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First Passage Events and Multivariate Regular Variation for Dependent Lévy Processes with Applications in Insurance

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Zusammenfassung

Diese Arbeit untersucht das Ereignis der Erstüberschreitung einer konstanten Barriere durch eine Summe von abhängigen Komponenten eines allgemeinen multivariaten Lévyprozeßes mittels eines Sprungs. Für $d = 2$ charakterisieren wir dieses Ereignis mit der gemeinsamen Verteilung von fünf Größen: die Zeitspanne zwischen Erstüberschreitung und dem letzten Maximum, die Zeit des letzten Maximums, der Überschuß, der Unterschuß und der Unterschuß des letzten Maximums. Die Abhängigkeit zwischen den Sprungkomponenten eines multivariaten Lévyprozeßes wird dabei mit einem sogenannten Pareto-Lévymaß modelliert, das zum Erstenmal für allgemeine Lévyprozesse betrachtet wird. Die Beziehung zwischen einem Lévymaß und seinem Pareto-Lévymaß wird dabei detailliert untersucht, wobei explizite Beispiele mit graphischen Darstellungen gegeben werden. Desweiteren werden Bedingungen an die eindimensionalen Rand-Lévymaße und das Pareto-Lévymaß formuliert, so daß das multivariate Lévymaß regulär variierend ist. Schließlich werden die Resultate auf einen spektral positiven Versicherungsrisikoprozeß angewendet.

Abstract

This thesis deals with the first upwards passage event of the sum of dependent components of a general multivariate Lévy process when a constant barrier is passed by a jump. For $d = 2$ we characterize this event by the joint distribution of five quantities: the time relative to the time of the previous maximum, the time of the previous maximum, the overshoot, the undershoot and the undershoot of the previous maximum. The dependence between the jump components of a multivariate Lévy process is modelled by a so-called Pareto Lévy measure which is considered the first time for general Lévy process. The relationship between a Lévy measure and its Pareto Lévy measure is investigated in detail where explicit examples with graphical representations are given. Furthermore, we prove conditions on the one-dimensional Lévy measures and the Pareto Lévy measure such that the multivariate Lévy measure is regularly varying. Finally, the results are applied to a spectrally positive insurance risk process.

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πάντων χρημάτων μέτρον ἐστὶν ἄνθρωπος,
τῶν μὲν ὄντων ὡς ἔστιν,
τῶν δὲ οὐκ ὄντων ὡς οὐκ ἔστιν.

Protagoras

Contents

Introduction

First passage events

The analysis of first passage events deals with the probability that a stochastic process excesses a given barrier and, in more detail, with the question how this passage happens for the first time. This subject has applications in a variety of areas as e. g. queuing theory, cf. [6], or option pricing, cf. [3]. In particular, risk theory encourages the interest in this theory during the last years since the first passage event of a risk process has the meaning of the ruin of an insurance company or bank and ruin probability is often used as a risk measure. For a detailed representation of the classical model of risk theory by Cramér and Lundberg and results we refer to the excellent monographs [7, 25].

Historically, first passage events have been studied for random walks where a rich mathematical theory exists. Considering a random walk Z with

$$Z_0 := 0 \quad and \quad Z_n := \sum_{i=1}^{d} \xi_i, \; n \in \mathbb{N},$$

for independent and identically distributed (i. i. d) random variables (r. v. s) $(\xi_i)_{i\in\mathbb{N}}$, the successive maxima of Z and the corresponding times form a bivariate renewal process, the so-called *ascending ladder process*. Similarly, the *descending ladder process* which is defined as the ascending ladder process of the dual random walk $-Z$, corresponds with the successive minima of Z. Applying Wiener-Hopf techniques to integral equations fundamental fluctuation identities for random walks are proven which yield the so-called *Wiener-Hopf factorization* where we refer to [12, 31, 53, 59, 60] and the excellent monographs [27, 61]. This result is the basis of the ladder theory since it relates the distributions of the ascending and descending ladder processes to that of the underlying random walk Z and it is the fundament of first passage results for random walks. The results for random walks can easily be extended to compound Poisson processes (CPPes) by considering CPPes at their jump times and exploiting the embedded random walk structure. Also first passage results for more general Lévy processes are often based on results for random walks since they are proven by approximation in means of a discrete-time skeleton, cf. [28, 54]. Another more elegant approach for investigating first passage events for general Lévy processes is given in [30] where the sample paths of the continuous-time Lévy processes are emphasized by using Poisson point processes of excursions from the maximum, cf. [39]. The main aspect hereby is that the fluctuations of a Lévy process X are investigated by introducing an exponentially distributed random time e_q, independent of X. Then the path of X on $[0, e_q]$ can be split at the maximum, i. e. the path of X can be decomposed on $[0, e_q]$ into two independent parts, the path before and after the time when X reaches its maximum on $[0, e_q]$. For a well-explained description of this fact in terms of excursions we refer to [49], Sections 6.3 and 6.4. In this way the Wiener-Hopf factorization for general Lévy processes can proven nicely. As for random walks this result relates the distributions of the ladder processes to that of the underlying Lévy process. The ladder processes for Lévy processes also correspond to the maxima and minima of the process as for random walks, but their construction is a more complex task where we refer to the monographs [13], Section IV, and [49], Section 6.2.

Applying such a decomposition method yields the so-called *quintuple law* [22], Theorem 3, which describes the first passage of a general Lévy process. This results characterizes the first upwards passage event over a constant barrier, caused by a jump, detailed with the common distribution of five quantities: the time of first passage relative to the time of the previous maximum, the time of the previous maximum, the overshoot, the undershoot and the undershoot of the previous maximum. Employing the Wiener-Hopf factorization for Lévy processes, the path of the Lévy process which causes the first passage is decomposed in three independent parts: the path before the last maximum before the passage, the path between the last maximum and the first passage and the jump which causes the passage. The first two quantities are given in terms of the potential measure of the ladder processes of the Lévy process and the third part is given by the jump measure.

In this thesis we especially investigate first passage events for Lévy processes which are the sum of dependent components of a general multivariate Lévy process. Therefore, we extend the classical quintuple law with regarding to dependence, cf. Theorem 3.2.4. This approach is motivated by recent insurance and operational risk models where the total risk process of an insurance company or bank is the sum of a multivariate risk portfolio where the dependence between different business lines and risk types is crucial. As considering the sum of Lévy processes, additionally to classical first passage events, the questions arise which components cause the first passage and how dependence affects this event. Further, as in the classical risk theory, we are interested in the asymptotic behaviour of the ruin probability which is also affected by the dependence between the components, cf. [37, 47] and, for the multivariate case, [17, 18, 38].

In the one-dimensional case we have two approaches to investigate first passage events: approximation by a discrete-time skeleton or using the ladder theory for Lévy processes. With regarding to dependence only the second approach is appropriate as I shall briefly explain. Following the classical approach one reduces the sum of CPPes to the sum of random walks and the dependence between the components is modelled by means of a *distributional copula* coupling the distributions of the single random walks, cf. [40, 52]. With regarding to dependence this is a rather crude method since the dependence between the jump times of the components is an important fact of the dependence structure of a Lévy process and by construction the random walks of the components almost surely (a. s.) jump together. Further, as the original time structure gets lost, modelling the dependence by a distributional copula does not distinguish single and common jumps which adulterates the original dependence structure of the jump sizes. In particular, applying this approach one can not analyse which component caused the passage. More sophisticated, we can model the sum of CPPes as the sum of random walks allowing that components may have jumps of size zero to keep the time structure. Then we have to model the dependence between the jump times, between the sizes of single and common jumps, separately, and between the jump sizes and the jump times. We see that this is a rather extensive method even for CPPes as we have to describe the whole original dependence structure by means of a distributional which is hardly possible for general Lévy processes. Consequently, in order to investigate the fluctuations of the sum of general Lévy processes with regarding to dependence one has to apply the theory of ladder processes.

Hence, for our analysis we proceed as in [22] using ascending and descending ladder processes. Further, to identify the ruin causing components we employ a decomposition of the sample paths of the Lévy process according to its jump behaviour which is not trivial as the Lévy process may have a. s. sample paths of unbounded variation. In this way, we obtain our quintuple law Theorem 3.2.4 which extends the quintuple law in [22] regarding dependence. Although it seems to be just a theoretical result characterizing the first passage event in terms of the potential measures of the ladder processes, we give two situation where these quantities can be determined concretely in Section 3.3. Moreover, we conclude from our quintuple law an asymptotic result for the ruin event of a spectrally positive insurance risk process in Section 3.5.

Modelling the dependence for Lévy processes

As already mentioned above, the dependence between the components of a d-dimensional r. v. can be modelled by a so-called *distributional copula* $C_D : [0,1]^d \to [0,1]$ due to Sklar's Theorem, [52], Theorem 2.3.3. A distributional copula of a r. v. $(X^1, \dots, X^d)$ defines a distribution whose one-dimensional margins are the uniform distributions on $[0,1]$. If the components X^i are continuous then a copula is the distribution function (d. f.) of the r. v. $(F_1(X^1), \dots, F_d(X^d))$, corresponding to a transformation to uniform margins. The interest in this modelling approach increased rapidly during the last years since dependence can be modelled independently from the marginal distributions and due to the uniform margins calculations are quite handy, cf. [1, 2, 4, 45]. Nevertheless, in [51] copulas have been criticized since a transformation to uniform margins is not reasonable in general, especially for considering extremes. The copula approach was advanced in [48] to a Pareto measure which is a distribution whose one-dimensional margins are standard Pareto distributions. A Pareto measure is related to the distribution defined by a copula by componentwise inversion and has due to the Pareto margins a better probabilistic interpretation for limit theory and heavy-tail analysis.

For Lévy processes with finite Lévy measures distributional copulas can be applied to model the dependence between the jumps sizes, between the jump times and between the jump times and the jump sizes, cf. [2]. But as mentioned above, by modelling just one of these dependence structures, information about the original the dependence may get lost and modelling all dependence structures may be very extensive. Moreover, this approach only works for finite Lévy measures.

The first concept to model the dependence structure of a general Lévy process is defined in [42] by the notion of a Lévy copula which was already considered for particular Lévy processes in [62, 63, 20]. We shall briefly explain the advantage of this approach. The distribution of a general d-dimensional Lévy process $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\geq 0}$ is uniquely determined by its characteristic triplet $(\boldsymbol{\gamma}, A, \Pi)$ and thus the dependence structure of $\boldsymbol{X}$ is characterized by the dependence structure of the r. v. $\boldsymbol{X}_t$ for some fixed $t > 0$. So in principle, one can model the dependence between the components of $\boldsymbol{X}$ by the distributional copula $C_{D,t}$ of the r. v. $\boldsymbol{X}_t$. As discussed in [42] this approach has two critical points:

- For given infinitely divisible one-dimensional laws the choice of copulas which yield an infinitely divisible d-dimensional law depends on the margins and can not be clarified in general.
- The distributional copula $C_{D,t}$ of $\boldsymbol{X}_t$ depends on t and for $s \neq t$ the copula $C_{D,s}$ of $\boldsymbol{X}_s$ can in general not be calculated only from $C_{D,t}$ since one needs also the marginal distributions at time s and t. Further, if $C_{D,s}$ can be calculated from $C_{D,t}$ then only with large numerical effort.

Consequently, it is more useful to model the dependence structure time-independently using the characteristic triplet. According to the Lévy-Itô decomposition, see [58], Theorem 19.2, the Gaussian and the jump part of $\boldsymbol{X}$ are independent processes where the dependence structure of the Gaussian part is entirely determined by the Gaussian covariance matrix A. Therefore, the dependence structure of the jump part is uniquely determined by the Lévy measure. In order to formulate a version of Sklar's Theorem for Lévy measures, one has to pay attention to the fact that a general Lévy measure may a singularity at the origin. Therefore, the analog notion to a d. f. or a right tail of a d. f. for Lévy measures may consider the Lévy measure only on sets that are always bounded away from zero. Thus, in [42] the notion of a tail integral of a Lévy measure is defined on $(\mathbb{R}\setminus\{0\})^d$. Analogously to distributional copulas they define Lévy copulas $\widehat{C} : (-\infty, \infty]^d \to (-\infty, \infty]$ as function that define a measure with Lebesgue margins. Thereby, in [42], Theorem 3.6, they formulate a version of Sklar's Theorem for Lévy measures that describes the relation between a Lévy measure, its margins and the Lévy copula in terms of the marginal tail integrals. The big advantage of Lévy copulas is that by modelling the Lévy measure Π the whole dependence structure of the jump part is modelled, contrary to the copula approach above, cf. Remark 3.4.5. On the other hand, due to the Lebesgue margins the measure that models the dependence of Lévy measure is not a Lévy measure itself. In [9] they propose to apply a componentwise inversion to the measure of a Lévy copula to obtain a Lévy measure whose margins are the Lévy measures of 1-stable Lévy processes. This Lévy measure is also proposed in [48] and since it parallels the notion a Pareto measure for Lévy measure they called it *Pareto Lévy measure*. Both papers are restricted to spectrally positive Lévy measures. Thus in this thesis Pareto Lévy measures are considered the first time for general Lévy processes and so we shall explain and visualize them intensively in Chapter 1. In opposite to Lévy copulas, modelling the dependence of a Lévy measure with a Lévy measure is a self-contained approach. A further advantage, especially for higher dimensions, is that calculation of the marginals a Pareto Lévy measure is easier then the calculation of the corresponding Lévy copula margin. Moreover, since the margins of a Pareto Lévy measure are the Lévy measures of an 1-stable Lévy process, Pareto Lévy measures can be better applied in the theory of multivariate regular variation considered in Section 2 than Lévy copulas. For $d = 2$ one does not have to calculate margins due to the standardized one-dimensional margins and Lévy copulas are notationally easier than Pareto Lévy measures. Therefore, we formulate our results in Section 3.4 in terms of Lévy copulas.

Regular variation for Lévy processes

In a series of papers Hult and Lindskog [33, 34, 35, 36] define and investigate regular variation of measures and additive processes which apply in particular to Lévy measures and Lévy processes. Their concept of regular variation of a stochastic process with càdlàg sample paths is for a Lévy process $\boldsymbol{X}$ equivalent to regular variation of the random vector $\boldsymbol{X}_1$ and its Lévy measure; cf. [36], Lemma 2.1. Since regular variation of a random vector $\boldsymbol{X}_1$ is well understood, cf. [55, 56], it seems that all such results can be translated to the corresponding Lévy measure. Of course, this is in principle true, but we argue that a new sight of the dynamic of the Lévy process $\boldsymbol{X}$ can be gained by investigating regular variation of the Lévy measure itself.

In this thesis we consider regular variation with regarding to dependence and investigate the relation between the regular variation of a Lévy measure and the regular variation of its Pareto Lévy measure. Further, regularly varying Lévy measure are of special interest in the context of risk theory since for heavy-tailed claims we have the so-called *non-Cramér case* where the ruin probability does not decay exponentially fast to 0.

A general outline

This thesis is divided into three chapters which are based on the papers [24, 23]. Throughout we assume that all stochastic quantities in this thesis are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Each chapter starts with an introduction including an outline. In the following we present an overview to the thesis, summarized from the introduction of each chapter.

Chapter 1. We present in the first chapter the notion of a Pareto Lévy measure and the main theorem for modelling the dependence between the jumps of a multivariate Lévy process. Since Pareto Lévy measures are strongly related to Lévy copulas we give in Section 1.1 a detailed presentation of the notion of a Lévy copula and of Sklar's Theorem for general Lévy processes which we use in this thesis. Since the relation between Pareto Lévy measures and Lévy copulas corresponds to the relation between copulas and Pareto measures for random vectors and due to the use of copulas in Section 3.1, we briefly summarize these approaches to model the dependence between r. v. s in Section 1.1.1. In Section 1.1.2 we define the concept of Pareto Lévy measures for general Lévy processes and prove the basic results for dependence modelling. Furthermore, we describe two approaches for graphical representation of the dependence structure in Section 1.2 and apply them to the examples in Section 1.3.

Chapter 2. We investigate regular variation of multivariate Lévy processes with respect to the dependence structure modelled by a Pareto Lévy measure. In Section 2.1 we formulate conditions on the one-dimensional marginal Lévy measure and the Pareto Lévy measure such that the multivariate Lévy measure is regularly varying and vice versa. In Section 2.2 we apply this result to the four examples given in Section 1.3.

Chapter 3. We investigate the first upwards passage event for the sum of a bivariate Lévy process and prove fluctuation identities under the aspect of dependence. For motivation and better understanding of the decomposition for our quintuple law, we first formulate the quintuple law for the sum of a bivariate random walk in

Section 3.1. The general quintuple law for the sum of a bivariate Lévy process is proven in Section 3.2. In Section 3.3 we consider two situations where all quantities of the quintuple law can be identified concretely. We calculate explicit quantities in Section 3.4 for different dependence structures which are modelled by a Lévy copula. In Section 3.5 we apply our results to insurance risk theory and obtain a detailed description of the ruin event regarding dependence.

Remarks on notation

The Borel-σ-algebra of a topological space $\mathbb{T}$ is denoted by $\mathcal{B}(\mathbb{T})$. For a set $B \in \mathcal{B}(\mathbb{T})$, let B°, $\overline{B}$ and $\partial B = \overline{B} \setminus B^\circ$ be the interior, the closure and the boundary of B, respectively.

For $a, b \in \mathbb{R}$ we write $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. For vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$, we mean by $\boldsymbol{a} \leq \boldsymbol{b}$ that the inequation holds componentwise and $(\boldsymbol{a}, \boldsymbol{b}] := (a_1, b_1] \times \cdots \times (a_d, b_d]$ denotes a left-open right-closed interval in $\mathbb{R}^d$. Furthermore, we set $\overline{\mathbb{R}} := [-\infty, \infty]$, $\boldsymbol{0} := (0, \ldots, 0)$ and $\boldsymbol{\infty} := (\infty, \ldots, \infty)$.

Chapter 1

Dependence modelling for multivariate Lévy processes

In this chapter we present the notion of a Pareto Lévy measure for modelling the dependence structure between the components of a general multivariate Lévy process. At first, we briefly summarize basic knowledge about Lévy processes and refer to the excellent monographs [13, 49, 58].

We recall that a stochastic process $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\geq 0}$ in $\mathbb{R}^d$ is called *Lévy process* if it has the following properties:

(1) $\boldsymbol{X}_0 = \mathbf{0}$ almost surely (a. s.)

(2) $\boldsymbol{X}$ has independent increments, i. e. for all $n \geq 1$ and $0 \leq t_0 < t_1 \cdots < t_n$ the random vectors (r. v. s) $\boldsymbol{X}_{t_0}, \boldsymbol{X}_{t_1} - \boldsymbol{X}_{t_0}, \dots, \boldsymbol{X}_{t_n} - \boldsymbol{X}_{t_{n-1}}$ are independent.

(3) $\boldsymbol{X}$ has stationary increments, i. e. the distribution of $\boldsymbol{X}_{s+t} - \boldsymbol{X}_s$ does not depend on s.

(4) $\boldsymbol{X}$ is stochastically continuous, i. e. for every $t \geq 0$ and $\epsilon > 0$,

$$\lim_{s\to t} \mathbb{P}(|\boldsymbol{X}_s - \boldsymbol{X}_t| > \epsilon) = 0.$$

(5) $\boldsymbol{X}$ is càdlàg, i. e. the sample paths of $\boldsymbol{X}$ are a. s. right-continuous and have left limits.

Stochastic processes satisfying properties (1)–(3) are also often called Lévy processes but such processes are reducible to the Lévy processes defined above, see [13] and [58], Notes of Chapter 2.

A Lévy process $(\boldsymbol{X}_t)_{t\geq 0}$ is characterized by the *Lévy-Khintchine representation* of the characteristic function

$$\mathbb{E}\left[e^{i(\boldsymbol{z},\boldsymbol{X}_t)}\right] = e^{-t\Psi(\boldsymbol{z})}, \quad t \geq 0, \boldsymbol{z} \in \mathbb{R}^d,$$

with

$$\Psi(\boldsymbol{z}) = i(\boldsymbol{\gamma}, \boldsymbol{z}) + \frac{1}{2}\boldsymbol{z}^\top A\boldsymbol{z} + \int_{\mathbb{R}^d} \left(1 - e^{i(\boldsymbol{z},\boldsymbol{x})} + i(\boldsymbol{z},\boldsymbol{x})1_{\{|\boldsymbol{x}|\leq 1\}}\right) \Pi(\mathrm{d}\boldsymbol{x}), \tag{1.0.1}$$

where $(\cdot,\cdot)$ denotes the inner product and $|\cdot|$ an arbitrary norm in $\mathbb{R}^d$ and 1_B represents the indicator function of the set B. The quantities $(\boldsymbol{\gamma}, A, \Pi)$ are called the *characteristic triplet*, where $\boldsymbol{\gamma} \in \mathbb{R}^d$, the *Gaussian covariance matrix* A is a symmetric non-negative definite $d \times d$ matrix, and the *Lévy measure* Π is a measure on $\mathbb{R}^d$ satisfying

$$\Pi(\{\boldsymbol{0}\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} \left(|\boldsymbol{x}|^2 \wedge 1\right) \Pi(\mathrm{d}\boldsymbol{x}) < \infty.$$

Important classes are the *spectrally one-sided* Lévy processes which have only positive or negative jumps, and specifically *subordinators* which are Lévy processes whose components have a. s. non-decreasing paths.

Since the dependence structure of the Gaussian part of a Lévy process $\boldsymbol{X}$ is determined entirely by its covariance matrix A and the continuous part and the jump part of $\boldsymbol{X}$ are independent, it remains to describe the dependence structure of the jump part of $\boldsymbol{X}$ which is characterized by the Lévy measure Π. Therefore, we consider in this thesis only the dependence structure of Lévy measures and model it by a reference Lévy measure with standardized one-dimensional margins, the so-called Pareto Lévy measure which is proposed in [48] for spectrally positive Lévy processes.

A particular role is played by α-stable Lévy processes $(\boldsymbol{X}_t)_{t\geq 0}$ which are Lévy processes such that $\boldsymbol{X}_1 = (X_1^1, \dots, X_1^d)$ is a stable r. v. with index $\alpha \in (0,2]$, i. e. for every $a > 0$ there is $\alpha \in (0,2]$ and $\boldsymbol{c} \in \mathbb{R}^d$ such that

$$\mathbb{E}\left[e^{i(\boldsymbol{z},\boldsymbol{X}_1)}\right]^a = \mathbb{E}\left[e^{i(a^{1/\alpha}\boldsymbol{z},\boldsymbol{X}_1)}\right] e^{i(\boldsymbol{c},\boldsymbol{z})}, \quad \boldsymbol{z} \in \mathbb{R}^d,$$

cf. [58], Definitions 13.1 and 13.2, Proposition 13.5, Theorem 13.11 and 13.15 and Definition 13.16. The following result shows how α-stability for $\alpha \in (0,2)$ is characterized by the characteristic triplet.

Theorem 1.0.1 ([58], Theorem 14.3)
Let $\boldsymbol{X}$ be Lévy process in $\mathbb{R}^d$ with characteristic triplet $(\boldsymbol{\gamma}, A, \Pi)$ and $\alpha \in (0,2)$. The following statements are equivalent:

(1) $\boldsymbol{X}_1$ is α-stable.

(2) $A = \mathbf{0}$ and Π is homogeneous of degree α *or* α-homogeneous, *i. e. for all $t > 0$ it holds*
$$t^{-\alpha}\Pi(B) = \Pi(tB) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d).$$

(3) $A = \mathbf{0}$ and there is a finite measure $\lambda_{\mathbb{S}}$ on the unit sphere $\mathbb{S} := \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| = 1\}$ such that
$$\Pi(B) = \int_{\mathbb{S}} \int_0^\infty 1_B(r\xi) r^{-\alpha-1} \,\mathrm{d}r \; \lambda_{\mathbb{S}}(\mathrm{d}\xi) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d).$$
The probability measure $\mu_{\mathbb{S}} := \lambda_{\mathbb{S}}/\lambda_{\mathbb{S}}(\mathbb{S})$ we call the spectral measure *of* Π.

For $d = 1$ an α-stable Lévy process has an absolutely continuous Lévy measure
$$\Pi(\mathrm{d}x) = \begin{cases} c_1 x^{-\alpha-1}\,\mathrm{d}x & \text{on } (0,\infty), \\ c_2 |x|^{-\alpha-1}\,\mathrm{d}x & \text{on } (-\infty,0), \end{cases}$$
with $c_1 \geq 0, c_2 \geq 0$ and $c_1 + c_2 > 0$; see [58], p. 80. For $c_1 = c_2 = 1$ we call the Lévy process *standard 1-stable* and its Lévy measure *standard 1-homogeneous*.

Pareto Lévy measures are multivariate Lévy measures whose one-dimensional margins are standard 1-homogeneous. A version of Sklar's Theorem for Lévy measures, see Theorem 1.1.10, states that the dependence structure of a Lévy measure can be modelled by a Pareto Lévy measure, independently of the marginal Lévy measures. In [20, 42, 62, 64] the authors propose Sklar's Theorem for Lévy measures in terms of Lévy copulas which are strongly related to Pareto Lévy measures. They use slightly different definitions of the fundamental notion of the tail integral and, consequently, their formulations of Sklar's Theorem for Lévy measures differ. Therefore, we give in Section 1.1 a detailed presentation of the notion of a Lévy copula and of Sklar's Theorem for general Lévy processes which we use in this thesis. Since the relation between Pareto Lévy measures and Lévy copulas corresponds to the relation between copulas and Pareto measures and due to the use of copulas in Section 3.1, we briefly summarize these approaches to model the dependence between r. v. s in Section 1.1.1. In Section 1.1.2 we define the concept of Pareto Lévy measures for general

Lévy processes and prove the basic results for dependence modelling. Furthermore, we describe two approaches for graphical representation of the dependence structure in Section 1.2 and apply them to the examples in Section 1.3.

1.1 Lévy copulas and Pareto Lévy measures

In this section we state the basic notions and results for dependence modelling used in this thesis. At first, we briefly summarize the essential facts of dependence modelling for r. v. s.

1.1.1 Copulas and Pareto measures

Let $\boldsymbol{X} = (X^1, \dots, X^d)$ be a d-dimensional r. v. with distribution function (d. f.) F, i. e.

$$F(x_1, \dots, x_d) := \mathbb{P}(X^1 \le x_1, \dots, X^d \le x_d)$$

and one-dimensional marginal d. f. s $F_i(x) := \mathbb{P}(X^i \le x)$ for $i = 1, \dots, d$. The classical approach to model the dependence between the components X^i is by using a (distributional) copula. The following definitions can be found in [52] where we use the notation as in [42].

Definition 1.1.1 (F-volume, d-increasing, [42], Definitions 2.1 and 2.2)
*Let $F : S \to \overline{\mathbb{R}}$ for some subset $S \subseteq \overline{\mathbb{R}}^d$. For $\boldsymbol{a}, \boldsymbol{b} \in S$ with $\boldsymbol{a} \le \boldsymbol{b}$ and $\overline{(\boldsymbol{a}, \boldsymbol{b}]} \subset S$, the F-*volume *of $(\boldsymbol{a}, \boldsymbol{b}]$ is defined by*

$$V_F((\boldsymbol{a}, \boldsymbol{b}]) := \sum_{\boldsymbol{u} \in \{a_1, b_1\} \times \dots \times \{a_d, b_d\}} (-1)^{N(\boldsymbol{u})} F(\boldsymbol{u}),$$

*where $N(\boldsymbol{u}) := \#\{k : u_k = a_k\}$. F is called d-*increasing *if $V_F((\boldsymbol{a}, \boldsymbol{b}]) \ge 0$ for all $(\boldsymbol{a}, \boldsymbol{b}] \in S$ with $\boldsymbol{a} \le \boldsymbol{b}$ and $\overline{(\boldsymbol{a}, \boldsymbol{b}]} \subset S$.*

Thereby, the notion of a (distributional) copula is defined as follows.

Definition 1.1.2 (Copula, [52], Definitions 2.10.5 and 2.10.6)
A function $C_D : [0, 1]^d \to [0, 1]$ is called (distributional) copula *if*

(1) $C_D(u_1, \dots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \dots, d\}$,

(2) C_D *is d-increasing,*

(3) $C_D(\underbrace{1,\dots,1}_{i-1},u,1,\dots,1) = u$ *for every* $i \in \{1,\dots,d\}$, $u \in [0,1]$.

The central result for dependence modelling by copulas is stated by Sklar's Theorem.

Theorem 1.1.3 (Sklar's Theorem, [52], Theorem 2.10.9)
Let F be a d-dimensional d. f. with margins $F_1,\dots,F_d$. Then there exists a (distributional) copula C_D such that

$$F(x_1,\dots,x_d) = C_D(F_1(x_1),\dots,F_d(x_d)), \quad (x_1,\dots,x_d) \in \overline{\mathbb{R}}^d. \tag{1.1.1}$$

If $F_1,\dots,F_d$ are all continuous, then C_D is unique; otherwise, C_D is uniquely determined on $\prod_{i=1}^d \mathrm{Ran}\, F_i$.
Conversely, if C_D is a (distributional) copula and $F_1,\dots,F_d$ are d. f. s, then the function defined by (1.1.1) is a d-dimensional d. f. with margins $F_1,\dots,F_d$.

Let $\boldsymbol{X} = (X^1,\dots,X^d)$ be a r. v. with d. f. F and copula C_D. Using the continuity of copulas, see [52], Theorem 2.10.7, Equation (1.1.1) can be reformulated with a copula $\widehat{C}_D$ of $-\boldsymbol{X}$ in terms of the right tails of $\boldsymbol{X}$, given by

$$\overline{F}(x_1,\dots,x_d) := \mathbb{P}(X^1 > x_1,\dots,X^d > x_d)$$

and $\overline{F}_i(x) := \mathbb{P}(X^i > x)$ for $i = 1,\dots,d$, such that

$$\overline{F}(x_1,\dots,x_d) = \widehat{C}_D(\overline{F}_1(x_1),\dots,\overline{F}_d(x_d)), \quad (x_1,\dots,x_d) \in \overline{\mathbb{R}}^d. \tag{1.1.2}$$

$\widehat{C}_D$ is called the *survival copula* of the r. v. $\boldsymbol{X}$ or the d. f. F. If all F_i are continuous then the copula C_D is the d. f. of the r. v. $\big(F_1(X^1),\dots,F_d(X^d)\big)$ and the survival copula $\widehat{C}_D$ is the d. f. of the r. v. $\big(\overline{F}_1(X^1),\dots,\overline{F}_d(X^d)\big)$, both corresponding with a transformation of the distribution to have uniform one-dimensional margins. Further, in [48] they propose for continuous margins F_i to consider the distribution of the r. v. $\big(1/\overline{F}_1(X^1),\dots,1/\overline{F}_d(X^d)\big)$, the so-called *Pareto measure*, which is a transformation to standard Pareto distributed r. v. s. The advantage of the Pareto measure is the stronger probabilistic interpretation for limit theory and heavy-tail analysis.

The idea of Pareto measures can be extended to general distributions as follows. Since copulas are continuous in every variable, see [52], Theorem 2.10.7, $\widehat{C}_D$ defines by its $\widehat{C}_D$-volume $V_{\widehat{C}_D}$ a unique probability measure whose one-dimensional marginals are the uniform distribution on $(0,1)$. As proposed in [9] we define the inversion map $Q : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}^d$ as

$$Q(x_1, \dots, x_d) = \left(x_1^{-1}, \dots, x_d^{-1}\right), \tag{1.1.3}$$

where we set $1/0 := \infty$, $-1/0 := -\infty$, $1/\infty := 0$ and $1/-\infty := 0$. Q restricted to $[0,1]^d$ is bijective and the image of $V_{\widehat{C}_D}$ under Q, given by $V_{\widehat{C}_D} \circ Q$, defines a d-dimensional distribution Γ_D with standard Pareto margins. With the right tail $\overline{\Gamma}_D$ relation (1.1.2) becomes

$$\overline{F}(x_1, \dots, x_d) = \overline{\Gamma}_D \left(\frac{1}{\overline{F}_1(x_1)}, \dots, \frac{1}{\overline{F}_d(x_d)} \right), \quad (x_1, \dots, x_d) \in \overline{\mathbb{R}}^d. \tag{1.1.4}$$

We call Γ_D *Pareto measure* and its right tail $\overline{\Gamma}_D$ is called *Pareto copula.*

1.1.2 Lévy copulas and Pareto Lévy measures: Definitions and basic results

Now let $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\geq 0} = (X_t^1, \dots, X_t^d)_{t\geq 0}$ be a d-dimensional Lévy process with Lévy measure Π. If Π is a finite measure we can model the dependence between the jump sizes by a copula or a Pareto measure, but in general Lévy measures may have a singularity at zero. Consequently, for a general approach to model the dependence the analogue of a d. f. or a right tail for Lévy measures has to be bounded away from zero and is defined as follows.

As in [42] we set for $x \in \overline{\mathbb{R}}$

$$\mathcal{I}(x) := \begin{cases} (x, \infty), & x \geq 0, \\ (-\infty, x], & x < 0, \end{cases} \tag{1.1.5}$$

and

$$\operatorname{sgn}(x) := 1_{\{x \geq 0\}} - 1_{\{x<0\}}.$$

Definition 1.1.4 (Tail integral of a Lévy measure)
Let $\boldsymbol{X}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure Π. The tail integral *of $\boldsymbol{X}$ or Π is the function $\overline{\Pi} : (\overline{\mathbb{R}} \setminus \{0\})^d \to \mathbb{R}$ defined as*

$$\overline{\Pi}(x_1, \dots, x_d) := \prod_{j=1}^{d} \operatorname{sgn}(x_j) \Pi \left(\prod_{i=1}^{d} \mathcal{I}(x_i) \right).$$

In [42], Definition 3.3, the tail integral is defined on $(\mathbb{R}\backslash\{0\})^d$. Since $\lim_{x_i\to\pm\infty}\overline{\Pi}(x_1,\dots,x_d)=0$ for all $i\in\{1,\dots,d\}$ and $\mathcal{I}(x)=\emptyset$ for $x\in\{-\infty,\infty\}$ our extension to $(\overline{\mathbb{R}}\setminus\{0\})^d$ is continuous and corresponds to the extension of the one-dimensional tail integral used in the proof of Theorem 3.6 in [42]. The main aspect in the definition of the tail integral is that a Lévy measure Π is always considered on sets bounded away from zero.

By definition (1.1.5) tail integrals are on $(\mathbb{R}\setminus\{0\})^d$ right-continuous functions and $(-1)^d\overline{\Pi}$ is d-increasing. However, the tail integral does not determine the Lévy measure uniquely because it does not specify its mass on $\mathbb{R}^d\setminus(\mathbb{R}\setminus\{0\})^d$. Therefore, we additionally need the marginal tail integrals. For a set I we define $|I|$ as its cardinality.

Definition 1.1.5 (Margins of a Lévy process/Lévy measure/tail integral)
Let $\boldsymbol{X}=(X^1,\dots,X^d)=(X^1_t,\dots,X^d_t)_{t\geq0}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure Π and $I\subseteq\{1,\dots,d\}$ a non-empty index set. We define the following quantities:

*(1) The I-*margin *of $\boldsymbol{X}$ is the Lévy process $X^I:=(X^i)_{i\in I}$.*

*(2) Π_I denotes the Lévy measure of X^I and is the I-*marginal Lévy measure*. It is given by*

$$\Pi_I(B)=\Pi\big(\{\boldsymbol{x}\in\mathbb{R}^d:(x_i)_{i\in I}\in B\}\big),\quad B\in\mathcal{B}(\mathbb{R}^{|I|}\setminus\{\mathbf{0}\}).$$

*(3) The I-*marginal tail integral *of $\boldsymbol{X}$ is given by $\overline{\Pi}_I:(\overline{\mathbb{R}}\setminus\{0\})^{|I|}\to\mathbb{R}$ with*

$$\overline{\Pi}_I((x_i)_{i\in I})=\prod_{i\in I}\operatorname{sgn}(x_i)\,\Pi_I\Big(\prod_{i\in I}\mathcal{I}(x_i)\Big).$$

To simplify notation, we denote one-dimensional margins by X^i, Π_i and $\overline{\Pi}_i$.

By [42], Lemma 3.5, the set of all marginal tail integrals $\{\overline{\Pi}_I : I\subseteq\{1,\dots,d\}\}$ determines the Lévy measure Π uniquely and vice versa. Moreover, we shall need for Lévy copulas the following definition of I-margins of a d-increasing function on $(-\infty,\infty]^d$.

Definition 1.1.6 ([42], Definition 2.4)
Let $F:(-\infty,\infty]^d\to(-\infty,\infty]$ be a d-increasing function such that $F(u_1,\dots,u_d)=0$ if $u_i=0$ for at least one $i\in\{1,\dots,d\}$. For every non-empty index set $I\subseteq$

*$\{1,\dots,d\}$, the I-*margin *of F is the function $F_I : (-\infty,\infty]^{|I|} \to (-\infty,\infty]$, defined by*

$$F_I\left((u_i)_{i\in I}\right) := \lim_{a\to\infty} \sum_{(u_i)_{i\in I^c}\in\{-a,\infty\}^{|I^c|}} F(u_1,\dots,u_d) \prod_{i\in I^c} \operatorname{sgn}(u_i),$$

where $I^c := \{1,\dots,d\} \setminus I$.

In analogy to (survival) copulas Kallsen and Tankov, [42], define Lévy copulas for general Lévy processes as follows.

Definition 1.1.7 (Lévy copula, [42], Definition 3.1)
A function $\widehat{C} : (-\infty,\infty]^d \to (-\infty,\infty]$ is called Lévy copula *if*

(1) $\widehat{C}(u_1,\dots,u_d) \neq \infty$ for $(u_1,\dots,u_d) \neq (\infty,\dots,\infty)$,

(2) $\widehat{C}(u_1,\dots,u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1,\dots,d\}$,

(3) $\widehat{C}$ is d-increasing,

(4) $\widehat{C}_{\{i\}}(u) = u$ for every $i \in \{1,\dots,d\}, u \in \mathbb{R}$.

Since Lévy copulas are right-continuous in every variable separately, see [42], Lemma 3.2, and d-increasing, according to [43], Section 4.5, there exists a unique measure $\mu_{\widehat{C}}$ on $(-\infty,\infty]^d \setminus \{\boldsymbol{\infty}\}$ such that for the $\widehat{C}$-volume $V_{\widehat{C}}$ of a Lévy copula $\widehat{C}$ and $\boldsymbol{a},\boldsymbol{b} \in (-\infty,\infty]^d \setminus \{\boldsymbol{\infty}\}$ with $\boldsymbol{a} \le \boldsymbol{b}$ we have

$$\mu_{\widehat{C}}((\boldsymbol{a},\boldsymbol{b}]) = V_{\widehat{C}}((\boldsymbol{a},\boldsymbol{b}]).$$

Due to Definition 1.1.7 (4) the one-dimensional margins of $\mu_{\widehat{C}}$ are the Lebesgue measure and so $\mu_{\widehat{C}}$ is no Lévy measure. The inversion map Q given in (1.1.3) restricted to $(-\infty,\infty]^d$ is bijective and applying Q to $\mu_{\widehat{C}}$, the concatentation $\Gamma := \mu_{\widehat{C}} \circ Q^{-1}$ is a measure on $\mathcal{B}((-\infty,\infty]^d \setminus \{\mathbf{0}\})$. Due to properties (1) and (2) in Definition 1.1.7, the measure Γ is finite outside neighbourhoods of the origin and $\Gamma((-\infty,\infty] \setminus \mathbb{R}^d) = 0$. Since we have $\overline{\Gamma}_i(x) = x^{-1}$ for $x \neq 0$ for the one-dimensional margins, Γ is a Lévy measure with one-dimensional 1-homogeneous margins. Analogously to the Pareto measure for distributions, Γ is our reference Lévy measure which was proposed for spectrally positive Lévy processes in [48].

Definition 1.1.8 (Pareto Lévy measure, Pareto Lévy copula)
A d-dimensional Lévy measure Γ is called Pareto Lévy measure (PLM) *if it has standard 1-homogeneous one-dimensional margins, i. e. $\Gamma_i(\mathrm{d}x_i) = |x_i|^{-2}\,\mathrm{d}x_i$ on $\mathbb{R} \setminus \{0\}$ for $i = 1,\dots,d$. The tail integral $\overline{\Gamma}$ is called* Pareto Lévy copula (PLC).

Note that a PLM is not in general a 1-homogeneous Lévy measure, although its one-dimensional margins are 1-homogeneous Lévy measures.

Remark 1.1.9 (Relation between Lévy copula and Pareto Lévy measure) Every Lévy copula $\widehat{C}$ defines uniquely a PLM Γ given for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ with $\mathbf{0} \notin [\boldsymbol{x}, \boldsymbol{y})$ as

$$\Gamma([\boldsymbol{x}, \boldsymbol{y})) = \sum_{\boldsymbol{u} \in \{1/y_1, 1/x_1\} \times \cdots \times \{1/y_d, 1/x_d\}} (-1)^{N(\boldsymbol{u})} \widehat{C}(\boldsymbol{u}), \tag{1.1.6}$$

where $\boldsymbol{u} = (u_1, \ldots, u_d) \in (-\infty, \infty]^d$ and $N(\boldsymbol{u}) := \#\{k : u_k = 1/y_k\}$. Furthermore, for the PLC $\overline{\Gamma}$ it holds

$$\overline{\Gamma}(x_1, \ldots, x_d) = \widehat{C}\left(\frac{1}{x_1}, \ldots, \frac{1}{x_d}\right), \quad (x_1, \ldots, x_d) \in (\overline{\mathbb{R}} \setminus \{0\})^d. \tag{1.1.7}$$

Conversely, every PLM Γ defines uniquely a Lévy copula $\widehat{C}$ defining $\widehat{C}$ on $\mathbb{R}^d$ by (1.1.7) and on $(-\infty, \infty]^d \setminus \mathbb{R}^d$ we set for $h \in \{1, \ldots, d\}$ and $x_i < \infty$ for $i > h$

$$\widehat{C}(\underbrace{\infty, \ldots, \infty}_{h}, x_{h+1}, \ldots, x_d) = \prod_{i=h+1}^{d} \operatorname{sgn}(x_i) \Gamma\left([0, \infty) \times \cdots \times [0, \infty) \times \prod_{i=h+1}^{d} \mathcal{I}\left(\frac{1}{x_i}\right)\right).$$

Consequently, for a PLM Γ with corresponding Lévy copula $\widehat{C}$ the following assertions are equivalent:

(1) Γ is 1-homogeneous.

(2) $\widehat{C}$ is homogeneous in the sense that for all $t > 0$ it holds

$$\widehat{C}(tu_1, \ldots, tu_d) = t\widehat{C}(u_1, \ldots, u_d), \quad (u_1, \ldots, u_d) \in \mathbb{R}^d.$$

The following result has been proven for Lévy copulas in [42], Theorem 3.6. Although we already know the relation between Lévy copulas and Pareto Lévy measures, we sketch the proof of the first part of Sklar's Theorem again for PLMs since we shall need the explicit construction of a PLM for a given Lévy measure in the sequel.

Theorem 1.1.10 (Sklar's Theorem for Pareto Lévy measures)

(1) Let $\boldsymbol{X}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure Π. Then there exists a PLM Γ such that

$$\overline{\Pi}_I\left((x_i)_{i \in I}\right) = \overline{\Gamma}_I\left(\left(\frac{1}{\overline{\Pi}_i(x_i)}\right)_{i \in I}\right), \quad (x_i)_{i \in I} \in (\overline{\mathbb{R}} \setminus \{0\})^{|I|}, \tag{1.1.8}$$

for every non-empty index set $I \subseteq \{1, \ldots, d\}$. The PLM Γ is unique on $\prod_{i=1}^d \overline{Ran\ (1/\overline{\Pi}_i)}$ and we call Γ a PLM of $\boldsymbol{X}$.

(2) Let Γ be a d-dimensional PLM and $\overline{\Pi}_i$ for $i = 1, \dots, d$ one-dimensional tail integrals of arbitrary Lévy processes. Then there exists a Lévy process $\boldsymbol{X}$ in $\mathbb{R}^d$ whose components have tail integrals $\overline{\Pi}_1, \dots, \overline{\Pi}_d$ and whose marginal tail integrals satisfy Equation (1.1.8) for every non-empty set $I \subseteq \{1, \dots, d\}$ and every $(x_i)_{i \in I} \in (\overline{\mathbb{R}} \setminus \{0\})^{|I|}$. The Lévy measure Π of $\boldsymbol{X}$ is uniquely determined by Γ and $\overline{\Pi}_1, \dots, \overline{\Pi}_d$.

Proof.
(1) Recall the following tools from [42], Theorem 3.6, and our extended definition of the tail integral. For $x \in (-\infty, \infty]$ and $i = 1, \dots, d$ we define

$$\dot{\overline{\Pi}}_i(x) := \begin{cases} \overline{\Pi}_i(x) & \text{for } x \neq 0, \\ \infty & \text{for } x = 0, \end{cases}$$

and

$$\Delta\overline{\Pi}_i(x) := \begin{cases} \lim_{\xi \uparrow x} \overline{\Pi}_i(\xi) - \overline{\Pi}_i(x) = \Pi_i(\{x\}) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Since Π_i may have atoms or may be finite we construct an atomless infinite measure m on $\mathcal{B}(((-\infty, \infty]^d \setminus \{\boldsymbol{0}\}) \times [0,1]^d \times \mathbb{R})$. Denote by Π^* the extension of Π to $(-\infty, \infty]^d \setminus \{\boldsymbol{0}\}$ given by $\Pi^*(B) := \Pi(B \cap \mathbb{R}^d)$, by λ the Lebesgue measure on $\mathbb{R}$ and by $\delta_{\boldsymbol{x}}$ the Dirac measure with mass on $\boldsymbol{x}$. Then we set

$$\begin{aligned} m \; &:= \; \Pi^* \otimes \lambda|_{[0,1]^d} \otimes \delta_0 \qquad (1.1.9) \\ &\quad + \sum_{i=1}^{d} \delta_{(\underbrace{0,\dots,0}_{i-1},\infty,0,\dots,0)} \otimes \delta_{(\underbrace{0,\dots,0}_{d})} \otimes \lambda|_{\left((-\infty,-\Pi_i((-\infty,0)))\cup(\Pi_i((0,\infty)),\infty)\right)}. \end{aligned}$$

For $B \in \mathcal{B}(\mathbb{R}^d \setminus \{\boldsymbol{0}\})$ we define

$$\begin{aligned} \Gamma(B) := m\Bigg(\Bigg\{&(x_1, \dots, x_d, y_1, \dots, y_d, z) \in \; ((-\infty, \infty]^d \setminus \{\boldsymbol{0}\}) \times [0,1]^d \times \mathbb{R} : \\ &\left(\frac{1}{\dot{\overline{\Pi}}_i(x_i) + y_i \Delta\overline{\Pi}_i(x_i) + z}\right)_{i=1,\dots,d} \in B\Bigg\}\Bigg). \qquad (1.1.10) \end{aligned}$$

Note that we use $1/(\dot{\overline{\Pi}}_i(x_i) + y_i \Delta\overline{\Pi}_i(x_i) + z)$ in (1.1.10) instead of $\dot{\overline{\Pi}}_i(x_i) + y_i \Delta\overline{\Pi}_i(x_i) + z$ as in [42], Equation (3.5), corresponding to componentwise inversion: $x \mapsto 1/x$. As in the proof of Theorem 3.6, [42], one shows that $\overline{\Gamma}_i(x) = x^{-1}$ for $x \neq 0$ and the assertion follows. □

Note that Sklar's Theorem in Equation (3.2) of [42] holds with our Definition 1.1.4 of tail integrals also for $(x_i)_{i\in I} \in (\overline{\mathbb{R}} \setminus \{0\})^{|I|}$ due to property (2) of Definition 1.1.7. For the representation (1.1.8) in terms of tail integrals we need the extended notion of a tail integral for vectors $(x_i)_{i\in I}$ with $\overline{\Pi}_I((x_i)_{i\in I}) = 0$.

Theorem 1.1.10 gives the relationship between a Lévy measure Π and its transformed PLM Γ in terms of all marginal tail integrals. In the next results we formulate this relationship for sets in the generating semi-algebra of rectangular sets in order to get a representation of (1.1.8) in terms of Γ, Π and the one-dimensional margins Π_i. For one-dimensional tail integrals we define (recall the possible singularity in 0)

$$\overline{\Pi}_i(x+) := \lim_{\beta \downarrow x} \overline{\Pi}_i(\beta) \quad \text{and} \quad \overline{\Pi}_i(x-) := \lim_{\beta \uparrow x} \overline{\Pi}_i(\beta) \quad \text{for } x \in \mathbb{R}. \tag{1.1.11}$$

Since PLCs are defined quadrantwise special care has to be taken for hyperplanes through the coordinate axes. The following result presents the Lévy measure Π in terms of the PLM Γ and the one-dimensional marginal tail integrals $\overline{\Pi}_i$, $i = 1, \ldots, d$. The proof is given in the Appendix.

Proposition 1.1.11
Let Π be the Lévy measure defined by (1.1.8) with PLM Γ and one-dimensional Lévy measures $\Pi_i, i = 1, \ldots, d$. With $\overline{\Pi}_i(0) := \overline{\Pi}_i(0+)$ the following assertions hold.

(1) For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $\mathbf{0} \notin \prod_{i=1}^d (a_i, b_i]$ it holds

$$\Pi\left(\prod_{i=1}^d (a_i, b_i]\right) = \Gamma\left(\prod_{i=1}^d \left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right]\right). \tag{1.1.12}$$

(2) Let $\emptyset \neq K \subset \{1, \ldots, d\}$. Define

$$A_i := \begin{cases} \left[\frac{1}{\overline{\Pi}_i(0-)}, \frac{1}{\overline{\Pi}_i(0+)}\right], & \text{if } \overline{\Pi}_i(0-) < 0, \overline{\Pi}_i(0+) > 0, \\ \left[0, \frac{1}{\overline{\Pi}_i(0+)}\right], & \text{if } \overline{\Pi}_i(0-) = 0, \overline{\Pi}_i(0+) > 0, \\ \left[\frac{1}{\overline{\Pi}_i(0-)}, 0\right], & \text{if } \overline{\Pi}_i(0-) < 0, \overline{\Pi}_i(0+) = 0. \end{cases} \tag{1.1.13}$$

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $\mathbf{0} \notin \prod_{i\in K}\{0\} \times \prod_{i\notin K}(a_i, b_i]$ it holds

$$\Pi\left(\prod_{i\in K}\{0\} \times \prod_{i\notin K}(a_i, b_i]\right) = \Gamma\left(\prod_{i\in K} A_i \times \prod_{i\notin K}\left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right]\right). \tag{1.1.14}$$

The next result is a direct consequence of the constructions (1.1.9) and (1.1.10) and presents the PLM Γ defined in (1.1.10) in terms of the Lévy measure Π and its one-dimensional marginal tail integrals $\overline{\Pi}_i$ for $i = 1, \dots, d$.

Proposition 1.1.12
Let Π be a Lévy measure with one-dimensional margins Π_i for $i = 1, \dots, d$. For the PLM Γ defined in (1.1.10) the following assertions hold.

(1) Define

$$\mathcal{D}_i := \mathcal{I}\left(\frac{1}{\overline{\Pi}_i(0-)}\right) \cup \mathcal{I}\left(\frac{1}{\overline{\Pi}_i(0+)}\right) \cup \{0\}. \tag{1.1.15}$$

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $(\boldsymbol{a}, \boldsymbol{b}] \subset \prod_{i=1}^d \mathcal{D}_i$ it holds

$$\begin{aligned}\Gamma((\boldsymbol{a}, \boldsymbol{b}]) =& \Pi \otimes \lambda|_{[0,1]^d}\Bigg(\Bigg\{(\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^d \setminus \{\boldsymbol{0}\}) \times [0,1]^d : \\ & \frac{1}{\dot{\overline{\Pi}}_i(x_i) + y_i \Delta\overline{\Pi}_i(x_i)} \in (a_i, b_i] \text{ for } i = 1, \dots, d\Bigg\}\Bigg),\end{aligned} \tag{1.1.16}$$

where $\dot{\overline{\Pi}}_i$ and $\Delta\Pi_i$ are defined as in the proof of Sklar's Theorem 1.1.10.

(2) For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $(\boldsymbol{a}, \boldsymbol{b}] \subset \mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i$ it holds

$$\Gamma((\boldsymbol{a}, \boldsymbol{b}]) = \sum_{i=1}^d \underbrace{\delta_0 \otimes \cdots \otimes \delta_0}_{i-1} \otimes \Gamma_i \otimes \underbrace{\delta_0 \otimes \cdots \otimes \delta_0}_{d-i}((\boldsymbol{a}, \boldsymbol{b}]), \tag{1.1.17}$$

where $\Gamma_i(\mathrm{d}x) = |x|^{-2}\mathrm{d}x$ for $x \in \mathbb{R} \setminus \{0\}$.

With these propositions we receive the following results which extend [9], Proposition 1, for general Lévy processes. The first result was formulated for Lévy copulas in Theorem 4.6 of [42].

Theorem 1.1.13
Let $\alpha \in (0, 2)$ and Π be a Lévy measure with one-dimensional margins $\Pi_i, i = 1, \dots, d$ and PLM Γ defined in (1.1.10).

(1) Π is α-homogeneous if and only if all Π_i are α-homogeneous and Γ is 1-homogeneous.

(2) *Let Π_i be α-homogeneous for $i = 1, \ldots, d$. Then Π is self-decomposable if and only if Γ is self-decomposable.*

Proof.
We treat only (2) since (1) is obvious. If Π_i is α-homogeneous then it holds $\overline{\Pi}_i(x_i) = \text{sgn}(x_i)k_i|x_i|^{-\alpha}$ for $x_i \neq 0$ and some $k_i > 0$. By [58], Theorem 15.8, Π is self-decomposable if and only if $\Pi(t^{-1}B) \geq \Pi(B)$ for all $B \in \mathcal{B}(\mathbb{R}^d)$ and $t > 1$. With Proposition 1.1.11 and 1.1.12 the equivalence holds. □

The advantage of working with a PLM instead of a Lévy copula is that a PLM is a Lévy measure and modelling dependence of a Lévy measure in terms of a PLM is a self-contained approach. The Lévy process corresponding to a PLM extends the class of stable processes in a natural way. Moreover, the I-margins of a PLM, given in Definition 1.1.5, are easier to calculate than the I-margins of a Lévy copula, given in Definition 1.1.6. Nevertheless, in the bivariate situation of Section 3.4 where we explicitly calculate the tail integral of the sum process $X^1 + X^2$ we formulate our results in terms of Lévy copulas since for $d = 2$ Lévy copulas are notationally easier to deal with.

1.2 Graphical representation of the dependence structure

In this section we describe two approaches for visualizing the dependence structure modelled by a PLM.

1.2.1 Spectral measure

For a stable r. v. the spectral measure characterizes the dependence between the marginals, see [57], which remains true for multivariate regularly varying r. v. in the limit, see Definition 4.1.3. Consequently, the spectral density has been a popular graphical tool for stable and regularly varying distributions and processes, at least in two dimensions. A 1-homogeneous PLM is the Lévy measure of a standard 1-stable Lévy process and, therefore, we consider its spectral measure where we reduce the situation to $d = 2$ for presentation purposes.

By Theorem 1.0.1, the spectral measure $\mu_{\mathbb{S}}$ of a 1-homogeneous PLM Γ is on $\mathcal{B}(\mathbb{S})$ given by

$$\mu_{\mathbb{S}}(\cdot) = \frac{\Gamma(\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| > 1, \boldsymbol{x}/|\boldsymbol{x}| \in \cdot\})}{\Gamma(\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| > 1\})}. \tag{1.2.1}$$

Moreover, let $\boldsymbol{X}$ be a Lévy process with Lévy measure Γ. By Lemma 4.1.4 the r. v. $\boldsymbol{X}_1$ is regularly varying and has the same spectral measure as Γ. Therefore, Equation (1.2.1) also means

$$\mu_{\mathbb{S}}(\cdot) = \lim_{t\to\infty} \mathbb{P}(\boldsymbol{X}_1/|\boldsymbol{X}_1| \in \cdot \mid |\boldsymbol{X}_1| > t)\ .$$

We see that $\mu_{\mathbb{S}}$ measures the dependence between extreme values and that it depends on the chosen norm $|\cdot|$. For the Euclidean 2-norm $|\cdot|_2$ with unit circle $\mathbb{S}_2 := \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1|^2 + |x_2|^2 = 1\}$ we get

$$\mu_{\mathbb{S}_2}(\cdot) = \lim_{t\to\infty} \mathbb{P}\left(\boldsymbol{X}_1/|\boldsymbol{X}_1|_2 \in \cdot \mid \sqrt{|X_1^1|^2 + |X_1^2|^2} > t\right)$$

and $\mu_{\mathbb{S}_2}$ describes the dependence between the components given their sum is extreme. Considering the 1-norm $|\cdot|_1$ with unit circle $\mathbb{S}_1 := \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| + |x_2| = 1\}$ yields a similar interpretation of $\mu_{\mathbb{S}_1}$. Applying the maximum-norm $|\cdot|_\infty$ the unit circle is given as $\mathbb{S}_\infty := \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| \vee |x_2| = 1\}$ and the spectral measure becomes

$$\mu_{\mathbb{S}_\infty}(\cdot) = \lim_{t\to\infty} \mathbb{P}(\boldsymbol{X}_1/|\boldsymbol{X}_1|_\infty \in \cdot \mid |X_1^1| \vee |X_1^2| > t).$$

Thus, $\mu_{\mathbb{S}_\infty}$ measures the dependence between the components under the condition that at least one of them is extreme, see [50], Section 5.2.

As uniform parametrization of the unit circle of all three norms we use polar coordinates, i. e. we apply the transformation $T : [0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$ defined by $T(r, \phi) = (r\cos(\phi), r\sin(\phi))$. Further, we define the arcs of the unit circles $S^2_{\rho_1,\rho_2} := \{(\cos(\phi), \sin(\phi)) : \phi \in [\rho_1, \rho_2]\}$, $S^1_{\rho_1,\rho_2} := \{(\cos(\phi), \sin(\phi))/|(\cos(\phi), \sin(\phi))|_1 : \phi \in [\rho_1, \rho_2]\}$ and $S^\infty_{\rho_1,\rho_2} := \{(\cos(\phi), \sin(\phi))/|(\cos(\phi), \sin(\phi))|_\infty : \phi \in [\rho_1, \rho_2]\}$. Transformation of a 1-homogeneous PLM Γ leads to a decomposition into a radial and an angular part given for $(r, \phi) \in [0,\infty) \times [0, 2\pi)$ as

$$\Gamma \circ T(\mathrm{d}r, \mathrm{d}\phi) = r^{-2}\mathrm{d}r\, \Gamma^\phi(\mathrm{d}\phi) \tag{1.2.2}$$

and the spectral measures result as

$$\mu_{\mathbb{S}_2}([\rho_1,\rho_2]) = \frac{\Gamma(\{(r\cos(\phi), r\sin(\phi)) : r>1, (\cos(\phi),\sin(\phi)) \in S^2_{\rho_1,\rho_2}\})}{\Gamma(\{(r\cos(\phi), r\sin(\phi)) : r>1\})}$$

$$= \frac{\int_{\phi\in[\rho_1,\rho_2]} \Gamma^{\phi}(\mathrm{d}\phi)}{\int_{\phi\in[0,2\pi)} \Gamma^{\phi}(\mathrm{d}\phi)}, \tag{1.2.3}$$

$$\mu_{\mathbb{S}_1}([\rho_1,\rho_2]) = \frac{\int_{\phi\in[\rho_1,\rho_2]} |(\cos(\phi),\sin(\phi))|_1 \; \Gamma^{\phi}(\mathrm{d}\phi)}{\int_{\phi\in[0,2\pi)} |(\cos(\phi),\sin(\phi))|_1 \; \Gamma^{\phi}(\mathrm{d}\phi)} \tag{1.2.4}$$

and

$$\mu_{\mathbb{S}_\infty}([\rho_1,\rho_2]) = \frac{\int_{\phi\in[\rho_1,\rho_2]} |(\cos(\phi),\sin(\phi))|_\infty \; \Gamma^{\phi}(\mathrm{d}\phi)}{\int_{\phi\in[0,2\pi)} |(\cos(\phi),\sin(\phi))|_\infty \; \Gamma^{\phi}(\mathrm{d}\phi)}. \tag{1.2.5}$$

We present the spectral measures in two ways. At first, we plot the density $\mu_{\mathbb{S}_\cdot}(\mathrm{d}\phi)/\mathrm{d}\phi$ on $[0,2\pi)$. Secondly, we take an idea from [10] and visualize $\mu_{\mathbb{S}_\cdot}$ as graph such that the included area between two angles ρ_1, ρ_2 and a solid curve ($s(\rho)$ for $\rho\in[\rho_1,\rho_2]$) represents the probability $\mu_{\mathbb{S}_\cdot}([\rho_1,\rho_2])$. This representation shows the directions in which the mass of Γ is distributed and we shall call these graphs *Basrak graphs*.

For a regularly varying PLM we consider the spectral measure of its 1-homogeneous limit measure, see Definition 2.0.1 which represents the dependence structure between extremes.

1.2.2 Pareto Lévy copula

Whereas a spectral measure describes a PLM on sets $\{\boldsymbol{x}\in\mathbb{R}^d : |\boldsymbol{x}|_\cdot > 1, \boldsymbol{x}/|\boldsymbol{x}| \in S^{\cdot}_{\rho_1\rho_2}\}$, a PLC describes a PLM on rectangle sets of $(\mathbb{R}\setminus\{0\})^2$. Note that for $d=2$ a PLM is uniquely determined by its PLC since the one-dimensional margins are standardized. Thus the PLC values on $\mathbb{S}_2\cap(\mathbb{R}\setminus\{0\})^2$ characterize a PLM outside the unit circle $\mathbb{S}_2$ as the spectral measure $\mu_{\mathbb{S}_2}$ due to

$$|\overline{\Gamma}(x_1,x_2)| = \Gamma(\mathcal{I}(x_1)\times\mathcal{I}(x_2)) \quad \text{for } (x_1,x_2)\in\mathbb{S}_2\cap(\mathbb{R}\setminus\{0\})^2.$$

Further, for an 1-homogeneous or regularly varying PLM the PLC describes the dependence structure of joint extremes as a spectral measure, see Section 2.1. This gives a new possibility of visualizing the dependence structure. We represent a PLM by weighting the points of $\mathbb{S}_2\cap(\mathbb{R}\setminus\{0\})^2$ by its absolute PLC value such that the

Euclidean distance of a point (x_1, x_2) to the origin corresponds to $\Gamma(\mathcal{I}(x_1) \times \mathcal{I}(x_2))$. Although we look at different sets as in Section 1.2.1, we see where the mass of Γ is distributed outside the unit circle. Contrary to spectral measures, PLCs always exist and, consequently, this graphical approach can also be applied for non-homogeneous and non-regularly varying PLMs.

1.3 Examples

In this section we give examples for dependence structures which are considered throughout this thesis and visualize them in the ways described in Section 1.2.

1.3.1 Independence Pareto Lévy measure

By [58], Exercise 12.10, the components of a Lévy process $\boldsymbol{X} = (X^1, \dots, X^d)$ with characteristic triplet $(\boldsymbol{\gamma}, A, \Pi)$ are independent, i. e. $X_t^1, \dots, X_t^d$ are independent for all $t > 0$, if and only if A is diagonal and Π is supported by the union of the coordinate axes $\{x\boldsymbol{e}_i : x \in \mathbb{R}, i = 1, \dots, d\}$ where $\boldsymbol{e}_i$ denote the unit vectors in $\mathbb{R}^d$. This motivates the following definition.

Definition 1.3.1 (Independence of jumps of a Lévy process)
Let $\boldsymbol{X}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure Π. Its jumps are said to be independent *if Π is supported by the union of the coordinate axes.*

From (1.1.14) we directly conclude that a Lévy process $\boldsymbol{X}$ has independent jumps if and only if the *independence PLM*

$$\Gamma_\perp(\mathrm{d}x_1, \dots, \mathrm{d}x_d) := \sum_{i=1}^d \Gamma_i(\mathrm{d}x_i) \prod_{j \neq i} \delta_0(\mathrm{d}x_j), \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}, \tag{1.3.1}$$

is a PLM of $\boldsymbol{X}$ where Γ_i denotes the Lévy measure of a one-dimensional standard 1-stable Lévy process. The set of all marginal tail integrals $\{\overline{\Gamma}_{\perp,I} : I \subseteq \{1, \dots, d\}\}$ is given for $(x_1 \dots, x_{|I|}) \in (\overline{\mathbb{R}} \setminus \{0\})^{|I|}$ by

$$\overline{\Gamma}_{\perp,I}(x_1, \dots, x_{|I|}) = \begin{cases} 0, & \text{if } |I| > 1, \\ \dfrac{1}{x}, & \text{if } |I| = 1. \end{cases}$$

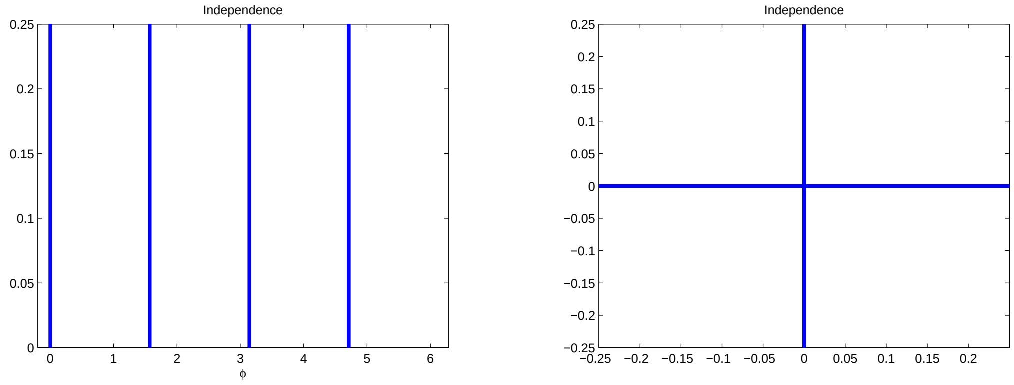

Figure 1.1: Spectral density of the independence PLM; left: atoms of the spectral measure $\mu_{\mathbb{S}}$ of the independence PLM in $[0, 2\pi)$ with uniform weights 0.25 on $0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$; right: the same density as Basrak graph

By (1.1.7) the corresponding *independence Lévy copula* is given by

$$\widehat{C}_{\perp}(u_1, \dots, u_d) = \sum_{i=1}^{d} u_i \prod_{i \neq i} 1_{\{\infty\}}(u_j), \quad \boldsymbol{u} \in (-\infty, \infty]^d, \tag{1.3.2}$$

which was proven in [42], Proposition 4.1. Obviously, $\Gamma_{\perp}$ is 1-homogeneous and since $\Gamma_{\perp}$ has mass only on the coordinate axes, its spectral measure for $d = 2$ is for all three norms given by

$$\mu_{\mathbb{S}}(\mathrm{d}\phi) \;=\; \frac{1}{4}\delta_0(\mathrm{d}\phi) + \frac{1}{4}\delta_{\frac{1}{2}\pi}(\mathrm{d}\phi) + \frac{1}{4}\delta_{\pi}(\mathrm{d}\phi) + \frac{1}{4}\delta_{\frac{3}{2}\pi}(\mathrm{d}\phi), \quad \phi \in [0, 2\pi).$$

Figure 1.1 shows the four atoms of the spectral measure $\mu_{\mathbb{S}}$ and in Figure 1.7 the representation of $\Gamma_{\perp}$ reduces to the point $(0, 0)$ as the PLC $\overline{\Gamma}_{\perp}$ on $\mathbb{S}_2 \cap (\mathbb{R} \setminus \{0\})^2$ is equal to zero.

1.3.2 Complete dependence Pareto Lévy measure

For the notion of complete dependence we recall the definition of ordered and strictly ordered sets.

Definition 1.3.2 (Strictly ordered set)
A subset $S \subset \mathbb{R}^d$ is called ordered*, if for every two vectors $\boldsymbol{x}, \boldsymbol{y} \in S$, either $x_k \leq y_k$ for all $k = 1, \dots, d$ or $x_k \geq y_k$ for all $k = 1, \dots, d$. S is called* strictly ordered*, if for*

every two different vectors $\boldsymbol{x}, \boldsymbol{y} \in S$, either $x_k < y_k$ for all $k = 1, \dots, d$ or $x_k > y_k$ for all $k = 1, \dots, d$.

Random variables $X^1, \dots, X^d$ are called *comonotonic* if their common distribution is supported by an ordered set. Then their copula and their survival copula are given by $C_{D,\|}(u_1, \dots, u_d) := u_1 \wedge \cdots \wedge u_d$ for $\boldsymbol{u} \in [0,1]$. Additionally, if $X^1, \dots, X^d$ are continuous and comonotonic, their common distribution is supported by a strictly ordered set. In this case every r. v. X^i is a. s. a strictly increasing function of every other, see [52], Theorem 2.10.14, and we call $X^1, \dots, X^d$ *completely dependent.* Complete dependence between jumps of a Lévy process means that the jumps of all components are a. s. determined by the jumps of every single component. Therefore, all components a. s. jump together and complete dependence is defined as below.

Definition 1.3.3 (Complete (positive) dependence, [42], Definition 4.2) *Let $\boldsymbol{X}$ be a Lévy process in $\mathbb{R}^d$. Set*

$$K := \{x \in \mathbb{R}^d : \operatorname{sgn}(x_1) = \cdots = \operatorname{sgn}(x_d)\} = [0,\infty)^d \cup (-\infty, 0)^d. \tag{1.3.3}$$

Its jumps are said to be completely (positive) dependent *if one of the following equivalent statements holds.*

(1) There is a strictly ordered set $S \subset K$ such that for almost all sample paths $\Delta \boldsymbol{X}_t := \boldsymbol{X}_t - \boldsymbol{X}_{t-} \in S$ for $t > 0$.

(2) There is a strictly ordered set $S \subset \big((0,\infty)^d \cup (-\infty,0)^d\big)$ such that $\Pi(\mathbb{R}^d \setminus S) = 0$.

Note that in Definition 1.3.3 (2) the condition $S \subset \big((0,\infty)^d \cup (-\infty,0)^d\big)$ can not be replaced by $S \subset K$. This can easily be seen by the example $\Pi(\mathrm{d}x_1, \mathrm{d}x_2) = \delta_{(0,1)}(\mathrm{d}x_1, \mathrm{d}x_2) + \delta_{(1,2)}(\mathrm{d}x_1, \mathrm{d}x_2)$ where the jumps of X^2 determine the jumps of X^1 but not vice versa.

Since the margins of a PLM are standardized the only PLM which is supported by a strictly ordered set $S \subset \big((0,\infty)^d \cup (-\infty,0)^d\big)$ is the *complete dependence PLM* $\Gamma_\|$ given by

$$\Gamma_\|(\mathrm{d}x_1, \dots, \mathrm{d}x_d) = |x_1|^{-2}\, \mathrm{d}x_1 1_{\{x_1 = \cdots = x_d\}}, \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}. \tag{1.3.4}$$

$\Gamma_{\|}$ is concentrated on $(\mathbb{R}\setminus\{0\})^d$ and so $\Gamma_{\|}$ is characterized by the corresponding PLC $\overline{\Gamma}_{\|}$, given for $(x_1\dots,x_d)\in(\overline{\mathbb{R}}\setminus\{0\})^d$ as

$$\overline{\Gamma}_{\|}(x_1,\dots,x_d)=\frac{1}{|x_1|\vee\dots\vee|x_d|}1_K((x_1,\dots,x_d))\prod_{j=1}^{d}\operatorname{sgn}(x_j). \tag{1.3.5}$$

By Relation (1.1.7) the corresponding *complete dependence Lévy copula* is given by

$$\widehat{C}_{\|}(u_1,\dots,u_d)=|u_1|\wedge\dots\wedge|u_d|1_K((u_1,\dots,u_d))\prod_{i=1}^{d}\operatorname{sgn}(u_i),\ \boldsymbol{u}\in(-\infty,\infty]^d, \tag{1.3.6}$$

see [42], Equation (4.3), and we reformulate Theorem 4.4 of [42] for PLMs.

Theorem 1.3.4
Let $\boldsymbol{X}$ be a Lévy process in $\mathbb{R}^d$ whose Lévy measure is supported by an ordered set $S\subset K$ where K is defined in (1.3.3). Then the complete dependence PLM $\Gamma_{\|}$, *given in (1.3.4), is a PLM of $\boldsymbol{X}$.*
Conversely, if $\Gamma_{\|}$ is a PLM of $\boldsymbol{X}$, then the Lévy measure of $\boldsymbol{X}$ is supported by an ordered subset of K. If, in addition, all marginal Lévy measures are infinite measures and have no atoms, then $\Gamma_{\|}$ is the unique PLM of $\boldsymbol{X}$ and the jumps of $\boldsymbol{X}$ are completely dependent.

For finite margins without atoms we obtain the following result.

Proposition 1.3.5
Let $\boldsymbol{X}$ be a Lévy process in $\mathbb{R}^d$ with absolutely continuous and finite Lévy measure Π and PLM $\Gamma_{\|}$. We set $\lambda^+:=\Pi([0,\infty)^d)$, $\lambda^-:=\Pi((-\infty,0]^d)$, $\lambda_i^-:=\Pi_i((-\infty,0))$ and $\lambda_i^+:=\Pi_i((0,\infty))$ for all $i=1,\dots,d$. Then the following statements hold.

(1) The jumps of $\boldsymbol{X}$ are completely dependent if and only if $\lambda_i^+=\lambda^+$ and $\lambda_i^-=\lambda^-$ for all $i=1,\dots,d$, the positive jump sizes are completely dependent, the negative jump size are completely dependent and the distribution of the jump sizes is concentrated on K, given in (1.3.3).

(2) If there is an index $i\in\{1,\dots,d\}$ with $\lambda_i^+>\max_{j\neq i}\lambda_j^+$ (or $\lambda_i^->\max_{j\neq i}\lambda_j^-$), then X^i has single positive jumps smaller than $\overline{\Pi}_i^{-1}\big(\max_{j\neq i}\lambda_j^+\big)$ (or single negative jumps bigger than $\overline{\Pi}_i^{-1}(-\max_{j\neq i}\lambda_j^-)$).

Proof.

(1) We denote the jump sizes by ΔX^i. Due to symmetry, we just consider positive jumps. If Π is supported by a strictly ordered set then

$$\lambda_i^+ = \lim_{x\downarrow 0} \overline{\Pi}_i(x) = \lim_{x\downarrow 0} \overline{\Pi}(x,\ldots,x) = \lambda^+.$$

With

$$\lambda^+ \mathbb{P}\left(\Delta X^1 > x_1, \ldots, \Delta X^d > x_d\right) = \overline{\Pi}(x_1,\ldots,x_d) \quad \text{for } (x_1,\ldots,x_d) \in (0,\infty)^d,$$

and Theorem 1.3.4 the assertion holds.

(2) Let $\lambda_i^+ > \max_{j\neq i} \lambda_j^+$. From Equation (1.1.14) we obtain for $0 < a_i \leq b_i < \overline{\Pi}_i^{-1}\left(\max_{j\neq i} \lambda_j^+\right)$ that

$$\Pi\left(\prod_{k<i}\{0\} \times (a_i,b_i] \times \prod_{k>i}\{0\}\right) = \Gamma_{\|}\left(\prod_{k<i} A_k \times \left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right] \times \prod_{k>i} A_k\right) > 0$$

where the sets A_k are defined in (1.1.13). □

Example 1.3.6

Let $\boldsymbol{X} = (X^1, X^2)$ be a spectrally positive CPP with jump size d. f. $F_1(x) = F_2(x) =$ expo(1) and PLM $\Gamma_{\|}$. Figure 1.2 shows simulated sample paths of $\boldsymbol{X}$ for the two situations of Proposition 1.3.5. For $\lambda_1 = \lambda_2$ the sample paths are equal since X^1 and X^2 jump at the same time by the same size. For $\lambda_1 = 50$ and $\lambda_2 = 10$ the component X^1 has additional single jumps smaller than $\overline{F}_1^{-1}(\lambda_2/\lambda_1) = -\ln(0.2)$.

For $d = 2$ the 1-homogeneous PLM $\Gamma_{\|}$ has mass only on $\{\boldsymbol{x} \in (\mathbb{R}\setminus\{\boldsymbol{0}\})^2 : x_1 = x_2\}$ and for all three norms the spectral measure is given by

$$\mu_{\mathbb{S}}(\mathrm{d}\phi) \;=\; \frac{1}{2}\delta_{\frac{1}{4}\pi}(\mathrm{d}\phi) + \frac{1}{2}\delta_{\frac{5}{4}\pi}(\mathrm{d}\phi), \quad \phi \in [0, 2\pi).$$

Figure 1.3 shows the two atoms of the spectral measure $\mu_{\mathbb{S}}$ and in Figure 1.7 the PLC $\overline{\Gamma}_{\|}$ is visualized on $\mathbb{S}_2 \cap (\mathbb{R}\setminus\{0\})^2$.

1.3.3 Archimedean Pareto Lévy measures

In [42], Section 6, they proved that parametric families of Lévy copulas can be constructed by a generator analogously to Archimedean copulas, cf. [52], Section 4.

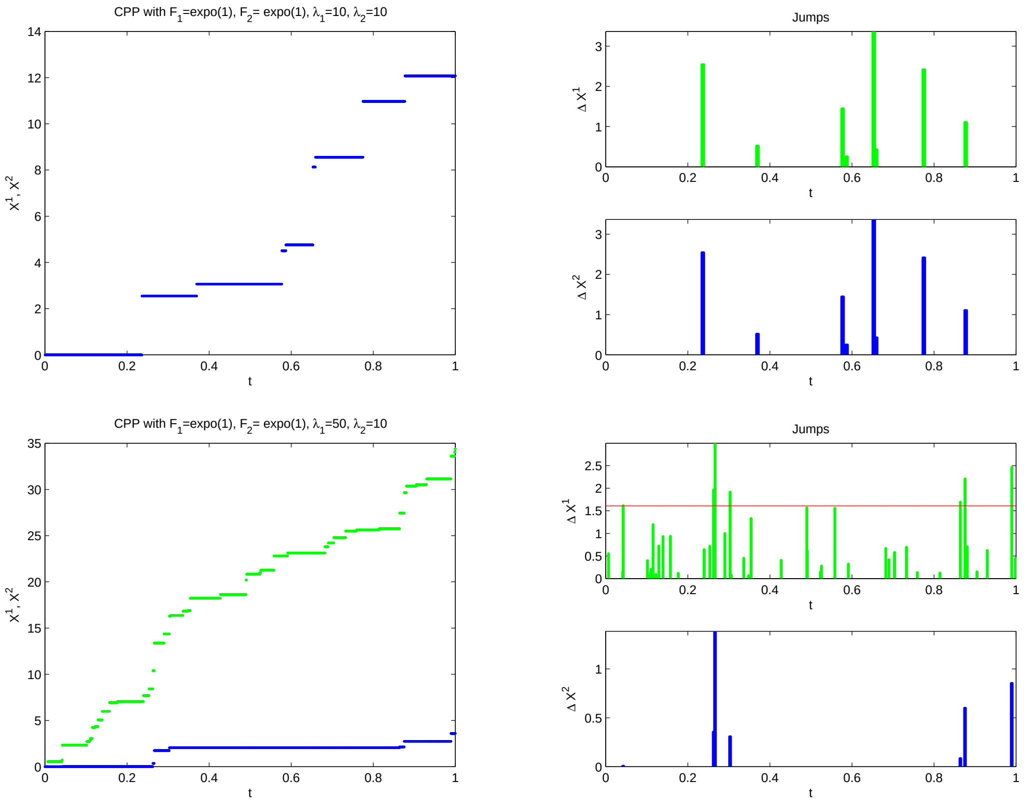

Figure 1.2: CPP (X^1, X^2) of Example 1.3.6; left: sample paths; right: jump times and jump sizes. For $\lambda_1 = \lambda_2$ the jumps are completely dependent. For $\lambda_1 > \lambda_2$ the component X^1 has single jumps smaller than $-\ln(0.2) \approx 1.6$.

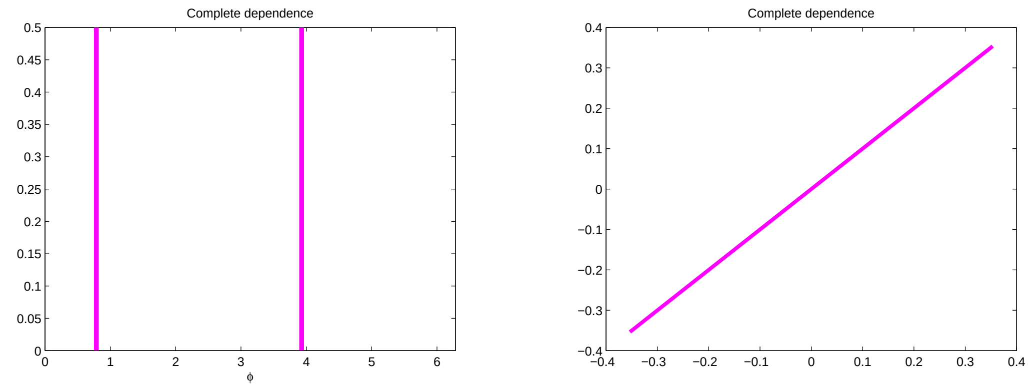

Figure 1.3: Spectral density of the complete dependence PLM; left: atoms of the spectral measure $\mu_{\mathbb{S}}$ of the complete dependence PLM in $[0, 2\pi)$ with uniform weights 0.5 on $\frac{1}{4}\pi, \frac{5}{4}\pi$; right: the same density as Basrak graph

We briefly summarize the construction of these so-called *Archimedean Lévy copulas* below.

Let $\varphi : [-1,1] \to [-\infty,\infty]$ be a strictly increasing continuous function with $\varphi(1) = \infty$, $\varphi(0) = 0$ and $\varphi(-1) = -\infty$, having derivatives of order up to d on $(-1,0)$ and $(0,1)$, satisfying for all $k = 1,\dots,d$,

$$\frac{\partial^k \varphi(u)}{\partial u^k} \geq 0, \quad u \in (0,1), \quad \text{and} \quad (-1)^k \frac{\partial^k \varphi(u)}{\partial u^k} \leq 0, \quad u \in (-1,0).$$

Set $\tilde{\varphi}(u) := 2^{d-2}\left(\varphi(u) - \varphi(-u)\right)$ for $u \in [-1,1]$. Then

$$\widehat{C}(u_1,\dots,u_d) = \varphi\left(\prod_{i=1}^{d} \tilde{\varphi}^{-1}\left(\frac{1}{u_i}\right)\right), \quad (u_1\dots,u_d) \in (-\infty,\infty]^d,$$

defines a Lévy copula, see [42], Theorem 6.1, which by (1.1.6) defines an *Archimedean Pareto Lévy measure*. For an Archimedean PLM Γ it holds $\Gamma(\mathbb{R}^d \setminus (\mathbb{R}\setminus\{0\})^d) = 0$ since Archimedean Lévy copulas are left-continuous in ∞. Thus an Archimedean PLM is uniquely defined by its PLC $\overline{\Gamma}$. Note that with Equation (1.1.14) a Lévy measure Π with an Archimedean PLM may have mass on $\mathbb{R}^d \setminus (\mathbb{R}\setminus\{0\})^d$.

Example 1.3.7 (Clayton PLM)
Setting

$$\varphi(x) = (-\log|x|)^{-1/\theta}\left(\eta 1_{\{x>0\}} - (1-\eta)1_{\{x<0\}}\right), \quad x \in [-1,1], \theta > 0, \eta \in (0,1),$$

yields the *Clayton Lévy copula family* given for $\theta > 0$ and $\eta \in [0,1]$ by

$$\begin{aligned}\widehat{C}_{\eta,\theta}(u_1,\dots,u_d) =& 2^{2-d}\left(\sum_{i=1}^{d}|u_i|^{-\theta}\right)^{-1/\theta} \\ & \left(\eta 1_{\{u_1\cdots u_d \geq 0\}} - (1-\eta)1_{\{u_1\cdots u_d<0\}}\right), \quad \boldsymbol{u} \in (-\infty,\infty]^d,\end{aligned}$$

see [42], Example 6.2. For $\eta = 1$ the two components always jump in the same direction, for $\eta = 0$ in opposite direction. The parameter θ models the degree of dependence: for $\eta = 1$ and $\theta \uparrow \infty$ we obtain the complete dependence model and for $\eta = 1$ and $\theta \downarrow 0$ the independence model. For $d = 2$ the Clayton Lévy copula becomes

$$\widehat{C}_{\eta,\theta}(u_1,u_2) = \left(|u_1|^{-\theta} + |u_2|^{-\theta}\right)^{-1/\theta}\left(\eta 1_{\{u_1u_2\geq 0\}} - (1-\eta)1_{\{u_1u_2<0\}}\right) \tag{1.3.7}$$

which was frequently used, e. g. in [17, 18, 20, 26].

The corresponding *Clayton Pareto Lévy copula* is for $\theta > 0$ and $\eta \in [0,1]$ given as

$$\begin{aligned}\overline{\Gamma}_{\eta,\theta}(x_1,\ldots,x_d) =& 2^{2-d}\left(\sum_{i=1}^{d}|x_i|^{\theta}\right)^{-1/\theta}\\ &\left(\eta 1_{\{x_1\cdots x_d>0\}} - (1-\eta)1_{\{x_1\cdots x_d<0\}}\right), \quad \boldsymbol{x}\in(\overline{\mathbb{R}}\setminus\{0\})^d, \qquad (1.3.8)\end{aligned}$$

and so Clayton PLMs are obviously 1-homogeneous. For $d = 2$ the density of a Clayton PLM $\Gamma_{\eta,\theta}$ is given by

$$\begin{aligned}\Gamma_{\eta,\theta}(\mathrm{d}x_1,\mathrm{d}x_2) =& (1+\theta)\left(|x_1|^{\theta}+|x_2|^{\theta}\right)^{-1/\theta-2}|x_1|^{\theta-1}|x_2|^{\theta-1}\\ &\left(\eta 1_{\{x_1x_2>0\}} + (1-\eta)1_{\{x_1x_2<0\}}\right)\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad (x_1,x_2)\in\mathbb{R}^2.\end{aligned}$$

By polar coordinate transformation as in Equation (1.2.2) we get

$$\begin{aligned}\frac{\Gamma^{\phi}_{\eta,\theta}(\mathrm{d}\phi)}{\mathrm{d}\phi} =& (1+\theta)\left(|\cos(\phi)|^{\theta}+|\sin(\phi)|^{\theta}\right)^{-1/\theta-2}|\cos(\phi)|^{\theta-1}|\sin(\phi)|^{\theta-1}\\ &\left(\eta 1_{\{\cos(\phi)\sin(\phi)>0\}} + (1-\eta)1_{\{\cos(\phi)\sin(\phi)<0\}}\right), \quad \phi\in[0,2\pi), \qquad (1.3.9)\end{aligned}$$

and with Equation (1.2.3) the spectral density with respect to the 2-norm results as

$$\frac{\mu_{\mathbb{S}_2}(\mathrm{d}\phi)}{\mathrm{d}\phi} = \frac{1}{\int_0^{2\pi}\ \Gamma^{\phi}_{\eta,\theta}(\mathrm{d}\phi)}\frac{\Gamma^{\phi}_{\eta,\theta}(\mathrm{d}\phi)}{\mathrm{d}\phi} \qquad (1.3.10)$$

which is visualized in Figure 1.4. For the 1-norm the spectral density becomes with Equation (1.2.5)

$$\frac{\mu_{\mathbb{S}_1}(\mathrm{d}\phi)}{\mathrm{d}\phi} = \frac{1}{\int_0^{2\pi}|(\cos(\phi),\sin(\phi))|_1\ \Gamma^{\phi}_{\eta,\theta}(\mathrm{d}\phi)}\frac{|(\cos(\phi),\sin(\phi))|_1\ \Gamma^{\phi}_{\eta,\theta}(\mathrm{d}\phi)}{\mathrm{d}\phi} \qquad (1.3.11)$$

plotted in Figure 1.5. The spectral density with respect to the maximum-norm is according to Equation (1.2.5) given as

$$\frac{\mu_{\mathbb{S}_\infty}(\mathrm{d}\phi)}{\mathrm{d}\phi} = \frac{1}{\int_0^{2\pi}|(\cos(\phi),\sin(\phi))|_\infty\ \Gamma^{\phi}_{\eta,\theta}(\mathrm{d}\phi)}\frac{|(\cos(\phi),\sin(\phi))|_\infty\ \Gamma^{\phi}_{\eta,\theta}(\mathrm{d}\phi)}{\mathrm{d}\phi}, \qquad (1.3.12)$$

visualized in Figure 1.6.

Comparing the different spectral measures we see that $\mu_{\mathbb{S}_2}$ and $\mu_{\mathbb{S}_1}$ are similar, but for the same θ-values $\mu_{\mathbb{S}_2}$ has more mass near the axes than $\mu_{\mathbb{S}_1}$. The spectral measure $\mu_{\mathbb{S}_\infty}$ is the most concentrated near the axes and, in particular, its density is not differentiable on $\phi = \pi/4$ and $\phi = 5\pi/4$.

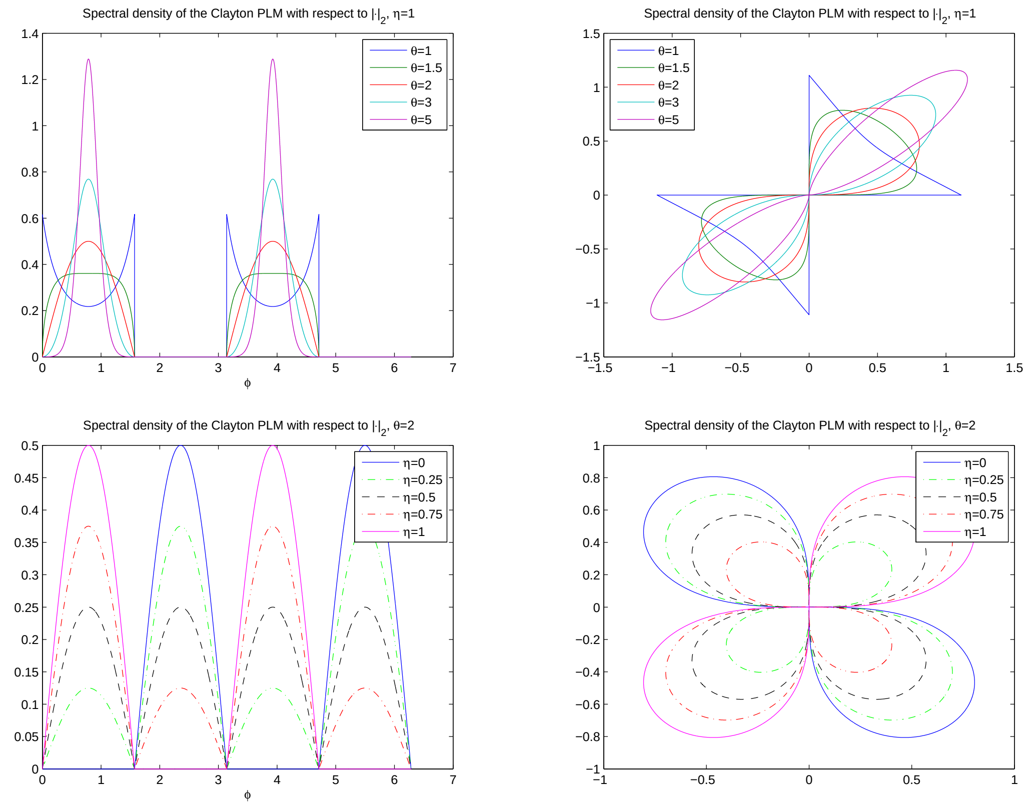

Figure 1.4: Spectral density of Clayton PLM with respect to $|\cdot|_2$ given in (1.3.10) for different parameter values for $\theta > 0$ and $\eta \in [0, 1]$; left: spectral density $\mu_{\mathbb{S}_2}(\mathrm{d}\phi)/\mathrm{d}\phi$ on $[0, 2\pi)$; right: the same spectral density as Basrak graph

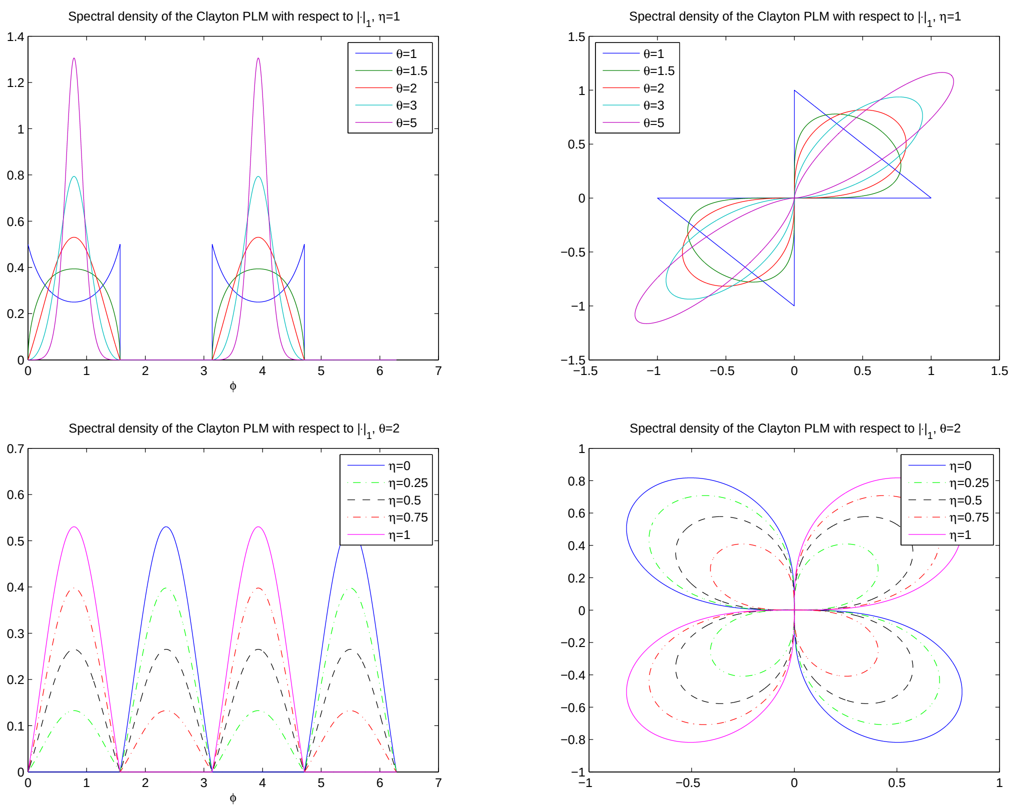

Figure 1.5: Spectral density of Clayton PLM with respect to $|\cdot|_1$ given in (1.3.11) for different parameter values for $\theta > 0$ and $\eta \in [0, 1]$; left: spectral density $\mu_{\mathbb{S}_1}(\mathrm{d}\phi)/\mathrm{d}\phi$ on $[0, 2\pi)$; right: the same spectral density as Basrak graph

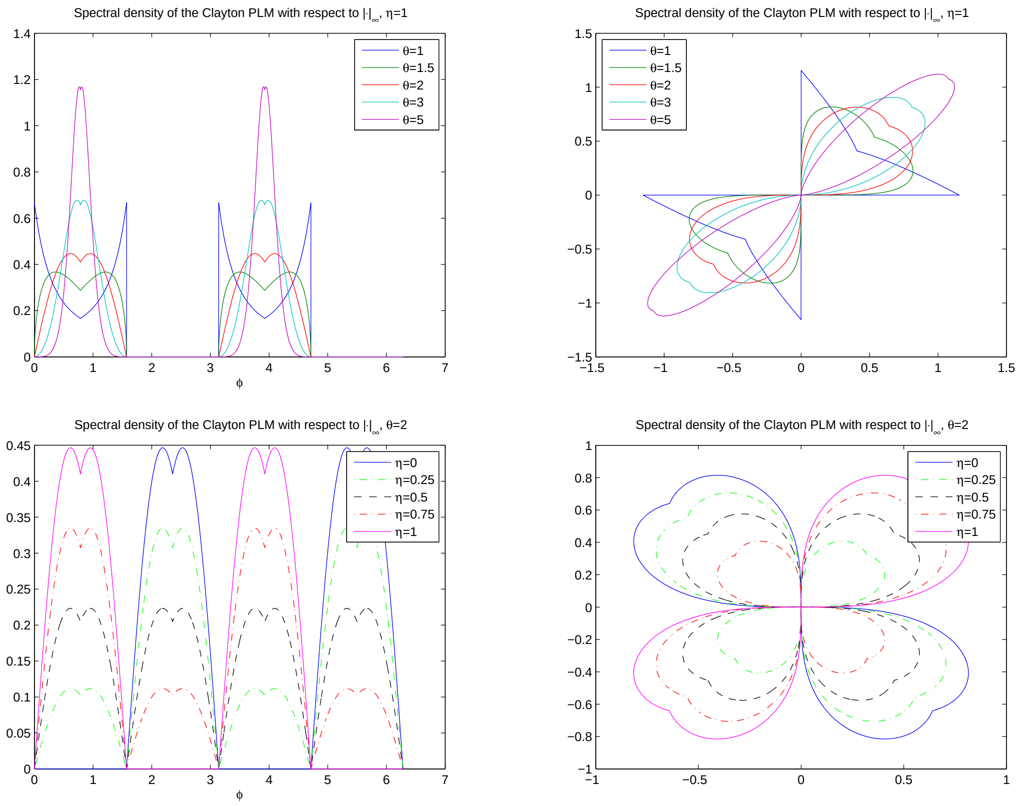

Figure 1.6: Spectral density of Clayton PLM with respect to $|\cdot|_\infty$ given in (1.3.12) for different parameter values for $\theta > 0$ and $\eta \in [0,1]$; left: spectral density $\mu_{\mathbb{S}_\infty}(\mathrm{d}\phi)/\mathrm{d}\phi$ on $[0, 2\pi)$; right: the same spectral density as Basrak graph

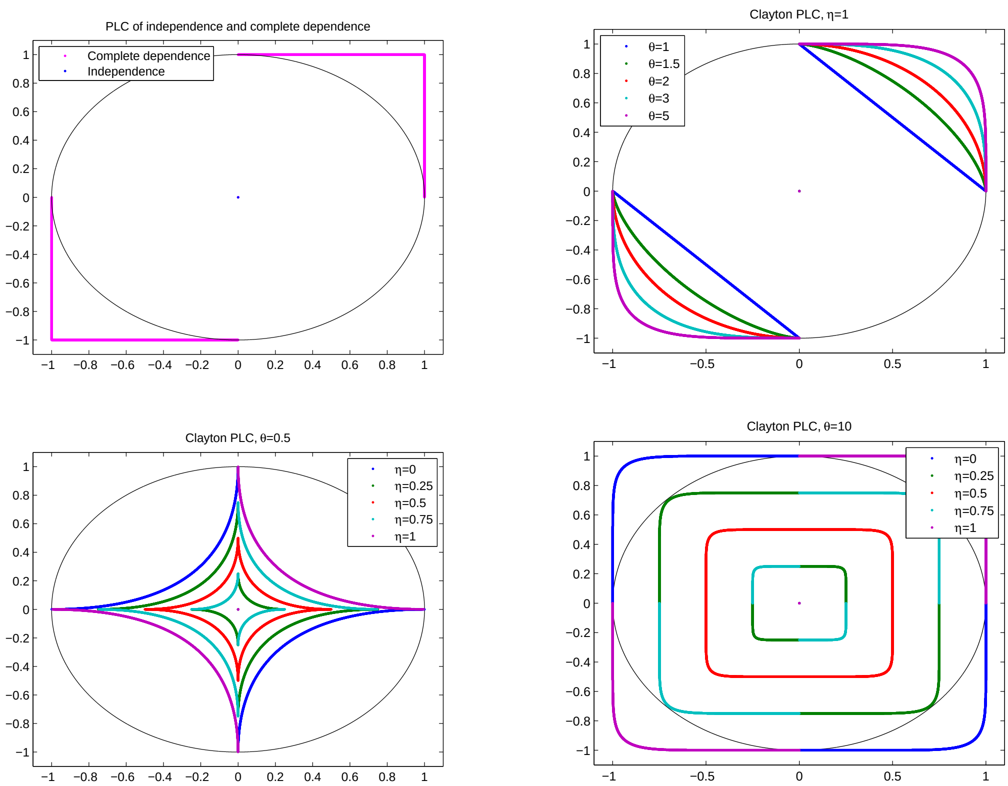

Figure 1.7: Independence PLC $\overline{\Gamma}_{\perp}$, complete dependence $\overline{\Gamma}_{\parallel}$ and Clayton PLC $\overline{\Gamma}_{\eta,\theta}$ for different parameter values for θ and $\eta \in [0,1]$; For $\eta = 1$ and increasing θ the Clayton PLM becomes the complete dependence PLM and for $\eta = 1$ and decreasing θ the independence PLM.

Furthermore, we visualize the Clayton PLC $\overline{\Gamma}_{\eta,\theta}$ given in (1.3.8) for $d = 2$ in Figure 1.7. For $\eta = 1$ comparing the upper pictures shows that increasing θ yields complete dependence since the mass near $\pi/4$ and $5\pi/4$ increase as shown in Figures 1.4, 1.5 and 1.6. For decreasing θ the PLC is attracted to the origin since the mass of $\Gamma_{\eta,\theta}$ moves to the axes and the PLM becomes the independence model. Different parameter values for η show how the mass of $\Gamma_{\eta,\theta}$ is distributed over the four quadrants.

Example 1.3.8 (A non-homogeneous Archimedean PLM)
Setting

$$\varphi(x) = \zeta \frac{|x|}{1-|x|} \left(\eta 1_{\{x>0\}} - (1-\eta) 1_{\{x<0\}}\right), \quad x \in [-1,1], \zeta > 0, \eta \in (0,1),$$

yields the Archimedean Lévy copula given for $\zeta > 0$ and $\eta \in [0,1]$ as

$$\begin{aligned}\widehat{C}_{\eta,\zeta}(u_1,\dots,u_d) =& \frac{\zeta \prod_{i=1}^d |u_i|}{\prod_{i=1}^d (|u_i|+\zeta) - \prod_{i=1}^d |u_i|} \\ & \left(\eta 1_{\{u_1 \cdots u_d \geq 0\}} - (1-\eta) 1_{\{u_1 \cdots u_d < 0\}}\right), \quad \boldsymbol{u} \in (-\infty,\infty]^d,\end{aligned} \tag{1.3.13}$$

which becomes for $d=2$

$$\widehat{C}_{\eta,\zeta}(u_1,u_2) = \frac{|u_1 u_2|}{|u_1|+|u_2|+\zeta} \left(\eta 1_{\{u_1 u_2 \geq 0\}} - (1-\eta) 1_{\{u_1 u_2 < 0\}}\right). \tag{1.3.14}$$

The corresponding Archimedean PLC is given for $\zeta > 0$ and $\eta \in [0,1]$ as

$$\begin{aligned}\overline{\Gamma}_{\eta,\zeta}(x_1,\dots,x_d) =& \frac{\zeta}{\prod_{i=1}^d (1+\zeta|x_i|) - 1} \\ & \left(\eta 1_{\{x_1 \cdots x_d > 0\}} - (1-\eta) 1_{\{x_1 \cdots x_d < 0\}}\right), \quad \boldsymbol{x} \in (\overline{\mathbb{R}} \setminus \{0\})^d\end{aligned}$$

and becomes for $d=2$

$$\overline{\Gamma}_{\eta,\zeta}(x_1,x_2) = \frac{1}{|x_1|+|x_2|+\zeta|x_1 x_2|} \left(\eta 1_{\{x_1 x_2 > 0\}} - (1-\eta) 1_{\{x_1 x_2 < 0\}}\right).$$

For $\eta = 1$ we obtain with $\zeta \downarrow 0$ the Clayton PLM given in Example 1.3.7 with Parameter $\theta = 1$ and with $\zeta \uparrow \infty$ independence. Obviously, this PLM is not 1-homogeneous, although it has 1-homogeneous one-dimensional margins. Additionally, in Example 2.2.4 we shall show that $\Gamma_{\eta,\zeta}$ is not regularly varying and, therefore, a spectral measure does not even exist in the limit.

For $d=2$ the non-homogeneous PLM $\Gamma_{\eta,\zeta}$ has the density

$$\begin{aligned}\Gamma_{\eta,\zeta}(\mathrm{d}x_1,\mathrm{d}x_2) =& \frac{2+\zeta|x_1|+\zeta|x_2|+\zeta^2|x_1 x_2|}{(|x_1|+|x_2|+\zeta|x_1 x_2|)^3} \\ & \left(\eta 1_{\{x_1 x_2 > 0\}} + (1-\eta) 1_{\{x_1 x_2 < 0\}}\right) \mathrm{d}x_1 \mathrm{d}x_2, \quad (x_1,x_2) \in \mathbb{R}^2,\end{aligned}$$

and transformation to polar coordinates yields

$$\Gamma_{\eta,\zeta} \circ T(\mathrm{d}r,\mathrm{d}\phi) = r^{-2}\, \mathrm{d}r\, \Gamma^{r,\phi}_{\eta,\zeta}(r,\mathrm{d}\phi) \quad r \in [0,\infty), \phi \in [0,2\pi),$$

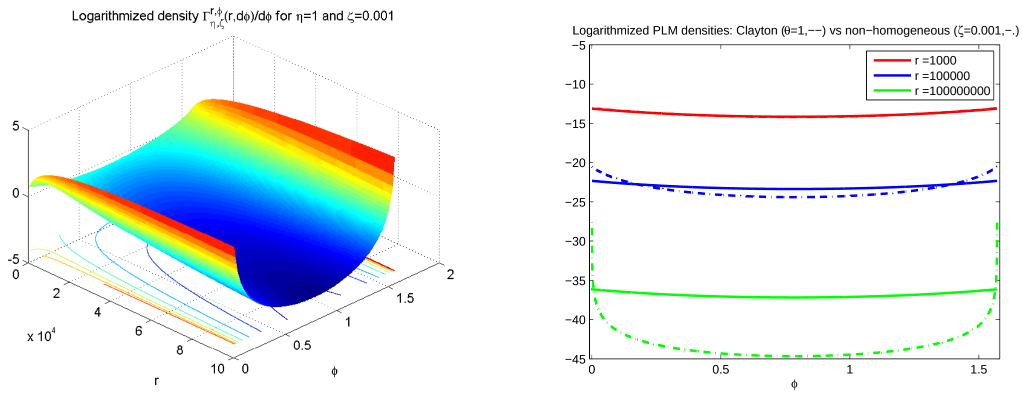

Figure 1.8: left: Logarithmized density $\Gamma^{r,\phi}_{\eta,\zeta}(r, \mathrm{d}\phi)/\mathrm{d}\phi$ given in (1.3.15) for $\eta = 1$ and $\zeta = 0.001$; For increasing radius r the density decreases near $\pi/4$ and increases near 0 and $\pi/2$. right: Comparison of logarithmized densities of the Clayton PLM for $\eta = 1$ and $\theta = 1$ and the non-homogeneous PLM for $\eta = 1$ and $\zeta = 0.001$; For small r the densities are almost identical. For increasing r the non-homogeneous density decreases rapidly near $\pi/4$ and more slowly near 0 and $\pi/2$, whereas the homogeneous density decreases uniformly for all angles.

where

$$\frac{\Gamma^{r,\phi}_{\eta,\zeta}(\mathrm{d}\phi)}{\mathrm{d}\phi} = \frac{2 + \zeta r|\cos(\phi)| + \zeta r|\sin(\phi)| + \zeta^2 r^2 |\cos(\phi)\sin(\phi)|}{(|\cos(\phi)| + |\sin(\phi)| + \zeta r|\cos(\phi)\sin(\phi)|)^3} \left(\eta 1_{\{\cos(\phi)\sin(\phi)>0\}} + (1-\eta)1_{\{\cos(\phi)\sin(\phi)<0\}}\right). \tag{1.3.15}$$

Contrary to Equation (1.3.9), the density $\Gamma^{r,\phi}_{\eta,\zeta}(\mathrm{d}\phi)/\mathrm{d}\phi$ depends on the radius r. Consequently, for increasing r the density $\Gamma^{r,\phi}_{\eta,\zeta}(\mathrm{d}\phi)/\mathrm{d}\phi$ increases for angles near $\phi \in \{0, \pi/2, \pi, 3\pi/2\}$ and decreases for angles near $\phi \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ as shown in the left picture of Figure 1.8. So for increasing radius the density of the non-homogeneous PLM decreases strongly for angles near $\phi \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ and weakly for angles near $\phi \in \{0, \pi/2, \pi, 3\pi/2\}$, whereas the density of the homogeneous Clayton PLM decreases uniformly for all angles. For $\eta = 1$ both densities are compared in the right picture of Figure 1.8. For a small radius r both densities are almost identical, but for increasing r the non-homogeneous density decreases rapidly near $\pi/4$ and more slowly near 0 and $\pi/2$, whereas the homogeneous density decreases uniformly for all angles. Therefore, the non-homogeneous PLM $\Gamma_{\eta,\zeta}$ has, for extreme values, more mass near the axes than the 1-homogeneous Clayton PLM. This effect we shall see again in Section 3.4.2.

Since there is no spectral measure we visualize the non-homogeneous PLM by its PLC $\overline{\Gamma}_{\eta,\zeta}$ in Figure 1.9. Comparing with Figure 1.7 we see that for small values of ζ the PLM $\Gamma_{\eta,\zeta}$ is similar to the Clayton PLM $\Gamma_{\eta,\theta}$ for $\theta = 1$. On the other side, we see that for decreasing parameter ζ the PLC values become smaller and the PLM converges to independence.

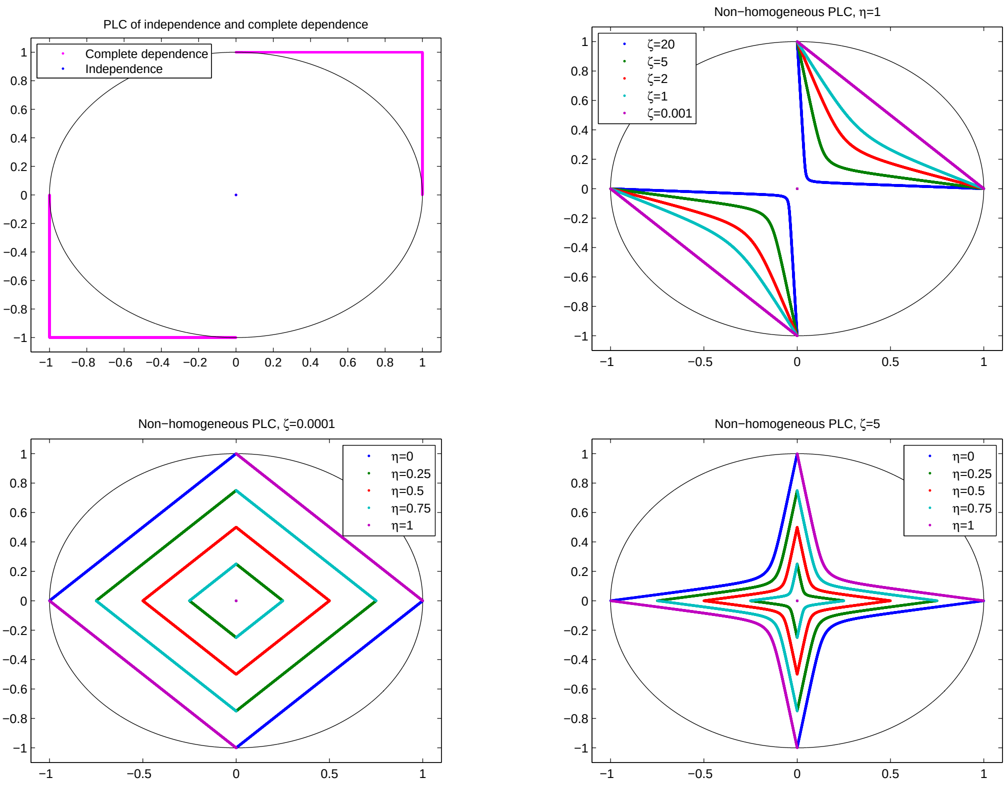

Figure 1.9: Independence PLC $\overline{\Gamma}_{\perp}$, complete dependence PLC $\overline{\Gamma}_{\parallel}$ and the non-homogeneous PLC $\overline{\Gamma}_{\eta,\zeta}$ for different parameter values for $\zeta > 0$ and $\eta \in [0,1]$; For $\eta = 1$ and for $\eta = 1$ and increasing ζ the non-homogeneous PLM becomes the independence PLM and for decreasing ζ the Clayton PLM with parameter $\theta = 1$.

1.3.4 Further construction of Pareto Lévy measures

Obviously, every convex linear combination of PLMs Γ^i, i. e.

$$\sum_{i=1}^{n} \alpha_i \Gamma^i \quad \text{with} \quad \sum_{i=1}^{n} \alpha_i = 1,$$

defines a PLM. This parallels the result of copulas, cf. [52], Exercise 2.3. for Lévy measures. Figure 1.10 shows for $\eta = 1$ the PLC of the PLMs defined by $\frac{1}{2}\Gamma_{\|} + \frac{1}{2}\Gamma_{1,\theta}$ (complete dependence + Clayton) and $\frac{1}{2}\Gamma_{\|} + \frac{1}{2}\Gamma_{1,\zeta}$ (complete dependence + non-homogeneous), respectively.

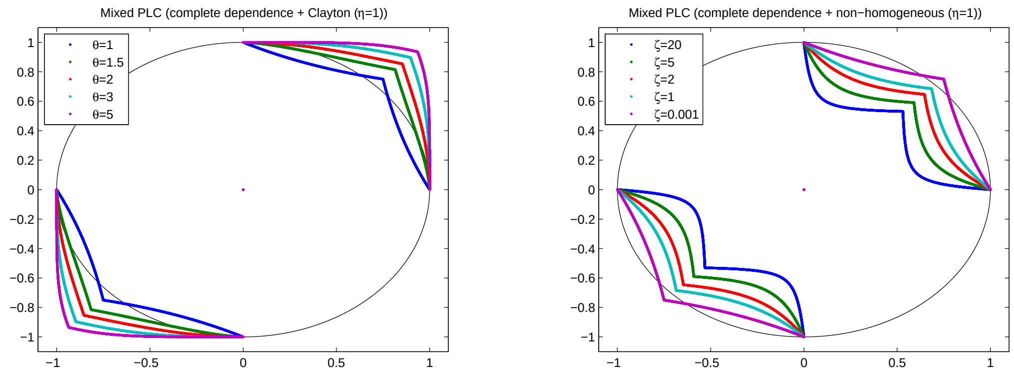

Figure 1.10: PLC of mixed PLMs; left: $\frac{1}{2}\Gamma_{\|}+\frac{1}{2}\Gamma_{1,\theta}$ (complete dependence + Clayton); right: $\frac{1}{2}\Gamma_{\|} + \frac{1}{2}\Gamma_{1,\zeta}$ (complete dependence + non-homogeneous)

Chapter 2

Multivariate regular variation of Lévy measures

In this chapter we investigate the dependence between the jumps of a multivariate regularly varying Lévy process modelled by a PLM. Regular variation of a Lévy process $\boldsymbol{X}$ is equivalent to regular variation of the r. v. $\boldsymbol{X}_1$ and its Lévy measure; cf. Lemma 4.1.4. The notion of multivariate regular variation of r. v. s has been in the focus of multivariate extreme value theory for years; cf. [55, 56]. Classical definitions and concepts of regular variation are postponed to the Appendix as not to disturb the flow of arguments. As in the case of r. v. s regular variation of Lévy measures is formulated in terms of vague convergence of Radon measures on the one-point uncompactification $\mathbb{E} := \overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$, equipped with the usual topology such that

$$\mathcal{B}(\mathbb{E}) \cap (\mathbb{R}^d \setminus \{\mathbf{0}\}) = \mathcal{B}(\mathbb{R}^d) \cap (\mathbb{R}^d \setminus \{\mathbf{0}\})$$

and the Borel sets of $\mathbb{R}^d$ bounded away from $\mathbf{0}$ are relatively compact in $\mathbb{E}$. Therefore, we consider in this section Lévy measures w. l. o. g. on $\mathbb{E}$ by setting

$$\Pi(B) := \Pi(B \cap \mathbb{R}^d) \quad \text{for } B \in \mathcal{B}(\mathbb{E}). \tag{2.0.1}$$

Regular variation in all generality has been investigated in [35] and for the notion of one-point uncompactification we refer to [56].

Definition 2.0.1 ([35], Section 3)
A Lévy measure Π *on* $\mathbb{E}$ *is called* regularly varying *if one of the following equivalent definitions (1) or (2) holds.*

(1) *There exists a norming sequence* $\{c_n\}_{n\in\mathbb{N}}$ *of positive numbers with* $c_n \uparrow \infty$ *as* $n \to \infty$ *and a non-zero Radon measure* μ *on* $\mathcal{B}(\mathbb{E})$ *with* $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ *such that*

$$n\Pi(c_n \cdot) \xrightarrow{v} \mu(\cdot) \quad \text{as } n \to \infty \tag{2.0.2}$$

where $\xrightarrow{v}$ *denotes vague convergence on* $\mathcal{B}(\mathbb{E})$*. Then the limit measure* μ *is necessarily homogeneous of some degree* $\alpha > 0$*. If* Π *is regularly varying with index* α*, norming sequence* $\{c_n\}_{n\in\mathbb{N}}$ *and limit measure* μ *then we write* $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$.

(2) *There exists a finite non-zero measure* $\mu_{\mathbb{S}}$ *on* $\mathcal{B}(\mathbb{S})$ *such that for all* $u > 0$

$$\frac{\Pi(\{\boldsymbol{x} \in \mathbb{E} : |\boldsymbol{x}| > tu, \boldsymbol{x}/|\boldsymbol{x}| \in \cdot\})}{\Pi(\{\boldsymbol{x} \in \mathbb{E} : |\boldsymbol{x}| > t\})} \xrightarrow{w} u^{-\alpha}\mu_{\mathbb{S}}(\cdot) \quad \text{as } t \to \infty\,, \tag{2.0.3}$$

where $\xrightarrow{w}$ *denotes weak convergence on* $\mathcal{B}(\mathbb{S})$*. We call* $\mu_{\mathbb{S}}$ *the* spectral measure *of* Π.

We did not specify the norm $|\cdot|$ since regular variation does not depend on the choice of the norm. This can be seen by the relation between regular variation for Lévy measures and r. v. s, given by Lemma 4.1.4, and the fact that the chosen norm does not affect the regular variation of r. v. s.

The next lemma is well-known, but in order to keep this thesis self-contained we give its proof in the Appendix.

Lemma 2.0.2
Let Π *be a* d*-dimensional Lévy measure. If* $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$ *then for* $x > 0$ *and for all* $i = 1, \dots, d$*, it holds*

$$n\overline{\Pi}_i(c_n x) \to \overline{\mu}_i(1)x^{-\alpha} \quad \text{and} \quad n\overline{\Pi}_i(-c_n x) \to \overline{\mu}_i(-1)x^{-\alpha} \quad \text{as } n \to \infty, \tag{2.0.4}$$

where $\mu_i(B) := \mu(\{\boldsymbol{x} \in \mathbb{E} : x_i \in B\})$ *for* $B \in \mathcal{B}(\overline{\mathbb{R}}\setminus\{0\})$ *and* $\overline{\mu}_i(1), |\overline{\mu}_i(-1)| \in [0, \infty)$*. Furthermore, there exists an index* $i_* \in \{1, \dots, d\}$ *such that* $\overline{\mu}_{i_*}(1) - \overline{\mu}_{i_*}(-1) > 0$ *and* $\Pi_{i_*} \in \mathrm{RV}(\alpha, c_n, \mu_{i_*})$.

By Lemma 2.0.2 multivariate regular variation of a Lévy measure Π implies regular variation of at least one of the one-dimensional marginal Lévy measures Π_i. In Section 2.1 we shall extend this result with respect to the PLM of the regularly varying Lévy measure. Further, we prove the converse result and give conditions on the margins and the PLM such that the Lévy measure is regularly varying. In Section 2.2 we apply this result to the four examples of Section 1.3.

2.1 Multivariate regular variation and Pareto Lévy measures

To prove the converse of Lemma 2.0.2 we assume w. l. o. g. that $\Pi_1 \in \mathrm{RV}(\alpha, c_n, \mu_1)$. We also assume that the following tail balance conditions hold for $x > 0$ and for all $i = 1, \ldots, d$:

$$n\overline{\Pi}_i(c_n x) \to p_i^+ x^{-\alpha} \quad \text{and} \quad -n\overline{\Pi}_i(-c_n x) \to p_i^- x^{-\alpha} \quad \text{as } n \to \infty, \tag{2.1.1}$$

where $p_i^+, p_i^- \in [0, \infty)$. For $x \in \overline{\mathbb{R}}$ we define

$$p_i^{\mathrm{sgn}(x)} := \begin{cases} p_i^+\,, & \text{if } x \geq 0, \\ p_i^-\,, & \text{if } x < 0. \end{cases}$$

The result below corresponds to [48], Theorem 3.1, and extends [9], Theorem 1, for general Lévy measures.

Theorem 2.1.1
Let Π be the d-dimensional Lévy measure defined in (1.1.8) with PLM Γ and one-dimensional Lévy measures $\Pi_i, i = 1, \ldots, d$. Suppose that $\Pi_1 \in \mathrm{RV}(\alpha, c_n, \mu_1)$ and that the tail balance conditions (2.1.1) for the margins hold. Furthermore, suppose that $\Gamma \in \mathrm{RV}(1, n, \nu)$. Then $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$ where for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ and for $i = 1, \ldots, d$

$$\widetilde{a}_i := \begin{cases} 0, & \text{if } a_i = 0, \\ \dfrac{\mathrm{sgn}(a_i)}{p_i^{\mathrm{sgn}(a_i)}}|a_i|^\alpha, & \text{if } a_i \neq 0, p_i^{\mathrm{sgn}(a_i)} > 0, \\ \infty, & \text{if } a_i > 0, p_i^+ = 0, \\ -\infty, & \text{if } a_i < 0, p_i^- = 0, \end{cases} \tag{2.1.2}$$

and $\widetilde{b}_i$ is defined analogously, we have

$$\mu((\boldsymbol{a}, \boldsymbol{b}]) = \nu\left(\prod_{i=1}^d (\widetilde{a}_i, \widetilde{b}_i]\right). \tag{2.1.3}$$

Furthermore,

$$\lim_{t\to\infty} \frac{\overline{\Pi}(t, \ldots, t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(1, \ldots, 1)}{\overline{\mu}_1(1)} = \frac{1}{p_1^+}\overline{\nu}\left(\frac{1}{p_1^+}, \ldots, \frac{1}{p_d^+}\right), \tag{2.1.4}$$

$$\lim_{t\to-\infty} \frac{\overline{\Pi}(t, \ldots, t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(-1, \ldots, -1)}{\overline{\mu}_1(-1)} = \frac{-1}{p_1^-}\overline{\nu}\left(\frac{-1}{p_1^-}, \ldots, \frac{-1}{p_d^-}\right). \tag{2.1.5}$$

Proof.
First we show that $\{n\Pi(c_n\cdot)\}_{n\in\mathbb{N}}$ is relatively compact in the vague topology. Since Π is a Lévy measure for the ball $B_{\mathbf{0},r} = \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x} - \mathbf{0}| < r\}$ we get

$$\sup_{n\in\mathbb{N}} n\Pi\left(c_n(\mathbb{R}^d \setminus B_{\mathbf{0},r})\right) < \infty \quad \text{for all } r > 0$$

and by [41], Theorem 15.7.5, the sequence $\{n\Pi(c_n\cdot)\}_{n\in\mathbb{N}}$ is relatively compact. So there are subsequential vague limits and by [35], Theorem 2.8, we have to show convergence for sets in a determining class. The sets $\{(\boldsymbol{a},\boldsymbol{b}] : \boldsymbol{a},\boldsymbol{b} \in \mathbb{E}, \boldsymbol{a} \le \boldsymbol{b}\}$ are an $\cap$-stable generator of $\mathcal{B}(\mathbb{E})$, but by extension (2.0.1) it is enough to investigate convergence on the sets $\{(\boldsymbol{a},\boldsymbol{b}] : \boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^d \setminus \{\mathbf{0}\}, \boldsymbol{a} \le \boldsymbol{b}\}$. Consequently, we have to show that $n\Pi(c_n(\boldsymbol{a},\boldsymbol{b}]) \to \mu((\boldsymbol{a},\boldsymbol{b}])$ as $n \to \infty$ for all sets $(\boldsymbol{a},\boldsymbol{b}]$ with $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $\mathbf{0} \notin \overline{(\boldsymbol{a},\boldsymbol{b}]}$ and

$$\mu(\partial(\boldsymbol{a},\boldsymbol{b}]) = \mu\left(\bigcup_{k=1}^{d} \prod_{i<k} [a_i, b_i] \times \{a_k, b_k\} \times \prod_{i>k} [a_i, b_i]\right) = 0$$

where μ is a non-zero Radon measure with $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ and homogeneous of degree α. For $\boldsymbol{a},\boldsymbol{b} \in \mathbb{E}$ and the weights p_i given in (2.1.1) we define the index sets

$$\begin{aligned}
K_1 &:= \{i : a_i b_i \neq 0, p_i^{\operatorname{sgn}(a_i)} p_i^{\operatorname{sgn}(b_i)} > 0\},\\
K_2 &:= \{i : a_i b_i \neq 0, p_i^{\operatorname{sgn}(a_i)} > 0, p_i^{\operatorname{sgn}(b_i)} = 0\},\\
K_3 &:= \{i : a_i b_i \neq 0, p_i^{\operatorname{sgn}(a_i)} = 0, p_i^{\operatorname{sgn}(b_i)} > 0\},\\
K_4 &:= \{i : a_i b_i > 0, p_i^{\operatorname{sgn}(a_i)} = p_i^{\operatorname{sgn}(b_i)} = 0\},\\
K_5 &:= \{i : a_i < 0 < b_i, p_i^{\operatorname{sgn}(a_i)} = p_i^{\operatorname{sgn}(b_i)} = 0\},\\
K_6 &:= \{i : a_i = 0, p_i^{\operatorname{sgn}(b_i)} > 0\},\\
K_7 &:= \{i : a_i = 0, p_i^{\operatorname{sgn}(b_i)} = 0\},\\
K_8 &:= \{i : b_i = 0, p_i^{\operatorname{sgn}(a_i)} > 0\},\\
K_9 &:= \{i : b_i = 0, p_i^{\operatorname{sgn}(a_i)} = 0\}.
\end{aligned} \tag{2.1.6}$$

Moreover, we set for $\boldsymbol{a},\boldsymbol{b} \in \mathbb{E}$ with $\mathbf{0} \notin \overline{(\boldsymbol{a},\boldsymbol{b}]}$

$$\begin{aligned}
&\mu((\boldsymbol{a},\boldsymbol{b}]) \qquad\qquad (2.1.7)\\
&:= \nu\Bigg(\prod_{i\in K_1}\left(\frac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}}|a_i|^\alpha, \frac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}}|b_i|^\alpha\right] \times \prod_{i\in K_2}\left(\frac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}}|a_i|^\alpha, \infty\right)\\
&\quad \times \prod_{i\in K_3}\left(-\infty, \frac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}}|b_i|^\alpha\right] \times \prod_{i\in K_4}\emptyset \times \prod_{i\in K_5}(-\infty,\infty) \times \prod_{i\in K_6}\left(0, \frac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}}|b_i|^\alpha\right]\\
&\quad \times \prod_{i\in K_7}(0,\infty) \times \prod_{i\in K_8}\left(\frac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}}|a_i|^\alpha, 0\right] \times \prod_{i\in K_9}(-\infty, 0]\Bigg).
\end{aligned}$$

Consider sets $(\boldsymbol{a},\boldsymbol{b}]$ with $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^d\setminus\{\boldsymbol{0}\}$, $\boldsymbol{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]}$ and $\mu(\partial(\boldsymbol{a},\boldsymbol{b}])=0$. From relation (1.1.12) we obtain

$$
\begin{aligned}
&n\Pi(c_n(\boldsymbol{a},\boldsymbol{b}]) \qquad\qquad (2.1.8)\\
&= n\Gamma\Bigg(n\prod_{i\in K_1,K_2,K_3,K_4,K_5}\left(\frac{1}{n\overline{\Pi}_i(c_na_i)},\frac{1}{n\overline{\Pi}_i(c_nb_i)}\right]\\
&\quad\times\prod_{i\in K_6,K_7}\left(\frac{1}{n\overline{\Pi}_i(0+)},\frac{1}{n\overline{\Pi}_i(c_nb_i)}\right]\times\prod_{i\in K_8,K_9}\left(\frac{1}{n\overline{\Pi}_i(c_na_i)},\frac{1}{n\overline{\Pi}_i(0+)}\right]\Bigg).
\end{aligned}
$$

From the definition of the p_i in (2.1.1) we conclude for $n\to\infty$ that

$$
\begin{aligned}
\left(\tfrac{1}{n\overline{\Pi}_i(c_na_i)},\tfrac{1}{n\overline{\Pi}_i(c_nb_i)}\right] &\to \left(\tfrac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}}|a_i|^\alpha,\tfrac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}}|b_i|^\alpha\right] =: B_1 && \text{for } i\in K_1,\\
\left(\tfrac{1}{n\overline{\Pi}_i(c_na_i)},\tfrac{1}{n\overline{\Pi}_i(c_nb_i)}\right] &\to \left(\tfrac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}}|a_i|^\alpha,\infty\right) := B_2 && \text{for } i\in K_2,\\
\left(\tfrac{1}{n\overline{\Pi}_i(c_na_i)},\tfrac{1}{n\overline{\Pi}_i(c_nb_i)}\right] &\to \left(-\infty,\tfrac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}}|b_i|^\alpha\right] =: B_3 && \text{for } i\in K_3,\\
\left(\tfrac{1}{n\overline{\Pi}_i(c_na_i)},\tfrac{1}{n\overline{\Pi}_i(c_nb_i)}\right] &\to \emptyset =: B_4 && \text{for } i\in K_4,\\
\left(\tfrac{1}{n\overline{\Pi}_i(c_na_i)},\tfrac{1}{n\overline{\Pi}_i(c_nb_i)}\right] &\to (-\infty,\infty) =: B_5 && \text{for } i\in K_5,\\
\left(\tfrac{1}{n\overline{\Pi}_i(0+)},\tfrac{1}{n\overline{\Pi}_i(c_nb_i)}\right] &\to \left(0,\tfrac{\operatorname{sgn}(b_i)}{p_i^{\operatorname{sgn}(b_i)}}|b_i|^\alpha\right] =: B_6 && \text{for } i\in K_6,\\
\left(\tfrac{1}{n\overline{\Pi}_i(0+)},\tfrac{1}{n\overline{\Pi}_i(c_nb_i)}\right] &\to (0,\infty) =: B_7 && \text{for } i\in K_7,\\
\left(\tfrac{1}{n\overline{\Pi}_i(c_na_i)},\tfrac{1}{n\overline{\Pi}_i(0+)}\right] &\to \left(\tfrac{\operatorname{sgn}(a_i)}{p_i^{\operatorname{sgn}(a_i)}}|a_i|^\alpha,0\right] =: B_8 && \text{for } i\in K_8,\\
\left(\tfrac{1}{n\overline{\Pi}_i(c_na_i)},\tfrac{1}{n\overline{\Pi}_i(0+)}\right] &\to (-\infty,0] =: B_9 && \text{for } i\in K_9.
\end{aligned}
$$

Furthermore,

$$
\boldsymbol{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]}\quad\Rightarrow\quad\boldsymbol{0}\notin\overline{\prod_{i=1}^d\left(\frac{1}{n\overline{\Pi}_i(c_na_i)},\frac{1}{n\overline{\Pi}_i(c_nb_i)}\right]}\quad\Rightarrow\quad\boldsymbol{0}\notin\prod_{i=1}^9\overline{B}_i
$$

and

$$
\mu(\partial(\boldsymbol{a},\boldsymbol{b}])=0\quad\Rightarrow\nu\left(\partial\prod_{i=1}^9 B_i\right)=0\quad\Rightarrow\quad\nu\left(\partial\prod_{i=1}^d\left(\frac{1}{n\overline{\Pi}_i(c_na_i)},\frac{1}{\overline{\Pi}_i(c_nb_i)}\right]\right)=0\,.
$$

Since $n\Gamma(n\cdot)\xrightarrow{v}\nu(\cdot)$ as $n\to\infty$ and ν has no atoms on the considered sets applying Propositions 4.2.1 and 4.2.2 yields that expression (2.1.8) converges to μ defined in (2.1.7). Consequently, relation (2.1.3) holds.

The properties of μ can easily be seen. μ is an α-homogeneous Radon measure on $\mathcal{B}(\mathbb{E})$ with $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ since ν is a 1-homogeneous Lévy measure. Moreover, μ is a non-zero measure because the one-dimensional margin μ_1 is a non-zero measure. Since $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$ and $\Pi_1 \in \mathrm{RV}(\alpha, c_n, \mu_1)$ it holds with (2.1.3)

$$\lim_{t\to\infty} \frac{\overline{\Pi}(t,\ldots,t)}{\overline{\Pi}_1(t)} = \lim_{n\to\infty} \frac{\overline{\Pi}(c_n,\ldots,c_n)}{\overline{\Pi}_1(c_n)} = \frac{\overline{\mu}(1,\ldots,1)}{\overline{\mu}_1(1)} = \frac{1}{p_1^+}\overline{\nu}\left(\frac{1}{p_1^+},\ldots,\frac{1}{p_d^+}\right)$$

and Equation (2.1.5) follows analogously. □

In the situation of Theorem 2.1.1 we obtain with Lemma 4.1.4 for the corresponding Lévy process $(\boldsymbol{X}_t)_{t\geq 0}$ that $(X_1^1,\ldots,X_1^d)$ is regularly varying. Moreover, with [11], Theorem 1.1, we get for linear combinations of $\boldsymbol{X}_1$ the following result.

Corollary 2.1.2

Suppose that the situation of Theorem 2.1.1 holds. Let $(\boldsymbol{X}_t)_{t\geq 0} = (X_t^1,\ldots,X_t^d)_{t\geq 0}$ *be a Lévy process with Lévy measure* $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$*. Then for all* $\boldsymbol{s} \in \mathbb{R}^d$ *the limit*

$$\lim_{n\to\infty} n\mathbb{P}((\boldsymbol{s},\boldsymbol{X}_1) > c_n) = w(\boldsymbol{s}) \quad \textit{exists}$$

and there exists one $\boldsymbol{s}_* \neq \boldsymbol{0}$ *with* $w(\boldsymbol{s}_*) > 0$*. If, additionally,* $\boldsymbol{X}$ *is spectrally positive then all linear combinations* $(\boldsymbol{s},\boldsymbol{X})$*,* $\boldsymbol{s} \in \mathbb{R}^d$*, are regularly varying with index* α*.*

Note that the r. h. s. in (2.1.4) and (2.1.5) is independent of the index α, i. e. the limits are defined by the dependence structure given by the PLM and the weights of the marginal Lévy measures given in (2.1.1). Further, note that (2.1.4) and (2.1.5) define a notion of tail dependence, well-known from extreme value theory for multivariate regularly varying random vectors; see [40].

Definition 2.1.3 (Tail integral dependence coefficient)

Let Π *be a Lévy measure with PLM* Γ*. We define the* upper tail integral dependence coefficient *as*

$$\Lambda_U := \lim_{t\to\infty} t\overline{\Gamma}(t,\ldots,t)$$

and lower tail integral dependence coefficient *as*

$$\Lambda_L := \lim_{t\to-\infty} |t\overline{\Gamma}(t,\ldots,t)|.$$

If $\Lambda_U > 0$ *we call* Γ upper tail integral dependent*. Similarly, if* $\Lambda_L > 0$ *the PLM* Γ *is called* lower tail integral dependent*. If* Γ *is upper (lower) tail integral dependent* *and the conditions (2.1.1) hold with $p_i^+ > 0$ ($p_i^- > 0$) for all $i = 1, \dots, d$ then we call Π upper (lower) tail integral dependent.*

Note that Λ_U and Λ_L always exists since due to the standardized one-dimensional margins we have

$$|t\overline{\Gamma}(t, \dots, t)| \leq 1 \quad \text{for } t \neq 0.$$

Furthermore, note that regular variation does not imply tail integral dependence and that tail integral dependence does not imply regular variation since a tail integral defines a PLM only on $(\mathbb{R} \setminus \{0\})^d$.

[42], Theorem 5.1, gives the relation how a Lévy copula of a Lévy process $\boldsymbol{X}$ is determined by the distributional copulas of the r. v. s $(X_t^1, \dots, X_t^d)$ which yields the following interpretation of the tail integral dependence coefficients.

Proposition 2.1.4
Let $\boldsymbol{X} = (X_t^1, \dots, X_t^d)$ be a Lévy process in $\mathbb{R}^d$ with infinite and absolutely continuous one-dimensional Lévy measures and PLM Γ. Denote by $\widehat{C}_{D,t}^{(\sigma_1,\dots,\sigma_d)} : [0,1]^d \to [0,1]$ the survival copula of $(\sigma_1 X_t^1, \dots, \sigma X_t^d)$ for $t > 0$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d) \in \{-1,1\}^d$. Then the following relations holds:

$$\Lambda_U = \lim_{t\to\infty} \lim_{s\to 0} \frac{t}{s} \widehat{C}_{D,s}^{(\sigma_1,\dots,\sigma_d)} \left(\frac{s}{t}, \dots, \frac{s}{t}\right)$$

and

$$\Lambda_L = \lim_{t\to-\infty} \lim_{s\to 0} \frac{t}{s} \widehat{C}_{D,s}^{(\sigma_1,\dots,\sigma_d)} \left(\frac{s}{t}, \dots, \frac{s}{t}\right).$$

Proof.
Denote by $\widehat{C}$ the Lévy copula corresponding to the PLM Γ. According to [42], Theorem 5.1, we obtain with Relation (1.1.7) for $(x_1, \dots, x_d) \in (\mathbb{R} \setminus \{0\})^d$ and $t > 0$ that

$$\begin{aligned} \overline{\Gamma}(tx_1, \dots, tx_d) &= \widehat{C}\left(\frac{1}{tx_1}, \dots, \frac{1}{tx_d}\right) \\ &= \lim_{s\to 0} \frac{1}{s} \widehat{C}_{D,s}^{(\sigma_1,\dots,\sigma_d)} \left(\frac{s}{t|x_1|}, \dots, \frac{s}{t|x_d|}\right) \prod_{i=1}^d \operatorname{sgn}(x_i) \end{aligned}$$

and the assertion follows. □

Note that $\widehat{C}_{D,s}^{(\sigma_1,\dots,\sigma_d)}$ depends on s and so the limit for $s \to 0$ is not a tail copula, cf. [45].

The following result is a converse of Theorem 2.1.1 and extends Lemma 2.0.2.

Theorem 2.1.5

Let Π be a d-dimensional Lévy measure with one-dimensional margins $\Pi_i, i = 1,\dots,d$ and Γ the PLM given in (1.1.10). Suppose that $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$. Then the tail balance conditions (2.1.1) *hold, there exists an index $i_* \in \{1,\dots,d\}$ such that $\Pi_{i_*} \in \mathrm{RV}(\alpha, c_n, \mu_{i_*})$ and $\Gamma \in \mathrm{RV}(1, n, \nu)$. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $(\boldsymbol{a}, \boldsymbol{b}] \subset \prod_{i=1}^d \mathcal{D}_i$, with $\mathcal{D}_i$ defined in (1.1.15), the relation between μ and ν is given as*

$$\nu((\boldsymbol{a}, \boldsymbol{b}]) = \mu\left(\prod_{i=1}^d (\hat{a}_i, \hat{b}_i]\right), \tag{2.1.9}$$

where for $i = 1,\dots,d$

$$\hat{a}_i := \begin{cases} 0, & \text{if } a_i = 0, \\ \operatorname{sgn}(a_i)\left(p_i^{\operatorname{sgn}(a_i)} a_i\right)^{1/\alpha}, & \text{if } a_i \neq 0, p_i^{\operatorname{sgn}(a_i)} > 0, \\ \infty, & \text{if } a_i > 0, p_i^+ = 0, \\ -\infty, & \text{if } a_i < 0, p_i^- = 0. \end{cases} \tag{2.1.10}$$

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $(\boldsymbol{a}, \boldsymbol{b}] \subset \mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i$ it holds $\nu = \Gamma$.

Proof.

By Lemma 2.0.2, the tail balance conditions (2.1.1) hold with $p_i^+ := \mu_i(1)$ and $p_i^- := -\overline{\mu}_i(-1)$ and there exists at least one index i_* such that $p_{i_*}^+ + p_{i_*}^- > 0$. Analogously to the proof of Theorem 2.1.1 we have to show that $n\Gamma(n(\boldsymbol{a}, \boldsymbol{b}]) \to \nu((\boldsymbol{a}, \boldsymbol{b}])$ as $n \to \infty$ for all sets $(\boldsymbol{a}, \boldsymbol{b}]$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$, $\boldsymbol{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$ and $\nu(\partial(\boldsymbol{a}, \boldsymbol{b}]) = 0$, where ν is a non-zero 1-homogeneous Radon measure with $\nu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$. Recall the definition of the sets $\mathcal{D}_i$ in (1.1.15). By relation (1.1.17) we have that Γ is 1-homogeneous on $\mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i$ and so we define $\nu((\boldsymbol{a}, \boldsymbol{b}]) := \Gamma((\boldsymbol{a}, \boldsymbol{b}])$ for sets $(\boldsymbol{a}, \boldsymbol{b}] \subset (\mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i)$. Further, we define ν on $\mathcal{B}(\mathbb{E})$ by

$$\nu((\boldsymbol{a}, \boldsymbol{b}]) := \mu(\{(x_1,\dots,x_d) \in \mathbb{R}^d \setminus \{\boldsymbol{0}\} : \ \widetilde{x}_i \in (a_i, b_i] \text{ for } i = 1,\dots,d\})$$

for $(\boldsymbol{a}, \boldsymbol{b}] \subset \prod_{i=1}^d \mathcal{D}_i$ where $\tilde{x}_i$ is defined as in (2.1.2), and by $\nu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) := 0$. ν is a non-zero 1-homogeneous Radon measure since μ is an α-homogeneous Lévy measure and Γ is 1-homogeneous on $\mathbb{R}^d \setminus \prod_{i=1}^d \mathcal{D}_i$. Moreover, ν is a non-zero measure because μ_{i_*} is a non-zero measure and $p_{i_*}^+ + p_{i_*}^- > 0$.

Suppose that $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ with $\boldsymbol{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$ and $\nu(\partial(\boldsymbol{a}, \boldsymbol{b}]) = 0$. With relation (1.1.16) we obtain for $(\boldsymbol{a}, \boldsymbol{b}] \subset \prod_{i=1}^{d} \mathcal{D}_i$ that

$$
\begin{aligned}
&n\Gamma(n(\boldsymbol{a}, \boldsymbol{b}]) \qquad (2.1.11)\\
&= n\Pi \otimes \lambda|_{[0,1]^d}\Bigg(\Bigg\{(c_n x_1, \dots, c_n x_d, y_1, \dots, y_d) \in (\mathbb{R}^d \setminus \{\mathbf{0}\}) \times [0,1]^d : \\
&\qquad \frac{1}{n\dot{\overline{\Pi}}_i(c_n x_i) + n y_i \Delta\overline{\Pi}_i(c_n x_i)} \in (a_i, b_i] \text{ for } i = 1, \dots, d\Bigg\}\Bigg).
\end{aligned}
$$

With

$$\lim_{n\to\infty} n\Delta\overline{\Pi}_i(c_n x_i) = \lim_{n\to\infty} n\left(\lim_{\xi \uparrow c_n x_i} \overline{\Pi}_i(\xi) - \overline{\Pi}_i(c_n x_i)\right) = 0$$

it holds

$$\lim_{n\to\infty} \frac{1}{n\dot{\overline{\Pi}}_i(c_n x_i) + y_i \Delta\overline{\Pi}_i(c_n x_i)} = \begin{cases} 0, & \text{if } x_i = 0 \\ \frac{\operatorname{sgn}(x_i)}{p_i^{\operatorname{sgn}(x_i)}}|x_i|^\alpha, & \text{if } x_i \neq 0, p_i^{\operatorname{sgn}(x_i)} > 0, \\ \infty, & \text{if } x_i > 0, p_i^+ = 0, \\ -\infty, & \text{if } x_i < 0, p_i^- = 0. \end{cases}$$

We see that $\nu(\partial(\boldsymbol{a}, \boldsymbol{b}]) = 0$ holds if and only if $\mu(\partial(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}]) = 0$, and $\mathbf{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$ implies $\mathbf{0} \notin \overline{(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}]}$. Hence, with Propositions 4.2.1 and 4.2.3 we obtain the convergence of (2.1.11) as $n \to \infty$

$$
\begin{aligned}
n\Gamma(n(\boldsymbol{a}, \boldsymbol{b}]) &\to \mu\left(\{(x_1, \dots, x_d) \in \mathbb{R}^d \setminus \{\mathbf{0}\} : \widetilde{x}_i \in (a_i, b_i] \text{ for all i}\}\right) \lambda_{[0,1]^d}([0,1]^d) \\
&= \nu((\boldsymbol{a}, \boldsymbol{b}]),
\end{aligned}
$$

and relation (2.1.9) follows. □

2.2 Examples

To simplify notation we consider the case $d = 2$. Moreover, we assume that we are in the framework of Theorem 2.1.1 with $\Pi_1 \in \mathrm{RV}(\alpha, c_n, \mu_1)$, $p_1^+, p_1^- > 0$ and $0 \le p_2^+, p_2^-$, i.e. $\overline{\mu}_i(x) = \operatorname{sgn}(x) p_i^{\operatorname{sgn}(x)} |x|^{-\alpha}$ for $x \neq 0$.

Example 2.2.1 (Independence PLM, continuation of Section 1.3.1)
Since $\Gamma_\perp$ is 1-homogeneous we get $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$. Further, with

$$\Gamma_\perp(\mathrm{d}x_1, \mathrm{d}x_2) = \delta_0(\mathrm{d}x_1)\,|x_2|^{-2}\,\mathrm{d}x_2 + |x_1|^{-2}\,\mathrm{d}x_1\,\delta_0(\mathrm{d}x_2), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{\mathbf{0}\},$$

the limit measure μ is supported on the axes. For $x_1 \in \mathbb{R}\setminus\{0\}$ we get $\mu(\mathcal{I}(x_1)\times\{0\}) = \mu_1(\mathcal{I}(x_1)) = p_1^{\mathrm{sgn}(x_1)}|x_1|^{-\alpha}$, and for $x_2 \in \mathbb{R}\setminus\{0\}$ we achieve

$$\mu(\{0\}\times\mathcal{I}(x_2)) = \begin{cases} \mathrm{sgn}(x_2)p_2^{\mathrm{sgn}(x_2)}|x_2|^{-\alpha}, & p_2^{\mathrm{sgn}(x_2)} > 0, \\ 0, & p_2^{\mathrm{sgn}(x_2)} = 0. \end{cases}$$

Therefore, the limit measure μ of Π results in

$$\begin{aligned}\mu(\mathrm{d}x_1, \mathrm{d}x_2) =& p_1^{\mathrm{sgn}(x_1)} \frac{\alpha}{|x_1|^{\alpha+1}}\,\mathrm{d}x_1\delta_0(\mathrm{d}x_2) \\ &+ \delta_0(\mathrm{d}x_1)\, p_2^{\mathrm{sgn}(x_2)}\frac{\alpha}{|x_2|^{\alpha+1}}\,\mathrm{d}x_2, \quad (x_1,x_2)\in\mathbb{R}^2\setminus\{\mathbf{0}\}\end{aligned}$$

and the limits in (2.1.4) and (2.1.5) are equal to 0.

Example 2.2.2 (Complete dependence PLM, continuation of Section 1.3.2) With Definition (1.3.5) and relation (2.1.3) we have

$$\mu(\mathcal{I}(x_1)\times\mathcal{I}(x_2)) = \left(p_1^{\mathrm{sgn}(x_1)}|x_1|^{-\alpha} \wedge p_2^{\mathrm{sgn}(x_2)}|x_2|^{-\alpha}\right) 1_K((x_1,x_2)), \quad (x_1,x_2)\in(\mathbb{R}\setminus\{0\})^2.$$

For $x_1 \in \mathbb{R}\setminus\{0\}$ we get

$$\begin{aligned}\mu(\mathcal{I}(x_1)\times\{0\}) &= \mu_1(\mathcal{I}(x_1)) - \lim_{x_2\uparrow 0}\mu(\mathcal{I}(x_1)\times\mathcal{I}(x_2)) - \lim_{x_2\downarrow 0}\mu(\mathcal{I}(x_1)\times\mathcal{I}(x_2)) \\ &= \begin{cases} p_1^{\mathrm{sgn}(x_1)}|x_1|^{-\alpha}, & \text{if } p_2^{\mathrm{sgn}(x_1)} = 0, \\ 0, & \text{if } p_2^{\mathrm{sgn}(x_1)} > 0. \end{cases}\end{aligned}$$

We see that μ can have mass on the coordinate axes, although $\Gamma_{\|}$ has not. Analogously for $x_2 \in \mathbb{R}\setminus\{0\}$ with $p_1^+, p_1^- > 0$ it holds $\mu(\{0\}\times\mathcal{I}(x_2)) = 0$. Since $\Gamma_{\|}$ has support on $\{(x_1,x_2)\in(\mathbb{R}\setminus\{0\})^2 : x_1 = x_2\}$, μ has support on $\{(x_1,x_2)\in(\mathbb{R}\setminus\{0\})^2 : x_2 = (p_2^{\mathrm{sgn}(x_2)}/p_1^{\mathrm{sgn}(x_1)})^{1/\alpha}x_1\}$. Finally, the limit measure μ becomes

$$\mu(\mathrm{d}x_1,\mathrm{d}x_2) = p_1^{\mathrm{sgn}(x_1)}\frac{\alpha}{|x_1|^{\alpha+1}} 1_{\left\{x_2 = \left(p_2^{\mathrm{sgn}(x_2)}/p_1^{\mathrm{sgn}(x_1)}\right)^{1/\alpha}x_1\right\}}\mathrm{d}x_1, \quad x_1\in\mathbb{R}\setminus\{0\}.$$

The limits in (2.1.4) and (2.1.5) are given as

$$\lim_{t\to\infty}\frac{\overline{\Pi}(t,t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(1,1)}{\overline{\mu}_1(1)} = \frac{p_1^+ \wedge p_2^+}{p_1^+} \quad \text{and} \quad \lim_{t\to-\infty}\frac{\overline{\Pi}(t,t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(-1,-1)}{\overline{\mu}_1(-1)} = -\frac{p_1^- \wedge p_2^-}{p_1^-}.$$

Example 2.2.3 (Clayton PLM, continuation of Example 1.3.7)
For the Clayton PLM defined in (1.3.8) we have $\Gamma_{\eta,\theta}(\mathbb{R}^2 \setminus (\mathbb{R} \setminus \{0\})^2) = 0$. Consequently, we receive for $x_1 \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned}
\mu(\mathcal{I}(x_1) \times \{0\}) &= \lim_{\epsilon \uparrow 0} \mu(\mathcal{I}(x_1) \times (\epsilon, 0]) \qquad (2.2.1)\\
&= \begin{cases} \Gamma_{\eta,\theta}\left(\mathcal{I}\left(\frac{\operatorname{sgn}(x_1)}{p_1^{\operatorname{sgn}(x_1)}}|x_1|^\alpha\right) \times \{0\}\right), & \text{if } p_2^- > 0, \\ \Gamma_{\eta,\theta}\left(\mathcal{I}\left(\frac{\operatorname{sgn}(x_1)}{p_1^{\operatorname{sgn}(x_1)}}|x_1|^\alpha\right) \times (-\infty, 0]\right), & \text{if } p_2^- = 0, \end{cases}\\
&= \begin{cases} 0, & \text{if } p_2^- > 0, \\ \lim_{\epsilon \uparrow 0} \Gamma_{\eta,\theta}\left(\mathcal{I}\left(\frac{\operatorname{sgn}(x_1)}{p_1^{\operatorname{sgn}(x_1)}}|x_1|^\alpha\right) \times \mathcal{I}(\epsilon)\right), & \text{if } p_2^- = 0, \end{cases}\\
&= \begin{cases} 0, & \text{if } p_2^- > 0, \\ p_1^{\operatorname{sgn}(x_1)}|x_1|^{-\alpha}\left(\eta 1_{\{x_1<0\}} + (1-\eta)1_{\{x_1>0\}}\right), & \text{if } p_2^- = 0, \end{cases}
\end{aligned}$$

and for $x_2 \in \mathbb{R} \setminus \{0\}$ we have

$$\begin{aligned}
\mu(\{0\} \times \mathcal{I}(x_2)) &= \lim_{\epsilon \uparrow 0} \mu((\epsilon, 0] \times \mathcal{I}(x_2)) = \lim_{\epsilon \uparrow 0} \Gamma\left(\left(\frac{-1}{p_1^-}|\epsilon|^\alpha, 0\right] \times \mathcal{I}\left(\widetilde{x}_2\right)\right)\\
&= \Gamma_{\eta,\theta}\left(\{0\} \times \mathcal{I}(\widetilde{x}_2)\right) = 0.
\end{aligned}$$

Let $x_1, x_2 \in \mathbb{R} \setminus \{0\}$. Then for $p_2^{\operatorname{sgn}(x_2)} > 0$ it follows

$$\begin{aligned}
\mu(\mathcal{I}(x_1) \times \mathcal{I}(x_2)) &= \left((p_1^{\operatorname{sgn}(x_1)})^{-\theta}|x_1|^{\alpha\theta} + \left(p_2^{\operatorname{sgn}(x_2)}\right)^{-\theta}|x_2|^{\alpha\theta}\right)^{-1/\theta}\\
&\qquad \left(\eta 1_{\{x_1 x_2 > 0\}} + (1-\eta)1_{\{x_1 x_2 < 0\}}\right),
\end{aligned}$$

and for $p_i^{\operatorname{sgn}(x_2)} = 0$ we get $\mu(\mathcal{I}(x_1) \times \mathcal{I}(x_2)) = 0$. Moreover, the density of the limit measure is given as

$$\begin{aligned}
\mu(\mathrm{d}x_1, \mathrm{d}x_2) &= \alpha^2(1+\theta)\left((p_1^{\operatorname{sgn}(x_1)})^{-\theta}|x_1|^{\alpha\theta} + (p_2^{\operatorname{sgn}(x_2)})^{-\theta}|x_2|^{\alpha\theta}\right)^{-1/\theta-2}\\
&\quad (p_1^{\operatorname{sgn}(x_1)})^{-\theta}(p_2^{\operatorname{sgn}(x_2)})^{-\theta}|x_1|^{\alpha\theta-1}|x_2|^{\alpha\theta-1}1_{\{p_2^{\operatorname{sgn}(x_2)}>0\}}\\
&\quad \left(\eta 1_{\{x_1 x_2 > 0\}} + (1-\eta)1_{\{x_1 x_2 < 0\}}\right)\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad + \alpha p_1^{\operatorname{sgn}(x_1)}|x_1|^{-\alpha-1}\,\mathrm{d}x_1\,1_{\{p_2^-=0\}}\,\delta_0(\mathrm{d}x_2), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}.
\end{aligned}$$

The limits in (2.1.4) and (2.1.5) result in

$$\lim_{t \to \infty} \frac{\overline{\Pi}(t,t)}{\overline{\Pi}_1(t)} = \frac{\overline{\mu}(1,1)}{\overline{\mu}(1)} = \begin{cases} \dfrac{\eta\left((p_1^+)^{-\theta} + (p_2^+)^{-\theta}\right)^{-1/\theta}}{p_1^+}, & \text{if } p_2^+ > 0, \\ 0\,, & \text{if } p_2^+ = 0, \end{cases}$$

and

$$\lim_{t\to-\infty}\frac{\overline{\Pi}(t,t)}{\overline{\Pi}_1(t)}=\frac{\overline{\mu}(-1,-1)}{\overline{\mu}(-1)}=\begin{cases}\dfrac{-\eta\left((p_1^-)^{-\theta}+(p_2^-)^{-\theta}\right)^{-1/\theta}}{p_1^-}, & \text{if } p_2^->0,\\ 0, & \text{if } p_2^-=0.\end{cases}$$

So if $\eta>0$ and $p_2^+>0$ ($p_2^->0$), then we always have upper (lower) tail dependence.

Example 2.2.4 (Non-homogeneous PLM, continuation of Example 1.3.8) The non-homogeneous PLM $\Gamma_{\eta,\zeta}$ has standard 1-homogeneous margins and so for every norming sequence c_n such that $\Gamma_i\in\mathrm{RV}(1,c_n,\nu_i)$ it holds $\lim_{n\to\infty}n/c_n=\lim_{n\to\infty}n\overline{\Gamma}_i(c_n)=\nu_i((1,\infty))<\infty$. Let c_n be a norming sequence such that $\Gamma_i\in\mathrm{RV}(1,c_n,\nu_i)$. Then by (1.3.13) we get for $x_1,x_2\neq 0$ that

$$n\Gamma_{\eta,\zeta}(c_n\mathcal{I}(x_1),c_n\mathcal{I}(x_2))=\frac{1}{\frac{c_n}{n}|x_1|+\frac{c_n}{n}|x_2|+c_n\frac{c_n}{n}\zeta|x_1x_2|}\to 0\quad\text{as } n\to\infty.$$

Consequently, the only possible limit measure for $\Gamma_{\eta,\zeta}$ is the independence PLM $\Gamma_\perp$. But for $x_1\in\mathbb{R}\setminus\{0\}$ the set $B:=\mathcal{I}(x_1)\times\{0\}$ is relatively compact with $\Gamma_\perp(\partial B)=0$ and we obtain, on the one hand, $n\Gamma_{\eta,\zeta}(c_nB)=0$ and, on the other hand, $\Gamma_\perp(B)=|x_1|^{-1}\neq 0$. Thus $\Gamma_{\eta,\zeta}$ is not only non-homogeneous, but also non-regularly varying. By Theorem 2.1.5 a Lévy measure Π with PLM $\Gamma_{\eta,\zeta}$ can not be regularly varying.

Chapter 3

First upwards passage event for sums of dependent Lévy processes

In this chapter we derive new fluctuation identities for sums of Lévy processes and study, in particular, the influence of jump dependence. We shall need the following definitions, and we shall consider them for a sum $X = X^1 + X^2$ of the components of a bivariate Lévy process (X^1, X^2).

We define the *running suprema* and *running infima of* X for $t > 0$

$$\overline{X}_t := \sup_{u \le t} X_u \quad \text{and} \quad \underline{X}_t := \inf_{u \le t} X_u \,, \tag{3.0.1}$$

and the *first upwards passage time* over and the *first downwards passage time* under a fixed barrier $x \in \mathbb{R}$ by

$$\tau_x^+ := \inf\{t > 0 : X_t > x\} \quad \text{and} \quad \tau_x^- := \inf\{t > 0 : X_t < x\}. \tag{3.0.2}$$

Further, we define the *time of the previous maximum of* X and the *time of the previous minimum of* X before time $t > 0$

$$\overline{G}_t := \sup\{s < t : \overline{X}_s = X_s\} \quad \text{and} \quad \underline{G}_t := \sup\{s < t : \underline{X}_s = X_s\}. \tag{3.0.3}$$

More precisely, we investigate the following quantities for a sum $X = X^1 + X^2$ of a bivariate Lévy process (X^1, X^2), which characterize first upwards passage of X over a fixed barrier, when it jumps over it:

(1) $\tau_x^+ - \overline{G}_{\tau_x^+ -}$ the time of first passage relative to the time of the previous maximum,

(2) $\overline{G}_{\tau_x^+ -}$ the time of the previous maximum,

(3) $X_{\tau_x^+} - x$ the overshoot,

(4) $x - X_{\tau_x^+ -}$ the undershoot, and

(5) $x - \overline{X}_{\tau_x^+ -}$ the undershoot of the previous maximum.

The common distribution of these five quantities is called the *quintuple law*.

Results for arbitrary Lévy processes can be found in the monographs [13], Chapter VI, and [49], Chapters 6 and 7. For an illustration see Figure 7.1 of [49].

To see which jump of (X^1, X^2) causes the first passage of the sum $X = X^1 + X^2$ and where the dependence affects this event, we decompose the jumps of (X^1, X^2) in single, common, positive and negative jumps. For motivation and better understanding of the decomposition for our quintuple law, we first formulate the quintuple law for the sum of a bivariate random walk in Section 3.1. The general quintuple law for the sum of a bivariate dependent Lévy process is proven in Section 3.2. In Section 3.3 we consider two situations where all quantities of the quintuple law can be identified concretely. We calculate explicit quantities in Section 3.4 where the dependence is modelled by a Lévy copula, and give four examples for different dependence structures. In Section 3.5 we apply our results to insurance risk theory and obtain a detailed description of the ruin event.

3.1 The quintuple law for the sum of a bivariate random walk

Consider a bivariate random walk $(Z_n^1, Z_n^2)_{n \in \mathbb{N}_0}$ starting in $(Z_0^1, Z_0^2) = 0$ and

$$Z_n^1 = \sum_{i=1}^n \xi_i^1 \quad \text{and} \quad Z_n^2 = \sum_{i=1}^n \xi_i^2 \,, \quad n \in \mathbb{N}\,,$$

where $(\xi_n^1, \xi_n^2)_{n \in \mathbb{N}}$ are independent and identically distributed (i. i. d.) with bivariate d. f. F and margins F_1 and F_2, respectively. We are interested in first upwards passage across $x \geq 0$ of their sum

$$Z_0 = 0 \quad \text{and} \quad Z_n = \sum_{i=1}^n (\xi_i^1 + \xi_i^2)\,, \quad n \in \mathbb{N}\,, \tag{3.1.1}$$

where $(\xi_n^1 + \xi_n^2)_{n\in\mathbb{N}}$ are i. i. d. with d. f. F_+. For $i = 1, 2$ we allow F_i to have an atom at zero with the consequence that the random walks can have jumps of size 0 and so one marginal random walk can jump without the other. We separate the jumps of Z according to their origin and their sign and decompose Z for each $n \in \mathbb{N}$ into components as follows:

$$\begin{aligned} Z_n &= P_n^1 + P_n^2 + P_n^3 + P_n^4 + P_n^5 \\ &\quad + \sum_{i=1}^{n} \xi_i^1 1_{\{\xi_i^1<0,\xi_i^2=0\}} + \sum_{j=1}^{n} \xi_j^2 1_{\{\xi_j^1=0,\xi_j^2<0\}} + \sum_{k=1}^{n} (\xi_k^1 + \xi_k^2) 1_{\{\xi_k^1<0,\xi_k^2<0\}} , \end{aligned} \tag{3.1.2}$$

where $P^1, \dots, P^5$ are those components, where upwards passage can happen; more precisely,

$$P_n^1 = \sum_{i=1}^{n} \xi_i^1 1_{\{\xi_i^1>0,\xi_i^2=0\}} , \qquad P_n^2 = \sum_{i=1}^{n} \xi_i^2 1_{\{\xi_i^1=0,\xi_i^2>0\}} ,$$

$$P_n^3 = \sum_{i=1}^{n} (\xi_i^1 + \xi_i^2) 1_{\{\xi_i^1>0,\xi_i^2>0\}} , \qquad P_n^4 = \sum_{i=1}^{n} (\xi_i^1 + \xi_i^2) 1_{\{\xi_i^1>0,\xi_i^2<0\}} ,$$

$$P_n^5 = \sum_{i=1}^{n} (\xi_i^1 + \xi_i^2) 1_{\{\xi_i^1<0,\xi_i^2>0\}} ,$$

and the increments ΔP^k have d. f. s F_{P^k} for $k = 1, \dots, 5$. Further, we define the analogous quantities to (3.0.1)–(3.0.3): the *running maxima of* Z by

$$\overline{Z}_n := \max_{k\le n} Z_k , \quad n \in \mathbb{N}_0 ,$$

the *first strictly upwards passage time of* Z over a fixed barrier $x \in \mathbb{R}$

$$\mathcal{T}_x^+ := \min\{n \in \mathbb{N} : Z_n > x\}$$

and the *time of the previous maximum of* Z before time $n \in \mathbb{N}$

$$\overline{G}^n := \max\{k \le n : Z_k = \overline{Z}_n\} . \tag{3.1.3}$$

The quantities (1)–(5) from the introduction of this chapter are given for the random walk Z by

(1) $\mathcal{T}_x^+ - 1 - \overline{G}^{\mathcal{T}_x^+ -1}$ the number of time points between the previous maximum and the first passage,

(2) $\overline{G}^{\mathcal{T}_x^+ -1}$ the time of the previous maximum,

(3) $Z_{\mathcal{T}_x^+} - x$ the overshoot,

(4) $x - Z_{\mathcal{T}_x^+ - 1}$ the undershoot, and

(5) $x - \overline{Z}_{\mathcal{T}_x^+ - 1}$ the undershoot of the previous maximum.

To describe the fluctuations of Z, we use its ladder processes, which are defined as follows.

Definition 3.1.1 (Ladder processes of a random walk)
Let $(Z_n)_{n\in\mathbb{N}_0}$ be random walk in $\mathbb{R}$. Let $(L_n)_{n\in\mathbb{N}_0}$ denote the number of times a maximum is reached after n steps, given by

$$L_0 := 0 \quad \text{and} \quad L_n := \#\{i \le n : Z_i \ge \overline{Z}_{i-1}\}, \quad n \in \mathbb{N}.$$

The bivariate weakly ascending ladder process $(L_n^{-1}, H_n)_{n\in\mathbb{N}_0}$ *of Z is defined by*

$$L_0^{-1} := 0, \quad L_n^{-1} := \inf\{k \ge 1 : L_k = n\} \quad \text{and} \quad H_0 := 0, \quad H_n := Z_{L_n^{-1}}, \quad n \in \mathbb{N},$$

where we set $\inf \emptyset := +\infty$. Let $(\widehat{L}_n^*)_{n\in\mathbb{N}_0}$ denote the number of times a new minimum is reached after n steps, given by

$$\widehat{L}_0^* := 0 \quad \text{and} \quad \widehat{L}_n^* := \#\{i \le n : -Z_i > \overline{-Z_{i-1}}\}, \quad n \in \mathbb{N}.$$

The bivariate strictly descending ladder process $(\widehat{L}_n^{-1*}, \widehat{H}_n^*)_{n\in\mathbb{N}_0}$ *of Z is defined by*

$$\widehat{L}_0^{-1*} := 0, \quad \widehat{L}_n^{-1*} := \inf\{k \ge 1 : \widehat{L}_k^* = n\} \quad \text{and} \quad \widehat{H}_0^* := 0, \quad \widehat{H}_n^* := -Z_{\widehat{L}_n^{-1*}}, \ n \in \mathbb{N}.$$

The weakly ascending time L_n^{-1} is the number of steps that Z requires to achieve n maxima, where the same maximum can be reached again, and the weakly ascending ladder height H_n is the n-th maximum of Z. The strictly descending ladder time $\widehat{L}_n^{-1*}$ describes the number of steps that Z requires to reach n new minima and $\widehat{H}_n$ is the values of the n-th new minima of Z. Since $(L_n^{-1}, H_n)_{n\in\mathbb{N}_0}$ and $(\widehat{L}_n^{-1,*}, \widehat{H}_n^*)_{n\in\mathbb{N}_0}$ are (possibly killed) renewal processes, their associated *potential measures* are given as

$$\begin{aligned} U(j, \mathrm{d}x) &= \sum_{n=0}^{\infty} \mathbb{P}(L_n^{-1} = j, H_n \in \mathrm{d}x), \qquad (3.1.4) \\ \widehat{U}^*(i, \mathrm{d}x) &= \sum_{n=0}^{\infty} \mathbb{P}(\widehat{L}_n^{-1*} = i, \widehat{H}_n^* \in \mathrm{d}x). \end{aligned}$$

Thereby, the quintuple law can be formulated for the random walk situation.

Theorem 3.1.2 (Quintuple law for the sum of a bivariate random walk)
Let Z be a random walk as in (3.1.1) and let $x > 0$ be a constant barrier. For $u > 0$, $y \in [0, x]$, $v \geq y$ and $i, j \in \mathbb{N}_0$ we have

$$\begin{aligned}\mathbb{P}\Big(&\mathcal{T}_x^+ - 1 - \overline{G}^{\mathcal{T}_x^+ -1} = i, \overline{G}^{\mathcal{T}_x^+ -1} = j, Z_{\mathcal{T}_x^+} - x \in \mathrm{d}u,\\ &x - Z_{\mathcal{T}_x^+ -1} \in \mathrm{d}v, x - \overline{Z}_{\mathcal{T}_x^+ -1} \in \mathrm{d}y, \Delta Z_{\mathcal{T}_x^+} = \Delta P^k_{\mathcal{T}_x^+}\Big) \\ &= F_{P^k}(v + \mathrm{d}u)\, \widehat{U}^*(i, \mathrm{d}v - y)\, U(j, x - \mathrm{d}y), \quad k = 1, \dots, 5,\end{aligned} \tag{3.1.5}$$

where F_{P^k} are the d. f. s of the increments of P^k defined in (3.1.2).

Proof.
The decomposition (3.1.2) in combination with the proof of Theorem 4 of [22] yields the following. The left-hand side (l. h. s.) of (3.1.5) is equal to

$$\begin{aligned}\mathbb{P}\Big(&Z_n \leq x - y, 0 \leq n < j, Z_j \in x - \mathrm{d}y, Z_{j+m} < x - y, 1 \leq m < i,\\ &Z_{j+i} \in x - \mathrm{d}v, Z_{j+i+1} \in x + \mathrm{d}u, \Delta Z_{\mathcal{T}_x^+} = \Delta P^k_{\mathcal{T}_x^+}\Big)\\ = \ &\mathbb{P}\left(Z_n \leq x - y, 0 \leq n < j, Z_j \in x - \mathrm{d}y\right)\\ &\times \mathbb{P}\left(Z_m < 0, 1 \leq m < i, Z_i \in y - \mathrm{d}v\right) \mathbb{P}\left(Z_1 \in v + \mathrm{d}u, Z_1 = P_1^k\right).\end{aligned}$$

By duality, we have

$$\mathbb{P}\left(Z_m < 0, 1 \leq m < i, Z_i \in y - \mathrm{d}v\right) = \widehat{U}^*(i, \mathrm{d}v - y),$$

and with $\mathbb{P}\left(Z_1 \in v + \mathrm{d}u, Z_1 = P_1^k\right) = F_{P^k}(v + \mathrm{d}u)$, the right-hand side (r. h. s.) of (3.1.5) results. □

For the barrier $x = 0$ we have $\overline{Z}_{\mathcal{T}_0^+ -1} = 0$ a. s. and analogously to Theorem 3.1.2 we get the following result.

Corollary 3.1.3
Let Z be a random walk as in (3.1.1) and let $x = 0$ be a constant barrier. For $u > 0$, $v \geq 0$ and $i, j \in \mathbb{N}_0$ we have

$$\begin{aligned}\mathbb{P}\Big(&\mathcal{T}_0^+ - 1 - \overline{G}^{\mathcal{T}_0^+ -1} = i, \overline{G}^{\mathcal{T}_0^+ -1} = j, Z_{\mathcal{T}_0^+} \in \mathrm{d}u, -Z_{\mathcal{T}_0^+ -1} \in \mathrm{d}v, \Delta Z_{\mathcal{T}_0^+} = \Delta P^k_{\mathcal{T}_0^+}\Big)\\ &= F_{P^k}(v + \mathrm{d}u)\, \widehat{U}^*(i, \mathrm{d}v)\, U(j, \{0\}), \quad k = 1, \dots, 5,\end{aligned}$$

where F_{P^k} are the d. f. s of the increments of P^k defined in (3.1.2).

Remark 3.1.4
Instead of $\overline{G}^n$ given in (3.1.3) we can also consider $\overline{G}_n^* := \min\{k \geq n : Z_k = \overline{Z}_n\}$. Then Theorem 3.1.2 and Corollary 3.1.3 analogously hold for the strictly (instead of weakly) ascending and the weakly (instead of strictly) descending ladder processes.

To study the influence of different dependence structures, we specify the dependence structure between the random walks Z^1 and Z^2 by a copula C on the increments ξ^1 and ξ^2 as described in Section 1.1.1. Then we find expressions for F_{P^k} and also for the d. f. F_+ of the sum $\xi^1 + \xi^2$, which makes the quintuple law of Theorem 3.1.2 and Corollary 3.1.3 precise in reference of the chosen copula. In the following result we only consider the situation where both random walks always jump together. If F_1, F_2 have atoms in 0, then we can decompose the random walks as in (3.1.2) and the result applies by observing that the absolutely continuous parts of F_1 and F_2 may have total mass smaller than 1.

Theorem 3.1.5
Let Z be a random walk as in (3.1.1). Suppose that F_i for $i = 1, 2$ are absolutely continuous and the dependence between Z^1 and Z^2 is modelled by a twice continuously differentiable copula C. Then $P^1 = P^2 = 0$ a. s. and F_{P^k} for $k = 3, 4, 5$ of Theorem 3.1.2 are given for $z > 0$ by

$$F_{P^3}(z) = \int_0^z \left[\frac{\partial C(u,v)}{\partial u}\Big|_{u=F_1(x_1)} \right]_{F_2(0)}^{F_2(z-x_1)} F_1(\mathrm{d}x_1),$$

$$F_{P^4}(z) = \int_{-\infty}^0 \left[\frac{\partial C(u,v)}{\partial v}\Big|_{v=F_2(x_2)} \right]_{F_1(0)}^{F_1(z-x_2)} F_2(\mathrm{d}x_2),$$

$$F_{P^5}(z) = \int_{-\infty}^0 \left[\frac{\partial C(u,v)}{\partial u}\Big|_{u=F_1(x_1)} \right]_{F_2(0)}^{F_2(z-x_1)} F_1(\mathrm{d}x_1).$$

Proof.
Since F_1, F_2 are absolutely continuous, all increments of Z^1 and Z^2 are non-zero and $P^1 = P^2 = 0$ a. s.. From (1.1.1) we obtain for $x_1, x_2 \in \mathbb{R}$

$$F(\mathrm{d}x_1, \mathrm{d}x_2) = \frac{\partial^2 C(u,v)}{\partial u \partial v}\Big|_{u=F_1(x_1), v=F_2(x_2)} F_1(\mathrm{d}x_1) F_2(\mathrm{d}x_2).$$

Furthermore, for $z > 0$

$$F_{P^3}(z) = \int_0^z \int_0^{z-x_1} F(\mathrm{d}x_1, \mathrm{d}x_2)$$

and the expressions for F_{P^4} and F_{P^5} follow analogously. □

The potential measures U and $\widehat{U}^*$ in Theorem 3.1.2 can be identified only in special cases as, for example, when Z has only positive jumps.

Recall the n-fold convolution $F_i^{n*}(\mathrm{d}x)$ of a probability measure $F_i(\mathrm{d}x)$, where $F_i^{0*}(\mathrm{d}x) = \delta_0(\mathrm{d}x)$ is the Dirac-measure in 0 and $F^{1*} = F$.

Theorem 3.1.6
Let Z be a random walk as in (3.1.1). Suppose that F_i for $i = 1, 2$ are absolutely continuous with $F_1(0) = F_2(0) = 0$. Further, let the dependence between ξ^1 and ξ^2 be modelled by a twice continuously differentiable copula C. Then $\widehat{U}^*(\{0\}, \{0\}) = 1$ and, for $j \in \mathbb{N}_0$ and $x \geq 0$,

$$U(j, \mathrm{d}x) = F_{P^3}^{j*}(\mathrm{d}x)\,,$$

where F_{P^3} is given in Theorem 3.1.5.

Proof.
Z reaches a new maximum with every jump. So in (3.1.4) we have $\mathbb{P}(T_n = j, H_n \in \mathrm{d}x) = 1_{\{n=j\}}\mathbb{P}(H_j \in \mathrm{d}x)$ and H_j is the sum of j independent jumps with d. f. F_{P^3}.
□

3.2 The quintuple law for the sum of a bivariate Lévy process

For an arbitrary bivariate Lévy process (X^1, X^2) with characteristic triplet (γ, A, Π) we consider the sum process $X = X^1 + X^2$ which is again a Lévy process; see [58], Proposition 11.10. The proofs of our results rely on the Lévy-Itô decomposition of (X^1, X^2) into two independent parts, corresponding to (1.0.1),

$$\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} + \begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix}, \quad t \geq 0,$$

where (W^1, W^2) is the Gaussian part of (X^1, X^2) with characteristic triplet $(\gamma, A, 0)$. The Lévy process (S^1, S^2) is the jump part of (X^1, X^2) with Lévy measure Π and is represented by

$$\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} = \int_{(0,t]} \int_{|\boldsymbol{x}|>1} \boldsymbol{x} \, J(\mathrm{d}\boldsymbol{x}, \mathrm{d}s) \\ + \lim_{\epsilon \downarrow 0} \int_{(0,t]} \int_{\epsilon < |\boldsymbol{x}| \le 1} (\boldsymbol{x} \, J(\mathrm{d}\boldsymbol{x}, \mathrm{d}s) - \boldsymbol{x} \, \Pi(\mathrm{d}\boldsymbol{x}) \, \mathrm{d}s), \quad t \ge 0, \tag{3.2.1}$$

see [58], Theorem 19.2. The convergence in the second term on the r. h. s. is a. s. and uniform on compacts for $t \in [0, \infty)$. The measure J is a Poisson random measure with intensity measure $\Pi(\mathrm{d}\boldsymbol{x}) \, \mathrm{d}s$ on $\mathbb{R}^2 \times (0, \infty)$. We investigate the first upwards passage either by a jump of the sum process X or equivalently by a jump of the sum process $S = S^1 + S^2$.

Analogously to the random walk in Section 3.1 we want to decompose the paths of (S^1, S^2) according to their jump behaviour in single, common, positive and negative jumps. This causes no problems, if (S^1, S^2) a. s. has sample paths of bounded variation; see [49], Exercise 2.8. But (S^1, S^2) may a. s. have sample paths of unbounded variation, so that – according to [15], relation (31.32) – a pathwise decomposition is not possible. In this case we truncate for arbitrary $0 < \epsilon < 1$ all jumps smaller than ϵ and consider first the truncated process for $t \ge 0$

$$\begin{pmatrix} S_t^{1,\epsilon} \\ S_t^{2,\epsilon} \end{pmatrix} = \int_{(0,t]} \int_{|\boldsymbol{x}|>1} \boldsymbol{x} \, J(\mathrm{d}\boldsymbol{x}, \mathrm{d}s) \\ + \int_{(0,t]} \int_{\epsilon < |\boldsymbol{x}| \le 1} (\boldsymbol{x} \, J(\mathrm{d}\boldsymbol{x}, \mathrm{d}s) - \boldsymbol{x} \, \Pi(\mathrm{d}\boldsymbol{x}) \, \mathrm{d}s) \tag{3.2.2}$$

which is a CPP with drift $(D_{S^{1,\epsilon}}, D_{S^{2,\epsilon}}) = -\int_{\epsilon < |\boldsymbol{x}| \le 1} \boldsymbol{x} \, \Pi(\mathrm{d}\boldsymbol{x})$ and Lévy measure $\Pi \, 1_{\{|\boldsymbol{x}| > \epsilon\}}$. For $\epsilon \downarrow 0$ the process $(S_t^{1,\epsilon}, S_t^{2,\epsilon})_{t \ge 0}$ a. s. converges to $(S_t^1, S_t^2)_{t \ge 0}$ and the convergence is locally uniform in $t \in [0, \infty)$; see Lemma 20.7 of [58]. As $(S^{1,\epsilon}, S^{2,\epsilon})$ a. s. has sample paths of bounded variation, we can decompose $(S^{1,\epsilon}, S^{2,\epsilon})$ in independent components. We denote by $S^{1,\epsilon,+}$ the process of single positive jumps of $S^{1,\epsilon}$; i. e. for $t > 0$

$$S_t^{1,\epsilon,+} = \int_{(0,t]} \int_{x_1 > 1} x_1 \, J((\mathrm{d}x_1, \{0\}), \mathrm{d}s) \\ + \int_{(0,t]} \int_{\epsilon < x_1 \le 1} (x_1 \, J((\mathrm{d}x_1, \{0\}), \mathrm{d}s) - x_1 \, \Pi(\mathrm{d}x_1, \{0\}) \, \mathrm{d}s)$$

and by $S^{1,\epsilon,-}$ the single negative jumps of $S^{1,\epsilon}$, i. e. for $t > 0$

$$\begin{aligned} S_t^{1,\epsilon,-} &= \int_{(0,t]} \int_{x_1<-1} x_1 \, J((\mathrm{d}x_1, \{0\}), \mathrm{d}s) \\ &+ \int_{(0,t]} \int_{-1\le x_1<-\epsilon} (x_1 \, J((\mathrm{d}x_1, \{0\}), \mathrm{d}s) - x_1 \, \Pi(\mathrm{d}x_1, \{0\}) \, \mathrm{d}s) \, . \end{aligned}$$

$S^{2,\epsilon,+}$ and $S^{2,\epsilon,-}$ are defined analogously for $S^{2,\epsilon}$.

The processes $S^{1,\epsilon,ij}$ and $S^{2,\epsilon,ij}$ for $i, j \in \{+, -\}$ are the dependent jump parts of $(S^{1,\epsilon}, S^{2,\epsilon})$, where e. g. $S^{1,\epsilon,++}$ denotes the positive jumps of $S^{1,\epsilon}$ which happen together with positive jumps of $S^{2,\epsilon}$; i. e. for $t > 0$

$$\begin{aligned} S_t^{1,\epsilon,++} &= \int_{(0,t]} \int_{x_1>1} x_1 \, J((\mathrm{d}x_1, (0,\infty)), \mathrm{d}s) \\ &+ \int_{(0,t]} \int_{\epsilon<x_1\le 1} (x_1 \, J((\mathrm{d}x_1, (0,\infty)), \mathrm{d}s) - x_1 \, \Pi(\mathrm{d}x_1, (0,\infty)) \, \mathrm{d}s) \, . \end{aligned}$$

Analogously, $S^{2,\epsilon,++}$ denotes the positive jumps of $S^{2,\epsilon}$ which happen together with positive jumps of $S^{1,\epsilon}$. The notations $S^{1,\epsilon,+-}$, $S^{2,\epsilon,+-}$ and $S^{1,\epsilon,--}$, $S^{2,\epsilon,--}$ should be clear now.

The Lévy measures of these processes are for $B \in \mathcal{B}((0,\infty))$ given by

$$\Pi_{1,\epsilon,+}(B) = \Pi((B \cap (\epsilon,\infty)) \times \{0\}) \text{ and } \Pi_{2,\epsilon,+}(B) = \Pi(\{0\} \times (B \cap (\epsilon,\infty))) \quad (3.2.3)$$

with analogous definitions for $\Pi_{1,\epsilon,-}$ and $\Pi_{2,\epsilon,-}$. For $i, j \in \{+, -\}$ and $B \in \mathcal{B}(\mathbb{R}^2)$ we define

$$B_1 := \{x \in \mathbb{R} : (x, 0) \in B\} \quad \text{and} \quad B_2 := \{y \in \mathbb{R} : (0, y) \in B\}.$$

Then the Lévy measure of the joint jumps is given by

$$\begin{aligned} \Pi_{1,\epsilon,ij,2,\epsilon,ij}(B) &= \Pi(B \cap \{|x| > \epsilon\}) - \Pi((B_1 \cap \{|x_1| > \epsilon\}) \times \{0\}) \\ &\quad -\Pi(\{0\} \times (B_2 \cap \{|x_2| > \epsilon\})). \qquad (3.2.4) \end{aligned}$$

This implies the following Lévy-Itô decomposition for the components

$$\begin{aligned} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} &= \begin{pmatrix} W^1 + \lim_{\epsilon\downarrow 0} S^{1,\epsilon} \\ W^2 + \lim_{\epsilon\downarrow 0} S^{2,\epsilon} \end{pmatrix} \\ &= \begin{pmatrix} W^1 + \lim_{\epsilon\downarrow 0} (S^{1,\epsilon,+} + S^{1,\epsilon,-} + S^{1,\epsilon,++} + S^{1,\epsilon,+-} + S^{1,\epsilon,-+} + S^{1,\epsilon,--}) \\ W^2 + \lim_{\epsilon\downarrow 0} (S^{2,\epsilon,+} + S^{2,\epsilon,-} + S^{2,\epsilon,++} + S^{2,\epsilon,+-} + S^{2,\epsilon,-+} + S^{2,\epsilon,--}) \end{pmatrix} \end{aligned}$$

and for the sum process

$$\begin{aligned} X &= X^1 + X^2 = W^1 + W^2 + S^1 + S^2 \\ &= W^1 + W^2 \\ &\quad + \lim_{\epsilon \downarrow 0} \left(P^{1,\epsilon} + P^{2,\epsilon} + P^{3,\epsilon} + P^{4,\epsilon} + P^{5,\epsilon} + S^{1,\epsilon,-} + S^{2,\epsilon,-} + S^{\epsilon,--} \right), \quad (3.2.5) \end{aligned}$$

where $W^1 + W^2$ denotes the Gaussian part of X which is independent of the jump component, and in (3.2.5) we have set $S^{\epsilon,--} := S^{1,\epsilon,--} + S^{2,\epsilon,--}$. Then we summarize, using analogous notation to (3.1.2):

$$\begin{aligned} &P^{1,\epsilon} := S^{1,\epsilon,+}, && P^{2,\epsilon} := S^{2,\epsilon,+}, && P^{3,\epsilon} := S^{1,\epsilon,++} + S^{2,\epsilon,++}, \\ &P^{4,\epsilon} := S^{1,\epsilon,+-} + S^{2,\epsilon,+-}, && P^{5,\epsilon} := S^{1,\epsilon,-+} + S^{2,\epsilon,-+}, \end{aligned}$$

which are all independent Lévy processes since they a. s. never jump together; see [58], Exercise 12.10. Since all processes in (3.2.5) are independent we can let $\epsilon \downarrow 0$ componentwise. According to [58], Lemma 20.7, we have

$$\lim_{\epsilon \downarrow 0} P^{k,\epsilon} =: P^k \quad \text{a. s.} \tag{3.2.6}$$

where the convergence is uniform on compacts for $t \in [0, \infty)$ for $k = 1, \ldots, 5$. The Lévy measures $\Pi_{P^{k,\epsilon}}$ converge to Π_{P^k} in the sense that

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} f(x)\, \Pi_{P^{k,\epsilon}}(\mathrm{d}x) = \int_{\mathbb{R}} f(x)\, \Pi_{P^k}(\mathrm{d}x) \tag{3.2.7}$$

for all bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ vanishing on a neighbourhood of 0; see [58], Theorem 8.7.

Further, the quintuple law is derived as a consequence of the Wiener-Hopf factorization where we refer to the monograph of [49]. Therefore, we need the ladder processes of X which are based on the notion of local time at the maximum and the regularity of zero.

Definition 3.2.1 (Regularity of zero, [49], Definition 6.4)
For a Lévy process X, the point 0 is said to be regular *(resp.* irregular*) for an open or closed set B if*

$$\mathbb{P}\left(\inf\{t > 0 : X_t \in B\} = 0\right) = 1 \quad (\textit{resp. } 0).$$

According to [49], Definition 6.1 and Theorem 6.7(ii), the notion of local time at the maximum is defined as follows.

Definition 3.2.2 (Local time at the maximum and minimum)
Let X be a Lévy process in $\mathbb{R}$.

(1) *If 0 is regular for $[0,\infty)$ then the* local time at the maximum $L = (L_t)_{t\geq 0}$ *is a continuous, non-decreasing, $[0,\infty)$-valued, $\mathcal{F}$-adapted process with the following properties:*

   (i) *The support of the Stieltjes measure $\mathrm{d}L_t$ is the closure of the random set times $\{t \geq 0 : \overline{X}_t = X_t\}$ and is finite for every $t \geq 0$.*

   (ii) *For every $\mathcal{F}$-stopping time T with $\overline{X}_T = X_T$ a. s. on $\{T < \infty\}$, the shifted process*
$$\big(X_{T+t} - X_T, \overline{X}_{T+t} - X_{T+t}, L_{T+t} - L_T\big)_{t\geq 0}$$
   is independent of $\mathcal{F}_T$ under $\mathbb{P}(\cdot|T<\infty)$ and has the same law as $\big(X, \overline{X} - X, L\big)$ under $\mathbb{P}$.

(2) *If 0 is irregular for $[0,\infty)$then we define the local time at the maximum as*
$$L_t := \sum_{i=0}^{n_t} e_\lambda^i, \tag{3.2.8}$$
where $\{e_\lambda^i\}_{i\in\mathbb{N}_0}$ are independent and exponentially distributed r. v. s with parameter λ and
$$n_t := \#\{0 < s \leq t : \overline{X}_s = X_s\}.$$

The local time at the maximum for the dual process $-X$ we call local time at the minimum.

In the regular case (1) there is always a local time at the maximum unique up to a multiplicative constant, by [49], Theorem 6.7(i). Further, the right-continuous, non-adapted process given in (3.2.8) satisfies the properties (i) and (ii) of Definition 3.2.2; see [49], Theorem 6.7(ii). According to [49], p. 147, we choose for spectrally negative processes the local time at the maximum as $\overline{X}$.

Finally, the ladder processes of a Lévy process are given by the following definition.

Definition 3.2.3 (Ladder process of a Lévy process, [49], p. 147)
Let X be a Lévy process in $\mathbb{R}$ and L a local time at the maximum with $L_\infty := \lim_{t\to\infty} L_t$. The inverse local time process
$$L_t^{-1} := \begin{cases} \inf\{s > 0 : L_s > t\}, & t < L_\infty, \\ \infty, & \textit{else}, \end{cases}$$

is called the ascending ladder time process *of* X *and*

$$H_t := \begin{cases} X_{L_t^{-1}}, & t < L_\infty, \\ \infty, & \text{else}, \end{cases}$$

is called the ascending ladder height process *of* X. *The bivariate process* $(L_t^{-1}, H_t)_{t\geq 0}$ *is the* ascending ladder process. *The ascending ladder process* $(\widehat{L}_t^{-1}, \widehat{H}_t)_{t\geq 0}$ *of* $-X$ *is called the* descending ladder process *of* X.

The range of the ascending ladder time L^{-1} corresponds to the time at which X reaches maxima and so the range of the ascending ladder height H corresponds to the set of maxima of X. Recall from [13], Proposition VI.4, and [49], p. 158, that with the exception of a CPP all local extrema of X are distinct, i. e. its maxima are obtained at unique times. This means that we only need to distinguish between *weak* and *strict* ladder processes for CPPes, for all other processes weak and strict ladder processes coincide. Thus, we exclude CPPes in the following and treat them separately in Remark 3.2.7. The following situation holds for every Lévy process X which is not a CPP.

$(L_t^{-1}, H_t)_{t\geq 0}$ and $(\widehat{L}_t^{-1}, \widehat{H}_t)_{t\geq 0}$ are (possibly killed) bivariate subordinators and their joint Laplace exponents κ and $\widehat{\kappa}$ are for $\alpha, \beta \geq 0$ defined by the identities

$$e^{-\kappa(\alpha,\beta)} = \mathbb{E}\left[e^{-\alpha L_1^{-1} - \beta H_1} 1_{\{1 < L_\infty\}}\right] \quad \text{and} \quad e^{-\widehat{\kappa}(\alpha,\beta)} = \mathbb{E}\left[e^{-\alpha \widehat{L}_1^{-1} - \beta \widehat{H}_1} 1_{\{1 < \widehat{L}_\infty\}}\right].$$

If $L_\infty < \infty$ a. s., then L_∞ is exponentially distributed with *killing rate* $q > 0$, where $q > 0$ if and only if $\lim_{t\to\infty} X_t = -\infty$ a. s.. By Equations (6.15) and (6.16) of [49] we can also write for $\beta \in [0, \infty) + i\mathbb{R}$

$$\kappa(0, \beta) = q + \xi(\beta) = q + D_H \beta + \int_{(0,\infty)} (1 - e^{-\beta x})\, \Pi_H(\mathrm{d}x), \tag{3.2.9}$$

where $D_H = -\gamma_H - \int_{|x|\leq 1} x \Pi_H(\mathrm{d}x)) \geq 0$ is the drift of H and Π_H its Lévy measure. Note that the function $\xi(\cdot)$ is the Laplace exponent of an unkilled subordinator. Similar notation is used for $\widehat{\kappa}(0, \beta)$ by replacing q, ξ, D_H and Π_H by $\widehat{q}$, $\widehat{\xi}$, $D_{\widehat{H}}$ and $\Pi_{\widehat{H}}$. We also recall that whenever $q > 0$ we have $\widehat{q} = 0$.

Associated with the ascending and descending ladder processes are the *bivariate potential measures* on $[0, \infty)^2$

$$\mathcal{U}(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty \mathbb{P}(L_t^{-1} \in \mathrm{d}s, H_t \in \mathrm{d}x)\, \mathrm{d}t, \tag{3.2.10}$$

$$\widehat{\mathcal{U}}(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty \mathbb{P}(\widehat{L}_t^{-1} \in \mathrm{d}s, \widehat{H}_t \in \mathrm{d}x)\, \mathrm{d}t. \tag{3.2.11}$$

Since a local time at the maximum is defined only up to a multiplicative constant, see Definition 3.2.2, the exponent κ can only be defined up to a multiplicative constant, which is then also inherited by $\mathcal{U}$.

Now we are ready to state the general quintuple law for the sum of a bivariate Lévy process.

Theorem 3.2.4 (Quintuple law for the sum of Lévy processes)
Let X be a Lévy process as in (3.2.5). Suppose that X is not a CPP and $\Pi_1((0,\infty)) > 0$, $\Pi_2((0,\infty)) > 0$. Consider the first upwards passage of X over a constant barrier $x > 0$. Then there exists a normalization of local time at the maximum, given by the identity

$$q = \kappa(q,0)\widehat{\kappa}(q,0) \quad \text{for } q > 0, \tag{3.2.12}$$

such that for $u > 0, y \in [0,x], v \geq y, s \geq 0, t \geq 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\tau_x^+ - \overline{G}_{\tau_x^+-} \in \mathrm{d}t, \overline{G}_{\tau_x^+-} \in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u,\\
&\qquad x - X_{\tau_x^+-} \in \mathrm{d}v, x - \overline{X}_{\tau_x^+-} \in \mathrm{d}y, \Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+}\Big)\\
&= \Pi_{P^k}(\mathrm{d}u + v)\widehat{\mathcal{U}}(\mathrm{d}t, \mathrm{d}v - y)\mathcal{U}(\mathrm{d}s, x - \mathrm{d}y), \quad k = 1, \dots, 5,
\end{aligned} \tag{3.2.13}$$

where P^k is defined in (3.2.6).

Proof.
Case 1: $S^1 + S^2$ is of bounded variation.
Let m, k, f, g and h be positive continuous functions with compact support satisfying $f(0) = g(0) = h(0) = 0$. The condition $f(0) = g(0) = h(0) = 0$ is to exclude from calculation the case of first passage by creeping, i. e. the event $\{X_{\tau_x^+} = x\}$ because we consider only the case, when the overshoot $X_{\tau_x^+} - x$ is a. s. positive. Since $S^1 + S^2$ is of bounded variation we decompose it as in (3.2.5) into

$$S^1 + S^2 = P^1 + P^2 + P^3 + P^4 + P^5 + S^{1,-} + S^{2,-} + S^{--}.$$

Let J_{1+2} denote the Poisson random measure associated with the jumps of $S^1 + S^2$. It holds

$$J_{1+2} = J_{P^1} + J_{P^2} + J_{P^3} + J_{P^4} + J_{P^5} + J_{S^{1,-}} + J_{S^{2,-}} + J_{S^{--}},$$

where J_{P^k} denotes the Poisson random measure associated with the jumps of P^k given in (3.2.5). As P^1, P^2, P^3, P^4, P^5, $S^{1,-}$, $S^{2,-}$ and S^{--} are independent, J_{P^1},

J_{P^2}, J_{P^3}, J_{P^4}, J_{P^5}, $J_{S^{1,-}}$, $J_{S^{2,-}}$ and $J_{S^{--}}$ have disjoint support with probability one. J_{P^k} has intensity measure $\Pi_{P^k}(\mathrm{d}x)\,\mathrm{d}t$ and analogously to Step 1 of the proof of Theorem 3 in [22] we obtain with the compensation formula, [49], Theorem 4.4, for $k = 1, \dots, 5$

$$\mathbb{E}\left[m(\tau_x^+ - \overline{G}_{\tau_x^+-})k(\overline{G}_{\tau_x^+-})f(X_{\tau_x^+} - x)g(x - X_{\tau_x^+-})h(x - \overline{X}_{\tau_x^+-})1_{\left\{\Delta X_{\tau_x^+}=\Delta P^k_{\tau_x^+}\right\}}\right]$$
$$= \widehat{\mathbb{E}}_x\left[\int_0^{\tau_0^-} m(t - \underline{G}_t)k(\underline{G}_t)w_k(X_t)h(\underline{X}_t)\,\mathrm{d}t\right], \tag{3.2.14}$$

where $w_k(z) = g(z)\int_{(z,\infty)} f(u - z)\,\Pi_{P^k}(\mathrm{d}u)$ and $\widehat{\mathbb{E}}$ refers to the dual process $-X$. Using identity (7) in [22], which is based on the Wiener-Hopf factorization and the normalization (3.2.12), the expectation in (3.2.14) is equal to

$$\int_{\phi\in[0,\infty)}\int_{t\in[0,\infty)}\int_{\xi\in[0,x]}\int_{s\in[0,\infty)} m(t)k(s)h(x-\xi)w_k(x+\phi-\xi)\,\mathcal{U}(\mathrm{d}s,\mathrm{d}\xi)\,\widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}\phi).$$

This results in the following first identity and, proceeding by substitution of variables, we obtain

$$\begin{aligned}
&\int_{u>0,y\in[0,x],v\geq y,s\geq 0,t\geq 0} m(t)k(s)f(u)g(v)h(y)\\
&\qquad\mathbb{P}\Big(\tau_x^+ - \overline{G}_{\tau_x^+-}\in \mathrm{d}t, \overline{G}_{\tau_x^+-}\in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u,\\
&\qquad\qquad x - X_{\tau_x^+-}\in \mathrm{d}v, x - \overline{X}_{\tau_x^+-}\in \mathrm{d}y, \Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+}\Big)\\
&= \int_{\phi\in[0,\infty)}\int_{t\in[0,\infty)}\int_{\xi\in[0,x]}\int_{s\in[0,\infty)} m(t)k(s)h(x-\xi)w_k(x+\phi-\xi)\,\mathcal{U}(\mathrm{d}s,\mathrm{d}\xi)\,\widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}\phi)\\
&= \int_{\phi\in[0,\infty)}\int_{t\in[0,\infty)}\int_{y\in[0,x]}\int_{s\in[0,\infty)} m(t)k(s)h(y)w_k(y+\phi)\,\mathcal{U}(\mathrm{d}s,x-\mathrm{d}y)\,\widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}\phi)\\
&= \int_{y\in[0,x]}\int_{s\in[0,\infty)}\int_{v\in(y,\infty)}\int_{t\in[0,\infty)} m(t)k(s)h(y)w_k(v)\,\widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}v-y)\,\mathcal{U}(\mathrm{d}s,x-\mathrm{d}y)\\
&= \int_{y\in[0,x]}\int_{s\in[0,\infty)}\int_{v\in[y,\infty)}\int_{t\in[0,\infty)} m(t)k(s)h(y)g(v)\int_{(v,\infty)} f(\eta-v)\,\Pi_{P^k}(d\eta)\\
&\qquad\qquad \widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}v-y)\,\mathcal{U}(\mathrm{d}s,x-\mathrm{d}y)\\
&= \int_{y\in[0,x]}\int_{s\in[0,\infty)}\int_{v\in[y,\infty)}\int_{t\in[0,\infty)} m(t)k(s)h(y)g(v)\int_{(0,\infty)} f(u)\,\Pi_{P^k}(\mathrm{d}u+v)\\
&\qquad\qquad \widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}v-y)\,\mathcal{U}(\mathrm{d}s,x-\mathrm{d}y).
\end{aligned}$$

Case 2: $S^1 + S^2$ is of unbounded variation.
We start with the truncated process $(S^{1,\epsilon}, S^{2,\epsilon})$ for $\epsilon > 0$ as given in (3.2.2). Then $S^{1,\epsilon} + S^{2,\epsilon}$ is of bounded variation and we can decompose its sample paths according to its jump behaviour like in (3.2.5). The unbounded variation of X implies that 0 is regular for $(0,\infty)$ and $(-\infty,0)$; see [49], Theorem 6.5(i). Therefore, $\mathcal{U}(\{0\},\{0\}) = \widehat{\mathcal{U}}(\{0\},\{0\}) = 0$. We apply the result of Case 1 above dropping the point 0 from integration, which yields

$$\begin{aligned}
&\int_{u>0,y\in[0,x],v\geq y,u+v>\epsilon,s\geq 0,t\geq 0} m(t)k(s)f(u)g(v)h(y) \\
&\qquad \mathbb{P}\Big(\tau_x^+ - \overline{G}_{\tau_x^+-} \in \mathrm{d}t, \overline{G}_{\tau_x^+-} \in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+-} \in \mathrm{d}v, \\
&\qquad\qquad x - \overline{X}_{\tau_x^+-} \in \mathrm{d}y, \Delta X_{\tau_x^+} = \Delta P^{k,\epsilon}_{\tau_x^+}\Big) \\
&= \int_{\phi\in(0,\infty)} \int_{t\in(0,\infty)} \int_{\xi\in(0,x]} \int_{s\in(0,\infty)} m(t)k(s)h(x-\xi)g(x+\phi-\xi) \\
&\qquad \int_{(x+\phi-\xi,\infty)} f(\eta-(x+\phi-\xi))\, \Pi_{P^{k,\epsilon}}(\mathrm{d}\eta)\, \mathcal{U}(\mathrm{d}s,\mathrm{d}\xi)\, \widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}\phi) \\
&= \int_{y\in[0,x)} \int_{s\in(0,\infty)} \int_{v\in(y,\infty)} \int_{t\in(0,\infty)} m(t)k(s)h(y)g(v) \int_{(v,\infty)} f(\eta - v)\, \Pi_{P^{k,\epsilon}}(\mathrm{d}\eta) \\
&\qquad \widehat{\mathcal{U}}(\mathrm{d}t, \mathrm{d}v - y)\, \mathcal{U}(\mathrm{d}s, x - \mathrm{d}y).
\end{aligned}$$

For $\epsilon \downarrow 0$ the processes $P^{k,\epsilon}$ converge a. s. to P^k and, hence, they also converge in distribution. Moreover, their Lévy measures converge in the sense of (3.2.7) and for all $v > 0$ the function $\widetilde{f}(\eta) := f(\eta - v)1_{\{\eta>v\}}$ is bounded and continuous and vanishes on $[0,v]$. So for all $v > 0$

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \int_{(v,\infty)} f(\eta-v)\, \Pi_{P^{k,\epsilon}}(\mathrm{d}\eta) &= \lim_{\epsilon\downarrow 0} \int_{(0,\infty)} \widetilde{f}(\eta)\, \Pi_{P^{k,\epsilon}}(\mathrm{d}\eta) = \int_{(0,\infty)} \widetilde{f}(\eta)\, \Pi_{P^k}(\mathrm{d}\eta) \\
&= \int_{(v,\infty)} f(\eta - v)\, \Pi_{P^k}(\mathrm{d}\eta) = \int_{(0,\infty)} f(u)\, \Pi_{P^k}(\mathrm{d}u + v)
\end{aligned}$$

and (3.2.13) follows. □

For the barrier $x = 0$ the situation simplifies by considering the two possible situations:

(R) 0 is regular for $(0,\infty)$; i. e. $\tau_0^+ = 0$ a.s., or

(I) 0 is irregular for $(0,\infty)$; i. e. $\tau_0^+ > 0$ a. s..

Let X be a Lévy process with characteristic triplet (γ_+, a, Π_+). Since we still exclude that X is CPP, (R) holds if and only if (see [49], Theorem 6.5)

- $S^1 + S^2$ is of unbounded variation, or
- $S^1 + S^2$ is of bounded variation and
  - X has a Gaussian component (i. e. $a > 0$), or
  - X has no Gaussian component (i. e. $a = 0$), but
    - X has drift $D_X = -\gamma_+ - \int_{|x|\leq 1} x\,\Pi_+(\mathrm{d}x) > 0$, or
    - X has drift $D_X = 0$ and $\displaystyle\int_0^1 \frac{x}{\int_0^x \Pi_+((-\infty,-y))\mathrm{d}y}\,\Pi_+(\mathrm{d}x) = \infty$.

(I) holds if and only if S^1+S^2 is of bounded variation, X has no Gaussian component and either

- $D_X < 0$, or
- $D_X = 0$ and $\displaystyle\int_0^1 \frac{x}{\int_0^x \Pi_+((-\infty,-y))\,\mathrm{d}y}\Pi_+(\mathrm{d}x) < \infty$.

Corollary 3.2.5
Let X be a Lévy process as in (3.2.5). Suppose that X is not a CPP and $\Pi_1((0,\infty)) > 0$, $\Pi_2((0,\infty)) > 0$. Consider the first passage of X over the barrier $x = 0$. Then $-\overline{X}_{\tau_0^+-} = 0$ a. s. and $\overline{G}_{\tau_0^+-} = 0$ a. s..
(1) If (R) holds, then all quantities of the quintuple law are a. s. equal to zero, i. e.

$$-\overline{X}_{\tau_0^+-} = \overline{G}_{\tau_0^+-} = \tau_0^+ = X_{\tau_0^+} = -X_{\tau_0^+-} = 0 \quad \text{a. s..}$$

(2) If (I) holds, then there exists a normalization of local time at the maximum, given by (3.2.12), such that for $u > 0, t \geq 0, v \geq 0$,

$$\begin{aligned}&\mathbb{P}\left(\tau_0^+ \in \mathrm{d}t, X_{\tau_0^+} \in \mathrm{d}u, -X_{\tau_0^+-} \in \mathrm{d}v, \Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+}\right)\\ &= \Pi_{P^k}(\mathrm{d}u + v)\widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}v)\mathcal{U}(\{0\},\{0\}), \quad k = 1,\ldots,5,\end{aligned} \tag{3.2.15}$$

where P^k is defined in (3.2.6).

Proof.
Since X is not a CPP, its maxima are obtained at unique times, so $\overline{G}_{\tau_0^+-} = \sup\{s <$

$\tau_0^+ : X_s = 0\} = 0$ a. s.. The proof for case (I) is analogous to Case 1 of the proof of Theorem 3.2.4. □

Remark 3.2.6 (First passage by creeping)
In Theorem 3.2.4 and Corollary 3.2.5 we investigated the first passage of X caused by a jump in one or both components X^1 and X^2. However, X may also creep over the barrier $x \in \mathbb{R}$, in which case

$$\mathbb{P}(X_{\tau_x^+} = x) > 0$$

holds. According to [13], Theorem VI.19, this is equivalent to

$$D_H = \lim_{\beta \uparrow \infty} \frac{\kappa(0, \beta)}{\beta} > 0 \,.$$

If X is of bounded variation, then X creeps upwards if and only if $D_X > 0$, see [21], Section 6.4, and [49], Theorem 7.11. The linear drift D_X is deterministic and so dependence between the jumps does not affect the creeping of X. If X has a Gaussian component, then from $A = 2D_H D_{\widehat{H}}$ (see [21], Corollary 4.4(i)) $D_H > 0$ follows. So dependence between the jumps does not affect that X can creep. If X is of unbounded variation, but has no Gaussian component, then X creeps upwards if and only if for its Lévy measure Π_+ holds

$$\begin{aligned}
&\int_0^1 \frac{x\Pi_+([x,\infty))}{\int_{-x}^0 \left(\int_{-1}^u \Pi_+((-\infty,z])\mathrm{d}z\right)\mathrm{d}u}\,\mathrm{d}x \\
=&\int_0^1 \frac{x\left(\Pi_{P^1}+\Pi_{P^2}+\Pi_{P^3}+\Pi_{P^4}+\Pi_{P^5}\right)([x,\infty)}{\int_{-x}^0 \left(-\int_{-1}^u \overline{\Pi}_{S^{1,-}}(z)+\overline{\Pi}_{S^{2,-}}(z)+\overline{\Pi}^-_{P^4}(z)+\overline{\Pi}^-_{P^5}(z)+\overline{\Pi}_{S^{--}}(z)\;\mathrm{d}z\right)\mathrm{d}u}\,\mathrm{d}x < \infty,
\end{aligned}$$

where P^k is defined in (3.2.6). So only in this case the dependence between the jumps can influence the possibility of creeping.

Remark 3.2.7 (Quintuple law for CPPes)
Only if X is a CPP, then X can visit the same maxima at distinct times, see [13], Proposition VI.4, and [49], p. 158. When we work with the *weakly* ascending ladder process (L^{-1}, H) and the *strictly* descending ladder process $(\widehat{L}^{-1*}, \widehat{H}^*)$ as in Section 3.1, we consider the *last time of the previous maximum of* X before time t defined by $\overline{G}$ in (3.0.3) and the *first time of the previous minimum of* X before time t; i. e.

$$\underline{G}_t^* := \inf\{s < t : X_s = \underline{X}_t\}, \tag{3.2.16}$$

see [22], Theorem 4, Remark 5, p. 98, and [49], pp. 167–168 and p. 194. If X is not a CPP, then the definition of $\underline{G}^*$ coincides with the definition of $\underline{G}$ in (3.0.3). The quintuple law of Theorem 3.2.4 holds also for CPPes with $\widehat{\mathcal{U}}$ replaced by $\widehat{\mathcal{U}}^*$ to indicate that this is the potential measure of the strictly descending ladder process as in Section 3.1. The result of Corollary 3.2.5 changes, since $\overline{G}_{\tau_0^+-} > 0$ a. s., and we obtain for $u > 0, t \geq 0, s > 0, v \geq 0$,

$$\begin{aligned}&\mathbb{P}\left(\tau_0^+ - \overline{G}_{\tau_0^+-} \in \mathrm{d}t, \overline{G}_{\tau_0^+-} \in \mathrm{d}s, X_{\tau_0^+} \in \mathrm{d}u, -X_{\tau_0^+-} \in \mathrm{d}v, \Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+}\right)\\&= \Pi_{P^k}(\mathrm{d}u + v)\widehat{\mathcal{U}}^*(\mathrm{d}t, \mathrm{d}v)\mathcal{U}(\mathrm{d}s, \{0\}), \quad k = 1, \dots, 5,\end{aligned}$$

where P^k is defined in (3.2.6). The proof of the quintuple law for CPPes is analogous to Case 1 of the proof of Theorem 3.2.4. The only subtlety is in the Wiener-Hopf factorization, where we have to assign the mass given by the probabilities $\mathbb{P}(X_t = 0)$ for $t \geq 0$ to one or the other of the integrals, which define κ and $\widehat{\kappa}$; see Equations (6.19) and (6.20) in [49]. With the definition of $\underline{G}^*$ in (3.2.16) we assign the mass to κ; cf. [49], pp. 167–168.

3.3 Two explicit situations

Whereas it is comparably easy to understand the influence of the last jump of the Lévy process since it is independent of the past, it is a rather complex task to trace the influence of the dependence within the potential measures $\mathcal{U}$ and $\widehat{\mathcal{U}}$ given in (3.2.10) and (3.2.11). They can be identified explicitly only for special Lévy processes. The ladder processes depend on the chosen local times at the maxima and minima, respectively, which in general can not be written as functionals of the path of X. In this section we present two situations of the quintuple law of Theorem 3.2.4 and Corollary 3.2.5 where we can calculate the potential measures explicitly. We investigate them further in Section 3.4 with regarding to the effect of the dependence between the jumps. We consider only cases where 0 is irregular for $(0, \infty)$ and X is spectrally positive. These two conditions are satisfied only in two situations which we discuss in Sections 3.3.1 and 3.3.2, respectively.

3.3.1 Spectrally positive compound Poisson process

Let (S^1, S^2) be a spectrally positive CPP and X be given as

$$X_t = S_t^1 + S_t^2, \quad t \geq 0, \tag{3.3.1}$$

and let $\lambda_+ > 0$ denote the jump intensity of X and F_+ the d. f. of the i. i. d. jump sizes of X; note that both are determined by the marginal frequencies, marginal jump sizes and the dependence structure. Due to $X_t = \overline{X}_t$ a. s. and $\overline{G}_t = \sup\{s < t : \overline{X}_s = X_s\} = t$ a. s. for all $t \geq 0$, for all $x \geq 0$ we have $x - \overline{X}_{\tau_x^+ -} = x - X_{\tau_x^+ -}$ a. s. and $\overline{G}_{\tau_x^+ -} = \tau_x^+$ a. s. and the quintuple law reduces to a triple law. Recall the definition of the convolution F_+^{n*} before Theorem 3.1.6.

Theorem 3.3.1
Suppose X is given as in (3.3.1). Consider the first passage of X over a constant barrier $x > 0$. Then for $u > 0, y \in [0, x], s > 0$ it holds for $k = 1, 2, 3$,

$$\begin{aligned} &\mathbb{P}\left(\tau_x^+ \in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+ -} \in \mathrm{d}y, \Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+}\right) \\ &= \Pi_{P^k}(\mathrm{d}u + y) \sum_{n=0}^{\infty} \frac{(\lambda_+ s)^n}{n!} e^{-\lambda_+ s} \,\mathrm{d}s\, F_+^{n*}(x - \mathrm{d}y). \end{aligned} \tag{3.3.2}$$

By construction of P^k as being independent, the d. f. F_+ has representation

$$F_+ = \frac{1}{\lambda_+} \sum_{k=1}^{3} \lambda_{P^k} F_{P^k}\,, \tag{3.3.3}$$

where the λ_{P^k} denote the jump intensities of P^k defined in (3.2.6).

Proof.
According to Theorem 3.2.4 and Remark 3.2.7, the l. h. s of of (3.3.2) is given by

$$\Pi_{P^k}(\mathrm{d}u + y)\widehat{\mathcal{U}}^*(\{0\}, \{0\})\mathcal{U}(\mathrm{d}s, x - \mathrm{d}y)$$

where $\widehat{\mathcal{U}}^*$ denotes the potential measure of the strictly descending ladder process $(\widehat{L}^{-1*}, \widehat{H}^*)$. Since X is of bounded variation and the point 0 is regular for $[0, \infty)$ and irregular for $(-\infty, 0)$, with [49], Theorem 6.8, we choose the local time at the maximum as $L_t = \int_0^t 1_{\{X_s = \overline{X}_s\}}\,\mathrm{d}s = t$ and the weakly ascending ladder process results as $(L_t^{-1}, H_t)_{t \geq 0} = (t, X_t)_{t \geq 0}$ with potential measure

$$\mathcal{U}(\mathrm{d}s, x - \mathrm{d}y) = \mathbb{P}\left(X_s \in x - \mathrm{d}y\right)\,\mathrm{d}s = \sum_{n=0}^{\infty} \frac{(\lambda_+ s)^n}{n!} e^{-\lambda_+ s}\,\mathrm{d}s\, F_+^{n*}(x - \mathrm{d}y).$$

Since X is a.s. increasing, the strictly descending ladder process is killed, i. e. $\widehat{L}_\infty^{-1*} \stackrel{\mathrm{d}}{=}$ expo$(\widehat{q})$ for some $\widehat{q} > 0$, see [49], Theorem 6.10(ii), and we obtain

$$(\widehat{L}_t^{-1*}, \widehat{H}_t^*) = \begin{cases} (0,0), & t < \widehat{L}_\infty^{-1*}, \\ (\infty,\infty), & t \geq \widehat{L}_\infty^{-1*}, \end{cases}$$

and $\widehat{\mathcal{U}}^*(\{0\},\{0\}) = \widehat{q}^{-1}$. The normalization condition (3.2.12) yields $\widehat{q} = 1$ and Equation (3.3.2) results. □

For the barrier $x = 0$ the result is reduced even further.

Corollary 3.3.2

Suppose X is given as in (3.3.1). Consider the first passage of X over the barrier $x = 0$. Then $-X_{\tau_0^+-} = 0$ a. s. and for $u > 0$, $s > 0$ it holds

$$\mathbb{P}\left(\tau_0^+ \in \mathrm{d}s, X_{\tau_0^+} \in \mathrm{d}u, \Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+}\right) = \Pi_{P^k}(\mathrm{d}u)\, e^{-\lambda s}\, \mathrm{d}s.$$

3.3.2 Subordinator with negative drift and finite mean

Let (S^1, S^2) be a driftless subordinator and X be given as

$$X_t = S_t - ct = S_t^1 + S_t^2 - ct, \quad t \geq 0, \tag{3.3.4}$$

with negative drift $D_X = -c < 0$. We denote its Lévy measure as Π_+ and recall the characteristic exponent of X from (1.0.1) which is given by

$$\Psi_X(\theta) = \Psi_S(\theta) + ic\theta = \int_0^\infty \left(1 - e^{i\theta x}\right) \Pi_+(\mathrm{d}x) + ic\theta, \quad \theta \in \mathbb{R}. \tag{3.3.5}$$

Further, we suppose

$$0 < \mathbb{E}[S_1] = \mu_S = \int_0^\infty x\Pi_+(\mathrm{d}x) < c < \infty, \tag{3.3.6}$$

such that $\lim_{t\to\infty} X_t = -\infty$ a. s..

Under these conditions the ascending ladder process (L^{-1}, H) of X is a killed bivariate CPP, and we denote its jump size distribution by $F_{\mathcal{L}^{-1}\mathcal{H}}(\mathrm{d}s, \mathrm{d}x)$. We denote by $F^{n*}_{\mathcal{L}^{-1}\mathcal{H}}$ the n-fold bivariate convolution of $F_{\mathcal{L}^{-1}\mathcal{H}}$, where $F^{0*}_{\mathcal{L}^{-1}\mathcal{H}}(\mathrm{d}s, \mathrm{d}z) = \delta_{(0,0)}(\mathrm{d}s, \mathrm{d}z)$ is the Dirac-measure in $(0,0)$ and $F^{1*}_{\mathcal{L}^{-1}\mathcal{H}} = F_{\mathcal{L}^{-1}\mathcal{H}}$. Since L^{-1} and H always jump together the convolution is taken componentwise; i. e. $F^{n*}_{\mathcal{L}^{-1}\mathcal{H}}$ is the distribution of n independent jumps with bivariate d. f. $F_{\mathcal{L}^{-1}\mathcal{H}}$.

Theorem 3.3.3

Suppose X is given as in (3.3.4), and that (3.3.6) holds. Consider the first passage of X over a constant barrier $x > 0$. Then for $u > 0$, $y \in [0, x]$, $v \geq y$, $s \geq 0$, $t \geq 0$ it holds for $k = 1, 2, 3$

$$\mathbb{P}\Big(\tau_x^+ - \overline{G}_{\tau_x^+-} \in \mathrm{d}t, \overline{G}_{\tau_x^+-} \in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+-} \in \mathrm{d}v, \\ x - \overline{X}_{\tau_x^+-} \in \mathrm{d}y, \Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+}\Big) \tag{3.3.7}$$
$$= \Pi_{P^k}(\mathrm{d}u + v)\, \mathbb{P}\left(\tau^-_{-(v-y)} \in \mathrm{d}t\right) \mathrm{d}v\, \frac{1}{c} \sum_{n=0}^{\infty} \left(\frac{\mu_S}{c}\right)^n F^{n*}_{\mathcal{L}^{-1}\mathcal{H}}(\mathrm{d}s, x - \mathrm{d}y)$$

where the bivariate jump size d. f. $F_{\mathcal{L}^{-1}\mathcal{H}}$ is given by

$$F_{\mathcal{L}^{-1}\mathcal{H}}(\mathrm{d}t, \mathrm{d}h) = \frac{1}{\mu_S} \int_0^\infty \Pi_+(\mathrm{d}h + \theta)\, \mathbb{P}(\tau^-_{-\theta} \in \mathrm{d}t)\, \mathrm{d}\theta. \tag{3.3.8}$$

Proof.

To calculate $\mathcal{U}$ and $\widehat{\mathcal{U}}$ in Theorem 3.2.4 explicitly, we have to specify the local time at maximum and at minimum such that the normalization condition (3.2.12) is satisfied. Since X is spectrally positive we choose the local time at the minimum as

$$\widehat{L}_t = \overline{-X}_t = c \int_0^t 1_{\{\underline{X}_s = X_s\}}\, \mathrm{d}s$$

where $\underline{X}$ is defined in (3.0.1). The unkilled descending ladder process is for $t \geq 0$ given by

$$\left(\widehat{L}^{-1}_t, \widehat{H}_t\right) = \left(\inf\{s > 0 : \underline{X}_s < -t\}, \widehat{X}_{\widehat{L}^{-1}_t}\right) = \left(\tau^-_{-t}, t\right). \tag{3.3.9}$$

Thus, by (3.2.11) we obtain

$$\widehat{\mathcal{U}}(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty \mathbb{P}(\widehat{L}^{-1}_t \in \mathrm{d}s, \widehat{H}_t \in \mathrm{d}x)\, \mathrm{d}t = \mathbb{P}(\widehat{L}^{-1}_x \in \mathrm{d}s)\, \mathrm{d}x \\ = \mathbb{P}(\tau^-_{-x} \in \mathrm{d}s)\, \mathrm{d}x, \tag{3.3.10}$$

$\widehat{\mathcal{U}}([0, \infty), \mathrm{d}x) = \mathrm{d}x$ and $\widehat{\kappa}(0, \beta) = \beta$, see (3.2.9). When the normalization condition (3.2.12) is satisfied, by Vigon [66], Proposition 3.3, it follows for $z > 0$

$$\overline{\Pi}_{\mathcal{H}}(z) = \int_z^\infty \overline{\Pi}_+(x)\, \widehat{\mathcal{U}}([0, \infty), \mathrm{d}x) = \int_z^\infty \overline{\Pi}_+(x)\, \mathrm{d}x. \tag{3.3.11}$$

Due to the irregularity of 0 for $[0, \infty)$ and following Definition 3.2.2 (2), we choose the local time at the maximum as

$$L_t = \sum_{k=0}^{n_t} e_\zeta^{(k)} \quad \text{with} \quad n_t = \#\{0 < s \leq t : \overline{X}_s = X_s\},$$

for an arbitrary parameter $\zeta > 0$ and i. i. d. $e_\zeta^{(k)} \stackrel{\mathrm{d}}{=} \mathrm{expo}(\zeta)$. Further, due to condition (3.3.6) the ascending ladder process is killed, i. e. there is a bivariate CPP $(\mathcal{L}^{-1}, \mathcal{H})$ with jump intensity ζ and $q > 0$ such that

$$\{(L_t^{-1}, H_t) : t < L_\infty\} \stackrel{\mathrm{d}}{=} \{(\mathcal{L}_t^{-1}, \mathcal{H}_t) : t < e_q\}$$

and $(L_t^{-1}, H_t) = (\infty, \infty)$ for $t \geq L_\infty \stackrel{\mathrm{d}}{=} e_q$. $\mathcal{H}$ is a CPP with intensity ζ and with (3.3.11) the normalization condition (3.2.12) is satisfied if and only if $\zeta = \Pi_{\mathcal{H}}(\mathbb{R}) = \mu_S$. From (3.2.9) we obtain

$$\kappa(0, -i\theta) = q + \int_0^\infty \left(1 - e^{i\theta x}\right) \Pi_{\mathcal{H}}(\mathrm{d}x)$$

and with (3.3.5) and the Wiener-Hopf factorization, see [49], Equation (6.21), it results

$$\kappa(0, -i\theta) = k' \frac{\Psi_X(\theta)}{\widehat{\kappa}(0, i\theta)} = \frac{k'}{i\theta} \left(ic\theta + \int_0^\infty \left(1 - e^{i\theta x}\right) \Pi_+(\mathrm{d}x) \right).$$

Since $\mathcal{H}$ is of bounded variation and $\lim_{x \downarrow 0} x \overline{\Pi}_+(x) = 0$ by (3.3.11), partial integration results in

$$\kappa(0, -i\theta) = k' \left((c - \mu_S) + \int_0^\infty \left(1 - e^{i\theta x}\right) \overline{\Pi}_+(x) \, \mathrm{d}x \right).$$

From (3.3.11) we conclude $k' = 1$ and $q = c - \mu_S$. Since $e_q \stackrel{\mathrm{d}}{=} \mathrm{expo}(q)$ with $q = c - \mu_S$ is independent of $(\mathcal{L}^{-1}, \mathcal{H})$ and $\mathcal{N}_t = \#\{0 < s \leq t : \Delta \mathcal{H}_t \neq 0\}$ is a Poisson process with intensity $\zeta = \mu_S$, we get with $F_{\mathcal{L}^{-1}\mathcal{H}} = \frac{1}{\mu_S} \Pi_{\mathcal{L}^{-1}\mathcal{H}}$ for $s \geq 0, x \geq 0$ that

$$\begin{aligned}
\mathcal{U}(\mathrm{d}s, \mathrm{d}x) &= \int_0^\infty \mathbb{P}(t < e_q, \mathcal{L}_t^{-1} \in \mathrm{d}s, \mathcal{H}_t \in \mathrm{d}x) \, \mathrm{d}t \\
&= \int_0^\infty e^{-qt} \sum_{n=0}^\infty \mathbb{P}(\mathcal{L}_t^{-1} \in \mathrm{d}s, \mathcal{H}_t \in \mathrm{d}x | \, \mathcal{N}_t = n) \mathbb{P}(\mathcal{N}_t = n) \, \mathrm{d}t \\
&= \int_0^\infty e^{-qt} \sum_{n=0}^\infty F_{\mathcal{L}^{-1}\mathcal{H}}^{n*}(\mathrm{d}s, \mathrm{d}x) \frac{(t\mu_S)^n}{n!} e^{-t\mu_S} \, \mathrm{d}t \\
&= \frac{1}{c} \sum_{n=0}^\infty \left(\frac{\mu_S}{c}\right)^n F_{\mathcal{L}^{-1}\mathcal{H}}^{n*}(\mathrm{d}s, \mathrm{d}x)
\end{aligned} \tag{3.3.12}$$

where the last equation results from the evaluation of the exponential integral; see e. g. [29], p. 362. Finally, from the quintuple law (3.2.13) with (3.3.10) and (3.3.12) we obtain for $u > 0$, $y \in [0,x]$, $v \geq y$, $s \geq 0$, $t \geq 0$ and for $k = 1,2,3$

$$\begin{aligned}
&\mathbb{P}\Big(\tau_x^+ - \overline{G}_{\tau_x^+-} \in \mathrm{d}t, \overline{G}_{\tau_x^+-} \in \mathrm{d}s, X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+-} \in \mathrm{d}v, \\
&\qquad\qquad\qquad x - \overline{X}_{\tau_x^+-} \in \mathrm{d}y, \Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+}\Big) \\
&= \Pi_{P^k}(\mathrm{d}u + v)\,\widehat{\mathcal{U}}(\mathrm{d}t, \mathrm{d}v - y)\,\mathcal{U}(\mathrm{d}s, x - \mathrm{d}y) \\
&= \Pi_{P^k}(\mathrm{d}u + v)\,\mathbb{P}\left(\tau^-_{-(v-y)} \in \mathrm{d}t\right) 1_{\{v-y\geq 0\}}\,\mathrm{d}v\,\frac{1}{c}\sum_{n=0}^{\infty}\left(\frac{\mu_S}{c}\right)^n F^{n*}_{\mathcal{L}^{-1}\mathcal{H}}(\mathrm{d}s, x - \mathrm{d}y).
\end{aligned}$$

According to [22], Corollary 6, we have

$$\Pi_{\mathcal{L}^{-1}\mathcal{H}}(\mathrm{d}t, \mathrm{d}h) = \int_{[0,\infty)} \Pi_+(\mathrm{d}h + \theta)\,\widehat{\mathcal{U}}(\mathrm{d}t, \mathrm{d}\theta)$$

and with (3.3.10) and the normalization condition Expression (3.3.8) holds. □

If we are only interested in the space variables, we can integrate out the time quantities in the above quintuple law and obtain the following.

Corollary 3.3.4
In the situation of Theorem 3.3.3, for $u > 0$, $y \in [0,x]$, $v \geq y$ it holds for $k = 1,2,3$

$$\begin{aligned}
&\mathbb{P}\left(X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+-} \in \mathrm{d}v, x - \overline{X}_{\tau_x^+-} \in \mathrm{d}y, \Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+}\right) \\
&= \Pi_{P^k}(\mathrm{d}u + v)\,\mathrm{d}v\,\frac{1}{c}\sum_{n=0}^{\infty}\left(\frac{\mu_S}{c}\right)^n F^{n*}_{\mathcal{H}}(x - \mathrm{d}y), \qquad (3.3.13)
\end{aligned}$$

and

$$\mathbb{P}(\Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+}) = \frac{1}{c}\int_0^x \int_y^\infty \overline{\Pi}_{P^k}(v)\,\mathrm{d}v \sum_{n=0}^{\infty}\left(\frac{\mu_S}{c}\right)^n F^{n*}_{\mathcal{H}}(x - \mathrm{d}y), \qquad (3.3.14)$$

$$\mathbb{P}(\tau_x^+ < \infty) = \left(1 - \frac{\mu_S}{c}\right)\sum_{n=1}^{\infty}\left(\frac{\mu_S}{c}\right)^n \overline{F^{n*}_{\mathcal{H}}}(x). \qquad (3.3.15)$$

The d.f. $F_{\mathcal{H}}$ is for $z > 0$ defined as

$$F_{\mathcal{H}}(\mathrm{d}z) = \frac{1}{\mu_S}\overline{\Pi}_+(z)\,\mathrm{d}z = \frac{1}{\mu_S}\left(\overline{\Pi}_{P^1} + \overline{\Pi}_{P^2} + \overline{\Pi}_{P^3}\right)(z)\,\mathrm{d}z. \qquad (3.3.16)$$

Here $F^{0*}_{\mathcal{H}}(\mathrm{d}z) = \delta_0(\mathrm{d}z)$ and for $n \in \mathbb{N}$

$$F^{n*}_{\mathcal{H}}(\mathrm{d}z) = \frac{1}{\mu_S^n}\overline{\Pi}_+^{n\otimes}(z)\,\mathrm{d}z$$

with $\overline{\Pi}_+^{1\otimes} = \overline{\Pi}_+$ and $\overline{\Pi}_+^{2\otimes}(z) := \int_0^z \overline{\Pi}_+(z-y)\overline{\Pi}_+(y)\,\mathrm{d}y$.

Proof.
Integrating out time in Equation (3.3.7) yields (3.3.13). The relation (3.3.16) follows from (3.3.11) with the normalization condition and the decomposition (3.2.5). The identity (3.3.14) results from (3.3.13) by integrating out u, v and y. By integrating out s, t and v in the quintuple law in [22], Theorem 3, it holds

$$\mathbb{P}\left(X_{\tau_x^+} - x \in \mathrm{d}u, x - \overline{X}_{\tau_x^+} \in \mathrm{d}y\right) = \mathcal{U}(x-\mathrm{d}y)\,\Pi_{\mathcal{H}}(\mathrm{d}u+y)$$

and with (3.3.12) we obtain

$$\begin{aligned}
\mathbb{P}(\tau_x^+ < \infty) &= \int_0^x \overline{\Pi}_{\mathcal{H}}(y)\mathcal{U}(x-\mathrm{d}y) \\
&= \sum_{n=0}^{\infty}\left(\frac{\mu_S}{c}\right)^{n+1}\int_0^x \overline{F}_{\mathcal{H}}(y)\,F_{\mathcal{H}}^{n*}(x-\mathrm{d}y) \\
&= \sum_{n=0}^{\infty}\left(\frac{\mu_S}{c}\right)^{n+1}\left(F_{\mathcal{H}}^{n*}(x) - F_{\mathcal{H}}^{(n+1)*}(x)\right) \\
&= \left(1-\frac{\mu_S}{c}\right)\sum_{n=1}^{\infty}\left(\frac{\mu_S}{c}\right)^{n}\overline{F_{\mathcal{H}}^{n*}}(x).
\end{aligned}$$

□

Remark 3.3.5
When the jump part S in (3.3.4) is a CPP with jump size d. f. F_+ and $\mathbb{E}[\Delta S] = \mu_{\Delta S}$ then, under the conditions of Theorem 3.3.3, for $x > 0$

$$F_{\mathcal{H}}(x) = \frac{1}{\mu_{\Delta S}}\int_0^x \overline{F}_+(z)\,\mathrm{d}z$$

and (3.3.15) is the well-known *Pollaczek-Khintchine formula.*

Corollary 3.3.6
Suppose X is given as in (3.3.4) and (3.3.6) holds. Consider the first passage of X over the barrier $x = 0$. Then for $u > 0$, $v \geq 0$ and $t > 0$

$$\begin{aligned}
&\mathbb{P}\left(\tau_0^+ \in \mathrm{d}t, X_{\tau_0^+} \in \mathrm{d}u, -X_{\tau_0^+-} \in \mathrm{d}v, \Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+}\right) \\
&= \Pi_{P^k}(\mathrm{d}u+v)\,\mathbb{P}(\tau_{-v}^- \in \mathrm{d}t)\,\frac{1}{c}\,\mathrm{d}v, \quad k = 1,2,3. \qquad (3.3.17)
\end{aligned}$$

Further,

$$\mathbb{P}\left(X_{\tau_0^+} \in \mathrm{d}u, -X_{\tau_0^+-} \in \mathrm{d}v, \Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+}\right) = \Pi_{P^k}(\mathrm{d}u+v)\,\frac{1}{c}\,\mathrm{d}v \qquad (3.3.18)$$

$$\mathbb{P}\left(\Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+}\right) = \frac{\mu_{P^k}}{c} \qquad (3.3.19)$$

$$\mathbb{P}(\tau_0^+ < \infty) = \frac{\mu_S}{c}. \qquad (3.3.20)$$

Proof.
For the barrier $x = 0$ we obtain under the normalization condition (3.2.12) with (3.2.15) and (3.3.12) that

$$\begin{aligned}&\mathbb{P}\left(\tau_0^+ \in \mathrm{d}t, X_{\tau_0^+} \in \mathrm{d}u, -X_{\tau_0^+-} \in \mathrm{d}v, \Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+}\right)\\ &= \Pi_{P^k}(\mathrm{d}u+v)\,\widehat{\mathcal{U}}(\mathrm{d}t,\mathrm{d}v)\,\mathcal{U}(\{0\},\{0\})\\ &= \Pi_{P^k}(\mathrm{d}u+v)\,\mathbb{P}(\tau_{-v}^- \in \mathrm{d}t)\,\mathrm{d}v\,\frac{1}{c}.\end{aligned}$$

The identities (3.3.18) and (3.3.19) follow from (3.3.17) by integrating out t, u and v. (3.3.20) results by summing up (3.3.19) for $k = 1, 2, 3$. □

The identities (3.3.18) and (3.3.19) are generalizations of Theorem 2.2(i) in [38] where only independence is treated. Comparing (3.3.15) and (3.3.20), we see that the first upwards passage for the barrier $x = 0$ is not affected by the dependence contrary to barriers $x > 0$.

3.4 Dependence modelled by a Lévy copula

Our main interest is studying the effect of dependence between the jumps of X^1 and X^2 for the quintuple law. This dependence affects the bivariate potential measures $\mathcal{U}, \widehat{\mathcal{U}}$ and also the factors Π_{P^k}. In this section we calculate these three quantities for the situations in Section 3.3 when the dependence is modelled by a Lévy copula as defined in Section 1.1. Finally, we apply our results to the four dependence structures discussed in Section 1.3.

3.4.1 Calculating the quantities in the quintuple law

Analogously to Definition 1.1.4 we define the tail integrals $\overline{\Pi}_{P^k}$ for $k = 1, \ldots, 5$ of the single and joint jump components. For better differentiation between the positive and the negative tail integral we denote for spectrally both-sided Lévy measures Π_i the positive tail integral by $\overline{\Pi}_i^+$ and the negative by $\overline{\Pi}_i^-$. The following result shows the influence of a specific Lévy copula on the tail integrals, where we set $\overline{\Pi}_2^+(0) := \Pi_2((0,\infty))$ in (3.4.3).

Theorem 3.4.1
Suppose that the jump parts S^1 and S^2, given in (3.2.1), have absolutely continuous Lévy measures Π_i and the dependence between their jumps is modelled by a twice continuously differentiable Lévy copula $\widehat{C}$. Then the tail integrals in Theorem 3.2.4 and Corollary 3.2.5, respectively, are given for $z > 0$

$$\overline{\Pi}_{P^1}(z) = \overline{\Pi}_1^+(z) - \lim_{y\downarrow 0} \widehat{C}\left(\overline{\Pi}_1^+(z), \overline{\Pi}_2^+(y)\right) + \lim_{y\uparrow 0} \widehat{C}\left(\overline{\Pi}_1^+(z), \overline{\Pi}_2^-(y)\right) \tag{3.4.1}$$

$$\overline{\Pi}_{P^2}(z) = \overline{\Pi}_2^+(z) - \lim_{x\downarrow 0} \widehat{C}\left(\overline{\Pi}_1^+(x), \overline{\Pi}_2^+(z)\right) + \lim_{x\uparrow 0} \widehat{C}\left(\overline{\Pi}_1^-(x), \overline{\Pi}_2^+(z)\right) \tag{3.4.2}$$

$$\overline{\Pi}_{P^3}(z) = \int_0^\infty \frac{\partial \widehat{C}(u,v)}{\partial u}\Big|_{u=\overline{\Pi}_1^+(x), v=\overline{\Pi}_2^+((z-x)\vee 0)} \Pi_1(\mathrm{d}x) \tag{3.4.3}$$

$$\overline{\Pi}_{P^4}^+(z) = \int_z^\infty \left[\frac{\partial \widehat{C}(u,v)}{\partial u}\Big|_{u=\overline{\Pi}_1^+(x)}\right]_{\lim_{a\uparrow 0}\overline{\Pi}_2^-(a)}^{\overline{\Pi}_2^-(z-x)} \Pi_1(\mathrm{d}x)$$

$$\overline{\Pi}_{P^5}^+(z) = \int_z^\infty \left[\frac{\partial \widehat{C}(u,v)}{\partial v}\Big|_{v=\overline{\Pi}_2^+(y)}\right]_{\lim_{a\uparrow 0}\overline{\Pi}_1^-(a)}^{\overline{\Pi}_1^-(z-y)} \Pi_2(\mathrm{d}y).$$

If $\widehat{C}$ is left-continuous in the second coordinate in ∞ and $\Pi_2((0,\infty)) = \Pi_2((-\infty,0)) = \infty$, then (3.4.1) reduces to $\Pi_{P^1} = 0$. If $\widehat{C}$ is left-continuous in the first coordinate in ∞ and $\Pi_1((0,\infty)) = \Pi_1((-\infty,0)) = \infty$, then (3.4.2) reduces to $\Pi_{P^2} = 0$.

Proof.
By Theorem 1.1.10 and (1.1.7) the tail integral of (S^1, S^2) for $i, j \in \{+,-\}$ is given by

$$\overline{\Pi}(x_1, x_2) = \widehat{C}\left(\overline{\Pi}_1^i(x_1), \overline{\Pi}_2^j(x_2)\right), \quad x_1, x_2 \in \mathbb{R}\setminus\{0\}.$$

So we get for $z > 0$, using (3.2.3),

$$\begin{aligned}\overline{\Pi}_{P^1}(z) &= \lim_{\epsilon\downarrow 0} \Pi_{P^{1,\epsilon}}((z,\infty)) = \lim_{\epsilon\downarrow 0} \Pi((z,\infty)\times\{0\}) 1_{\{z>\epsilon\}} = \Pi((z,\infty)\times\{0\}) \\ &= \Pi((z,\infty)\times\mathbb{R}) - \lim_{y\downarrow 0}\Pi((z,\infty)\times(y,\infty)) - \lim_{y\uparrow 0}\Pi((z,\infty)\times(-\infty,y)) \\ &= \Pi_1((z,\infty)) - \lim_{y\downarrow 0}\overline{\Pi}(z,y) + \lim_{y\uparrow 0}\overline{\Pi}(z,y) \\ &= \overline{\Pi}_1^+(z) - \lim_{y\downarrow 0}\widehat{C}\left(\overline{\Pi}_1^+(z), \overline{\Pi}_2^+(y)\right) + \lim_{y\uparrow 0}\widehat{C}\left(\overline{\Pi}_1^+(z), \overline{\Pi}_2^-(y)\right).\end{aligned}$$

Analogous calculations give $\overline{\Pi}_{P^2}$.

For the common jump measures we obtain by (3.2.4) for $z > 0$

$$\overline{\Pi}_{P^3}(z) = \Pi\left(\{(x,y) \in (0,\infty) \times (0,\infty) : x + y > z\}\right) \tag{3.4.4}$$

$$\overline{\Pi}^+_{P^4}(z) = \Pi\left(\{(x,y) \in (0,\infty) \times (-\infty,0) : x + y > z\}\right) \tag{3.4.5}$$

$$\overline{\Pi}^+_{P^5}(z) = \Pi\left(\{(x,y) \in (-\infty,0) \times (0,\infty) : x + y > z\}\right). \tag{3.4.6}$$

Since $\widehat{C}$ can be continuously differentiated twice, we obtain by relation (1.1.7) and Equation (1.1.8) on $(\mathbb{R} \setminus \{0\})^2$ the density (cf. [20], Proposition 5.8)

$$\Pi(\mathrm{d}x, \mathrm{d}y) = \left.\frac{\partial^2 \widehat{C}(u,v)}{\partial u \partial v}\right|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2(y)} \Pi_1(\mathrm{d}x)\, \Pi_2(\mathrm{d}y). \tag{3.4.7}$$

So the r. h. s. of (3.4.4) is given by

$$\begin{aligned}
\int_0^\infty \int_{(z-x)\vee 0}^\infty \Pi(\mathrm{d}x, \mathrm{d}y) &= \int_0^\infty \int_{(z-x)\vee 0}^\infty \left.\frac{\partial^2 \widehat{C}(u,v)}{\partial u \partial v}\right|_{u=\overline{\Pi}^+_1(x), v=\overline{\Pi}^+_2(y)} \Pi_2(\mathrm{d}y)\, \Pi_1(\mathrm{d}x) \\
&= \int_0^\infty \int_0^{\overline{\Pi}^+_2((z-x)\vee 0)} \left.\frac{\partial^2 \widehat{C}(u,v)}{\partial u \partial v}\right|_{u=\overline{\Pi}^+_1(x), v=s} \mathrm{d}s\, \Pi_1(\mathrm{d}x) \\
&= \int_0^\infty \left[\left.\frac{\partial \widehat{C}(u,v)}{\partial u}\right|_{u=\overline{\Pi}^+_1(x)} \right]_0^{\overline{\Pi}^+_2((z-x)\vee 0)} \Pi_1(\mathrm{d}x) \\
&= \int_0^\infty \left.\frac{\partial \widehat{C}(u,v)}{\partial u}\right|_{u=\overline{\Pi}^+_1(x), v=\overline{\Pi}^+_2((z-x)\vee 0)} \Pi_1(\mathrm{d}x),
\end{aligned}$$

since $\widehat{C}(u,0) = 0$ for all $u \in (-\infty, \infty]$. The r. h. s. of (3.4.5) is given by

$$\begin{aligned}
\int_z^\infty \int_{z-x}^0 \Pi(\mathrm{d}x, \mathrm{d}y) &= \int_z^\infty \int_{z-x}^0 \left.\frac{\partial^2 \widehat{C}(u,v)}{\partial u \partial v}\right|_{u=\overline{\Pi}^+_1(x), v=\overline{\Pi}^-_2(y)} \Pi_2(\mathrm{d}y)\, \Pi_1(\mathrm{d}x) \\
&= \int_z^\infty \int_{\lim_{a\uparrow 0} \overline{\Pi}^-_2(a)}^{\overline{\Pi}^-_2(z-x)} \left.\frac{\partial \widehat{C}(u,v)}{\partial u \partial v}\right|_{u=\overline{\Pi}^+_1(x), v=s} \mathrm{d}s\, \Pi_1(\mathrm{d}x) \\
&= \int_z^\infty \left[\left.\frac{\partial \widehat{C}(u,v)}{\partial u}\right|_{u=\overline{\Pi}^+_1(x)} \right]_{\lim_{a\uparrow 0} \overline{\Pi}^-_2(a)}^{\overline{\Pi}^-_2(z-x)} \Pi_1(\mathrm{d}x).
\end{aligned}$$

The r. h. s. of (3.4.6) is given by

$$\int_z^\infty \int_{z-y}^0 \Pi(\mathrm{d}x, \mathrm{d}y) = \int_z^\infty \int_{z-y}^0 \left.\frac{\partial^2 \widehat{C}(u,v)}{\partial u \partial v}\right|_{u=\overline{\Pi}^-_1(x), v=\overline{\Pi}^+_2(y)} \Pi_1(\mathrm{d}x)\, \Pi_2(\mathrm{d}y).$$

□

The following result is a simple consequence of Theorem 3.4.1.

Corollary 3.4.2 ([18], Proposition 2.16)
Assume that the conditions of Theorem 3.4.1 hold and that S^1 and S^2 are spectrally positive. Then the tails (3.4.1), (3.4.2) and (3.4.3) reduce to

$$\begin{aligned}\overline{\Pi}_{P^1}(z) &= \overline{\Pi}_1(z) - \lim_{y\downarrow 0} \widehat{C}\left(\overline{\Pi}_1(z), \overline{\Pi}_2(y)\right),\\ \overline{\Pi}_{P^2}(z) &= \overline{\Pi}_2(z) - \lim_{x\downarrow 0} \widehat{C}\left(\overline{\Pi}_1(x), \overline{\Pi}_2(z)\right),\\ \overline{\Pi}_{P^3}(z) &= \int_0^\infty \frac{\partial \widehat{C}(u,v)}{\partial u}\bigg|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2((z-x)\vee 0)} \Pi_1(\mathrm{d}x).\end{aligned}$$

If $\widehat{C}$ is left-continuous in the second coordinate in ∞ and $\Pi_2((0,\infty)) = \infty$, then $\Pi_{P^1} = 0$. If $\widehat{C}$ is left-continuous in the first coordinate in ∞ and $\Pi_1((0,\infty)) = \infty$, then $\Pi_{P^2} = 0$.

The following lemma shows that for a left-continuous and homogeneous Lévy copula $\widehat{C}$ positive single jumps always have a lighter tail than the corresponding component.

Lemma 3.4.3
Assume that the conditions of Corollary 3.4.2 hold. If $\widehat{C}$ is left-continuous in the j-th coordinate in ∞ and homogeneous, then for $i = 1, 2$, such that $i \neq j$,

$$\overline{\Pi}_{P^i}(z) = o(\overline{\Pi}_i(z)), \quad z \to \infty. \tag{3.4.8}$$

Proof.
With Corollary 3.4.2 we get by homogeneity,

$$\frac{\overline{\Pi}_{P^1}(z)}{\overline{\Pi}_1(z)} = 1 - \lim_{y\downarrow 0} \widehat{C}\left(1, \frac{\overline{\Pi}_2(y)}{\overline{\Pi}_1(z)}\right), \quad z \geq 0,$$

which is equal to 1 for all $z \geq 0$ in the case of independence. Otherwise, by left-continuity in ∞,

$$\lim_{z\to\infty} \frac{\overline{\Pi}_{P^1}(z)}{\overline{\Pi}_1(z)} = 1 - \lim_{z\to\infty}\lim_{y\downarrow 0} \widehat{C}\left(1, \frac{\overline{\Pi}_2(y)}{\overline{\Pi}_1(z)}\right) = 1 - \widehat{C}(1,\infty) = 0.$$

The proof for P^2 is analogous. □

Now we apply our results to the situations of Section 3.3.

Theorem 3.4.4
Suppose that the jump parts S^1 and S^2, given in (3.2.1), have absolutely continuous Lévy measures Π_i and the dependence between their jumps is modelled by a twice continuously differentiable Lévy copula $\widehat{C}$.

(1) *In the situation of Section 3.3.1, when S^1 and S^2 are spectrally positive CPPes with jump intensities λ_1, λ_2 and jump size d. f. s F_1, F_2, Theorem 3.3.1 and Corollary 3.3.2 hold with*

$$\lambda_+ = \lambda_1 + \lambda_2 - \widehat{C}(\lambda_1, \lambda_2), \tag{3.4.9}$$

and for $z > 0$

$$\begin{aligned}\overline{F}_+(z) =& \frac{1}{\lambda_+}\Big(\lambda_1\overline{F}_1(z) - \widehat{C}\left(\lambda_1\overline{F}_1(z), \lambda_2\right) + \lambda_2\overline{F}_2(z) - \widehat{C}\left(\lambda_1, \lambda_2\overline{F}_2(z)\right) \\ &+ \lambda_1 \int_0^\infty \frac{\partial\widehat{C}(u,v)}{\partial u}\bigg|_{u=\lambda_1\overline{F}_1(x), v=\lambda_2\overline{F}_2((z-x)\vee 0)} F_1(\mathrm{d}x)\Big).\end{aligned} \tag{3.4.10}$$

(2) *In the situation of Section 3.3.2, when $S = S^1 + S^2$ is a subordinator, Corollary 3.3.4 holds for $F_{\mathcal{H}}(z)$ given for $z > 0$ by*

$$\begin{aligned}F_{\mathcal{H}}(\mathrm{d}z) =& \frac{1}{\mu_S}\Big(\overline{\Pi}_1(z) - \lim_{y\downarrow 0}\widehat{C}\left(\overline{\Pi}_1(z), \overline{\Pi}_2(y)\right) \\ &+ \overline{\Pi}_2(z) - \lim_{x\downarrow 0}\widehat{C}\left(\overline{\Pi}_1(x), \overline{\Pi}_2(z)\right) \\ &+ \int_0^\infty \frac{\partial\widehat{C}(u,v)}{\partial u}\bigg|_{u=\overline{\Pi}_1(x), v=\overline{\Pi}_2((z-x)\vee 0)} \Pi_1(\mathrm{d}x)\Big)\,\mathrm{d}z.\end{aligned} \tag{3.4.11}$$

Proof.
(1) Equation (3.4.9) holds by

$$\begin{aligned}\lambda_+ &= \Pi_+((0,\infty)) = \Pi([0,\infty)^2) \\ &= \Pi_1((0,\infty)) + \Pi_2((0,\infty)) - \Pi((0,\infty)^2) = \lambda_1 + \lambda_2 - \lim_{x\downarrow 0}\overline{\Pi}(x,x)\end{aligned}$$

and Equation (3.4.10) results from (3.3.3) with Corollary 3.4.2.
(2) Equation (3.3.16) and Corollary 3.4.2 yield relation (3.4.11). □

Remark 3.4.5 (Comparison of random walk and Lévy process modelling)
Let (X^1, X^2) be a spectrally positive CPP (without drift) with marginal intensities λ_1, λ_2 and absolutely continuous marginal jump size d. f. s F_1, F_2. Denote by

$(W_n^i)_{n\in\mathbb{N}_0}$ the arrival times of the jumps of X^i. We use the embedded random walk structure for defining $Z_n^i := X^i_{W_n^i}$. Then the d. f. of the increments of Z^i is equal to F_i and Z^1 and Z^2 always jump together, since F_i has no atom at zero. If we model the dependence between the jumps by a distributional copula C_D as suggested in Section 1.1.1, then with Theorem 3.1.5 the tail of the jump size d. f. F_Z of $Z = Z^1 + Z^2$ is given by

$$\begin{aligned}\overline{F}_Z(z) &= \int_0^\infty \int_{(z-x)\vee 0}^\infty \left.\frac{\partial^2 C_D(u,v)}{\partial u \partial v}\right|_{u=F_1(x),v=F_2(y)} F_2(\mathrm{d}y)\, F_1(\mathrm{d}x)\\ &= \int_0^\infty \left(1 - \left.\frac{\partial C_D(u,v)}{\partial u}\right|_{u=F_1(x),v=F_2((z-x)\vee 0)}\right) F_1(\mathrm{d}x).\end{aligned}$$

Rewriting this expression in terms of the distributional survival copula $\widehat{C}_D(u,v) := u + v - 1 + C_D(1-u, 1-v)$, see [52], Equation (2.6.2), yields

$$\overline{F}_Z(z) = \int_0^\infty \left.\frac{\partial \widehat{C}_D(u,v)}{\partial u}\right|_{u=\overline{F}_1(x),v=\overline{F}_2((z-x)\vee 0)} F_1(\mathrm{d}x). \tag{3.4.12}$$

When we consider, however, the Lévy process (X^1, X^2) and use a Lévy copula $\widehat{C}$, then the tail $\mathbb{P}(\Delta X^1 + \Delta X^2 > z)$ is given by (3.4.10). Comparing (3.4.12) and (3.4.10), the most apparent differences are the first four summands in (3.4.10). These summands represent the possibility of single jumps of X^i. They are missing in (3.4.12) since the random walks Z^1 and Z^2 always jump together by construction. But also the last integrals in (3.4.10) and (3.4.12) differ which represent the common jumps of X^1 and X^2. Furthermore, a distributional survival copula $\widehat{C}_D : [0,1]^2 \to [0,1]$ and a Lévy copula $\widehat{C} : (-\infty,\infty]^2 \to (-\infty,\infty]$ which both respectively describe the same dependence structure, are in general not identical. E. g. with the independence distributional survival copula $\widehat{C}_D(u,v) = uv$ Equation (3.4.12) becomes for $z > 0$

$$\overline{F}_Z(z) = \int_0^z \overline{F}_2(z-x)\, F_1(\mathrm{d}x) + \overline{F}_1(z).$$

On the other hand, the independence Lévy copula $\widehat{C}_\perp$ given in (1.3.2) yields in Equation (3.4.10) that

$$\overline{F}_+(z) = \frac{\lambda_1 \overline{F}_1(z) + \lambda_2 \overline{F}_2(z)}{\lambda_1 + \lambda_2}.$$

Consequently, a Lévy copula approach and a (survival) copula approach produce even for a bivariate CPP (without drift) different results. Furthermore, for a CPP with a linear drift the increments of the corresponding random walk are the sum of the original jump size and an independent exponentially distributed r. v. (inter jump times times drift). Thus, the dependence induced by a (survival)copula differs from the dependence given by a Lévy copula. The advantage of a Lévy copula is that a Lévy copula describes the dependence not only between the jump sizes, but also between the jump times. Moreover, by applying a copula to increments of the random walk model, single and common jumps of the CPP are treated equally which alienates the originally dependence structure between the jump sizes.

3.4.2 Examples for different dependence structures

We present four examples for different dependence structures, modelled by a Lévy copula $\widehat{C}$, and characterize completely all quantities in the quintuple law of Theorem 3.3.1 and Corollary 3.3.4.

Independence

If S^1 and S^2 are independent, then S^1 and S^2 a. s. never jump together and $\widehat{C}_\perp$, given in (1.3.2), is a Lévy copula of (S^1, S^2). Therefore, $P^1 = S^{1,+} = S^1$, $P^2 = S^{2,+} = S^2$ and $P^3 = S^{1,++} + S^{2,++} = 0$ and we obtain

$$\Pi_+(\mathrm{d}z) = (\Pi_1 + \Pi_2)(\mathrm{d}z) = (\Pi_{P^1} + \Pi_{P^2})(\mathrm{d}z).$$

In the situation of Section 3.3.1, when the jump parts S^1 and S^2 are spectrally positive CPPes with intensities λ_1 and λ_2 and absolutely continuous jump size d. f. s F_1 and F_2, we get in Theorem 3.3.1 and in Corollary 3.3.2 the identities $\lambda_+ = \lambda_1 + \lambda_2$ and

$$F_+(\mathrm{d}z) = \frac{1}{\lambda}\Pi_+(\mathrm{d}z) = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}F_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}F_2\right)(\mathrm{d}z).$$

In the situation of Section 3.3.2, when X is a subordinator with negative drift, Corollary 3.3.4 holds with

$$F_{\mathcal{H}}(\mathrm{d}z) = \frac{1}{\mu_S}\left(\overline{\Pi}_1(z) + \overline{\Pi}_2(z)\right)\,\mathrm{d}z.$$

For this dependence structure Equations (3.3.18) and (3.3.19) are the results of [38], Theorem 2.2(i), for $d = 2$.

Complete dependence

If the jumps of (S^1, S^2) are completely dependent, then S^1 and S^2 a. s. jump together and $\widehat{C}_{\|}$, given in (1.3.6), is a Lévy copula of (S^1, S^2). Thus, $P^1 = P^2 = 0$ and $P^3 = S^1 + S^2 = S$ and in Section 3.3.1 we receive in Theorem 3.3.1 and Corollary 3.3.2 the identities $\lambda_+ = \lambda_1 = \lambda_2$ and

$$F_+(\mathrm{d}z) = \mathbb{P}(\Delta S^1 + \Delta S^2 \in \mathrm{d}z),$$

where the jump sizes ΔS^1 and ΔS^2 are completely dependent by Proposition 1.3.5. In Corollary 3.3.4, we get $F_{\mathcal{H}}(\mathrm{d}z) = 1/\mu_S\, \overline{\Pi}_{P^3}(z)\mathrm{d}z$. An easy example for complete dependence is $S^1 \equiv S^2$, then $F(\mathrm{d}z) = F_1(\mathrm{d}z/2)$ and

$$F_{\mathcal{H}}(\mathrm{d}z) = \frac{1}{\mu_S} \overline{\Pi}_1(z/2)\, \mathrm{d}z.$$

Clayton Lévy copula

Suppose that the dependence between the jump parts S^1 and S^2 is given by a Clayton Lévy copula defined in (1.3.7) for $\theta > 0$ and $\eta \in [0, 1]$, which is left-continuous in ∞. For $i = 1, 2$ let the Lévy measures Π_i be absolutely continuous. Further, we define for $i = 1, 2$

$$\Pi_i((0, \infty)) = \lambda_i^+ \leq \infty \quad \text{and} \quad \Pi_i((-\infty, 0)) = \lambda_i^- \leq \infty \tag{3.4.13}$$

and distinguish the following cases, where $i, j \in \{1, 2\}$ and $i \neq j$:

(a) $\lambda_j^+ < \infty$ and $\lambda_j^- < \infty$,

(b) $\lambda_j^+ < \infty$, $\lambda_j^- = \infty$ and $\eta \neq 0$,

(c) $\lambda_j^- < \infty$, $\lambda_j^+ = \infty$ and $\eta \neq 1$,

(d) $\lambda_j^+ = \infty$ and $\lambda_j^- = \infty$.

Then the quintuple laws of Theorem 3.2.4 and Corollary 3.2.5 hold, where for $z > 0$

$$\overline{\Pi}_{P^i}(z) = \begin{cases} \overline{\Pi}_i^+(z) - \eta\left(\overline{\Pi}_i^+(z)^{-\theta} + (\lambda_j^+)^{-\theta}\right)^{-1/\theta} - (1-\eta)\left(\overline{\Pi}_i^+(z)^{-\theta} + (\lambda_j^-)^{-\theta}\right)^{-1/\theta}, \\ \eta\left(\overline{\Pi}_i^+(z) - \left(\overline{\Pi}_i^+(z)^{-\theta} + (\lambda_j^+)^{-\theta}\right)^{-1/\theta}\right), \\ (1-\eta)\left(\overline{\Pi}_i^+(z) - \left(\overline{\Pi}_i^+(z)^{-\theta} + (\lambda_j^-)^{-\theta}\right)^{-1/\theta}\right), \\ 0, \end{cases}$$

and each line corresponds to the cases (a)-(d), respectively. Moreover,

$$\begin{aligned} \overline{\Pi}_{P^3}(z) =& \eta\int_0^\infty \left(\overline{\Pi}_1^+(x)^{-\theta} + \overline{\Pi}_2^+((z-x)\vee 0)^{-\theta}\right)^{-1/\theta-1} \overline{\Pi}_1^+(x)^{-\theta-1}\Pi_1(\mathrm{d}x) \\ \overline{\Pi}_{P^4}^+(z) =& (1-\eta)\int_z^\infty \Bigg(\left(\overline{\Pi}_1^+(x)^{-\theta} + (\lambda_2^-)^{-\theta}\right)^{-1/\theta-1} \\ & - \left(\overline{\Pi}_1^+(x)^{-\theta} + \left|\overline{\Pi}_2^-(z-x)^{-\theta}\right|\right)^{-1/\theta-1}\Bigg)\overline{\Pi}_1^+(x)^{-\theta-1}\Pi_1(\mathrm{d}x) \\ \overline{\Pi}_{P^5}^+(z) =& (1-\eta)\int_z^\infty \Bigg(\left((\lambda_1^-)^{-\theta} + \overline{\Pi}_2^+(y)^{-\theta}\right)^{-1/\theta-1} \\ & - \left(\left|\overline{\Pi}_1^-(z-y)\right|^{-\theta} + \overline{\Pi}_2^+(y)^{-\theta}\right)^{-1/\theta-1}\Bigg)\overline{\Pi}_2^+(y)^{-\theta-1}\Pi_2(\mathrm{d}y). \end{aligned}$$

In both situations of Section 3.3 the jump part S has only positive jumps and we must have $\eta = 1$ in (1.3.7). Then the tail integral of X is for $z > 0$ given by

$$\begin{aligned} \overline{\Pi}_+(z) &= \overline{\Pi}_{P^1}(z) + \overline{\Pi}_{P^2}(z) + \overline{\Pi}_{P^3}(z) \\ &= \overline{\Pi}_1(z) - \left(\overline{\Pi}_1(z)^{-\theta} + \lambda_2^{-\theta}\right)^{-1/\theta} + \overline{\Pi}_2(z) - \left(\overline{\Pi}_2(z)^{-\theta} + \lambda_1^{-\theta}\right)^{-1/\theta} \\ &\quad + \int_0^\infty \left(\overline{\Pi}_1(x)^{-\theta} + \overline{\Pi}_2((z-x)\vee 0)^{-\theta}\right)^{-1/\theta-1} \overline{\Pi}_1(x)^{-\theta-1}\,\Pi_1(\mathrm{d}x). \end{aligned} \tag{3.4.14}$$

If $\lambda_1 = \infty$, then we see at (3.4.14) that $\Pi_{P^2} = 0$ holds, i. e. S^2 has no single jumps. So if Π_{S^1} and Π_{S^2} are infinite measures, then there are infinitely many common jumps and no single jumps. If $\lambda_1 = \infty$ and $\lambda_2 < \infty$, then the intensity rate of the common jumps reduce to $\Pi((0,\infty)\times(0,\infty)) = \lim_{a\to\infty}\widehat{C}_\theta(a,\lambda_2) = \lambda_2$. If (S^1, S^2) is a CPP, then we get the result of [18], Proposition 3.1. In Section 3.3.1, Theorem 3.3.1 holds with

$$\lambda_+ = \lambda_1 + \lambda_2 - \left(\lambda_1^{-\theta} + \lambda_2^{-\theta}\right)^{-1/\theta} \quad \text{and} \quad \overline{F}_+(z) = \frac{1}{\lambda_+}\overline{\Pi}_+(z), \quad z > 0,$$

and in Section 3.3.2, Corollary 3.3.4 holds with $F_{\mathcal{H}}(\mathrm{d}z) = 1/\mu_S\,\overline{\Pi}_+(\mathrm{d}z)$. Since for all $u, v > 0$,

$$\begin{aligned} \frac{\partial \widehat{C}_\theta}{\partial \theta}(u,v) &= \theta^{-2}\left(u^{-\theta}+v^{\theta}\right)^{-1/\theta-1}\Big(u^{-\theta}\left(\ln(u^{-\theta}+v^{-\theta})+\theta\ln(u)\right) \\ &\quad +v^{-\theta}\left(\ln(u^{-\theta}+v^{-\theta}+\theta\ln(v))\right)\Big) \geq 0 \end{aligned}$$

and $\overline{\Pi}_{P^1}(z) = \overline{\Pi}_1(z) - \widehat{C}_\theta(\overline{\Pi}_1(z), \lambda_2)$, increasing the dependence parameter θ yields that the tail integrals of the single jump components and the intensity λ_+, i. e. the expected number of jumps of X per unit time, decrease as shown in Figure 3.1. This effect of increasing θ corresponds with the results in Example 1.3.7 for the Clayton PLM where the representation of the spectral density and of the PLC showed that increasing θ causes that the mass of $\Gamma_{1,\theta}$ moves to the diagonal.

According to Lemma 3.4.3, using a Clayton Lévy copula the single jumps P^1 and P^2 are always lighter-tailed than S^1 and S^2, respectively. For equal marginal Lévy measures this implies that asymptotically for large z the joint jumps P^3 dominate.

This can be seen in the special case of two CPPes with the same marginal Lévy measures, which are exponential, i. e. $\Pi_1(\mathrm{d}x) = \Pi_2(\mathrm{d}x) = ae^{-ax}\mathrm{d}x$ for some $a > 0$ and $\theta = 1$. For $z > 0$ we get an explicit expression for the Lévy measure of the sum (cf. [18], Example 3.11) as

$$\begin{aligned} \overline{\Pi}_+(z) &= \frac{3+2e^{az}+e^{-az}}{(e^{az}+1)(e^{-az}+1)} + \frac{1}{2}e^{-1/2az}\left(\arctan e^{1/2az} - \arctan e^{-1/2az}\right) \\ &\sim e^{-az}\left(1+\frac{\pi}{2}e^{-\frac{1}{2}az}\right) \quad \text{as } z \to \infty. \end{aligned}$$

Non-homogeneous Archimedean Lévy copula

Now we consider the positive Archimedean Lévy copula given in (1.3.7). As in (3.4.13) we define λ_i^+, λ_i^- and distinguish the same four situations. Then the quintuple law of Theorem 3.2.4 and Corollary 3.2.5 hold, where for $z > 0$ and $i \neq j$

$$\overline{\Pi}_{P^i}(z) = \begin{cases} \overline{\Pi}_i^+(z) - \eta\frac{\overline{\Pi}_i^+(z)\lambda_j^+}{\overline{\Pi}_i^+(z)+\lambda_j^+ + \zeta} - (1-\eta)\frac{\overline{\Pi}_i^+(z)\lambda_j^-}{\overline{\Pi}_i^+(z)+\lambda_j^-+\zeta}, \\ \eta\overline{\Pi}_i^+(z)\frac{\overline{\Pi}_i^+(z)+\zeta}{\overline{\Pi}_i^+(z)+\lambda_j^++\zeta}, \\ (1-\eta)\overline{\Pi}_i^+(z)\frac{\overline{\Pi}_i^+(z)+\zeta}{\overline{\Pi}_i^+(z)+\lambda_j^-+\zeta}, \\ 0, \end{cases}$$

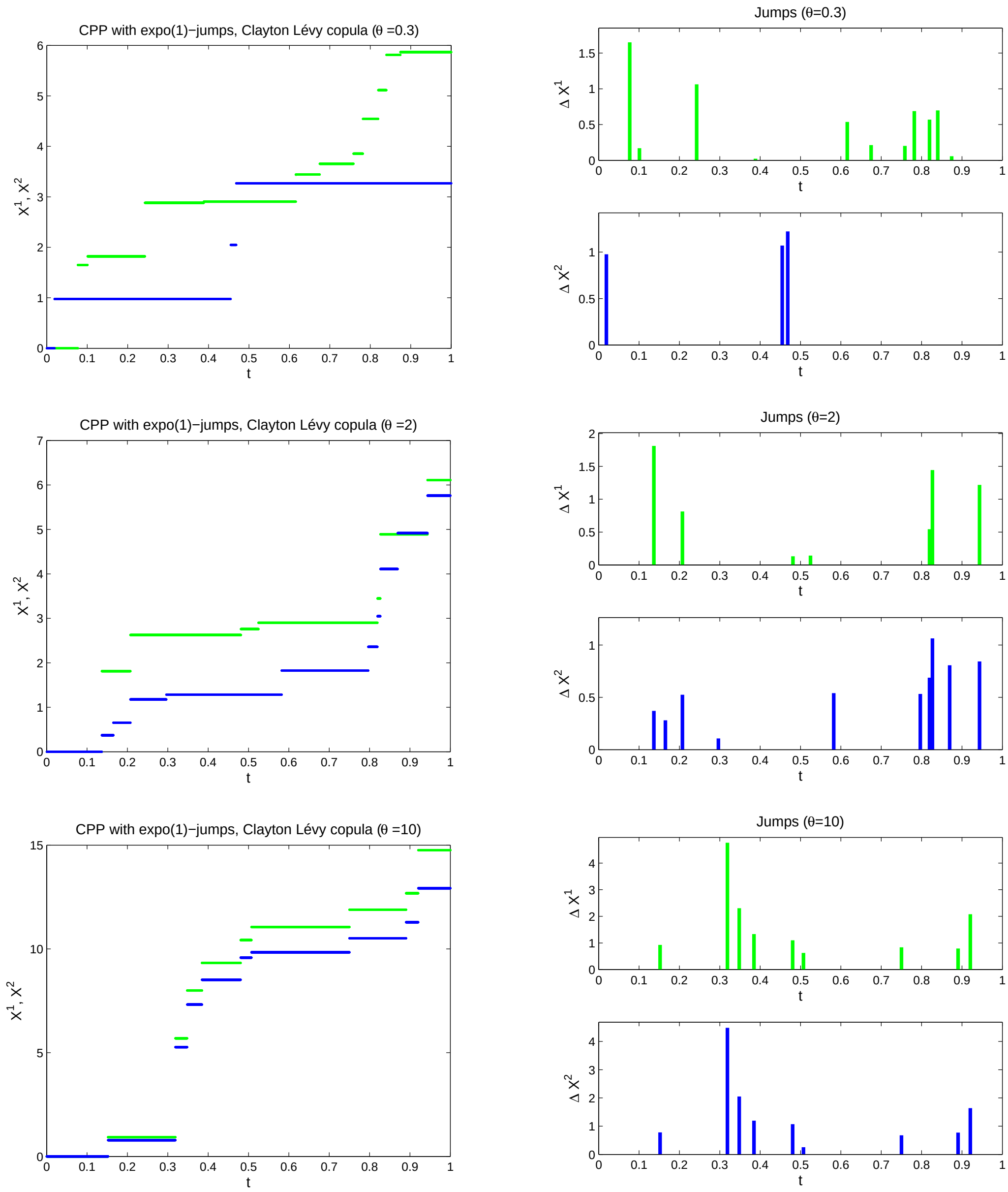

Figure 3.1: Spectrally positive CPP (X^1, X^2) with expo(1)-distributed jump sizes and dependence modelled by a Clayton Lévy copula for $\theta = 0.3$, $\theta = 2$ and $\theta = 10$; left: sample paths; right: jump times and jump sizes. When θ increases, then the number of single jumps $\Delta P^1, \Delta P^2$, cf. (3.2.6), decreases and the number of common jumps ΔP^3 increases. Further, for increasing θ, the dependence between jump sizes of X^1 and X^2 increases.

and each line corresponds to the cases (a)-(d), respectively. Moreover,

$$\overline{\Pi}_{P^3}(z) = \eta \int_0^\infty \frac{\overline{\Pi}_2^+(0 \vee (z-x))^2 + \zeta \overline{\Pi}_2^+(0 \vee (z-x))}{\left(\overline{\Pi}_1^+(x) + \overline{\Pi}_2^+(0 \vee (z-x)) + \zeta\right)^2} \Pi_1(\mathrm{d}x)$$

$$\overline{\Pi}_{P^4}^+(z) = (1-\eta) \int_z^\infty \left(\frac{(\lambda_2^-)^2 + \zeta \lambda_2^-}{\left(\overline{\Pi}_1^+(x) + \lambda_2^- + \zeta\right)^2} - \frac{\left|\overline{\Pi}_2^-(z-x)\right|^2 + \zeta \left|\overline{\Pi}_2^-(z-x)\right|}{\left(\overline{\Pi}_1^+(x) + \left|\overline{\Pi}_2^-(z-x)\right| + \zeta\right)^2} \right) \Pi_1(\mathrm{d}x)$$

$$\overline{\Pi}_{P^5}^+(z) = (1-\eta) \int_z^\infty \left(\frac{(\lambda_1^-)^2 + \zeta \lambda_1^-}{\left(\lambda_1^- + \overline{\Pi}_2^+(y) + \zeta\right)^2} - \frac{\left|\overline{\Pi}_1^-(z-y)\right|^2 + \zeta \left|\overline{\Pi}_1^-(z-y)\right|}{\left(\left|\overline{\Pi}_1^-(z-y)\right| + \overline{\Pi}_2^+(y) + \zeta\right)^2} \right) \Pi_2(\mathrm{d}y)$$

In the spectrally positive situations of Section 3.3 we must have again $\eta = 1$ and it follows for $z > 0$

$$\begin{aligned} \overline{\Pi}_+(z) =& \overline{\Pi}_1(z) \left(1 - \frac{\lambda_2}{\overline{\Pi}_1(z) + \lambda_2 + \zeta}\right) + \overline{\Pi}_2(z) \left(1 - \frac{\lambda_1}{\overline{\Pi}_2(z) + \lambda_1 + \zeta}\right) \\ &+ \int_0^\infty \frac{\overline{\Pi}_2(0 \vee (z-x))^2 + \zeta \overline{\Pi}_2(0 \vee (z-x))}{\left(\overline{\Pi}_1(x) + \overline{\Pi}_2(0 \vee (z-x)) + \zeta\right)^2} \Pi_1(\mathrm{d}x). \end{aligned} \tag{3.4.15}$$

In Section 3.3.1, Theorem 3.3.1 and Corollary 3.3.2 hold with

$$\lambda_+ = \lambda_1 + \lambda_2 - \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 + \zeta} \quad \text{and} \quad \overline{F}_+(z) = \frac{1}{\lambda} \overline{\Pi}_+(z), \quad z > 0,$$

and in Section 3.3.2 Corollary 3.3.4 holds with $F_{\mathcal{H}}(\mathrm{d}z) = 1/\mu_S \overline{\Pi}_+(z)\,\mathrm{d}z$. Contrary to the Clayton Lévy copula, increasing the dependence parameter ζ yields that the tail integrals of the single jumps increase and the tail integrals of the common jumps decreases. Further, the jump intensity λ_+ increases due to more single jumps which can be seen in Figure 3.2. This result corresponds with the investigation of the non-homogeneous PLM in Example 1.3.8 where the PLC representation illustrated how the mass of $\Gamma_{1,\zeta}$ for increasing ζ moves to the coordinate axes.

Since Lemma 3.4.3 does not cover the asymptotic of the single jumps for this non-homogeneous Lévy copula for finite measures, we calculate the following fraction explicitly.

$$\begin{aligned} \frac{\overline{\Pi}_{P^1}(z)}{\overline{\Pi}_1(z)} &= 1 - \lim_{y \downarrow 0} \frac{\overline{\Pi}_2(y)}{\overline{\Pi}_1(z) + \overline{\Pi}_2(y) + \zeta} \\ &= 1 - \frac{\lambda_2}{\overline{\Pi}_1(z) + \lambda_2 + \zeta} \to \frac{\zeta}{\lambda_2 + \zeta}, \quad z \to \infty. \end{aligned}$$

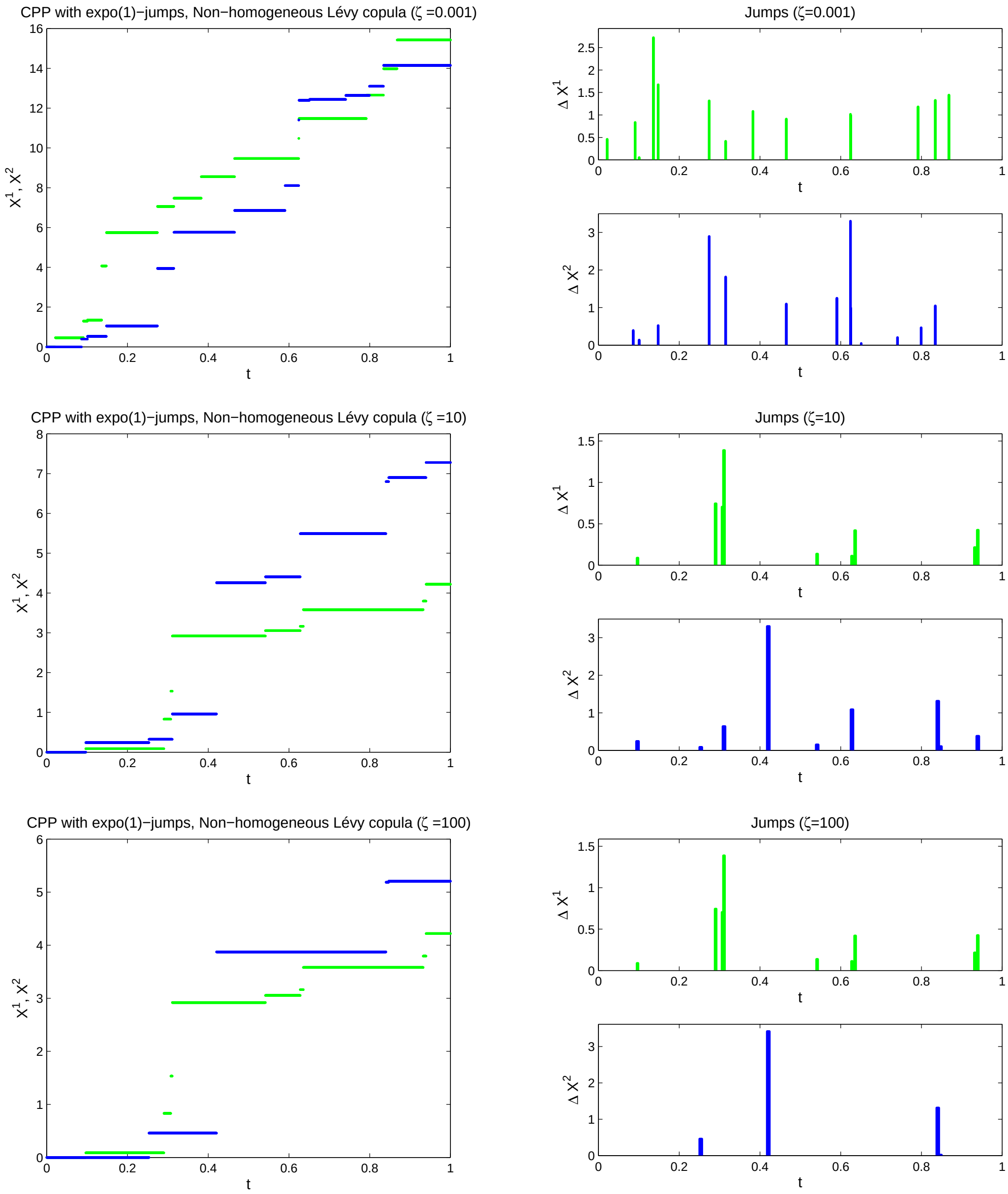

Figure 3.2: Spectrally positive CPP (X^1, X^2) with expo(1)-distributed jump sizes and dependence modelled by the non-homogeneous Lévy copula for $\zeta = 0.001$, $\zeta = 10$ and $\zeta = 100$; left: sample paths; right: jump times and jump sizes. When ζ increases, then the number of single jumps $\Delta P^1, \Delta P^2$, cf. (3.2.6), increases and the number of common jumps ΔP^3 decreases. Further, for increasing ζ, the dependence between jump sizes of X^1 and X^2 decreases.

Contrary to the homogeneous Lévy copulas the single jump Lévy measures are tail-equivalent to the Lévy measures of the components. Consequently, the Lévy measure of the sum process can be dominated by the common jumps, but it does not have to, as shown in following three examples for marginal Lévy measures.

Proposition 3.4.6
Suppose that (X^1, X^2) is a spectrally positive Lévy process with the non-homogeneous Lévy copula $\widehat{C}_{1,\zeta}, \zeta > 2$, given in (1.3.14). Further, suppose that the marginal Lévy measures are exponential distributions, i. e. $\overline{\Pi}_1(x) = \overline{\Pi}_2(x) = e^{-ax}$ for some $a > 0$ and $x > 0$. Then for $z > 0$ we get

$$\overline{\Pi}_{P^1}(z) = \overline{\Pi}_{P^2}(z) = e^{-az}\frac{e^{-az}+\zeta}{e^{-az}+1+\zeta} \sim \frac{\zeta}{1+\zeta}e^{-az}, \quad z \to \infty,$$

and

$$\begin{aligned}
&\overline{\Pi}_{P^3}(z)\\
&= \frac{1}{1+e^{az}(1+\zeta)} + \frac{e^{-az}\zeta(1-e^{-az})}{(4e^{-az}-\zeta^2)(1+\zeta+e^{-az})}\\
&\quad + \frac{e^{-az}(2e^{-az}-\zeta^2)}{(4e^{-az}-\zeta^2)\sqrt{\zeta^2-4e^{-az}}}\ln\left(\frac{(2+\zeta-\sqrt{\zeta^2-4e^{-az}})(2e^{-az}+\zeta+\sqrt{\zeta^2-4e^{-az}})}{(2+\zeta+\sqrt{\zeta^2-4e^{-az}})(2e^{-az}+\zeta-\sqrt{\zeta^2-4e^{-az}})}\right)\\
&\sim \frac{az}{\zeta}e^{-az} \quad \text{as } z \to \infty.
\end{aligned}$$

Consequently, for exponential marginals and $\zeta > 2$ the common jumps dominate the tail integral $\overline{\Pi}_+(z)$ asymptotically for large z as in the situation for the Clayton Lévy copula with exponential marginals and parameter $\theta = 1$.

As heavy-tailed example we consider standard Pareto margins.

Proposition 3.4.7
Suppose that (X^1, X^2) is a spectrally positive Lévy process with the non-homogeneous Lévy copula $\widehat{C}_{1,\zeta}, \zeta > 0$, given in (1.3.14). Further, suppose that the marginal Lévy measures are standard Pareto distributions, i. e. $\overline{\Pi}_1(x) = \overline{\Pi}_2(x) = x^{-1}$ for $x \geq 1$. Then we get for $z > 1$

$$\overline{\Pi}_{P^1}(z) = \overline{\Pi}_{P^2}(z) = \frac{\zeta+z^{-1}}{1+z(1+\zeta)} \sim \frac{\zeta}{1+\zeta}z^{-1} \quad \text{as } z \to \infty,$$

and for $z > 2$

$$\begin{aligned}\overline{\Pi}_{P^3}(z) &= \frac{2z^2\zeta + 6z - 2z\zeta - 4}{(4+z\zeta)(-\zeta+z\zeta+z)z} \\ &\quad + \frac{2(2+z\zeta)}{(4+z\zeta)z\sqrt{z\zeta(4+z\zeta)}} \ln\left(\left|\frac{z\zeta - 2\zeta + \sqrt{z\zeta(4+z\zeta)}}{z\zeta - 2\zeta - \sqrt{z\zeta(4+z\zeta)}}\right|\right) \\ &\sim \frac{2}{1+\zeta} z^{-1} \quad \text{as } z \to \infty.\end{aligned}$$

Contrary to the light-tailed example in Proposition 3.4.6, the common jumps do not dominate the tail integral $\overline{\Pi}_+(z)$ for large z. This result can be reasoned by the corresponding PLM. In Example 1.3.8 we saw that for increasing values the mass of the non-homogeneous PLM $\Gamma_{1,\zeta}$ decreases inside the quadrant stronger than near the axes. By relation (1.1.14) we obtain for a Lévy measure Π with PLM $\Gamma_{1,\zeta}$ and heavy-tailed margins that the mass of Π on the axes may decrease more slowly than inside the quadrant. Thus, the single jumps of Π can also be heavy-tailed.

Finally, we consider the tail behaviour of Π_+ for infinite marginal Lévy measures.

Proposition 3.4.8
Suppose that (X^1, X^2) *is a spectrally positive Lévy process with the non-homogeneous Lévy copula* $\widehat{C}_{1,\zeta}, \zeta > 2$, *given in (1.3.7). Further, suppose that the marginal Lévy measures are standard 1-homogeneous, i. e.* $\overline{\Pi}_1(x) = \overline{\Pi}_2(x) = x^{-1}$ *for* $x > 0$. *Then we have* $P^1 = P^2 = 0$ *a. s. and for* $z > 0$

$$\begin{aligned}\overline{\Pi}_{P^3}(z) &= \frac{6+2z\zeta}{z(4+z\zeta)} \\ &\quad + \frac{4+2z\zeta}{z(4+z\zeta)\sqrt{z\zeta(4+z\zeta)}} \ln\left(\left|\frac{z\zeta + \sqrt{z\zeta(4+z\zeta)}}{z\zeta - \sqrt{z\zeta(4+z\zeta)}}\right|\right) \\ &\sim 2z^{-1} \quad \text{as } z \to \infty.\end{aligned}$$

Let (X^1, X^2) be a Lévy process, whose Lévy measure is given by the non-homogeneous PLM $\Gamma_{1,\zeta}$. By Example 2.2.4, (X^1, X^2) can not be regularly varying. Although, Proposition 3.4.8 shows that $X^1 + X^2$ is regularly varying. Note that this does not contradict [11], Theorem 1.1, where it was proven that a r. v. $\boldsymbol{X}$ is regularly varying if and only if *every* linear combination $(\boldsymbol{t}, \boldsymbol{X}), \boldsymbol{t} \in \mathbb{R}^d$, is regularly varying.

3.5 Applications in insurance risk theory

In this section we apply our results to insurance risk theory where we refer to the monographs [7, 25].

In the classical Cramér-Lundberg insurance risk model the claims arriving within the interval $(0, t], t > 0$, are modelled as a spectrally positive CPP and with a premium rate $c > 0$ the risk process is given as

$$X_t = S_t - ct, \quad t \geq 0,$$

describing the net balance of the insurance company. Starting with an initial capital $x \geq 0$ ruin of the company occurs if the first hitting time τ_x^+ given in (3.0.2) is finite. Supposing throughout this section that the net profit condition

$$\lim_{t \to \infty} X_t = -\infty \quad \text{a. s.} \tag{3.5.1}$$

holds, the probability of first upwards passage over the barrier x decreases to 0 as $x \to \infty$ which can be considered as risk measure. For the Cramér case we refer to [25], Section 1.2. and for the non-Cramér case to [25], Section 1.4. Further, the way ruin happens is of interest and was investigated in [5, 8].

In [18] this model was extended to a d-dimensional risk portfolio of an insurance company where the risk processes X^i of the business lines may be dependent. For $d = 2$ and dependence modelled by a Clayton Lévy copula they investigate the company's total risk process $X = X^1 + X^2$ and prove the asymptotic behaviour of the ruin probability for Pareto and exponentially distributed jump sizes of X^i. More generally, our results from Section 3.3.2 and 3.4 give a very precise description of the ruin event for every barrier $x \geq 0$ when the jump process S is the sum of subordinators and the dependence structure is modelled by one the four Lévy copula examples.

Further, the one-dimensional risk model has been generalized in [46, 47] by investigating the ruin event for general spectrally positive Lévy processes when $x \to \infty$. Invoking our quintuple law 3.2.4 we obtain asymptotic results on the ruin event with regarding to dependence when $X = X^1 + X^2$ is the sum of general spectrally positive Lévy processes that may contain a Gaussian part.

Since X is spectrally positive we can choose as in Section 3.3.2 the descending ladder process $(\widehat{L}^{-1}, \widehat{H})$ such that under the normalization condition (3.2.12) we obtain Relation (3.3.11), i. e.

$$\overline{\Pi}_{\mathcal{H}}(u) = \int_u^\infty \overline{\Pi}_+(z)\,\mathrm{d}z, \quad u > 0. \tag{3.5.2}$$

This implies that the integral in (3.5.2) and so $\mathbb{E}[X_1]$ is finite.

We will first answer the question which business line is most likely to cause ruin. Recall the definition of the P^k in (3.2.6) and the representation of their tail integrals in Corollary 3.4.2.

For this result we require $\Pi_{\mathcal{H}}$ to be subexponential. We say that $\Pi_{\mathcal{H}}$ is subexponential if $\Pi_{\mathcal{H}}$ is tail-equivalent to an infinitely divisible distribution $F \in \mathcal{S}$ on $[0,\infty)$, i. e. $\overline{F}(x) > 0$ for every $x \geq 0$ and

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{F(x)} = 2.$$

Then we write $\Pi_{\mathcal{H}} \in \mathcal{S}$. If Π_+ is a finite measure with $\int_0^\infty \overline{\Pi}_+(x)\,\mathrm{d}x < \infty$ and infinite support, this is implied by the d. f. of the increments of $S = S^1 + S^2$ belonging to the class $\mathcal{S}^*$ as introduced in [44].

We recall that subexponential distributions or d. f. s in $\mathcal{S}^*$ can belong to the maximum domain of attraction of the Fréchet distribution, $\mathrm{MDA}(\Phi_\alpha)$ for some $\alpha > 0$, or of the Gumbel distribution, $\mathrm{MDA}(\Lambda)$. The first class covers the regular variation case, the second class contains subexponentials with lighter tails like lognormal or heavy-tailed Weibull distributions. For details see [25], Chapter 3.

In the multivariate regularly varying setting, considered in Chapter 2, we know from Corollary 2.1.2 that $S = S^1 + S^2$ is regularly varying, provided that at least one of the marginals is regularly varying in combination with a Lévy copula whose corresponding PLM is regularly varying with index 1.

We say that Π_+ is in the maximum domain of attraction of the Gumbel distribution if Π_+ is tail-equivalent to a infinitely divisible distribution $F \in \mathrm{MDA}(\Lambda)$ and write $\Pi_+ \in \mathrm{MDA}(\Lambda)$. Recall that a distribution is in the maximum domain of attraction of the Gumbel distribution if and only is there is a function a satisfying $a'(x) \to 0$ such that

$$\lim_{x\to\infty} \frac{\overline{F}(x + a(x)u)}{\overline{F}(x)} = e^{-u} \quad \text{for all } u > 0.$$

Theorem 3.5.1
Suppose that (X^1, X^2) is a spectrally positive Lévy process such that $X = X^1 + X^2$ satisfies the net profit condition (3.5.1). Assume for the Lévy measure Π_+ of X that either
(i) $\Pi_+ \in \mathrm{RV}(\alpha, c_n, \mu_+)$ for $\alpha > 1$ or
(ii) $\Pi_+ \in MDA(\Lambda) \cap \mathcal{S}$ and $\Pi_{\mathcal{H}} \in \mathcal{S}$.
Then the ruin probability is subexponential, i. e. $\mathbb{P}(\tau_\cdot^+ < \infty) \in \mathcal{S}$.
In case (i), $\mathbb{P}(\tau_\cdot^+ < \infty)$ is regularly varying with index $\alpha - 1$.
Let $a(x) \sim \int_x^\infty \overline{\Pi}_+(z)\mathrm{d}z/\overline{\Pi}_+(x)$ as $x \to \infty$ and suppose that the Lévy copula satisfies (3.4.8). Then for $k = 1, 2$ and $u, v > 0$ we have

$$\begin{aligned}\lim_{x\to\infty} \mathbb{P}\left(\frac{X_{\tau_x^+} - x}{a(x)} > u, \frac{-X_{\tau_x^+ -}}{a(x)} > v, \Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+} \,\middle|\, \tau_x^+ < \infty\right) &= 0 \quad (3.5.3)\\ \lim_{x\to\infty} \mathbb{P}(\Delta X_{\tau_x^+} = \Delta P^k_{\tau_x^+} \,|\, \tau_x^+ < \infty) &= 0 \quad (3.5.4)\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to\infty} \mathbb{P}\left(\frac{X_{\tau_x^+} - x}{a(x)} > u, \frac{-X_{\tau_x^+ -}}{a(x)} > v, \Delta X_{\tau_x^+} = \Delta P^3_{\tau_x^+} \,\middle|\, \tau_x^+ < \infty\right) &= \mathrm{GPD}(u+v)\\ \lim_{x\to\infty} \mathbb{P}(\Delta X_{\tau_x^+} = \Delta P^3_{\tau_x^+} \,|\, \tau_x^+ < \infty) &= 1. \quad (3.5.5)\end{aligned}$$

In case (i), $\mathrm{GPD}(u+v) = \left(1 + \frac{u+v}{\alpha}\right)^{-\alpha}$ and $a(x) \sim x/\alpha$; in case (ii), $\mathrm{GPD}(u+v) = e^{-(u+v)}$.

Proof.
From [47], Lemma 3.5, we have for $\Pi_{\mathcal{H}} \in \mathcal{S}$ the relation

$$\lim_{x\to\infty} \frac{\mathbb{P}(\tau_x^+ < \infty)}{\overline{\Pi}_{\mathcal{H}}(x)} = \mathcal{U}([0,\infty)) = \frac{1}{|\mathbb{E}[X_1]|}. \quad (3.5.6)$$

In case (i) the assumption $\Pi_+ \in \mathrm{RV}(\alpha, c_n, \mu_+)$ and applying Karamata's Theorem (cf. [16], Theorem 1.5.11(ii)) to (3.5.2) yields that $\Pi_{\mathcal{H}}$ is regularly varying with index $\alpha - 1$ and so $\Pi_{\mathcal{H}} \in \mathcal{S}$. So the first assertion results.

From Theorem 3.2.4 it follows for $u^*, v^* > 0$

$$\begin{aligned}&\mathbb{P}\left(X_{\tau_x^+} - x > u^*, x - X_{\tau_x^+ -} > v^*, \Delta X_{\tau_x^+} = \Delta P^1_{\tau_x^+}\right)\\ &= \int_{y\in[0,x]} \int_{v\in[v^*\vee y,\infty)} \int_{u\in[u^*,\infty)} \Pi_{P^1}(\mathrm{d}u + v)\,\mathrm{d}v\,\mathcal{U}(x - \mathrm{d}y)\\ &= \int_{y\in[0,x]} \int_{v\in[v^*\vee y,\infty)} \overline{\Pi}_{P^1}(u^* + v)\,\mathrm{d}v\,\mathcal{U}(x - \mathrm{d}y)\\ &= \int_{y\in[0,x]} \overline{\mu}_1(u^* + (v^* \vee y))\,\mathcal{U}(x - \mathrm{d}y),\end{aligned}$$

where $\overline{\mu}_1(z) := \int_z^\infty \overline{\Pi}_{P^1}(s)\,\mathrm{d}s$. For $u, v > 0$, defining $u^* := a(x)u, v^* := x + a(x)v$ we have,

$$\mathbb{P}\left(\frac{X_{\tau_x^+} - x}{a(x)} > u, \frac{-X_{\tau_x^+ -}}{a(x)} > v, \Delta X_{\tau_x^+} = \Delta P^1_{\tau_x^+}\right) = \mathcal{U}([0,x))\,\overline{\mu}_1(x + a(x)(u+v)).$$

With (3.5.6) we obtain

$$\begin{aligned}
&\lim_{x\to\infty} \mathbb{P}\left(\frac{X_{\tau_x^+} - x}{a(x)} > u, \frac{-X_{\tau_x^+ -}}{a(x)} > v, \Delta X_{\tau_x^+} = \Delta P^1_{\tau_x^+} \,\middle|\, \tau_x^+ < \infty\right) \\
&= \lim_{x\to\infty} \frac{\mathcal{U}([0,x))\,\overline{\mu}_1\,(x + a(x)(u+v))}{\mathcal{U}([0,\infty))\,\overline{\Pi}_{\mathcal{H}}(x)} \leq \lim_{x\to\infty} \frac{\mathcal{U}([0,x))\,\overline{\mu}_1\,(x)}{\mathcal{U}([0,\infty))\,\overline{\Pi}_{\mathcal{H}}(x)}. \qquad (3.5.7)
\end{aligned}$$

Now recall that

$$\frac{\overline{\mu}_1\,(x)}{\overline{\Pi}_{\mathcal{H}}(x)} = \frac{\int_x^\infty \overline{\Pi}_{P^1}(s)\,ds}{\int_x^\infty \overline{\Pi}_+(s)\,ds}. \qquad (3.5.8)$$

Since with (3.4.8) we receive that

$$\frac{\overline{\Pi}_{P^1}(x)}{\overline{\Pi}_+(x)} \leq \frac{\overline{\Pi}_{P^1}(x)}{\overline{\Pi}_1(x)} \to 0 \quad \text{as } x \to \infty,$$

the r. h. s of (3.5.8) tends to 0 as $x \to \infty$ by l'Hospital's Lemma and, hence, the right hand bound of (3.5.7) is 0. The analogous result holds for P^2. This implies the assertions (3.5.3) and (3.5.4). By [46], Theorem 1 and 2, it holds for $u, v > 0$

$$\lim_{x\to\infty} \mathbb{P}\left(\frac{X_{\tau_x^+} - x}{a(x)} > u, \frac{-X_{\tau_x^+ -}}{a(x)} > v \,\middle|\, \tau_x^+ < \infty\right) = \mathrm{GPD}(u+v)$$

and the last two equations result. □

Remark 3.5.2

(i) Theorem 3.5.1 generalizes the CPP situation in [18], Corollary 3.6, where the ruin probability was calculated for Pareto distributed jump sizes and a Clayton Lévy copula.

(ii) By [47], Remark 4.3(iii), ruin can asymptotically occur for subexponential $\Pi_{\mathcal{H}}$ only by a jump. In the situation of Theorem 3.5.1, Relation (3.5.5) means that ruin occurs asymptotically only by a common jump, i. e. a claim that applies to both business lines at the same time.

Finally, we investigate the ruin event for the barrier $x = 0$.

Corollary 3.5.3

Suppose that (X^1, X^2) is a spectrally positive Lévy process such that 0 is irregular for $(0,\infty)$ with respect to $X = X^1 + X^2$.

(1) If the dependence is modelled by a Clayton Lévy copula $\widehat{C}_{1,\theta}$ defined in (1.3.7) then

$$\lim_{\theta\to\infty} \mathbb{P}\left(\Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+} \mid \tau_0^+ < \infty\right) = \begin{cases} 0, & \text{for } k = 1,2, \\ 1, & \text{for } k = 3. \end{cases}$$

(2) If the dependence is modelled by the non-homogeneous Archimedean Lévy copula $\widehat{C}_{1,\zeta}$ defined in (1.3.14) then

$$\lim_{\zeta\to\infty} \mathbb{P}\left(\Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+} \mid \tau_0^+ < \infty\right) = \begin{cases} \dfrac{\int_0^\infty \overline{\Pi}_k(z)\,\mathrm{d}z}{\mu_S}, & \text{for } k = 1,2, \\ 0, & \text{for } k = 3. \end{cases}$$

Proof.

If 0 is irregular for $(0,\infty)$, we get with Corollary 3.2.5

$$\mathbb{P}\left(\Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+}\right) = \int_0^\infty \overline{\Pi}_{P^k}(z)\,\mathrm{d}z\,\mathcal{U}(\{0\})$$

and $\mathbb{P}(\tau_0^+ < \infty) = \mu_S \mathcal{U}(\{0\})$ where S denotes the jump part of X and $\mu_S = \mathbb{E}[S_1]$. Note that $\mathcal{U}(\{0\}) > 0$, if 0 is irregular for $(0,\infty)$. So

$$\mathbb{P}\left(\Delta X_{\tau_0^+} = \Delta P^k_{\tau_0^+} \middle| \tau_0^+ < \infty\right) = \frac{\mu_{P^k}}{\mu_S}$$

where $\mu_{P^k} = \int_0^\infty \overline{\Pi}_{P^k}(z)\,\mathrm{d}z$. From Section 3.4.2 we know that increasing the dependence parameter θ of the Clayton Lévy copula lowers the tail integral of the single jump components, i. e. $\lim_{\theta\to\infty} \overline{\Pi}_{P^k}(z) = 0$ for $k = 1,2$. Furthermore, increasing the dependence parameter ζ of the non-homogeneous Lévy copula lowers the tail integral of the common jump, i. e. $\lim_{\zeta\to\infty} \overline{\Pi}_{P^3}(z) = 0$ and $\lim_{\zeta\to\infty} \overline{\Pi}_{P^k}(z) = \overline{\Pi}_k$ for $k = 1,2$. □

Chapter 4

Appendix

4.1 Basic definitions and results of regular variation

In this Appendix we summarize some definitions and concepts of regular variation used in this thesis.

Theorem 4.1.1 (Portmanteau theorem, [14], p. 11)
Let $\mathbb{P}, (\mathbb{P}_n)_{n\in\mathbb{N}}$ be probability measures. The following are equivalent:

(1) $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$ *as* $n \to \infty$.

(2) $\lim_{n\to\infty} \mathbb{P}(B) = \mathbb{P}(B)$ *for all sets* B *with* $\mathbb{P}(\partial B) = 0$.

For a Radon measure μ on $\mathbb{E}$, i. e. $\mu(K) < \infty$ for all compact sets $K \in \mathcal{B}(\mathbb{E})$, one considers vague convergence.

Theorem 4.1.2 ([35], Theorem 2.4)
Let $\mu, (\mu_n)_{n\in\mathbb{N}}$ non-negative Radon measure on $\mathbb{E}$. The following are equivalent:

(1) $\mu_n \xrightarrow{v} \mu$ *as* $n \to \infty$.

(2) $\mu_n(B) \to \mu(B)$ *for all relatively compact sets* B, *i. e.* $\mathbf{0} \notin \overline{B}$, *and* $\mu(\partial B) = 0$.

Definition 4.1.3 (Multivariate regular variation for r. v. s)
A d-dimensional r. v. $\boldsymbol{X} = (X^1, \dots, X^d)$ in $\mathbb{R}^d$ and its distribution are regularly varying *if one of the following equivalent definitions (1) or (2) holds.*

(1) There exists a sequence $(c_n)_{n\in\mathbb{N}}$ *of positive numbers with* $c_n \uparrow \infty$ *as* $n \to \infty$ *and a non-zero Radon measure* μ *on* $\mathcal{B}(\mathbb{E})$ *with* $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ *such that*

$$n\mathbb{P}\left(c_n^{-1}\boldsymbol{X} \in \cdot\right) \xrightarrow{v} \mu(\cdot),$$

where $\xrightarrow{v}$ *denotes vague convergence on* $\mathcal{B}(\mathbb{E})$. *Then the limit measure* μ *is necessarily homogeneous of degree* $\alpha > 0$.

(2) There exists a r. v. $\boldsymbol{\Theta}$ *with values in* $\mathbb{S}$ *such that for all* $t > 0$

$$\frac{\mathbb{P}(|\boldsymbol{X}| > tu, \boldsymbol{X}/|\boldsymbol{X}| \in \cdot)}{\mathbb{P}(|\boldsymbol{X}| > u)} \xrightarrow{w} t^{-\alpha}\,\mathbb{P}(\boldsymbol{\Theta} \in \cdot) \quad \text{as } u \to \infty,$$

where $\xrightarrow{v}$ *denotes vague convergence on* $\mathcal{B}(\mathbb{S})$. *The distribution of* $\boldsymbol{\Theta}$ *is called the* spectral measure *of* $\boldsymbol{X}$.

We did not specify the norm $|\cdot|$, since the regular variation does not depend on the choice of the norm, see [32], Lemma 2.1.

The relation between regular variation of Lévy process and regular variation of their Lévy measures is given by the following result.

Theorem 4.1.4 ([36], Lemma 2.1)
Let $\boldsymbol{X}$ *be a Lévy process with Lévy measure* Π. *The following statements are equivalent:*

(1) $\boldsymbol{X}_1 \in \mathrm{RV}(\alpha, c_n, \mu)$.

(2) $\Pi \in \mathrm{RV}(\alpha, c_n, \mu)$.

(3) $\boldsymbol{X} \in \mathrm{RV}(\alpha, c_n, m)$ *with* $m_t = t\mu$ *for every* $t \in [0, 1]$.

4.2 Auxiliary results and technical proofs

In this Appendix we give auxiliary results and proofs used in this thesis.

4.2.1 Chapter 1

In this Section we give the technical proof of Proposition 1.1.11.

Proof of Proposition 1.1.11

(1) Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$. Assume that $a_i b_i \leq 0$ for at most $k \in \{0, \dots, d\}$ indices. We prove (1.1.12) by induction on $k = 0, \dots, d$.

Let $k = 0$, i. e. $a_i b_i > 0$ for all $i = 1, \dots, d$. Then with Definition 1.1.4 and (1.1.8) we obtain

$$\begin{aligned}\Pi\left(\prod_{i=1}^{d}(a_i, b_i]\right) &= V_{(-1)^d\overline{\Pi}}\left(\prod_{i=1}^{d}(a_i, b_i]\right) = V_{(-1)^d\overline{\Gamma}}\left(\prod_{i=1}^{d}\left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right]\right) \\ &= \Gamma\left(\prod_{i=1}^{d}\left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right]\right).\end{aligned}$$

Suppose (1.1.12) holds for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $a_i b_i \leq 0$ for at most k indices. W. l. o. g. we assume that $a_i b_i \leq 0$ for $i = 1, \dots, k+1$. If $a_{k+1} = 0$, then the induction hypothesis results in

$$\begin{aligned}&\Pi\left(\prod_{i=1}^{d}(a_i, b_i]\right) \\ &= \lim_{\epsilon\downarrow 0}\Pi\left(\prod_{i<k+1}(a_i, b_i] \times (\epsilon, b_{k+1}] \times \prod_{i>k+1}(a_i, b_i]\right) \\ &= \lim_{\epsilon\downarrow 0}\Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right] \times \left(\frac{1}{\overline{\Pi}_i(\epsilon)}, \frac{1}{\overline{\Pi}_i(b_i)}\right] \times \prod_{i>k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right]\right) \\ &= \Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right] \times \left(\frac{1}{\overline{\Pi}_i(0+)}, \frac{1}{\overline{\Pi}_i(b_i)}\right] \times \prod_{i>k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)}, \frac{1}{\overline{\Pi}_i(b_i)}\right]\right).\end{aligned}$$

If $a_{k+1} \neq 0$, i. e. $a_{k+1} < 0$ and $b_{k+1} \geq 0$, then with induction hypothesis we get

$$\begin{aligned}&\Pi\left(\prod_{i=1}^{d}(a_i, b_i]\right) \\ &= \Pi_{\{1,\dots,d\}\setminus\{k+1\}}\left(\prod_{i\neq k+1}(a_i, b_i]\right) - \lim_{\beta\downarrow b_{k+1}}\Pi\left(\prod_{i<k+1}(a_i, b_i] \times (\beta, \infty) \times \prod_{i>k+1}(a_i, b_i]\right) \\ &\quad - \Pi\left(\prod_{i<k+1}(a_i, b_i] \times (-\infty, a_k] \times \prod_{i\in I, i>k+1}(a_i, b_i]\right)\end{aligned}$$

$$
\begin{aligned}
&= \Gamma_{\{1,\dots,d\}\setminus\{k+1\}}\left(\prod_{i\neq k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right)\\
&\quad - \lim_{\beta\downarrow b_{k+1}}\Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\left(\frac{1}{\overline{\Pi}_{k+1}(\beta)},\infty\right)\times\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right)\\
&\quad - \Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\left(-\infty,\frac{1}{\overline{\Pi}_{k+1}(a_{k+1})}\right]\times\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right)\\
&= \Gamma\left(\prod_{i<k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\times\left(\frac{1}{\overline{\Pi}_{k+1}(a_{k+1})},\frac{1}{\overline{\Pi}_{k+1}(b_{k+1}+)}\right]\right.\\
&\qquad\qquad\qquad\left.\times\prod_{i>k+1}\left(\frac{1}{\overline{\Pi}_i(a_i)},\frac{1}{\overline{\Pi}_i(b_i)}\right]\right).
\end{aligned}
$$

Recall that for $b_{k+1} > 0$ we have by right-continuity of the tail integral $\overline{\Pi}_{k+1}(b_{k+1}+) = \overline{\Pi}_{k+1}(b_{k+1})$. If $b_{k+1} = 0$, then $\overline{\Pi}_{k+1}(b_{k+1}+) = \overline{\Pi}_{k+1}(0+)$.
(2) We prove (1.1.14) analogously by induction on $|K| = 1,\dots,d-1$. For $|K| = 1$ we assume w. l. o. g. that $K = \{1\}$. Sklar's Theorem 1.1.10 implies

$$
\begin{aligned}
&\Pi\left(\{0\}\times\prod_{i=2}^{d}\mathcal{I}(x_i)\right)\\
&= \Pi_{\{2,\dots,d\}}\left(\prod_{i=2}^{d}\mathcal{I}(x_i)\right) - \lim_{\epsilon\downarrow 0}\Pi\left(\mathcal{I}(\epsilon)\times\prod_{i=2}^{d}\mathcal{I}(x_i)\right) - \lim_{\epsilon\uparrow 0}\Pi\left(\mathcal{I}(\epsilon)\times\prod_{i=2}^{d}\mathcal{I}(x_i)\right)\\
&= \Gamma_{\{2,\dots,d\}}\left(\prod_{i=2}^{d}\mathcal{I}\left(\frac{1}{\overline{\Pi}_i(x_i)}\right)\right) - \Gamma\left(\mathcal{I}\left(\frac{1}{\overline{\Pi}_1(0+)}\right)\times\prod_{i=2}^{d}\mathcal{I}\left(\frac{1}{\overline{\Pi}_i(x_i)}\right)\right)\\
&\quad - \Gamma\left(\left(-\infty,\frac{1}{\overline{\Pi}_1(0-)}\right)\times\prod_{i=2}^{d}\mathcal{I}\left(\frac{1}{\overline{\Pi}_i(x_i)}\right)\right)\\
&= \Gamma\left(\left[\frac{1}{\overline{\Pi}_1(0-)},\frac{1}{\overline{\Pi}_1(0+)}\right]\times\prod_{i=2}^{d}\mathcal{I}\left(\frac{1}{\overline{\Pi}_i(x_i)}\right)\right).
\end{aligned}
$$

With induction on $|K|$ Equations (1.1.13) and (1.1.14) result. □

4.2.2 Chapter 2

In this section we prove Lemma 2.0.2 and give auxiliary results for the proofs of Theorem 2.1.1 and 2.1.5.

Proof of Lemma 2.0.2
Suppose $i \in \{1, \dots, d\}$ and $A \in \mathcal{B}(\overline{\mathbb{R}} \setminus \{0\})$ with $0 \notin \overline{A}$ and $\mu_i(\partial A) = \mu(\{\boldsymbol{x} \in \mathbb{E} : x_i \in \partial A\}) = 0$. Note that $\mathbf{0} \notin \overline{\{\boldsymbol{x} \in \mathbb{R}^d : x_i \in A\}}$ since $0 \notin \overline{A}$, and that

$$\mu(\partial\{\boldsymbol{x} \in \mathbb{E} : x_i \in A\}) \;=\; \mu(\{\boldsymbol{x} \in \mathbb{E} : x_i \in \partial A\}) = 0\,.$$

Consequently,

$$\begin{aligned} n\Pi_i(c_n A) =& n\Pi(c_n\{\boldsymbol{x} \in \mathbb{E} : x_i \in A\}) \\ &\to \mu(\{\boldsymbol{x} \in \mathbb{E} : x_i \in A\}) = \mu_i(A) < \infty. \end{aligned}$$

Setting $A = (x, \infty)$ and $A = (-\infty, -x]$ for $x > 0$, the homogeneity of μ yields (2.0.4). Since μ is a non-zero measure, there is at least one index i_* such that μ_{i_*} is a non-zero measure, i. e. $\overline{\mu}_{i_*}(1) - \overline{\mu}_{i_*}(-1) > 0$ and $\Pi_{i_*} \in \mathrm{RV}(\alpha, c_n, \mu_{i_*})$. □

The following propositions are auxiliary results for the proofs of Theorem 2.1.1 and 2.1.5. They parallel the equivalence of weak convergence and convergence with respect to the Lévy distance, see [65], on the level of Lévy measures and vague convergence. We prove this results to keep this thesis self-contained.

Proposition 4.2.1

Let M and $(M_n)_{n\in\mathbb{N}}$ be measures on $\mathcal{B}(\mathbb{E})$ and $\boldsymbol{a}, \boldsymbol{b}, (\boldsymbol{a}_n)_{n\in\mathbb{N}}, (\boldsymbol{b}_n)_{n\in\mathbb{N}} \in \mathbb{E}$ with $\mathbf{0} \notin \overline{(\boldsymbol{a}, \boldsymbol{b}]}$ and $\mathbf{0} \notin \overline{(\boldsymbol{a}_n, \boldsymbol{b}_n]}$ for all n. Suppose

(1) $\boldsymbol{a}_n \to \boldsymbol{a}$ and $\boldsymbol{b}_n \to \boldsymbol{b}$ *as* $n \to \infty$,

(2) $M(\partial(\boldsymbol{a}, \boldsymbol{b}]) = 0$,

(3) $\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{E},\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]},M(\partial(\boldsymbol{a},\boldsymbol{b}])=0} |M_n((\boldsymbol{a}, \boldsymbol{b}]) - M((\boldsymbol{a}, \boldsymbol{b}])| \to 0$ *as* $n \to \infty\,.$

Then $M_n((\boldsymbol{a}_n, \boldsymbol{b}_n]) \to M((\boldsymbol{a}, \boldsymbol{b}])$ *as* $n \to \infty$.

Proof.
For $n \to \infty$ we have

$$\begin{aligned} &|M_n((\boldsymbol{a}_n, \boldsymbol{b}_n]) - M((\boldsymbol{a}, \boldsymbol{b}])| \\ \le\; &|M_n((\boldsymbol{a}_n, \boldsymbol{b}_n]) - M((\boldsymbol{a}_n, \boldsymbol{b}_n])| + |M((\boldsymbol{a}_n, \boldsymbol{b}_n]) - M((\boldsymbol{a}, \boldsymbol{b}])| \\ \le\; &\underbrace{\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{E},\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]},M(\partial(\boldsymbol{a},\boldsymbol{b}])=0} |M_n((\boldsymbol{a}, \boldsymbol{b}]) - M((\boldsymbol{a}, \boldsymbol{b}])|}_{(3)\to 0} + \underbrace{|M((\boldsymbol{a}_n, \boldsymbol{b}_n]) - M((\boldsymbol{a}, \boldsymbol{b}])|}_{(1)+(2)\to 0} \to 0. \end{aligned}$$

□

Proposition 4.2.2

Suppose the situation of Theorem 2.1.1. Then the following holds

$$\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{E},\,\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]},\,\nu(\partial(\boldsymbol{a},\boldsymbol{b}])=0} |n\Gamma(n(\boldsymbol{a},\boldsymbol{b}]) - \nu((\boldsymbol{a},\boldsymbol{b}])| \to 0 \quad \text{as } n\to\infty.$$

Proof.
For $\boldsymbol{a},\boldsymbol{b}\in\mathbb{E}$ with $\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]}$ and $\nu(\partial(\boldsymbol{a},\boldsymbol{b}])=0$ define $g_n(\boldsymbol{a},\boldsymbol{b}) := |n\Gamma(n(\boldsymbol{a},\boldsymbol{b}]) - \nu((\boldsymbol{a},\boldsymbol{b}])|$. g_n is continuous on $\mathbb{E}^2$, since Γ and ν have no atoms. So for $\epsilon>0$ the sequence $S_n := \{\boldsymbol{x},\boldsymbol{y}\in\mathbb{E} : g_n(\boldsymbol{x},\boldsymbol{y})<\epsilon\}$ are open sets. g_n is decreasing for $n\to\infty$ and converges pointwise to 0. Therefore, S_n is ascending and $(S_n)_{n\in\mathbb{N}}$ is an open cover of $\mathbb{E}^2$. Since $\mathbb{E}^2$ is compact in the here used topology, see one-point uncompactification [56], p.171, there exists an $N\in\mathbb{N}$ such that $S_N=\mathbb{E}^2$. So for every $n>N$ and every $(\boldsymbol{x},\boldsymbol{y})\in\mathbb{E}^2$ we get $|n\Gamma(n(\boldsymbol{x},\boldsymbol{y}]) - \nu((\boldsymbol{x},\boldsymbol{y}])| = g_n(\boldsymbol{x},\boldsymbol{y})<\epsilon$, where N does not depend on $(\boldsymbol{x},\boldsymbol{y})$. □

Proposition 4.2.3

Suppose the situation of Theorem 2.1.5. Then the following holds

$$\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathbb{E},\,\mathbf{0}\notin\overline{(\boldsymbol{a},\boldsymbol{b}]},\,\mu(\partial(\boldsymbol{a},\boldsymbol{b}])=0} |n\Pi(c_n(\boldsymbol{a},\boldsymbol{b}]) - \mu((\boldsymbol{a},\boldsymbol{b}])| \to 0 \quad \text{as } n\to\infty.$$

The proof is analogous to the proof of Proposition 4.2.2.

4.2.3 Chapter 3

Proof of Proposition 3.4.6
The first expression holds with Corollary 3.4.2 obviously. It remains to calculate $\overline{\Pi}_{P^3}$:

$$\begin{aligned}\overline{\Pi}_{P^3}(z) =& a\int_0^z \frac{e^{-2a(z-x)}+\zeta e^{-a(z-x)}}{(e^{-ax}+e^{-a(z-x)}+\zeta)^2}e^{-ax}dx \\ &+ a\int_z^\infty \frac{1+\zeta}{(e^{-ax}+1+\zeta)^2}e^{-ax}dx =: I(z)+II(z).\end{aligned}$$

We substitute $y = e^{ax}$ and calculate both integrals for $\zeta > 2$ separately with [19], Tabelle 1.1.3.3.45 and 1.1.2.2.48.

$$II(z) = \int_{e^{az}}^{\infty} \frac{1+\zeta}{(1+y(1+\zeta))^2} dy = \frac{1}{1+e^{az}(1+\zeta)} \sim \frac{1}{1+\zeta} e^{-az}, \quad z \to \infty,$$

and

$$\begin{aligned} I(z) &= \int_1^{e^{az}} \frac{e^{-2az}y^2 + \zeta e^{-az} y}{(1+e^{-az}y^2+\zeta y)^2} dy \\ &= \frac{e^{-az}\zeta(1-e^{-az})}{(4e^{-az}-\zeta^2)(1+\zeta+e^{-az})} \\ &\quad + \frac{e^{-az}(2e^{-az}-\zeta^2)}{(4e^{-az}-\zeta^2)\sqrt{\zeta^2-4e^{-az}}} \ln\left(\frac{(2+\zeta-\sqrt{\zeta^2-4e^{-az}})(2e^{-az}+\zeta+\sqrt{\zeta^2-4e^{-az}})}{(2+\zeta+\sqrt{\zeta^2-4e^{-az}})(2e^{-az}+\zeta-\sqrt{\zeta^2-4e^{-az}})} \right). \end{aligned}$$

With

$$\frac{(2e^{-az}+\zeta+\sqrt{\zeta^2-4e^{-az}})}{(2e^{-az}+\zeta-\sqrt{\zeta^2-4e^{-az}})} \sim e^{az}\frac{\zeta^2}{1+\zeta} \quad \text{as } z \to \infty$$

and l'Hospital's Lemma it holds

$$\ln\left(\frac{(2e^{-az}+\zeta+\sqrt{\zeta^2-4e^{-az}})}{(2e^{-az}+\zeta-\sqrt{\zeta^2-4e^{-az}})} \right) \sim \ln\left(e^{az}\frac{\zeta^2}{1+\zeta} \right) = az + \ln\left(\frac{\zeta^2}{1+\zeta} \right) \quad \text{as } z \to \infty.$$

This yields the asymptotic result for $\overline{\Pi}_{P^3}$. □

Proof of Proposition 3.4.7

As in Proposition 3.4.6 the first expression results easily from Corollary 3.4.2 and it remains again to calculate $\overline{\Pi}_{P^3}(z)$. For $z > 2$ we receive

$$\begin{aligned} \overline{\Pi}_{P^3}(z) &= \int_0^{\infty} \frac{\overline{\Pi}_{S^2}(0 \vee (z-x))^2 + \zeta \overline{\Pi}_{S^2}(0 \vee (z-x))}{\left(\overline{\Pi}_{S^1}(x) + \overline{\Pi}_{S^2}(0 \vee (z-x)) + \zeta\right)^2} \Pi_{S^1}(\mathrm{d}x) \\ &= \int_1^{z-1} \frac{(z-x)^{-2} + \zeta(z-x)^{-1}}{\left(x^{-1} + (z-x)^{-1} + \zeta\right)^2} x^{-2}\,\mathrm{d}x + \int_{z-1}^{\infty} \frac{1+\zeta}{\left(x^{-1}+1+\zeta\right)^2} x^{-2}\,\mathrm{d}x \end{aligned}$$

and with [19], Tabelle 1.1.3.3.41 and 1.1.2.2.45 straight calculations yield

$$\begin{aligned}
&\int_1^{z-1} \frac{(z-x)^{-2}+\zeta(z-x)^{-1}}{\left(x^{-1}+(z-x)^{-1}+\zeta\right)^2} x^{-2}\,\mathrm{d}x \\
&\quad= \int_1^{z-1} \frac{1+\zeta z-\zeta x}{(-\zeta x^2+z\zeta x+z)^2}\,\mathrm{d}x \\
&\quad= \frac{2z-4+z^2\zeta-2z\zeta}{(4+z\zeta)(-\zeta+z\zeta+z)z} \\
&\qquad+ \frac{2(2+z\zeta)}{(4+z\zeta)z\sqrt{z\zeta(4+z\zeta)}} \ln\left(\left|\frac{z\zeta-2\zeta+\sqrt{z\zeta(4+z\zeta)}}{z\zeta-2\zeta-\sqrt{z\zeta(4+z\zeta)}}\right|\right)
\end{aligned}$$

and

$$\int_{z-1}^{\infty} \frac{1+\zeta}{\left(x^{-1}+1+\zeta\right)^2} x^{-2}\,\mathrm{d}x = \frac{1}{-\zeta+z\zeta+z}.$$

Since

$$\begin{aligned}
\left|\frac{z\zeta-2\zeta+\sqrt{z\zeta(4+z\zeta)}}{z\zeta-2\zeta-\sqrt{z\zeta(4+z\zeta)}}\right| &= z\frac{z}{|4z\zeta-4\zeta^2+4z\zeta^2|}\left(\zeta-\frac{2\zeta}{z}+\sqrt{\zeta\left(\frac{4}{z}+\zeta\right)}\right)^2 \\
&\sim z\frac{\zeta}{1+\zeta} \quad \text{as } z\to\infty,
\end{aligned}$$

we get with l'Hospital's Lemma that

$$\ln\left(\left|\frac{z\zeta-2\zeta+\sqrt{z\zeta(4+z\zeta)}}{z\zeta-2\zeta-\sqrt{z\zeta(4+z\zeta)}}\right|\right) \sim \ln\left(z\frac{\zeta}{1+\zeta}\right) \quad \text{as } z\to\infty$$

and the asymptotic of $\overline{\Pi}_{P^3}$ follows. □

Proof of Proposition 3.4.8

Since $\widehat{C}_{1,\zeta}$ is left-continuous in both coordinates in ∞ and $\Pi_1((0,\infty)) = \Pi_2((0,\infty)) = \infty$, with Corollary 3.4.2 it results $\overline{\Pi}_{P^1}(z) = \overline{\Pi}_{P^2}(z) = 0$ for all $z > 0$. Furthermore, for $z > 0$ we achieve

$$\begin{aligned}
\overline{\Pi}_{P^3}(z) &= \int_0^{\infty} \frac{\overline{\Pi}_2(0\vee(z-x))^2+\zeta\overline{\Pi}_2(0\vee(z-x))}{\left(\overline{\Pi}_1(x)+\overline{\Pi}_2(0\vee(z-x))+\zeta\right)^2}\,\Pi_1(\mathrm{d}x) \\
&= \int_0^{z} \frac{(z-x)^{-2}+\zeta(z-x)^{-1}}{\left(x^{-1}+(z-x)^{-1}+\zeta\right)^2 x^2}\,\mathrm{d}x + \int_z^{\infty}\frac{1}{x^2}\,\mathrm{d}x \\
&=(1+\zeta z)\int_0^{z}\frac{1}{(-\zeta x^2+z\zeta x+z)^2}\,\mathrm{d}x - \zeta\int_0^{z}\frac{x}{(-\zeta x^2+z\zeta x+z)^2}\,\mathrm{d}x + z^{-1}
\end{aligned}$$

and with [19], Tabelle 1.1.3.3.41 and 1.1.2.2.45, it results

$$\begin{aligned}\overline{\Pi}_{P^3}(z) =& \frac{6+2z\zeta}{z(4+z\zeta)} \\ &+ \frac{4+2z\zeta}{z(4+z\zeta)\sqrt{z\zeta(4+z\zeta)}} \ln\left(\left|\frac{z\zeta+\sqrt{z\zeta(4+z\zeta)}}{z\zeta+\sqrt{z\zeta(4+z\zeta)}}\right|\right)\end{aligned}$$

The asymptotic behaviour of $\overline{\Pi}_{P^3}$ follows as in Proposition 3.4.7. □

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Index

List of Abbreviations and Symbols

$a \vee b$, $a \wedge b$	$\max\{a, b\}$, $\min\{a, b\}$
a. s.	almost sure(ly)
$\mathcal{B}(\cdot)$	Borel-σ-algebra
$\overline{B}$, B°, ∂B	closure, interior and boundary of the set B
$\lvert \cdot \rvert$, $\#$	cardinality of a set
C_D, $\widehat{C}_D$	(distributional) copula, survival copula
$\widehat{C}$	Lévy copula
$\widehat{C}_I$	I-margin of Lévy copula
$\widehat{C}_\perp$, $\widehat{C}_\Vert$	independence/complete dependence Lévy copula
$\widehat{C}_{\eta,\theta}$, $\widehat{C}_{\eta,\zeta}$	Clayton/non-homogeneous Lévy copula
CPP	compound Poisson process
$\mathcal{D}^i$	$\mathcal{I}(1/\overline{\Pi}_i(0-)) \cup \mathcal{I}(1/\overline{\Pi}_i(0+)) \cup \{0\}$
$\delta_{\boldsymbol{x}}$	Dirac measure with mass on $\boldsymbol{x} \in \mathbb{R}^d$
d. f.	distribution function
$\mathbb{E}$	$[-\infty, \infty]^d \setminus \{\mathbf{0}\}$
$\mathbb{E}[\cdot]$	expectation operator
$\mathrm{expo}(q)$	exponential distribution with parameter q
F, F_i, F_+	d-dimensional d. f., one-dimensional d. f., one-dimensional d. f. of the sum
F^{n*}	n-fold convolution
$\overline{F}$, $\overline{F}_i$, $\overline{F}_+$	right tail of the d. f. F, F_i, F_+
$\overline{G} = (\overline{G}_t)_{t\geq 0}$, $\underline{G} = (\underline{G}_t)_{t\geq 0}$	time of the previous maximum/minimum of X
$(\boldsymbol{\gamma}, A, \Pi)$	characteristic triplet
Γ_D	Pareto measure
$\overline{\Gamma}_D$	Pareto copula = right tail of Pareto measure
Γ	Pareto Lévy measure
$\overline{\Gamma}$	Pareto Lévy copula = tail integral of PLM

$\Gamma_\perp$, $\Gamma_\parallel$	independence/complete dependence PLM
$\Gamma_{\eta,\theta}$, $\Gamma_{\eta,\zeta}$	Clayton/non-homogeneous PLM
$\mathcal{I}(x)$	$(-\infty, x]$ for $x < 0$ and (x, ∞) for $x \geq 0$
i. i. d.	independent and identically distributed
1_B	indicator function of the set B
$\boldsymbol{\infty}$	$(\infty, \dots, \infty)$
$(\cdot, \cdot)$	inner product
J	Poisson random measure of Lévy process $\boldsymbol{X} = (X^1, X^2)$
κ, $\widehat{\kappa}$	Laplace exponent of the ascending/descending ladder process
K	$\{x \in \mathbb{R}^d : \operatorname{sgn}(x_1) = \dots = \operatorname{sgn}(x_d)\}$
$(\widehat{L}^{-1}, \widehat{H}) = (\widehat{L}_t^{-1}, \widehat{H}_t)_{t \geq 0}$	descending ladder process of X
$(L^{-1}, H) = (L_t^{-1}, H_t)_{t \geq 0}$	ascending ladder process of X
λ_i^+, λ_i^-	$\overline{\Pi}_i(0+)$, $\overline{\Pi}_i(0-)$
Λ_U, Λ_L	upper/lower tail integral dependence coefficient
$L = (L_t)_{t \geq 0}$	local time at the maximum of X
l. h. s.	left-hand side
$\mathrm{MDA}(\Phi_\alpha)$, $\mathrm{MDA}(\Lambda)$	maximum domain of attraction of the Fréchet distribution/Gumbel distribution
$\mu_{\mathbb{S}.}$	spectral measure on unit sphere $\mathbb{S}.$ of a 1-homogeneous or regularly varying PLM
$\mu_{\widehat{C}}$	measure defined by a Lévy copula
$\lvert \cdot \rvert.$	$1, 2$-or ∞-norm
$\mathbb{P}$	probability measure
$P^1 = (P_t^1)_{t \geq 0}$, $P^2 = (P_t^2)_{t \geq 0}$	process of the single positive jumps of X^1/X^2
$P^3 = (P_t^3)_{t \geq 0}$	process whose jumps are the sum of common positive jumps of X^1 and X^2
$P^4 = (P_t^4)_{t \geq 0}$	process whose jumps are the sum of the positive jumps of X^1 and the negative jumps of X^2 which happen at the same time
$P^5 = (P_t^5)_{t \geq 0}$	process whose jumps are the sum of the negative jumps of X^2 and the positive jumps of X^2 which happen at the same time
Π, Π_i, Π_+	Lévy measure of Lévy process $\boldsymbol{X} = (X^1, \dots, X^d)/X^i/X := \sum_{i=1}^d X^i$
Π_I	I-margin of Lévy measure Π

Π_{P^i}	Lévy measure of the jump process P^i
$\overline{\Pi}$	tail integral of the Lévy measure Π
PLC	Pareto Lévy copula
PLM	Pareto Lévy measure
Q	inversion map
$\overline{\mathbb{R}}$	$[-\infty, \infty]$
Ran	range
r. v.	random vector/variable
$\mathrm{RV}(\alpha, c_n, \mu)$	regular variation with index α, norming sequence c_n and limit measure μ
r. h. s.	right-hand side
$(S^1, S^2) = (S_t^1, S_t^2)_{t \geq 0}$	jump part of Lévy process (X^1, X^2)
$(S^{1,\epsilon}, S^{2,\epsilon}) = (S_t^{1,\epsilon}, S_t^{2,\epsilon})_{t \geq 0}$	truncated jump part of (X^1, X^2) with jumps $\geq \epsilon$
$\mathbb{S}$	unit sphere in $\mathbb{R}^d$
$\mathcal{S}$	subexponential distributions
$\mathrm{sgn}(x)$	$1_{\{x \geq 0\}} - 1_{\{x<0\}}$
$S = (S_t^1 + S_t^2)_{t \geq 0}$	jump part of $X = X^1 + X^2$
$S^{\cdot}_{\rho_1,\rho_2}$	arc $\{(\cos(\phi), \sin(\phi))/\lvert(\cos(\phi), \sin(\phi))\rvert. : \phi \in [\rho_1, \rho_2]\}$ of the unit circle $\mathbb{S}$.
τ_x^+, τ_x^-	first upwards/downwards passage time of X
$\mathcal{U}, \widehat{\mathcal{U}}$	potential measure of the ascending/descending ladder process
$\xrightarrow{v}$	vague convergence
V_F	F-volume
$(W^1, W^2) = (W_t^1, W_t^2)_{t \geq 0}$	Gaussian part of Lévy process (X^1, X^2)
$\xrightarrow{w}$	weak convergence
w. l. o. g.	without loss of generality
$\boldsymbol{X} = (X_t^1, \dots, X_t^d)_{t \geq 0}$	d-dimensional Lévy process
$X^i = (X_t^i)_{t \geq 0}$	one-dimensional Lévy process
$\overline{X} = (\overline{X}_t)_{t \geq 0}$, $\underline{X} = (\underline{X}_t)_{t \geq 0}$	running suprema/infima of X
$X = (\sum_{i=1}^d X_t^i)_{t \geq 0}$	sum of d Lévy processes
$\mathbf{0}$	$(0, \dots, 0) \in \mathbb{R}^d$