).$$

The goal of this section is to provide two-scale convergence methods that allow to analyze the limiting behavior of $\mathcal{I}^{\varepsilon,h}$ (and related functionals) as both fine-scales simultaneously tend to zero. As we have seen in the discussion in Section 3.3, a key step in homogenization is a decent understanding of the two-scale behavior of the involved quantities. In the setting addressed here, this means to analyze the two-scale convergence behavior of sequences of the form $(\nabla_{2,h} u_h)$.

A simplified version of our main result (see Theorem 6.3.3 below) is the following: Suppose that $h/\varepsilon \to \gamma$ with $\gamma \in (0, \infty)$. We are going to show that if (u_h) is a weakly convergent sequence in $W^{1,2}(\Omega; \mathbb{R}^3)$ with limit u and the sequence of scaled gradients $(\nabla_{2,h} u_h)$ is bounded in $L^2(\Omega; \mathbb{M}(3))$, then u can be identified with a map in $W^{1,2}(\omega; \mathbb{R}^3)$ and any weak two-scale cluster point of $(\nabla_{2,h} u_h)$ can be written as a sum of the gradient of u and a scaled gradient in the form

$$\left(\; \partial_{y_1} u_0(x,y) \; \middle| \; \partial_{y_2} u_0(x,y) \; \middle| \; \tfrac{1}{\gamma} \partial_3 u_0(x,y) \; \right) \tag{6.3}$$

where u_0 denotes an additional auxiliary function that is Y-periodic with respect to its y-components. Additionally, we prove that the characterization is "sharp" in the following sense: Whenever we have a scaled gradient in the form (6.3) and a map $u \in W^{1,2}(\omega; \mathbb{R}^3)$, then we can construct a weakly convergent sequence with limit u such that the associated sequence of scaled gradients strongly two-scale converges to the sum of ∇u and the scaled gradient. We also obtain similar results for the cases $\gamma = 0$ and $\gamma = \infty$ where the fine-scales separate.

In the result above we use a slight variant of the notion of two-scale convergence that only captures oscillations in the $\hat{x}$-components. In the next section we introduce this variant and briefly elaborate on its relation to the standard notion of two-scale convergence. In Section 6.3 we prove the two-scale characterization result for scaled gradients in a more general setting, including the degenerated cases where $\gamma \in \{0, \infty\}$. In last section of this chapter we study the asymptotic behavior of the energy $\mathcal{I}^{\varepsilon,h}$ in the situation where g is convex and compute the Γ-limit as h and ε simultaneously tend to zero.

Some notation. Throughout this chapter $\mathbb{E}$ denotes a finite dimensional Euclidean space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$.

Let $n, m \in \mathbb{N}$ with $m \leq n$, set $Y := [0,1)^m$ and let Ω be an open subset of $\mathbb{R}^n$. We decompose each point $x \in \mathbb{R}^n$ according to $x = (\hat{x}, \bar{x})$ with $\hat{x} \in \mathbb{R}^m$ and $\bar{x} \in \mathbb{R}^{n-m}$, and call $\hat{x}$ the in-plane and $\bar{x}$ out-of-plane component of $x \in \mathbb{R}^n$.

6.2. Two-scale convergence suited for in-plane oscillations

In the following we present a slight variant of two-scale convergence that is suited for sequences in $L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$ which feature oscillations only in the first $m \leq n$ components.

Definition 6.2.1. For each measurable map $u : \Omega \to \mathbb{E}$ and $\varepsilon > 0$ we define

$$\mathcal{T}_\varepsilon^m u : \mathbb{R}^n \times Y \to \mathbb{E}, \qquad (\mathcal{T}_\varepsilon^m u)(x,y) = \begin{cases} u(\varepsilon\lfloor \hat{x}/\varepsilon \rfloor + \varepsilon y, \bar{x}) & \text{if } (\varepsilon\lfloor \hat{x}/\varepsilon \rfloor + \varepsilon y, \bar{x}) \in \Omega \\ 0 & \text{else.} \end{cases}$$

Lemma 6.2.2. *Let $p \in [1, \infty]$. The operator $\mathcal{T}_\varepsilon^m : L^p(\Omega;\mathbb{E}) \to L^p(\mathbb{R}^n \times Y;\mathbb{E})$ is a linear (nonsurjective) isometry.*

Proof. Let $v : \mathbb{R}^n \to \mathbb{E}$ denote the extension of u to $\mathbb{R}^n$ by zero and define

$$V(\hat{x}) := \int_{\mathbb{R}^{n-m}} |v(\hat{x}, \bar{x})|^p \, \mathrm{d}\bar{x}.$$

Then

$$\|u\|^p_{L^p(\Omega;\mathbb{E})} = \int_{\mathbb{R}^n} |v(x)|^p \, \mathrm{d}x = \int_{\mathbb{R}^m} V(\hat{x}) \, \mathrm{d}\hat{x} = \iint_{\mathbb{R}^m \times Y} V(\varepsilon\lfloor \hat{x}/\varepsilon \rfloor + \varepsilon y) \, \mathrm{d}y \, \mathrm{d}\hat{x}.$$

The last identity holds due to Lemma 9.1.3. The right hand side is equal to

$$\|\mathcal{T}_\varepsilon^m u\|^p_{L^p(\Omega \times Y;\mathbb{E})} .$$

□

Definition 6.2.3. Let $p \in [1, \infty)$ and let (ε) denote an arbitrary vanishing sequence of positive numbers. For any sequence of measurable functions $u_\varepsilon : \Omega \to \mathbb{E}$ and any measurable function $u : \Omega \times Y \to \mathbb{E}$ we say that

> (u_ε) strongly (weakly) two-scale converges to u in $L^p(\Omega \times Y;\mathbb{E})$
> (with respect to (ε)-oscillations in the first m-components)

whenever $(\mathcal{T}_\varepsilon^m u_\varepsilon)$ strongly (weakly) converges to u in $L^p(\mathbb{R}^n \times Y;\mathbb{E})$. We use the following notation

$$u_\varepsilon \xrightarrow{2} u \qquad \text{strongly two-scale in } L^p(\Omega \times Y;\mathbb{E}),$$
$$u_\varepsilon \xrightharpoonup{2} u \qquad \text{weakly two-scale in } L^p(\Omega \times Y;\mathbb{E}).$$

Remark 6.2.4. Despite the potential danger of confusion, we use the same notation for the new variant and the standard notion of two-scale convergence introduced in Section 2. The fact that the line

$$u_\varepsilon \xrightarrow{2} u \qquad \text{strongly two-scale in } L^p(\Omega \times Y;\mathbb{E})$$

means two-scale convergence with respect to oscillations in the first m-components is encoded in the dimension m of the periodicity cell Y. The notation is also justified by the trivial observation that both notions coincide if Y is the usual n-dimensional unit cube.

The following result establishes a link between both notions of two-scale convergence. In particular, it shows that the variant discussed here, is included in the standard notion of two-scale convergence:

Proposition 6.2.5. *Let $p \in (1, \infty)$. Set*

$$Z := [0,1)^n = \{ (y,z) \,:\, y \in Y,\, z \in [0,1)^{n-m} \}$$

and consider a sequence $(u_h) \subset L^p(\Omega;\mathbb{E})$.

(1) *If (u_h) weakly (strongly) two-scale converges to a function u_Z in $L^p(\Omega{\times}Z;\mathbb{E})$, then (u_h) weakly (strongly) two-scale converges to the function*

$$u(x,y) := \int_{(0,1)^{n-m}} u_Z(x,y,z)\,\mathrm{d}z$$

in $L^p(\Omega{\times}Y;\mathbb{E})$ in the sense of Definition 6.2.3.

(2) *If (u_h) weakly (strongly) two-scale converges to a function u in $L^p(\Omega{\times}Y;\mathbb{E})$ in the sense of Definition 6.2.3, then from any subsequence of (u_h) we can extract a subsequence that weakly (strongly) two-scale converges to a function u_Z in $L^p(\Omega{\times}Z;\mathbb{E})$ in the sense of Definition 4.1.2 with*

$$u(x,y) = \int_{(0,1)^{n-m}} u_Z(x,y,z)\,\mathrm{d}z.$$

Proof. (1) is trivial. In order to prove (2), first note that a weakly (strongly) two-scale convergent sequence in $L^p(\Omega{\times}Y;\mathbb{E})$ is bounded. In view of the two-scale compactness (see Proposition 2.1.4) we can pass to a subsequence (not relabeled) such that

$$u_h \xrightarrow{2} u_Z \qquad \text{weakly two-scale in } L^p(\Omega{\times}Z;\mathbb{E})$$

for a suitable map $u_Z \in L^p(\Omega{\times}Z;\mathbb{E})$; thus, (1) implies that $\int_{(0,1)^{n-m}} u_Z\,\mathrm{d}z = u$.

Now suppose that (u_ε) is even strongly two-scale convergent in $L^2(\Omega{\times}Y;\mathbb{E})$, i.e. we additionally have (see Lemma 2.1.5)

$$\|u_\varepsilon\|_{L^p(\Omega;\mathbb{E})} \to \|u\|_{L^p(\Omega\times Y;\mathbb{E})}\,.$$

Then

$$\|u\|^p_{L^p(\Omega\times Y;\mathbb{E})} = \iint_{\Omega\times Y} \left| \int_{(0,1)^{n-m}} u_Z\,\mathrm{d}z \right|^p \mathrm{d}y\,\mathrm{d}x \le \|u_Z\|^p_{L^p(\Omega\times Z;\mathbb{E})}$$
$$\le \liminf_{\varepsilon\to 0} \|u_\varepsilon\|^p_{L^p(\Omega;\mathbb{E})} = \|u\|^p_{L^p(\Omega\times Y;\mathbb{E})}\,.$$

Since $L^p(\Omega\times Z;\mathbb{E})$ is a uniformly convex Banach space, weak convergence combined with convergence of the norm yields strong convergence and we can infer that $\mathcal{T}_\varepsilon^n u_\varepsilon$ strongly converges to u_Z. □

In view of the previous proposition it is clear that all the results for two-scale convergence in the sense of Definition 4.1.2 generalize in an obvious way to the variant of two-scale convergence introduced here.

6.3. Two-scale limits of scaled gradients.

In this section we consider cylindrical domains of the form

$$\Omega = \omega\times S \qquad \text{with } \omega \subset \mathbb{R}^m \text{ and } S \subset \mathbb{R}^{n-m}$$

and assume that ω and S are open and bounded sets with Lipschitz boundary. As before we set $Y := [0,1)^m$.

Definition 6.3.1 (Scaled gradients)**.** Let $h, \gamma > 0$ and $m \in \mathbb{N}$ with $m \leq n$.

(a) For $u \in W^{1,2}(\Omega;\mathbb{E})$ we define the scaled gradient

$$\begin{aligned} \nabla_{m,h}\, u(x) &:= \left(\ \nabla_{\hat{x}}\ u(x) \ \middle|\ \tfrac{1}{h}\nabla_{\bar{x}}\, u(x) \ \right) \\ \text{where} \qquad \nabla_{\hat{x}}\ u(x) &:= \left(\ \partial_1 u(x) \ \middle|\ \cdots \ \middle|\ \partial_m u(x) \ \right), \\ \nabla_{\bar{x}}\, u(x) &:= \left(\ \partial_{m+1} u(x) \ \middle|\ \cdots \ \middle|\ \partial_n u(x) \ \right). \end{aligned}$$

(b) For $\psi \in L^2(\omega; W^{1,2}(S\times Y;\mathbb{E})$ we define

$$\begin{aligned} \widetilde{\nabla}_{m,\gamma}\psi(x,y) &:= \Big(\ \nabla_y\, \psi(x,y) \ \Big|\ \tfrac{1}{\gamma}\nabla_{\bar{x}}\, \psi(x,y) \ \Big) \\ \text{where} \qquad \nabla_y\, \psi(x,y) &:= \left(\ \partial_{y_1}\psi(x) \ \middle|\ \cdots \ \middle|\ \partial_{y_m}\psi(x) \ \right) \\ \nabla_{\bar{x}}\, \psi(x,y) &:= \left(\ \partial_{m+1}\psi(x,y) \ \middle|\ \cdots \ \middle|\ \partial_n\psi(x,y) \ \right). \end{aligned}$$

(c) We define the function space

$$W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{E}) := \Big\{ w \in W^{1,2}(S\times Y;\mathbb{E}) \ : \ w(x,y) \text{ is } Y\text{-periodic in its second variable.} \Big\}$$

We call $\widetilde{\nabla}_{m,\gamma}w$ with $w \in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{E})$ an auxiliary gradient.

Remark 6.3.2. The gradients $\nabla_{m,h}\, u(x)$ as well as $\widetilde{\nabla}_{m,\gamma}u(x)$ are "row-vectors" in $\mathbb{E}^n$. In applications we typically have $\mathbb{E} = \mathbb{R}^d$. In this case we identify the gradients above with matrices in $\mathbb{R}^{n\times d}$ as it is usual.

Theorem 6.3.3. *Let* $\varepsilon : (0,\infty) \to (0,\infty)$ *satisfy*

$$\lim_{h\to 0} \varepsilon(h) = 0 \quad \text{and} \quad \lim_{h\to 0} \frac{h}{\varepsilon(h)} = \gamma \quad \text{with} \quad \gamma \in [0,\infty].$$

Let (u_h) *be a weakly convergent sequence in* $W^{1,2}(\Omega;\mathbb{E})$ *with limit* u *and suppose that*

$$\limsup_{h\to 0} \int_\Omega |\nabla_{m,h}\, u_h|^2 \,\mathrm{d}x < \infty.$$

Then u *is independent of* $\bar{x}$ *and can be identified with a map in* $W^{1,2}(\omega;\mathbb{E})$*. Moreover:*

(1) Let $\gamma \in \{0,\infty\}$*. Then there exist maps*

$$\begin{cases} u_0 \in L^2(\omega; W^{1,2}_{\text{per},0}(Y;\mathbb{E})) \text{ and } \bar{u} \in L^2(\omega\times Y; W^{1,2}(S;\mathbb{E})) & \text{if } \gamma = 0 \\ u_0 \in L^2(\Omega; W^{1,2}_{\text{per},0}(Y;\mathbb{E})) \text{ and } \bar{u} \in L^2(\omega; W^{1,2}(S;\mathbb{E})) & \text{if } \gamma = \infty \end{cases}$$

and a subsequence (not relabeled) such that

$$\nabla_{m,h}\, u_h \xrightarrow{2} \left(\nabla_{\hat{x}}\, u + \nabla_y\, u_0 \;\middle|\; \nabla_{\bar{x}}\, \bar{u} \right)$$

weakly two-scale in $L^2(\Omega\times Y;\mathbb{E}^n)$ *with respect to* $(\varepsilon(h))$*-oscillations in the first* m*-components (see Definition 6.2.3).*

(2) Let $\gamma \in (0,\infty)$*. Then there exist a map*

$$w_0 \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{E}))$$

and a subsequence (not relabeled) such that

$$\nabla_{m,h}\, u_h \xrightarrow{2} \left(\nabla_{\hat{x}}\, u(\hat{x}) \;\middle|\; 0 \right) + \widetilde{\nabla}_{m,\gamma} w_0(x,y)$$

weakly two-scale in $L^2(\Omega\times Y;\mathbb{E}^n)$ *with respect to* $(\varepsilon(h))$*-oscillations in the first* m*-components (see Definition 6.2.3).*

Proof. For brevity we always write ε instead of $\varepsilon(h)$. For $x \in \mathbb{R}^n$ and $y \in Y$ we use the notation

$$x = (x_1, \cdots, x_n) \quad \text{and} \quad y = (y_1, \cdots, y_m).$$

Moreover, ∂_k and ∂_{y_k} refer to the derivative with respect to the coordinate x_k and y_k, respectively. We denote the standard inner product in $\mathbb{R}^k$, $k \in \mathbb{N}$, by $\langle a,\, b\rangle$. We only consider the case $\mathbb{E} = \mathbb{R}$. The case for a general d-dimensional Euclidean space is then recovered by applying the subsequent analysis to each of the d components separately. Additionally, we assume without loss of generality that $\mathcal{H}^{n-m}(S) = 1$.

<u>*Step 1.*</u> We define the maps

$$\hat{u}_h(\hat{x}) := \int_S u_h(\hat{x},\bar{x})\,\mathrm{d}\bar{x} \quad \text{and} \quad \mathring{u}_h(x) := u_h(x) - \hat{u}_h(\hat{x}).$$

Each map $\mathring{u}_h$ has vanishing mean value (w.r.t. $\bar{x}$). Thus, we can apply Poincaré's inequality and deduce that

$$\int_S |\mathring{u}_h|^2 \, \mathrm{d}\bar{x} \leq c' \int_S |\nabla_{\bar{x}} \mathring{u}_h|^2 \, \mathrm{d}\bar{x} = c' h^2 \int_S \left|\tfrac{1}{h} \nabla_{\bar{x}} u_h\right|^2 \, \mathrm{d}\bar{x}$$

where $c' > 0$ only depends on the geometry of S. Integration over ω on both sides leads to the estimate

$$\limsup_{h\to 0} \frac{1}{h^2} \int_\Omega |\mathring{u}_h|^2 \, \mathrm{d}x \leq c' \limsup_{h\to 0} \int_\Omega |\nabla_{m,h} u_h|^2 \, \mathrm{d}x. \tag{6.4}$$

By assumption, the right hand side is finite, and therefore $\mathring{u}_h$ strongly converges to 0 in $L^2(\Omega)$. On the other side, the sequence $(\hat{u}_h)$ weakly converges to $\int_S u \, \mathrm{d}\bar{x}$ in $W^{1,2}(\omega)$. Because $u_h = \mathring{u}_h + \hat{u}_h$ and since $(\mathring{u}_h)$ vanishes in the limit, we deduce that $u = \int_S u \, \mathrm{d}\bar{x}$ and the first statement follows.

Step 2. We claim that there exist maps

$$\hat{u}_0 \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y)), \qquad \mathring{u}_0 \in L^2(\Omega; W^{1,2}_{\mathrm{per},0}(Y)) \text{ with } \int_S \mathring{u}_0(x,y) \, \mathrm{d}\bar{x} = 0$$
$$\text{and } \bar{u} \in L^2(\omega\times Y; W^{1,2}(S))$$

such that

$$\nabla_{\hat{x}} \hat{u}_h \xrightarrow{2} \nabla_{\hat{x}} u(\hat{x}) + \nabla_y \hat{u}_0(\hat{x}, y) \qquad \text{weakly two-scale in } L^2(\omega\times Y; \mathbb{R}^m) \tag{6.5}$$

$$\nabla_{\hat{x}} \mathring{u}_h \xrightarrow{2} \nabla_y \mathring{u}_0(x, y) \qquad \text{weakly two-scale in } L^2(\Omega\times Y; \mathbb{R}^m) \tag{6.6}$$

$$\frac{1}{h} \nabla_{\bar{x}} u_h \xrightarrow{2} \nabla_{\bar{x}} \bar{u}(x, y) \qquad \text{weakly two-scale in } L^2(\Omega\times Y; \mathbb{R}^{n-m}) \tag{6.7}$$

for a subsequence. (6.5) and (6.6) follow from Proposition 2.1.14.

In order to prove (6.7) consider a test function $\Psi \in C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Y; \mathbb{R}^{n-m})$ with

$$\mathrm{div}_{\bar{x}} \, \Psi(x,y) := \sum_{k=1}^{n-m} \partial_{m+k} \Psi_k(x,y) = 0.$$

Since $(\frac{1}{h} \nabla_{\bar{x}} u_h)_h$ is bounded in $L^2(\Omega; \mathbb{R}^{n-m})$ (at least for a subsequence), we have

$$\frac{1}{h} \nabla_{\bar{x}} u_h \xrightarrow{2} U \qquad \text{weakly two-scale in } L^2(\Omega\times Y; \mathbb{R}^{n-m})$$

for a suitable map $U \in L^2(\Omega\times Y; \mathbb{R}^{n-m})$ and a subsequence (not relabeled). With partial integration we find that

$$\frac{1}{h} \int_\Omega \langle \nabla_{\bar{x}} u_h(x), \, \Psi(x, \hat{x}/\varepsilon) \rangle \, \mathrm{d}x = -\frac{1}{h} \int_\Omega u_h(x) \, (\mathrm{div}_{\bar{x}} \, \Psi)(x, \hat{x}/\varepsilon) \, \mathrm{d}x = 0.$$

On the left we can pass to the limit $h \to 0$ by means of two-scale convergence and obtain

$$\iint_{\Omega\times Y} \langle U(x,y),\, \Psi(x,y)\rangle \,\mathrm{d}y\,\mathrm{d}x = 0$$
$$\text{for all } \Psi \in C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Y;\mathbb{R}^{n-m}) \text{ with } \operatorname{div}_{\bar{x}} \Psi = 0.$$

In particular, we have for almost every $\hat{x} \in \omega$ and $y \in Y$

$$\int_S \langle U(\hat{x},\bar{x},y),\, \Psi(\bar{x})\rangle \,\mathrm{d}\bar{x} = 0 \qquad \text{for all } \Psi \in \mathcal{V}(S) := \{\, \Psi \in C_c^\infty(S;\mathbb{R}^n) \,:\, \operatorname{div}\, \Psi = 0 \,\}.$$

Hence, Lemma 6.3.4 implies that U is a gradient of a map in $L^2(\omega\times Y; W^{1,2}(S))$.

Step 3. We consider the case $\gamma = 0$, i.e. $h/\varepsilon \to 0$. It remains to prove that

$$\nabla_y\, \mathring{u}_0 = 0. \tag{6.8}$$

Let $\Psi \in C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Y;\mathbb{R}^m))$. Then partial integration yields

$$\int_\Omega \langle \nabla_{\hat{x}}\, \mathring{u}_h(x),\, \Psi(x, \hat{x}/\varepsilon)\rangle \,\mathrm{d}x = -\int_\Omega \Big\langle \mathring{u}_h(x),\, \Big((\operatorname{div}_{\hat{x}} \Psi)(x,\hat{x}/\varepsilon) + \frac{1}{\varepsilon}(\operatorname{div}_y \Psi)(x,\hat{x}/\varepsilon) \Big)\Big\rangle \,\mathrm{d}x \tag{6.9}$$

The modulus of the right hand side is bounded by

$$C_\Psi \,\frac{1}{\varepsilon}\, \|\mathring{u}_h\|_{L^2(\Omega;\mathbb{R}^n)}$$

where C_Ψ is a positive constant that only depends on the test function Ψ. Hence, in view of estimate (6.4) and due to the assumption that $h/\varepsilon(h) \to 0$, we find that

$$\int_\Omega \langle \nabla_{\hat{x}}\, \mathring{u}_h(x),\, \Psi(x,\hat{x}/\varepsilon)\rangle \,\mathrm{d}x \to 0.$$

On the other hand, the integral on the left hand side in (6.9) converges to

$$\iint_{\Omega\times Y} \langle \nabla_y\, \mathring{u}_0(x,y),\, \Psi(x,y)\rangle \,\mathrm{d}x\,\mathrm{d}y$$

which must be equal to zero. Hence, identity (6.8) follows, because Ψ is an arbitrary function in the space $C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Y;\mathbb{R}^m))$ which is dense in $L^2(\Omega\times Y;\mathbb{R}^m)$.

Step 4. We consider the case $\gamma = \infty$, i.e. $\varepsilon/h \to 0$. We only have to prove that we can choose the map $\bar{u}$ independent of $y \in Y$. To this end, let k and l be indices with $m < k \le n$ and $1 \le l \le m$. For clarity we set $\pi_h(x) := (x, \hat{x}/\varepsilon(h))$. Let $\psi \in C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Y))$ and consider the integral

$$\int_\Omega \Big\langle \frac{1}{h}\partial_k u_h,\, \varepsilon\partial_l(\psi\circ\pi_h)\Big\rangle \,\mathrm{d}x. \tag{6.10}$$

The sequence $(\frac{1}{h}\partial_k u_h)$ is weakly two-scale convergent to $\partial_k \bar{u}$, while $(\varepsilon\partial_l(\psi\circ\pi_h))$ strongly two-scale converges to $\partial_{y_l}\psi(x,y)$ (see Proposition 2.1.16). Hence, we can pass to the limit in (6.10) by means of Proposition 2.1.12 and deduce

$$\int_\Omega \left\langle \frac{1}{h}\partial_k u_h,\, \varepsilon\partial_l(\psi\circ\pi_h)\right\rangle \mathrm{d}x \to \iint_{\Omega\times Y} \langle \partial_k\bar{u}(x,y),\, \partial_{y_l}\psi(x,y)\rangle \,\mathrm{d}y\,\mathrm{d}x. \tag{6.11}$$

On the other hand, we can interchange the derivatives in (6.10) by partial integration and get

$$\int_\Omega \left\langle \frac{1}{h}\partial_k u_h,\, \varepsilon\partial_l(\psi\circ\pi_h)\right\rangle \mathrm{d}x = \frac{\varepsilon}{h}\int_\Omega \langle \partial_l u_h,\, \partial_k(\psi\circ\pi_h)\rangle\,\mathrm{d}x.$$

In contrast to ∂_l, ∂_k is a derivative with respect to an "out-of-plane" direction. Hence, the modulus of the integral on the right hand side is bounded by

$$\frac{\varepsilon}{h}C_\Psi \left\|\nabla_{m,h}\, u_h\right\|_{L^2(\Omega;\mathbb{R}^n)},$$

where C_Ψ is a constant that is independent of h. Because of $\frac{\varepsilon}{h}\to 0$, the previous estimate and (6.11) yield

$$\iint_{\Omega\times Y} \langle \partial_k\bar{u}(x,y),\, \partial_{y_l}\psi(x,y)\rangle\,\mathrm{d}y\,\mathrm{d}x = 0 \qquad \text{for all } \psi\in C_c^\infty(\Omega;C^\infty_{\mathrm{per}}(Y)).$$

This implies that the map $\partial_k\bar{u}(x,y)$ is independent of y_l (the lth component of $y\in Y$). The previous reasoning holds for all $m<k\leq n$ and $1\leq l\leq m$; thus, we see that $\nabla_{\bar{x}}\,\bar{u}$ is independent of $y\in Y$, and in particular we have

$$\nabla_{\bar{x}}\,\bar{u} = \nabla_{\bar{x}}\left(\int_Y \bar{u}(\cdot,y)\,\mathrm{d}y\right),$$

which completes the proof in the case $\varepsilon/h\to 0$.

<u>*Step 5.*</u> We prove the statement in the case $h/\varepsilon\to\gamma\in(0,\infty)$. To this end we consider test functions

$$\Psi = \left(\ \Psi_1 \,\middle|\, \cdots \,\middle|\, \Psi_n\ \right) \in C_c^\infty(\Omega;C^\infty_{\mathrm{per}}(Y;\mathbb{R}^n)) \qquad \text{with} \qquad \widetilde{\mathrm{div}}_\gamma\Psi = 0 \tag{6.12}$$

where

$$\widetilde{\mathrm{div}}_\gamma\Psi(x,y) = \sum_{k=1}^m \partial_{y_k}\Psi_k(x,y) + \frac{1}{\gamma}\sum_{k=m+1}^n \partial_k\Psi_k(x,y).$$

Note that $\widetilde{\mathrm{div}}_\gamma$ is the sum of the divergence with respect to the fast variable y and the divergence with respect to $\bar{x}$ (the out-of-plane component of x) scaled by γ^{-1}. Moreover, we use the convention to decompose Ψ according to

$$\Psi = \left(\ \widehat{\Psi} \,\middle|\, \bar{\Psi}\ \right)$$

with

$$\widehat{\Psi} = \left(\ \Psi_1 \,\middle|\, \cdots \,\middle|\, \Psi_m\ \right) \quad \text{and} \quad \bar{\Psi} = \left(\ \Psi_{m+1} \,\middle|\, \cdots \,\middle|\, \Psi_n\ \right).$$

Step 5a. We pass to a subsequence (not relabeled) with

$$\nabla_{m,h}\,\mathring{u}_h \xrightarrow{\;2\;} U \qquad \text{weakly two-scale in } L^2(\Omega\times Y;\mathbb{R}^n)$$

where $U \in L^2(\Omega\times Y;\mathbb{R}^n)$ and claim that

$$\iint_{\Omega\times Y} \langle U(x,y),\, \Psi(x,y)\rangle \,\mathrm{d}y\,\mathrm{d}x = 0 \qquad \text{for all } \Psi \text{ as in (6.12).} \tag{6.13}$$

This can be seen as follows. Set $\pi_h(x) := (x, \hat{x}/\varepsilon(h))$. Then (by partial integration) we obtain

$$\begin{aligned}(6.14)\quad &\int_\Omega \langle \nabla_{m,h}\,\mathring{u}_h,\, \Psi\circ\pi_h\rangle \,\mathrm{d}x \\ &= -\int_\Omega \Big\langle \mathring{u}_h,\, (\mathrm{div}_{\hat{x}}\,\widehat{\Psi})\circ\pi_h\Big\rangle \,\mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega \Big\langle \mathring{u}_h,\, \Big(\mathrm{div}_y\,\widehat{\Psi} + \frac{\varepsilon}{h}\,\mathrm{div}_{\bar{x}}\,\bar{\Psi}\Big)\circ\pi_h\Big\rangle \,\mathrm{d}x\end{aligned}$$

The first integral on the right hand side converges to zero because of estimate (6.4). Due to $\widetilde{\mathrm{div}}_\gamma\Psi = 0$, we can rewrite the second integral on the right hand side according to

$$\frac{1}{\varepsilon}\int_\Omega \Big\langle \mathring{u}_h,\, \Big(\mathrm{div}_y\,\widehat{\Psi} + \frac{\varepsilon}{h}\,\mathrm{div}_{\bar{x}}\,\bar{\Psi}\Big)\circ\pi_h\Big\rangle \,\mathrm{d}x = \tfrac{h}{\varepsilon}\left(\tfrac{\varepsilon}{h} - \gamma^{-1}\right)\int_\Omega \Big\langle \frac{1}{h}\mathring{u}_h,\, (\mathrm{div}_{\bar{x}}\,\bar{\Psi})\circ\pi_h\Big\rangle \,\mathrm{d}x.$$

Now estimate (6.4) implies that the lim sup of the integral on the right is bounded. Moreover, its prefactor converges to zero because of $h/\varepsilon \to \gamma$. Hence, the integral on the left hand side of (6.14) converges to zero. On the other hand, it converges to $\iint_{\Omega\times Y} U(x,y)\Psi(x,y)\,\mathrm{d}y\,\mathrm{d}x$ by means of two-scale convergence and (6.13) is proved.

Step 5b. Equation (6.13) and the change of coordinates

$$V(x,y) := U(\hat{x}, \tfrac{1}{\gamma}\bar{x}, y)$$

lead to the formula

$$\iint_{(\gamma S)\times Y} \langle V(\hat{x},\bar{x},y),\, \Psi(\bar{x},y)\rangle \,\mathrm{d}y\,\mathrm{d}\bar{x} = 0 \tag{6.15}$$

which is valid for almost every $\hat{x}\in\omega$ and for all (scaled) test functions

$$\Psi \in C_c^\infty(\gamma S; C^\infty_{\mathrm{per}}(Y;\mathbb{R}^n)) \quad \text{with} \quad \mathrm{div}_y\,\widehat{\Psi} + \mathrm{div}_{\bar{x}}\,\bar{\Psi} = 0.$$

In particular, test functions in the space $\mathcal{V}(\gamma S\times Y)$ (defined in Lemma 6.3.4) are admissible and we deduce with Lemma 6.3.4 that there exists a map w in the space $L^2(\omega; W^{1,2}((\gamma S)\times Y))$ with

$$V(x,y) = \widetilde{\nabla} w(x,y) := \big(\ \nabla_y\, w(x,y)\ \big|\ \nabla_{\bar{x}}\, w(x,y)\ \big).$$

Hence, the retransformation to $\Omega\times Y$

$$w_0(x,y) := w(\hat{x}, \gamma\bar{x}, y)$$

is a map in $L^2(\omega; W^{1,2}(S\times Y))$ that satisfies $\widetilde{\nabla}_{m,\gamma}\, w_0 = U$. It remains to prove that $w_0(x,y)$ is Y-periodic in its y-component. To this end, we consider the test function

$$\Psi(x,y) := \big(\ \psi(x,y)\ \big|\ 0\ \big|\ \cdots\ \big|\ 0\ \big)$$

where $\psi(x,y) \in C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Y))$ is independent of y_1 (i.e. $\partial_{y_1}\psi = 0$). Due to (6.13) and $U = \widetilde{\nabla}_{m,\gamma}\, w_0$, we have

$$0 = \iint_{\Omega\times Y} \partial_{y_1} w_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and partial integration yields the equation

$$0 = \int_\Omega \int_{(0,1)^{m-1}} \big(\, w_0(x,1,\tilde{y}) - w_0(x,0,\tilde{y})\,\big)\psi(x,\tilde{y})\,\mathrm{d}\tilde{y}\,\mathrm{d}x \qquad \text{with} \qquad \tilde{y} := (y_2, ..., y_m).$$

This implies that $w(x,y)$ is $(0,1)$-periodic with respect to the component y_1. The same reasoning yields periodicity of w with respect to y_k for all $k \in \{1,...,m\}$ and the proof is complete. □

Lemma 6.3.4 (See Theorem 3.4 in Girault and Raviart [GR79])**.** *Let A be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Define*

$$\mathcal{V}(A) := \{\, \Psi \in C_c^\infty(A;\mathbb{R}^n)\ :\ \operatorname{div}\, \Psi = 0\,\}.$$

Let $\mathcal{H}(A)$ denote the closure of $\mathcal{V}(A)$ in $L^2(A;\mathbb{R}^n)$ and $\mathcal{H}^\perp(A)$ the orthogonal complement of $\mathcal{H}(A)$ in $L^2(A;\mathbb{R}^n)$. Then

$$L^2(A;\mathbb{R}^n) = \mathcal{H}(A) \oplus \mathcal{H}^\perp(A) \qquad \textit{and} \qquad \mathcal{H}^\perp(A) = \{\, \nabla u\ :\ u \in W^{1,2}(A)\,\}.$$

In particular, if $U \in L^2(A;\mathbb{R}^n)$ satisfies

$$\int_A \langle U(x),\ \Psi(x)\rangle\ \mathrm{d}x = 0 \qquad \textit{for all } \Psi \in \mathcal{V}(A),$$

then $U = \nabla u$ for a suitable map in $W^{1,2}(A)$.

6.3.1. Recovery sequences for auxiliary gradients

Proposition 6.3.5. *Let $\varepsilon : (0,\infty) \to (0,\infty)$ satisfy*

$$\lim_{h\to 0} \varepsilon(h) = 0 \qquad \textit{and} \qquad \lim_{h\to 0} \frac{h}{\varepsilon(h)} = \gamma \qquad \textit{with} \qquad \gamma \in [0,\infty].$$

(1) Let $\gamma \in \{0,\infty\}$, $u \in W^{1,2}(\omega;\mathbb{E})$ and consider maps

$$\begin{cases} u_0 \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y;\mathbb{E}))\ \textit{and}\ \bar{u} \in L^2(\omega\times Y; W^{1,2}(S;\mathbb{E})) & \textit{if } \gamma = 0 \\ u_0 \in L^2(\Omega; W^{1,2}_{\mathrm{per},0}(Y;\mathbb{E}))\ \textit{and}\ \bar{u} \in L^2(\omega; W^{1,2}(S;\mathbb{E})) & \textit{if } \gamma = \infty \end{cases}$$

Then there exists a weakly convergent sequence (u_h) *in* $W^{1,2}(\Omega;\mathbb{E})$ *with limit* u *such that*

$$\nabla_{m,h}\, u_h \xrightarrow{2} \left(\ \nabla_{\hat{x}}\ u + \nabla_y\ u_0 \ \middle|\ \nabla_{\bar{x}}\,\bar{u}\ \right)$$

strongly two-scale in $L^2(\Omega{\times}Y;\mathbb{E}^n)$ *with respect to* $(\varepsilon(h))$*-oscillations in the first* m*-components.*

(2) *Let* $\gamma \in (0,\infty)$, $u \in W^{1,2}(\omega;\mathbb{E})$ *and*

$$w_0 \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S{\times}Y;\mathbb{E})).$$

Then there exists a weakly convergent sequence (u_h) *in* $W^{1,2}(\Omega;\mathbb{E})$ *with limit* u *such that*

$$\nabla_{m,h}\, u_h \xrightarrow{2} \left(\ \nabla_{\hat{x}}\, u(\hat{x}) \ \middle|\ 0\ \right) + \widetilde{\nabla}_{m,\gamma} w_0(x,y)$$

strongly two-scale in $L^2(\Omega{\times}Y;\mathbb{E}^n)$ *with respect to* $(\varepsilon(h))$*-oscillations in the first* m*-components.*

Proof. For brevity we always write ε instead of $\varepsilon(h)$ and only consider the case $\mathbb{E} = \mathbb{R}$.

Step 1. We consider the case $\gamma \in \{0,\infty\}$. Let u, u_0 and $\bar{u}$ be given according to the proposition. By density arguments, we can find for each $\delta > 0$ maps

$$\begin{cases} u_0^{(\delta)} \in C_c^\infty(\omega; C^\infty_{\text{per}}(Y)) \text{ and } \bar{u}^{(\delta)} \in C_c^\infty(\omega{\times}Y; C^\infty(\overline{S})) & \text{if } \gamma = 0 \\ u_0^{(\delta)} \in C_c^\infty(\Omega; C^\infty_{\text{per}}(Y)) \text{ and } \bar{u}^{(\delta)} \in C_c^\infty(\omega; C^\infty(\overline{S})) & \text{if } \gamma = \infty \end{cases}$$

such that

$$\left\| \nabla_y\ u_0^{(\delta)} - \nabla_y\, u_0 \right\|_{L^2(\Omega\times Y;\mathbb{R}^m)} + \left\| \nabla_{\bar{x}}\ \bar{u}^{(\delta)} - \nabla_{\bar{x}}\ \bar{u} \right\|_{L^2(\Omega\times Y;\mathbb{R}^{n-m})} \leq \delta. \tag{6.16}$$

Define the doubly indexed sequence

$$u_{\delta,h}(x) := u(\hat{x}) + \varepsilon u_0^{(\delta)}(x, \hat{x}/\varepsilon) + h\bar{u}^{(\delta)}(x, \hat{x}/\varepsilon).$$

desired Then for each $\delta > 0$, the sequence $(u_{\delta,h})$ belongs to $W^{1,2}(\Omega)$ and strongly converges to u in $L^2(\Omega)$ as $h \to 0$. Moreover, it is easy to check that

$$\nabla_{m,h}\, u_{\delta,h} \xrightarrow{2} \left(\ \nabla_{\hat{x}}\ u + \nabla_y\ u_0^{(\delta)} \ \middle|\ \nabla_{\bar{x}}\,\bar{u}^{(\delta)}\ \right)$$

strongly two-scale in $L^2(\Omega{\times}Y;\mathbb{R}^n)$ as $h \to 0$ (see Lemma 2.1.9 and Proposition 2.1.16). Our aim is to construct the desired sequence by selecting a suitable diagonal sequence. To this end, let

$$U := \left(\ \nabla_{\hat{x}}\ u + \nabla_y\ u_0 \ \middle|\ \nabla_{\bar{x}}\,\bar{u}\ \right), \qquad U^{(\delta)} := \left(\ \nabla_{\hat{x}}\ u + \nabla_y\ u_0^{(\delta)} \ \middle|\ \nabla_{\bar{x}}\,\bar{u}^{(\delta)}\ \right)$$

and extend both maps to the domain $\mathbb{R}^n{\times}Y$ by zero. Set

$$c_{\delta,h} := \left\| u - u_{\delta,h} \right\|_{L^2(\Omega)} + \left\| \mathcal{T}_\varepsilon^m(\nabla_{m,h}\ u_{\delta,h}) - U \right\|_{L^2(\mathbb{R}^n\times Y;\mathbb{R}^n)}.$$

Then we have

$$0 \le \limsup_{h\to 0} c_{\delta,h} \le \left\| U^{(\delta)} - U \right\|_{L^2(\Omega\times Y;\mathbb{R}^n)}$$

and in view of (6.16) we deduce that

$$\limsup_{\delta\to 0} \limsup_{h\to 0} c_{\delta,h} = 0.$$

This allows us to apply Attouch's diagonalization argument (see Lemma A.2.1) and we see that there exists a diagonal sequence $\delta(h)$ with $\delta(h) \to 0$ and $c_{\delta(h),h} \to 0$ as h tends to zero. This implies that the sequence

$$u_h := u_{\delta(h),h}$$

converges to u strongly in $L^2(\Omega)$ and that the corresponding sequence of scaled gradients strongly two-scale converges to the desired limit. Because the boundedness of $(\nabla_{m,h} u_h)$ implies also boundedness of (∇u_h) as a sequence in $L^2(\Omega;\mathbb{R}^n)$, we eventually find that (u_h) is also weakly convergent to u in $W^{1,2}(\Omega)$.

Step 2. The proof for $\gamma \in (0,\infty)$ is quite similar to the previous one. Let u and w_0 be given according to the proposition. For each $\delta > 0$ choose a map

$$w_0^{(\delta)} \in C_c^\infty(\omega; C^\infty(\overline{S}; C^\infty_{\text{per}}(Y)))$$

with

$$\left\| \widetilde{\nabla}_{m,\gamma}\, w_0^{(\delta)} - \widetilde{\nabla}_{m,\gamma}\, w_0 \right\|_{L^2(\Omega\times Y;\mathbb{R}^n)} < \delta$$

and define

$$u_{\delta,h}(x) := u(\hat{x}) + \varepsilon w_0^{(\delta)}(x, \hat{x}/\varepsilon).$$

By a reasoning similar to Step 1, one can show that the claimed sequence can be recovered by selecting a suitable diagonal sequence $(u_{\delta(h),h})$. □

Remark 6.3.6. A close look to the proof of the previous proposition reveals that the constructed sequences $(u_h) \subset W^{1,2}(\Omega)$ with $u_h \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$ satisfy the boundary condition

$$u_h - u \in \{\, v \in W^{1,2}(\Omega) \,:\, v\big|_{(\partial\omega)\times S} = 0 \,\}.$$

6.3.2. A Korn inequality for the space of auxiliary gradients

Define

$$\mathcal{V} := \left\{ u \in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^n) \,:\, \iint_{S\times Y} u \,\mathrm{d}y\,\mathrm{d}\bar{x} = 0 \right\}.$$

Let $\gamma \in (0,\infty)$ and consider a weakly converging sequence (u_h) in $W^{1,2}(\Omega;\mathbb{R}^n)$ with limit u. Theorem 6.3.3 revealed that whenever the sequence of scaled gradients

$(\nabla_{m,h}\, u_h)$ weakly two-scale converges in $L^2(\Omega{\times}Y;\mathbb{M}(n))$, then the two-scale limit of the scaled gradient can be written in the form

$$\left(\ \nabla_{\hat{x}}\, u(\hat{x}) \ \middle|\ 0\ \right) + \widetilde{\nabla}_{m,\gamma} u_0(x,y)$$

where the auxiliary map u_0 belongs to the space $L^2(\omega;\mathcal{V})$. In the following we prove that an inequality of Korn-type holds in the space $\mathcal{V}$. To this end, we utilize the standard Korn inequality in the following version (e.g. see [CC05])

Theorem 6.3.7. *Let $U \subset \mathbb{R}^n$ be an open, bounded domain with Lipschitz boundary. Set*

$$\mathcal{R} := \left\{ r \in W^{1,2}(U;\mathbb{R}^n) \,:\, r(x) = Ax + b \ \textit{ with } \ A \in \mathbb{M}_{\text{skew}}(n),\, b \in \mathbb{R}^n \right\}.$$

Then there exists a constant C_U that only depends on the domain U such that

$$\inf_{r \in \mathcal{R}} \|u - r\|_{W^{1,2}(U;\mathbb{R}^n)} \leq C_U \,\|\mathrm{sym}\, \nabla u\|_{L^2(U;\mathbb{R}^n)}$$

for all $u \in W^{1,2}(U;\mathbb{R}^n)$.

In other words, the previous result states that the seminorm $u \mapsto \|\mathrm{sym}\,\nabla u\|_{L^2(U;\mathbb{R}^n)}$ is a norm on the quotient space $W^{1,2}(U;\mathbb{R}^n)$ modulus $\mathcal{R}$. The following theorem adapts the Korn inequality to functions in $\mathcal{V}$ in situations where the gradient is replaced by a scaled gradient:

Theorem 6.3.8. *There exists a constant c_γ such that for all*

$$u \in \mathcal{V} := \left\{ u \in W^{1,2}_{Y\text{-per}}(S{\times}Y;\mathbb{R}^n) \,:\, \iint_{S\times Y} u \,\mathrm{d}y\, \mathrm{d}\bar{x} = 0 \right\}$$

the estimate

$$\inf_{r \in \mathcal{R}(m)} \|u + r\|_{W^{1,2}(S\times Y;\mathbb{R}^n)} \leq c_\gamma \left\| \mathrm{sym}\, \widetilde{\nabla}_{m,\gamma} u \right\|_{L^2(S\times Y;\mathbb{R}^n)} \leq c_\gamma^2 \,\|u\|_{W^{1,2}(S\times Y;\mathbb{R}^n)}$$

is valid, where

$$\mathcal{R}(m) \ := \ \left\{ \bar{r} \ \in \ \mathcal{V} \ : \ \bar{r}(\bar{y},x) \ = \ \begin{pmatrix} \mathbf{0}_m \\ \bar{A}(\bar{x} - c_S) \end{pmatrix} \textit{ with } \bar{A} \ \in \ \mathbb{M}_{\text{skew}}(n{-}m) \right\}.$$

Above $\mathbf{0}_m$ denotes the zero vector in $\mathbb{R}^m$ and $c_S := \frac{1}{\mathcal{H}^{n-m}(S)} \int_S \bar{x}\,\mathrm{d}\bar{x}$.

Proof. Without loss of generality we assume that $\mathcal{H}^{n-m}(S) = 1$. We only have to prove the estimate

$$\inf_{r \in \mathcal{R}(m)} \|u + r\|_{W^{1,2}(S\times Y;\mathbb{R}^n)} \leq c_\gamma \left\| \mathrm{sym}\, \widetilde{\nabla}_{m,\gamma} u \right\|_{L^2(S\times Y;\mathbb{R}^n)}.$$

For convenience we set $Z := Y \times S$, $z := (y, \bar{x})$ and $c_Z := \fint_Z z \,\mathrm{d}z$.

Step 1. The set

$$\mathcal{R} := \left\{ r \in W^{1,2}(Z;\mathbb{R}^n) \,:\, r(z) = A(z - c_Z) + b \text{ with } A \in \mathbb{M}_{\text{skew}}(n),\, b \in \mathbb{R}^n \right\}.$$

is a finite dimensional (and therefore closed) subspace of the Hilbert space $W^{1,2}(Z;\mathbb{R}^n)$. Note that the linear space $\mathcal{R}(m)$ is contained in $\mathcal{R}$. We decompose each matrix A in $\mathbb{M}_{\text{skew}}(n)$ according to

$$\bar{A} := \sum_{i,j=m+1}^{n} A_{\{i,j\}}(e_i \otimes e_j) \quad \text{and} \quad \widehat{A} := A - \bar{A}.$$

Thus, $\bar{A}$ is the "lower-right" $(n-m)\times(n-m)$-sub matrix of A and can be identified with a matrix in $\mathbb{M}_{\text{skew}}(n-m)$. Note that

$$Az = \widehat{A}z + \begin{pmatrix} \mathbf{0}_m \\ \bar{A}\bar{x} \end{pmatrix} \qquad \text{where } z = (y, \bar{x}).$$

Now it is easy to check that

$$\mathcal{R}^\star(m) := \left\{ r \in \mathcal{R} \,:\, \hat{r}(y,\bar{x}) = \widehat{A}(z - c_Z) + b \text{ with } A \in \mathbb{M}_{\text{skew}}(n),\, b \in \mathbb{R}^n \right\}.$$

is the orthogonal complement of $\mathcal{R}(m)$ in $\mathcal{R}$ with respect to the inner product in $W^{1,2}(Z;\mathbb{R}^n)$, i.e. $\mathcal{R} = \mathcal{R}(m) \oplus \mathcal{R}^\star(m)$.

Step 2. We prove the case $\gamma = 1$. The standard Korn inequality says that there exists a constant C_Z that only depends on the geometry of S and Y so that for every $u \in W^{1,2}(Z;\mathbb{R}^n)$ we have

$$\inf_{r\in\mathcal{R}} \|u - r\|_{W^{1,2}(Z;\mathbb{R}^n)} \le C_Z \|\operatorname{sym}\nabla u\|_{L^2(Z;\mathbb{R}^n)} = C_Z \iint_{S\times Y} \left|\operatorname{sym}\widetilde{\nabla}_{m,\gamma} u\right|^2 \mathrm{d}y\,\mathrm{d}\bar{x}. \tag{6.17}$$

Let $u \in \mathcal{V}$, and let $r \in \mathcal{R}$ denote the orthogonal projection of u on $\mathcal{R}$ in $W^{1,2}(Z;\mathbb{R}^n)$. Then

$$\begin{aligned} \|u\|^2_{W^{1,2}(Z;\mathbb{R}^n)} &= \|u - r\|^2_{W^{1,2}(Z;\mathbb{R}^n)} + \|r\|^2_{W^{1,2}(Z;\mathbb{R}^n)} \\ &\text{and} \qquad \|u - r\|_{W^{1,2}(Z;\mathbb{R}^n)} = \inf_{p\in\mathcal{R}} \|u - p\|_{W^{1,2}(Z;\mathbb{R}^n)}. \end{aligned} \tag{6.18}$$

Set $v := u - r$ and decompose r according to

$$r = \hat{r} + \bar{r} \qquad \text{with} \quad \hat{r} \in \mathcal{R}^\star(m) \quad \text{and} \quad \bar{r} \in \mathcal{R}(m).$$

We claim that

$$\|\hat{r}\|_{W^{1,2}(Z;\mathbb{R}^n)} \le c' \|v\|_{W^{1,2}(Z;\mathbb{R}^n)}.$$

Here and below, c' and c'' denote positive constants that may change from line to line, but can be chosen only depending on the geometry of S and Y. The assertion above can be justified as follows: Because $\hat{r} \in \mathcal{R}^\star(m)$, there exist $A \in \mathbb{M}_{\text{skew}}(n)$ and $b \in \mathbb{R}^n$ such that

$$\hat{r}(z) = \hat{A}z + b.$$

The maps u and $\bar{r}$ have vanishing mean value, and therefore we infer that

$$-b = \int_Z -\hat{r}\,\mathrm{d}z = \int_Z v\,\mathrm{d}z \;\Rightarrow\; |b| \leq c' \|v\|_{W^{1,2}(Z;\mathbb{R}^n)}$$

On the other hand, the Y-periodicity of u implies that

$$\sum_{k=1}^m |v(\bar{x}, y+e_k) - v(\bar{x},y)| = \sum_{k=1}^m |r(\bar{x}, y+e_k) - r(\bar{x},y)| = \sum_{k=1}^m \left|\hat{A}_{\{:,k\}}\right|$$

where $\hat{A}_{\{:,k\}}$ denotes the kth column of the matrix $\hat{A}$. The left hand side can be controlled by the norm of v, while the right hand side controls the modulus of $\hat{A}$, because A is skew-symmetric and $\hat{A}$ is constructed by deleting the "lower-right" $(n-m)\times(n-m)$-submatrix $\bar{A}$. As a consequence, we obtain the estimate

$$\|\hat{r}\|_{W^{1,2}(Z;\mathbb{R}^n)} \leq c'(|b| + \left|\hat{A}\right|) \leq c'' \|v\|_{W^{1,2}(Z;\mathbb{R}^n)}.$$

Now the orthogonality $\hat{r} \perp \bar{r}$, (6.18), (6.17) and the previous estimate imply that

$$\begin{aligned}\|u\|_{W^{1,2}(Z;\mathbb{R}^n)} &= \|u-r\|^2_{W^{1,2}(Z;\mathbb{R}^n)} + \|\hat{r}\|^2_{W^{1,2}(Z;\mathbb{R}^n)} + \|\bar{r}\|^2_{W^{1,2}(Z;\mathbb{R}^n)} \\ &\leq c' \iint_{S\times Y} \left|\operatorname{sym}\widetilde{\nabla}_{m,\gamma} u\right|^2 \mathrm{d}y\,\mathrm{d}\bar{x} + \|\bar{r}\|^2_{W^{1,2}(Z;\mathbb{R}^n)}.\end{aligned}$$

Because $\mathcal{R}(m) \oplus \mathcal{R}^\star(m)$ is an orthogonal decomposition of $\mathcal{R}$, we infer that $\bar{r}$ is the orthogonal projection of u on $\mathcal{R}(m)$. As a consequence, we have

$$\begin{aligned}\inf_{\bar{p}\in\mathcal{R}(m)} \|u-\bar{p}\|^2_{W^{1,2}(Z;\mathbb{R}^n)} &= \|u-\bar{r}\|^2_{W^{1,2}(Z;\mathbb{R}^n)} \\ &= \|u\|^2_{W^{1,2}(Z;\mathbb{R}^n)} - \|\bar{r}\|^2_{W^{1,2}(Z;\mathbb{R}^n)} \leq c' \iint_{S\times Y} \left|\operatorname{sym}\widetilde{\nabla}_{m,\gamma} u\right|^2 \mathrm{d}y\,\mathrm{d}\bar{x}\end{aligned}$$

and the proof for $\gamma = 1$ is complete.

Step 3. Let $\gamma \in (0,\infty)$. Define

$$w(\bar{x}, y) := u(\tfrac{1}{\gamma}\bar{x}, y).$$

Then $w \in W^{1,2}((\gamma S)\times Y;\mathbb{R}^n)$ and we have

$$\widetilde{\nabla}_{m,1} w(\bar{x},y) = (\widetilde{\nabla}_{m,\gamma} u)(\tfrac{1}{\gamma}\bar{x}, y).$$

Moreover, we can apply the Korn inequality derived in the previous step to the scaled map w and obtain

$$\inf_{r\in\mathcal{R}_\gamma(m)} \|w-r\|^2_{W^{1,2}((\gamma S)\times Y;\mathbb{R}^n)} \leq c' \int_{(\gamma S)\times Y} \left|\mathrm{sym}(\widetilde{\nabla}_{m,\gamma}u)(\tfrac{1}{\gamma}\bar{x},y)\right|^2 \mathrm{d}y\,\mathrm{d}\bar{x}.$$

where

$$\mathcal{R}_\gamma(m) := \left\{ r_\gamma(\bar{x}) := \begin{pmatrix} \mathbf{0}_m \\ \bar{A}(\bar{x}-c_{\gamma S}) \end{pmatrix} \,:\, \bar{A}\in\mathbb{M}_{\mathrm{skew}}(n-m) \right\}.$$

Now the change of coordinates $(\gamma\bar{x},y)\mapsto(\bar{x},y)$ and the observation that

$$\bar{x}\mapsto r_\gamma(\gamma\bar{x}) \in \mathcal{R}(m) \qquad \text{for all } r_\gamma\in\mathcal{R}_\gamma(m)$$

lead to

$$\inf_{r\in\mathcal{R}(m)} \|u\|^2_{W^{1,2}(S\times Y;\mathbb{R}^n)} \leq C_\gamma \int_{S\times Y} \left|\mathrm{sym}(\widetilde{\nabla}_{m,\gamma}u)(\bar{x},y)\right|^2 \mathrm{d}y\,\mathrm{d}\bar{x}$$

for a constant C_γ that only depends on γ and the geometry of S and Y. □

Corollary 6.3.9. *For each $u\in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^n)$ there exists $w\in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^n)$ such that*

$$\mathrm{sym}\,\widetilde{\nabla}_{m,\gamma}u(\bar{x},y) = \mathrm{sym}\,\widetilde{\nabla}_{m,\gamma}w(\bar{x},y) \qquad \textit{almost everywhere}$$

and

$$\|w\|_{W^{1,2}(S\times Y;\mathbb{R}^n)} \leq c_\gamma \left\|\mathrm{sym}\,\widetilde{\nabla}_{m,\gamma}u\right\|_{L^2(S\times Y;\mathbb{R}^n)}$$

where c_γ is the constant from Theorem 6.3.8.

Proof. This directly follows from the previous theorem and the fact that

$$\{\,\bar{r}(\bar{x})+b \,:\, \bar{r}\in\mathcal{R}(m),\, b\in\mathbb{R}^n\,\} \subset W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^n).$$

□

6.4. Homogenization and dimension reduction of a convex energy

In this section we demonstrate how the developed two-scale convergence methods can be used to analyze the asymptotic behavior of the functional $\mathcal{E}^{\varepsilon,h}$ defined in the introduction (see equation (6.2)) when both fine-scales ε, h simultaneously tend to zero.

As in the introduction, let $\Omega := \omega\times S$ and $\Omega_h := \omega\times hS$ with $\omega\subset\mathbb{R}^2$ and $S := (-\frac{1}{2},\frac{1}{2})$ and suppose that ω is an open and bounded domain with Lipschitz boundary. Moreover, we set $Y := [0,1)^2$.

We consider a measurable integrand $g : \mathbb{R}^2\times\mathbb{M}(3) \to [0,\infty)$ and suppose that g is Y-periodic, convex and lower semicontinuous in the sense of Definition 3.1.4. Moreover, we suppose that g satisfies the growth condition

$$\frac{1}{c}|F|^2 - c \le g(y,F) \le c(|F|^2+1) \qquad \text{for all } F\in\mathbb{M}(3) \text{ and a.e. } y\in Y. \tag{6.19}$$

We have seen in the introduction that (instead of $\mathcal{E}^{\varepsilon,h}$) we can equivalently study the functional

$$\mathcal{I}^{\varepsilon,h}(u) := \begin{cases} \displaystyle\int_\Omega g(\hat{x}/\varepsilon, \nabla_{2,h}\, u(x))\,\mathrm{d}x & \text{if } u\in W^{1,2}(\Omega;\mathbb{R}^3)\\ +\infty & \text{if } u\in L^2(\Omega;\mathbb{R}^3)\setminus W^{1,2}(\Omega;\mathbb{R}^3).\end{cases}$$

In order to describe the limiting behavior of $(\mathcal{I}^{\varepsilon,h})$ we define for $\gamma\in(0,\infty)$ and pairs

$$(u,u_0)\in\mathcal{W} := \Big\{u\in W^{1,2}(\Omega;\mathbb{R}^3)\ :\ u \text{ is independent of } x_3\Big\}\times L^2(\omega;\mathcal{V})$$
$$\text{where } \mathcal{V} := \left\{ u\in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^3)\ :\ \iint_{S\times Y} u\,\mathrm{d}y\,\mathrm{d}\bar{x} = 0\right\}$$

the two-scale functional

$$\mathcal{I}^\gamma(u,u_0) := \iint_{\Omega\times S} g(y,\nabla u(x) + \widetilde{\nabla}_{2,\gamma}u_0(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

Theorem 6.4.1. *Let (h) denote an arbitrary vanishing sequence of positive numbers and let $\varepsilon:(0,\infty)\to(0,\infty)$ satisfy*

$$\lim_{h\to 0}\varepsilon(h) = 0 \quad \textit{and} \quad \lim_{h\to 0}\frac{h}{\varepsilon(h)} = \gamma \quad \textit{with} \quad \gamma\in(0,\infty).$$

(1) Let (u_h) be an arbitrary sequence in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that

$$\limsup_{h\to 0}\left\{\left|\int_\Omega u_h\,\mathrm{d}x\right| + \mathcal{I}^{\varepsilon(h),h}(u_h)\right\} < \infty. \tag{6.20}$$

Then there exist a subsequence (not relabeled) and a pair

$$(u,u_0)\in\mathcal{W}$$

such that

$$(\star)\qquad \begin{cases} u_h\to u & \textit{strongly in } L^2(\Omega;\mathbb{R}^3)\\ \nabla u_h \xrightarrow{2} \nabla u + \widetilde{\nabla}_{2,\gamma}u_0 & \textit{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(3)).\end{cases}$$

(2) Suppose that $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *converges to a pair*

$$(u, u_0) \in \mathcal{W}$$

in the sense of $(\star)$. *Then*

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u_h) \geq \mathcal{I}^{\gamma}(u, u_0).$$

(3) For any pair $(u, u_0) \in \mathcal{W}$ *there exists a sequence* (u_h) *in* $W^{1,2}(\Omega;\mathbb{R}^3)$ *converging to* (u, u_0) *in the sense of* $(\star)$ *such that*

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u_h) = \mathcal{I}^{\gamma}(u, u_0).$$

Proof. We prove statement (1). In view of the growth condition (6.19) and Poincaré-Wirtinger inequality, it is easy to check that assumption (6.20) allows us to extract a subsequence that weakly converges in $W^{1,2}(\Omega;\mathbb{R}^3)$ and satisfies the assumptions of Theorem 6.3.3. Thus, statement (1) is an immediate consequence of Theorem 6.3.3.

In virtue of Theorem 6.3.3 and Proposition 6.3.5, the lower bound statement (2) and the recovery sequence statement (3) can be proved with the approach similar to the one used in the proof of Theorem 3.3.1. For this reason we omit the proof and refer to Section 3.3. □

Remark 6.4.2. Analogously to the convex homogenization example in Section 3.3 we define for $u \in L^2(\Omega;\mathbb{R}^3)$ the functional

$$\mathcal{I}^{\gamma}_{\text{hom}}(u) := \begin{cases} \inf\limits_{u_0 \in L^2(\omega;\mathcal{V})} \mathcal{I}^{\gamma}(u, u_0) & \text{if } u \in W^{1,2}(\Omega;\mathbb{R}^3) \text{ and independent of } x_3 \\ +\infty & \text{else.} \end{cases}$$

In the same way as in Section 3.3, one can show that $\mathcal{I}^{\gamma}_{\text{hom}}$ is the Γ-limit of the sequence $(\mathcal{I}^{\varepsilon(h),h})$ as $h \to 0$ with respect to strong convergence in $L^2(\Omega;\mathbb{R}^3)$. We like to remark that $\mathcal{I}^{\gamma}_{\text{hom}}$ still depends on γ which captures the asymptotic behavior of the ratio between thickness h and size of the material microstructure ε.

Moreover, one can show that for all $u \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $u(\hat{x}, x_3)$ independent of x_3, we have

$$\mathcal{I}^{\gamma}_{\text{hom}}(u) = \int_{\omega} g^{\gamma}_{\text{hom}}(\nabla u(\hat{x}))\,\mathrm{d}\hat{x}$$

where g^{γ}_{hom} denotes the homogenized integrand and is defined according to

$$g^{\gamma}_{\text{hom}}(F) := \inf_{\varphi \in \mathcal{V}} \iint\limits_{S\times Y} g(y, F + \widetilde{\nabla}_{2,\gamma}\varphi(x_3, y))\,\mathrm{d}y\,\mathrm{d}x_3.$$

Remark 6.4.3. The previous theorem and the previous remark remain valid if the integrand g only satisfies the growth- and coercivity condition of Korn-type

$$\frac{1}{c}\left|\operatorname{sym} F\right|^2 - c \leq g(y,F) \leq c(\left|F\right|^2 + 1) \qquad \text{for all } F \in \mathbb{M}(3) \text{ and a.e. } y \in Y$$

and the condition

$$g(y,F) = g(y, \operatorname{sym} F)$$

as it is the case in linear elasticity. In this setting, the developed Korn inequality (see Theorem 6.3.8) and Corollary 6.3.9 guarantee that the minimization problems in the definition of $\mathcal{I}^{\gamma}_{\text{hom}}$ and g^{γ}_{hom} admit minimizers in $L^2(\omega;\mathcal{V})$ and $\mathcal{V}$, respectively.

Part III.

Dimension reduction and homogenization of slender elastic bodies in the bending regime

7. Rigorous derivation of a homogenized theory for planar rods from nonlinear 2d elasticity

7.1. Introduction and main result

In this chapter we derive a homogenized, planar rod theory from two-dimensional elasticity in a scaling regime which is associated to bending deformations. Our starting point is an elastic body that occupies in its undeformed configuration the slender, two-dimensional domain

$$\Omega_h := \omega \times S_h \quad \text{with} \quad \omega := (0, L) \text{ and } S_h := (-{}^h\!/_2, {}^h\!/_2).$$

We suppose that the elastic body consists of a composite material featuring a laterally, periodic microstructure with period ε. We are interested in the limiting behavior of the elastic body opposed to forces and subject to boundary conditions as h and ε simultaneously converge to zero

To this end, we introduce the energy

$$\mathcal{E}^{\varepsilon,h}(v; f) := \frac{1}{h} \int_{\Omega_h} W({}^{x_1}\!/_{\varepsilon}, \nabla v(x)) - \langle f(x),\, v(x) \rangle \,\mathrm{d}x,$$

where the deformation v is a map from Ω_h to $\mathbb{R}^2$ and $f : \Omega_h \to \mathbb{R}^2$ is a given vector field representing the applied force. The elastic potential $W : \mathbb{R} \times \mathbb{M}(2) \to [0, \infty]$ is supposed to be a measurable integrand that is $[0, 1)$-periodic in its first component and vanishes for rotations. The precise assumptions on W are presented below.

We are going to prove that, as h and ε simultaneously tend to zero, the scaled energy $\frac{1}{h^2}\mathcal{E}^{\varepsilon,h}$ Γ-converges to a limiting energy that only depends on one-dimensional deformations and that is finite only for bending deformations, i.e. maps in $W^{2,2}_{\text{iso}}(\omega; \mathbb{R}^2)$. In this case the limiting energy is quadratic in the curvature of the bending deformation and its stiffness coefficient emerges from a subtle relaxation and homogenization mechanism depending on the elastic material parameters, but also on the limiting behavior of the fine-scale ratio ${}^h\!/_{\varepsilon}$. More precisely, we distinguish the three fine-scale coupling regimes

$${}^h\!/_{\varepsilon} \to \gamma \quad \text{with} \quad \gamma = 0, \qquad \gamma = \infty \quad \text{and} \quad \gamma \in (0, \infty).$$

We are going to see that each fine-scale coupling regime leads to a different cell problem, which determines the effective coefficient appearing in the limiting problem.

As a corollary of the Γ-convergence result, we see that (almost) minimizers of $\mathcal{E}^{\varepsilon,h}$ converge to minimizers of the limiting energy. This observation turns out to be compatible with one-sided boundary conditions and justifies to call the limiting energy an effective theory that — despite being much simpler than the initial one — captures the essential behavior of the problem for small h and ε.

We now give a precise formulation of the main result. Let us start with some remarks on the fine-scales. Since we are interested in a limiting process where both fine-scales ε and h simultaneously tend to zero, we assume that the fine-scales are coupled in the following sense:

$$\begin{cases} (h) := (h_j)_{j\in\mathbb{N}} & \text{is a vanishing sequence of positive numbers} \\ \varepsilon : h \mapsto \varepsilon(h) & \text{is a map from } (0,\infty) \text{ to } (0,\infty) \text{ with } \lim_{h\to 0}\varepsilon(h) = 0. \\ \lim_{j\to\infty} \dfrac{h_j}{\varepsilon(h_j)} = \gamma & \text{with } \gamma \in [0,\infty]. \end{cases} \tag{7.1}$$

Unless indicated otherwise, we just write h and ε to refer to elements of the sequences (h) and $(\varepsilon(h))$, respectively. In particular, if both parameters appear simultaneously in a formula, then we use the convention $\varepsilon = \varepsilon(h)$. For instance we may write $\mathcal{E}^{\varepsilon,h}$ instead of $\mathcal{E}^{\varepsilon(h),h}$. In the case $\gamma = 0$ or $\gamma = \infty$, we say that the fine-scales separate in the limit.

For $\gamma \in [0,\infty]$ we define the limiting energy:

$$\mathcal{E}_\gamma(u,\hat{f}) := \begin{cases} q_\gamma \displaystyle\int_\omega \boldsymbol{\kappa}_{(u)}^2 \,\mathrm{d}x_1 - \int_\omega \left\langle \hat{f},\, u\right\rangle \mathrm{d}x_1 & \text{if } u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2) \\ +\infty & \text{else.} \end{cases}$$

Here, $\boldsymbol{\kappa}_{(u)}$ denotes the curvature of the bending deformation u and q_γ denotes the effective bending stiffness, which we will specify below.

In our main result we consider sequences of deformations (v_h) with $v_h \in W^{1,2}(\Omega_h;\mathbb{R}^2)$. In order to equip those sequences with a common topology, we associate to each v_h the cross-sectional average

$$\hat{v}_h(x_1) := \frac{1}{h}\int_{S_h} v_h(x_1,x_2)\,\mathrm{d}x_2, \qquad x_1 \in \omega.$$

Thereby, all functions $\hat{v}_h$ belong to the same Sobolev space $W^{1,2}(\omega;\mathbb{R}^2)$ and we can state the subsequent Γ-convergence result in the topology of $W^{1,2}(\omega;\mathbb{R}^2)$.

We say that the sequence (v_h) satisfies the *one-sided boundary condition* associated to (u_0, n_0) with $u_0, n_0 \in \mathbb{R}^2$, $|n_0| = 1$ if for each h

$$v_h(0,x_2) = u_0 + x_2 n_0 \qquad \text{for almost every } x_2 \in S_h. \tag{7.2}$$

The resulting limiting boundary condition for a bending deformation $u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2)$ reads

$$u(0) = u_0 \quad \text{and} \quad n_{(u)}(0) = n_0. \tag{7.3}$$

Here, $n_{(u)}(0)$ denotes the unit normal vector of the curve parametrized by u at the point $u(0)$ (see below for the precise definition).

Regarding the forces, we consider vector fields $f_h \in L^2(\Omega_h;\mathbb{R}^2)$ that converge in the following sense

$$\frac{1}{h^2} g_h \rightharpoonup f \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^2), \qquad \Omega := \omega \times S \tag{7.4}$$

where $g_h(x_1,x_2) := f_h(x_1,hx_2)$ and f is a map in $L^2(\Omega;\mathbb{R}^2)$. For instance, whenever f_h is independent of the cross-sectional direction x_2 and $(\frac{1}{h^2}f_h)$ is weakly convergent in $L^2(\omega;\mathbb{R}^2)$ this condition is satisfied.

At last, we present the precise conditions on the elastic potential. Let $Y := [0,1)$ denote the reference cell of periodicity. We assume that $W : \mathbb{R}\times\mathbb{M}(2) \to [0,\infty]$ is a measurable integrand that is Y-periodic in its first variable and satisfies the following conditions:

(W1) W is frame indifferent, i.e.

$$W(y,RF) = W(y,F) \qquad \text{for all } R \in SO(2),\ F \in \mathbb{M}(2) \text{ and a.e. } y \in \mathbb{R}$$

(W2) The identity is a natural state, i.e.

$$W(y,Id) = 0 \qquad \text{for a.e. } y \in \mathbb{R}$$

(W3) W is non-degenerate, i.e. there exists $C > 0$ such that

$$W(y,F) \geq C\,\mathrm{dist}^2(F,SO(2)) \qquad \text{for all } F \in \mathbb{M}(2) \text{ and a.e. } y \in \mathbb{R}$$

(W4) W admits a quadratic Taylor expansion at the identity, i.e.

$$\exists Q \in \mathfrak{Q}(Y;2)\ :\ \limsup_{\substack{F\to 0\\ F\neq 0}} \operatorname*{ess\,sup}_{y\in Y} \frac{|W(y,F)-Q(y,F)|}{|F|^2} = 0.$$

In the latter condition, $\mathfrak{Q}(Y;m)$ denotes the set of all measurable integrands

$$Q\,:\, Y\times\mathbb{M}(m) \to [0,\infty)$$

that are Y-periodic in the first, quadratic in the second variable and bounded in the sense that

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \sup_{|F|=1} Q(y,F) < +\infty.$$

Finally, we define the effective limiting coefficients according to

$$q_\gamma := \begin{cases} \dfrac{1}{12} \min\limits_{\alpha\in W^{1,2}_{\mathrm{per},0}(Y)} \displaystyle\int_Y \min_{d\in\mathbb{R}^2} Q\left(y,\ (1+\partial_y\alpha)(e_1\otimes e_1) + d\otimes e_2\right) \mathrm{d}y & \text{if } \gamma = 0,\\[2ex] \dfrac{1}{12} \min\limits_{\substack{\varphi\in W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^2),\\ d\in\mathbb{R}^2}} \displaystyle\int_Y Q\left(y,\ (e_1+\partial_y\varphi)\otimes e_1 + d\otimes e_2\right) \mathrm{d}y & \text{if } \gamma = \infty,\\[2ex] \min\limits_{w\in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2)} \displaystyle\iint_{S\times Y} Q(y, x_2(e_1\otimes e_1) + \widetilde{\nabla}_{1,\gamma} w)\,\mathrm{d}y\,\mathrm{d}x_2 & \text{if } \gamma \in (0,\infty). \end{cases} \tag{7.5}$$

The scaled gradient $\widetilde{\nabla}_{1,\gamma}$ and the function space $W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2)$ are defined in Section 6.3. In Section 7.4.5 we analyze the minimization problems related to the definition above.

We state our main result:

Theorem 7.1.1. *Let (u_0, n_0) denote admissible boundary data according to (7.2). Consider forces $f_h \in L^2(\Omega_h;\mathbb{R}^2)$ converging to $f \in L^2(\Omega;\mathbb{R}^2)$ in the sense of (7.4) and set $\hat{f} := \int_S f \,\mathrm{d}x_2$.*

(1) For every sequence (v_h), $v_h \in W^{1,2}(\Omega_h;\mathbb{R}^2)$, satisfying the one-sided boundary condition associated to (u_0, n_0) and having equibounded energy, i.e.

$$\limsup_{h\to 0} \frac{1}{h^2}\mathcal{E}^{\varepsilon,h}(v_h; f_h) < \infty,$$

there exists a map $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ satisfying the limiting boundary condition (7.3) and

$$\hat{v}_h \to u \qquad \text{strongly in } W^{1,2}(\omega;\mathbb{R}^2)$$

for a subsequence (not relabeled).

(2) For any sequence (v_h), $v_h \in W^{1,2}(\Omega_h;\mathbb{R}^2)$, satisfying the one-sided boundary condition associated to (u_0, n_0) and any map $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ with

$$\hat{v}_h \to u \qquad \text{strongly in } W^{1,2}(\omega;\mathbb{R}^2)$$

we have

$$\liminf_{h\to 0} \frac{1}{h^2}\mathcal{E}^{\varepsilon,h}(v_h; f_h) \geq \mathcal{E}_\gamma(u; \hat{f}).$$

(3) For each $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ satisfying the limiting boundary condition associated to (u_0, n_0) there is a sequence (v_h), $v_h \in W^{1,2}(\Omega_h;\mathbb{R}^2)$, satisfying the one-sided boundary condition associated to (u_0, n_0),

$$\hat{v}_h \to u \quad \text{strongly in } W^{1,2}(\omega;\mathbb{R}^2) \qquad \text{and} \qquad \lim_{h\to 0}\frac{1}{h^2}\mathcal{E}^{\varepsilon,h}(v_h; f_h) = \mathcal{E}_\gamma(u, \hat{f}).$$

As a consequence, we immediately obtain convergence of (almost) minimizers:

Corollary 7.1.2. *Consider forces (f_h) that converge to f in the sense of (7.4). Suppose that (v_h) satisfies the one-sided boundary condition (7.2) and is a sequence of almost minimizers, i.e.*

$$\limsup_{h\to 0}\frac{1}{h^2}\left|\mathcal{E}^{\varepsilon,h}(v_h; f_h) - \inf\left\{\,\mathcal{E}^{\varepsilon,h}(v; f_h)\;:\; v \in W^{1,2}(\Omega_h;\mathbb{R}^2) \text{ satisfies } (7.2)\,\right\}\right| = 0.$$

Then up to a subsequence $(\hat{v}_h)$ strongly converges in $W^{1,2}(\omega;\mathbb{R}^2)$ to $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ where u is a minimizer of the energy $\mathcal{E}_\gamma(v;\hat{f})$ among all deformations $v \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ satisfying the boundary condition (7.2).

Theorem 7.1.1 yields Γ-convergence of the functionals $(\mathcal{E}^{\varepsilon,h})$ to the limiting energy $\mathcal{E}_\gamma$. Indeed, statement (2) establishes the lower bound inequality, while (3) proves existence of recovery sequences. The first statement proves equi-coercivity of the functionals and justifies the sequential characterization of Γ-convergence.

It is well-known that Γ-convergence is robust with respect to continuous perturbations. This suggests that the results above remain valid for more general applied forces.

A close look at the cell problems determining the effective coefficients q_0 and q_∞ reveals that we trivially have the relation $q_0 \leq q_\infty$. Hence, elastic rods with a microstructure that is small — even compared to the thickness of the rod — tend to be stiffer than such rods with a relatively large microstructure.

The effective stiffness coefficient q_γ is determined by the linear cell problem (7.5). In the case where the fine-scales separate in the limit, i.e. $\gamma = 0$ or $\gamma = \infty$, the cell problem lives on the one-dimensional domain Y. In contrast, in the intermediate cases, where ${}^h\!/\!_\varepsilon$ converges to a finite ratio $\gamma \in (0, \infty)$, the cell problem lives on the two-dimensional domain $S\times Y$. This is in accordance with the observation that for $\gamma \in (0, \infty)$ the thickness and the material fine-scale remain "comparable" and lateral oscillations on scale ε couple with the cross-sectional behavior. We would like to remark that for special (but still inhomogeneous) materials (for instance isotropic materials with vanishing Poisson's ratio) the limiting energies $\mathcal{E}_0$, $\mathcal{E}_\gamma$ and $\mathcal{E}_\infty$ are equal.

What is the "right" topology for the Γ-convergence statement? In fact, there are several answers to this question. First, we observe that the domains of the deformations entering the functionals $\mathcal{E}^{\varepsilon,h}$ depend on the parameter h, and therefore a common topology for all deformations is not given a priori. A possible and — from the point of view of applications — natural choice is the topology induced by strong convergence of the cross-sectional averages in $W^{1,2}(\omega;\mathbb{R}^2)$. Similar topologies have been used in the field of dimension reduction by Acerbi et al. [ABP91] or Conti [CDD$^+$03] for instance. Nevertheless, from the mathematical point of view it is more convenient to consider a scaled formulation of the problem that allows us to work on the fixed domain $\Omega := \omega\times S$ with $S := (-1/2, 1/2)$. In the next section we are going to develop this scaled formulation, state the scaled analogon to Theorem 7.1.1 and argue that both procedures lead to the same result. In the proof of the main result we solely work with the scaled formulation.

A brief outline of this chapter

In the next section we develop and discuss the scaled formulation of the problem and prove the equivalence to the original formulation presented above.

The proof of the main result is contained in Section 7.4. The results derived in this section are presented in a quite general manner and can be applied to settings more general than the one considered in Theorem 7.1.1. In particular, in Section 7.4.2 we derive a powerful *two-scale characterization of nonlinear limiting strains* that emerge from sequences with finite bending energy. In Section 7.4.4 we demonstrate a *general scheme to approximate bending deformations* with simultaneous consideration of the two-scale behavior of the associated nonlinear strain (which turns out to be essential for the limiting behavior of the energy). At the beginning of Section 7.4 we give a more detailed summary about our approach.

We complete our analysis with the results derived in Section 7.5 and Section 7.6. In particular, in Section 7.5 we prove *strong two-scale convergence of the nonlinear strain* for low energy sequences. In Section 7.6 we analyze the effective limiting coefficients q_γ in the case where the fine-scales separate, i.e. for $\gamma \in \{0, \infty\}$. We prove that q_0 and q_∞ can equivalently be computed by consecutively modifying the elastic potential W by operations that are related to dimension reduction and homogenization. In this sense, we *justify the interpretation* that the cases $\gamma = 0$ and $\gamma = \infty$ correspond to the homogenization of the model with reduced dimension and the dimension reduction of the homogenized model, respectively. This insight is not trivial but involves the fact that linearization and homogenization commute in elasticity — as we proved in a previous chapter.

As a complement, in Section 7.7 we elaborate again on the two-scale characterization of the limiting strain and prove that the characterization is sharp. Based on this insight we present an analysis that derives a homogenized rod theory under quite general assumptions covering layered and prestressed materials. The derivation is rigorous and stated in the spirit of two-scale Γ-convergence (see [MT07]). The findings in this section rely on the results derived in Section 7.4 and Section 5.2 and the proof is remarkably short. In some sense, in this section (although not explicitly) we summarize the detailed analysis of the previous sections from a larger perspective and emphasize the general strategy.

Eventually, we would like to remark that Section 7.3 is devoted to a qualitative discussion of the problem. There, we perform an ansatz based analysis of the situation with the aim to offer an intuitive and easily accessible understanding of the emerging relaxation phenomena.

We conclude this introduction by fixing **some notation** that is specific to this chapter:

Rotations. For $\alpha \in \mathbb{R}$ we define the rotation

$$\mathcal{R}(\alpha) := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

which is exactly the clockwise rotation in $\mathbb{R}^2$ by angle α.

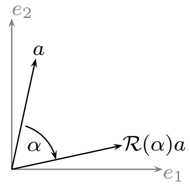

Figure 7.1.: Rotation.

Note that

$$\mathcal{R}(\alpha)\mathcal{R}(\beta) = \mathcal{R}(\alpha+\beta) = \mathcal{R}(\beta)\mathcal{R}(\alpha)$$

for all $\alpha, \beta \in \mathbb{R}$. (Thus, the map $\mathcal{R}$ is a continuous homeomorphism from the additive group $\mathbb{R}/(2\pi\mathbb{Z})$ to the multiplicative group $SO(2)$, which is commutative in contrast to $SO(n)$ for $n > 2$). It is easy to check that $\mathcal{R}\colon\mathbb{R} \to SO(2)$ is smooth and satisfies

$$\partial_\alpha \mathcal{R}(\alpha) = \mathcal{R}(\alpha+\pi/2) = \mathcal{R}(\pi/2)\mathcal{R}(\alpha).$$

In the sequel we frequently encounter the elementary identity

$$\mathcal{R}(\alpha)^{\mathrm{T}}\partial_1\mathcal{R}(\alpha) = \partial_1\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is valid for all sufficiently smooth maps $\alpha : \omega \to \mathbb{R}$.

Bending deformations, planar curves and curvature. In this chapter, we refer to maps

$$u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2) := \left\{ u \in W^{2,2}(\omega;\mathbb{R}^2) \;:\; |\partial_1 u(x_1)| = 1 \qquad \text{for a.e. } x_1 \in \omega \right\}$$

as bending deformations. We associate to each $u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2)$ a signed curvature, a tangent- and a normal field in the following unique way

$$t_{(u)} := \partial_1 u, \quad n_{(u)} := -\mathcal{R}(\pi/2)t_{(u)} \quad \text{and}$$
$$\boldsymbol{\kappa}_{(u)} := \left\langle t_{(u)}, \partial_1 n_{(u)} \right\rangle = -\left\langle n_{(u)}, \partial^2_{11} u_{(u)} \right\rangle.$$

Furthermore, we set $R_{(u)} := \left(\, t_{(u)} \,\middle|\, n_{(u)} \, \right)$ and call $R_{(u)}$ the *frame* associated to u. Note that the sign of $\boldsymbol{\kappa}_{(u)}$ is chosen in such a way that a curve with negative curvature bends in the direction of its normal (see Figure 7.2).

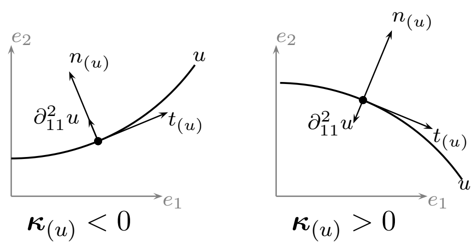

Figure 7.2.: Bending deformation.

Two-scale convergence. Throughout this chapter, two-scale convergence is understood in the sense of Definition 6.2.3 with $m = 1$, i.e. with respect to $(\varepsilon(h))$-oscillations in direction x_1.

7.2. The scaled formulation

In this section we discuss the scaling procedure leading to the scaled version of Theorem 7.1.1. To this end we introduce the fixed domain with unit thickness

$$\Omega := \Omega_1 = \omega \times (-1/2, 1/2)$$

and the scaling transformation

$$\pi_h : \Omega \to \Omega_h, \qquad (x_1, x_2) \mapsto (x_1, h\, x_2).$$

To each deformation $v_h : \Omega_h \to \mathbb{R}^2$ we associate the map $u_h := v_h \circ \pi_h$ and call u_h the scaled deformation associated to v_h. We refer to v_h as the rescaled deformation associated to u_h. Obviously, if $v_h \in W^{1,2}(\Omega_h; \mathbb{R}^2)$, then $u_h \in W^{1,2}(\Omega; \mathbb{R}^2)$ and we have

$$\nabla v_h = \nabla(u_h \circ \pi_h^{-1}) = \nabla_h u_h \qquad \text{where } \nabla_h u_h := \left(\partial_1 u_h \;\middle|\; \tfrac{1}{h}\partial_2 u_h \right).$$

This procedure allows us to rewrite the total energy $\mathcal{E}^{\varepsilon,h}$ in terms of the new coordinates as

$$\mathcal{E}^{\varepsilon,h}(v_h;\, f_h) = \int_\Omega W(x_1/\varepsilon,\, \nabla_h u_h(x)) - \langle g_h(x),\, u_h(x)\rangle \, \mathrm{d}x$$

where $g_h := f_h \circ \pi_h$. This suggests to define the functional

$$\mathcal{I}^{\varepsilon,h}(u_h) := \begin{cases} \dfrac{1}{h^2} \displaystyle\int_\Omega W(x_1/\varepsilon,\, \nabla_h u(x))\, \mathrm{d}x & \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^2), \\ +\infty & \text{else,} \end{cases}$$

which is the elastic part of the total energy in the scaled setting. Note that whenever we have a sequence $(u_h) \subset W^{1,2}(\Omega; \mathbb{R}^2)$ with equibounded energy, i.e.

$$\limsup_{h \to 0} \mathcal{I}^{\varepsilon,h}(u_h) < \infty,$$

then due to the non-degeneracy condition (W3) this sequence satisfies

$$\text{(FBE)} \qquad \limsup_{h\to 0} \frac{1}{h^2} \int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(2))\, \mathrm{d}x < \infty.$$

Following [FJM02] we call a sequence that fulfills (FBE) a sequence with *finite bending energy*. As we will see, this property is the key assumption in the subsequent compactness result, which roughly speaking states that a sequence with finite bending energy converges to a bending deformation. It is important to note that this property is also compatible with the total energy in the unscaled formulation:

Lemma 7.2.1. *Let (u_0, n_0) and (f_h) be defined according to Theorem 7.1.1. Consider a sequence (v_h) such that $v_h \in W^{1,2}(\Omega_h; \mathbb{R}^2)$ satisfies the one-sided boundary condition associated to (u_0, n_0) and has equibounded energy, i.e.*

$$\limsup_{h\to 0} \frac{1}{h^2} \mathcal{E}^{\varepsilon,h}(v_h; f_h) < \infty.$$

Then $u_h := v_h \circ \pi_h$ has finite bending energy.

Proof. In a first step we prove that

$$\|u_h\|_{L^2(\Omega;\mathbb{R}^2)} \leq c' \, \|\nabla_h u_h\|_{L^2(\Omega_h;\mathbb{M}(2))} + c'' \qquad \text{for all } 0 < h. \tag{7.6}$$

Here and below, c', c'' denote generic positive constants that may change from line to line, but can be chosen independent of h. For the proof of this estimate we define the map

$$w_h(x) := u_0 + hx_2 n_0.$$

Since (v_h) satisfies the one-sided boundary condition, the difference $u_h - w_h$ belongs (for all $h>0$) to the subspace of maps in $W^{1,2}(\Omega;\mathbb{R}^2)$ that vanish on the set $\{0\}\times S$. Consequently, the Poincaré-Friedrichs inequality implies that

$$\begin{aligned}\|u_h\|_{L^2(\Omega;\mathbb{R}^2)} &\leq \|u_h - w_h\|_{L^2(\Omega;\mathbb{R}^2)} + \|w_h\|_{L^2(\Omega;\mathbb{R}^2)} \\ &\leq c' \, \|\nabla_h u_h - \nabla_h w_h\|_{L^2(\Omega;\mathbb{M}(2))} + \|w_h\|_{L^2(\Omega;\mathbb{R}^2)} \, .\end{aligned}$$

Because of $\limsup_{h\to 0}(\|w_h\|_{L^2(\Omega;\mathbb{R}^2)} + \|\nabla_h w_h\|_{L^2(\Omega;\mathbb{M}(2))}) < \infty$, this already implies (7.6).

The remaining proof is a slight variant of arguments used in [Con03]. We introduce the scaled force $g_h := f_h \circ \pi_h$. Since

$$\mathcal{E}^{\varepsilon,h}(v_h;\, f_h) \geq h^2 \mathcal{I}^{\varepsilon,h}(u_h) - \int_\Omega \langle g_h(x),\, u_h(x)\rangle \,\mathrm{d}x$$

and $h^2\mathcal{I}^{\varepsilon,h}(u_h) \geq C \int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(2)) \,\mathrm{d}x$ due to (W3), it is sufficient to analyze the force term. By assumption the sequence $(h^{-2} g_h)$ is bounded in $L^2(\Omega;\mathbb{R}^2)$, and therefore we have

$$\begin{aligned}\left|\int_\Omega \langle g_h,\, u_h\rangle \,\mathrm{d}x\right| &\leq h^2 c' \, \|u_h\|_{L^2(\Omega;\mathbb{R}^2)} \\ &\leq h^2 (c' \, \|\nabla_h u_h\|_{L^2(\Omega;\mathbb{M}(2))} + c'')\end{aligned}$$

for all $h > 0$. We apply the inequality $1/2\, |F|^2 \leq \operatorname{dist}^2(F, SO(2)) + 2$ and deduce that

$$\|\nabla_h u_h\|_{L^2(\Omega;\mathbb{M}(2))} \leq c' \left(\int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(2)) \,\mathrm{d}x + 1 \right).$$

Consequently

$$\mathcal{E}^{\varepsilon,h}(v_h;\, f_h) \geq (C - h^2 c') \int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(2)) \,\mathrm{d}x - h^2 c''.$$

Thus, we obtain

$$\limsup_{h\to 0} \frac{1}{h^2} \int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(2)) \,\mathrm{d}x \leq \frac{1}{C} \limsup_{h\to 0} \frac{1}{h^2} \mathcal{E}^{\varepsilon,h}(v_h;\, f_h) + c'' < \infty.$$

□

Remark 7.2.2. If we drop the requirement that (v_h) satisfies the one-sided boundary condition, then the statement does not remain valid in general. Nevertheless, in the case without boundary condition, we obtain a similar result for forces f_h with $\int_{\Omega_h} f_h \,\mathrm{d}x = 0$. Alternatively, we could also consider the assumption that the averages $\frac{1}{h}\int_{\Omega_h} v_h \,\mathrm{d}x$ are uniformly bounded.

In order to present the analogon to Theorem 7.1.1 for the scaled setting, we define for $u \in W^{2,2}_{\mathrm{iso}}(\omega, \mathbb{R}^2)$ and $\gamma \in \{0, \infty\}$ the functionals

$$\mathcal{I}_\gamma(u) := \inf\left\{ \frac{1}{12} \iint\limits_{\omega\times Y} Q\left(y, \boldsymbol{\kappa}_{(u)}(e_1 \otimes e_1) + \begin{pmatrix} \partial_y \alpha & g \\ g & c \end{pmatrix} \right) \mathrm{d}y\,\mathrm{d}x_1 \;:\; (\alpha, g, c) \in \mathbb{X}_\gamma \right\}$$

where

$$\begin{aligned} \mathbb{X}_0 &:= \{\, (\alpha, g, c) \;:\; \alpha \in L^2(\omega; W^{1,2}_{\mathrm{per}}(Y)),\; g, c \in L^2(\omega\times Y) \,\} \\ \mathbb{X}_\infty &:= \{\, (\alpha, g, c) \;:\; \alpha \in L^2(\omega; W^{1,2}_{\mathrm{per}}(Y)),\; g \in L^2(\omega\times Y),\, c \in L^2(\omega) \,\}. \end{aligned}$$

For $\gamma \in (0, \infty)$ and $u \in W^{2,2}_{\mathrm{iso}}(\omega; \mathbb{R}^2)$ we define

$$\mathcal{I}_\gamma(u) := \inf\left\{ \iint\limits_{\Omega\times Y} Q\left(y, \big(a(x_1) + x_2 \boldsymbol{\kappa}_{(u)}\big)(e_1 \otimes e_1) + \widetilde{\nabla}_{1,\gamma} w_0(x, y) \right) \mathrm{d}y\,\mathrm{d}x \;: \right.$$
$$\left. a \in L^2(\omega),\; w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y; \mathbb{R}^2)) \right\}.$$

We extend the functionals $\mathcal{I}_\gamma$ to $L^2(\Omega; \mathbb{R}^2)$ by setting $\mathcal{I}_\gamma(u) = +\infty$ if $u \notin W^{2,2}_{\mathrm{iso}}(\omega; \mathbb{R}^2)$.

Remark 7.2.3. The scaled gradient $\widetilde{\nabla}_{1,\gamma} w = (\partial_y w \,|\, \frac{1}{\gamma}\partial_2 w)$ as well as the function space $W^{1,2}_{Y\text{-per}}(S\times Y; \mathbb{R}^2)$ are defined in Section 6.3.

Remark 7.2.4. In Section 7.4.5 we are going to show that

$$\mathcal{I}_\gamma(u) = \mathcal{E}_\gamma(u; 0)$$

for all $u \in W^{2,2}_{\mathrm{iso}}(\omega; \mathbb{R}^2)$.

We now state the main result in the scaled setting:

Theorem 7.2.5.

(1) *(Compactness). For every sequence (u_h) in $W^{1,2}(\Omega; \mathbb{R}^2)$ with finite bending energy there exists a map $u \in W^{2,2}_{\mathrm{iso}}(\omega; \mathbb{R}^2)$ such that*

$$\begin{aligned} u_h - u_{\Omega,h} &\to u && \text{strongly in } L^2(\Omega; \mathbb{R}^2) \\ \nabla_h u_h &\to R_{(u)} && \text{strongly in } L^2(\Omega; \mathbb{M}(2)) \end{aligned}$$

for a suitable subsequence (not relabeled). Here, $u_{\Omega,h} := \frac{1}{\mathcal{H}^n(\Omega)}\int_\Omega u_h \,\mathrm{d}x$ denotes the integral average of u_h.

(2) *(Lower bound). For every sequence (u_h) in $W^{1,2}(\Omega;\mathbb{R}^2)$ with $u_h \rightharpoonup u$ weakly in $L^2(\Omega;\mathbb{R}^2)$ we have*
$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) \geq \mathcal{I}_\gamma(u).$$

(3) *(Upper bound). For each map $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ there is a sequence (u_h) in $W^{1,2}(\Omega;\mathbb{R}^2)$ with*
$$\begin{aligned} u_h &\to u && \textit{strongly in } L^2(\Omega;\mathbb{R}^2)\\ \nabla_h u_h &\to R_{(u)} && \textit{strongly in } L^2(\Omega;\mathbb{M}(2)) \end{aligned}$$
such that
$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) = \mathcal{I}_\gamma(u).$$
Additionally, we can choose the sequence (u_h) in such a way that the boundary condition
$$u_h(0,x_2) = u(0) + hx_2 n_{(u)}(0)$$
is satisfied by each u_h.

(For the proof see page 119).

Remark 7.2.6. In the previous theorem the limiting deformation u is a map from ω to $\mathbb{R}^2$, while the sequence (u_h) consists of maps from Ω to $\mathbb{R}^2$. Therefore, to be absolutely precise we have to identify in the convergence statements above the map $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ with the trivial extension to Ω, i.e. with the map
$$\widetilde{u} : \Omega \to \mathbb{R}^2, \qquad \widetilde{u}(x_1,x_2) := u(x_1).$$
If necessary, in the sequel we are going to use this identification without further indication.

The compactness part of the theorem above can be improved: The following proposition shows that the one-sided boundary condition is stable for sequences with finite bending energy.

Proposition 7.2.7. *Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^2)$ satisfying the scaled one-sided boundary condition*
$$u_h(0,x_2) = u_0 + hx_2 n_0$$
with $u_0, n_0 \in \mathbb{R}^2$, $|n_0| = 1$. Suppose that (u_h) has finite bending energy. Then there is a map $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ that satisfies the limiting boundary condition
$$u(0) = u_0, \qquad n_{(u)}(0) = n_0$$
and (u_h) converges to u up to a subsequence in the following sense
$$\begin{aligned} u_h &\to u && \textit{strongly in } L^2(\Omega;\mathbb{R}^2)\\ \nabla_h u_h &\to R_{(u)} && \textit{strongly in } L^2(\Omega;\mathbb{M}(2)). \end{aligned}$$

(For the proof see page 131).

The previous results can be stated in the language of Γ-convergence:

Corollary 7.2.8. *For $u \in L^2(\Omega;\mathbb{R}^2)$ define the functionals*

$$\mathcal{I}^{\varepsilon,h}_{bc}(u) := \begin{cases} \mathcal{I}^{\varepsilon,h}(u) & \text{if } u \in W^{1,2}(\Omega;\mathbb{R}^2) \text{ satisfies } u(0,x_2){=}u_0{+}hx_2n_0,\\ +\infty & \text{else,}\end{cases}$$

$$\mathcal{I}_{\gamma,bc}(u) := \begin{cases} \mathcal{I}_\gamma(u) & \text{if } u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2) \text{ satisfies } u(0){=}u_0,\ n_{(u)}(0){=}n_0,\\ +\infty & \text{else.}\end{cases}$$

The family $(\mathcal{I}^{\varepsilon,h}_{bc})$ is equi-coercive and Γ-converges to $\mathcal{I}_{\gamma,bc}$ with respect to the strong topology in $L^2(\Omega;\mathbb{R}^2)$.

Proof. The equi-coercivity is a direct consequence of Proposition 7.2.7. Consequently, the sequential characterization of Γ-convergence is valid (see Proposition 4.2.7) and the convergence statement directly follows from (2) and (3) in Theorem 7.2.5. □

The proof of the previous theorem and proposition is the main subject of Section 7.4. Here, we only argue that both imply the validity of Theorem 7.1.1. The key observation in this context is the following:

Lemma 7.2.9. *Let (u_0, n_0) and (f_h) be defined as in Theorem 7.1.1. Let (v_h), $v_h \in W^{1,2}(\Omega_h;\mathbb{R}^2)$, be a sequence that satisfies the one-sided boundary condition associated to (u_0, n_0) and has finite energy, i.e.*

$$\limsup_{h\to 0} \frac{1}{h^2}\mathcal{E}^{\varepsilon,h}(v_h; f_h) < \infty.$$

Then the following statements are equivalent:

(1) $(\hat{v}_h)$ converges to u weakly in $L^2(\omega;\mathbb{R}^2)$

(2) $(\hat{v}_h)$ converges to u strongly in $W^{1,2}(\omega;\mathbb{R}^2)$

(3) $(u_h := v_h \circ \pi_h)$ converges to u strongly in $L^2(\Omega;\mathbb{R}^2)$.

Proof. From Lemma 7.2.1 we deduce that (u_h) has finite bending energy. Now the statement directly follows from the compactness result, Proposition 7.2.7 and the uniqueness of the limit u. □

In virtue of this observation, it is clear that Theorem 7.1.1 immediately follows from the theorems in this section.

7.3. A qualitative picture

In this section we study the asymptotic behavior of the energy $\mathcal{I}^{\varepsilon,h}$ based on certain classes of ansatzes. The aim of the discussion is to offer an intuitive and easily accessible understanding of the emerging relaxation effects related to dimension reduction and homogenization. As already remarked in the introduction, the classical, ansatz based approaches to the derivation of elastic rod and plate theories can be viewed as the attempt to regard the lower-dimensional theories as constrained versions of three-dimensional elasticity in the situation where the three-dimensional body is slender and subjected to additional constitutive restrictions. Although, the framework of Γ-convergence is ansatz free, we follow in the sequel the philosophy of the classical approach to gain a qualitative picture and understanding of the problem. In the following we are content with formal results. Nevertheless, we would like to remark that the results presented below can be made rigorous by the methods that we are going to introduce in Section 7.4.4.

Our analysis consists of the following steps:

(1) For a given (one-dimensional) bending deformation $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ we construct a sequence of two-dimensional deformations $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ in such a way that

   i. (u_h) is an extension of u obeying a certain *ansatz*; and

   ii. (u_h) approximates u in the sense that

$$\begin{aligned} u_h &\to u && \text{strongly in } L^2(\Omega;\mathbb{R}^2)\\ \nabla_h u_h &\to R_{(u)} && \text{strongly in } L^2(\Omega;\mathbb{M}(2)). \end{aligned}$$

(2) We evaluate the associated limiting energy

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h). \tag{7.7}$$

It turns out that (7.7) can be written as a function of $\boldsymbol{\kappa}_{(u)}$ (the curvature of u) and certain free parameters which are specific to the ansatz that we use in the construction of (u_h).

For the computation of (7.7) we employ the following strategy: Because W is frame indifferent and admits a quadratic Taylor expansion at Id, we see that

$$\begin{aligned} \frac{1}{h^2}W(x_1/\varepsilon, F) =& \frac{1}{h^2}W\left(x_1/\varepsilon, Id + h\frac{R^{\mathrm{T}}F - Id}{h}\right)\\ =& Q\left(x_1/\varepsilon, \frac{R^{\mathrm{T}}F - Id}{h}\right) + \text{ higher order terms} \end{aligned}$$

for all $F \in \mathbb{M}(2)$ and rotations $R \in SO(2)$. This suggests to consider the quantity

$$E^{\mathrm{ap}}_h = \frac{R_h^{\mathrm{T}}\,\nabla_h u_h - Id}{h}$$

where $R_h : \omega \to SO(2)$ is a rotation field. We choose R_h in such a way that it is close to $R_{(u)}$, the frame associated to u. In this case E_h^{ap} can be regarded as an approximation of the nonlinear strain

$$E_h := \frac{\sqrt{\nabla_h u_h^{\mathrm{T}} \nabla_h u_h} - Id}{h},$$

which plays a crucial role in finite elasticity. We expect that

$$\frac{1}{h^2}\int_\Omega W(x_1/\varepsilon,\, \nabla_h u_h(x))\,\mathrm{d}x = \int_\Omega Q(x_1/\varepsilon,\, E_h^{\mathrm{ap}}(x))\,\mathrm{d}x \;+\; \text{higher order terms.}$$

Indeed, one can show (see Theorem 5.2.1) that a sufficient condition for the validity of this expansion is the boundedness of the sequence (E_h^{ap}) in the sense that

$$\limsup_{h\to 0}\, \operatorname*{ess\,sup}_{x\in\Omega} \left|E_h^{\mathrm{ap}}(x)\right| < \infty. \tag{7.8}$$

All ansatzes that we discuss below satisfy this condition, and therefore we are going to compute the limiting energy in (7.7) by passing to the limit in the quadratic functional

$$E_h^{\mathrm{ap}} \mapsto \int_\Omega Q(x_1/\varepsilon, E_h^{\mathrm{ap}}(x))\,\mathrm{d}x. \tag{7.9}$$

(3) The limiting energy (7.7) depends not only on $\boldsymbol{\kappa}_{(u)}$, but also on free parameters specific to the ansatz. By minimizing (7.7) with respect to these free parameters we gain an *effective limiting energy* which is optimal within the ansatz class. This procedure leads to a relaxed limiting energy of the form

$$\frac{\tilde{q}}{12}\int_\omega \boldsymbol{\kappa}_{(u)}^2\,\mathrm{d}x_1,$$

where $\boldsymbol{\kappa}_{(u)}$ is the curvature of the bending deformation u and $\tilde{q}$ is a stiffness coefficient depending on the ansatz and the elastic properties of the material.

We would like to remark that in virtue of the sequential characterization of Γ-convergence each of the relaxed energies, that we derive by the procedure described above, naturally yields an upper bound to the rigorous Γ-limit of $(\mathcal{I}^{\varepsilon,h})$.

For simplicity, we restrict our analysis to the case where W corresponds to an isotropic material and satisfies (W1)-(W4). In this case, the quadratic form in the expansion (W4) takes the form

$$Q(y,F) = \lambda(y)(\operatorname{tr} F)^2 + 2\mu(y)\left|\operatorname{sym} F\right|^2.$$

We suppose that λ and μ are smooth, Y-periodic functions with $\inf_{y\in Y}\{\,\mu,\, \lambda{+}\mu\,\}{>}0$. This condition implies that Q restricted to symmetric matrices is strictly convex. In analogy to three-dimensional elasticity we define

$$\nu := \frac{\lambda}{\lambda+2\mu} \qquad \text{and} \qquad E := 4\frac{\mu(\lambda+\mu)}{\lambda+2\mu},$$

which are the two-dimensional versions of Poisson's ratio and Young's modulus. Moreover, we follow the convention to write $\overline{\lambda}$ to denote the mean value $\int_Y \lambda(y)\,\mathrm{d}y$ and use the same notation for any map in $L^1(Y)$.

7.3.1. Ansatzes ignoring oscillations

Ansatz 1 (Standard Cosserat ansatz). Let us consider the sequence

$$u_h^{(1)}(x) := u(x_1) + hx_2\, n_{(u)}(x_1), \qquad x \in \Omega$$

corresponding to the situation where the mid line is purely bended and each fiber orthogonal to the mid line remains perpendicular to the mid line and unstretched. In the literature this ansatz is called the standard nonlinear Cosserat ansatz. (The prefactor h originates from the upscaling of the slender domain Ω_h to Ω). An elementary calculation shows that

$$E_h^{(1)} := \frac{R_{(u)}^{\mathrm{T}}\, \nabla_h u_h^{(1)} - Id}{h} = x_2 \boldsymbol{\kappa}_{(u)}(x_1)(e_1 \otimes e_1). \tag{7.10}$$

We plug this expression into the quadratic functional (7.9) and obtain

$$\frac{1}{12}\int_\omega (\lambda(x_1/\varepsilon) + 2\mu(x_1/\varepsilon))\boldsymbol{\kappa}_{(u)}^2(x_1)\,\mathrm{d}x_1.$$

Since the Lamé constants are periodic, we can pass to the limit $h \to 0$ and formally obtain the limiting energy

$$\frac{\tilde{q}^{(1)}}{12}\int_\omega \boldsymbol{\kappa}^2(x_1)\,\mathrm{d}x_1 \quad \text{with} \quad \tilde{q}^{(1)} := \overline{\lambda} + 2\overline{\mu}.$$

Figure 7.3 is a visualization of Ansatz 1. The plot shows a rod that is deformed by the map

$$v^{(1)}(x) = u(x_1) + x_2 n_{(u)}(x_1)$$

where u is a bending deformation with linearly growing curvature. The deformation $v^{(1)}$ is exactly the rescaled deformation corresponding to $u_h^{(1)}$ from Ansatz 1. The deformed mid line of the rod, i.e. the curve associated to u, is represented by the bold red line. The local strain is indicated by the deformed grid. Moreover, the coloring represents the locally stored energy. We see that the stored energy increases towards the longitudinal boundaries with a rate proportional to the curvature of the mid line. This is not surprising, since (7.10) reveals that at each material point $(x_1, x_2) \in \Omega$ the rod is stretched (resp. compressed) in the lateral direction by an amount proportional to $hx_2\boldsymbol{\kappa}_{(u)}(x_1)$. This can also be observed in the enlarged plot in Figure 7.3.

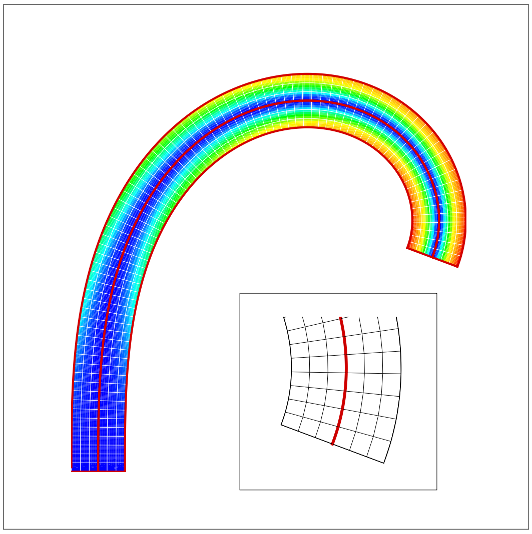

Figure 7.3.: Ansatz 1

The fibers orthogonal to the mid line of the rod are not stretched as (7.10) shows. As a consequence, we have a certain symmetry in Figure 7.3: The deformed mid line $\{ u(x_1) : x_1 \in \omega \}$ remains in the "middle" of the deformed rod.

We claim that allowing stretch in the cross-sectional direction (and breaking the symmetry) leads to lower energies. This can be seen as follows: Consider a sample of an isotropic elastic material that is uniformly extended in direction e_1, for instance by the map $x \mapsto x + (\alpha x_1, x_2)$ with $\alpha > 0$. We observe that such a deformation not only induces stress in direction e_1 but also "transverse stress" in the perpendicular direction. If we allow the material to relax by considering deformations $x \mapsto x + (\alpha x_1, \beta x_2)$ where $\beta \in \mathbb{R}$ is a free parameter, then we observe that the sample "contracts" in the perpendicular direction in order to minimize the amount of "transversal stress" (and thus, the elastic energy). This effect is called Poisson's effect and the ratio $-\alpha/\beta$ for optimal β is given by Poisson's ratio ν. A similar analysis of the stress tensor corresponding to Ansatz 1 reveals that the Cauchy stress vector in the cross-sectional direction does not vanish (for $x_2 \neq 0$ and $\boldsymbol{\kappa}_{(u)}(x_1) \neq 0$). In order to incorporate Poisson's effect, we are going to extend Ansatz 1 by allowing stretching of the fibers orthogonal to the mid line.

Ansatz 2 (Director correction). We consider the sequence

$$u_h^{(2)}(x) := u^{(1)}(x) + h w_h^{(2)}(x), \qquad w_h^{(2)}(x) := h R_{(u)}(x_1) d(x)$$

where $d : \Omega \to \mathbb{R}^2$ is a smooth map. Therefore, the deformation of fibers orthogonal to the mid line is determined by the director field

$$n_{(u)}(x_1) + w_h^{(2)}(x) = R_{(u)}(e_2 + h d(x)).$$

In contrast to Ansatz 1, we see that the fibers are possibly inhomogeneously stretched and do not need to remain perpendicular to the mid line after deformation. In view of this, the map $w_h^{(2)}$ can be called a corrector term that renders the deviation of the director to the normal field. We would like to remark that Ansatz 2 is an adaption of the recovery sequence used in [FJM02] to the rod setting.

As before, we compute

$$E_h^{(2)} := \frac{R_{(u)}^{\mathrm{T}} \nabla_h u_h^{(2)} - Id}{h} = x_2 \boldsymbol{\kappa}_{(u)} (e_1 \otimes e_1) + \partial_2 d \otimes e_2 + O(h)$$

and formally obtain the limiting energy

$$\int_{\Omega} (\bar{\lambda} + 2\bar{\mu}) \Big(x_2^2 \boldsymbol{\kappa}_{(u)}^2 + (\partial_2 d_2)^2 \Big) + 2\bar{\lambda} x_2 \boldsymbol{\kappa}_{(u)} \partial_2 d_2 + \bar{\mu} (\partial_2 d_1)^2 \, \mathrm{d}x$$

where $d = (d_1, d_2)^{\mathrm{T}}$. A simple calculation shows that the right hand side is minimized for $d = d^\star$ with

$$d_1^\star = 0, \qquad d_2^\star = -\boldsymbol{\kappa}_{(u)} \widetilde{\nu} \, \frac{x_2^2}{2} \quad \text{where} \quad \widetilde{\nu} := \frac{\bar{\lambda}}{\bar{\lambda} + 2\bar{\mu}}.$$

This corresponds to the relaxed limiting energy

$$\frac{\tilde{q}^{(2)}}{12} \int_{\omega} \boldsymbol{\kappa}_{(u)}^2 (x_1) \, \mathrm{d}x_1 \quad \text{with} \quad \tilde{q}^{(2)} := 4 \frac{\bar{\mu}(\bar{\lambda} + \bar{\mu})}{\bar{\lambda} + 2\bar{\mu}}$$

which is recovered by the "optimal" sequence

$$u_h^{(2\star)}(x) = u^{(1)}(x) - \frac{h^2 x_2^2}{2} \boldsymbol{\kappa}_{(u)} \widetilde{\nu} \, n_{(u)}.$$

Figure 7.4 visualizes the situation from Figure 7.3 adapted to Ansatz 2. The rod is deformed by the map

$$v^{(2)}(x) = u(x_1) + x_2 n_{(u)}(x_1) - \tfrac{1}{2} x_2^2 \widetilde{\nu} \boldsymbol{\kappa}_{(u)}(x_1) n_{(u)}(x_1)$$

which is exactly the rescaled deformation corresponding to $u_h^{(2\star)}$. Compared with Ansatz 1, we see that the strain energy is drastically reduced. The enlarged section shows that the segments above the mid line are contracted in the lateral direction, while the segments below the mid line are elongated.

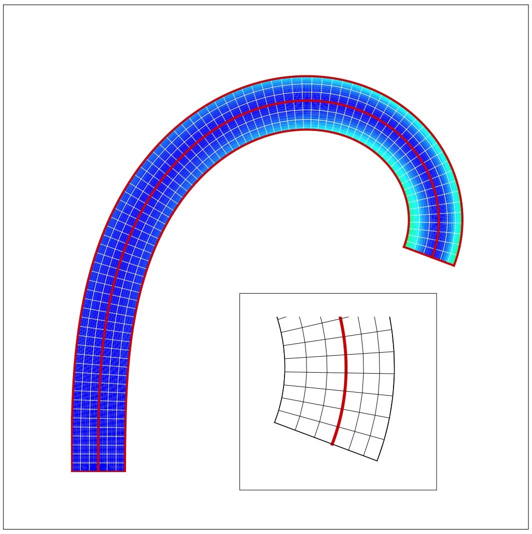

Figure 7.4.: Ansatz 2

In the isotropic case (as discussed here) the optimal corrector contracts the rod in a direction perpendicular to the mid line. Analytically, this corresponds to the equation $d_1^\star = 0$. We would like to emphasize that this is a specific property of isotropic materials and does not hold in general.

In the case where λ and μ are constant, and therefore effects due to homogenization are absent, it turns out that Ansatz 2 is already optimal in the sense that it recovers the energy given by the rigor Γ-limit. In this case $\tilde{q}^{(2)}$ and $\tilde{\nu}$ are equal to Young's modulus and Poisson's ratio respectively.

7.3.2. Ansatzes featuring oscillations

The ansatzes discussed so far ignore effects due to rapid oscillations of the material. Clearly, for a composite material consisting of a strong and a very weak component it is energetically preferable if most of the stress emerges in the weak component. Consequently, if the composite has a microstructure on scale ε, it is natural to expect a deformation with low energy also to oscillate on scale ε. In the following, we study two different types of ansatzes that feature oscillations.

Ansatz 3 (Oscillating curvature). The idea of the first oscillating ansatz is the following: First, we construct a bending deformation that on the one hand, allows for small oscillations in its curvature on scale and magnitude ε, and on the other hand has a macroscopic shape that equals u. In a second step, we extend this oscillating bending deformation to a two-dimensional deformation by applying the standard Cosserat ansatz introduced above. This procedure leads to the sequence

$$u_h^{(3)}(x) := u(0) + \int_0^{x_1} R_\varepsilon(s) t_{(u)}(s)\,\mathrm{d}s + h x_2 R_\varepsilon(x_1) n_{(u)}(x_1)$$

where $R_\varepsilon : \omega \to SO(2)$ is a smooth rotation field which is close to Id and periodic on scale ε. One can check that (u_h) approximates u in the sense of (1) ii. if $R_\varepsilon \to Id$ strongly in $L^2(\omega; \mathbb{M}(2))$. A straightforward computation shows that

$$\nabla_h u_h^{(3)} = R_\varepsilon R_{(u)} + h x_2 (\, \partial_1 R_\varepsilon n_{(u)} + R_\varepsilon \partial_1 n_{(u)}) \otimes e_1$$

and

$$E_h^{(3)} := \frac{(R_\varepsilon R_h)^{\mathrm{T}}\, \nabla_h u_h^{(2)} - Id}{h} = x_2 \left(\boldsymbol{\kappa}_{(u)}(x_1) e_1 + R_{(u)}^{\mathrm{T}} (R_\varepsilon^{\mathrm{T}} \partial_1 R_\varepsilon) n \right) \otimes e_1.$$

We specify the rotation field R_ε by setting

$$R_\varepsilon(x_1) := \mathcal{R}(\varepsilon(\alpha \circ \pi_\varepsilon)(x_1)) \quad \text{where} \quad \alpha \in C^\infty(\omega; C^\infty_{\mathrm{per}}(Y)) \quad \text{and} \quad \pi_\varepsilon(x_1) := (x_1, {}^{x_1}\!/\varepsilon).$$

Recall that $\mathcal{R}(\beta)$ denotes the unique clockwise rotation by angle β; thus, we see that

$$R_\varepsilon^{\mathrm{T}} \partial_1 R_\varepsilon = \big(\varepsilon(\partial_1 \alpha) \circ \pi_\varepsilon + (\partial_y \alpha) \circ \pi_\varepsilon \,\big) \mathcal{R}(\pi/2),$$

and since $\mathcal{R}(\pi/2) n_{(u)} = -t_{(u)}$, we arrive at

$$E_h^{(3)} = x_2 \left(\boldsymbol{\kappa}_{(u)}(x_1) - \big(\varepsilon(\partial_1 \alpha) \circ \pi_\varepsilon + (\partial_y \alpha) \circ \pi_\varepsilon \,\big) \right) e_1 \otimes e_1.$$

As a consequence, in virtue of Lemma 2.1.9 the sequence $(E_h^{(3)})$ strongly two-scale converges to the map $x_2(\boldsymbol{\kappa}_{(u)}(x_1) - \partial_y \alpha(x_1, y))(e_1 \otimes e_1)$ and by applying Lemma 3.2.1

we (formally) obtain the limiting energy

$$\frac{1}{12}\iint\limits_{\omega\times Y} Q(y,(\boldsymbol{\kappa}_{(u)}(x_1)-\partial_y\alpha(x_1,y))(e_1\otimes e_1))\,\mathrm{d}y\,\mathrm{d}x_1$$
$$=\frac{1}{12}\iint\limits_{\omega\times Y}(\lambda+2\mu)(\,\boldsymbol{\kappa}_{(u)}-\partial_y\alpha\,)^2\,\mathrm{d}y\,\mathrm{d}x_1.$$

The right hand side is minimized for the function

$$\alpha^\star(x_1,y):=\boldsymbol{\kappa}_{(u)}(x_1)\int_0^y 1-\rho(\tau)\,\mathrm{d}\tau \quad \text{with} \quad \rho(y):=\tfrac{1}{\lambda(y)+2\mu(y)}\left(\int_Y \tfrac{1}{\lambda(y)+2\mu(y)}\,\mathrm{d}y\right)^{-1}.$$

This corresponds to the relaxed limiting energy

$$\frac{\tilde q^{(3)}}{12}\int_\omega \boldsymbol{\kappa}^2_{(u)}(x_1)\,\mathrm{d}x_1 \quad \text{with} \quad \tilde q^{(3)}:=\left(\int_Y \frac{1}{\lambda+2\mu}\,\mathrm{d}y\right)^{-1}.$$

The harmonic mean in the definition of $\tilde q^{(3)}$ is a typical average which emerges in the context of homogenization.

Figure 7.5 is a visualization of Ansatz 3. It shows a laterally periodic rod consisting of stiff and soft components. The rod is deformed by a two-dimensional deformation that is associated to a bending deformation with constant positive curvature. The deformation is constructed according to Ansatz 3 by the procedure described above in a situation where $h \sim \varepsilon$.

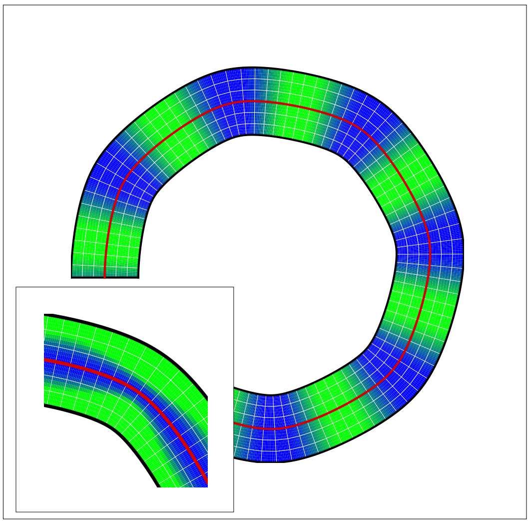

Figure 7.5.: Ansatz 3

In the larger plot in Figure 7.5 the coloring indicates the strength of the material, where blue means soft and green means stiff. The smaller plot in Figure 7.5 shows an enlarged section of the deformed rod. Here, the coloring indicates the locally stored energy.

As before, the bold red line represents the deformed mid line of the rod. As expected, the curvature of the deformed mid line is not constant, but is larger in the weak (blue) parts of the rod and close to zero in the stiff (green) parts of the rod. Because of this inhomogeneous bending, the lateral oscillation of the local stress is quite mild (as it is illustrated in the enlarged section in Figure 7.5).

Because Ansatz 1 is a special case of Ansatz 3 (corresponding to $\alpha = 0$), we have $\tilde q^{(3)} \leq \tilde q^{(1)}$. Qualitatively speaking, the higher the contrast of the elastic components, the larger is the difference between Ansatz 1 and Ansatz 3.

We can easily extend Ansatz 3 by adding the corrector $w_h^{(2)}$ from Ansatz 2. It turns out that for isotropic materials this extended ansatz is already rich enough to recover the Γ-limit of $(\mathcal{I}^{\varepsilon,h})$ corresponding to the fine-scale coupling $h/\varepsilon \to \infty$.

Ansatz 4 (Oscillating director correction). For periodic materials it is reasonable that the relaxation related to Poisson's effect is resolved on the level of the material fine-scale. For this reason, we extend Ansatz 3 and allow oscillations of the director field. In the sequel we have to suppose that

$$\lim_{h\to 0} \frac{h}{\varepsilon} = 0. \tag{7.11}$$

Let $R_\varepsilon : \omega \to SO(2)$ denote the rotation field introduced in the previous ansatz and define

$$w_h^{(4)}(x) := \frac{h\, x_2^2}{2} R_\varepsilon R_{(u)} \left(\phi \circ \pi_\varepsilon\right) \quad \text{where} \quad \phi \in C_0^\infty(\omega; C^\infty_{\text{per}}(Y;\mathbb{R}^2)).$$

We compute

$$(R_\varepsilon R_{(u)})^{\mathrm{T}} \nabla_h w_h^{(4)} = x_2((\phi \circ \pi_\varepsilon)\otimes e_2) + \frac{h}{\varepsilon}\frac{x_2^2}{2}\left((\partial_y \phi \circ \pi_\varepsilon)\otimes e_1\right) + O(h).$$

In view of (7.11), we see that the second term on the right hand side is of higher order and vanishes uniformly in the limit. Now we consider the deformation $u_h^{(4)} := u^{(3)} + h w_h^{(4)}$ and compute

$$\begin{aligned} E_h^{(4)} := \frac{(R_\varepsilon R_{(u)})^{\mathrm{T}} \nabla_h u_h^{(4)} - Id}{h} \\ = x_2\left(\boldsymbol{\kappa}_{(u)} - (\partial_y \alpha \circ \pi_\varepsilon)\right)(e_1 \otimes e_1) + x_2((\phi\circ\pi_\varepsilon)\otimes e_2) + O(h). \end{aligned}$$

The sequence $(E_h^{(4)})$ strongly two-scale converges and (formally) we obtain the limiting energy

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(E_h^{(4)}) = \frac{1}{12}\iint_{\omega\times Y} Q\left(y, \begin{bmatrix} \boldsymbol{\kappa}_{(u)} - \partial_y \alpha & \phi_1 \\ 0 & \phi_2 \end{bmatrix}\right) \mathrm{d}y\, \mathrm{d}x_1$$

where $\phi = (\phi_1, \phi_2)^{\mathrm{T}}$. Minimization over all admissible ϕ and α yields the relaxed limiting energy

$$\frac{\tilde{q}^{(4)}}{12}\int_\omega \boldsymbol{\kappa}_{(u)}^2\, \mathrm{d}x_1 \quad \text{with} \quad \tilde{q}^{(4)} = \left(\int_Y \frac{1}{E(y)}\, \mathrm{d}y\right)^{-1}$$

which is recovered by the optimal parameters

$$\alpha^\star(x_1, y) := \boldsymbol{\kappa}_{(u)}(x_1)\int_0^y 1 - \frac{\tilde{q}^{(4)}}{E(\tau)}\, \mathrm{d}\tau \quad \text{and} \quad \phi_1^\star = 0,\ \phi_2^\star(x_1,y) = -\tilde{q}^{(4)}\frac{\nu}{E(y)}\boldsymbol{\kappa}_{(u)}(x_1).$$

Figure 7.6 is the equivalent to Figure 7.5 for Ansatz 4. Additional to the inhomogeneous lateral bending, we see that also the fibers orthogonal to the mid line are inhomogeneously stretched in order to compensate for Poisson's effect. The enlarged plot in Figure 7.6 shows that this effect is coupled to the curvature of the mid line as well as to the stiffness of the material. Moreover, the enlarged plot shows that in comparison to Figure 7.5, the energy has drastically reduced.

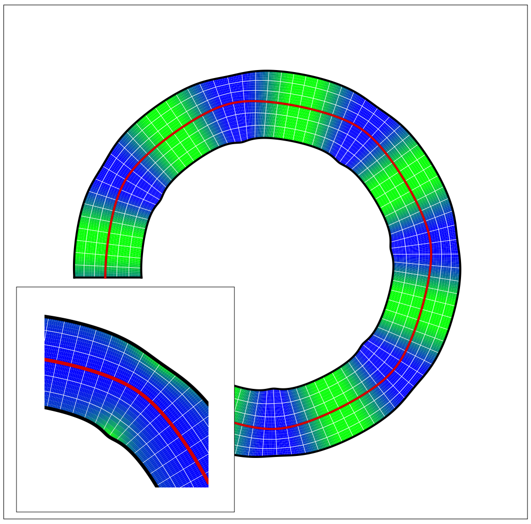

Figure 7.6.: Ansatz 4

Ansatz 4 entails the previous ansatzes as special cases. Consequently, we have $\min\{\tilde{q}^{(1)}, \tilde{q}^{(2)}, \tilde{q}^{(3)}\} \geq \tilde{q}^{(4)}$. It turns out that Ansatz 4 is rich enough to recover the Γ-limit of $(\mathcal{I}^{\varepsilon,h})$ corresponding to the fine-scale coupling $h/\varepsilon \to \infty$. In this case (and for isotropic materials) $\tilde{q}^{(4)}$ is exactly the harmonic mean of Young's modulus.

7.4. Proof of the main result

This section is devoted to the proof of Theorem 7.2.5. We briefly outline our strategy:

(1) **(Compactness).** In the first part of Section 7.4.1 we show that a sequence (u_h) of deformations with *finite bending energy* is relatively compact. More precisely, we show that up to subsequence $(u_{h'})$ and $(\nabla_{h'} u_{h'})$ strongly converge and the sequence of associated nonlinear strain

$$E_h := \frac{\sqrt{\nabla_h u_h^{\mathrm{T}} \nabla_h u_h} - Id}{h}$$

weakly two-scale converges. Moreover, the compactness result (summarized in Theorem 7.4.2 below) reveals that the limiting deformation can be identified with a bending deformation, i.e. a map in $W^{2,2}_{\text{iso}}(\omega; \mathbb{R}^2)$.

This insight relies on an approximation scheme that allows to approximate the scaled gradient $\nabla_h u_h$ by a map R_h from ω to $SO(2)$ with an error controlled by the L^2-distance of $\nabla_h u_h$ to $SO(2)$.

In [FJM02] (for elastic plates) and in [MM03] (for elastic rods) an approximation of this type with a piecewise constant map R_h has been derived by applying the geometric rigidity estimate (see Theorem 7.4.6). In our setting, it is necessary (in particular for the analysis in the next step) to develop an approximation scheme that allows to choose the map R_h in $W^{1,2}(\omega; SO(2))$ and to control the two-scale convergence behavior of $\partial_1 R_h$ as $h \to 0$. More precisely, we show that in the cases $h \sim \varepsilon$ and $h \gg \varepsilon$ the sequence R_h can be constructed in such a way that $\partial_1 R_h$ does "not carry oscillations" in the two-scale sense. This is done in Proposition 7.4.4 and Proposition 7.4.12.

(2) **(Two-scale characterization of the limiting strain).** Theorem 7.4.10 establishes a link between the limiting deformation u and the two-scale limit E of the sequence (E_h). It turns out that E admits a presentation in the form

$$E(x, y) = x_2 \boldsymbol{\kappa}_{(u)}(x_1)(e_1 \otimes e_1) + G(x, y)$$

where $\boldsymbol{\kappa}_{(u)}$ is the curvature of the limiting deformation and G a "relaxation profile" that captures the oscillation properties of the sequence (E_h). We are going to see that the general structure of the map G depends in a subtle way on the limiting behavior of the ratio between h and ε.

Generally speaking, the key insight in the proof of this result is a decomposition of the form

$$u_h = \Big(v_h(x_1) + h x_2 n_{(v_h)}(x_1) \Big) + h w_h$$

where v_h is a one-dimensional map, close to a bending deformation that is constructed on the basis of the approximation R_h, and w_h is a corrector term. The map in the braces can be interpreted as the extension of v_h by a standard Cosserat Ansatz (cf. Section 7.3). Since the construction of v_h is quite explicit, we can easily characterize the contribution of the term in the braces to the limiting strain E. On the other side, we can characterize the contribution of the corrector term w_h by means of the *two-scale characterization of scaled gradients* (see Theorem 6.3.3).

(3) **(Lower bound).** The proof of the lower bound part of Theorem 7.2.5 consists of two stages: First, we derive a *lower bound by means of the nonlinear limiting strain* E (see Lemma 7.4.13). The derivation is mainly based on the *simultaneous homogenization and linearization* result (Theorem 5.2.1) combined with the compactness part of the main Theorem 7.2.5. Secondly, we combine the derived lower bound with the *two-scale characterization of the limiting strain*. Eventually, some minor modifications of the resulting lower bound complete the proof of Theorem 7.2.5 (2).

(4) **(Upper bound).** In Section 7.4.4 we construct recovery sequences in three steps. First, we present a construction for smooth data (see Proposition 7.4.14). Secondly, we prove in Lemma 7.4.17 that $C^\infty_{\mathrm{iso}}(\overline{\omega}; \mathbb{R}^2)$ is dense in $W^{2,2}_{\mathrm{iso}}(\omega; \mathbb{R}^2)$ and eventually, we lift the smooth construction to the general case by a diagonal sequence construction that is due to H. Attouch.

(5) **(Cell problem).** The program (1) - (4) proves Γ-convergence of $(\mathcal{I}^{\varepsilon,h})$ to the functional $\mathcal{I}_\gamma$, which is implicitly defined by means of an relaxation procedure. In Section 7.4.5 we show that $\mathcal{I}_\gamma$ can be identified with the energy $\mathcal{E}_\gamma(\cdot; 0)$. Therefore, we analyze the linear cell problem that determines the effective coefficient q_γ and that establishes the link between $\mathcal{I}_\gamma$ and $\mathcal{E}_\gamma(\cdot; 0)$.

7.4.1. Compactness

In this section we prove Proposition 7.2.7 and the compactness part of Theorem 7.2.5. The section is outlined as follows. We start with a short discussion pointing out the challenge in the proof of the compactness result. In the main part of this section (see page 123 et seq.) we present a careful approximation of the scaled gradient by piecewise affine maps in $W^{1,2}(\omega;\mathbb{M}(2)) \cap L^\infty(\omega;SO(2))$. Eventually, we elaborate on the one-sided boundary condition and prove Proposition 7.2.7 (see page 131 et seq.).

Below, c_0, c_1, c_2 denote generic positive constants which may change from line to line, but can be chosen independent of h. Furthermore, we assume without loss of generality that $h \leq 1$. Let us consider a sequence (u_h) of deformations in $W^{1,2}(\Omega;\mathbb{R}^2)$ with finite bending energy. We denote the integral average of u_h by $u_{\Omega,h}$. Thus, the Poincaré-Wirtinger inequality implies that

$$\|u_h - u_{\Omega,h}\|^2_{W^{1,2}(\Omega;\mathbb{R}^2)} \leq c_0 \int_\Omega |\nabla u_h|^2 \, \mathrm{d}x.$$

Because of the non-degeneracy condition (W3) and the inequality

$$\forall F \in \mathbb{M}(2) \, : \, \operatorname{dist}^2(F, SO(2)) \geq \frac{1}{2} |F|^2 - \sqrt{2},$$

we see that

$$\|u_h - u_{\Omega,h}\|^2_{W^{1,2}(\Omega;\mathbb{R}^2)} \leq c_0 \int_\Omega |\nabla_h u_h|^2 \, \mathrm{d}x \leq c_1 \, h^2 \mathcal{I}^{\varepsilon(h),h}(u_h) + c_2. \tag{7.12}$$

Hence, the sequence $(u_h - u_{\Omega,h})$ is bounded in $W^{1,2}(\Omega;\mathbb{R}^2)$ and we obtain

$$\begin{aligned} u_h - u_{\Omega,h} &\rightharpoonup u && \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^2) \\ \nabla_h u_h &\rightharpoonup (\partial_1 u \,|\, d) && \text{weakly in } L^2(\Omega;\mathbb{M}(2)). \end{aligned}$$

for a subsequence (not relabel) and suitable maps $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ and $d \in L^2(\Omega;\mathbb{R}^2)$. The boundedness of $(\frac{1}{h}\partial_2 u_h)$ in $L^2(\Omega;\mathbb{M}(2))$ implies that $\partial_2 u_h$ strongly converges to zero; thus, u only depends on x_1 and can be identified with a map in $W^{1,2}(\omega;\mathbb{R}^2)$. Furthermore, the compactness of the embedding $W^{1,2}(\Omega;\mathbb{R}^2) \subset L^2(\Omega;\mathbb{R}^2)$ yields strong convergence of $u_h - u_{\Omega,h}$ to u in L^2.

We recall that the scaled nonlinear strain associated to u_h is defined by

$$E_h := \frac{\sqrt{\nabla_h u_h^{\mathrm{T}} \, \nabla_h u_h} - Id}{h}.$$

Because of the estimate $\operatorname{dist}^2(F, SO(2)) \geq \left|\sqrt{F^{\mathrm{T}}F} - Id\right|^2$, we immediately deduce that (E_h) is bounded in $L^2(\Omega;\mathbb{M}_{\mathrm{sym}}(2))$, and therefore is weakly two-scale relatively compact. So far we have proved the following statement:

Lemma 7.4.1. *Let $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ be a sequence with finite bending energy. There are maps*

$$u \in W^{1,2}(\omega;\mathbb{R}^2), \qquad U \in L^2(\Omega,\mathbb{M}(2)) \quad \textit{with} \quad Ue_1 = \partial_1 u$$
$$\textit{and} \quad E \in L^2(\Omega{\times}Y;\mathbb{M}(2))$$

such that

$$\begin{aligned} u_h - u_{\Omega,h} &\to u && \textit{weakly in } W^{1,2}(\Omega;\mathbb{R}^2)\\ \nabla_h u_h &\rightharpoonup U && \textit{weakly in } L^2(\Omega;\mathbb{M}(2))\\ E_h &\xrightarrow{2} E && \textit{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{M}(2)) \end{aligned}$$

for a subsequence (not relabeled). Here, $u_{\Omega,h} := \frac{1}{\mathcal{H}^n(\Omega)}\int_\Omega u_h\,\mathrm{d}x$ denotes the integral average of u_h.

In order to complete the proof of the compactness result, it remains to show that $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ or equivalently that $U \in W^{1,2}(\omega;\mathbb{M}(2))$ and $U(x_1) \in SO(2)$ for almost every $x_1 \in \omega$. It turns out that this is a hard problem and cannot be verified by elementary methods like those used in the discussion so far. The difficulty is caused by the non-convexity of the set $SO(2)$. We elaborate on this statement in the following lines using some arguments borrowed from [Con03].

Because the sequence (u_h) has finite bending energy, we can assume without loss of generality that

$$\int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(2))\,\mathrm{d}x \le h^2 c_0. \tag{7.13}$$

Let us assume for a moment that $\nabla_h u_h$ strongly converges to U in $L^2(\Omega;\mathbb{M}(2))$. Then (7.13) implies that $U(x) \in SO(2)$ for almost every $x \in \Omega$ and we deduce that u is an isometric immersion. However, if the sequence $(\nabla_h u_h)$ converges to U only in the weak topology, then in general only the weaker estimate

$$\int_\Omega \theta(U)\,\mathrm{d}x \le \liminf_{h\to 0}\int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(2))\,\mathrm{d}x = 0$$

holds, where $\theta(\cdot)$ is the convex hull of the map $\mathbb{M}(2) \ni F \mapsto \operatorname{dist}^2(F, SO(2))$. A direct calculation (see [Con03]) shows that the null set of θ is not $SO(2)$, but the set of $2{\times}2$ matrices F satisfying $F^{\mathrm{T}}F \le Id$.

In the next paragraph we overcome this failure by introducing an approximation scheme that enables us to approximate $\nabla_h u_h$ by a map

$$R_h \in W^{1,2}(\omega;SO(2)) := W^{1,2}(\omega;\mathbb{M}(2)) \cap L^\infty(\omega;SO(2))$$

in such way that

$$\left\|\frac{\nabla_h u_h - R_h}{h}\right\|^2_{L^2(\Omega;\mathbb{M}(2))} + \|\partial_1 R_h\|^2_{L^2(\omega;\mathbb{M}(2))}$$

is bounded by a constant independent of h, provided (u_h) has finite bending energy. As a consequence, we obtain the following result:

Theorem 7.4.2. *Let $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ be a sequence with finite bending energy. There is a bending deformation $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ and a map $E \in L^2(\Omega\times Y;\mathbb{M}_{\text{sym}}(2))$ such that*

$$u_h - u_{\Omega,h} \to u \qquad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^2) \tag{7.14}$$

$$\nabla_h u_h \to R_{(u)} \qquad \text{strongly in } L^2(\Omega;\mathbb{M}(2)) \tag{7.15}$$

$$E_h \xrightarrow{2} E \qquad \text{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(2)) \tag{7.16}$$

for a subsequence (not relabeled). Here, $u_{\Omega,h} := \frac{1}{\mathcal{H}^n(\Omega)}\int_\Omega u_h \,\mathrm{d}x$ denotes the integral average of u_h.

(For the proof see page 130).

Remark 7.4.3. In view of the non-degeneracy condition (W3) it is clear that any sequence (u_h) with equibounded energy has finite bending energy; thus, the previous theorem immediately implies the validity of the compactness part (statement (1)) of Theorem 7.2.5.

Approximation of the scaled gradient based on geometric rigidity

In the first part of this paragraph we present a careful approximation for scaled deformation gradients of maps in

$$W^{1,2}((0,L)\times S;\mathbb{R}^n) \qquad \text{with} \qquad S \subset \mathbb{R}^{n-1}$$

by piecewise constant maps from $(0,L)$ to $SO(n)$. We prove this statement for dimensions $n \geq 2$. In particular, in the current and the subsequent chapter we apply the result with $n = 2, 3$.

In the second part (see Proposition 7.4.7) we consider two-dimensional deformations $u \in W^{1,2}(\Omega;\mathbb{R}^2)$. We carefully regularize the piecewise constant approximation of the corresponding scaled gradient $\nabla_h u$. In this way we obtain a scheme that allows to approximate the scaled gradient $\nabla_h u$ by a map in $W^{1,2}(\omega;SO(2))$, the derivative of which is coherent to an ϵ-lattice for a given small parameter ϵ. We would like to remark that the approximation scheme is tailor-made for sequences of deformations with finite bending energy and plays an important role, not only in the proof of the compactness result, but also for the two-scale characterization of the limiting strain.

Proposition 7.4.4. *Let S denote an open and bounded Lipschitz domain in $\mathbb{R}^{n-1}$. Set $\omega := (0,L)$ and $\Omega := \omega\times S$. Suppose that $0 < h, \epsilon < L$ satisfy*

$$\gamma_0 \leq \frac{h}{\epsilon} \leq \frac{1}{\gamma_0}$$

for a positive number γ_0. To any $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ we can assign a ϵ-coherent map $R : \omega \to SO(n)$ such that

$$\|R - \nabla_{1,h}\, u\|^2_{L^2(\Omega;\mathbb{M}(n))} + \epsilon \operatorname{Var}_2 R \le C \int_\Omega \operatorname{dist}^2(\nabla_{1,h}\, u(x), SO(n))\, \mathrm{d}x.$$

The constant C only depends on γ_0 and the geometry of S.

Remark 7.4.5. In the proposition above $\nabla_{1,h}\, u$ is the scaled gradient introduced in Definition 6.3.1. In the two-dimensional case (as considered in this chapter) we have $\nabla_{1,h}\, u = \nabla_h u$. Moreover, we like to remind that the notion of a coherent map is defined in Section 2.2.

The result is a refinement of a similar statement in [FJM02] and is based on the geometric rigidity estimate by G. Friesecke, R.D. James and S. Müller:

Theorem 7.4.6 (Geometric rigidity [FJM02])**.** *Let A be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$. There exists a constant $C(A)$ with the following property: For each $v \in W^{1,2}(A;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\int_A |\nabla v(x) - R|^2 \,\mathrm{d}x \le C(A) \int_A \operatorname{dist}^2(\nabla v(x), SO(n))\, \mathrm{d}x.$$

The constant $C(A)$ can be chosen uniformly for a family of domains which are bilipschitz equivalent with controlled Lipschitz constants. The constant $C(A)$ is invariant under dilations.

The rough idea is the following: In a first step, we cover Ω by a union of cylindrical sets of the form

$$U(\xi) = [\xi, \xi + \epsilon) \times S \qquad \text{with} \qquad \xi \in \epsilon\mathbb{Z}.$$

Because in general the smallest union of such sets covering Ω is larger than Ω, we carefully extend u to this slightly larger domain. Next, we approximate $\nabla_h u$ on each cylinder by a constant rotation minimizing the L^2-distance to the gradient. In this way we obtain the piecewise constant, ϵ-coherent approximation R. By applying the geometric rigidity theorem to each cylindrical segment separately, we find that

$$\int_\Omega |R - \nabla_{1,h}\, u|^2 \,\mathrm{d}x \le C \int_\Omega \operatorname{dist}^2(\nabla_{1,h}\, u, SO(n))\, \mathrm{d}x,$$

where the constant C only depends on the geometry of S and the ratio ϵ/h. Furthermore, it turns out that the difference $|R(\xi + \epsilon) - R(\xi)|$ (i.e. the variation between neighboring segments) can also be estimated by means of the geometric rigidity theorem. This allows us to estimate the quadratic variation of R.

Proof. In the following we use the convention to decompose any point $x \in \mathbb{R}^n$ according to

$$x = (x_1, \bar{x}) \qquad \text{with } x_1 \in \mathbb{R} \text{ and } \bar{x} \in \mathbb{R}^{n-1}.$$

<u>*Step 1.*</u> Set $\hat\omega := \{\, x_1 \in \mathbb{R} \,:\, \operatorname{dist}(x_1,\omega) < 2\gamma_0 h \,\}$ and $\widehat\Omega := \hat\omega \times S$. We claim that there exists an extension $u^{\rm ex} \in W^{1,2}(\widehat\Omega;\mathbb{R}^n)$ of u such that

$$\int_{\widehat\Omega} \operatorname{dist}^2(\nabla_{1,h}\, u^{\rm ex}(x), SO(n))\,\mathrm{d}x \le (1+\gamma_0 C_S)\int_\Omega \operatorname{dist}^2(\nabla_{1,h}\, u(x), SO(n))\,\mathrm{d}x \tag{7.17}$$

where the constant C_S only depends on the geometry of S. We postpone the proof to Step 5.

<u>*Step 2.*</u> For convenience we set

$$U(\xi) := [\xi, \xi+\epsilon)\times S \quad \text{and} \quad \widehat U(\xi) := [\xi-\epsilon, \xi+\epsilon)\times S \qquad \text{for all } \xi \in \epsilon\mathbb{Z}.$$

Let $\mathcal{L}$ denote the smallest subset of $\epsilon\mathbb{Z}$ such that $\Omega \subset \bigcup_{\xi\in\mathcal{L}} U(\xi)$. Note that by construction, for all $\xi \in \mathcal{L}$ the cylindrical sets $U(\xi)$ and $\widehat U(\xi)$ are contained in $\widehat\Omega$.

We choose discrete maps

$$\mathfrak{r} : \mathcal{L} \to SO(n) \quad \text{and} \quad \hat{\mathfrak{r}} : \mathcal{L} \to SO(n)$$

such that

$$\mathfrak{r}(\xi) \in \operatorname*{argmin}_{R\in SO(n)} \int_{U(\xi)} |\nabla_{1,h}\, u^{\rm ex}(x) - R|^2 \,\mathrm{d}x$$
$$\hat{\mathfrak{r}}(\xi) \in \operatorname*{argmin}_{R\in SO(n)} \int_{\widehat U(\xi)} |\nabla_{1,h}\, u^{\rm ex}(x) - R|^2 \,\mathrm{d}x$$

for all $\xi \in \mathcal{L}$ and define the ϵ-coherent map

$$R : \omega \to SO(n), \qquad R(x) := \sum_{\xi\in\mathcal{L}} 1_{U(\xi)}(x)\,\mathfrak{r}(\xi).$$

<u>*Step 3.*</u> We claim that

$$\int_\Omega |R - \nabla_{1,h}\, u|^2 \,\mathrm{d}x \le C \int_\Omega \operatorname{dist}^2(\nabla_{1,h}\, u, SO(n))\,\mathrm{d}x. \tag{7.18}$$

Here and hereafter, C denotes a constant that may change from line to line, but can be chosen only depending on γ_0 and the geometry of S.

To this end, first note that

$$\int_\Omega |R - \nabla_{1,h}\, u|^2 \,\mathrm{d}x \le \sum_{\xi\in\mathcal{L}} \int_{U(\xi)} |\mathfrak{r}(\xi) - \nabla_{1,h}\, u^{\rm ex}(x)|^2 \,\mathrm{d}x \tag{7.19}$$

because $u^{\rm ex}$ is an extension of u and Ω is contained in the union of the sets $U(\xi)$, $\xi \in \mathcal{L}$. For the sequel, it is convenient to introduce the rescaled map

$$v : \hat\omega\times hS \to \mathbb{R}^n, \qquad v := u^{\rm ex}\circ \pi_h \quad \text{where} \quad \pi_h(x_1,\bar x) := (x_1, \tfrac{1}{h}\bar x).$$

It is easy to check that $\nabla v = \nabla_{1,h}\, u^{\rm ex} \circ \pi_h$, and therefore

$$\int_{U(\xi)} |\mathfrak{r}(\xi) - \nabla_{1,h}\, u^{\rm ex}|^2 \, {\rm d}x = \frac{1}{h} \int_{(\xi,\xi+\epsilon)\times hS} |\mathfrak{r}(\xi) - \nabla v|^2 \, {\rm d}x \qquad \text{for all } \xi \in \mathcal{L}.$$

We apply the geometric rigidity estimate (Theorem 7.4.6) to the integral on the right and obtain

$$\int_{U(\xi)} |\mathfrak{r}(\xi) - \nabla_{1,h}\, u^{\rm ex}|^2 \, {\rm d}x \le \frac{c'}{h} \int_{(\xi,\xi+\epsilon)\times hS} {\rm dist}^2(\nabla v, SO(n))\, {\rm d}x \\ = c' \int_{U(\xi)} {\rm dist}^2(\nabla_{1,h}\, u^{\rm ex}, SO(n))\, {\rm d}x \tag{7.20}$$

where c' only depends on the geometry of the cylindrical integration domain. In a similar manner we obtain the estimate

$$\int_{\widehat{U}(\xi)} |\widehat{\mathfrak{r}}(\xi) - \nabla_{1,h}\, u^{\rm ex}|^2 \, {\rm d}x \le c' \int_{\widehat{U}(\xi)} {\rm dist}^2(\nabla_{1,h}\, u^{\rm ex}, SO(n))\, {\rm d}x. \tag{7.21}$$

Since

$$\left\{ (\xi,\xi+\epsilon)\times hS,\ (\xi,\xi+2\epsilon)\times hS \,:\, \xi \in \mathbb{R},\ h,\epsilon > 0 \,\text{with}\, \gamma_0 \le \tfrac{h}{\epsilon} \le \tfrac{1}{\gamma_0} \right\}$$

is a family of cylindrical domains which are Bilipschitz equivalent with controlled Lipschitz constant, we can choose the constant c' in (7.20) and (7.21) in such a way that it only depends on the geometry of S and on γ_0.

By combining (7.19) and (7.20), we see that

$$\int_\Omega |R - \nabla_{1,h}\, u|^2 \, {\rm d}x \le c' \int_{\widehat{\Omega}} {\rm dist}^2(\nabla_{1,h}\, u^{\rm ex}(x), SO(n))\, {\rm d}x.$$

In virtue of (7.17), estimate (7.18) follows.

<u>*Step 4.*</u> We estimate the variation of R. Since ω is a one-dimensional interval and R a ϵ-coherent map, we can rewrite the variation of R as follows:

$${\rm Var}_2\, R = \sum_{\xi \in \mathcal{L}\setminus \min \mathcal{L}} |\mathfrak{r}(\xi) - \mathfrak{r}(\xi-\epsilon)|^2 \,.$$

We estimate each term of the sum on the right hand side separately. To this end, let $\xi \in \mathcal{L} \setminus \min \mathcal{L}$. By construction $U(\xi)$ as well as $U(\xi-\epsilon)$ are contained in $\widehat{U}(\xi)$. This motivates the following calculation:

$$\begin{aligned}
|\mathfrak{r}(\xi) - \mathfrak{r}(\xi-\epsilon)|^2 &\le 2\Big(|\mathfrak{r}(\xi) - \widehat{\mathfrak{r}}(\xi)|^2 + |\widehat{\mathfrak{r}}(\xi) - \mathfrak{r}(\xi-\epsilon)|^2 \Big) \\
&\le \frac{2}{\epsilon\,|S|} \left(\int_{U(\xi)} |\mathfrak{r}(\xi) - \widehat{\mathfrak{r}}(\xi)|^2 \, {\rm d}x + \int_{U(\xi-\epsilon)} |\widehat{\mathfrak{r}}(\xi) - \mathfrak{r}(\xi-\epsilon)|^2 \, {\rm d}x \right) \\
&\le \frac{4}{\epsilon\,|S|} \left(\int_{U(\xi)} |\mathfrak{r}(\xi) - \nabla_{1,h}\, u^{\rm ex}(x)|^2 \, {\rm d}x + \int_{\widehat{U}(\xi)} |\widehat{\mathfrak{r}}(\xi) - \nabla_{1,h}\, u^{\rm ex}(x)|^2 \, {\rm d}x + \right. \\
&\qquad \left. \int_{\widehat{U}(\xi)} |\widehat{\mathfrak{r}}(\xi) - \nabla_{1,h}\, u^{\rm ex}(x)|^2 \, {\rm d}x + \int_{U(\xi-\epsilon)} |\mathfrak{r}(\xi-\epsilon) - \nabla_{1,h}\, u^{\rm ex}(x)|^2 \, {\rm d}x \right)
\end{aligned}$$

where $|S|$ denotes the $n-1$-dimensional Hausdorff measure of S.

By applying estimates (7.20) and (7.21) to the integrals on the right hand side, we obtain

$$|\mathfrak{r}(\xi)-\mathfrak{r}(\xi-\epsilon)|^2 \le \frac{16\,c'}{\epsilon\,|S|}\int_{\widehat{U}(\xi)} \operatorname{dist}^2(\nabla_{1,h}\,u^{\mathrm{ex}}(x),SO(n))\,\mathrm{d}x.$$

By summing over all $\xi\in\mathcal{L}\setminus\min\mathcal{L}$ we arrive at

$$\operatorname{Var}_2(R)\le \frac{32\,c'}{\epsilon\,|S|}\int_{\widehat{\Omega}} \operatorname{dist}^2(\nabla_{1,h}\,u^{\mathrm{ex}}(x),SO(n))\,\mathrm{d}x,$$

because each $x\in\widehat{\Omega}$ is contained in at most two of the sets $\{\,\widehat{U}(\xi)\;:\;\xi\in\mathcal{L}\,\}$. In view of Step 1, we deduce that

$$\epsilon\operatorname{Var}_2(R)\le C\int_{\Omega}\operatorname{dist}^2(\nabla_{1,h}\,u(x),SO(n))\,\mathrm{d}x.$$

Step 5. It remains to prove the claim in Step 1. First, we extend u to the "left", more precisely to the domain

$$\widehat{\Omega}^- := (-2hK,L)\times S$$

where $K\in\mathbb{N}$ is the smallest number larger then γ_0. To this end, let v denote the rescaled deformation from Step 3 and choose a rotation R_0 satisfying

$$R_0\in\operatorname*{argmin}_{R\in SO(n)}\int_{(0,h)\times hS}|\nabla v-R|^2\;\mathrm{d}x.$$

For $x\in(0,h)\times hS$ we set $w(x):=v(x)-R_0x$ and extend w to the domain $\mathbb{R}\times hS$ by reflection and periodicity:

$$w^{\mathrm{ex}}(x_1,\bar{x}):=\begin{cases} w(x_1+2hk,\bar{x}) & \text{if } \exists k\in\mathbb{Z} \text{ such that } x_1+2hk\in(0,h)\\ w(-x_1+2hk,\bar{x}) & \text{if } \exists k\in\mathbb{Z} \text{ such that } -x_1+2hk\in(0,h).\end{cases}$$

By construction the map $w^{\mathrm{ex}}(x_1,\bar{x})$ is $2h$-periodic in x_1 and belongs to $W^{1,2}(I\times hS;\mathbb{R}^n)$ for all bounded intervals $I\subset\mathbb{R}$. Moreover, we have $w^{\mathrm{ex}}(x)=v(x)-R_0x$ for all $x\in(0,h)\times hS$ and

$$\int_{(-2Kh,0)\times hS}|\nabla w^{\mathrm{ex}}|^2\;\mathrm{d}x = 2K\int_{(0,h)\times hS}|\nabla w|^2\;\mathrm{d}x = 2K\int_{(0,h)\times hS}|\nabla v-R_0|^2\;\mathrm{d}x.$$

Note that we can control the right hand side by means of the geometric rigidity estimate, i.e.

$$\int_{(-2Kh,0)\times hS}|\nabla w^{\mathrm{ex}}|^2\;\mathrm{d}x\le c'\,2K\int_{(0,h)\times hS}\operatorname{dist}^2(\nabla v(x),SO(n))\,\mathrm{d}x, \tag{7.22}$$

where c' only depends on the geometry of S. We extend u to the domain $\widehat{\Omega}^-$ according to

$$u^{\mathrm{ex}}(x):=\begin{cases} u(x) & \text{if } x\in\Omega\\ w^{\mathrm{ex}}(x_1,h\bar{x})+R_0(x_1,h\bar{x}) & \text{if } x\in\widehat{\Omega}^-\setminus\Omega.\end{cases}$$

By construction u^{ex} belongs to $W^{1,2}(\widehat{\Omega}^-;\mathbb{R}^n)$ and

$$\begin{aligned}\int_{\widehat{\Omega}^-\setminus\Omega} \operatorname{dist}^2(\nabla_{1,h}\, u^{\text{ex}}(x), SO(n))\,\mathrm{d}x &\le \int_{\widehat{\Omega}^-\setminus\Omega} |\nabla_{1,h}\, u^{\text{ex}}(x) - R_0|^2\,\mathrm{d}x\\ &= \int_{(-2Kh,0)\times S} |(\nabla w^{\text{ex}})(x_1, h\bar{x})|^2\,\mathrm{d}x = \frac{1}{h}\int_{(-2Kh,0)\times hS} |\nabla w^{\text{ex}}|^2\,\mathrm{d}x.\end{aligned}$$

We can estimate the right hand side by (7.22) and deduce that

$$\begin{aligned}\int_{\widehat{\Omega}^-\setminus\Omega} \operatorname{dist}^2(\nabla_{1,h}\, u^{\text{ex}}(x), SO(n))\,\mathrm{d}x &\le 2K\,\frac{c'}{h}\int_{(0,h)\times S} \operatorname{dist}^2(\nabla v(x), SO(n))\,\mathrm{d}x\\ &= 2K\,c'\int_{(0,h)\times S} \operatorname{dist}^2(\nabla_{1,h}\, u(x), SO(n))\,\mathrm{d}x.\end{aligned}$$

In summary, we have

$$\int_{\widehat{\Omega}^-} \operatorname{dist}^2(\nabla_{1,h}\, u^{\text{ex}}(x), SO(n))\,\mathrm{d}x \le (1+2Kc')\int_{\Omega} \operatorname{dist}^2(\nabla_{1,h}\, u(x), SO(n))\,\mathrm{d}x.$$

Eventually, we extend u^{ex} in the very same way to the "right", i.e. to the domain $(-2Kh, L+2Kh)\times S$. Since $\widehat{\Omega}$ is contained in $(-2Kh, L+2Kh)\times S$, the claim follows. □

Proposition 7.4.7. *Consider the situation in Proposition 7.4.4 for dimension $n = 2$. Then there exists a piecewise affine map $\alpha \in W^{1,2}(\omega)$ such that $\partial_1\alpha$ is a piecewise constant, ϵ-coherent map and*

$$\|\mathcal{R}(\alpha) - \nabla_h u\|^2_{L^2(\Omega;\mathbb{M}(n))} + \epsilon^2\,\|\partial_1\alpha\|^2_{L^2(\omega)} \le C\int_\Omega \operatorname{dist}^2(\nabla_h u, SO(n))\,\mathrm{d}x.$$

The constant C only depends on Ω and γ_0.

Proof. Let $R : \omega \to SO(2)$ denote the approximation from Proposition 7.4.4. The idea of the proof is the following: First, we represent the rotation field R by means of the corresponding rotation angles. In this way we obtain a piecewise constant map $\tilde{\alpha}$ satisfying $\mathcal{R}(\tilde{\alpha}(x_1)) = R(x_1)$ for all $x_1 \in \omega$.

Secondly, we carefully regularize $\tilde{\alpha}$ by applying Proposition 2.2.7 where we utilize the fact that $\tilde{\alpha}$ is coherent to a ϵ-lattice.

To this end, we set $\mathcal{L}_{\epsilon,c} := \epsilon\mathbb{Z} + c$. Because R is ϵ-coherent due to Proposition 7.4.4, we can find a translation $c \in [0,\epsilon)$ and a discrete map $\mathfrak{r} : \mathcal{L}_{\epsilon,c} \to SO(2)$ satisfying

$$R(x_1) = \sum_{\xi\in\mathcal{L}_{\epsilon,c}} 1_{[\xi,\xi+\epsilon)\cap\omega}(x_1)\mathfrak{r}(\xi)$$

for almost every $x_1 \in \omega$. Now we choose a map $\mathfrak{a} : \mathcal{L}_{\epsilon,c} \to \mathbb{R}$ with the properties

$$\mathcal{R}(\mathfrak{a}(\xi)) = \mathfrak{r}(\xi), \qquad |\mathfrak{a}(\xi) - \mathfrak{a}(\xi+\epsilon)| \le \pi \quad \text{and} \quad \mathfrak{a}(c) \in [0,2\pi) \tag{7.23}$$

for all $\xi \in \mathcal{L}_{\epsilon,c}$. This is possible, because the map $\mathcal{R} : \mathbb{R} \to SO(2)$ is surjective and 2π-periodic. The corresponding map $\tilde{\alpha} : \omega \to \mathbb{R}$ defined by

$$\tilde{\alpha}(x_1) := \sum_{\xi \in \mathcal{L}_{\epsilon,c}} 1_{[\xi,\xi+\epsilon)\cap\omega}(x_1)\mathfrak{a}(\xi)$$

is ϵ-coherent and satisfies $\mathcal{R}(\tilde{\alpha}(x_1)) = R(x_1)$ for almost every $x_1 \in \omega$. In view of Lemma 7.4.8 below, (7.23) immediately implies that

$$c^{(1)}\operatorname{Var}_2(R) \le \operatorname{Var}_2(\tilde{\alpha}) \le c^{(2)}\operatorname{Var}_2(R). \tag{7.24}$$

By applying Proposition 7.4.4 (with $h = \epsilon$) we obtain a map $\alpha \in W^{1,2}(\omega)$ where $\partial_1\alpha$ is ϵ-coherent. Moreover, α satisfies the estimate

$$\int_\omega |\tilde{\alpha} - \alpha|^2 + \epsilon^2 |\partial_1\alpha|^2 \,\mathrm{d}x_1 \le C\epsilon\operatorname{Var}_2(\tilde{\alpha}). \tag{7.25}$$

Now for one thing, the right hand side is controlled by the variation of R due to (7.24) and in view of Proposition 7.4.4 eventually by $\int_\Omega \operatorname{dist}^2(\nabla_h u, SO(2))\,\mathrm{d}x$. For another thing, the first integral on the left hand side controls the L^2-distance between $\mathcal{R}(\alpha)$ and $\mathcal{R}(\tilde{\alpha})$. In summary, we obtain the estimate

$$\begin{aligned}\int_\omega |\mathcal{R}(\tilde{\alpha}) - \mathcal{R}(\alpha)|^2 + \epsilon^2|\partial_1\alpha|^2\,\mathrm{d}x \le c'\int_\omega |\tilde{\alpha}-\alpha|^2 + \epsilon^2|\partial_1\alpha|^2\,\mathrm{d}x_1 \\ \le c'\int_\Omega \operatorname{dist}^2(\nabla_h u, SO(2))\,\mathrm{d}x.\end{aligned}$$

Here and below, c' denotes a generic constant, which may change from line to line, is independent of ϵ, h and u, but may depend on the geometry of Ω and γ_0. Eventually, we compute

$$\begin{aligned}\int_\omega |\mathcal{R}(\alpha) - \nabla_h u|^2\,\mathrm{d}x &\le 2\int_\omega |\mathcal{R}(\alpha) - \mathcal{R}(\tilde{\alpha})|^2 + |\mathcal{R}(\tilde{\alpha}) - \nabla_h u|^2\,\mathrm{d}x \\ &\le c'\left(\epsilon\operatorname{Var}_2(\tilde{\alpha}) + \int_\Omega \operatorname{dist}^2(\nabla_h u, SO(2))\,\mathrm{d}x\right) \\ &\le c'\int_\Omega \operatorname{dist}^2(\nabla_h u, SO(2))\,\mathrm{d}x\end{aligned}$$

and the proof is complete. □

Lemma 7.4.8. *There exist constants $c^{(1)}, c^{(2)} > 0$ such that*

$$c^{(1)}|\mathcal{R}(\alpha) - \mathcal{R}(\beta)|^2 \le |\alpha - \beta|^2 \le c^{(2)}|\mathcal{R}(\alpha) - \mathcal{R}(\beta)|^2$$

for all $\alpha, \beta \in \mathbb{R}$ with $|\alpha - \beta| \le \pi$.

Proof. Set $\xi := \alpha - \beta$. A straightforward calculation shows that

$$|\mathcal{R}(\alpha) - \mathcal{R}(\beta)|^2 = |\mathcal{R}(\xi) - Id|^2 = 4(1 - \cos\xi).$$

Set $f_\lambda(\xi) := 1 - \cos\xi - \lambda\xi^2$. Since $1 - \cos\xi$ behaves as $1/2\,\xi^2$ for ξ close to 0, one can show that $f_\lambda(\xi) \ge 0$ for all $\xi \in [\pi, \pi]$, provided $\lambda > 0$ is small enough. On the other side, we have $f_\lambda(\xi) \le 0$ for all $\xi \in \mathbb{R}$, provided $\lambda > 0$ is large enough. □

Proof of Theorem 7.4.2

In view of Lemma 7.4.1 we already now that

$$u_h - p_h \to u \qquad \text{strongly in } L^2(\Omega;\mathbb{R}^2) \quad \text{and weakly in } W^{1,2}(\Omega;\mathbb{R}^2) \tag{7.26}$$

for a map $u \in W^{1,2}(\omega;\mathbb{R}^2)$ and a suitable subsequence (not relabeled). It only remains to prove that u is a bending deformation, i.e. $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ and that (after possibly passing to a further subsequence) we have

$$\nabla_h u_h \to R_{(u)} \qquad \text{strongly in } L^2(\Omega;\mathbb{R}^2) \tag{7.27}$$

where $R_{(u)} \in W^{1,2}(\omega;\mathbb{M}(2)) \cap L^\infty(\omega;SO(2))$ denotes the frame associated to u.

For convenience we set

$$e_h := \frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(2))\,\mathrm{d}x.$$

According to Proposition 7.4.7, we associate to each deformation u_h a map $\alpha_h \in W^{1,2}(\omega)$ satisfying

$$\left\|\frac{\mathcal{R}(\alpha_h) - \nabla_h u_h}{h}\right\|^2_{L^2(\Omega;\mathbb{M}(2))} + \|\partial_1\alpha_h\|^2_{L^2(\omega)} \le c' e_h. \tag{7.28}$$

Here and below, c' denotes a generic constant which may change from line to line, but is independent of h. Let $\alpha_{\omega,h}$ denote the integral average of α_h over ω. Poincaré's inequality and the previous estimate imply that

$$\begin{aligned}\int_\omega |\alpha_h|^2 + |\partial_1\alpha_h|^2\,\mathrm{d}x_1 &\le 2\int_\omega \alpha^2_{\omega,h} + |\alpha_h - \alpha_{\omega,h}|^2\,\mathrm{d}x_1 + \int_\omega |\partial_1\alpha_h|^2\,\mathrm{d}x_1\\ &\le c'\left(\alpha^2_{\omega,h} + \int_\omega |\partial_1\alpha_h|^2\,\mathrm{d}x_1\right) \le c'\left(\alpha^2_{\omega,h} + e_h\right).\end{aligned}$$

Since the map $\mathcal{R} : \mathbb{R} \to SO(2)$ is 2π-periodic, we can assume without loss of generality that $\alpha_{\omega,h} \in [0, 2\pi)$; thereby, the previous estimate implies

$$\limsup_{h\to 0} \|\alpha_h\|^2_{W^{1,2}(\omega)} \le c'(1 + \limsup_{h\to 0} e_h).$$

Because (u_h) is a sequence with finite bending energy, the right hand side is bounded. Hence, there exists a subsequence of (α_h) (not relabeled) that converges to a map $\alpha \in W^{1,2}(\omega)$ weakly in $W^{1,2}(\omega)$ and strongly in $L^2(\omega)$ due to compact embedding. But this implies that $\alpha_h \to \alpha$ pointwise almost everywhere, and consequently we also have $\mathcal{R}(\alpha_h) \to \mathcal{R}(\alpha)$ — initially pointwise, but then also strongly in $L^2(\omega;\mathbb{M}(2))$ for a subsequence by dominated convergence. In view of estimate (7.28) we obtain the claimed convergence (7.27). Moreover, (7.26) and (7.27) imply that $\partial_1 u = \mathcal{R}(\alpha)e_1$ and since the map $x_1 \mapsto \mathcal{R}(\alpha(x_1))$ belongs to $W^{1,2}(\omega;\mathbb{M}(2)) \cap L^\infty(\omega;SO(2))$, we deduce that $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$. □

Compatibility with the boundary condition

Below, we prove Proposition 7.2.7 which says that the one-sided boundary condition is stable for sequences with finite bending energy. To this end, we prove that the estimate in Proposition 7.4.7 is strong enough to pass to the limit of $\nabla_h u_h$ in traces. The proof is adapted from [FJM02]. In particular, the subsequent trace estimate can be found in [FJM02]:

Lemma 7.4.9. *There exists a positive constant c_0 such that for all $w \in W^{1,2}(\Omega;\mathbb{R}^2)$ and all $0 < h \leq 1$ there exists $c \in \mathbb{R}^2$ with*

$$\int_S |w(0,x_2) - c|^2 \,\mathrm{d}x_2 \leq c_0 h \int_{(0,h)\times S} |\nabla_h w(x)|^2 \,\mathrm{d}x.$$

Proof. Due to the Poincaré inequality there exists a positive constant c_0 such that

$$\int_S \left|f(0,x_2) - c\right|^2 \,\mathrm{d}x_2 \leq c_0 \int_{(0,1)\times S} |\nabla f(x)|^2 \,\mathrm{d}x$$

for all $f \in W^{1,2}((0,1)\times S;\mathbb{R}^2)$ and $c = \int_{(0,1)\times S} f \,\mathrm{d}x$. By specifying

$$f(x_1,x_2) := w(hx_1,x_2)$$

we obtain

$$\int_S |w(0,x_2) - c|^2 \,\mathrm{d}x_2 \leq c_0 \int_{(0,1)\times S} |h(\nabla_h w)(hx_1,x_2)|^2 \,\mathrm{d}x = c_0 h \int_{(0,h)\times S} |\nabla_h w|^2 \,\mathrm{d}x.$$

□

Proof of Proposition 7.2.7. We associate to each u_h a piecewise affine map α_h according to Proposition 7.4.7 with $\epsilon = h$. Since (u_h) has finite bending energy, we can apply Theorem 7.4.2 and pass to a subsequence (not relabeled) such that

$$\begin{cases} u_h - u_{\Omega,h} \to u & \text{strongly in } L^2(\Omega;\mathbb{R}^2) \\ \nabla_h u_h \to R_{(u)} & \text{strongly in } L^2(\Omega;\mathbb{M}(2)) \end{cases} \tag{7.29}$$

and (due to Proposition 7.4.7)

$$\sup_h \left(\left\| \frac{\mathcal{R}(\alpha_h) - \nabla_h u_h}{h} \right\|^2_{L^2(\Omega;\mathbb{M}(2))} + \|\partial_1 \alpha_h\|^2_{L^2(\omega)} \right) < \infty. \tag{7.30}$$

The map $\mathcal{R}\colon \mathbb{R} \to SO(2)$ is invariant under translations by $2k\pi$, $k \in \mathbb{Z}$. Therefore, we can choose the sequence (α_h) in such a way that it is bounded in $W^{1,2}(\omega)$. Now (7.29) and (7.30) immediately imply that (α_h) weakly converges to a map α in $W^{1,2}(\omega)$ with $\mathcal{R}(\alpha) = R_{(u)}$.

We prove that the limit u satisfies the boundary condition by comparing (u_h) with a suitable ansatz $(\tilde{u}_h)$. Let us define

$$\tilde{u}_h(x) := u_0 + \int_0^{x_1} \mathcal{R}(\alpha_h(s))e_1 \, \mathrm{d}s + hx_2 \, \mathcal{R}(\alpha(x_1))e_2,$$

$$w_h := u_h - \tilde{u}_h \quad \text{ and } \quad \bar{w}_h(x_1) := \int_S w_h(x_1, x_2) \, \mathrm{d}x_2.$$

Note that $(\tilde{u}_h)$ converges to $u_0 + \int_0^{x_1} R_{(u)}(s)e_1 \, \mathrm{d}s$, while $(\bar{u}_h)$ converges to u. Since $\bar{w}_h(0) = 0$ for all h, the Poincaré-Friedrichs inequality implies that

$$\|\bar{w}_h\|^2_{W^{1,2}(\omega;\mathbb{R}^2)} \leq c' \int_\omega |\partial_1 \bar{w}_h|^2 \, \mathrm{d}x_1.$$

Here and below, c' denotes a generic positive constant that may change from line to line, but can be chosen independent of h. On the other hand, we have $\partial_1 \int_S \tilde{u}_h \, \mathrm{d}x_2 = \mathcal{R}(\alpha_h)e_1$ which leads to the estimate

$$\int_\omega |\partial_1 \bar{w}_h|^2 \, \mathrm{d}x_1 \leq \int_\Omega |\partial_1 u_h - \mathcal{R}(\alpha_h)e_1|^2 \, \mathrm{d}x \leq c' h^2,$$

and we deduce that $\bar{w}_h \to 0$ strongly in $W^{1,2}(\omega;\mathbb{R}^2)$. As a consequence, we infer that the limits of $(\bar{u}_h)$ and $(\int_S \tilde{u}_h \, \mathrm{d}x_2)$ are equal, which means that

$$u(x_1) = u_0 + \int_0^{x_1} R_{(u)}(s)e_1 \, \mathrm{d}s.$$

Hence, it remains to prove that $R_{(u)}(0)e_2 = n_0$: We apply the previous lemma to the map $\frac{1}{h}w_h$ and get

$$\int_S \left|\tfrac{1}{h} w_h(0, x_2) - c_h\right|^2 \leq c_0 h^{-1} \int_{(0,h)\times S} |\nabla_h w_h(x)|^2 \, \mathrm{d}x.$$

Because of

$$\nabla_h w_h = \nabla_h u_h - \mathcal{R}(\alpha_h) - hx_2 \partial_1 \alpha_h \mathcal{R}(\alpha_h + \pi/2) e_2 \otimes e_1,$$

estimate (7.30) implies that the right hand side of the previous inequality is bounded by $c'h$ for a suitable constant c'. On the other hand, we observe that $\frac{1}{h} w_h(0, x_2) = x_2(n_0 - \mathcal{R}(\alpha_h(0))e_2)$ and thus,

$$\frac{1}{12} |n_0 - \mathcal{R}(\alpha_h(0))e_2|^2 \leq \int_S |x_2(n_0 - \mathcal{R}(\alpha_h(0))e_2) - c_h|^2 \, \mathrm{d}x_2 \leq c'h$$

where we used that $\int_S x_2 c_h \, \mathrm{d}x_2 = 0$. Because of $\mathcal{R}(\alpha_h) \rightharpoonup R_{(u)}$ weakly in $W^{1,2}(\omega; \mathbb{M}(2))$, we obtain

$$R_{(u)}(0)e_2 = \lim_{h\to 0} \mathcal{R}(\alpha_h(0))e_2 = n_0.$$

□

7.4.2. Two-scale characterization of the limiting strain

Let us consider a sequence of scaled deformations (u_h) in $W^{1,2}(\Omega;\mathbb{R}^2)$. We associate to each u_h the scaled nonlinear strain $E_h \in L^2(\Omega;\mathbb{M}_{\text{sym}}(2))$ by

$$E_h := \frac{\sqrt{\nabla_h u_h^{\mathrm{T}} \nabla_h u_h} - Id}{h}. \tag{7.31}$$

Furthermore, we associate to the sequence (u_h) the set of limiting strains

(7.32)

$$\mathfrak{E} := \Big\{ E \in L^2(\Omega{\times}Y;\mathbb{M}(2)) \,:\, E_h \xrightarrow{2} E \quad \text{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{M}(2)) \text{ for a subsequence (not relabeled)} \Big\}.$$

If the sequence (u_h) has finite bending energy and converges to a map u in $L^2(\Omega;\mathbb{R}^2)$, then Theorem 7.4.2 reveals that u is a bending deformation, i.e. $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$, and additionally we infer that $\mathfrak{E}$ is non-empty. In this section we establish a link between the limiting deformation u and a limiting strain E in $\mathfrak{E}$. Generally speaking, we are going to see that E admits a presentation of the form

$$E(x,y) = x_2 \boldsymbol{\kappa}_{(u)}(x_1)(e_1{\otimes}e_1) + G(x,y)$$

where $\boldsymbol{\kappa}_{(u)}$ is the curvature of the limiting deformation and G a "relaxation profile" that captures the oscillation properties of the sequence (E_h). We are going to see that the general structure of the profile G depends in a subtle way on the ratio γ; recall that by assumption (7.1) we have $\lim_{h\to 0}\frac{h}{\varepsilon} = \gamma$ with $\gamma \in [0,+\infty]$. The main result in this section is the following:

Theorem 7.4.10. *Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^2)$ with finite bending energy, let $\mathfrak{E}$ be defined as in (7.32) and suppose that (u_h) converges to u in the sense of (7.14). Then $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ and each E in $\mathfrak{E}$ can be represented in the form*

$$E(x,y) = \big(\, a(x_1) + x_2(\boldsymbol{\kappa}_{(u)}(x_1) + \partial_y \alpha_0(x_1,y))\,\big)(e_1{\otimes}e_1) + \operatorname{sym} G(x,y)$$

where

$$a \in L^2(\omega) \qquad \text{and} \qquad \begin{cases} \alpha \in L^2(\omega; W^{1,2}_{\text{per},0}(Y)) & \text{if } \gamma = 0 \\ \alpha = 0 & \text{else,} \end{cases}$$

and G is a map in $L^2(\Omega{\times}Y;\mathbb{M}(2))$ that satisfies

$$G = \begin{cases} \big(\ \partial_y w_0 \,\big|\, \partial_2 \bar{w}\ \big) & \text{if } \gamma \in \{0,\infty\} \\ \widetilde{\nabla}_{1,\gamma} w_0 := \Big(\ \partial_y w_0 \,\Big|\, \tfrac{1}{\gamma}\partial_2 w_0\ \Big) & \text{else} \end{cases}$$

where

$$\begin{cases} w_0 \in L^2(\omega; W^{1,2}_{\text{per},0}(Y;\mathbb{R}^2)) \text{ and } \bar{w} \in L^2(\omega{\times}Y; W^{1,2}(S;\mathbb{R}^2)) & \text{if } \gamma = 0, \\ w_0 \in L^2(\Omega; W^{1,2}_{\text{per},0}(Y;\mathbb{R}^2)) \text{ and } \bar{w} \in L^2(\omega; W^{1,2}(S;\mathbb{R}^2)) & \text{if } \gamma = \infty, \\ w_0 \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S{\times}Y;\mathbb{R}^2)) & \text{else.} \end{cases}$$

(For the proof see page 137).

Remarks

1. In virtue of the two-scale characterization of scaled gradients (see Theorem 6.3.3), we see that the map $G : \Omega\times Y \to \mathbb{M}(2)$ can be obtained as the weak two-scale limit of a sequence $(\nabla_h w_h)$, where $(w_h) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ weakly converges to 0 and has the property that $(\nabla_h w_h)$ is bounded as a sequence in $L^2(\Omega;\mathbb{M}(2))$.

2. If $\gamma \in (0,\infty)$, then the map G is a scaled gradient consisting of a derivative with respect to the fast variable y and a scaled derivative in the cross-sectional direction x_2. We would like to remark that the definition of the scaled gradient $\widetilde{\nabla}_{1,\gamma}$ and the space $W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2)$ are introduced in Section 6.3.

3. If we average $E \in \mathfrak{E}$ over Y, then we obtain a decomposition in the form

$$A(x_1) + x_2\boldsymbol{\kappa}_{(u)}(x_1)(e_1\otimes e_1) + \operatorname{sym} g(x)\otimes e_2 \tag{7.33}$$

   with $A \in L^2(\omega;\mathbb{M}_{\text{sym}}(2))$ and $g \in L^2(\Omega;\mathbb{R}^2)$, $\int_S g\,\mathrm{d}x_2 = 0$ for a.e. x_1. This is in accordance with [FJM02], where a similar decomposition was derived in the case of two-dimensional plates with homogeneous materials, i.e. $W(x,F)$ is independent of x. In virtue of Lemma 2.1.9 it is clear that $\int_Y E(x,y)\,\mathrm{d}y$ is the weak limit of a suitable subsequence of (E_h); thus, the goal of Theorem 7.4.10 is a precise understanding of the oscillations emerging in the nonlinear strain, taking into account the coupling of the fine-scales.

In the cases where the fine-scales separate in the limit, i.e. $\gamma = 0$ or $\gamma = \infty$, it is convenient to consider the projection of the limiting strain to the subspace of maps linear in x_2. Therefore, we define

$$\Pi_E(x_1,y) = \int_S E(x,y)2\sqrt{3}x_2\,\mathrm{d}x_2.$$

Since the function $x_2 \mapsto 2\sqrt{3}x_2$ is a unit vector in $L^2(S)$ and linear in x_2, the map

$$(x,y) \mapsto 2\sqrt{3}x_2\,\Pi_E(x_1,y)$$

is exactly the projection of E to the space of functions in $L^2(\Omega\times Y;\mathbb{M}(2))$ that are linear in x_2.

Corollary 7.4.11. *Consider the situation in Theorem 7.4.10. Let $E \in \mathfrak{E}$ and suppose that $\gamma \in \{0,\infty\}$. Then*

$$\Pi_E = \frac{1}{\sqrt{12}}\boldsymbol{\kappa}_{(u)}(e_1\otimes e_1) + \begin{pmatrix} \partial_y\alpha & g \\ g & c \end{pmatrix}$$

with

$$\alpha \in L^2(\omega;W^{1,2}_{\text{per},0}(Y)), \qquad g \in L^2(\omega\times Y) \quad \textit{and} \quad c \in \begin{cases} L^2(\omega\times Y) & \textit{if } \gamma = 0 \\ L^2(\omega) & \textit{if } \gamma = \infty. \end{cases}$$

Proof. The statement can be checked by a straightforward calculation. ☐

In the remaining part of this section we prove Theorem 7.4.10. The strategy is the following: We associate to each deformation u_h a function $\alpha_h \in W^{1,2}(\omega)$ such that the map $\mathcal{R}(\alpha_h(\cdot))$ is an approximation of $\nabla_h u_h$ in the sense of Proposition 7.4.7. In Proposition 7.4.12 (see below), we observe that $(\partial_1\alpha_h)$ weakly converges to the curvature of the limiting deformation u, and therefore the quantity $\boldsymbol{\kappa}_h^{\mathrm{ap}} := \partial_1\alpha_h$ can be interpreted as an approximating curvature field. Moreover, the approximation is tailored in such a way that the sequence $(\boldsymbol{\kappa}_h^{\mathrm{ap}})$ "does not carry oscillations" on scale ε if $\gamma > 0$.

In a second step, we consider the decomposition

$$u_h(x_1, x_2) = \Big(\bar{u}_h(x_1) + hx_2\mathcal{R}(\alpha_h(x_1))\Big) + hw_h(x_1, x_2)$$

where $\bar{u}_h$ denotes the cross-sectional average of u_h. The term in the braces can be regarded as an extension of the one-dimensional deformation $\bar{u}_h$ to a two-dimensional deformation by a Cosserat-like ansatz. Because the construction of this term is quite explicit, we can characterize its contribution to the limiting strain E. In this context it turns out that the major component of the nonlinear strain associated to the deformation in the braces is related to the approximating curvature field, and therefore to the curvature of the limiting deformation. On the other hand, the map w_h can be interpreted as a corrector of higher order and its contribution to E can be characterized by means of the two-scale characterization of scaled gradients (see Theorem 6.3.3). It is important to note that both characterizations are sensitive to the limiting behavior of the fine scale ratio h/ε. We would like to remark that in Section 7.7.1, we prove that the derived characterization is sharp.

Proposition 7.4.12. *Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^2)$ with finite bending energy.*

(1) There exists a sequence $(\alpha_h) \subset W^{1,2}(\omega)$ such that

$$\limsup_{h\to 0}\Big(\|\alpha_h\|_{W^{1,2}(\omega)} + \big\|E_h^{\mathrm{ap}}\big\|_{L^2(\Omega;\mathbb{M}(2))}\Big) < \infty$$

where $E_h^{\mathrm{ap}} := h^{-1}\Big(\mathcal{R}(\alpha_h)^T\nabla_h u_h - Id\Big)$.

(2) There holds $E_h - \operatorname{sym} E_h^{\mathrm{ap}} \xrightarrow{2} 0$ weakly two-scale in $L^2(\Omega\times Y;\mathbb{M}(2))$.

(3) If $u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2)$ is the limit of (u_h) in the sense of (7.14), then

$$\partial_1\alpha_h \rightharpoonup \boldsymbol{\kappa}_{(u)} \qquad \text{weakly in } L^2(\omega).$$

Moreover, if $\gamma > 0$, then we can choose (α_h) such that

$$\partial_1\alpha_h \xrightarrow{2} \boldsymbol{\kappa}_{(u)} \qquad \text{weakly two-scale in } L^2(\omega\times Y)$$

is additionally satisfied.

Proof of Proposition 7.4.12. To each deformation u_h we associate a map α_h in $W^{1,2}(\omega)$ by applying Proposition 7.4.7 where we specify the free scale parameter $\epsilon = \epsilon(h)$ according to

$$\epsilon(h) := \begin{cases} h & \text{if } \gamma \in \{0, \infty\} \\ \varepsilon(h) & \text{else.} \end{cases}$$

We are going to see that this choice guarantees that the sequence $(\partial_1 \alpha_h)$ is constant on scale ε if $\gamma > 0$. We define maps $R_h : \omega \to SO(2)$ and $E_h^{\mathrm{ap}} : \Omega \to \mathbb{M}(2)$ according to

$$R_h(x_1) := \mathcal{R}(\alpha_h(x_1)) \quad \text{and} \quad E_h^{\mathrm{ap}}(x) := h^{-1}(R_h(x_1)^{\mathrm{T}} \nabla_h u_h(x) - Id) \tag{7.34}$$

where $\mathcal{R}(\cdot)$ denotes the clockwise rotation in $\mathbb{R}^2$ (see page 105). Because R_h is "close" to $\nabla_h u_h$, the map E_h^{ap} can be interpreted as an approximation of the nonlinear strain. By definition $\mathcal{R}(\cdot)$ is invariant under translations by $2k\pi$ with $k \in \mathbb{Z}$; thus, we can assume without loss of generality that $\alpha_h(0) \in [0, 2\pi)$, and consequently there exists a constant c' independent of h such that

$$\|\alpha_h\|_{W^{1,2}(\omega)} \leq c'(\|\partial_1 \alpha_h\|_{L^2(\omega)} + 1).$$

Because (u_h) has finite bending energy, the estimate in Proposition 7.4.7 immediately implies that

$$\limsup_{h \to 0} \left(\frac{\epsilon(h)}{h} \|\partial_1 \alpha_h\|_{L^2(\omega)} + \left\|E_h^{\mathrm{ap}}\right\|_{L^2(\Omega;\mathbb{M}(2))} \right) < \infty. \tag{7.35}$$

By construction, the ratio ${}^{\epsilon(h)}/_{h}$ is bounded (either by $\gamma < \infty$ or by 1); consequently, the previous estimate implies statement (1).

We prove statement (2). Because (E_h^{ap}) and (E_h) are bounded in $L^2(\Omega; \mathbb{M}(2))$, we can pass to a subsequence (not relabeled) such that

$$\begin{aligned} E_h^{\mathrm{ap}} &\xrightharpoonup{2} E^{\mathrm{ap}} && \text{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(2)), \\ E_h - \operatorname{sym} E_h^{\mathrm{ap}} &\xrightharpoonup{2} D && \text{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(2)) \end{aligned}$$

with $E^{\mathrm{ap}}, D \in L^2(\Omega \times Y; \mathbb{M}(2))$. It is sufficient to prove that $D = 0$.

We have $\nabla_h u_h = R_h(Id + E_h^{\mathrm{ap}})$, and therefore

$$E_h = \frac{\sqrt{\nabla_h u_h^{\mathrm{T}} \nabla_h u_h} - Id}{h} = \frac{\sqrt{(Id + hE_h^{\mathrm{ap}})^{\mathrm{T}}(Id + hE_h^{\mathrm{ap}})} - Id}{h}.$$

Now Corollary 2.3.4 implies that

$$E_h \xrightharpoonup{2} \operatorname{sym} E^{\mathrm{ap}} \qquad \text{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(2))$$

and since the linear map $\mathbb{M}(2) \ni A \mapsto \operatorname{sym} A \in \mathbb{M}(2)$ is continuous with respect to weak two-scale convergence, we deduce that $D = 0$.

We prove statement (3). Suppose that $u_h - u_{\Omega,h} \to u$ strongly in $L^2(\Omega;\mathbb{R}^2)$. In view of the compactness result (see Theorem 7.4.2), we infer that $\partial_1 u_h \to \partial_1 u$ and $R_h e_1 - \partial_1 u_h \to 0$ strongly in $L^2(\Omega;\mathbb{R}^2)$, and consequently we have

$$R_h \to R_{(u)} \qquad \text{strongly in } L^2(\omega;\mathbb{M}(2)). \tag{7.36}$$

On the other side, the sequence (α_h) is bounded in $W^{1,2}(\omega)$; thus, we can pass to a subsequence (not relabeled) such that

$$\alpha_h \rightharpoonup \alpha \qquad \text{weakly in } W^{1,2}(\omega).$$

But this implies that $R_h \rightharpoonup \mathcal{R}(\alpha)$ weakly in $W^{1,2}(\omega;\mathbb{M}(2))$ and in view of (7.36) we obtain $\mathcal{R}(\alpha) = R_{(u)}$. Moreover, we have

$$\partial_1\alpha_h\, e_1 = \partial_1\alpha_h\, \mathcal{R}(\pi/2)e_2 = \mathcal{R}(\alpha_h)^{\mathrm{T}}\partial_1\mathcal{R}(\alpha_h)e_2.$$

Now the right hand side equals $R_h^{\mathrm{T}}\partial_1 R_h e_2$ and converges (as a product of a strongly and weakly convergent sequence) to $R_u^{\mathrm{T}}\partial_1 R_u e_2 = \boldsymbol{\kappa}_{(u)}\, e_1$. Because this reasoning is valid for arbitrary subsequences, we obtain

$$\partial_1\alpha_h \rightharpoonup \boldsymbol{\kappa}_{(u)} \qquad \text{weakly in } L^2(\omega)$$

for the entire sequence.

In the following, we suppose that $\gamma > 0$. In view of Proposition 2.1.14 we already know that

$$\partial_1\alpha_h \xrightharpoonup{2} \boldsymbol{\kappa}_{(u)} + \partial_y\alpha_0 \qquad \text{weakly two-scale in } L^2(\omega{\times}Y)$$

for a subsequence (not relabeled) and a suitable map $\alpha_0 \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y))$. We show that $\partial_y\alpha_0 = 0$. The condition $\gamma > 0$ corresponds to the scalings $h \sim \varepsilon$ and $h \gg \varepsilon$. Since α_h is constructed by an approximation scheme involving piecewise constant maps that are coherent to a lattice with a scale comparable or larger than ε, it is natural to expect that the approximation is too rough to capture oscillations on the finer scale ε. Indeed, by construction each map $\partial_1\alpha_h$ is $\epsilon(h)$-coherent with $\epsilon(h) = \varepsilon(h)$ or $\epsilon(h) \gg \varepsilon(h)$; hence, Lemma 2.2.4 and Lemma 2.2.3 imply that $\partial_y\alpha_0 = 0$. □

Proof of Theorem 7.4.10. We choose sequences (α_h) and (E_h^{ap}) according to Proposition 7.4.12 and define

$$\bar{u}_h(x_1) := \int_S u_h(x)\,\mathrm{d}x_2 \qquad \text{and} \qquad w_h := \frac{u_h - \bar{u}_h}{h} - x_2\mathcal{R}(\alpha_h)e_2.$$

Obviously, the map w_h belongs to $W^{1,2}(\Omega;\mathbb{R}^2)$ and has vanishing mean value. Therefore, we can apply the Poincaré-Wirtinger inequality and obtain the estimate

$$\begin{aligned}\int_S |w_h|^2\,\mathrm{d}x_2 \le c'\int_S |\partial_2 w_h|^2\,\mathrm{d}x_2 &= c'h^2\int_S \left|\frac{\frac{1}{h}\partial_2 u_h - \mathcal{R}(\alpha_h)e_2}{h}\right|^2\,\mathrm{d}x_2\\ &\le c'h^2\int_S \left|\mathcal{R}(\alpha_h)^{\mathrm{T}}E_h^{\mathrm{ap}}\,e_2\right|^2\,\mathrm{d}x_2.\end{aligned}$$

Here and below, c' and c'' are positive constants which may change from line to line, but can be chosen independent of h. By construction (see Proposition 7.4.12) the sequence (E_h^{ap}) is bounded; thus, integrating the previous estimate over ω leads to

$$\int_\Omega |w_h|^2 \, \mathrm{d}x \le c' \int_\Omega |\partial_2 w_h|^2 \, \mathrm{d}x \le c'' h^2. \tag{7.37}$$

Furthermore, $(\partial_1 w_h)$ is bounded as can be seen by the following reasoning:

$$\partial_1 w_h = \frac{\partial_1 u_h - \partial_1 \bar{u}_h}{h} - x_2 \partial_1 \alpha_h \mathcal{R}(\alpha_h + \pi/2) e_2.$$

Because the second term on the right hand side is bounded due to Proposition 7.4.12, we only have to estimate the first term:

$$\begin{aligned} v_h := \frac{\partial_1 u_h - \partial_1 \bar{u}_h}{h} &= \frac{\partial_1 u_h - \mathcal{R}(\alpha_h) e_1}{h} - \int_S \frac{\partial_1 u_h(x) - \mathcal{R}(\alpha_h(x_1)) e_1}{h} \, \mathrm{d}x_2 \\ &= \mathcal{R}(\alpha_h) \left(E_h^{\mathrm{ap}} e_1 - \int_S E_h^{\mathrm{ap}} e_1 \, \mathrm{d}x_2 \right) \end{aligned}$$

and again the boundedness of E_h^{ap} implies that (v_h) is a bounded sequence in $L^2(\Omega, \mathbb{R}^2)$. So far, we have shown that

$$\begin{cases} w_h \rightharpoonup 0 & \text{weakly in } W^{1,2}(\Omega; \mathbb{R}^2), \\ (\nabla_h w_h) & \text{is bounded in } L^2(\Omega; \mathbb{M}(2)). \end{cases} \tag{7.38}$$

Now let $E \in \mathfrak{E}$. Since $(\partial_1 \alpha_h)$ and $(\nabla_h w_h)$ are bounded, we can pass to a subsequence (not relabeled) such that

$$\begin{aligned} E_h &\xrightarrow{2} E && \text{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(2)) \\ E_h^{\mathrm{ap}} &\xrightarrow{2} E^{\mathrm{ap}} && \text{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(2)) \\ \nabla_h w_h &\xrightarrow{2} G && \text{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(2)) \\ \partial_1 \alpha_h &\xrightarrow{2} \boldsymbol{\kappa}_{(u)} + \partial_y \alpha_0 && \text{weakly two-scale in } L^2(\omega \times Y) \\ \mathcal{R}(\alpha_h) &\to R_{(u)} && \text{strongly in } L^2(\omega; \mathbb{M}(2)), \end{aligned}$$

where $E^{\mathrm{ap}}, G \in L^2(\Omega \times Y; \mathbb{M}(2))$ and $\alpha_0 \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y))$ depend on the choice of the specific subsequence. Note that the convergence properties of $(\partial_1 \alpha_h)$ and $\mathcal{R}(\alpha_h)$ are justified due to the compactness result of Theorem 7.4.2 and statement (3) of Proposition 7.4.12.

By rewriting the definition of w_h we see that $u_h = \bar{u}_h + h\big(w_h + x_2 \mathcal{R}(\alpha_h) e_2\big)$, and consequently

$$\nabla_h u_h = \left(\; \partial_1 \bar{u}_h \; \middle| \; \mathcal{R}(\alpha_h) e_2 \; \right) + h \, \nabla_h w_h + h x_2 \partial_1 \mathcal{R}(\alpha_h) e_2 \otimes e_1.$$

Multiplication by $\mathcal{R}(\alpha_h)^{\mathrm{T}}$ yields

$$\mathcal{R}(\alpha_h)^{\mathrm{T}} \nabla_h u_h = Id + h \mathcal{R}(\alpha_h)^{\mathrm{T}} \nabla_h w_h + h x_2 \partial_1 \alpha_h (e_1 \otimes e_1) + h \left(\int_S E_h^{\mathrm{ap}} e_1 \, \mathrm{d}x_2 \right) \otimes e_1. \tag{7.39}$$

Here, we used the identity

$$\mathcal{R}(\alpha_h)^{\mathrm{T}}\partial_1\bar{u}_h = \mathcal{R}(\alpha_h)^{\mathrm{T}}\int_S \mathcal{R}(\alpha_h)e_1 + (\partial_1 u_h - \mathcal{R}(\alpha_h)e_1)\,\mathrm{d}x_2 = e_1 + h\int_S E_h^{\mathrm{ap}} e_1\,\mathrm{d}x_2.$$

Now we subtract Id on both sides of (7.39), divide the equation by h and pass to the limit. In this way we obtain the equation

$$E^{\mathrm{ap}} = R_{(u)}^{\mathrm{T}} G + x_2(\boldsymbol{\kappa}_{(u)} + \partial_y\alpha_0)(e_1\otimes e_1) + A\otimes e_1 \tag{7.40}$$

where $A := \int_S E^{\mathrm{ap}} e_1\,\mathrm{d}x_2$.

Our next goal is to identify $R_{(u)}^{\mathrm{T}}G + A\otimes e_1$ by means of the two-scale characterization result for scaled gradients. As a first step, we consider the sequence

$$w_h^\star := \mathcal{R}(\alpha_h)^{\mathrm{T}} w_h.$$

Because of (7.37) and (7.38), we have

$$\nabla_h w_h^\star \xrightarrow{2} R_{(u)}^{\mathrm{T}} G \qquad \text{weakly two-scale}$$

and the sequence $(w_h^\star)$ satisfies the properties in (7.38) as well. For this reason, we can apply Theorem 6.3.3 (with $n = 2$ and $m = 1$) and conclude that

$$R_{(u)}^{\mathrm{T}} G = \begin{cases} \left(\ \partial_y w_0^\star \ \middle|\ \partial_2 \bar{w}^\star\ \right) & \text{if } \gamma\in\{0,\infty\} \\ \widetilde{\nabla}_{1,\gamma} w_0^\star & \text{else} \end{cases}$$

where

$$\begin{cases} w_0^\star \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^2)) \text{ and } \bar{w}^\star\in L^2(\omega{\times}Y; W^{1,2}(S;\mathbb{R}^2)) & \text{if } \gamma = 0 \\ w_0^\star \in L^2(\Omega; W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^2)) \text{ and } \bar{w}^\star\in L^2(\omega; W^{1,2}(S;\mathbb{R}^2)) & \text{if } \gamma = \infty \\ w_0^\star \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S{\times}Y;\mathbb{R}^2)) & \text{else.} \end{cases} \tag{7.41}$$

Next, we are going to show that parts of the matrix $A\otimes e_1$ can be represented as an auxiliary gradient in the form (7.41) as well. In order to do so, we set

$$\bar{A}(x_1) := \int_Y A\,\mathrm{d}y, \qquad \mathring{A}(x_1,y) := A - \bar{A}$$

and denote the first and second entry of $\bar{A}$ by $\bar{A}_1$ and $\bar{A}_2$, respectively. Define the map

$$g(x_1,y) := \int_0^y \mathring{A}(x_1,s)\,\mathrm{d}s - \int_Y\int_0^y \mathring{A}(x_1,s)\,\mathrm{d}s\,\mathrm{d}y.$$

Then $g \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^2))$ and satisfies $\partial_y g = \mathring{A}$. As a consequence, also the modified maps

$$\begin{cases} w_0(x,y) := w_0^\star(x,y) + g(x_1,y), \quad \bar{w}(x,y) := \bar{w}^\star(x,y) + (x_2\bar{A}_2(x_1))e_1 & \text{if } \gamma\in\{0,\infty\} \\ w_0(x,y) := w_0^\star(x,y) + g(x_1,y) + (x_2\bar{A}_2(x_1))e_1 & \text{else} \end{cases}$$

belong to the appropriate function spaces as described in (7.41) and it is easy to check that

$$\widetilde{G} := \begin{cases} \left(\partial_y w_0 \mid \partial_2 \bar{w} \right) & \text{if } \gamma \in \{0, \infty\} \\ \widetilde{\nabla}_{1,\gamma} w_0 & \text{else} \end{cases}$$

satisfies

$$\operatorname{sym} \widetilde{G} + \bar{A}_1(e_1 \otimes e_1) = \operatorname{sym} \left[R_{(u)}^{\mathrm{T}} W + A \otimes e_1 \right].$$

Because of $E = \operatorname{sym} E^{\mathrm{ap}}$ (see Proposition 7.4.12), we can rewrite equation (7.40) and obtain

$$E = x_2(\boldsymbol{\kappa}_{(u)} + \partial_y \alpha_0 + \bar{A}_1)(e_1 \otimes e_1) + \operatorname{sym} \widetilde{G}.$$

This completes the proof in the case $\gamma = 0$. For $\gamma > 0$ it remains to check that $\partial_y \alpha_0 = 0$. But this is exactly the statement of Proposition 7.4.12 (3). □

7.4.3. Lower bound

In this section we prove the lower bound part of Theorem 7.2.5 (see page 142 et seq.). As a preliminary result, we derive a lower bound for the limit inferior of $(\mathcal{I}^{\varepsilon,h})$ by means of the limiting strain.

Lemma 7.4.13. *Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^2)$. Then*

$$\liminf_{h \to 0} \mathcal{I}^{\varepsilon,h}(u_h) \geq \inf_{E \in \mathfrak{E}} \iint_{\Omega \times Y} Q(y, E(x,y))\,\mathrm{d}y\,\mathrm{d}x$$

where $\mathfrak{E}$ is defined according to (7.32).

Proof. For convenience we set

$$e_h := \mathcal{I}^{\varepsilon,h}(u_h) = \frac{1}{h^2} \int_{\Omega} W({}^{x_1}/_{\varepsilon(h)}, \nabla_h u_h(x))\,\mathrm{d}x.$$

We only have to consider the case

$$\liminf_{h \to 0} e_h < \infty.$$

Furthermore, we can pass to a subsequence (not relabeled) in such a way that $\lim_{h\to 0} e_h$ is well defined and equal to the left hand side of the previous equation. Due to the non-degeneracy condition (W3) the sequence (u_h) has finite bending energy and Theorem 7.4.2 implies that

$$E_h \xrightarrow{2} E \qquad \text{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(2))$$

for a suitable subsequence (not relabeled) and $E \in \mathfrak{E}$.

We define the set

$$\Omega_h := \{\, x \in \Omega \,:\, \operatorname{dist}^2(\nabla_h u_h(x), SO(2)) \geq \delta \,\},$$

where we choose $\delta > 0$ in such a way that $\det \nabla_h u_h(x) > 0$ for all $x \in \Omega_h$. This is possible, because $\det(\cdot)$ is continuous and equal to 1 on $SO(2)$. Note that

$$|\Omega \setminus \Omega_h| \leq \int_\Omega \frac{\operatorname{dist}^2(\nabla_h u_h(x), SO(2))\,\mathrm{d}x}{\delta}\,\mathrm{d}x.$$

Because (u_h) has finite bending energy, the non-degeneracy condition (W3) leads to the estimate

$$|\Omega \setminus \Omega_h| \leq c' h^2$$

for some positive constant c' and we deduce that (1_{Ω_h}) converges to 1 boundedly in measure.

In virtue of the polar decomposition for matrices in $\mathbb{M}(2)$, we can factorize $\nabla_h u_h(x)$ for all $x \in \Omega_h$ according to

$$\nabla_h u_h(x) = \widetilde{R}_h(x)\sqrt{\nabla_h u_h(x)^{\mathrm{T}}\,\nabla_h u_h(x)} = \widetilde{R}_h(x)\big(Id + hE_h(x)\big)$$

where $\widetilde{R}_h$ is a suitable map from Ω_h to $SO(2)$. Because W is frame indifferent, the previous factorization implies that

$$W(x_1/\varepsilon, \nabla_h u_h(x)) = W(x_1/\varepsilon, Id + hE_h(x)) \qquad \text{for all } x \in \Omega_h.$$

By utilizing assumption (W2) and the previous observation, we find that

$$e_h \geq \frac{1}{h^2}\int_\Omega W(x_1/\varepsilon, Id + h\,1_{\Omega_h}(x)E_h(x))\,\mathrm{d}x. \tag{7.42}$$

The convergence of (1_{Ω_h}) to 1 allows us to apply Proposition 2.3.1 and we deduce that

$$1_{\Omega_h} E_h \xrightarrow{2} E \qquad \text{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(2)).$$

Now we can apply the simultaneous homogenization and linearization result (see Theorem 5.2.1) to equation (7.42) and obtain

$$\liminf_{h\to 0} e_h \geq \iint_{\Omega\times Y} Q(y, E(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

The proof is complete, because the right hand side is bounded from below by

$$\inf_{E\in\mathfrak{C}} \iint_{\Omega\times Y} Q(y, E(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

□

We continue with the proof of the lower bound part of Theorem 7.2.5. Essentially, we combine the previous lemma with the two-scale characterization of the limiting strain.

Proof of Theorem 7.2.5 (2). We only have to consider the case where

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) < \infty. \tag{7.43}$$

We pass to a subsequence (not relabeled) such that $\lim_{h\to 0}\mathcal{I}^{\varepsilon,h}(u_h)$ is well defined and equal to the left hand side of (7.43). Because the elastic potential W is non-degenerate, the sequence (u_h) has finite bending energy and we can apply Lemma 7.4.13, which yields the lower bound estimate

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) \geq \inf_{E\in\mathfrak{E}} \iint_{\Omega\times Y} Q\left(y, E(x,y)\right)\,\mathrm{d}y\,\mathrm{d}x \tag{7.44}$$

where $\mathfrak{E}$ denotes the set of weak two-scale cluster points of the sequence (E_h). In view of the compactness result of Theorem 7.4.2, the set $\mathfrak{E}$ is non-empty.

Step 1. We consider the cases $\gamma \in \{0,\infty\}$. Choose an arbitrary limiting strain $E\in\mathfrak{E}$ and consider the decomposition

$$E(x,y) = \overline{E}(x,y) + 2\sqrt{3}x_2\,\Pi_E(x_1,y).$$

This motivates to study the expansion

$$\begin{aligned} &\int_S Q\left(y, E(x_1,x_2,y)\right)\,\mathrm{d}x_2 \\ &= \int_S Q\left(y,\overline{E}(x_1,x_2,y)\right)\,\mathrm{d}x_2 + 2\int_S \left\langle \mathbb{L}(y)\overline{E}(x_1,x_2,y),\, 2\sqrt{3}x_2\,\Pi_E(x_1,y)\right\rangle\,\mathrm{d}x_2 \\ &\qquad + Q\left(y,\Pi_E(x_1,y)\right)\int_S (2\sqrt{3}x_2)^2\,\mathrm{d}x_2. \end{aligned} \tag{7.45}$$

The first integral on the right hand side is non-negative and the coupling term in the middle vanishes, because of

$$\int_S \overline{E}(x_1,x_2,y)\,2\sqrt{3}x_2\,\mathrm{d}x_2 = 0 \qquad \text{almost everywhere.}$$

As a consequence, we obtain the estimate

$$\iint_{\Omega\times Y} Q\left(y,E(x,y)\right)\,\mathrm{d}y\,\mathrm{d}x \geq \iint_{\omega\times Y} Q\left(y,\Pi_E(x_1,y)\right)\,\mathrm{d}y\,\mathrm{d}x_1. \tag{7.46}$$

By Proposition 6.2.5 we can characterize Π_E: There exist maps

$$\alpha\in L^2(\omega;W^{1,2}_{\mathrm{per},0}(Y)),\qquad g\in L^2(\omega\times Y)\quad\text{and}\quad c\in\begin{cases} L^2(\omega\times Y) & \text{if } \gamma=0\\ L^2(\omega) & \text{if } \gamma=\infty\end{cases}$$

such that

$$\Pi_E = \frac{1}{12}\left(\boldsymbol{\kappa}_{(u)}(e_1\otimes e_1) + \begin{pmatrix} \partial_y\alpha & g\\ g & c\end{pmatrix}\right) \qquad\text{with}\qquad (\alpha,g,c)\in\mathbb{X}_\gamma$$

(see page 108 for the definition of $\mathbb{X}_\gamma$). Hence, we deduce that (7.46) is bounded from below by $\mathcal{I}_\gamma(u)$. Because the limiting strain $E \in \mathfrak{E}$ was arbitrarily chosen, we finally obtain the lower bound

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) \geq \mathcal{I}_\gamma(u).$$

Step 2. For $\gamma \in (0,\infty)$ Theorem 7.4.10 implies that

$$\mathfrak{E} \subset \Big\{ \big(a(x_1) + x_2 \boldsymbol{\kappa}_{(u)}(x_1)\big)(e_1 \otimes e_1) + \operatorname{sym} \widetilde{\nabla}_{1,\gamma} w(x,y) \ : \\ w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S \times Y; \mathbb{R}^2)),\ a \in L^2(\omega) \Big\}$$

and the lim inf-inequality directly follows from (7.44). □

7.4.4. Upper bound

In this section we prove the upper bound part of Theorem 7.2.5. We construct recovery sequences (u_h) that converge to a given bending deformation $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$ in the following sense

$$\begin{aligned} u_h &\to u && \text{strongly in } L^2(\Omega;\mathbb{R}^2) \\ \nabla_h u_h &\to R_{(u)} && \text{strongly in } L^2(\Omega;\mathbb{M}(2)). \end{aligned} \tag{7.47}$$

Additionally, we take one-sided boundary conditions into account; namely, we are going to construct recovery sequences (u_h) that satisfy

$$u_h(0,x_2) = u(0) + h x_2 \partial_1 u(0) \qquad \text{for a.e. } x_2 \in S. \tag{7.48}$$

The outline of this section is the following: First, we present the construction for a smooth bending deformation and a smooth prescribed limiting strain. Secondly, we prove that arbitrary limiting deformations can be approximated in the strong topology of $W^{2,2}(\omega;\mathbb{R}^2)$ by smooth bending deformations with regard to one-sided boundary conditions. Finally, we lift the smooth construction to the general case by choosing suitable diagonal sequences.

Smooth construction

Proposition 7.4.14. *Let $u \in C^\infty_{\text{iso}}(\overline{\omega};\mathbb{R}^2)$ and*

$$\begin{cases} \alpha, g, c \in C^\infty_c(\omega; C^\infty_{\text{per}}(Y)) & \text{if } \gamma = 0, \\ \alpha, g \in C^\infty_c(\omega; C^\infty_{\text{per}}(Y)),\ c \in C^\infty_c(\omega) & \text{if } \gamma = \infty, \\ a \in C^\infty_c(\omega),\ w_0 \in C^\infty_c(\omega; C^\infty(\overline{S}; C^\infty_{\text{per}}(Y;\mathbb{R}^2))) & \text{if } \gamma \in (0,\infty). \end{cases}$$

Then there exists a sequence $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ such that:

(1) The deformation u_h converges to u in the sense of (7.47) and satisfies the one-sided boundary condition (7.48).

(2) The nonlinear strain E_h strongly two-scale converges to

$$E(x,y) := \begin{cases} x_2 \left(\boldsymbol{\kappa}_{(u)}(e_1 \otimes e_1) + \begin{pmatrix} \partial_y \alpha & g \\ g & c \end{pmatrix} \right) & \text{if } \gamma \in \{0,\infty\}, \\ \left(a + x_2\, \boldsymbol{\kappa}_{(u)}\right)(e_1 \otimes e_1) + \operatorname{sym} \widetilde{\nabla}_{1,\gamma} w(x,y) & \text{if } \gamma \in (0,\infty). \end{cases}$$

(3) The energy $\mathcal{I}^{\varepsilon,h}(u_h)$ converges to

$$\iint_{\Omega \times Y} Q\left(y, E(x,y)\right) \mathrm{d}y\, \mathrm{d}x.$$

In the proof of the proposition we distinguish the scaling regimes $\gamma \in \{0,\infty\}$ and $\gamma \in (0,\infty)$. We start with the smooth construction for $\gamma \in \{0,\infty\}$:

Lemma 7.4.15. *Suppose $\gamma \in \{0,\infty\}$. Let $u \in C^\infty_{\text{iso}}(\overline{\omega}; \mathbb{R}^2)$ and*

$$\begin{cases} \alpha, g, c \in C^\infty_c(\omega; C^\infty_{\text{per}}(Y)) & \text{if } \gamma = 0, \\ \alpha, g \in C^\infty_c(\omega; C^\infty_{\text{per}}(Y)),\ c \in C^\infty_c(\omega) & \text{if } \gamma = \infty. \end{cases}$$

Set $\pi_h(x_1) := (x_1, {}^{x_1}/_{\varepsilon(h)})$ and define

$$R_h(x_1) := \mathcal{R}\Big(\varepsilon(\alpha \circ \pi_h)(x_1)\Big) R_{(u)}(x_1),$$

$$\bar{g}(x_1) := \int_Y g(x_1, \xi)\, \mathrm{d}\xi, \qquad \varphi(x_1, y) := \int_0^y g(x_1,\xi) - \bar{g}(x_1)\, \mathrm{d}\xi$$

$$\text{and} \quad d_h(x) := \begin{cases} \dfrac{x_2^2}{2} \left[(g \circ \pi_h)(x_1)\, e_1 + (c \circ \pi_h)(x_1)\, e_2 \right] & \text{if } \gamma = 0 \\ \dfrac{x_2^2}{2} \left[\bar{g}(x_1) e_1 + c(x_1) e_2 \right] + \dfrac{\varepsilon\, x_2}{h} (\varphi \circ \pi_h)(x_1) e_2 & \text{if } \gamma = \infty. \end{cases}$$

Then the map

$$u_h(x) := u(0) + \int_0^{x_1} R_h(s) e_1 \, \mathrm{d}s + h x_2 R_h(x_1) e_2 + h^2 R_h(x_1) d_h(x)$$

satisfies the one-sided boundary condition

$$u_h(0, x_2) = u(0) + h x_2 n_{(u)}(0)$$

and the sequence (u_h) converges to u in the sense of (7.47). Moreover, we have

$$E_h^{\text{ap}} := \frac{R_h^T \nabla_h u_h - Id}{h} \xrightarrow{2} x_2 \left(\boldsymbol{\kappa}_{(u)}(e_1 \otimes e_1) + G \right) \quad \text{with} \quad \operatorname{sym} G = \operatorname{sym} \begin{pmatrix} \partial_y \alpha & g \\ 0 & c \end{pmatrix}$$

strongly two-scale in $L^2(\Omega \times Y; \mathbb{M}(2))$ and the sequence (E_h^{ap}) is uniformly bounded in $L^\infty(\Omega; \mathbb{M}(3))$.

Proof. Step 1. We start with some useful observations simplifying the subsequent calculation. By the chain rule we have

$$\partial_1(\alpha \circ \pi_h) = (\partial_1 \alpha) \circ \pi_h + \frac{1}{\varepsilon}(\partial_y \alpha) \circ \pi_h.$$

The same holds also for the functions $g \circ \pi_h$ and $c \circ \pi_h$. Moreover, we compute

$$\begin{aligned} R_h^{\mathrm{T}}(\partial_1 R_h) = R_{(u)}^{\mathrm{T}} \big(\mathcal{R}(\varepsilon\alpha \circ \pi_h)^{\mathrm{T}} \partial_1 \mathcal{R}(\varepsilon\alpha \circ \pi_h)\big)\, R_{(u)} + R_{(u)}^{\mathrm{T}} \partial_1 R_{(u)} \\ = \Big(\boldsymbol{\kappa}_{(u)} + \varepsilon \partial_1(\alpha \circ \pi_h) \Big)\, \mathcal{R}(\pi/2) \end{aligned}$$

and in view of Lemma 2.1.9 we deduce that
(7.49)

$$\begin{aligned} &\varepsilon \partial_1(\alpha \circ \pi_h) \xrightarrow{2} \partial_y \alpha && \text{strongly two-scale in } L^2(\omega \times Y), \\ &R_h^{\mathrm{T}}(\partial_1 R_h) \xrightarrow{2} (\boldsymbol{\kappa}_{(u)} + \partial_y \alpha)\, \mathcal{R}(\pi/2) && \text{strongly two-scale in } L^2(\omega \times Y; \mathbb{M}(2)). \end{aligned}$$

Furthermore, we compute

$$h\, \nabla_h d_h = \begin{cases} x_2 \begin{pmatrix} 0 & g \circ \pi_h \\ 0 & c \circ \pi_h \end{pmatrix} + \dfrac{x_2^2}{2} \begin{pmatrix} \frac{h}{\varepsilon}(\partial_y g) \circ \pi_h + h(\partial_1 g) \circ \pi_h & 0 \\ \frac{h}{\varepsilon}(\partial_y c) \circ \pi_h + h(\partial_1 c) \circ \pi_h & 0 \end{pmatrix} & \text{if } \gamma = 0, \\ x_2 \begin{pmatrix} 0 & \bar{g} \\ (\partial_y \varphi) \circ \pi_h & c \end{pmatrix} + \begin{pmatrix} \frac{h x_2^2}{2} \partial_1 \bar{g} & 0 \\ \frac{h x_2^2}{2} \partial_1 c + \frac{\varepsilon x_2}{h} (\partial_1 \varphi) \circ \pi_h & \frac{\varepsilon}{h} \varphi \circ \pi_h \end{pmatrix} & \text{if } \gamma = \infty. \end{cases}$$

In the previous equation we collected the terms of higher order in the second matrix in each line. Note that $\varphi(x,y)$ is periodic in its second component and fulfills $\partial_y \varphi = g - \bar{g}$ by construction. Because all quantities involved in the definition of d_h are sufficiently smooth, the previous computation leads to the convergence

$$h\, \nabla_h d_h \xrightarrow{2} \begin{cases} x_2 \begin{pmatrix} 0 & g \\ 0 & c \end{pmatrix} & \text{if } \gamma = 0, \\ x_2 \begin{pmatrix} 0 & \bar{g} \\ g - \bar{g} & c \end{pmatrix} & \text{if } \gamma = \infty \end{cases} \tag{7.50}$$

strongly two-scale in $L^2(\Omega \times Y; \mathbb{M}(2))$.

Step 2. It is easy to check that

$$\nabla_h u_h = R_h + h x_2\, (\partial_1 R_h e_2) \otimes e_1 + h R_h (h\, \nabla_h d_h) + h^2\, (\partial_1 R_h d_h) \otimes e_1$$

and

$$E_h^{\mathrm{ap}} = x_2 (R_h^{\mathrm{T}} \partial_1 R_h) e_2 \otimes e_1 + h\, \nabla_h d_h + \{\, h (R_h^{\mathrm{T}} \partial_1 R) d_h \otimes e_1 \,\}.$$

The term in the curly brackets is of higher order; thus, (7.49) and (7.50) imply that (E_h^{ap}) strongly two-scale converges as it is claimed in the lemma.

Moreover, the quantities involved in the construction above are sufficiently smooth to guarantee that

$$\left|\left(u(0)+\int_0^{x_1} R_h(s)e_1\,\mathrm{d}s\right)-u_h(x_1)\right|+|\nabla_h u_h(x)-R_h(x)|\leq c'h \quad \text{and} \quad \left|E_h^{\mathrm{ap}}(x)\right|\leq c'$$

for a suitable constant c' and all $x\in\Omega$.

Because $\varepsilon\alpha\circ\pi_h$ converges to 0 uniformly, the sequences (R_h) and $(\nabla_h u_h)$ strongly converge to $R_{(u)}$ and we deduce that u_h converges (in the sense of (7.47)) to the map

$$u(0)+\int_0^{x_1} R_{(u)}(s)e_1\,\mathrm{d}s = u(0)+\int_0^{x_1}\partial_1 u(s)\,\mathrm{d}s = u(x_1).$$

Because the functions α, c and g vanish near the boundary $\{0\}\times S$, the constructed sequences trivially satisfy the boundary condition. □

For $\gamma\in(0,\infty)$ the recovery sequence is constructed as follows:

Lemma 7.4.16. *Suppose that $\gamma\in(0,\infty)$. Let*

$$u\in C^\infty_{\mathrm{iso}}(\overline{\omega};\mathbb{R}^2),\qquad w\in C^\infty_c(\omega;C^\infty(\overline{S};C^\infty_{\mathrm{per}}(Y;\mathbb{R}^2))),\qquad a\in C^\infty_c(\omega)$$

and define

$$u_h(x):=u(x_1)+hx_2R_{(u)}(x_1)e_2+h\int_0^{x_1}R_{(u)}(s)e_1a(s)\,\mathrm{d}s+h\varepsilon\,w(x,{}^{x_1}\!/\!_{\varepsilon}).$$

Then the map u_h satisfies the one-sided boundary condition

$$u_h(0,x_2)=u(0)+hx_2n_{(u)}(0)$$

and the sequence (u_h) converges to u in the sense of (7.47). Moreover, the sequence

$$E_h^{\mathrm{ap}}:=\frac{R_{(u)}^T\,\nabla_h u_h - Id}{h}$$

is uniformly bounded in $L^\infty(\Omega;\mathbb{M}(2))$ and

$$E_h^{\mathrm{ap}}\xrightarrow{2}\big(a(x_1)+x_2\boldsymbol{\kappa}_{(u)}\big)(e_1\otimes e_1)+\widetilde{\nabla}_{1,\gamma}w(x,y)$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(2))$.

Proof. We only prove the convergence of (E_h^{ap}), since the other statements are obvious. Set $w^\varepsilon(x):=w(x,{}^{x_1}\!/\!_{\varepsilon})$. A simple computation shows that

$$E_h^{\mathrm{ap}}=\big(a+x_2\boldsymbol{\kappa}_{(u)}\big)(e_1\otimes e_1)+\varepsilon\,\nabla_h w^\varepsilon+\varepsilon(R_{(u)}^{\mathrm{T}}\partial_1R_{(u)})w^\varepsilon. \tag{7.51}$$

Note that

$$\varepsilon\,\nabla_h w^\varepsilon=(\partial_y w)(x,{}^{x_1}\!/\!_{\varepsilon})\otimes e_1+\frac{\varepsilon}{h}(\partial_2 w)(x,{}^{x_1}\!/\!_{\varepsilon})\otimes e_2+\text{higher order terms}.$$

Since $w(x,y)$ is periodic in y, the previous calculation implies that

$$\varepsilon \nabla_h w^\varepsilon \xrightarrow{2} \widetilde{\nabla}_{1,\gamma} w(x,y)$$

strongly two-scale (cf. Proposition 6.3.5). Combined with (7.51), we obtain the claimed convergence statement for the sequence (E_h^{ap}). □

Proof of Proposition 7.4.14. Let $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ and (E_h^{ap}) be defined according to Lemma 7.4.15 (if $\gamma \in \{0,\infty\}$) and Lemma 7.4.16 (if $\gamma \in (0,\infty)$) respectively. Then statement (1) is fulfilled and we have

$$E_h^{\mathrm{ap}} \xrightarrow{2} E^{\mathrm{ap}} \qquad \text{strongly two-scale in } L^2(\Omega{\times}Y;\mathbb{M}(2))$$

for a map $E^{\mathrm{ap}} \in L^2(\Omega{\times}Y;\mathbb{M}(2))$ that satisfies

$$\operatorname{sym} E^{\mathrm{ap}} = E.$$

By construction the sequence (E_h^{ap}) is uniformly bounded. This allows us to apply the simultaneous homogenization and linearization result (see Theorem 5.2.1), which yields

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) = \iint_{\Omega\times Y} Q(y, E^{\mathrm{ap}})\,\mathrm{d}y\,\mathrm{d}x.$$

Because the quadratic form Q vanishes for skew-symmetric matrices (see Lemma 5.2.4), we have

$$Q(y,E^{\mathrm{ap}}) = Q(y, \operatorname{sym} E^{\mathrm{ap}}) = Q(y,E)$$

and statement (3) follows. □

Approximation by smooth data

Lemma 7.4.17. *For all $u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2)$ there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset C^\infty_{\mathrm{iso}}(\overline{\omega};\mathbb{R}^2)$ satisfying the one-sided boundary condition*

$$u_k(0) = u(0) \quad \text{and} \quad \partial_1 u_k(0) = \partial_1 u(0)$$

such that

$$\begin{aligned} u_k &\to u && \text{strongly in } W^{2,2}(\omega;\mathbb{R}^2),\\ \boldsymbol{\kappa}_{(u_k)} &\to \boldsymbol{\kappa}_{(u)} && \text{strongly in } L^2(\omega). \end{aligned}$$

Proof. First, note that we can represent any $u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2)$ in the form

$$u(x_1) = u(0) + \int_0^{x_1} \mathcal{R}(\alpha(s))e_1\,\mathrm{d}s$$

where $\alpha(x_1) := \alpha_0 + \int_0^{x_1} \boldsymbol{\kappa}_{(u)}(s)\,\mathrm{d}s$ and $\alpha_0 \in [0,2\pi)$ satisfies $\mathcal{R}(\alpha_0)e_1 = \partial_1 u(0)$.

This suggests to approximate u by mollifying $\boldsymbol{\kappa}_{(u)}$: Let $(\boldsymbol{\kappa}_k)$ denote a sequence in $C_0^\infty(\omega)$ converging to $\boldsymbol{\kappa}_{(u)}$ in $L^2(\omega)$. We define

$$\alpha_k(x_1) := \alpha_0 + \int_0^{x_1} \boldsymbol{\kappa}_k(s)\,\mathrm{d}s \quad \text{and} \quad u_k(x_1) := u(0) + \int_0^{x_1} \mathcal{R}(\alpha_k(s))e_1\,\mathrm{d}s.$$

By construction (u_k) is a sequence in $C^\infty_{\mathrm{iso}}(\overline{\omega};\mathbb{R}^2)$ satisfying the one-sided boundary condition. Moreover, it is easy to check that

$$\alpha_k \to \alpha \qquad \text{strongly in } W^{1,2}(\omega) \text{ and pointwise.}$$

Hence, we immediately deduce that

$$\mathcal{R}(\alpha_k) \to \mathcal{R}(\alpha) \qquad \text{pointwise and strongly in } L^2(\omega; SO(2)),$$

where the latter follows due to dominated convergence. Since

$$\partial_1 u_k = \mathcal{R}(\alpha_k)e_1$$

this already implies that (u_k) converges to u in $W^{1,2}(\omega;\mathbb{R}^2)$.

It remains to prove that $\partial_{11}^2 u_k \to \partial_{11}^2 u$. Equivalently, we can show that

$$\partial_1\mathcal{R}(\alpha_k) \to \partial_1\mathcal{R}(\alpha) \qquad \text{strongly in } L^2(\omega;\mathbb{M}(2)). \tag{7.52}$$

A simple calculation leads to the characterization

$$\partial_1\mathcal{R}(\alpha_k) = \boldsymbol{\kappa}_k\,\mathcal{R}(\alpha_k + \pi/2).$$

Since $(\boldsymbol{\kappa}_k)$ as well as $\mathcal{R}(\alpha_k)$ converge in L^2, we deduce that $\partial_1\mathcal{R}(\alpha_k)$ converges to $\partial_1\mathcal{R}(\alpha)$ in $L^1(\omega,\mathbb{M}(2))$. Because of

$$|\partial_1\mathcal{R}(\alpha_k(x_1))|^2 \leq 2\,|\boldsymbol{\kappa}_k(x_1)|^2\,,$$

(7.52) follows by Vitali convergence theorem. □

Proof of the upper bound

Let $\gamma \in (0,\infty)$. In view of Proposition 7.4.18 there exist maps

$$a \in L^2(\omega) \quad \text{and} \quad w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2))$$

such that

$$\mathcal{I}_\gamma(u) = \iint_{\Omega\times Y} Q\left(y, \big(a(x_1) + x_2\boldsymbol{\kappa}_{(u)}\big)(e_1\otimes e_1) + \widetilde{\nabla}_{1,\gamma} w(x,y)\right)\mathrm{d}y\,\mathrm{d}x. \tag{7.53}$$

Because the inclusions

$$C_c^\infty(\omega) \subset L^2(\omega) \quad \text{and} \quad C_c^\infty(\omega; C^\infty(\bar{S}; C^\infty_{\mathrm{per}}(Y;\mathbb{R}^2)) \subset L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2))$$

are dense and due to Lemma 7.4.17, we can find for any $\delta > 0$ approximations

$$u^{(\delta)} \in C^\infty_{\text{iso}}(\overline{\omega};\mathbb{R}^2) \quad \text{with} \quad u^{(\delta)}(0) = u(0) \quad \text{and} \quad \partial_1 u^{(\delta)}(0) = \partial_1 u(0),$$
$$a^{(\delta)} \in C^\infty_c(\omega) \quad \text{and} \quad w^{(\delta)} \in C^\infty_c(\omega; C^\infty(\bar{S}; C^\infty_{\text{per}}(Y;\mathbb{R}^2))$$

such that

$$\left\| u^{(\delta)} - u \right\|_{W^{2,2}(\omega;\mathbb{R}^2)} + \left\| a^{(\delta)} - a \right\|_{L^2(\omega)} + \left\| w^{(\delta)} - w \right\|_{L^2(\omega;W^{1,2}(S\times Y;\mathbb{R}^2))} < \delta.$$

The integral functional on the right hand side in (7.53) is continuous (with respect to strong convergence in the appropriate function spaces). For this reason, we can choose the approximation above in such a way that

$$\left| \mathcal{I}_\gamma(u) - \iint_{\Omega\times Y} Q\left(y, \left(a^{(\delta)} + x_2 \boldsymbol{\kappa}_{(u^{(\delta)})}\right)(e_1 \otimes e_1) + \widetilde{\nabla}_{1,\gamma} w^{(\delta)} \right) \, \mathrm{d}y \, \mathrm{d}x \right| \leq \delta$$

is additionally satisfied.

Now we associate to each triplet $(u^{(\delta)}, a^{(\delta)}, w^{(\delta)})$ a sequence $(u_h^{(\delta)}) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ according to Lemma 7.4.16. Then for each $\delta > 0$ the sequence $(u_h^{(\delta)})_h$ converges to u_δ in the sense of (7.47) and fulfills the boundary condition (7.48); and

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h^{(\delta)}) = \iint_{\Omega\times Y} Q\left(y, \left(a^{(\delta)} + x_2 \boldsymbol{\kappa}_{(u^{(\delta)})}\right)(e_1 \otimes e_1) + \widetilde{\nabla}_{1,\gamma} w^{(\delta)} \right) \, \mathrm{d}y \, \mathrm{d}x.$$

Set

$$c_{\delta,h} := \left\| u - u_h^{(\delta)} \right\|_{L^2(\Omega;\mathbb{R}^2)} + \left\| R_{(u)} - \nabla_h u_h^{(\delta)} \right\|_{L^2(\omega;\mathbb{M}(2))} + \left| \mathcal{I}_\gamma(u) - \mathcal{I}^{\varepsilon,h}(u_h^{(\delta)}) \right|.$$

Then we have

$$\lim_{k\to\infty} \lim_{h\to 0} c_{k,h} = 0.$$

This allows us to extract a diagonal sequence $\delta(h)$ with $\lim_{h\to 0} c_{\delta(h),h} = 0$ due to an argument by H. Attouch (see Lemma A.2.1). Hence, the sequence $u_h := u_h^{(\delta(h))}$ converges to u in the sense of (7.47) and recovers the energy. Moreover, since each deformation $u_h^{(\delta)}$ satisfies the appropriate one-sided boundary condition, the same holds for the diagonal sequence (u_h) and the proof is complete.

The proof in the case $\gamma \in \{0, \infty\}$ is similar and omitted here. □

7.4.5. Cell formulas

In this section we prove that $\mathcal{I}_\gamma$ can be identified with $\mathcal{E}_\gamma(\cdot;0)$. Moreover, we analyze the cell problems that determine the effective stiffness coefficients q_γ. Recall that $\mathbb{X}_\gamma$ is defined for $\gamma \in \{0,\infty\}$ according to

$$\mathbb{X}_0 := \{ (\alpha, g, c) \, : \, \alpha \in L^2(\omega; W^{1,2}_{\text{per}}(Y)), \; g, c \in L^2(\omega\times Y) \}$$
$$\mathbb{X}_\infty := \{ (\alpha, g, c) \, : \, \alpha \in L^2(\omega; W^{1,2}_{\text{per}}(Y)), \; g \in L^2(\omega\times Y), \, c \in L^2(\omega) \}$$

(see page 108).

Proposition 7.4.18.

(1) Let $\gamma \in \{0, \infty\}$. For all $u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2)$ we have

$$\mathcal{I}_\gamma(u) = \mathcal{E}_\gamma(u;0)$$

and there exists a triplet

$$(\alpha, g, c) \in \mathbb{X}_\gamma$$

such that

$$\mathcal{I}_\gamma(u) = \frac{1}{12}\iint\limits_{\omega\times Y} Q\left(y, \boldsymbol{\kappa}_{(u)}(e_1 \otimes e_1) + \begin{pmatrix} \partial_y \alpha & g \\ g & c \end{pmatrix}\right) \mathrm{d}y\,\mathrm{d}x_1.$$

(2) Let $\gamma \in (0, \infty)$. For all $u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2)$ we have

$$\mathcal{I}_\gamma(u) = \mathcal{E}_\gamma(u;0)$$

and there exist unique maps

$$a \in L^2(\omega) \quad \text{and} \quad w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2)) \text{ with } \iint\limits_{S\times Y} w(\cdot, x_2, y)\,\mathrm{d}y\,\mathrm{d}x_2 = 0$$

such that

$$\mathcal{I}_\gamma(u) = \iint\limits_{\Omega\times Y} Q\Big(y, \big(a(x_1) + x_2\boldsymbol{\kappa}_{(u)}\big)(e_1\otimes e_1) + \widetilde{\nabla}_{1,\gamma} w_0(x,y)\Big)\,\mathrm{d}y\,\mathrm{d}x.$$

Proof. We only present the proof for $\gamma \in (0,\infty)$. The case $\gamma \in \{0,\infty\}$ can be justified in a similar manner. The strategy is the following:

(A) Fix a map in $\boldsymbol{\kappa} \in L^2(\omega)$. Show that the minimization problem

$$(7.54)\quad (a, w) \mapsto \iint\limits_{\Omega\times Y} Q\Big(y, \big(a(x_1) + x_2\boldsymbol{\kappa}\big)(e_1\otimes e_1) + \widetilde{\nabla}_{1,\gamma} w(x,y)\Big)\,\mathrm{d}y\,\mathrm{d}x$$

$$\text{with} \quad a \in L^2(\omega),\ w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2))$$

admits a solution $(a_{\boldsymbol{\kappa}}, w_{\boldsymbol{\kappa}})$.

Because $\alpha_{\boldsymbol{\kappa}}(x_1) \in \mathbb{R}$ and $w_{\boldsymbol{\kappa}}(x_1) \in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2)$ for almost every $x_1 \in \omega$, this already implies that $\mathcal{I}_\gamma(u) \geq \mathcal{E}_\gamma(u, 0)$.

(B) Fix $\boldsymbol{\kappa} \in \mathbb{R}$. Show that the minimization problem

$$(7.55)\quad (a, w) \mapsto \iint\limits_{S\times Y} Q\Big(y, \big(a + x_2\boldsymbol{\kappa}\big)(e_1\otimes e_1) + \widetilde{\nabla}_{1,\gamma} w(x_2,y)\Big)\,\mathrm{d}y\,\mathrm{d}x_2$$

$$\text{with} \quad a \in \mathbb{R},\ w \in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2))$$

admits a solution $(a^\star_{\boldsymbol{\kappa}},\, w^\star_{\boldsymbol{\kappa}})$ and prove that there exists a linear and continuous map

$$\Phi\,:\,\mathbb{R}\mapsto\mathbb{R}\times W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2),\qquad \Phi(\boldsymbol{\kappa}):=(a^\star_{\boldsymbol{\kappa}},w^\star_{\boldsymbol{\kappa}})$$

such that $(a^\star_{\boldsymbol{\kappa}},w^\star_{\boldsymbol{\kappa}}):=\Phi(\boldsymbol{\kappa})$ is a minimizer of (7.55).

Because the map Φ is measurable and linear, for any $\boldsymbol{\kappa}\in L^2(\omega)$ the map

$$x_1\mapsto\Phi(\boldsymbol{\kappa}(x_1))$$

is measurable and can be identified with a pair in $L^2(\omega)\times L^2(\omega;W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2))$. This implies that $\mathcal{I}_\gamma(u)\leq\mathcal{E}_\gamma(u;0)$.

Statement (A) can be proved by standard arguments from the direct method of the calculus of variations. Since Q is quadratic and has uniformly bounded coefficients, the functional in (7.54) is lower semicontinuous with respect to weak convergence in $L^2(\omega)$ and $L^2(\omega;W^{1,2}(S\times Y;\mathbb{R}^2))$. The crucial point is to prove the coercivity of the functional. To this end, we start with the observation that the integral in (7.54) is bounded from below by

$$c'\left(\|\boldsymbol{\kappa}\|^2_{L^2(\omega)}+\|a\|^2_{L^2(\omega)}+\left\|\operatorname{sym}\widetilde{\nabla}_{1,\gamma}w\right\|^2_{L^2(\Omega\times Y;\mathbb{M}(2))}\right).\tag{7.56}$$

Here and below, $c'>0$ denotes a constant that may change from line to line, but can be chosen independent of $\boldsymbol{\kappa}$, a and w. For the estimate above we used two facts: First, the maps

$$a(x_1)(e_1\otimes e_1),\qquad x_2\boldsymbol{\kappa}(x_1)(e_1\otimes e_1)\qquad\text{and}\qquad\operatorname{sym}\widetilde{\nabla}_{1,\gamma}w(x_1,x_2,y)$$

are orthogonal (for a.e. $x_1\in\omega$) with respect to the scalar product in the Hilbert space $L^2(S\times Y;\mathbb{M}(3))$. And secondly, we used that the non-degeneracy condition implies that

$$Q(y,F)\geq c'\,|\operatorname{sym}F|^2\qquad\text{for all } F\in\mathbb{M}(2)\quad\text{and a.e.}\quad y\in Y.$$

Moreover, it is obvious that the minimum of (7.54) can equivalently be computed on the class of functions

$$a\in L^2(\omega)\quad\text{and}\quad w\in L^2(\omega;\mathcal{W})$$
$$\text{with}\quad\mathcal{W}:=\left\{w\in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2))\;:\;\iint_{S\times Y}w(x_2,y)\,\mathrm{d}y\,\mathrm{d}x_2=0\right\}.$$

Theorem 6.3.8 shows that a Korn inequality holds on $\mathcal{W}$ (note that in Theorem 6.3.8 we have $\mathcal{R}(1)=\{0\}$). Consequently, estimate (7.56) implies the coercivity of the minimization problem and indeed the direct method yields existence of a minimizer. The uniqueness is a consequence of the strict convexity of Q for symmetric matrices.

For the very same reason also the minimization problem in (B) admits a minimizer. As before, we can equivalently compute the minimum on the class of functions

$$a\in\mathbb{R}\quad\text{and}\quad w\in\mathcal{W}.$$

Now the strict convexity of $Q(y,\cdot)$ on the subspace of symmetric matrices implies that (B) admits even a unique minimizer $(a^\star_{\boldsymbol{\kappa}}, w^\star_{\boldsymbol{\kappa}})$ with $a^\star_{\boldsymbol{\kappa}} \in \mathbb{R}$ and $w^\star_{\boldsymbol{\kappa}} \in \mathcal{W}$. Moreover, one can characterize this minimizer by a linear Euler-Lagrange equation. Consequently, the map $\Phi(\boldsymbol{\kappa}) := (a^\star_{\boldsymbol{\kappa}}, w^\star_{\boldsymbol{\kappa}})$ is indeed linear and because of $\Phi(0) = (0,0)$ also continuous. □

7.5. Strong two-scale convergence of the nonlinear strain for low energy sequences

In this section we consider low energy sequences and prove that the associated sequence of nonlinear strain **strongly** two-scale converges. The following theorem is an extension of a result in [FJM02] (originally for homogeneous plates) to the homogenization setting considered in this chapter.

Theorem 7.5.1. *Let $\gamma \in (0,\infty)$ and $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)$. Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^2)$ converging to u strongly in $L^2(\Omega;\mathbb{R}^2)$. Suppose that*

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) = \mathcal{I}_\gamma(u). \tag{7.57}$$

Then

$$E_h := \frac{\sqrt{\nabla_h u_h^T \nabla_h u_h} - Id}{h} \xrightarrow{2} \big(\, a^\star(x_1) + x_2 \boldsymbol{\kappa}_{(u)}(x_1)\,\big)(e_1 \otimes e_1) + \operatorname{sym} \widetilde{\nabla}_{1,\gamma} w^\star(x,y)$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(2))$, where the a pair $(a^\star, w^\star)$ with

$$a^\star \in L^2(\omega) \quad \text{and} \quad w^\star \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2)), \qquad \iint_{S\times Y} w^\star(\cdot, x_2, y)\,\mathrm{d}y\,\mathrm{d}x_2 = 0$$

is the unique minimizer of the minimization problem in Proposition 7.4.18 (2).

Proof. Step 1. Set

$$E_{\min}(x,y) := \big(\, a^\star(x_1) + x_2 \boldsymbol{\kappa}_{(u)}(x_1)\,\big)(e_1 \otimes e_1) + \operatorname{sym} \widetilde{\nabla}_{1,\gamma} w^\star(x,y)$$

and let $\mathfrak{C}$ (see (7.32)) denote the set of all weak two-cluster points of the sequence (E_h). The sequence (u_h) has finite bending energy, and therefore (E_h) is weakly two-scale relatively compact (see Theorem 7.4.2) and $\mathfrak{C}$ is non-empty. In view of the compactness part of Theorem 7.2.5 and due to the two-scale characterization of the limiting strain (see Theorem 7.4.10), we find that

$$E - E_{\min} = a(x_1)(e_1 \otimes e_1) + \operatorname{sym} \widetilde{\nabla}_{1,\gamma} w(x,y)$$

for suitable maps $a \in L^2(\omega)$ and $w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2))$. A close look at the definition of q_γ (see also Remark 7.2.4) reveals that

$$\inf_{E\in\mathfrak{C}} \iint_{\Omega\times Y} Q(y, E(x,y))\,\mathrm{d}y\,\mathrm{d}x \geq \mathcal{I}_\gamma(u). \tag{7.58}$$

On the other side, Lemma 7.4.13 and assumption (7.57) yield

$$\mathcal{I}_\gamma(u) = \lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) \geq \inf_{E\in\mathfrak{E}} \iint_{\Omega\times Y} Q(y, E(x,y))\,\mathrm{d}y\,\mathrm{d}x,$$

and consequently (7.58) actually holds with equality. Thereby, the uniqueness of the minimizing pair in Proposition 7.4.18 (2) implies that $E = E_{\min}$ for all $E \in \mathfrak{E}$, and thus,

$$E_h \xrightarrow{2} E_{\min} \qquad \text{weakly in } L^2(\Omega\times Y;\mathbb{R}^2).$$

<u>*Step 2.*</u> Let δ be a small positive paramter with the property that

$$\forall F \in \mathbb{M}(2) \,:\, \operatorname{dist}(F, SO(2)) < \delta \quad \Rightarrow \quad \det F > 0.$$

We consider the set

$$\Omega_h := \Big\{ x\in\Omega \,:\, \operatorname{dist}(\nabla_h u_h(x), SO(2)) < \delta \quad \text{and} \quad |E_h(x)| < h^{-1/2} \Big\}.$$

and aim to show that the measure of $\Omega\setminus\Omega_h$ vanishes as $h\to 0$. To this end, first note that the inequality

$$\forall F \in \mathbb{M}(2) \,:\, \left|\sqrt{F^{\mathrm{T}}F} - Id\right|^2 \leq \operatorname{dist}^2(F, SO(2)), \tag{7.59}$$

implies that

$$h\,|E_h|^2 + h^2 \frac{\operatorname{dist}^2(\nabla_h u_h, SO(2))}{\delta^2 h^2}\,\mathrm{d}x \leq c' h\, \frac{\operatorname{dist}^2(\nabla_h u_h(x), SO(2))}{h^2}$$

for all $h < 1$. Here and below, c' denotes a positive constant that may change from line to line, but can be chosen independent of h. By definition, for a.e. $x \in \Omega\setminus\Omega_h$ the left hand side is bigger than 1. Thus, the non-degeneracy of W leads to

$$\mathcal{H}^n(\Omega\setminus\Omega_h) \leq h\,c'\,\frac{1}{h^2}\int_\Omega W({x_1}/{\varepsilon}, \nabla_h u_h(x))\,\mathrm{d}x$$

and because the integral on the right hand side is finite, we obtain

$$\mathcal{H}^n(\Omega\setminus\Omega_h) \leq c'\,h. \tag{7.60}$$

<u>*Step 3.*</u> Recall the definition of the unfolding operator

$$\mathcal{T}_\varepsilon^1 \,:\, L^2(\Omega;\mathbb{M}(2)) \to L^2(\mathbb{R}^2\times Y;\mathbb{M}(2))$$

defined in Definition 6.2.1. We consider the maps

$$A_h := \mathcal{T}_\varepsilon^1(1_{\Omega_h} E_h), \qquad B_h := \mathcal{T}_\varepsilon^1 E_h - A_h = \mathcal{T}_\varepsilon^1(1_{\Omega\setminus\Omega_h} E_h)$$

and the sets

$$Z_h := \{\, (x,y) \in \mathbb{R}^2\times Y \,:\, (\varepsilon\lfloor {x_1}/{\varepsilon}\rfloor + y, x_2) \in \Omega_h \,\},$$
$$Z_h^C := \{\, (x,y) \in \mathbb{R}^2\times Y \,:\, (\varepsilon\lfloor {x_1}/{\varepsilon}\rfloor + y, x_2) \in \Omega\setminus\Omega_h \,\}.$$

By construction, the support of $\mathcal{T}_\varepsilon^h u_h$ is contained in $Z_h \cup Z_h^C$. With Lemma 9.1.3 we can rewrite the elastic energy of u_h by means of the unfolding operator $\mathcal{T}_\varepsilon^1$:

$$\begin{aligned}\mathcal{I}^{\varepsilon,h}(u_h) &= \frac{1}{h^2}\iint\limits_{\mathbb{R}^2\times Y} W(y,(\mathcal{T}_\varepsilon^1\,\nabla_h u_h)(x,y))\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{1}{h^2}\iint\limits_{Z_h} W(y,(\mathcal{T}_\varepsilon^1\,\nabla_h u_h)(x,y))\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{h^2}\iint\limits_{Z_h^C} W(y,(\mathcal{T}_\varepsilon^1\,\nabla_h u_h)(x,y))\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Now for a.e. $(x,y) \in Z_h$ the matrix $(\mathcal{T}_\varepsilon^1\,\nabla_h u_h)(x,y)$ has a positive determinant and there exists a rotation field $R : Z_h \to SO(2)$ such that

$$(\mathcal{T}_\varepsilon^1\,\nabla_h u_h)(x,y) = R(x,y)\,(\mathcal{T}_\varepsilon^1\sqrt{\nabla_h u_h^{\mathrm{T}}\,\nabla_h u_h})(x,y) \qquad \text{for a.e. } (x,y)\in Z_h.$$

Because W is frame indifferent, this implies that

$$W(y,(\mathcal{T}_\varepsilon^1\,\nabla_h u_h)(x,y)) = W(y, Id + hA_h(x,y)) \qquad \text{for a.e. } (x,y)\in Z_h.$$

By construction, we have $|hA_h|^2 \le h$, and therefore the quadratic expansion in assumption (W4) implies that

$$\frac{1}{h^2}\iint\limits_{Z_h} W(y, Id + hA_h(x,y))\,\mathrm{d}y\,\mathrm{d}x = \iint\limits_{Z_h} Q(y, A_h(x,y))\,\mathrm{d}y\,\mathrm{d}x + \mathrm{rest}_h^{(1)}$$

with $\left|\mathrm{rest}_h^{(1)}\right| \le c'h$. Set

$$\mathrm{rest}_h^{(2)} := \frac{1}{h^2}\iint\limits_{Z_h^C} W(y,(\mathcal{T}_\varepsilon^1\,\nabla_h u_h)(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

Then

$$\mathcal{I}^{\varepsilon,h}(u_h) = \iint\limits_{Z_h} Q(y, A_h(x,y))\,\mathrm{d}y\,\mathrm{d}x + \mathrm{rest}_h^{(1)} + \mathrm{rest}_h^{(2)}. \tag{7.61}$$

<u>*Step 4.*</u> We claim that

$$\lim_{h\to 0}\mathrm{rest}_h^{(2)} = 0. \tag{7.62}$$

This can be seen as follows: First, we have

$$A_h \rightharpoonup E_{\min} \qquad \text{weakly in } L^2(\mathbb{R}^2\times Y;\mathbb{M}(2))$$

where we set $E_{\min}(x,y) = 0$ for $(x,y) \in (\mathbb{R}^2\setminus\Omega)\times Y$. This follows from the definition of two-scale convergence (see Definition 6.2.3), Proposition 2.1.13 and the fact that

(7.60) implies that (1_{Ω_h}) converges to 1_Ω boundedly in measure. By applying the lower-semicontinuity of convex functionals, we deduce that

$$\begin{aligned}\mathcal{I}_\gamma(u) &= \limsup_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h)\\ &\geq \liminf_{h\to 0} \iint_{Z_h} Q(y, A_h(x,y))\,\mathrm{d}y\,\mathrm{d}x + \limsup_{h\to 0} \mathrm{rest}_h^{(1)} + \limsup_{h\to 0} \mathrm{rest}_h^{(2)}\\ &\geq \iint_{\Omega\times Y} Q(y, E_{\min}(x,y))\,\mathrm{d}y\,\mathrm{d}x + \limsup_{h\to 0} \mathrm{rest}_h^{(2)}.\end{aligned}$$

Now the integral on the right hand side is equal to $\mathcal{I}_\gamma(u)$ (see Proposition 7.4.18 (2)) and because $\mathrm{rest}_h^{(2)} \geq 0$ for all h, this already implies (7.62). In particular, we obtain

$$\lim_{h\to 0} \iint_{\mathbb{R}^2\times Y} Q(y, A_h(x,y))\,\mathrm{d}y\,\mathrm{d}x = \iint_{\mathbb{R}^2\times Y} Q(y, E_{\min}(x,y))\,\mathrm{d}y\,\mathrm{d}x = \mathcal{I}_\gamma(u). \tag{7.63}$$

Next, we prove that

$$B_h \to 0 \qquad \text{strongly in } L^2(\mathbb{R}^2\times Y; \mathbb{M}(2)). \tag{7.64}$$

We have $|B_h|^2 \leq \frac{1}{h^2}\,\mathrm{dist}^2(\mathcal{T}_\varepsilon^1 \nabla_h u_h, SO(2))$ due to (7.59). Because W is non-degenerate and the support of B_h is contained in Z_h^C, we arrive at

$$\iint_{\mathbb{R}^2\times Y} |B_h|^2\,\mathrm{d}y\,\mathrm{d}x = \iint_{Z_h^C} |B_h|^2\,\mathrm{d}y\,\mathrm{d}x \leq \mathrm{rest}_h^{(2)}.$$

The right hand side converges to zero; thus, (7.64) follows.

Step 5. In virtue of the decomposition $\mathcal{T}_\varepsilon^1 \nabla_h u_h = A_h + B_h$, it remains to show that

$$A_h \to E_{\min} \qquad \text{strongly in } L^2(\mathbb{R}^2\times Y; \mathbb{M}(2)). \tag{7.65}$$

The quadratic form $\mathbb{M}_{\mathrm{sym}}(2) \ni F \mapsto Q(y,F)$ is positive definite (see Lemma 5.2.4). In particular, there exists a positive constant c' such that

$$Q(y,A) - Q(y,E) - 2\,\langle \mathbb{L}(y)E,\, A-E\rangle \geq c'\,|A-E| \qquad \text{for all } A, E \in \mathbb{M}_{\mathrm{sym}}(2)$$

and a.e. $y \in Y$. We apply this inequality to the quadratic functional in (7.61):

$$\begin{aligned}c' \iint_{\mathbb{R}^2\times Y} |A_h - E_{\min}|^2\,\mathrm{d}y\,\mathrm{d}x &\leq \iint_{\mathbb{R}^2\times Y} Q(y, A_h(x,y)) - Q(y, E_{\min}(x,y))\,\mathrm{d}y\,\mathrm{d}x\\ &\quad - 2 \iint_{\mathbb{R}^2\times Y} \langle \mathbb{L}(y)E_{\min}(x,y),\, A_h(x,y) - E_{\min}(x,y)\rangle\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Now the first integral and second integral on the right hand side converge to zero because of (7.63) and the weak convergence of A_h to $E_{\min}$, respectively. □

7.6. Interpretation of the limiting models

In this part we consider the fine-scale coupling regimes

$$h \gg \varepsilon \quad \text{and} \quad h \ll \varepsilon,$$

which corresponds to the situation where the fine-scales separate in the limit. It is natural to expect that in these cases the limiting theories are related to the effective theories obtained by consecutively passing to the limits $\varepsilon \to 0$ and $h \to 0$, i.e. by first homogenizing the initial energy and then reducing the dimension and vice versa. In the following we justify this hypothesis. For simplicity we state this insight on the level of the integrands and additionally suppose that W is a Carathéodory function satisfying the quadratic growth and coercivity condition

$$\tfrac{1}{c}\,|\operatorname{sym} F|^2 - c \le W(y,F) \le c(1+|F|^2) \tag{7.66}$$

for a positive constant c that is independent of y and F.

Let $m \in \{1,2\}$. For convenience we introduce the class $\mathfrak{I}(Y;m)$ consisting of all Carathéodory functions from $Y \times \mathbb{M}(m)$ to $\mathbb{R}$ that are Y-periodic in the first variable and that satisfy the growth condition (7.66) for a suitable constant $c > 0$. Furthermore, we define the subclass $\mathcal{I}^2(Y;m)$ as the set of integrands $W \in \mathfrak{I}(Y;m)$ that admit a quadratic Taylor expansion in the sense that

$$\limsup_{\substack{G\to 0\\ G\neq 0}} \operatorname*{ess\,sup}_{y\in Y} \frac{|W(y, Id+G) - Q(y,G)|}{|G|^2} = 0$$

for a suitable quadratic integrand $Q \in \mathfrak{Q}(Y;m)$.

We like to remark that every integrand $W \in \mathfrak{I}(Y,2)$ that satisfies assumption (W1) – (W4) belongs to $\mathfrak{I}^2(Y,2)$.

We define the following maps:

$$\begin{aligned}
\mathsf{hom}: \mathfrak{I}(Y,m) \to \mathfrak{I}(Y,m), \qquad & (\mathsf{hom}\, W)(F) := W^{(\mathrm{mc})}_{\mathrm{hom}}(F)\\
\mathsf{lin}: \mathfrak{I}^2(Y,m) \to \mathfrak{I}(Y,m), \qquad & (\mathsf{lin}\, W)(y,F) := Q(y,F)\\
\mathsf{rel}: \mathfrak{I}(Y,2) \to \mathfrak{I}(Y,1), \qquad & (\mathsf{rel}\, W)(y,a) := \min_{d\in\mathbb{R}^2} W(y, a(e_1\otimes e_1) + d\otimes e_2)\\
\mathsf{dred}: \mathfrak{I}^2(Y,2) \to \mathfrak{I}(Y,1), \qquad & \mathsf{dred}\, W := \mathsf{rel}\circ \mathsf{lin}\, W.
\end{aligned}$$

In the definition above, $W^{(\mathrm{mc})}_{\mathrm{hom}}$ denotes the multi-cell homogenization formula

$$W^{(\mathrm{mc})}_{\mathrm{hom}}(F) := \inf_{k\in\mathbb{N}} \inf \left\{ \frac{1}{k^m} \int_{(0,k)^m} W(z_1, F + \nabla\varphi(z))\,\mathrm{d}z \;:\; \varphi \in W^{1,p}_{\mathrm{per}}((0,k)^m;\mathbb{R}^m) \right\}.$$

The operation hom is related to homogenization. More precisely, the Γ-limit of the integral functional

$$W^{1,2}(U;\mathbb{R}^m) \ni u \mapsto \int_U W(x_1/\varepsilon, \nabla u(x))\,\mathrm{d}x \qquad (\text{with } U \subset \mathbb{R}^m \text{ open})$$

for $\varepsilon \to 0$ is given by the functional

$$W^{1,2}(U;\mathbb{R}^m) \ni u \mapsto \int_U (\mathsf{hom}\, W)(\nabla u(x))\,\mathrm{d}x.$$

Furthermore, we have seen in Theorem 5.3.10 that the operation lin is related to linearization. In the following result we show that the operation dred is related to variational dimension reduction. Moreover, the following theorem reveals that the effective limiting coefficient that appears in the limiting functional $\mathcal{E}_\gamma$ can be computed by consecutively applying hom and dred to the energy density W.

Theorem 7.6.1. *Let $W : Y\times\mathbb{M}(2) \to \mathbb{R}$ be an integrand of class $\mathfrak{I}^2(Y,2)$ and let q_γ denote the effective limiting coefficients associated to W and defined in (7.5).*

(1) If $W(y,F)$ is independent of y, then

$$q_\gamma = \frac{1}{12}(\mathsf{dred}\, W)(1) \qquad \text{for all } \gamma \in [0,\infty].$$

(2) For $\gamma \in \{0,\infty\}$ we have

$$q_0 = \frac{1}{12}\,(\,\mathsf{hom}\circ\mathsf{dred}\;W\,)\,(1) \qquad \text{and} \qquad q_\infty = \frac{1}{12}\,(\,\mathsf{dred}\circ\mathsf{hom}\;W\,)\,(1).$$

Proof. Statement (1) directly follows by the definition of q_γ (see equation (7.5)). For the case $\gamma = 0$ in statement (2) we use the fact that

$$(\mathsf{hom}\circ\mathsf{dred}\, W)(1) = \inf_{\alpha\in W^{1,2}_{\mathrm{per},0}(Y)} \int_Y (\mathsf{dred}\, W)\big(1 + \partial_y\alpha(y)\big)\,\mathrm{d}y,$$

because $\mathsf{dred}\, W$ is a convex (even quadratic) integrand, and therefore the multi-cell homogenization formula is equal to the one-cell homogenization formula. Now a close look at the definition of q_0 proves the claimed identity.

For $\gamma = \infty$ we first observe that by definition

$$\mathsf{dred}\circ\mathsf{hom}\, W = \mathsf{rel}\circ\mathsf{lin}\circ\mathsf{hom}\, W.$$

Now the analysis in Section 5.1 (cf. (5.4) and [NS10]) revealed that homogenization and linearization commute, and therefore

$$\mathsf{dred}\circ\mathsf{hom}\, W = \mathsf{rel}\circ\mathsf{hom}\circ\mathsf{lin}\, W.$$

By assumption (W4), we have $\mathsf{lin}\, W = Q$; thus, the previous equation implies

$$(7.67)\quad (\mathsf{hom}\circ\mathsf{lin}\, W)(e_1\otimes e_1 + d\otimes e_2) = \inf_{\varphi\in W^{1,2}_{\mathrm{per},0}((0,1)^2;\mathbb{R}^2)} \int_{(0,1)^2} Q(z_1, e_1\otimes e_1 + d\otimes e_2 + \nabla_y\,\varphi(z))\,\mathrm{d}z$$

for all $d \in \mathbb{R}^2$. In the previous equation we used again the fact that the multi-cell homogenization formula reduces to a one-cell formula for convex integrands. In the next step we follow an idea of S. Müller [Mül]. By assumption the quadratic form $F \mapsto Q(y, F)$ is positive definite on $\mathbb{M}_{\mathrm{sym}}(2)$. As a consequence one can show that for each $d \in \mathbb{R}^2$ the minimization problem on the right hand side in (7.67) admits a unique minimizer $\varphi_d \in W^{1,2}_{\mathrm{per},0}((0,1)^2;\mathbb{R}^2)$, i.e.

$$(\mathsf{hom} \circ \mathsf{lin}\, W)(e_1\otimes e_1 + d\otimes e_2) = \int_{(0,1)^2} Q_W(z_1, e_1\otimes e_1 + d\otimes e_2 + \nabla_y \varphi_d(z))\, \mathrm{d}z.$$

Because φ_d is periodic and W only depends on the first component z_1, we see that the map $z \mapsto \varphi_d(z_1, z_2 + \lambda)$ with $\lambda \in \mathbb{R}$ is also a minimizer and due to the uniqueness of the minimizer it follows that

$$\varphi_d(z_1, z_2) = \varphi_d(z_1, z_2 + \lambda) \qquad \text{for all } \lambda \in \mathbb{R}.$$

Thus, φ_d is independent of z_2 and can be identified with a map in $W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^2)$. In summary, so far we have shown that

$$(\mathsf{dred} \circ \mathsf{hom}\, W)(1) = \min_{d\in\mathbb{R}^2} \inf_{\varphi\in W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^2)} \int_Y Q\left(y, e_1\otimes e_1 + d\otimes e_2 + \partial_y\varphi(y)\otimes e_1\right)\, \mathrm{d}y.$$

A comparison with equation (7.5) reveals that the right hand side is equal to q_∞. □

7.7. Advanced applications: Layered and prestressed materials

In this section, we demonstrate that the methods derived in the previous sections can be easily applied to more general settings. It is important to note that the compactness result (see Theorem 7.4.2) and the two-scale characterization of the nonlinear limiting strain (see Theorem 7.4.10) can be applied to arbitrary sequences with finite bending energy. In this sense, both results are independent of the specific form of the elastic potential W.

In the first part of this section, we show that the two-scale characterization of the nonlinear limiting strain is *sharp* in the sense that any admissible limiting strain can be recovered by a sequence of deformations in the strong two-scale sense. In the second part, we apply this insight to periodically layered and prestressed planar rods.

For simplicity, we only consider the case where h and ε are comparable, i.e. $\gamma \in (0,\infty)$.

7.7.1. Sharpness of the two-scale characterization of the limiting strain

Define

$$\mathfrak{G}_\gamma := \Big\{ \operatorname{sym}\Big[a(x_1)(e_1\otimes e_1) + \widetilde{\nabla}_{1,\gamma} w(x,y) \Big] \; : \\ a \in L^2(\omega), \quad w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2)) \Big\}.$$

Theorem 7.7.1. *Let $\gamma \in (0,\infty)$ and (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^2)$ with finite bending energy.*

(1) There exist a subsequence (not relabeled) and maps

$$u \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2) \qquad \text{and} \qquad G \in \mathfrak{G}_\gamma$$

such that

$$(\star) \quad \begin{cases} u_h - u_{\Omega,h} \to u & \text{strongly in } L^2(\Omega;\mathbb{R}^2) \\ \nabla_h u_h \to R_{(u)} & \text{strongly in } L^2(\Omega;\mathbb{M}(2)) \\ E_h \xrightarrow{2} E := x_2 \boldsymbol{\kappa}_{(u)}(e_1 \otimes e_1) + G & \text{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(2)). \end{cases}$$

Here $u_{\Omega,h} := \frac{1}{\mathcal{H}^n(\Omega)}\int_\Omega u_h \,\mathrm{d}x$ denotes the integral average of u_h.

(2) If $(\star)$ holds for the entire sequence, then there exists a sequence $(v_h) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned} v_h - u_h &\to 0 && \text{strongly in } L^2(\Omega;\mathbb{R}^2) \\ \nabla_h(v_h - u_h) &\to 0 && \text{strongly in } L^2(\Omega;\mathbb{M}(2)) \\ \widehat{E}_h := \frac{\sqrt{\nabla_h v_h^T \nabla_h v_h} - Id}{h} &\xrightarrow{2} E && \text{strongly two-scale in } L^2(\Omega\times Y;\mathbb{M}(2)) \end{aligned}$$

and

$$\limsup_{h\to 0} \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\mathrm{dist}^2(\nabla_h v_h(x), SO(2))}{h} + \left|\sqrt{h}\widehat{E}_h(x)\right|^2 \right) = 0. \tag{7.68}$$

(3) For each pair $(u,G) \in W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^2)\times\mathfrak{G}_\gamma$ there exists a sequence $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ with finite bending energy such that $(\star)$ holds.

Remark 7.7.2. Condition (7.68) guarantees on the one hand, that for sufficiently small h the deformation v_h satisfies $\det \nabla_h v_h > 0$, and therefore

$$W(\cdot,\nabla_h v_h) = W(\cdot, Id + h\widehat{E}_h)$$

for all frame indifferent integrands W. On the other hand, the condition is tailor-made for the simultaneous homogenization and linearization statement (see Theorem 5.2.1).

Proof. The first statement directly follows by combining the compactness part of Theorem 7.4.2 and the two-scale characterization of the limiting strain in Theorem 7.4.10. We prove statement (2). We choose maps $a \in L^2(\omega)$ and $w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^2))$ such that

$$E = x_2\boldsymbol{\kappa}_{(u)}(x_1)(e_1\otimes e_1) + \mathrm{sym}\left[a(x_1)(e_1\otimes e_1) + \widetilde{\nabla}_{1,\gamma} w(x,y) \right].$$

The construction of (v_h) is a slight extension of the one used for the recovery sequence in Theorem 7.2.5. As in the proof of the upper bound (see page 148), we assign to each positive δ maps

$$u^{(\delta)} \in C^{2,2}_{\text{iso}}(\overline{\omega};\mathbb{R}^2), \qquad a^{(\delta)} \in C^\infty_c(\omega) \quad \text{and} \quad w^{(\delta)} \in C^\infty_c(\omega; C^\infty(\overline{S}; C^\infty_{\text{per}}(Y;\mathbb{R}^2)))$$

such that

$$\left\| u^{(\delta)} - u \right\|_{W^{2,2}(\omega;\mathbb{R}^2)} + \left\| \boldsymbol{\kappa}_{(u^{(\delta)})} - \boldsymbol{\kappa}_{(u)} \right\|_{L^2(\omega)} + \left\| a^{(\delta)} - a \right\|_{L^2(\omega)} + \left\| w^{(\delta)} - w \right\|_{L^2(\omega;W^{1,2}(S\times Y;\mathbb{R}^2))} \leq \delta$$

and define a sequence $(v_h^{(\delta)}) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ according to Lemma 7.4.16. By construction we have

$$\begin{aligned} v_h^{(\delta)} &\to u^{(\delta)} && \text{strongly in } L^2(\Omega;\mathbb{R}^2) \\ \nabla_h v_h^{(\delta)} &\to R_{(u^{(\delta)})} && \text{strongly in } L^2(\Omega;\mathbb{M}(2)). \end{aligned}$$

We set

$$E_h^{(\delta)\,\text{ap}}(x) := \frac{R^{\mathrm{T}}_{(u^{(\delta)})}\, \nabla_h v_h^{(\delta)} - Id}{h} \qquad \text{and} \qquad \widehat{E}_h^{(\delta)} := \frac{\sqrt{(\nabla_h v_h^{(\delta)})^{\mathrm{T}}\, \nabla_h v_h^{(\delta)}} - Id}{h}.$$

As in Lemma 7.4.16, we deduce that

$$E_h^{(\delta)\,\text{ap}} \xrightarrow{2} x_2 \boldsymbol{\kappa}_{(u^{(\delta)})}(e_1 \otimes e_1) + a^{(\delta)}(e_1 \otimes e_1) + \widetilde{\nabla}_{1,\gamma} w^{(\delta)}$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(2))$. Furthermore, by construction we have

$$\limsup_{h\to 0}\, \operatorname*{ess\,sup}_{x\in\Omega} \left| E_h^{(\delta)\,\text{ap}}(x) \right| < \infty.$$

Note that

$$\sqrt{(\nabla_h v_h^{(\delta)})^{\mathrm{T}}\, \nabla_h v_h^{(\delta)}} = \sqrt{(Id + hE_h^{(\delta)\,\text{ap}})^{\mathrm{T}}(Id + hE_h^{(\delta)\,\text{ap}})}.$$

Thus, in view of Corollary 2.3.3 and Corollary 2.3.4 we infer that

$$\widehat{E}_h^{(\delta)} := \frac{\sqrt{(\nabla_h v_h^{(\delta)})^{\mathrm{T}}\, \nabla_h v_h^{(\delta)}} - Id}{h} \xrightarrow{2} x_2 \boldsymbol{\kappa}_{(u^{(\delta)})}(e_1 \otimes e_1) + a^{(\delta)}(e_1 \otimes e_1) + \widetilde{\nabla}_{1,\gamma} w^{(\delta)}$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(2))$.

Now we define the doubly indexed sequence

$$c_{\delta,h} := \left\| v_h^{(\delta)} - u \right\|_{L^2(\omega;\mathbb{R}^2)} + \left\| \nabla_h v_h^{(\delta)} - u \right\|_{L^2(\Omega;\mathbb{M}(2))} + \left\| \mathcal{T}_{\varepsilon(h)} \widehat{E}_h^{(\delta)} - E \right\|_{L^2(\mathbb{R}^2\times Y;\mathbb{M}(2))} + \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\operatorname{dist}^2(\nabla_h v_h^{(\delta)}(x), SO(2))}{h} + \left| \sqrt{h}\widehat{E}_h^{(\delta)}(x) \right|^2 \right).$$

In view of the previous discussion, it is easy to check that

$$0 \leq \limsup_{\delta\to 0} \limsup_{h\to 0} c_{\delta,h} = 0.$$

Hence, by applying Lemma A.2.1 we can pass to a diagonal sequence $\delta(h)$ such that

$$\lim_{h\to 0} \delta(h) = 0 \qquad \text{and} \qquad \lim_{h\to 0} c_{\delta(h),h} = 0.$$

We define the diagonal sequence

$$v_h := v_h^{(\delta(h))} + \int_\Omega u_h(x) - v_h^{(\delta(h))}(x)\,\mathrm{d}x.$$

It is straightforward to check that (v_h) fulfills the claimed properties and statement (2) follows. The same construction yields the existence of the sequence in statement (3). □

7.7.2. Application to layered, prestressed materials

Let $W_0 : \Omega\times\mathbb{R}\times\mathbb{M}(2) \to [0,\infty]$ be a measurable integrand and suppose that

($W_0$1) W is frame indifferent, i.e.

$$W(x,y,RF) = W(x,y,F) \qquad \text{for all } R \in SO(2),\ F \in \mathbb{M}(2)$$

($W_0$2) The identity is a natural state, i.e.

$$W(x,y,Id) = 0$$

($W_0$3) W is non-degenerate, i.e. there exists $C > 0$ such that

$$W(x,y,F) \geq C\,\mathrm{dist}^2(F,SO(2)) \qquad \text{for all } F \in \mathbb{M}(2)$$

($W_0$4) W admits a quadratic Taylor expansion at the identity, i.e.

$$\begin{cases} \text{there exists } \mathbb{L}_0 \in C(\overline{\Omega}; L^\infty_{\mathrm{per}}(Y;\mathbb{T}_{\mathrm{sym}}(2))) \text{ such that} \\ \displaystyle\limsup_{F\to 0} \operatorname*{ess\,sup}_{y\in Y} \frac{|W(x,y,F) - \langle \mathbb{L}_0(x,y)F,\,F\rangle|}{|F|^2} = 0. \end{cases}$$

We associate to $\mathbb{L}_0$ the quadratic integrand

$$Q_0 : \Omega\times Y\times\mathbb{M}(2) \to [0,\infty), \qquad Q_0(x,y,F) := \langle \mathbb{L}_0(x,y)F,\,F\rangle\,.$$

Moreover, let (B_h) be a sequence in $L^\infty(\Omega;\mathbb{M}_{\mathrm{sym}}(2))$ and suppose that

$$\text{(B)} \qquad \begin{cases} \displaystyle\limsup_{h\to 0} \operatorname*{ess\,sup}_{x\in\Omega} |B_h(x)| < \infty \\ B_h \xrightarrow{2} B_0 \qquad \text{strongly two-scale in } L^\infty(\Omega\times Y;\mathbb{M}_{\mathrm{sym}}(2)). \end{cases}$$

For each h and $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ we consider the energy

$$\mathcal{G}^h(u) := \frac{1}{h^2}\int_\Omega W_h(x,\nabla_h u(x))\,\mathrm{d}x$$

where the integrand $W_h\,\Omega\times\mathbb{M}(2)\to[0,\infty]$ is given as

$$W_h(x,F) := W(x, {x_1}/{\varepsilon(h)}, F(Id + hB_h(x))).$$

The energy $\mathcal{G}^h$ is an adaption and extension of the model considered by B. Schmidt in [Sch07] to the planar rods setting with laterally periodic materials. In contrast to the setting considered in the previous sections, the stored energy function W_h is allowed to vary in the "out-of-plane" direction x_2 (at least on the macroscopic scale). Thus, the model is capable to describe layered materials. The energy potential $W(y,F)$ considered in the previous sections is minimized for $F\in SO(2)$. In contrast, the position of the energy wells of W_h are mismatched due to the presence of the matrix field B_h. In particular, it is no longer obligatory that $W_h(x,F)$ is minimized for matrices in $SO(2)$. As a consequence, the case of slightly prestressed materials is covered by the setting above.

We define the two-scale limiting functional

$$\begin{aligned}&\mathcal{G}^\gamma : W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)\times\mathfrak{G}^\gamma\to[0,\infty),\\ &\qquad\mathcal{G}^\gamma(u,G) := \iint\limits_{\Omega\times Y} Q_0(x,y,x_2\boldsymbol{\kappa}_{(u)}(x_1)(e_1\otimes e_1) + G(x,y) + B_0(x,y))\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

The primary insight of this section is he following convergence result:

Theorem 7.7.3. *Let the fine-scales ε and h be coupled according to (7.1) and suppose that*

$$\lim_{h\to0}\frac{h}{\varepsilon(h)} = \gamma \qquad \textit{with } \gamma\in(0,\infty).$$

(1) (Compactness). Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^2)$ with equibounded energy, i.e.

$$\limsup_{h\to0}\mathcal{G}^h(u_h)<\infty.$$

Then there exist a pair $(u,G)\in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)\times\mathfrak{G}^\gamma$ and a subsequence (not relabeled) such that

$$(\star)\quad\begin{cases} u_h - u_{\Omega,h}\to u & \textit{strongly in } L^2(\Omega;\mathbb{R}^2)\\ \nabla_h u_h\to R_{(u)} & \textit{strongly in } L^2(\Omega;\mathbb{M}(2))\\ E_h \xrightharpoonup{2} x_2\boldsymbol{\kappa}_{(u)}(e_1\otimes e_1) + G & \textit{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(2)).\end{cases}$$

where $E_h := \frac{\sqrt{(\nabla_h u_h)^T\nabla_h u_h} - Id}{h}$. (Here, $u_{\Omega,h} := \frac{1}{\mathcal{H}^n(\Omega)}\int_\Omega u_h\,\mathrm{d}x$ denotes the integral average of u_h).

(2) (Lower bound). Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^2)$. Suppose that (u_h) converges to a pair $(u,G)\in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)\times\mathfrak{G}^\gamma$ in the sense of $(\star)$. Then

$$\liminf_{h\to0}\mathcal{G}^h(u_h)\geq\mathcal{G}^\gamma(u,G).$$

(3) (Upper bound). Let (u,G) be an arbitrary pair in $W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^2)\times\mathfrak{G}^\gamma$. Then there exists a sequence (u_h) in $W^{1,2}(\Omega;\mathbb{R}^2)$ converging to (u,G) in the sense of $(\star)$ and

$$\lim_{h\to 0}\mathcal{G}^h(u_h)=\mathcal{G}^\gamma(u,G).$$

Additionally, we can even realize convergence of (E_h) with respect to strong two-scale convergence in $L^2(\Omega\times Y;\mathbb{M}(2))$.

Proof. Step 1. (Compactness). In view of Theorem 7.7.1, it is sufficient to show that the equiboundedness of the energy implies that (u_h) has finite bending energy. In order to show this, we proceed as in [Sch07]. First, note that condition (B) implies that

$$\begin{aligned}\operatorname{dist}^2(F(Id+hB_h(x)),SO(2)) &\geq 2\left(\operatorname{dist}^2(F,SO(2))-h^2\,|FB_h(x)|^2\right)\\ &\geq c'\left(\operatorname{dist}^2(F,SO(2))-h^2\,|F)|^2\right).\end{aligned}$$

Here and below, c' denotes a positive constant that may change from line to line, but can be chosen independent of x and h. Because of the inequality $\frac{1}{2}\,|F|^2\leq\operatorname{dist}^2(F,SO(2))+2$, a rearrangement of the previous estimate implies that

$$\operatorname{dist}^2(F,SO(2))\leq\frac{c'}{1-2h^2}\left(\operatorname{dist}^2(F(Id+hB_h(x),SO(2))+h^2\right)$$

for all $h<1$. We substitute $F=\nabla_h u_h(x)$ and integrate both sides over Ω. Since the first term on the right hand side is controlled by $W_h(x,F)$ due to the non-degeneracy condition ($W_0$3), we find that

$$\limsup_{h\to 0}\frac{1}{h^2}\int_\Omega\operatorname{dist}^2(\nabla_h u_h(x),SO(2))\,\mathrm{d}x\leq c'\left(\limsup_{h\to 0}\mathcal{G}^h(u_h)+\mathcal{H}^n(\Omega)\right).$$

The right hand side is finite, and consequently the sequence (u_h) has finite bending energy.

Step 2. (Lower bound). For convenience, we set

$$E(x,y):=x_2\boldsymbol{\kappa}_{(u)}(x_1)(e_1\otimes e_1)+G(x,y).$$

We only have to consider the case where

$$\limsup_{h\to 0}\mathcal{G}^h(u_h)<\infty.$$

Thereby, (u_h) has finite bending energy as we have seen in Step 1. Thus, there exists a sequence (R_h) of measurable maps from ω to $SO(2)$ such that

$$E_h^{\text{ap}}(x):=\frac{R_h(x)^{\mathrm{T}}\,\nabla_h u_h(x)-Id}{h}$$

is bounded in $L^2(\Omega;\mathbb{M}(2))$ (for instance apply Proposition 7.4.4 with $\epsilon = h$.) As a bounded sequence, (E_h^{ap}) is weakly two-scale relatively compact and we can pass to a subsequence (that we do not relabel) such that

$$E_h^{\mathrm{ap}} \xrightarrow{2} E^{\mathrm{ap}} \qquad \text{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{M}(2)).$$

By construction, we have

$$\sqrt{\nabla_h u_h^{\mathrm{T}}\,\nabla_h u_h} = \sqrt{(Id + hE_h^{\mathrm{ap}})^{\mathrm{T}}(Id + hE_h^{\mathrm{ap}})}.$$

Now the assumption that E_h weakly two-scale converges to E, and Corollary 2.3.4 imply that $\operatorname{sym} E^{\mathrm{ap}} = E$.

A close look at the definition of W_h reveals that

$$\mathcal{G}^h(u_h) = \frac{1}{h^2}\int_\Omega W_0(x, {}^{x_1}\!/\!_{\varepsilon(h)}, \nabla_h u_h(x)(Id + hB_h(x))\,\mathrm{d}x.$$

The frame indifference of W_0 and application of the identity

$$\nabla_h u_h = R_h(Id + hE_h^{\mathrm{ap}})$$

lead to

$$\mathcal{G}^h(u_h) = \frac{1}{h^2}\int_\Omega W_0\Big(\,x, {}^{x_1}\!/\!_{\varepsilon(h)}, Id + h\left(\,E_h^{\mathrm{ap}} + B_h + hE_h^{\mathrm{ap}}B_h\right)\,\Big)\,\mathrm{d}x.$$

By assumption (B_h) is weakly two-scale convergent to B_0. As a consequence, we obtain

$$E_h^{\mathrm{ap}} + B_h + hE_h^{\mathrm{ap}}B_h \xrightarrow{2} E^{\mathrm{ap}} + B_0 \qquad \text{weakly two-scale in } L^2(\Omega{\times}Y;\mathbb{M}(2)).$$

This allows us to apply the simultaneous homogenization and linearization result (see Theorem 5.2.1) and we arrive at

$$\liminf_{h\to 0}\mathcal{G}^h(u_h) \geq \iint_{\Omega\times Y} Q_0(x,y,E^{\mathrm{ap}}(x,y) + B_0(x,y))\,\mathrm{d}y\,\mathrm{d}x. \tag{7.69}$$

In virtue of Lemma 5.2.4 and conditions ($W_0$1),($W_0$3) and ($W_0$4), we have

$$Q_0(x,y,F) = Q_0(x,y,\operatorname{sym} F) \qquad \text{for all } F \in \mathbb{M}(2).$$

Hence, the integral on the right of (7.69) is equal to $\mathcal{G}^\gamma(u,G)$. Because the previous reasoning is independent of the choice of the subsequence, the proof of the lower bound is complete.

<u>*Step 3.*</u> (Upper bound). First, recall that the polar factorization says that every $F \in \mathbb{M}(2)$ with $\det F > 0$ can be factorized as $F = R\sqrt{F^{\mathrm{T}}F}$ for a suitable rotation $R \in SO(2)$. Thus, we see that the frame indifference of W_0 allows us to rewrite $W_h(y,F)$ according to

$$W_h(x,F) = W_0\left(\,x, {}^{x_1}\!/\!_{\varepsilon(h)}, \left(Id + h\tfrac{\sqrt{F^{\mathrm{T}}F} - Id}{h}\right)(Id + hB_h(x))\right) \tag{7.70}$$

for all $x \in \Omega$ and $F \in \mathbb{M}(2)$ with $\det F > 0$. Now let (v_h) denote the sequence associated to the pair (u, G) according to Theorem 7.7.1 (2). In view of property (7.68), we deduce that

$$\det \nabla_h v_h(x) > 0 \qquad \text{for a.e. } x \in \Omega$$

provided h is sufficiently small. Hence, for small h the energy of v_h can be rewritten by means of (7.70) and we obtain

$$\mathcal{G}^h(v_h) = \frac{1}{h^2} \int_\Omega W_0\left(x, {}^{x_1}\!/\!_{\varepsilon(h)}, Id + h\left(\widehat{E}_h + B_h + hE_h B_h\right)\right) \mathrm{d}x$$

where $\widehat{E}_h$ is the nonlinear strain associated to v_h and is defined in Theorem 7.7.1 (2). The same theorem and assumption (B) imply that

$$\widehat{E}_h + B_h + hE_h B_h \xrightarrow{2} E + B_0 \qquad \text{strongly two-scale in } L^2(\Omega\times Y; \mathbb{M}(2)).$$

Because this sequence satisfies the assumption of Theorem 5.2.1 (2), we can applying this theorem and find that

$$\lim_{h\to 0} \mathcal{G}^h(v_h) = \mathcal{G}^\gamma(u, G).$$

□

Remark 7.7.4.

1. Using the notion introduced in [MT07] by A. Mielke et al. we can restate the previous result and say that $\mathcal{G}^\gamma$ is the two-scale Γ-limit of $(\mathcal{G}^h)$ with respect to the two-scale cross-convergence (⋆).
2. A natural next step is to analyze the reduced functional

   $$\mathcal{G}^\gamma_{\text{hom}}(u) := \inf_{G \in \mathfrak{G}^\gamma} \mathcal{G}^\gamma(u, G)$$

   and to prove that $\mathcal{G}^\gamma_{\text{hom}}$ is the Γ-limit of $(\mathcal{G}^h)$ in $L^2(\Omega; \mathbb{R}^2)$ with respect to strong convergence. The picture could be completed by deriving a cell formula that characterizes the homogenized integrand associated to $\mathcal{G}^\gamma_{\text{hom}}$.
3. We like to emphasize that the implication

   (u_h) has equibounded energy $\quad\Rightarrow\quad$ (u_h) has finite bending energy

   is crucial for the analysis. Generally speaking, we can say that our approach can be adapted to any situation that guarantees this implication.

8. Rigorous derivation of a homogenized Cosserat theory for inextensible rods from nonlinear 3d elasticity

8.1. Introduction and main result

In this chapter we derive a homogenized Cosserat theory for inextensible rods as Γ-limit of nonlinear, three-dimensional elasticity following the strategy introduced in the previous chapter. Our starting point is an elastic body that occupies a thin cylindrical domain of the form

$$\Omega_h := \omega \times (hS) \qquad \omega := (0, L) \quad \text{and} \quad S \subset \mathbb{R}^2$$

with thickness $h > 0$. We suppose that the body consists of an hyperelastic material that features a laterally periodic microstructure with period ε. This situation is described by the elastic energy

$$\mathcal{E}^{\varepsilon,h}(v) := \int_{\Omega_h} W(x_1/\varepsilon(h),\, \nabla v(x))\, \mathrm{d}x$$

where the deformation v is a map in $W^{1,2}(\Omega_h; \mathbb{R}^3)$ and the elastic potential $W(y, F)$ is assumed to be $[0, 1) =: Y$-periodic in its first variable.

We are interested in the asymptotic behavior of the energy as the fine-scale parameters h and ε simultaneously tend to zero. We are going to see that (under suitable conditions on W) the scaled energy $h^{-4}\mathcal{E}^{\varepsilon,h}$ Γ-converges to a functional $\mathcal{I}_\gamma$ that can be interpreted as a homogenized Cosserat theory for inextensible rods; in particular, this means that $\mathcal{I}_\gamma$ is a functional which is finite only for rod configuration, i.e. pairs $(u, \mathcal{R})$ with

$$u \in W^{2,2}_{\text{iso}}(\omega; \mathbb{R}^3), \qquad \mathcal{R} \in W^{1,2}(\omega; SO(3)) \quad \text{and} \quad \partial_1 u = \mathcal{R}e_1. \tag{8.1}$$

The rigorous derivation of such a theory in the setting where the material is homogeneous has been done by M.G. Mora and S. Müller in [MM03]. In the situation considered here, effects due to homogenization additionally come into play. As a consequence, the precise form of the limiting energy depends on the additional parameter γ which captures the limiting behavior of the fine-scale ratio h/ε.

As in the previous chapter, for the precise formulation of our main result, it is convenient to consider a scaled formulation of the problem. To this end, set

$$\Omega := \omega \times S$$

and assume that $S \subset \mathbb{R}^2$ is an open, bounded and connected domain with Lipschitz boundary. We decompose any point $x \in \mathbb{R}^3$ according to $x = (x_1, \bar{x})$ where $x_1 \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^2$ and introduce the *shape-vector*

$$d_S(\bar{x}) := x_2 e_2 + x_3 e_3 - \frac{1}{\mathcal{H}^2(S)} \int_S x_2 e_2 + x_3 e_3 \, \mathrm{d}\bar{x}. \tag{8.2}$$

Similarly to the previous chapter, we assume that the fine-scale parameters h, ε are coupled according to (7.1). In particular, we assume that $\varepsilon = \varepsilon(h)$ and $h/\varepsilon \to \gamma$ as $h \to 0$ with $\gamma \in [0, \infty]$.

The scaled version of $\mathcal{E}^{\varepsilon,h}$ is the following functional from $L^2(\Omega; \mathbb{R}^3)$ to $\bar{\mathbb{R}}$:

$$\mathcal{I}^{\varepsilon,h}(u) := \begin{cases} \dfrac{1}{h^2} \displaystyle\int_\Omega W(x_1/\varepsilon(h), \nabla_h u_h(x)) \, \mathrm{d}x & \text{if } u \in W^{1,2}(\Omega, \mathbb{R}^3) \\ +\infty & \text{else.} \end{cases}$$

Above, ∇_h denotes the scaled deformation gradient and is defined in Definition 8.1.4, i.e.

$$\nabla_h u(x) := \nabla_{1,h} \, u(x) = \left(\, \partial_1 u(x) \, \middle| \, \tfrac{1}{h} \nabla_{\bar{x}} \, u(x) \, \right).$$

We suppose that the elastic potential $W : \mathbb{R} \times \mathbb{M}(3) \to [0, \infty]$ is a non-negative integrand that is Y-periodic in its first variable and that satisfies the following conditions:

(W1) W is frame indifferent, i.e.

$$W(y, RF) = W(y, F) \qquad \text{for all } R \in SO(3),\ F \in \mathbb{M}(3) \text{ and a.e. } y \in Y$$

(W2) The identity is a natural state, i.e.

$$W(y, Id) = 0 \qquad \text{for a.e. } y \in Y$$

(W3) W is non-degenerate, i.e. there exists a positive constant C such that

$$W(y, F) \geq C \operatorname{dist}^2(F, SO(3)) \qquad \text{for all } F \in \mathbb{M}(3) \text{ and a.e. } y \in Y$$

(W4) W admits a quadratic Taylor expansion at the identity, i.e.

$$\exists Q \in \mathfrak{Q}(Y;3) \, : \, \limsup_{F \to 0} \operatorname*{ess\,sup}_{y \in Y} \frac{|W(y,F) - Q(y,F)|}{|F|^2} = 0.$$

Above, $\mathfrak{Q}(Y;3)$ denotes the set of all Y-periodic, quadratic integrands from $Y \times \mathbb{M}(3)$ to $\mathbb{R}$ with uniformly bounded coefficients (see page 101).

In order to present the limiting energy, we introduce the relaxed quadratic form

$$Q_\gamma : \mathbb{M}_{\text{skew}}(3) \to [0, \infty)$$

which is determined by the linear cell problem

$$Q_\gamma(\mathcal{K}) := \min \left\{ \iint_{S \times Y} Q\Big(y, (\mathcal{K} \, d_S(\bar{x})) \otimes e_1 + G(\bar{x}, y)\Big) \, \mathrm{d}y \, \mathrm{d}\bar{x} \, : \, G \in \mathcal{G}_\gamma \right\} \tag{8.3}$$

with

$$\mathcal{G}_0 := \Bigg\{ \Big[(\partial_y \Psi d_S)\otimes e_1 + \big(\partial_y w_0 \,|\, \nabla_{\bar{x}}\, \bar{w} \big) + a(e_1\otimes e_1) \Big] \;:\; \Psi \in W^{1,2}_{\mathrm{per},0}(Y;\mathbb{M}_{\mathrm{skew}}(3)),$$

$$a \in \mathbb{R},\; w_0 \in W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^3),\; \bar{w} \in L^2(Y;W^{1,2}(S;\mathbb{R}^3)) \Bigg\}$$

$$\mathcal{G}_\infty := \Bigg\{ \Big[\big(\partial_y w_0 \,|\, \nabla_{\bar{x}}\, \bar{w} \big) + a(e_1\otimes e_1) \Big] \;:\; a \in \mathbb{R},\; w_0 \in L^2(S;W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^3)),$$

$$\bar{w} \in W^{1,2}(S;\mathbb{R}^3) \Bigg\}.$$

For $\gamma \in (0,\infty)$ we define

$$\mathcal{G}_\gamma := \Bigg\{ \Big[\widetilde{\nabla}_{1,\gamma} w + a(e_1\otimes e_1) \Big] \;:\; a \in \mathbb{R},\; w \in W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^3) \Bigg\}.$$

Above, $\widetilde{\nabla}_{1,\gamma} w = \Big(\; \partial_y w \;\Big|\; \frac{1}{\gamma} \nabla_{\bar{x}}\, w \; \Big)$ is the auxiliary gradient defined in Section 6.3 (see page 81).

For $(u,\mathcal{R}) \in L^2(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{M}(3))$ we define the limiting energy

$$\mathcal{I}_\gamma(u,\mathcal{R}) := \begin{cases} \displaystyle\int_\omega Q_\gamma(\mathcal{K}_{\mathcal{R}}(x_1))\,\mathrm{d}x_1 & \text{if } (u,\mathcal{R}) \text{ is a rod-configuration} \\ & \text{(in the sense of Definition 8.1.4 below),} \\ +\infty & \text{else.} \end{cases}$$

In the definition above the quantity $\mathcal{K}_{\mathcal{R}}$ is related to the torsion and bending of the rod and precisely defined in the sequel.

We are now in position to state the main result:

Theorem 8.1.1.

(1) Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^3)$ with finite bending energy, i.e.

$$\text{(FBE)} \qquad \limsup_{h\to 0} \frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u(x), SO(3))\,\mathrm{d}x < \infty.$$

Then there exist a subsequence, a map $E \in L^2(\Omega\times Y;\mathbb{M}(3))$ and a rod configuration $(u,\mathcal{R})$, such that

$$\begin{aligned} u_h - u_{\Omega,h} &\to u && \text{strongly in } L^2(\Omega;\mathbb{R}^3)\\ \nabla_h u_h &\to \mathcal{R} && \text{strongly in } L^2(\Omega;\mathbb{M}(3))\\ E_h := \frac{\sqrt{\nabla_h u_h^T \nabla_h u_h} - Id}{h} &\xrightarrow{2} E && \text{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(3)) \end{aligned}$$

(Here, $u_{\Omega,h}$ denotes the integral average of u_h over Ω.)

(2) Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^3)$ that converges to the rod configuration $(u,\mathcal{R})$ in the following sense:

$$\begin{aligned} u_h &\to u && \text{strongly in } L^2(\Omega;\mathbb{R}^3)\\ \nabla_h u_h &\to \mathcal{R} && \text{strongly in } L^2(\Omega;\mathbb{R}^3). \end{aligned}$$

Then

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) \geq \mathcal{I}_\gamma(u,\mathcal{R}).$$

(3) Let $(u,\mathcal{R})$ be a rod configuration. Then there exists a sequence $(u_h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ that converges to the rod configuration $(u,\mathcal{R})$ in the following sense:

$$\begin{aligned} u_h &\to u && \text{strongly in } L^2(\Omega;\mathbb{R}^3)\\ \nabla_h u_h &\to \mathcal{R} && \text{strongly in } L^2(\Omega;\mathbb{R}^3) \end{aligned}$$

and

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) = \mathcal{I}_\gamma(u,\mathcal{R}).$$

Remark 8.1.2. The previous theorem is the analogon to Theorem 7.2.5 which is the core of the derivation of the planar rod theory from 2d elasticity. As in the previous chapter, one could also restate the previous result as a convergence statement for the initial energy $\mathcal{E}^{\varepsilon,h}$ and could take boundary conditions and forces into account. Moreover, it is possible to prove statements resembling Theorem 7.6.1, Theorem 7.7.1 and to consider more general settings, e.g. layered and prestressed rods. For the sake of brevity, we restrict our effort to the proof of Theorem 8.1.1 and refrain from lengthy explanations in the sequel.

The general strategy of the proof of Theorem 8.1.1 is related to the proof of the analogous statement (i.e. Theorem 7.2.5). One of the major changes originates from the observation that in contrast to the 2d case, the types of oscillations emerging from the bending of the mid line are more rich in the 3d situation considered here. In particular, not only oscillations related to curvature, but also oscillations related to torsion appear. Mathematically, in the 2d case, we repeatedly used the possibility to parametrize the manifold $SO(2)$ by a single parameter (the rotation angle), while in the 3d setting a more elaborated and less handy presentation of $SO(3)$ has to be utilized. As a consequence, in several parts of the proof we have to spend some additional effort. In particular, this becomes visible in the approximation of the scaled gradient by maps in $W^{1,2}(\omega; SO(3))$. In this context, we utilize the observation (see Lemma 8.2.3 and Lemma 2.2.3 below) that we can connect two matrices by a smooth path $\mathcal{R} \in C^\infty([s_1, s_2]; SO(3))$ in such a way that the modulus of the derivative is pointwise controlled by $(s_2 - s_1)^{-2} |A - B|^2$ and $\mathcal{R}^T \partial_1 \mathcal{R}$ is constant.

Remark 8.1.3. In view of Lemma 5.2.4, the quadratic form $Q(y, \cdot)$ is positive definite on the subspace of symmetric matrices and satisfies $Q(y, F) = Q(y, \operatorname{sym} F)$ for all $F \in \mathbb{M}(3)$. For this reason it is not difficult to show (e.g. by means of the direct methods of the calculus of variations) that the minimization problem (8.3) has a solution in $\mathcal{G}_\gamma$.

Moreover, we can find a subspace of $\mathcal{G}_\gamma$ with the property that the minimum in (8.3) is obtained by a unique element in that subspace and can be determined by a linear Euler-Lagrange equation. Now similarly to the reasoning in Section 7.4.5, we eventually can show that for rod configurations $(u, \mathcal{R})$ the limiting energy can be written in the form

$$\mathcal{I}_\gamma(u, \mathcal{R}) = \inf \left\{ \iint_{\Omega \times Y} Q\left(y, \left(\mathcal{K}_{\mathcal{R}}(x_1) d_S(\bar{x})\right) \otimes e_1 + G(x, y)\right) \mathrm{d}y \, \mathrm{d}x \; : \; G \in \mathfrak{G}_\gamma \right\} \tag{8.4}$$

where

$$\mathfrak{G}_0 := \Big\{ \left[(\partial_y \Psi d_S) \otimes e_1 + \left(\partial_y w_0 \,|\, \nabla_{\bar{x}} \bar{w} \right) + a(e_1 \otimes e_1) \right] \; : \; a \in L^2(\omega),\ w_0 \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y; \mathbb{R}^3)),\ \bar{w} \in L^2(\omega \times Y; W^{1,2}(S; \mathbb{R}^3)),\ \Psi \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y; \mathbb{M}_{\mathrm{skew}}(3))) \Big\}$$

$$\mathfrak{G}_\infty := \Big\{ \left[\left(\partial_y w_0 \,|\, \nabla_{\bar{x}} \bar{w} \right) + a(e_1 \otimes e_1) \right] \; : \; a \in L^2(\omega),\ w_0 \in L^2(\Omega; W^{1,2}_{\mathrm{per},0}(Y; \mathbb{R}^3)),\ \bar{w} \in L^2(\omega; W^{1,2}(S; \mathbb{R}^3)) \Big\}$$

and for $\gamma \in (0, \infty)$

$$\mathfrak{G}_\gamma := \left\{ \left[\widetilde{\nabla}_{1,\gamma} w + a(e_1 \otimes e_1) \right] \; : \; a \in L^2(\omega),\ w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S \times Y; \mathbb{R}^3)) \right\}.$$

We conclude this introduction with commenting on the **nonlinear Cosserat theory for rods** that we present in the spirit of [Ant05].

In the nonlinear Cosserat rod theory, a *rod configuration* is a pair $(u; \mathcal{R})$, where u and $\mathcal{R}$ are (at least) weakly differentiable maps from ω to $\mathbb{R}^3$ and $SO(3)$, respectively. The point $u(x_1) \in \mathbb{R}^3$ denotes the deformed position of the material point x_1 lying in a reference configuration on the mid line ω. Therefore, we call u the (rod) deformation. The columns of the matrix $\mathcal{R}(x_1)$ are unit vectors and characterize the orientation of the (infinitely small) cross-section of the rod through $u(x_1)$. We call $\mathcal{R}$ *Cosserat frame* and use the convention to denote its columns, which we refer to as *directors*, by t, d_2 and d_3. Usually, the first column t is related to $\partial_1 u$.

The central quantity in elastic rod theory is the product $\mathcal{K}_{\mathcal{R}}(x_1) := \mathcal{R}(x_1)^{\mathrm{T}} \partial_1 \mathcal{R}(x_1)$ which by construction is always a skew symmetric matrix. We define coefficients k_2, k_3, τ according to

$$\mathcal{K}_{\mathcal{R}} = \begin{pmatrix} 0 & k_2 & k_3 \\ -k_2 & 0 & \tau \\ -k_3 & -\tau & 0 \end{pmatrix}. \tag{8.5}$$

The components $k_2(x_1), k_3(x_1)$ and the component $\tau(x_1)$ are related to the flexure and the torsion of the rod configuration at the material point x_1, respectively. This suggests to call $\mathcal{K}_{\mathcal{R}}$ the *rod strain* associated to the rod configuration $(u, \mathcal{R})$.

In the case of *inextensible, unshearable rods*, which is the situation that emerges in our asymptotic analysis, we have the constraint

$$\partial_1 u = \mathcal{R} e_1.$$

In this case the rod deformation is a curve parametrized by arc length and $k_2(x_1)$ and $k_3(x_1)$ are related to curvature, as we will see in the sequel.

Definition 8.1.4. A *rod configuration* is a pair $(u; \mathcal{R})$ consisting of a bending deformation $u \in W^{2,2}_{\mathrm{iso}}(\omega; \mathbb{R}^3)$ and a Cosserat frame $\mathcal{R} \in W^{1,2}(\omega; \mathbb{M}(3))$ adapted to u, i.e.

$$\mathcal{R}(x_1) \in SO(3) \quad \text{and} \quad \partial_1 u(x_1) := t_{(u)}(x_1) = \mathcal{R}(x_1)\mathbf{e}_1$$

for almost every $x_1 \in \omega$.

If necessary, we identify (without indication) the bending deformation $u \in W^{2,2}_{\mathrm{iso}}(\omega; \mathbb{R}^3)$ with its constant extension to the domain Ω, i.e with the map $\hat{u}$ in $W^{1,2}(\Omega; \mathbb{R}^3)$ defined by $\hat{u}(x_1, \bar{x}) = u(x_1)$.

For hyperelastic rods the total elastic energy associated to a rod configuration $(u, \mathcal{R})$ is given by the integral

$$\int_\omega W(x_1; \mathcal{K}_{\mathcal{R}}(x_1))\, \mathrm{d}x_1$$

where $W : \omega \times \mathbb{M}_{\mathrm{skew}}(3) \to [0, \infty)$ is called the *stored energy function* of the material.

A classical example of a stored energy function for an hyperelastic, isotropic rod with quadratic growth is

$$W(x_1; \mathcal{K}) = \lambda_2(x_1)k_2^2 + \lambda_3(x_1)k_3^2 + \mu(x_1)\tau^2$$

where the material parameters λ_i and μ are non-negative and the strain coefficients (k_2, k_3, τ) are related to $\mathcal{K}$ according to (8.5). Note that in this case the reference configuration is stress free, and therefore a natural state.

8.2. Compactness and two-scale characterization of the nonlinear limiting strain

In this section we prove the compactness part of Theorem 8.1.1 and present a characterization of two-scale cluster points of the nonlinear strain associated to sequences with finite bending energy (see Theorem 8.2.1 below). Although we do not state it explicitly, it becomes clear in the proof of Theorem 8.1.1 (3) that the two-scale characterization is sharp and may be applied to more general situations as the one considered in Theorem 8.1.1.

As in the previous chapter, the compactness and characterization results rely on an approximation scheme that allows to approximate scaled gradients $\nabla_h u$ by a rotation field $\mathcal{R} \in W^{1,2}(\omega; SO(3))$ in such a way that the L^2 -distance between $\mathcal{R}$ and $\nabla_h u$, as well as the norm of $\partial_1 \mathcal{R}$ are bounded by

$$\int_\Omega \operatorname{dist}^2(\nabla_h u(x), SO(3))\,\mathrm{d}x$$

up to a constant prefactor. Additionally, if $\varepsilon \sim h$ or $\varepsilon \gg h$, we can construct the approximation in such a way that $\mathcal{R}^{\mathrm{T}} \partial_1 \mathcal{R}$ is ε-coherent or h-coherent, respectively. Later, when the approximation scheme is applied to sequences with finite bending energy, this property will guarantee that $\mathcal{R}^{\mathrm{T}} \partial_1 \mathcal{R})$ is constant on scale ε.

The two-scale characterization of emerging limiting strains is summarized in the following theorem:

Theorem 8.2.1. *Let (u_h) be a sequence in $W^{1,2}(\Omega; \mathbb{R}^3)$ with finite bending energy and suppose that (u_h) converges to a rod configuration $(u, \mathcal{R})$ in the sense of Theorem 8.1.1 (1). Consider the set*

$$\mathfrak{E} := \Big\{ E \in L^2(\Omega \times Y; \mathbb{M}(3)) \,:\, E_h \xrightarrow{2} E \quad \textit{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(3)) \\ \textit{for a subsequence (not relabeled)} \Big\}$$

where (E_h) denotes the sequence of nonlinear strains associated to (u_h). Then each limiting strain $E \in \mathfrak{E}$ can be represented as follows:

$$E(x, y) = \operatorname{sym}\Big[\Big(\big(\mathcal{K}_{\mathcal{R}}(x_1) + \partial_y \Psi(x_1, y)\big) d_S(\bar{x}) \Big) \otimes e_1 + W(x, y) \Big] + a(x_1)(e_1 \otimes e_1)$$

where

$$a \in L^2(\omega) \qquad \textit{and} \qquad \begin{cases} \Psi \in L^2(\omega; W^{1,2}_{\text{per},0}(Y;\mathbb{M}_{\text{skew}}(3))) & \textit{if } \gamma = 0 \\ \Psi = 0 & \textit{else.} \end{cases}$$

and W *is a map in* $L^2(\Omega{\times}Y;\mathbb{M}(3))$ *that satisfies*

$$W(x,y) = \begin{cases} \left(\,\partial_y w_0 \;\middle|\; \nabla_{\bar{x}}\, \bar{w}\, \right) & \textit{if } \gamma \in \{0,\infty\} \\ \widetilde{\nabla}_{1,\gamma} w_0 & \textit{else} \end{cases} \tag{8.6}$$

where

$$\begin{cases} w_0 \in L^2(\omega; W^{1,2}_{\text{per},0}(Y;\mathbb{R}^3)) \textit{ and } \bar{w} \in L^2(\omega{\times}Y; W^{1,2}(S;\mathbb{R}^3)) & \textit{if } \gamma = 0 \\ w_0 \in L^2(\Omega; W^{1,2}_{\text{per},0}(Y;\mathbb{R}^3)) \textit{ and } \bar{w} \in L^2(\omega; W^{1,2}(S;\mathbb{R}^3)) & \textit{if } \gamma = \infty \\ w_0 \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S{\times}Y;\mathbb{R}^3)) & \textit{else.} \end{cases} \tag{8.7}$$

(For the proof see page 178).

For the proof (which we postpone to Section 8.2.3) we use the following approximation scheme:

Proposition 8.2.2. *Let* h, ϵ *and* γ_0 *be positive parameters and suppose that*

$$\gamma_0 \leq \tfrac{h}{\epsilon} \leq \tfrac{1}{\gamma_0}.$$

To any $u \in W^{1,2}(\Omega;\mathbb{R}^3)$ *we can assign a map*

$$\mathcal{R} \in W^{1,2}(\omega;\mathbb{M}(3)) \cap L^\infty(\omega; SO(3))$$

such that $\mathcal{K}_{\mathcal{R}} = \mathcal{R}^T\partial_1\mathcal{R}$ *is* ϵ*-coherent and*

$$\|\mathcal{R} - \nabla_h u\|^2_{L^2(\Omega;\mathbb{M}(3))} + \epsilon^2 \, \|\partial_1 \mathcal{R}\|^2_{L^2(\omega;\mathbb{M}(3))} \leq C \int_\Omega \operatorname{dist}^2(\nabla_h u, SO(3))\,\mathrm{d}x.$$

The constant C *only depends on* Ω *and* γ_0.

(For the proof see page 175).

The proof is based on the geometric rigidity result in [FJM02] and utilizes the piecewise constant approximation scheme that we presented in Proposition 7.4.4. We postpone the proof to Section 8.2.2.

8.2.1. Proof of the Theorem 8.1.1: Compactness

Step 1. Without loss of generality assume that $h < 1$ and that

$$\sup_{h\in(0,1)} \frac{1}{h^2} \int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(3))\,\mathrm{d}x \leq C.$$

This implies that $(\nabla_h u_h)$ and (∇u_h) are bounded sequences in $L^2(\Omega;\mathbb{M}(3))$. Thus, the sequence $(u_h - u_{\Omega,h})$ (consisting of mean value free maps) is bounded in $W^{1,2}(\Omega;\mathbb{R}^3)$ due to Poincaré's inequality and we can pass to a subsequence (not relabeled) such that

$$u_h - u_{\Omega,h} \rightharpoonup u \qquad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^3)$$

for a suitable map $u \in W^{1,2}(\Omega;\mathbb{R}^3)$. Because of the compactness of the embedding $W^{1,2}(\Omega;\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3)$, the convergence also holds strongly in $L^2(\Omega;\mathbb{R}^3)$.

Step 2. To each u_h we assign a rotation field $\mathcal{R}_h \in W^{1,2}(\omega; SO(3))$ according to Proposition 8.2.2 where we specify the free parameter as $\epsilon = h$. We have

$$\left\| \frac{\mathcal{R}_h - \nabla_h u_h}{h} \right\|^2_{L^2(\Omega;\mathbb{M}(3))} + \|\partial_1 \mathcal{R}_h\|^2_{L^2(\omega;\mathbb{M}(3))} \leq C. \tag{8.8}$$

Set $\bar{\mathcal{R}}_h := \frac{1}{|\omega|}\int_\omega \mathcal{R}_h \,\mathrm{d}x_1$. Now (8.8) and Poincaré's inequality imply that $(\mathcal{R}_h - \bar{\mathcal{R}}_h)$ is bounded in $W^{1,2}(\omega;\mathbb{M}(3))$. Moreover, the sequence of matrices $(\bar{\mathcal{R}}_h)$ is bounded, therefore, we can pass to a subsequence (not relabeled) such that

$$\mathcal{R}_h \rightharpoonup \mathcal{R} \qquad \text{weakly in } W^{1,2}(\omega;\mathbb{M}(3))$$

for a suitable map $\mathcal{R} \in W^{1,2}(\omega;\mathbb{M}(3))$. Again, by compact embedding, we deduce that the latter convergence also holds strongly in $L^2(\omega;\mathbb{M}(3))$. This implies that $\mathcal{R}$ only takes values in $SO(3)$ and belongs to $W^{1,2}(\omega;SO(3)) := W^{1,2}(\omega;\mathbb{M}(3)) \cap L^\infty(\omega;SO(3))$.

Furthermore, (8.8) implies that $\nabla_h u_h - \mathcal{R}_h$ strongly converges to zero in $L^2(\Omega;\mathbb{M}(3))$ and we see that

$$\nabla_h u_h \to \mathcal{R} \qquad \text{strongly in } L^2(\omega;\mathbb{M}(3)).$$

In particular, we deduce that $(\partial_1 u_h)$ converges to $\mathcal{R}e_1$, and consequently $\partial_1 u = \mathcal{R}e_1$, which means that $u \in W^{2,2}_{\text{iso}}(\omega;\mathbb{R}^3)$. Thus, the pair $(u,\mathcal{R})$ is indeed a rod configuration.

Step 3. The weak two-scale relative compactness of (E_h) follows from the inequality

$$\operatorname{dist}^2(\nabla_h u_h(x), SO(3)) \geq c' \left| \sqrt{\nabla_h u_h^{\mathrm{T}}\, \nabla_h u_h} - Id \right|^2$$

and Proposition 2.1.4. □

8.2.2. Approximation of the scaled gradient

Proof of Proposition 8.2.2. Let $R : \omega \to SO(3)$ denote the piecewise constant, ϵ-coherent approximation from Proposition 7.4.4 satisfying

$$\|R - \nabla_h u\|^2_{L^2(\Omega;\mathbb{M}(3))} + \epsilon \operatorname{Var}_2 R \leq c' \int_\Omega \operatorname{dist}^2(\nabla_h u, SO(3)) \,\mathrm{d}x. \tag{8.9}$$

Here and below, c' and c'' denote generic constants that may change from line to line, but can be chosen only depending on γ_0 and the geometry of Ω.

Set $\mathcal{L}_{\epsilon,c} := \epsilon\mathbb{Z} + c$. Because R is ϵ-coherent, we can find a translation $c \in [0, \epsilon)$ and a discrete map $\mathfrak{r} : \mathcal{L}_{\epsilon,c} \to SO(3)$ with

$$R(x_1) = \sum_{\xi \in \mathcal{L}_{\epsilon,c}} 1_{[\xi,\xi+\epsilon)\cap\omega}(x_1)\mathfrak{r}(\xi)$$

for almost every $x_1 \in \omega$. For convenience, let $\mathcal{L}(\omega)$ denote the smallest subset of $\mathcal{L}_{\epsilon,c}$ such that

$$\omega \subseteq \bigcup_{\xi\in\mathcal{L}(\omega)} [\xi, \xi + \epsilon)$$

and set $\xi_{\max} := \max \mathcal{L}(\omega)$ and $\mathcal{L}^\star(\omega) := \mathcal{L}(\omega) \setminus \xi_{\max}$.

For each $\xi \in \mathcal{L}^\star(\omega)$ let $\mathcal{R}_\xi \in C^\infty([\xi, \xi + \epsilon]; \mathbb{M}(3))$ denote the map constructed in Lemma 8.2.3 with

$$\mathcal{R}_\xi(\xi) = \mathfrak{r}(\xi) \quad \text{and} \quad \mathcal{R}_\xi(\xi + \epsilon) = \mathfrak{r}(\xi + \epsilon).$$

Define

$$\mathcal{R}(x_1) := \sum_{\xi\in\mathcal{L}^\star(\omega)} 1_{[\xi,\xi+\epsilon)}(x_1)\mathcal{R}_\xi(x_1) + 1_{[\xi_{\max},\, \xi_{\max}+\epsilon)}(x_1)\mathfrak{r}(\xi_{\max}) \qquad \text{for } x_1 \in \omega.$$

Then $\mathcal{R}$ is a map in $W^{1,2}(\omega; \mathbb{M}(3)) \cap L^\infty(\omega; SO(3))$ and by construction $\mathcal{R}^{\mathrm{T}}\partial_1\mathcal{R}$ is ϵ-coherent. The estimate in Lemma 8.2.3 justifies the following computation:

$$\begin{aligned}\int_\omega |\partial_1\mathcal{R}(x_1)|^2 \,\mathrm{d}x_1 &\le \sum_{\xi\in\mathcal{L}^\star(\omega)} \int_{[\xi,\xi+\epsilon)} |\partial_1\mathcal{R}_\xi(x_1)|^2 \,\mathrm{d}x_1 \\ &\le c'\epsilon^{-1} \sum_{\xi\in\mathcal{L}^\star(\omega)} |\mathfrak{r}(\xi + \epsilon) - \mathfrak{r}(\xi)|^2 \le \frac{c'}{\epsilon} \operatorname{Var}_2(R),\end{aligned}$$

and in view of (8.9), we obtain

$$\epsilon^2 \int_\omega |\partial_1\mathcal{R}(x_1)|^2 \,\mathrm{d}x_1 \le c'\,\epsilon \operatorname{Var}_2 R \le c'' \int_\Omega \operatorname{dist}^2(\nabla_h u(x), SO(3))\,\mathrm{d}x.$$

It remains to show that

$$\int_\Omega |\mathcal{R}(x_1) - \nabla_h u(x)|^2 \,\mathrm{d}x \le c'' \int_\Omega \operatorname{dist}^2(\nabla_h u(x), SO(3))\,\mathrm{d}x.$$

In view of (8.9) the previous inequality is valid if we replace $\mathcal{R}$ by the piecewise constant approximation R. Hence, due to the triangle inequality, it is sufficient to show that

$$\int_\omega |\mathcal{R} - R|^2 \,\mathrm{d}x_1 \le c'' \int_\Omega \operatorname{dist}^2(\nabla_h u(x), SO(3))\,\mathrm{d}x.$$

In order to prove this, first note that for each $\xi \in \mathcal{L}^\star(\omega)$ we have (cf. Lemma 8.2.3)

$$\sup_{x_1 \in [\xi, \xi+\epsilon)} |\mathcal{R}(x_1) - R(x_1)| \leq \int_0^\epsilon |\partial_1 \mathcal{R}_\xi(\xi + \tau)| \, \mathrm{d}\tau \leq c' \, |\mathfrak{r}(\xi + \epsilon) - \mathfrak{r}(\xi)|$$

and consequently

$$\int_{(\xi,\xi+\epsilon)} |\mathcal{R}(x_1) - R(x_1)|^2 \, \mathrm{d}x_1 \leq c' \, \epsilon \, |\mathfrak{r}(\xi + \epsilon) - \mathfrak{r}(\xi)|^2 \, .$$

But this means that

$$\int_\omega |\mathcal{R} - R|^2 \, \mathrm{d}x_1 \leq \sum_{\xi \in \mathcal{L}^\star(\omega)} \int_{(\xi,\xi+\epsilon)} |\mathcal{R} - R|^2 \, \mathrm{d}x_1 \leq c' \, \epsilon \, \mathrm{Var}_2 \, R,$$

and again (8.9) implies that the right hand side is bounded by $c' \int_\Omega \mathrm{dist}^2(\nabla_h u, SO(3)) \, \mathrm{d}x$. □

In the proof above we utilize the subsequent construction, which allows to connect two rotations by a smooth path in $SO(3)$. Recall that the rotation in $SO(3)$ about an axis $a \in \mathbb{R}^3$, $|a| = 1$ by an angle α can be written by means of Rodrigues' rotation formula:

$$\mathrm{Rot}(a, \alpha) := Id + \sin \alpha N_a + (1 - \cos \alpha) N_a^2,$$

where N_a denotes the skew-symmetric matrix in $\mathbb{M}(3)$ determined by $N_a e = a \wedge e$ for all $e \in \mathbb{R}^3$.

Lemma 8.2.3. *Let $A, B \in SO(3)$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$. There exist a map $\mathcal{R} \in C^\infty([s_1, s_2]; SO(3))$ and a matrix $K \in \mathbb{M}_{\mathrm{skew}}(3)$ such that*

$$\mathcal{R}(s_1) = A, \qquad \mathcal{R}(s_2) = B \qquad \textit{and} \qquad \mathcal{R}(x_1)^T \partial_1 \mathcal{R}(x_1) = K \textit{ for all } x_1 \in [s_1, s_2].$$

Furthermore,

$$\sup_{x_1 \in [s_1, s_2]} |\partial_1 \mathcal{R}(x_1)|^2 \leq c' \frac{|A - B|^2}{(s_2 - s_1)^2}$$

where c' is a positive constant independent of s_1, s_2 and A, B.

Proof. Due to Lemma 8.2.4 below, there exist an rotation axis $a \in \mathbb{R}^3$ with $|a| = 1$ and a rotation angle $\alpha \in [0, \pi]$ such that $B = \mathrm{Rot}(a, \alpha) A$. Set $\pi(x_1) := \frac{x_1 - s_1}{s_2 - s_1}$ and define the rotation field

$$\mathcal{R}(x_1) := \mathrm{Rot}(a, \pi(x_1)\alpha) A.$$

By construction, the map $\mathcal{R}$ belongs to $C^\infty([s_1, s_2]; SO(3))$, and satisfies $\mathcal{R}(s_1) = A$ and $\mathcal{R}(s_2) = B$. Moreover, Rodrigues' formula reveals that

$$\partial_1 \mathcal{R}(x_1) = \frac{\alpha}{s_2 - s_1} \Big(\cos(\pi(x_1)\alpha) \, N_a + \sin(\pi(x_1)\alpha) \, N_a^2 \Big),$$

and consequently

$$|\partial_1 \mathcal{R}(x_1)|^2 \le 2\frac{\alpha^2}{(s_2-s_1)^2} \le c'\frac{|A-B|^2}{(s_2-s_1)^2}.$$

For the latter inequality we used the estimate from Lemma 8.2.4 below. Furthermore, a short computation shows that

$$\mathcal{R}(x_1)^{\mathrm{T}}\partial_1\mathcal{R}(x_1) = \frac{\alpha}{s_2-s_1}N_a \qquad \text{for all } x_1 \in [s_1,s_2].$$

Because $\frac{\alpha}{s_2-s_1}N_a$ is a constant skew-symmetric matrix, the proof is complete. □

Lemma 8.2.4. *Let $A, B \in SO(3)$. There exist $a \in \mathbb{R}^3$ with $|a| = 1$ and $\alpha \in [0,\pi]$ such that*

$$B = \mathrm{Rot}(a,\alpha)A \quad \textit{and} \quad \tfrac{1}{c}\,|A-B|^2 \le \alpha^2 \le c\,|A-B|^2$$

where c is a positive constant that is independent of A and B.

Proof. Without loss of generality we suppose that $A = Id$. By Euler's Rotation Theorem there exist a unit vector a with $Ba = a$. Thus, we can represent the rotation B by means of Rodrigues' rotation formula, i.e $B = \mathrm{Rot}(a,\alpha)$. By probably replacing a with $-a$, we can guarantee that $\alpha \in [0,\pi]$.

Let R denote a rotation in $SO(3)$ of the form $(a\,|\,r_2\,|\,r_3)$. In view of Rodrigues' formula, a straightforward computation shows that

$$(B-Id)R = \big(\sin\alpha\, r_3 - (1-\cos\alpha)\, r_2\big)\otimes e_2 - \big(\sin\alpha\, r_2 + (1-\cos\alpha)\, r_3\big)\otimes e_3.$$

Because r_2 and r_3 are orthogonal unit vectors, it is easy to check that

$$|B-Id|^2 = |(B-Id)R|^2 = 2(\sin\alpha^2 + (1-\cos\alpha)^2) = 4(1-\cos\alpha).$$

As in Lemma 7.4.8 we see that there exists a positive constant c such that

$$\tfrac{1}{c}(1-\cos\alpha) \le \alpha^2 \le c(1-\cos\alpha) \qquad \text{for all } \alpha \in [0,\pi],$$

which completes the proof. □

8.2.3. Two-scale characterization of the limiting strain

Proof of Theorem 8.2.1. Without loss of generality we assume that $\mathcal{H}^2(S) = 1$. Let $E \in \mathfrak{E}$. We pass to a subsequence (not relabeled) such that

$$\frac{1}{h^2}\int_\Omega \mathrm{dist}^2(\nabla_h u_h(x), SO(3))\,\mathrm{d}x \le c' \qquad \text{for each } u_h \tag{8.10}$$

and

$$E_h \xrightarrow{2} E \qquad \text{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(3)). \tag{8.11}$$

Here and below, c' denotes a positive constant that may change from line to line, but can be chosen independent of h.

Step 1. We start with the approximation scheme introduced in Section 8.2.2 applied in the following form: Let $\mathcal{R}_h \in W^{1,2}(\omega; SO(3))$ be the approximation of u_h constructed in Proposition 8.2.2 where we specify the free parameter ϵ according to

$$\epsilon = \begin{cases} h & \text{if } \gamma \in \{0,\infty\} \\ \varepsilon(h) & \text{else.} \end{cases}$$

The particular choice of the free parameter ϵ guarantees that

(8.12)
$$\left\| \frac{\mathcal{R}_h - \nabla_h u_h}{h} \right\|^2_{L^2(\Omega;\mathbb{M}(3))} + \|\partial_1 \mathcal{R}_h\|^2_{L^2(\omega;\mathbb{M}(3))} \leq c' \frac{1}{h^2} \int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(3))\, \mathrm{d}x.$$

Based on this approximation, we define the maps

$$\mathcal{K}_h(x_1) := \mathcal{R}_h(x_1)^{\mathrm{T}} \partial_1 \mathcal{R}_h(x_1), \qquad E_h^{\mathrm{ap}}(x) := \frac{\mathcal{R}_h(x_1)^{\mathrm{T}} \nabla_h u_h(x) - Id}{h}.$$

In view of (8.10) and (8.12), the sequence $(\mathcal{K}_h)$ and (E_h^{ap}) are bounded in $L^2(\Omega; \mathbb{M}(3))$ and $(\mathcal{R}_h)$ is relatively compact with respect to strong convergence in $L^2(\omega; \mathbb{R}^3)$. Hence, we can pass to a further subsequence (not relabeled) such that

$$\begin{aligned} &\mathcal{R}_h \to \mathcal{R} && \text{strongly in } L^2(\omega; \mathbb{M}(3)) \\ &\mathcal{K}_h \xrightharpoonup{2} \mathcal{K} && \text{weakly two-scale in } L^2(\omega \times Y; \mathbb{M}(3)) \\ &E_h^{\mathrm{ap}} \xrightharpoonup{2} E^{\mathrm{ap}} && \text{weakly two-scale in } L^2(\Omega \times Y; \mathbb{M}(3)) \end{aligned}$$

where the limit $E^{\mathrm{ap}} \in L^2(\Omega \times Y; \mathbb{M}(3))$ satisfies

$$\operatorname{sym} E^{\mathrm{ap}} = E$$

due to Corollary 2.3.3, and $\mathcal{K} \in L^2(\omega \times Y; \mathbb{M}(3))$ satisfies

$$\mathcal{K} = \begin{cases} \mathcal{K}_{\mathcal{R}} + \partial_y \Psi & \text{if } \gamma = 0 \text{ with } \Psi \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y; \mathbb{M}(3)_{\mathrm{skew}})) \\ \mathcal{K}_{\mathcal{R}} & \text{if } \gamma > 0. \end{cases}$$

due to Lemma 8.2.5 (see below).

Step 2. Consider the following maps

$$\hat{u}_h(x_1) := \int_S u_h(x_1, \bar{x})\, \mathrm{d}\bar{x}, \qquad \text{and} \qquad w_h(x) := \frac{u_h - \hat{u}_h}{h} - \mathcal{R}_h(x_1) d_S(\bar{x}).$$

(For the definition of d_S see page 168). We claim that

(8.13)
$$\|w_h\|_{L^2(\Omega;\mathbb{R}^3)} \leq c' h \quad \text{and} \quad \int_\Omega |\nabla_h w_h|^2\, \mathrm{d}x \leq c'.$$

This can be seen as follows: Because each map w_h has vanishing mean value with respect to $\bar{x}$, Poincaré's inequality yields

$$\int_S |w_h(x_1,\bar{x})|^2 \,\mathrm{d}\bar{x} \le c' \int_S |\nabla_{\bar{x}} w_h(x_1,\bar{x})|^2 \,\mathrm{d}\bar{x} \le c' \int_S \left| \frac{1}{h}\nabla_{\bar{x}}\, u_h(x) - \mathcal{R}_h(x_1)(e_2{\otimes}e_2 + e_3{\otimes}e_3)\right|^2 \mathrm{d}\bar{x}$$

and integration over ω leads to

$$\int_\Omega |w_h|^2 \,\mathrm{d}x \le c'h^2 \int_\Omega \left| \frac{\frac{1}{h}\nabla_{\bar{x}}\, u_h(x) - \mathcal{R}_h(x_1)(e_2{\otimes}e_2 + e_3{\otimes}e_3)}{h}\right|^2 \mathrm{d}x.$$

Now the right hand side is bounded by $c'h^2 \left\| E_h^{\mathrm{ap}} \right\|^2_{L^2(\Omega;\mathbb{M}(3))}$. In view of (8.12) and (8.10), this proves the first statement in (8.13). The previous estimates also reveal that $(\frac{1}{h}\nabla_{\bar{x}}\, w_h)$ is a bounded sequence in $L^2(\Omega;\mathbb{R}^{3\times 2})$. Hence, in order to establish the second statement in (8.13), it remains to argue that $(\partial_1 w_h)$ is a bounded sequence as well. To this end, we observe that

$$\|\partial_1 w_h\|_{L^2(\Omega;\mathbb{R}^3)} \le \left\| \frac{\partial_1 u_h - \partial_1 \hat{u}_h}{h} \right\|_{L^2(\Omega;\mathbb{R}^3)} + \|\partial_1 \mathcal{R}_h d_S\|_{L^2(\Omega;\mathbb{R}^3)}$$

Note that the second term on the right is uniformly bounded in h due to (8.12), while

$$\left\| \frac{\partial_1 u_h - \partial_1 \hat{u}_h}{h} \right\|^2_{L^2(\Omega;\mathbb{R}^3)} \le \int_\Omega \left| \frac{\partial_1 u_h - \mathcal{R}_h e_1}{h} \right|^2 \mathrm{d}x + \int_\omega \left| \int_S \frac{\partial_1 u_h - \mathcal{R}_h e_1}{h}\,\mathrm{d}\bar{x} \right|^2 \mathrm{d}x_1.$$

Obviously, the right hand side can be bounded by a constant uniform in h due (8.12), and (8.13) follows.

Step 3. We have

$$u_h = \hat{u}_h + h\mathcal{R}_h d_S + h w_h$$

and a straightforward computation yields

$$\nabla_h u_h = \mathcal{R}_h + (\partial_1 \hat{u}_h - \mathcal{R}_h e_1 + h\partial_1 \mathcal{R}_h d_S){\otimes}e_1 + h\,\nabla_h w_h$$
$$E_h^{\mathrm{ap}} = \frac{\mathcal{R}_h^{\mathrm{T}}\,\nabla_h u_h - Id}{h} = \left(\int_S E_h^{\mathrm{ap}} e_1 \,\mathrm{d}\bar{x} + \mathcal{K}_h d_S \right) {\otimes}e_1 + \mathcal{R}_h^{\mathrm{T}}\,\nabla_h\, w_h.$$

This allows us to identify the two-scale limit of (E_h^{ap}) term by term. The only non-obvious limit is the one of the sequence $(\mathcal{R}_h^{\mathrm{T}}\,\nabla_h w_h)$.

We claim that $(\mathcal{R}_h^{\mathrm{T}}\,\nabla_h w_h)$ weakly two-scale converges to a map $W \in L^2(\Omega{\times}Y;\mathbb{M}(3))$ that has the structure described in (8.6) and (8.7). This can be seen as follows: Set $\tilde{w}_h(x) := \mathcal{R}_h(x_1)^{\mathrm{T}} w_h(x)$. Then

$$\mathcal{R}_h^{\mathrm{T}}\,\nabla_h w_h = \nabla_h \tilde{w}_h - \partial_1 \mathcal{R}_h^{\mathrm{T}} w_h$$

and

$$\tilde{w}_h \rightharpoonup 0 \qquad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^3) \quad \text{and} \quad \|\nabla_h \tilde{w}_h\|_{L^2(\Omega;\mathbb{M}(3))} \leq c'.$$

Hence, we can apply the two-scale characterization result for scaled gradients (see Theorem 8.2.1) and deduce that $(\nabla_h \tilde{w}_h)$ weakly two-scale converges to a map W in the form described above. Because of (8.13), we have

$$\left\| \mathcal{R}_h^{\mathrm{T}} w_h \right\|_{L^2(\Omega;\mathbb{R}^3)} \leq c'\, h$$

and we deduce that W is also the weak two-scale limit of $(\mathcal{R}_h^{\mathrm{T}}\,\nabla_h w_h)$, which proves the claim.

So far we have show that

$$E^{\mathrm{ap}}(x,y) = \Big(\mathcal{K}(x_1,y)d_S(\bar{x}) + A(x_1,y)\Big)\otimes e_1 + W(x,y).$$

where $A(x_1,y) := \int_S E^{\mathrm{ap}}(x_1,\bar{\xi},y)e_1 \,\mathrm{d}\bar{\xi}$. We set

$$\bar{A}(x_1) := \int_Y A(x_1,y)\,\mathrm{d}y, \qquad \mathring{A} := A - \bar{A}.$$

Because $\int_Y \mathring{A}\,\mathrm{d}y = 0$, we can write $\mathring{A}$ as the y-derivative of a periodic map, i.e

$$\partial_y \psi = \mathring{A} \quad \text{with} \quad \psi \in L^2(\omega; W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^3)).$$

Furthermore, define

$$\bar{v}(x) := \big(\, a_2(x_1))x_2 + a_3(x_1))x_3\,\big)e_1,$$

where $a_k(x_1)$ denotes the kth component of the vector $\bar{A}(x_1)$. Then v is a map in $L^2(\omega; W^{1,2}(S;\mathbb{R}^3))$ with $\nabla_{\bar{x}}\, v = a_2(e_1\otimes e_2) + a_3(e_1\otimes e_3)$. By combining both functions we observe that

$$\begin{aligned}
\operatorname{sym}\big(\ \nabla_y \psi \ \big|\ \nabla_{\bar{x}}\, v\ \big) &= \operatorname{sym} \widetilde{\nabla}_{1,\gamma}\,(\psi + \gamma v) = \operatorname{sym}\left[\partial_y\psi\otimes e_1 + e_1\otimes \begin{pmatrix} 0 \\ a_2 \\ a_3 \end{pmatrix}\right] \\
&= \operatorname{sym}\left[\mathring{A}\otimes e_1 + e_1\otimes \begin{pmatrix} 0 \\ a_2 \\ a_3 \end{pmatrix}\right] = \operatorname{sym}(A\otimes e_1) - a_1(x_1)(e_1\otimes e_1).
\end{aligned}$$

Now a close look to (8.7) reveals that the map

$$\widetilde{W} := W + \mathring{A}\otimes e_1 + e_1\otimes \begin{pmatrix} 0 \\ a_2 \\ a_3 \end{pmatrix}$$

is in accordance with the structure described in (8.6) and (8.7). This allows us to represent E according to

$$E(x,y) = \operatorname{sym} E^{\mathrm{ap}}(x,y) = \operatorname{sym}\Big[\big(\mathcal{K}(x_1,y)d_S(\bar{x})\,\big)\otimes e_1 + \operatorname{sym}\widetilde{W}(x,y)\Big] + a_1(x_1)(e_1\otimes e_1)$$

as it is claimed in the theorem. □

Lemma 8.2.5. *Let $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence with finite bending energy and let $(\mathcal{R}_h)$ be defined as in the proof of Theorem 8.2.1. Then the sequence $(\mathcal{K}_h) \subset L^2(\omega;\mathbb{M}_{\text{skew}}(3))$ defined by*

$$\mathcal{K}_h := \mathcal{R}_h^T \partial_1 \mathcal{R}_h$$

is weakly two-scale relatively compact in $L^2(\omega\times Y;\mathbb{M}_{\text{skew}}(3))$ and any weak two-scale cluster point $\mathcal{K}$ satisfies

$$\mathcal{K}(x_1,y) - \mathcal{R}(x_1)^T\partial_1\mathcal{R}(x_1) \in \begin{cases} L^2(\omega; W^{1,2}_{\text{per},0}(Y;\mathbb{M}_{\text{skew}}(3))) & \text{if } \gamma = 0 \\ \{0\} & \text{else} \end{cases}$$

where $\mathcal{R}$ denotes the strong limit of $\mathcal{R}_h$.

Proof. In view of Proposition 8.2.2 and due to the choice of the free parameter ϵ whilst applying the proposition, we have

$$\|\partial_1\mathcal{R}_h\|^2_{L^2(\omega;\mathbb{M}(3))} \leq c'\frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u_h(x), SO(3))\,\mathrm{d}x.$$

Since (u_h) has finite bending energy, the limes superior of the right hand side is bounded. Moreover, the norm of $\mathcal{K}_h$ is controlled by the left hand side and consequently

$$\limsup_{h\to 0} \|\mathcal{K}_h\|^2_{L^2(\omega;\mathbb{M}(3))} < \infty.$$

Due to Proposition 2.1.4 the sequence $(\mathcal{K}_h)$ is weakly two-scale relatively compact.

Because $\mathcal{R}_h(x_1) \in SO(3)$ almost everywhere, we have

$$\partial_1\left(\mathcal{R}_h^{\mathrm{T}}\mathcal{R}_h\right) = \partial_1 Id = 0$$

and in view of the product rule, we deduce that

$$0 = \partial_1\mathcal{R}_h^{\mathrm{T}}\mathcal{R}_h + \mathcal{R}_h^{\mathrm{T}}\partial_1\mathcal{R}_h = \mathcal{K}_h^{\mathrm{T}} + \mathcal{K}_h.$$

Hence, $\mathcal{K}_h \in \mathbb{M}_{\text{skew}}(3)$ almost everywhere. As a direct consequence, the same holds for each weak two-scale cluster point.

Let $\mathcal{K} \in L^2(\omega\times Y;\mathbb{M}_{\text{skew}}(3))$ be an arbitrary two-scale cluster point of $(\mathcal{K}_h)$. Set

$$\mathcal{K}_0(x_1,y) := \mathcal{K}(x_1,y) - \int_Y \mathcal{K}(x_1,s)\,\mathrm{d}s$$

and define

$$\Psi(x,y) := \int_0^y \mathcal{K}_0(x,s)\,\mathrm{d}s - \int_Y\int_0^{\bar{y}} \mathcal{K}_0(x,s)\,\mathrm{d}s\,\mathrm{d}\bar{y}.$$

Then $\Psi \in L^2(\omega; W^{1,2}_{\text{per},0}(Y;\mathbb{M}_{\text{skew}}(3)))$ and $\partial_y\Psi = \mathcal{K}_0(x,y)$. By construction, in the case $\gamma > 0$ the map $\mathcal{K}_h$ is ϵ-coherent, where ϵ is either equal to ε or $\epsilon \gg h$. Thus, Lemma 2.2.3 and Lemma 2.2.4, respectively, imply that $\mathcal{K}$ is independent of y; and thus, $\Psi = 0$.

Because $(\mathcal{R}_h)$ converges to $\mathcal{R}$ weakly in $W^{1,2}(\omega;\mathbb{M}(3))$ and strongly in $L^2(\omega;\mathbb{M}(3))$, we immediately deduce that $\mathcal{K}_h$ weakly converges to $\mathcal{R}^{\mathrm{T}}\partial_1\mathcal{R}$, and therefore Lemma 2.1.11 implies that $\int_Y \mathcal{K}(x_1,s)\,\mathrm{d}s = \mathcal{R}^{\mathrm{T}}\partial_1\mathcal{R}$. □

8.3. Γ-convergence

8.3.1. Proof of Theorem 8.1.1: Lower bound

In order to prove the inequality

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) \geq \mathcal{I}_\gamma(u,\mathcal{R}) \tag{8.14}$$

it is sufficient to consider the case where the left hand side is finite. We can pass to a subsequence (not relabeled) such that $\lim_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h)$ exists and is equal to the left hand side of (8.14). This implies that (u_h) is a sequence with finite bending energy and we can pass to a further subsequence (not relabeled) such that

$$E_h \xrightharpoonup{2} E \qquad \text{weakly two-scale in } L^2(\Omega;\mathbb{M}(3))$$

by means of the compactness part of Theorem 8.1.1.

In exactly the same way as in the proof of Lemma 7.4.13, we observe that

$$\liminf_{h\to 0} \mathcal{I}^{\varepsilon,h}(u_h) \geq \iint_{\Omega\times Y} Q(y,E(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

Now Theorem 8.2.1 allows us to characterize the limiting strain E and we infer that the left hand side in (8.14) is bounded from below by

$$\iint_{\Omega\times Y} Q\left(y,\left(\big(\mathcal{K}_\mathcal{R}(x_1)+\partial_y\Psi(x_1,y)\big)d_S(\bar{x})\right)\otimes e_1 + W(x,y) + a(x_1)(e_1\otimes e_1)\right)\,\mathrm{d}y\,\mathrm{d}x$$

where a, Ψ and W belong to the sets defined in Theorem 8.2.1. Minimization over all admissible a, Ψ and W yields (8.14) (cf. Remark 8.1.3).

8.3.2. Proof of Theorem 8.1.1: Recovery sequence

The case $\gamma \in (0,\infty)$.

<u>*Step 1.*</u> Choose maps $a \in L^2(\omega)$ and $w \in L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^3))$ such that

$$\mathcal{I}_\gamma(u,\mathcal{R}) = \iint_{\Omega\times Y} Q\left(y,(\mathcal{K}_\mathcal{R} d_S)\otimes e_1 + \widetilde{\nabla}_{1,\gamma} w + a(e_1\otimes e_1)\right)\,\mathrm{d}y\,\mathrm{d}x$$

(cf. Remark 8.1.3). Because the inclusions

$$C^\infty_c(\omega) \subset L^2(\omega) \quad \text{and} \quad C^\infty_c(\omega; C^1(\overline{S}; C^\infty_{\text{per}}(Y;\mathbb{R}^3))) \subset L^2(\omega; W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^3))$$

are dense and in virtue of Lemma 8.3.1 (below), we can find for any $\delta > 0$ approximations

$$u^{(\delta)} \in C^2_{\text{iso}}(\overline{\omega};\mathbb{R}^3) \quad \text{and} \quad \mathcal{R}^{(\delta)} \in C^1(\overline{\omega}; SO(3)) \quad \text{with} \quad \partial_1 u^{(\delta)} = \mathcal{R}^{(\delta)} e_1$$

and maps

$$g^{(\delta)} \in C_c^\infty(\omega) \quad \text{and} \quad w^{(\delta)} \in C_c^\infty(\omega; C^1(\overline{S}; C^\infty_{\mathrm{per}}(Y;\mathbb{R}^3))$$

such that

$$\begin{aligned}(8.15)\quad & \left\|g^{(\delta)} - g\right\|_{L^2(\omega)} + \left\|w^{(\delta)} - w\right\|_{L^2(\omega;W^{1,2}(S\times Y;\mathbb{R}^3))} \\ & + \left\|u^{(\delta)} - u\right\|_{L^2(\omega;\mathbb{R}^3)} + \left\|\mathcal{R}^{(\delta)} - \mathcal{R}\right\|_{L^2(\omega;\mathbb{M}(3))} + \|\mathcal{K}_{\mathcal{R}^{(\delta)}} - \mathcal{K}_{\mathcal{R}}\|_{L^2(\omega;\mathbb{M}(3))} < \delta\end{aligned}$$

and

$$(8.16)\quad \left|\mathcal{I}_\gamma(u,\mathcal{R}) - \iint_{\Omega\times Y} Q\left(y, (\mathcal{K}_{\mathcal{R}^{(\delta)}} d_S)\otimes e_1 + \widetilde{\nabla}_{1,\gamma} w^{(\delta)} + a^{(\delta)}(e_1\otimes e_1)\right)\mathrm{d}y\,\mathrm{d}x\right| \leq \delta.$$

For the latter, we used the continuity of the quadratic integral functional above.

Step 2. Set $\pi_h(x) := (x, x_1/\varepsilon(h))$ and define

$$\begin{aligned}u_{\delta,h}(x) := u^{(\delta)}(x_1) + h\mathcal{R}^{(\delta)}(x_1)d_S(\bar{x}) + h\int_0^{x_1} \mathcal{R}^{(\delta)}(s)e_1 a^{(\delta)}(s)\,\mathrm{d}s \\ + h\varepsilon(h)\,\mathcal{R}^{(\delta)}(x_1)(w^{(\delta)}\circ\pi_h)(x).\end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}\nabla_h u_{\delta,h} &= \mathcal{R}^{(\delta)} + h\left(\ \partial_1\mathcal{R}^{(\delta)}d_S + \mathcal{R}^{(\delta)}e_1 a^{(\delta)} \mid 0 \mid 0\ \right) \\ &+ h\mathcal{R}^{(\delta)}\left(\ (\partial_y w^{(\delta)})\circ\pi_h \ \middle|\ \tfrac{\varepsilon(h)}{h}(\nabla_{\bar{x}}\, w^{(\delta)})\circ\pi_h\ \right) + h\varepsilon(h)\,\mathcal{R}^{(\delta)}\left(\ (\partial_1 w^{(\delta)})\circ\pi_h \mid 0 \mid 0\ \right).\end{aligned}$$

Now it is easy to check that for $h \to 0$

$$(8.17)\quad \begin{cases} u_{\delta,h} \to u_\delta & \text{strongly in } L^2(\Omega;\mathbb{R}^3) \\ \nabla_h u_{\delta,h} \to \mathcal{R}^{(\delta)} & \text{strongly in } L^2(\Omega;\mathbb{M}(3)) \end{cases}$$

and

$$E^{\mathrm{ap}}_{\delta,h} := \frac{(\mathcal{R}^{(\delta)})^{\mathrm{T}}\,\nabla_h u_{\delta,h} - Id}{h} \xrightarrow{2} \left(\mathcal{K}_{\mathcal{R}^{(\delta)}} d_S + a^{(\delta)}e_1\right)\otimes e_1 + \widetilde{\nabla}_{1,\gamma} w^{(\delta)}$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(3))$. Because all involved quantities are smooth, the sequence (E^{ap}_h) additionally satisfies

$$\limsup_{h\to 0}\operatorname*{ess\,sup}_{x\in\Omega}\left|hE^{\mathrm{ap}}_h(x)\right| = 0.$$

This allows us to apply Theorem 5.2.1 and we see that

$$\begin{aligned}(8.18)\quad \lim_{h\to 0}\frac{1}{h^2}\int_\Omega W(x_1/\varepsilon(h),\, Id + hE^{\mathrm{ap}}_{\delta,h}(x))\,\mathrm{d}x \\ = \iint_{\Omega\times Y} Q\left(y, (\mathcal{K}_{\mathcal{R}^{(\delta)}} d_S)\otimes e_1 + \widetilde{\nabla}_{1,\gamma} w^{(\delta)} + a^{(\delta)}(e_1\otimes e_1)\right)\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Because $\nabla_h u_{\delta,h} = \mathcal{R}^{(\delta)}(Id+hE^{\rm ap}_{\delta,h})$, the frame indifference of W implies that

$$\mathcal{I}^{\varepsilon(h),h}(u_{\delta,h}) = \frac{1}{h^2}\int_\Omega W({}^{x_1}/_{\varepsilon(h)},\, \nabla_h u_{\delta,h}(x))\,\mathrm{d}x = \frac{1}{h^2}\int_\Omega W({}^{x_1}/_{\varepsilon(h)},\, Id + hE^{\rm ap}_h(x))\,\mathrm{d}x.$$

Set

$$c_{\delta,h} \quad := \quad \left|\mathcal{I}^{\varepsilon(h),h}(u_{\delta,h}) - \mathcal{E}_\gamma(u,\mathcal{R})\right| \;+\; \left\|u^{(\delta)}-u\right\|_{L^2(\omega;\mathbb{R}^3)} \;+\; \|\nabla_h u_{\delta,h}-\mathcal{R}\|_{L^2(\omega;\mathbb{M}(3))}\,.$$

Then (8.15), (8.16), (8.17) and (8.18) imply that

$$\lim_{\delta\to 0}\limsup_{h\to 0} c_{\delta,h} = 0$$

and in view of Lemma A.2.1 we can extract a diagonal sequence $\delta(h)$ with $\delta(h)\to 0$ and $c_{\delta(h),h}\to 0$ as $h\to 0$. Thus, $u_h := u_{\delta(h),h}$ recovers the limiting energy and converges to the rod configuration $(u,\mathcal{R})$. □

The case $\gamma = \infty$.

Because the proof is quite similar to the one in the case $\gamma\in(0,\infty)$, we only explain the construction of the sequence for smooth data. To this end, we suppose that

$$u\in C^2_{\rm iso}(\overline{\omega};\mathbb{R}^3) \quad\text{and}\quad \mathcal{R}\in C^1(\overline{\omega};SO(3)) \quad\text{with}\quad \partial_1 u = \mathcal{R}^{(\delta)}e_1$$

and consider maps

$$a\in C^\infty_c(\omega),\qquad w_0\in C^\infty_c(\Omega;C^\infty_{\rm per}(Y;\mathbb{R}^3))\quad\text{and}\quad \bar{w}\in C^\infty_c(\omega;C^1(\bar{S};\mathbb{R}^3)).$$

Set $\pi_h(x) := (x, {}^{x_1}/_{\varepsilon(h)})$ and define

$$u^{(1)}_h(x) := u(x_1) + h\mathcal{R}(x_1)d_S(\bar{x}) + h\int_0^{x_1}\mathcal{R}(s)e_1 a(s)\,\mathrm{d}s$$
$$w^{(1)}_h(x) := \varepsilon(h)\mathcal{R}(x_1)(w_0\circ\pi_h)(x) + h\mathcal{R}(x_1)(\bar{w}\circ\pi_h)(x),$$

and the sequence

$$u_h := u^{(1)}_h + hw^{(1)}_h.$$

The convergence behavior of the sequence $(u^{(1)}_h)$ and the corresponding contribution to the limiting strain has already been studied along the lines of the proof for the case $\gamma\in(0,\infty)$. We briefly analyze the corrector sequence $(w^{(1)}_h)$. First, it is clear that $hw^{(1)}_h$ converges to 0 uniformly. Moreover, a simple calculation (and the application of Lemma 2.1.9) reveals that

$$\mathcal{R}^{\rm T}\,\nabla_h w_h \xrightarrow{2} \Big(\partial_y w_0(x,y)\,|\,\nabla_{\bar{x}}\,\bar{w}(x_1,y)\Big)$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(3))$. In summary, we have

$$\frac{\mathcal{R}^{\rm T}\,\nabla_h u_h - Id}{h} \xrightarrow{2} (\mathcal{K}_{\mathcal{R}}d_S)\otimes e_1 + a(e_1\otimes e_1) + \Big(\partial_y w_0\,|\,\nabla_{\bar{x}}\,\bar{w}\Big)$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(3))$. To complete the proof, we proceed as in the proof for the case $\gamma\in(0,\infty)$. □

The case $\gamma = 0$.

The proof is similar to the previous one. Therefore, we only explain the construction of the sequence for smooth data. Let

$$u \in C^2_{\text{iso}}(\overline{\omega};\mathbb{R}^3) \quad \text{and} \quad \mathcal{R} \in C^1(\overline{\omega}; SO(3)) \quad \text{with} \quad \partial_1 u = \mathcal{R}^{(\delta)} e_1$$

and consider maps

$$a \in C^\infty_c(\omega), \qquad w_0 \in C^\infty_c(\omega; C^\infty_{\text{per}}(Y;\mathbb{R}^3)) \quad \text{and} \quad \bar{w} \in C^1(\bar{\Omega}; C^\infty_{\text{per}}(Y;\mathbb{R}^3)).$$

Moreover, let

$$\Psi \in C^\infty_c(\omega; C^\infty_{\text{per}}(Y;\mathbb{M}_{\text{skew}}(3))).$$

Since Ψ is skew symmetric, there exist function $\tau, \kappa_2, \kappa_3 \in C^\infty_c(\omega; C^\infty_{\text{per}}(Y))$ such that

$$\Psi(x_1, y) = \begin{pmatrix} 0 & \kappa_2(x_1,y) & \kappa_3(x_1,y) \\ -\kappa_2(x_1,y) & 0 & \tau(x_1,y) \\ -\kappa_3(x_1,y) & -\tau(x_1,y) & 0 \end{pmatrix}.$$

For brevity we set $\tau^{(h)}(x_1) := \varepsilon(h)\tau(x_1, {}^{x_1}/_{\varepsilon(h)})$ and define $\kappa_2^{(h)}$ and $\kappa_3^{(h)}$ similarly. We define the map $\mathcal{R}^{(h)} : \omega \to SO(3)$ by

$$\mathcal{R}^{(h)}(x_1) := \mathcal{R}(x_1)\ \text{Rot}(e_1, \tau^{(h)}(x_1))\,\text{Rot}(e_2, \kappa_2^{(h)}(x_1))\,\text{Rot}(e_3, \kappa_3^{(h)}(x_1)).$$

Then an elementary, but tedious calculation shows that

$$(8.19) \qquad \begin{cases} \mathcal{R}^{(h)} \to \mathcal{R} & \text{uniformly} \\ {\mathcal{R}^{(h)}}^{\mathrm{T}} \partial_1 \mathcal{R}^{(h)} \xrightarrow{2} \mathcal{K}_{\mathcal{R}} + \partial_y \Psi & \text{strongly two-scale in } L^2(\omega{\times}Y;\mathbb{M}_{\text{skew}}(3)). \end{cases}$$

Now set

$$\begin{aligned} u_h^{(2)}(x) &:= u(0) + h\int_0^{x_1} \mathcal{R}^{(h)}(s) e_1 (1 + a(s))\,\mathrm{d}s + h\mathcal{R}^{(h)}(x_1) d_S(\bar{x}) \\ w_h^{(1)}(x) &:= \varepsilon(h)\mathcal{R}(x_1)(w_0 \circ \pi_h)(x) + h\mathcal{R}(x_1)(\bar{w}\circ\pi_h)(x), \end{aligned}$$

and consider the sequence

$$u_h := u_h^{(2)} + h w_h^{(1)}.$$

As before, $hw_h^{(1)}$ uniformly converges to zero and we have

$$\mathcal{R}^{\mathrm{T}}\,\nabla_h w_h^{(1)} \xrightarrow{2} \Big(\, \partial_y w_0(x_1,y) \,|\, \nabla_{\bar{x}}\, \bar{w}(x,y) \Big)$$

strongly two-scale in $L^2(\Omega{\times}Y;\mathbb{M}(3))$. Moreover, (8.19) implies that

$$\frac{{\mathcal{R}^{(h)}}^{\mathrm{T}}\,\nabla_h u_h^{(2)} - Id}{h} \xrightarrow{2} \Big(\big(\mathcal{K}_{\mathcal{R}}(x_1) + \partial_y \Psi(x_1,y)\big) d_S \Big) {\otimes} e_1 + a(x_1)(e_1{\otimes}e_1)$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(3))$. Since $\mathcal{R}^{(h)} - \mathcal{R} \to 0$, we can combine the last two convergence statements and arrive at

$$\frac{\mathcal{R}^{(h)\mathrm{T}}\nabla_h u_h - Id}{h} \xrightarrow{2} \Big(\big(\mathcal{K}_{\mathcal{R}}(x_1) + \partial_y\Psi(x_1,y)\big)d_S\Big)\otimes e_1 + a(x_1)(e_1\otimes e_1) + \Big(\,\partial_y w_0 \,|\, \nabla_{\bar{x}}\,\bar{w}\,\Big)$$

strongly two-scale in $L^2(\Omega\times Y;\mathbb{M}(3))$. To complete the proof, we proceed as in the case $\gamma \in (0,\infty)$. □

It remains to prove that arbitrary rod configurations can be approximated by smooth maps:

Lemma 8.3.1. *Let $\omega = (0,L)$. Let $(u,\mathcal{R})$ be a rod configuration. Then for all $\delta > 0$ there exists a smooth rod configuration $(u^{(\delta)},\mathcal{R}^{(\delta)})$ with*

$$u^{(\delta)} \in C^2_{\mathrm{iso}}(\overline{\omega};\mathbb{R}^3), \qquad \mathcal{R}^{(\delta)} \in C^1(\overline{\omega};SO(3)) \quad \text{and} \quad \partial_1 u^{(\delta)} = \mathcal{R}^{(\delta)}e_1,$$

such that $u^{(\delta)}(0) = u(0)$, $\mathcal{R}^{(\delta)}(0) = \mathcal{R}(0)$ and

$$\left\|u^{(\delta)} - u\right\|_{L^2(\omega;\mathbb{R}^3)} + \left\|\mathcal{R}^{(\delta)} - \mathcal{R}\right\|_{L^2(\omega;\mathbb{M}(3))} + \|\mathcal{K}_{\mathcal{R}^{(\delta)}} - \mathcal{K}_{\mathcal{R}}\|_{L^2(\omega;\mathbb{M}(3))} < \delta.$$

Proof. Without loss of generality let $\mathcal{R}(0) = Id$ and $u(0) = 0$. Set $\mathcal{K} := \mathcal{R}^{\mathrm{T}}\partial_1\mathcal{R}$. For all $k \in \mathbb{N}$ choose a map $\mathcal{K}^{(k)} \in C^\infty_c(\omega;\mathbb{M}_{\mathrm{skew}}(3))$ such that

$$\left\|\mathcal{K} - \mathcal{K}^{(k)}\right\|_{L^2(\omega;\mathbb{M}(3))} \le \frac{1}{k}.$$

Step 1. By the Picard-Lindelöf theorem one can show that (for all $k \in \mathbb{N}$) there exists a unique map $\mathcal{R}^{(k)} \in C^1(\overline{\omega};\mathbb{M}(3))$ that solves the system

$$\partial_1\mathcal{R}^{(k)}(x_1) = \mathcal{R}^{(k)}(x_1)\mathcal{K}^{(k)}(x_1) \qquad \text{for all } x_1 \in \overline{\omega} \quad \text{and} \quad \mathcal{R}^{(k)}(0) = Id.$$

Because $\mathcal{K}(x_1)$ is skew symmetric for all $x_1 \in \overline{\omega}$, one can show that the matrix $\mathcal{R}^{(k)}(x_1)$ is orthogonal for all $x_1 \in \overline{\omega}$. Now the continuity of the map $\mathcal{R}^{(k)}$, the initial value $\mathcal{R}^{(k)}(0) = Id \in SO(3)$ and the fact that $SO(3)$ is a maximally connected component of the set of orthogonal 3×3-matrices implies that $\mathcal{R}^{(k)} \in C^1(\overline{\omega};SO(3))$.

Step 2. We claim that

$$\mathcal{R}^{(k)} \rightharpoonup \mathcal{R} \qquad \text{weakly in } W^{1,2}(\omega;\mathbb{M}(3)).$$

The sequence $(\mathcal{R}^{(k)})$ is obviously bounded in $W^{1,2}(\omega;\mathbb{M}(3))$. Hence, we can pass to a subsequence that weakly converges to a map $\tilde{\mathcal{R}} \in W^{1,2}(\omega;\mathbb{M}(3))$. Due to the compactness of the embedding $W^{1,2}(\omega;\mathbb{M}(3)) \subset L^2(\omega;\mathbb{M}(3))$, the latter convergence holds strongly in $L^2(\omega;\mathbb{M}(3))$ as well. Moreover, we have

$$\mathcal{R}^{(k)}(x_1) = Id + \int_0^{x_1} \mathcal{R}^{(k)}(\xi)\mathcal{K}^{(k)}(\xi)\,\mathrm{d}\xi.$$

The strong convergence of $\mathcal{R}^k$ and $\mathcal{K}^k$ allows us to pass to the limit in the equation above and we deduce that

$$\tilde{\mathcal{R}}(x_1) = Id + \int_0^{x_1} \tilde{\mathcal{R}}(\xi)\mathcal{K}(\xi)\,\mathrm{d}\xi.$$

Note that the equation above is the integral version of an initial value problem which exhibits a unique continuous solution. Because of the embedding $W^{1,2}(\overline{\omega};\mathbb{M}(3)) \subset C(\overline{\omega};\mathbb{M}(3))$, both $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are solutions and the uniqueness implies $\mathcal{R} = \tilde{\mathcal{R}}$.

Step 3. Set

$$u^{(k)}(x_1) := u(0) + \int_0^{x_1} \mathcal{R}^{(k)}(\xi)e_1\,\mathrm{d}\xi.$$

Then $u^{(k)}(0) = u(0)$ and

$$\left\|u^{(k)} - u\right\|_{L^2(\omega;\mathbb{R}^3)} \le c' \left\|\partial_1 u^{(k)} - \partial_1 u\right\|_{L^2(\omega;\mathbb{R}^3)} \le c' \left\|\mathcal{R}^{(k)} - \mathcal{R}\right\|_{L^2(\omega;\mathbb{M}(3))}$$

for a suitable constant c'. Because $\mathcal{R}^{(k)} \to \mathcal{R}$ and $\mathcal{K}^{(k)} \to \mathcal{K}$ strongly in $L^2(\omega;\mathbb{M}(3))$, we deduce that

$$\left\|u^{(k)} - u\right\|_{L^2(\omega;\mathbb{R}^3)} + \left\|\mathcal{R}^{(k)} - \mathcal{R}\right\|_{L^2(\omega;\mathbb{M}(3))} + \left\|\mathcal{K}^{(k)} - \mathcal{K}\right\|_{L^2(\omega;\mathbb{M}(3))} \le \delta$$

if k is sufficiently large. □

9. Bounds for homogenized plate theories

9.1. Introduction and main result

In this section we present partial results concerning the derivation of homogenized nonlinear plate theories from three-dimensional elasticity for materials featuring periodic microstructures. The derivation relies on the one side, on the work in [FJM02] where elastic plates for materials without x-dependency have been considered. On the other side, our approach is based on the homogenization methods developed in the previous chapters. The main result is a (non-sharp) two-scale characterization of nonlinear limiting strains that emerge from sequences of deformations with finite bending energy. Furthermore, we prove a lower bound and an upper bound estimate for the limiting energy that — although in general being not optimal — already capture fine-scale properties of the limiting behavior and indicate the dependency of the limiting theory on the limiting ratio between the thickness of the plate and the size of the material microstructure.

In the following we describe the precise setting. Set $Y := [0,1)^2$ and let $\Omega := \omega \times S$ be a cylindrical domain in $\mathbb{R}^3$ where the mid plane ω is a bounded, regular, convex subset of $\mathbb{R}^2$ with Lipschitz boundary and $S := (-1/2, 1/2)$. In the sequel we decompose points $x \in \mathbb{R}^3$ according to

$$x = (\hat{x}, x_3) \quad \text{with} \quad \hat{x} \in \mathbb{R}^2 \quad \text{and} \quad x_2 \in \mathbb{R}$$

and call $\hat{x}$ and x_3 the in-plane and out-of-plane components of x, respectively.

We suppose that $W : \mathbb{R}^2 \times \mathbb{M}(3) \to [0, \infty)$ is a non-negative integrand that is Y-periodic in its first variable and satisfies conditions (W1) – (W4) from the previous chapter (see page 168). For positive parameters ε, h and $u \in L^2(\Omega; \mathbb{R}^3)$ we define the scaled elastic energy

$$\mathcal{I}^{\varepsilon,h}(u) := \begin{cases} \dfrac{1}{h^2} \displaystyle\int_\Omega W(\hat{x}/\varepsilon, \nabla_h u_h(x))\,\mathrm{d}x & \text{if } u \in W^{1,2}(\Omega, \mathbb{R}^3) \\ +\infty & \text{else.} \end{cases}$$

Above, the scaled deformation gradient is defined in Definition 8.1.4, i.e.

$$\nabla_h u := \nabla_{2,h}\, u = (\nabla_{\hat{x}}\, u \,|\, \tfrac{1}{h}\partial_3 u), \qquad \nabla_{\hat{x}}\, u = (\partial_1 u \,|\partial_2 u).$$

As in the previous chapters, we suppose that (h) and (ε) are coupled fine-scale sequences in the sense of (7.1). In particular, we suppose that $\varepsilon = \varepsilon(h)$ and

$$\lim_{h \to 0} \frac{h}{\varepsilon(h)} = \gamma \quad \text{with} \quad \gamma \in [0, \infty].$$

In order to describe the limiting energy, we introduce the space of bending deformations over ω by

$$\mathcal{A} := \Big\{ u \in W^{2,2}(\omega;\mathbb{R}^3) \,:\, |\partial_1 u(\hat{x})| = |\partial_2 u(\hat{x})| = 1, \\ \langle \partial_1 u(\hat{x}),\, \partial_2 u(\hat{x})\rangle = 0 \quad \text{for a.e. } \hat{x} \in \omega \Big\}.$$

To each bending deformation $u \in \mathcal{A}$ we associate a normal field, an associated frame and the second fundamental form according to

$$n_{(u)} := \partial_1 u \wedge \partial_2 u, \qquad R_{(u)} := (\,\partial_1 u\,|\partial_2 u\,|\,n_{(u)}\,), \qquad \mathrm{II}_{(u)} := \sum_{i,j=1}^{2} \big\langle \partial_i u,\, \partial_j n_{(u)} \big\rangle\, (e_i \otimes e_j).$$

In the definition above, we understand $\mathrm{II}_{(u)}(\hat{x})$ as a matrix in $\mathbb{M}(2)$ (i.e. $\{e_1, e_2\}$ denotes the canonical basis of $\mathbb{R}^2$).

Along the derivation of the homogenized rod theory in the previous chapter, we have seen that in the limiting process oscillations of quantities that are related to curvature occur. In the context of rods, these oscillations could be described by "periodic profiles" in the space $W^{1,2}_{\mathrm{per},0}((0,1);\mathbb{M}_{\mathrm{skew}}(3))$ (see Lemma 8.2.5). In the situation of plates, the structure of bending deformations, and thereby, also the structure of the admissible oscillations, is more complex. Roughly speaking, if $u \in \mathcal{A}$ is a bending deformation, then locally $\nabla_{\hat{x}}\, u$ is either constant or constant along a line segment, the end points of which touching the boundary of the mid plane $\partial\omega$ (for details we refer to [Kir01, Pak04, FJM06]). This suggests that an appropriate class of periodic profiles should reflect these constraints as well.

In this thesis we do not address this question. Instead, we introduce a space of oscillation profiles in a rather implicit way: To each bending deformation $u \in \mathcal{A}$ we associate the class

$$\Phi(u) := \Big\{ \phi \in L^2(\omega\times Y;\mathbb{M}(2)) \,:\, \exists (R_h) \subset W^{1,2}(\omega; SO(3)) \text{ such that} \\ R_h \rightharpoonup R_{(u)} \text{ weakly in } W^{1,2}(\omega;\mathbb{M}(3)) \text{ and} \\ \sum_{i,j=1}^{2} \big(\, \langle R_h e_i,\, \partial_j R_h e_3 \rangle\,\big)(e_i \otimes e_j) \xrightarrow{2} \phi + \mathrm{II}_{(u)} \\ \text{weakly two-scale in } L^2(\omega\times Y;\mathbb{M}(2)) \Big\}.$$

Our main result in this chapter is the following:

Theorem 9.1.1 (Compactness and (non-sharp) two-scale characterization of the limiting strain)**.** *There exists a positive constant δ_0 depending only on ω such that the following holds. Let (u_h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying*

$$\limsup_{h\to 0}\frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u(x), SO(3))\,\mathrm{d}x \le \delta_0.$$

Then there exist a subsequence (not relabeled) and maps

$$u\in\mathcal{A},\qquad \phi\in\Phi(u),\qquad A\in L^2(\omega\times Y;\mathbb{R}^{3\times 2})$$

and

$$W(x,y)=\begin{cases}\left(\ \nabla_y w_0\ \middle|\ \partial_3\bar{w}\ \right) & \text{if } \gamma\in\{0,\infty\}\\ \widetilde{\nabla}_{2,\gamma}w_0 & \text{else}\end{cases}$$

where

$$\begin{cases} w_0\in L^2(\omega;W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^3)) \text{ and } \bar{w}\in L^2(\omega\times Y;W^{1,2}(S;\mathbb{R}^3)) & \text{if } \gamma=0\\ w_0\in L^2(\Omega;W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^3)) \text{ and } \bar{w}\in L^2(\omega;W^{1,2}(S;\mathbb{R}^3)) & \text{if } \gamma=\infty\\ w_0\in L^2(\omega;W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^3)) & \text{else}\end{cases}$$

such that

$$\begin{cases} u_h\to u & \text{strongly in } L^2(\Omega;\mathbb{R}^3)\\ \nabla_h u_h\to R_{(u)} & \text{strongly in } L^2(\Omega;\mathbb{M}(3))\\ E_h:=\dfrac{\sqrt{\nabla_h u_h^T\nabla_h u_h}-Id}{h}\xrightarrow{2} E & \text{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(3))\end{cases}\tag{9.1}$$

where the nonlinear limiting strain is given by

$$E:=\operatorname{sym}\left[(A\,|0)+x_3\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{II}_{(u)}+\phi} & \begin{array}{c}0\\0\end{array}\\ \hline 0\quad 0 & & 0\end{array}\right)+W\right].$$

(For the proof see page 193).

The scaled derivative $\widetilde{\nabla}_{2,\gamma}$ and the function space $W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{R}^3)$ are defined in Section 6.3.

By combining the previous theorem with the simultaneous homogenization and linearization method, we obtain a lower bound for the limiting theory:

Lemma 9.1.2. *Let (u_h) be a sequence in $L^2(\Omega;\mathbb{R}^3)$ that strongly converges to a map $u\in\mathcal{A}$. Then*

$$\liminf_{h\to 0}\mathcal{I}^{\varepsilon(h),h}(u_h)\ge\min\left\{\frac{1}{12}\int_\omega Q^0_{\hom}(\mathrm{II}_{(u)}(\hat{x}))\,\mathrm{d}\hat{x},\ \delta_0\right\}$$

where

$$Q^0_{\hom}(\mathrm{II}) := \inf\left\{ \int_Y \min_{d\in\mathbb{R}^3} Q\left(y, \left(\begin{array}{c|c} \mathrm{II}+\phi(y) & \\ \nabla_y\,\alpha(y) & d \end{array}\right)\right) \mathrm{d}y \ : \ \phi\in\Phi(u),\ \alpha\in W^{1,2}_{\mathrm{per},0}(Y) \right\}.$$

(For the proof see page 194).

For the fine-scale coupling regime $h \gg \varepsilon$, we demonstrate the construction of recovery sequences:

Lemma 9.1.3. *Suppose that $\gamma = \infty$. Let $u \in \mathcal{A}$. Then there exists a sequence $(u_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ converging to u in the sense of (9.1) such that*

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u_h) = \frac{1}{12}\int_\omega Q^\infty_{\hom}(\mathrm{II}_{(u)}(\hat{x}))\,\mathrm{d}\hat{x}$$

where

$$Q^\infty_{\hom}(\mathrm{II}) := \inf\left\{ \int_Y Q\left(y, \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{II}} & \begin{array}{c}0\\0\end{array} \\ \hline 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{c|c} \nabla_y\, w_0(y) & \bar{w} \end{array}\right)\right) \mathrm{d}y \ : \ w_0\in W^{1,2}_{\mathrm{per},0}(Y),\ \bar{w}\in\mathbb{R}^3 \right\}.$$

(For the proof see page 194).

In the next section we prove the previous results. In Section 9.3 we discuss our findings and draft ideas and strategies that may lead to more complete results.

9.2. Proofs

The compactness part of the Theorem 8.1.1 is presented in [FJM02, FJM06] and relies on the following approximation:

Theorem 9.2.1 (see Theorem 6 & Remark 5 in [FJM06])**.** *In the situation of the previous theorem (with δ_0 sufficiently small) there exists a constant $C > 0$ and a sequence of maps $R_h : \omega \to SO(3)$ in $W^{1,2}(\omega;\mathbb{M}(3))$ such that*

$$\limsup_{h\to 0}\left\{ \left\| \frac{\nabla_h u_h - R_h}{h} \right\|^2_{L^2(\omega;\mathbb{M}(3))} + \|\partial_1 R_h\|^2_{L^2(\omega;\mathbb{M}(3))} + \|\partial_2 R_h\|^2_{L^2(\omega;\mathbb{M}(3))} \right\} \le C\delta_0.$$

Remark 9.2.2. The construction of the map R_h in the result above is done in two stages. First, based on the rigidity estimate (see Theorem 7.4.6) a map $\widetilde{R}_h \in W^{1,2}(\omega;\mathbb{M}(3))$ is constructed in such a way that $\left\|\widetilde{R}_h - \nabla_h u_h\right\|^2_{L^2}$, as well as $h^2\left\|\nabla_{\hat{x}}\widetilde{R}_h\right\|^2_{L^2}$ are controlled by the L^2-distance of $\nabla_h u_h$ to $SO(3)$. It turns out that for each $\hat{x}\in\omega$, the matrix $\widetilde{R}_h(\hat{x})$ lies in a small neighborhood of $SO(3)$, say U, provided $\frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u_h, SO(3))\,\mathrm{d}x < \delta_0$ and δ_0 is sufficiently small. As a consequence, it is reasonable to define the desired rotation field R_h as the pointwise projection of $\widetilde{R}_h$ to $SO(3)$. If the neighborhood U is small (i.e. δ_0 is sufficiently small) then the projection is a smooth map and $R_h \in W^{1,2}(\omega;\mathbb{M}(3))\cap L^\infty(\omega;SO(3))$.

Proof of Theorem 9.1.1. Without loss of generality we assume that $\mathcal{H}^2(S)=1$.

Step 1. In virtue of Theorem 9.2.1 we can pass to a subsequence (that we do not relabel) such that

(9.2)
$$\begin{cases} u_h \to u & \text{strongly in } L^2(\Omega;\mathbb{R}^3)\\ \nabla_h u_h \to R_{(u)} & \text{strongly in } L^2(\Omega;\mathbb{M}(3))\\ R_h \rightharpoonup R_{(u)} & \text{weakly in } W^{1,2}(\omega;\mathbb{M}(3))\\ E_h := \dfrac{\sqrt{\nabla_h u_h^T \nabla_h u_h} - Id}{h} \xrightharpoonup{2} E & \text{weakly two-scale in } L^2(\Omega\times Y;\mathbb{M}(3)). \end{cases}$$

This can be justified in the same way as we did in the proof of Theorem 8.2.1.

Step 2. Define
$$E_h^{\mathrm{ap}} := \frac{R_h^T \nabla_h u_h - Id}{h}.$$
Then Theorem 9.2.1 implies that (E_h^{ap}) is bounded in $L^2(\Omega;\mathbb{M}(3))$ and we can pass to a (further) subsequence such that $E_h^{\mathrm{ap}} \xrightharpoonup{2} E^{\mathrm{ap}}$ weakly two-scale in $L^2(\Omega\times Y;\mathbb{M}(3))$. In view of Corollary 2.3.4 we deduce that $\operatorname{sym} E^{\mathrm{ap}} = E$.

Step 3. Now define the sequence
$$w_h(x) := \frac{u_h(x) - \bar{u}_h(\hat{x})}{h} - x_3 R_h(\hat{x})e_3, \qquad \bar{u}_h(\hat{x}) := \int_S u_h(\hat{x},x_3)\,\mathrm{d}x_3.$$
As in the proof of Theorem 8.2.1 one can show that
$$w_h \to 0 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3) \quad \text{and} \quad \limsup_{h\to 0}\|\nabla_h w_h\|_{L^2(\Omega;\mathbb{M}(3))} < \infty.$$
As a consequence, we can apply the two-scale characterization of scaled gradients and pass to a further subsequence such that

$$\nabla_h w_h \xrightharpoonup{2} W \qquad \text{weakly two scale in } L^2(\Omega\times Y;\mathbb{M}(3)) \tag{9.3}$$

where W is a map that fulfills the requirements of the theorem.

Step 4. We rewrite u_h in terms of w_h:

$$u_h(x) = \bar{u}_h(\bar{x}) + h\,(\,x_3 R_h(\bar{x})e_3 + w_h(x))$$

and deduce that

$$E_h^{\rm ap} = (A_h\,|\,0) + x_3\,\big(\ R_h^{\rm T}\,\nabla_{\hat{x}}(R_h e_3)\ \big|\ 0\ \big) + \nabla_h w_h. \tag{9.4}$$

where $A_h(\hat{x})$ is the $3{\times}2$ matrix consisting of the first and second column of the averaged matrix $\int_S E_h^{\rm ap}(\hat{x},x_3)\,{\rm d}x_3$. Obviously, we can pass to a further subsequence such that the first term in (9.4) weakly two-scale converges to a map of the form

$$(A\,|\,0) \quad\text{with}\quad A\in L^2(\omega{\times}Y;\mathbb{R}^{3\times 2})$$

and the second term weakly two-scale converges to

$$x_3\left(\begin{array}{c|c} {\rm II}_{(u)} + \phi & \begin{matrix}0\\0\end{matrix} \\ \hline 0\quad 0 & 0\end{array}\right)$$

where ϕ denotes a suitable map in $\Phi(u)$. Because ${\rm sym}\,E^{\rm ap} = E$ and in virtue of (9.3) this completes the proof. □

Proof of Lemma 9.1.2. It is sufficient to consider the situation where

$$\liminf_{h\to 0}\mathcal{I}^{\varepsilon(h),h}(u_h)\le\delta_0.$$

In this case we can pass to a subsequence (not relabeled) with the property that $\lim_{h\to 0}\mathcal{I}^{\varepsilon(h),h}$ exists and is equal to the left hand side of the previous inequality. Hence, we can apply Theorem 9.1.1 and (after probably passing to a further subsequence) deduce that the convergence in (9.2) holds with

$$E := {\rm sym}\left[(A\,|0) + x_3\left(\begin{array}{c|c} {\rm II}_{(u)} + \phi & \begin{matrix}0\\0\end{matrix} \\ \hline 0\quad 0 & 0\end{array}\right) + W\right].$$

for suitable maps A, ϕ and W. It is straightforward to show that for almost every $\hat{x}\in\omega$ we have

$$\int_S E(\hat{x},x_3)x_3\,{\rm d}x_3 = \frac{1}{12}\left(\begin{array}{c|c} \begin{matrix}{\rm II} + \phi(y)\\ \hline \nabla_y\,\alpha(y)\end{matrix} & d(y)\end{array}\right)$$

for a suitable maps $\alpha\in W^{1,2}_{{\rm per},0}(Y)$, $\phi\in L^2_0(Y;\mathbb{M}_{\rm sym}(2))$ and $d\in L^2(Y;\mathbb{R}^3)$. Now the proof can be completed by proceeding as in the proof of the lower bound part of Theorem 7.1.1. □

Proof of Lemma 9.1.3. We first consider the case where $u\in\mathcal{A}\cap C^2(\overline{\omega};\mathbb{R}^3)$. Let $w_0\in C^\infty_c(\omega;C^\infty_{\rm per}(Y;\mathbb{R}^3))$ and $\bar{w}\in C^\infty_c(\omega;\mathbb{R}^3)$ and set $\pi_h(\hat{x}) := (\hat{x},\hat{x}/\varepsilon(h))$. We define the sequence

$$u_h(x) = u(\hat{x}) + hx_3\left(R_{(u)}(\hat{x})e_3 + \varepsilon(h)w_0\circ\pi_h(\hat{x})\right) + \frac{h^2x_3^2}{2}\bar{w}(\hat{x})$$

and set

$$G(\hat{x},y) := \operatorname{sym}\left[\left(\begin{array}{c|c} \mathrm{II}_{(u)}(\hat{x}) & \begin{matrix}0\\0\end{matrix} \\ \hline 0 \quad 0 & 0 \end{array}\right) + \left(\ \nabla_y w_0(\hat{x},y) \ \middle|\ \bar{w}(\hat{x})\ \right)\right].$$

It is easy to show that (u_h) converges to u in the sense of (9.1). Moreover, it is straightforward to check that

$$E_h := \frac{\sqrt{\nabla_h u_h^{\mathrm{T}} \nabla_h u_h} - Id}{h} \xrightarrow{2} x_3 G(\hat{x},y) \qquad \text{strongly two-scale in } L^2(\Omega\times Y;\mathbb{M}(3))$$

and $\limsup_{h\to 0} \operatorname{ess\,sup}_{x\in\Omega} |hE_h(x)| = 0$. Proceeding as in the proof of the upper bound part of Theorem 7.2.5, we see that the simultaneous linearization and homogenization result (see Theorem 5.2.1) implies that

$$\lim_{h\to 0} \mathcal{I}^{\varepsilon(h),h}(u_h) = \frac{1}{12} \iint_{\omega\times Y} Q(y, G(\hat{x},y))\,\mathrm{d}y\,\mathrm{d}\hat{x}.$$

Because the inclusions $\mathcal{A}\cap C^2(\overline{\omega};\mathbb{R}^3) \subset \mathcal{A}$ (see [Pak04]) and

$$C_c^\infty(\omega; C_{\mathrm{per}}^\infty(Y;\mathbb{R}^3)) \subset L^2(\omega; W_{\mathrm{per}}^{1,2}(Y;\mathbb{R}^3)) \quad \text{and} \quad C_c^\infty(\omega;\mathbb{R}^3) \subset L^2(\omega;\mathbb{R}^3)$$

are dense, we can construct the sequence for non-smooth data by the diagonal sequence construction that we already employed in the previous chapters. In particular, we can construct a sequence of deformations (u_h) in $W^{1,2}(\Omega;\mathbb{R}^3)$ that converges to u in the sense of (9.1) and that satisfies

$$E_h := \frac{\sqrt{\nabla_h u_h^{\mathrm{T}} \nabla_h u_h} - Id}{h} \xrightarrow{2} x_3 G^\star(\hat{x},y) \qquad \text{strongly two-scale in } L^2(\Omega\times Y;\mathbb{M}(3))$$

with

$$G^\star(\hat{x},y) := \operatorname{sym}\left[\left(\begin{array}{c|c} \mathrm{II}_{(u)}(\hat{x}) & \begin{matrix}0\\0\end{matrix} \\ \hline 0 \quad 0 & 0 \end{array}\right) + \left(\ \nabla_y w_0^\star(\hat{x},y) \ \middle|\ \bar{w}^\star(\hat{x})\ \right)\right]$$

where $w_0^\star \in L^2(\omega; W_{\mathrm{per}}^{1,2}(Y;\mathbb{R}^3))$ and $\bar{w}^\star \in L^2(\omega;\mathbb{R}^3)$ fulfill

$$\iint_{\omega\times Y} Q\left(y, \operatorname{sym}\left[\left(\begin{array}{c|c} \mathrm{II}_{(u)}(\hat{x}) & \begin{matrix}0\\0\end{matrix} \\ \hline 0 \quad 0 & 0 \end{array}\right) + \left(\ \nabla_y w_0^\star(\hat{x},y) \ \middle|\ \bar{w}^\star(\hat{x})\ \right)\right]\right)\mathrm{d}y = \int_\omega Q_{\mathrm{hom}}^\infty(\mathrm{II}_{(u)}(\hat{x}))\,\mathrm{d}\hat{x}.$$

Application of the simultaneous linearization and homogenization result (see Theorem 5.2.1) completes the proof. □

9.3. Discussion

Lemma 9.1.2 provides a lower bound for the lower Γ-limit of the sequence $(\mathcal{I}^{\varepsilon(h),h})$. We do not believe that this bound is sharp. In particular, the construction of the set $\Phi(u)$ is too rough to allow a precise identification of the limiting strains. Our experience with the analysis in the rod setting, suggests that the characterization should even hold, when we replace $\Phi(u)$ by the trivial set $\{0\}$ in the fine-scale coupling regimes $h \sim \varepsilon$ and $h \gg \varepsilon$, and the class

$$\Big\{ A \in L^2(\omega;\mathbb{M}(2)) \,:\, \exists (u_h) \subset \mathcal{A}, \qquad u_h \rightharpoonup u \qquad \text{weakly in } W^{2,2}(\omega;\mathbb{R}^3) \text{ and } \mathrm{II}_{(u_h)} \xrightarrow{2} \mathrm{II}_{(u)} + A \text{ weakly two-scale in } L^2(\omega{\times}Y;\mathbb{M}(2)) \Big\} \qquad \text{if } h \ll \varepsilon. \tag{9.5}$$

It is natural to expect that the latter class can be represented in a streamlined and more explicit form. Similarly to the analysis in the rod setting, a natural approach to justify this hypothesis is to develop a refined version of the approximation contained in Theorem 9.2.1. In contrast to the approximation scheme that we developed in the rod setting (see Proposition 8.2.2), Theorem 9.2.1 is not suited to capture and filter oscillations on the prescribed scale ε. As a consequence, Theorem 9.1.1 does not distinguish between oscillations that solely emerge due to curvature oscillations of the mid plane and those that stem from oscillations related to the director field. Note that the latter can be captured by the relaxation profile W, and therefore should be deleted from the set $\Phi(u)$.

Nevertheless, in the case where $\gamma = 0$ we believe that the homogenized quadratic form corresponding to the Γ-limit of the energy sequence is an improved version of Q^0_{hom} where $\Phi(u)$ is replaced by the set in (9.5).

With regard to the upper bound, Lemma 9.1.3 immediately provides an estimate for the upper Γ-limit of $(\mathcal{I}^{\varepsilon(h),h})$ in the case $\gamma = \infty$. In fact, we believe that the derived upper bound is already the Γ-limit of $(\mathcal{I}^{\varepsilon(h),h})$ (for $\gamma = \infty$). While recovery sequences for the fine-scale coupling regimes $\gamma \in (0, \infty)$ can be constructed in a similar way, we believe that the construction in the case $\gamma = 0$ is more interesting. In this case, oscillations that are related to oscillations of the mid plane come into play, and therefore the rigidity of bending deformations prohibits oscillations of the mid plane in arbitrary directions. In this context, we would like to remark that the homogenization problem of the nonlinear bending theory for plates seems to be open; in particular, the Γ-convergence properties of the functional

$$\mathcal{A} \ni u \mapsto \int_\omega \widetilde{Q}(\hat{x}/\varepsilon, \mathrm{II}_{(u)}(\hat{x}))\,\mathrm{d}\hat{x}$$

as $\varepsilon \to 0$ has not been studies yet. The analysis in Section 7.6 suggests that the Γ-limit corresponding to the energy above with

$$\widetilde{Q}(y, F) = \min_{d\in\mathbb{R}^3} Q\left(y, \sum_{i\in\{1,2\}} F_{\{i,j\}}(e_i{\otimes}e_j) + d{\otimes}e_3\right)$$

is similar to the Γ-limit of the sequence $(\mathcal{I}^{\varepsilon(h),h})$ in the case where $h \ll \varepsilon$.

A. Appendix

A.1. Poincaré and Korn inequalities

Proposition A.1.1 (Poincaré inequality)**.** *Let $1 \leq p < \infty$.*

(1) ***(Poincaré inequality.)*** *Let Ω be an open set in $\mathbb{R}^n$ with finite width (that is Ω lies between two prallel hyperplanes). Then there exist a constant C (depending only on p, n and the distance between the two planes) such that*

$$\int_\Omega |u|^p \, \mathrm{d}x \leq C \int_\Omega |\nabla u|^p \, \mathrm{d}x \qquad \text{for all } u \in W_0^{1,p}(\Omega).$$

(2) ***(Poincaré-Wirtinger inequality.)*** *Let Ω be a bounded and connected subset of $\mathbb{R}^n$ with Lipschitz boundary. Then there exists a constant C (depending only on p, n and Ω)*

$$\int_\Omega |u - u_\Omega|^p \, \mathrm{d}x \leq C \int_\Omega |\nabla u|^p \, \mathrm{d}x \qquad \text{for all } u \in W^{1,p}(\Omega)$$

where $u_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x$.

(3) ***(Poincaré-Friedrichs inequality.)*** *Let Ω be a bounded and connected subset of $\mathbb{R}^n$ with Lipschitz boundary. Let $\gamma \subset \partial\Omega$ a measurable subset with $\mathcal{H}^{n-1}(\gamma) > 0$. Then there exists a constant C (depending only on p, n, Ω and γ) such that*

$$\int_\Omega |u|^p \, \mathrm{d}x \leq C \int_\Omega |\nabla u|^p \, \mathrm{d}x \qquad \text{for all } u \in W_{\gamma,0}^{1,p}(\Omega)$$

Proposition A.1.2 (Korn inequality)**.** *Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$ with Lipschitz boundary. Set*

$$\mathcal{R}(\Omega) := \left\{ r \in W^{1,2}(\Omega; \mathbb{R}^n) \, : \, r(x) = Ax + b \ \text{ with } \ A \in \mathbb{M}_{\text{skew}}(n), \, b \in \mathbb{R}^n \right\}.$$

Then

$$W^{1,2}(\Omega; \mathbb{R}^n) = \left\{ u \in L^2(\Omega; \mathbb{R}^n) \, : \, \operatorname{sym} \nabla u \in L^2(\Omega; \mathbb{R}^n) \right\}$$

and there exists a constant C (depending only on n and Ω) such that

$$\|u\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \leq C \left(\|u\|_{L^2(\Omega;\mathbb{R}^n)} + \|\nabla u\|_{L^2(\Omega;\mathbb{M}(n))} \right) \leq C^2 \|u\|_{W^{1,2}(\Omega;\mathbb{R}^n)}$$
$$\inf_{r \in \mathcal{R}(\Omega)} \|u - r\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \leq C_U \|\operatorname{sym} \nabla u\|_{L^2(\Omega;\mathbb{R}^n)}$$

for all $u \in W^{1,2}(\Omega; \mathbb{R}^n)$.

Proposition A.1.3 (Korn inequality for periodic functions). *Let $Y := [0,1)^n$. There exists a constant C (depending only on n) such that*

$$\|u\|_{W^{1,2}(Y;\mathbb{R}^n)} \le C\,\|\operatorname{sym}\nabla u\|_{L^2(Y;\mathbb{M}(n))}$$

for all $u \in W^{1,2}_{\mathrm{per},0}(Y;\mathbb{R}^n)$.

A.2. Attouch's diagonalization lemma

Lemma A.2.1 (see Lemma 1.15 & Corollary 1.16 in [Att84]). *Let $(a_{k,j})_{k,j\in\mathbb{N}}$ be a doubly indexed sequence of real numbers. Then there exists a subsequence $(k(j))_{j\in\mathbb{N}}$ increasing to $+\infty$ such that*

$$\limsup_{j\to\infty} a_{k(j),j} \le \limsup_{k\to\infty}\limsup_{j\to\infty} a_{j,k}.$$

If the right hand side is equal to zero and $a_{k,j}$ nonnegative for all $j,k\in\mathbb{N}$, then

$$\lim_{j\to\infty} a_{k(j),j} = 0$$

A.3. Notation

Scalars, vectors, matrices and tensors. We denote the set of real numbers by $\mathbb{R}$ and the set of the extended real numbers $\mathbb{R}\cup\{+\infty,-\infty\}$ by $\bar{\mathbb{R}}$. Throughout this work, $\{e_1,e_2,\dots,e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. We write $a_{\{i\}}$ to refer to the ith component of the vector a. For vectors $a,b\in\mathbb{R}^n$ we write

$$\langle a,\, b\rangle = a^{\mathrm{T}}b \quad \text{and} \quad |a| = \sqrt{\langle a,\, a\rangle}$$

to denote the Euclidean inner product and the Euclidean norm in $\mathbb{R}^n$.

We denote the set of matrices with n rows and m columns by $\mathbb{M}(n,m) = \mathbb{R}^n\otimes\mathbb{R}^m$. The corresponding canonical basis can be written by means of the tensor products $e_i\otimes\tilde{e}_j$, where $\{e_1,e_2,\dots,e_n\}$ and $\{\tilde{e}_1,\tilde{e}_2,\dots,\tilde{e}_m\}$ are the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A,B\in\mathbb{M}(n,m)$. We write

$$\langle A,\, B\rangle = \operatorname{tr} A^{\mathrm{T}}B \quad \text{and} \quad |A| = \sqrt{\langle A,\, A\rangle}$$

to denote the matrix inner product and matrix norm, respectively. We use the abbreviation $A_{\{i,j\}}$ to refer to the component in the ith row and jth column; thus,

$$A = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{\{i,j\}}\,(e_j\otimes e_j).$$

In the special case $n=m$ we set $\mathbb{M}(n) := \mathbb{M}(n,n)$ and denote the trace and determinant of a matrix $A\in\mathbb{M}(n)$ by $\operatorname{tr} A$ and $\det A$. Moreover, we set $\operatorname{sym} A = \frac{1}{2}(A^{\mathrm{T}}+A)$ and skew $A = A -$ sym A and denote by $\mathbb{M}_{\text{sym}}(n)$ and $\mathbb{M}_{\text{skew}}(n)$ the set of symmetric and skew symmetric matrices in $\mathbb{M}(n)$, respectively. As usual we denote the set of rotations by $SO(n)$ and write Id for the unit matrix.

We denote the set of symmetric fourth order tensors over $\mathbb{R}^n$ by $\mathbb{T}_{\text{sym}}(n)$ and identify $\mathbb{T}_{\text{sym}}(n)$ with the set of linear maps $\mathbb{L} : \mathbb{M}(n) \to \mathbb{M}(n)$ satisfying

$$\forall A, B \in \mathbb{M}(n) \; : \; \langle \mathbb{L}A, \, B \rangle = \langle \mathbb{L}B, \, A \rangle \, .$$

If needed, we equip $\mathbb{T}_{\text{sym}}(n)$ with the norm

$$|\mathbb{L}| = \inf_{\{ \, A \in \mathbb{M}(n) \, : \, |A| = 1 \, \}} |\mathbb{L}A| \, .$$

Function spaces. Let Ω be a measurable subset of $\mathbb{R}^n$ with $n \in \mathbb{N}$, let $p \in [1, \infty]$ and let $\mathbb{E}$ and $\mathbb{X}$ be a finite dimensional Euclidean space and a Banach space, respectively. Let $\{e_1, ..., e_d\}$ be a fixed orthonormal basis of $\mathbb{E}$. We use the standard notation for Lebesgue- and Sobolev spaces. In particular, $W^{1,p}(\Omega; \mathbb{E})$ stands for the space of measurable maps $u : \Omega \to \mathbb{E}$ in $L^p(\Omega; \mathbb{E})$ with weak partial derivatives of order one in $L^p(\Omega; \mathbb{E})$. For any $u \in W^{1,p}(\Omega; \mathbb{E})$ we set

$$\nabla u := (\, \partial_1 u \, | \, \cdots \, | \, \partial_n u \,)$$

where $\partial_l u \in L^p(\Omega; \mathbb{E})$ is the weak derivative in direction e_l; thus, $\nabla u(\cdot) \in L^p(\Omega; \mathbb{E}^n)$. In particular, in the case $\mathbb{E} = \mathbb{R}^n$ we identify $\nabla u(x)$ with a matrix in $\mathbb{M}(n)$, the columns of which are given by the partial derivatives $\partial_l u$, $l \in \{1, ..., n\}$.

The space $L^p(\Omega; \mathbb{X})$ is understood in the sense of Bochner. Moreover, we use the abbreviation to write $L^p(\Omega; SO(d))$ to refer to the set

$$\Big\{ \, R \in L^p(\Omega; \mathbb{M}(d)) \; : \; R(x) \in SO(d) \text{ for a.e. } x \in \Omega \, \Big\}.$$

We set $W^{k,p}(\Omega; SO(d)) := L^p(\Omega; SO(d)) \cap W^{k,p}(\Omega; \mathbb{M}(d))$.

In this contribution, we frequently encounter **function spaces of periodic functions**. Let $Y := [0,1)^n$ denote the *reference cell of periodicity*. We say a continuous map $u : \mathbb{R}^n \to \mathbb{E}$ is Y*-periodic*, if $u(x + e) = u(x)$ for all $x \in \mathbb{R}^n$ and $e \in \mathbb{Z}^n$. We denote the set of Y-periodic functions with values in a metric space X by $C_{\text{per}}(Y; X)$. Likewise, we set

$$L^p_{\text{per}}(Y; \mathbb{X}) := \big\{ \, u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{X}) \; : \; u(x + e) = u(x) \text{ for a.e. } x \in \mathbb{R}^n \text{ and all } e \in \mathbb{Z}^n \, \big\}$$
$$W^{k,p}_{\text{per}}(Y; \mathbb{E}) := \big\{ \, u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n; \mathbb{E}) \; : \; u \in L^p_{\text{per}}(Y; \mathbb{E}) \, \big\}.$$

These spaces equipped with the norms of $L^p(Y; \mathbb{X})$ and $W^{k,p}(Y; \mathbb{E})$, respectively, are Banach spaces. Note that by definition, Y-periodic maps are defined for all $x \in \mathbb{R}^n$.

For the subspace of functions with **vanishing mean value** we use the abbreviations

$$L_0^p(\Omega;\mathbb{X}) := \left\{ u \in L^p(\Omega;\mathbb{X}) \,:\, \int_\Omega u \,\mathrm{d}x \right\}$$
$$W^{1,p}_{\mathrm{per},0}(Y;\mathbb{E}) := W^{1,p}_{\mathrm{per}}(Y;\mathbb{E}) \cap L_0^p(Y;\mathbb{E}).$$

For a Lipschitz domain Ω and a measurable subset Γ of $\partial\Omega$ we introduce the space $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$ as the space of functions $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ with $u = 0$ on Γ in the sense of trace. Usually, we suppose that Γ has positive $n-1$-dimensional Hausdorff measure and is sufficiently regular to guarantee that

$$W^{1,\infty}(\Omega;\mathbb{R}^n) \cap W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$$

is strongly dense in $W^{1,2}_{\Gamma,0}(\Omega;\mathbb{R}^n)$.

List of mathematical symbols

$\mathbb{R}$	set of real numbers
$\bar{\mathbb{R}}$	set of extended real numbers: $\mathbb{R} \cup \{-\infty, +\infty\}$
$\mathbb{R}^n$	set of n-dimensional vectors
$\mathbb{M}(n)$	set of $n \times n$ matrices
$\mathbb{M}_{\mathrm{sym}}(n)$	set of symmetric $n \times n$ matrices
$\mathbb{M}_{\mathrm{skew}}(n)$	set of skew symmetric $n \times n$ matrices
$\mathbb{T}_{\mathrm{sym}}(n)$	set of symmetric fourth order tensors over $\mathbb{R}$
$\mathbb{E}$	a finite dimensional Euclidean space
$e_1, e_2, \cdots$	canonical basis of $\mathbb{R}^n$ or $\mathbb{E}$
$\langle a,\, b \rangle$	inner product of a and b in $\mathbb{E}$ or $\mathbb{R}^{n \times m}$
$a \otimes b$	tensor or dyadic product of a, b in $\mathbb{E}$ or $\mathbb{R}^n$
$\lvert \cdot \rvert$	Euclidean norm in $\mathbb{E}$ and $\mathbb{R}^{n \times m}$
A^{T}	transposition of A
$\mathrm{tr}\, A$	trace of A
$\lvert A \rvert$	$= \sqrt{\mathrm{tr}\, A^{\mathrm{T}} A}$
$A_{\{i,j\}}$	entry in row i and column j of the matrix A
$\mathbb{X}$	Banach space
$\lVert \cdot \rVert_{\mathbb{X}}$	norm in the Banach space $\mathbb{X}$
$\xrightharpoonup{2}$	weak two-scale convergence, see Definition 2.1.1
$\xrightarrow{2}$	strong two-scale convergence, see Definition 2.1.1
$\xrightharpoonup{2\star}$	weak star two-scale convergence, see Definition 2.1.1

$\xrightarrow{\Gamma}$	Γ-convergence, see Definition 4.2.1
$\mathcal{K}_{\mathcal{R}}$	rod strain
$\mathcal{R}$	Cosserat frame of a rod configuration
$\mathrm{II}_{(u)}$	second fundamental form of a bending deformation $u \in \mathcal{A}$
$\mathcal{A}$	space of bending deformations for plates
$n_{(u)}$	normal field of a bending deformation u (see page 105)
$t_{(u)}$	tangent field of a bending deformation u (see page 105)
$\boldsymbol{\kappa}_{(u)}$	signed curvature of a bending deformation u (see page 105)
$R_{(u)}$	$= (t_{(u)} \vert n_{(u)})$ frame associated to a bending deformation u (see page 105)
$\mathcal{R}(\alpha)$	clock-wise rotation in $\mathbb{R}^2$ by angle α
$\mathrm{Rot}(a, \alpha)$	rotation in $SO(3)$ about axis a by angle α (see page 177)
$\mathrm{Var}_p\, u$	p-variation of a piecewise constant map u (see page 26)
$\nabla_{m,h}\, u$	scaled gradient, see Definition 6.3.1
$\widetilde{\nabla}_{m,\gamma}\psi$	scaled gradient, see Definition 6.3.1
$C(U;X)$	usual space of continuous functions from U to X
$C_c(U;X)$	usual space of continuous functions from U to X with compact support in U
$C^\infty(U;X)$	usual space of smooth functions from U to a metric space X
$\mathrm{supp}\, u$	support of u
$C_c^\infty(U;X)$	$= C^\infty(U;X) \cap C_c(U;X)$
$C_0(U;X)$	$= \overline{C_c(U;X)}$ with respect to the supremum norm
$C_0^\infty(U;X)$	$= C_0(U;X) \cap C^\infty(U;X)$
$C_{\mathrm{per}}(Y;X)$	space of $Y := [0,1)^d$-periodic maps $u \in C(\mathbb{R}^d;X)$
$C_{\mathrm{per}}^\infty(Y;X)$	$= C_{\mathrm{per}}(Y;X) \cap C^\infty(Y;X)$
$L^p(\Omega)$	usual Lebesgue space of scalar valued functions

$L^p(\Omega;\mathbb{X})$	usual Lebesgue space of maps with values in the Banach space $\mathbb{X}$ in the sense of Bochner
$L^p_0(\Omega;\mathbb{X})$	set of maps $u\in L^p(\Omega;\mathbb{X})$ with $\int_\Omega u\,\mathrm{d}x=0$
$L^p_{\mathrm{per}}(Y;\mathbb{X})$	set of $Y:=[0,1)^d$-periodic maps $u\in L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{X})$
$L^p_{\mathrm{per},0}(Y;\mathbb{X})$	set of $Y:=[0,1)^d$-periodic maps $u\in L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{X})$ with $\int_Y u\,\mathrm{d}y=0$
$L^p(\Omega;SO(d))$	set of maps $R\in L^p(\Omega;\mathbb{M}(d))$ with $R(x)\in SO(d)$ for a.e. $x\in\Omega$
$W^{k,p}(\Omega)$	usual Sobolev space of scalar valued functions
$W^{k,p}(\Omega;\mathbb{E})$	usual Sobolev space of maps with values in the Euclidean space $\mathbb{E}$
$W^{1,p}_0(\Omega;\mathbb{E})$	$=\overline{C^1_0(\Omega;\mathbb{E})}$ closure with respect to the norm in $W^{1,p}(\Omega;\mathbb{E})$
$W^{1,p}_{\Gamma,0}(\Omega;\mathbb{E})$	space of maps in $W^{1,2}(\Omega;\mathbb{E})$ with $u=0$ on Γ in the sense of trace.
$W^{k,p}_{\mathrm{per}}(Y;\mathbb{E})$	$=W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{E})\cap L^p_{\mathrm{per}}(Y;\mathbb{E})$
$W^{k,p}_{\mathrm{per},0}(Y;\mathbb{E})$	$=W^{k,p}_{\mathrm{per}}(Y;\mathbb{E})\cap L^p_0(Y;\mathbb{E})$
$W^{k,p}(\Omega;SO(d))$	$=L^p(\Omega;SO(d))\cap W^{k,p}(\Omega;\mathbb{M}(d))$
$W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^n)$	$=\{\,u\in W^{2,2}(\omega;\mathbb{R}^n)\,:\,\lvert\partial_1 u(x_1))\rvert=1$ for a.e. $x_1\,\}$ space of bending deformations (see page 105)
$W^{1,2}_{Y\text{-per}}(S\times Y;\mathbb{E})$	see Definition 6.3.1
$W^{2,2}_{\mathrm{iso}}(\omega;\mathbb{R}^3)$	bending deformation from ω to $\mathbb{R}^3$
$\mathfrak{Q}(Y;m)$	set of Y-periodic, quadratic integrands from $Y\times\mathbb{M}(m)$ to $\mathbb{R}$
$\mathcal{T}_\varepsilon$	unfolding operator, see Definition 2.1.1
$\mathcal{T}^m_\varepsilon$	unfolding operator for in-plane oscillations, see Definition 6.2.1
$\lfloor x\rfloor$	integer part of $x\in\mathbb{R}^n$, see page 14
Ω	subset of $\mathbb{R}^n$, $n\in\mathbb{N}$
Y	reference cell of periodicity $:=[0,1)^d$, $1\le d\le n$
$\mathcal{H}^n(A)$	n-dimensional Hausdorff measure of the set A

Bibliography

[ABP91] Emilio Acerbi, Giuseppe Buttazzo, and Danilo Percivale, *A variational definition of the strain energy for an elastic string*, Journal of Elasticity **25** (1991), 137–148.

[AJDH90] Todd Arbogast, Jr. Jim Douglas, and Ulrich Hornung, *Derivation of the double porosity model of single phase flow via homogenization theory*, SIAM Journal on Mathematical Analysis **21** (1990), no. 4, 823–836.

[All92] Grégoire Allaire, *Homogenization and two-scale convergence*, SIAM Journal on Mathematical Analysis **23** (1992), no. 6, 1482–1518.

[Ant05] Stuart S. Antman, *Nonlinear problems of elasticity*, second ed., Applied Mathematical Sciences, vol. 107, Springer, New York, 2005.

[App88] Jürgen Appell, *The superposition operator in function spaces - A survey*, Expositiones Mathematicae **6** (1988), 209.

[AT84] Rohan Abeyaratne and Nicolas Triantafyllidis, *An Investigation of Localization in a Porous Elastic Material Using Homogenization Theory*, Journal of Applied Mechanics **51** (1984), no. 3, 481–486.

[Att84] Hedy Attouch, *Variational convergence for functions and operators*, Pitman Advanced Pub. Program, 1984.

[BB06] Jean-Francois Babadjian and Margarida Baía, *3D-2D analysis of a thin film with periodic microstructure.*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics. **136** (2006), no. 2, 223–243.

[BD98] Andrea Braides and Anneliese Defranceschi, *Homogenization of multiple integrals*, Oxford Lecture Series in Mathematics and its Applications, vol. 12, The Clarendon Press Oxford University Press, New York, 1998.

[BF07] Margarida Baía and Irene Fonseca, *The limit behavior of a family of variational multiscale problems*, Indiana University Mathematics Journal **56** (2007), no. 1, 1–50.

[BFF00] Anrea Braides, Irene Fonseca, and Gilles A. Francfort, *3D-2D asymptotic analysis for inhomogeneous thin films*, Indiana University Mathematics Journal **49** (2000), no. 4, 1367–1404.

[BLM96] Alain Bourgeat, Stephan Luckhaus, and Andro Mikelić, *Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow*, SIAM Journal on Mathematical Analysis **27** (1996), no. 6, 1520–1543.

[BLP78] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou, *Asymptotic analysis for periodic structures*, Studies in Mathematics and its Applications, vol. 5, North-Holland Publishing Co., Amsterdam, 1978.

[Bra85] Andrea Braides, *Homogenization of some almost periodic coercive functional*, Rendiconti. Accademia Nazionale delle Scienze detta dei XL. Serie V. Memorie di Matematica. Parte I **9** (1985), no. 1, 313–321.

[CC05] Philippe G. Ciarlet and Patrick Ciarlet, Jr., *Another approach to linearized elasticity and a new proof of Korn's inequality*, Mathematical Models & Methods in Applied Sciences **15** (2005), no. 2, 259–271.

[CDD^{+}03] Sergio Conti, Antonio DeSimone, Georg Dolzmann, Stefan Müller, and Felix Otto, *Multiscale modeling of materials—the role of analysis*, Trends in nonlinear analysis, Springer, Berlin, 2003.

[CDDA06] Doina Cioranescu, Alain Damlamian, and Riccardo De Arcangelis, *Homogenization of quasiconvex integrals via the periodic unfolding method*, SIAM Journal on Mathematical Analysis **37** (2006), no. 5, 1435–1453 (electronic).

[CDG02] Doina Cioranescu, Alain Damlamian, and Georges Griso, *Periodic unfolding and homogenization*, Comptes Rendus Mathematique **335** (2002), no. 1, 99–104.

[CDG08] Doina Cioranescu, Alain Damlamian, and Georges Griso, *The periodic unfolding method in homogenization*, SIAM Journal on Mathematical Analysis **40** (2008), no. 4, 1585–1620.

[Con03] Sergio Conti, *Low-energy deformations of thin elastic sheets: isometric embeddings and branching patterns*, Habilitation thesis, Universität Leipzig, 2003.

[Dac08] Bernard Dacorogna, *Direct methods in the calculus of variations*, second ed., Applied Mathematical Sciences, vol. 78, Springer, New York, 2008.

[Dam05] Alain Damlamian, *An elementary introduction to periodic unfolding*, GAKUTO International Series. Mathematical Sciences and Applications. Gakkōtosho, Tokyo. **24** (2005), 119–136.

[DGDM83] Ennio. De Giorgi and Gianni Dal Maso, *Gamma-convergence and calculus of variations (in Mathematical Theories of Optimization)*, Lecture Notes in Mathematics, vol. 979, pp. 121–143, Berlin: Springer, 1983.

[DGF75] Ennio De Giorgi and Tullio Franzoni, *Su un tipo di convergenza variazionale*, Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali **58** (1975), no. 6, 842–850.

[DM93] Gianni Dal Maso, *An introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Applications, 8, Birkhäuser Boston Inc., Boston, MA, 1993.

[DMNP02] Gianni Dal Maso, Matteo Negri, and Danilo Percivale, *Linearized elasticity as* Γ*-limit of finite elasticity*, Set-Valued Analysis **10** (2002), no. 2-3, 165–183.

[FJM02] Gero Friesecke, Richard D. James, and Stefan Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity*, Communications on Pure and Applied Mathematics **55** (2002), no. 11, 1461–1506.

[FJM06] ———, *A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence*, Archive for Rational Mechanics and Analysis **180** (2006), no. 2, 183–236.

[FL07] Irene Fonseca and Giovanni Leoni, *Modern methods in the calculus of variations.* L^p *spaces.*, Springer Monographs in Mathematics. New York, NY: Springer. xiv, 2007.

[FM86] Gilles A. Francfort and François Murat, *Homogenization and optimal bounds in linear elasticity.*, Archive for Rational Mechanics and Analysis **94** (1986), 307–334.

[FMT09] Gilles A. Francfort, François Murat, and Luc Tartar, *Homogenization of monotone operators in divergence form with* x*-dependent multivalued graphs.*, Annali di Matematica Pura ed Applicata **4** (2009), 631–652.

[GMT93] Giuseppe Geymonat, Stefan Müller, and Nicolas Triantafyllidis, *Homogenization of non-linearly elastic materials, microscopic bifurcation and macroscopic loss of rank-one convexity*, Archive for Rational Mechanics and Analysis **122** (1993), no. 3, 231–290.

[GR79] Vivette Girault and Pierre-Arnaud Raviart, *Finite element approximation of the navier-stokes equations*, Lecture Notes in Mathematics, vol. 749, Springer-Verlag, Berlin, 1979.

[Kir01] Bernd Kirchheim, *Geometry and Rigidity of Microstructures*, Habilitation thesis, Universität Leipzig, 2001.

[LDR95] Hervé Le Dret and Annie Raoult, *The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity*, Journal de Mathématiques Pures et Appliquées. Neuvième Série **74** (1995), no. 6, 549–578.

[LNW02] Dag Lukkassen, Gabriel Nguetseng, and Peter Wall, *Two-scale convergence*, International Journal of Pure and Applied Mathematics **2** (2002), no. 1, 35–86.

[Mar78] Paolo Marcellini, *Periodic solutions and homogenization of nonlinear variational problems*, Annali di Matematica Pura ed Applicata. Serie Quarta **117** (1978), 139–152.

[MM03] Maria Giovanna Mora and Stefan Müller, *Derivation of the nonlinear bending-torsion theory for inextensible rods by* Γ*-convergence*, Calculus of Variations and Partial Differential Equations **18** (2003), no. 3, 287–305.

[MN10] Stefan Müller and Stefan Neukamm, *On the commutability of homogenization and linearization in finite elasticity.*, Preprint, 2010.

[MT07] Alexander Mielke and Aida M. Timofte, *Two-scale homogenization for evolutionary variational inequalities via the energetic formulation*, SIAM Journal on Mathematical Analysis **39** (2007), no. 2, 642–668 (electronic).

[Mül] Stefan Müller, *Personal communication.*

[Mül87] ______, *Homogenization of nonconvex integral functionals and cellular elastic materials*, Archive for Rational Mechanics and Analysis **99** (1987), no. 3, 189–212.

[Ngu89] Gabriel Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM Journal on Mathematical Analysis **20** (1989), no. 3, 608–623.

[NS10] Stefan Neukamm and Philipp E. Stelzig, *On the interplay of two-scale convergence and translations*, to appear in Asymptotic Analysis, 2010.

[OSI84] Olga Arsen'evna Olejnik, Aleksey Stanislavovich Shamaev, and Grigorii Andronikovich Iosif'yan, *Problèmes d'homogénéisation pour le système de l'élasticité linéaire à coefficients oscillant non-uniformément.*, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics **298** (1984), 273–276.

[Pak04] Mohammad Reza Pakzad, *On the Sobolev space of isometric immersions*, Journal of Differential Geometry **66** (2004), no. 1, 47–69.

[Per99] Danilo Percivale, *Thin elastic beams: the variational approach to St. Venant's problem*, Asymptotic Analysis **20** (1999), no. 1, 39–59.

[Sch07] Bernd Schmidt, *Plate theory for stressed heterogeneous multilayers of finite bending energy*, Journal de Mathématiques Pures et Appliqués **88** (2007), no. 1, 107.

[Shu00] Y.C. Shu, *Heterogeneous thin films of martensitic materials.*, Arch. Ration. Mech. Anal. **153** (2000), no. 1, 39–90.

[SP80] Enrique Sánchez-Palencia, *Nonhomogeneous media and vibration theory*, Lecture Notes in Physics, vol. 127, Springer-Verlag, Berlin, 1980.

[Tar77] Luc Tartar, *Cours Peccot au Collège de France*, Paris, 1977.

[Tar09] ———, *The general theory of homogenization. A personalized introduction.*, Lecture Notes of the Unione Matematica Italiana 7. Berlin: Springer. xvii, 2009.

[TM85] Nicolas Triantafyllidis and B.N. Maker, *On the comparison between microscopic and macroscopic instability mechanisms in a class of fiber-reinforced composites*, Journal of Applied Mechanics **52** (1985), 794–800.

[Vis06] Augustu Visintin, *Towards a two-scale calculus*, Control, Optimisation and Calculus of Variations **12** (2006), no. 3, 371.

[Vis07] ———, *Two-scale convergence of some integral functionals*, Calculus of Variations and Partial Differential Equations **29** (2007), no. 2, 239.

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