MULTIUSER DIVERSITY AND BEAMFORMING GAIN IN A FAST FADING MU-MISO DOWNLINK WITH MANY USERS

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ABSTRACT

Multiuser diversity is an inherent form of diversity present in time-varying systems with several users, like for instance the broadcast channel. An opportunistic scheduler has to be used in order to exploit this type of diversity. Besides the number of users and speed of the channel fluctuations, the degree of multiuser diversity depends on the dynamic range of the channel. A scheme that increases the effective dynamic range of the channel by deploying multiple antennas at the transmitter is called *opportunistic beamforming* It has been shown that the deployment of multiple transmit antennas with opportunistic beamforming increases the degree of multiuser diversity in several scenarios, including in slow fading and correlated fading scenarios. However, in this work we show that even in a fast fading scenario the deployment of multiple transmit antennas with opportunistic beamforming also results in having beamforming gain when the number of users is large. In the present work, we have considered a *multiuser multiple-output single-output* (MU-MISO) downlink channel in a single cell scenario.

1. INTRODUCTION

The use of multiple antennas in single user links, i.e. *single-user multiple-input single-output* (SU-MISO) systems, provides a remarkable increase in the spectral efficiency [1, 2]. Nevertheless, when considering *multiuser multiple-input single-output* (MU-MISO) systems, like the broadcast channel, this spectral efficiency can be further enhanced through *multiuser diversity* [3].

Multiuser diversity is inherent in any time varying point-to-multi-point link such as the downlink of a network with a base station and several users, i.e. a broadcast channel. This diversity is harnessed by the use of an opportunistic scheduler. The degree of multiuser diversity depends on several factors of the multiuser system such as the number of users, the speed of the channel fluctuations and the dynamic range of the channel fluctuations. An approach that increases the dynamic range with the use of multiple antennas at the transmitter is called *opportunistic beamforming* [4]. In opportunistic beamforming, random beamforming and an opportunistic scheduler are combined in order to increase the spectral efficiency producing a gain in several scenarios.

In a *slow fading* MU-MISO broadcast channel, with multiple antennas at the transmitter and single antenna users, opportunistic beamforming artificially increases the speed of the channel variations and hence increases the effective dynamic range of the channel fluctuations. With *correlated fading*, the dynamic range of the channel fluctuations, and hence the multiuser diversity, can also be increased with opportunistic beamforming. However, it has been stated that under *fast fading*, opportunistic beamforming with multiple transmit antennas provides no gain compared to the case when having an opportunistic scheduler with only a single transmit antenna [4]. Nevertheless, considering a more realistic fading model, in this work we show that with multiple transmit antennas, opportunistic beamforming does in fact increase the dynamic range of the channel as a consquence of having transmit antenna gain or beamforming gain when the number of users is large in a fast fading MU-MISO downlink.

To this end, this paper is organized as follows. In Section 2 the channel and system model is presented. A proposed fading distribution is motivated in Section 3. The beamforming gain wiht opportunistic beamforming in a fast fading MU-MISO downlink is computed in Section 4. Section 5 presents simulation results and finally, Section 6 concludes the paper.

2. CHANNEL AND SYSTEM MODEL

Consider a flat-fading downlink of a single cell with a base station equipped with N antennas and with K single-antenna users, as shown for one user in Fig. 1. Let us denote the random beamforming vector applied at the transmitter at time slot n as $\mathbf{w}[n] = [w_1[n], w_2[n], \cdots, w_N[n]]^{\mathrm{T}} = \in \mathbb{C}^N$, where $\mathbf{w}[n]$ is a unit norm vector, i.e. $\|\mathbf{w}[n]\|_2^2 = 1$. The received signal $y_k[n] \in \mathbb{C}$ for user k at time slot n is expressed

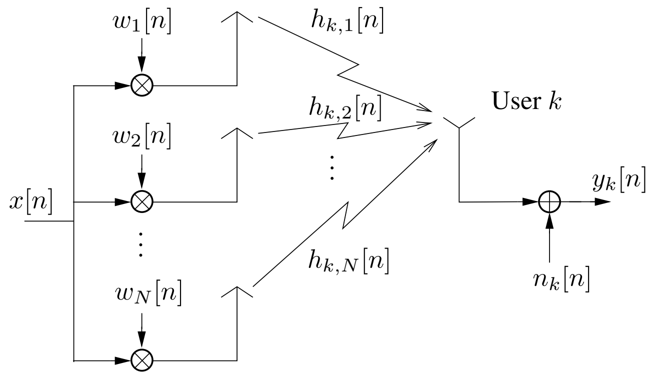

Fig. 1. MISO Channel Model for user k

as

$$y_k[n] = \sqrt{P_{\text{T}}} \cdot \mathbf{w}^{\text{T}}[n] \cdot \mathbf{h}_k[n] \cdot x_k[n] + v_k[n], \quad (1)$$
$$= \sqrt{P_{\text{T}}} \cdot h_{k,\text{eq}}[n] \cdot x_k[n] + v_k[n], \quad (2)$$

where $\mathbf{h}_k[n] = [h_{k,1}[n], h_{k,2}[n], \cdots, h_{k,N}[n]]^{\text{T}} \in \mathbb{C}^N$ is the MISO frequency flat fading channel vector from the N transmit antennas to user k at time slot n, i.e. $h_{k,i}[n]$ is the channel from the i-th transmit antenna to user k. Furthermore, $x_k[n] \in \mathbb{C}$ is the transmitted symbol for user k at time slot n and $v_k[n] \in \mathbb{C}$ is the additive white Gaussian noise (AWGN) at the receiver of user k at time slot n. For every user we assume that $v_k[n]$ is a zero-mean unit-variance complex Guassian random variable. Additionally, P_{T} is the transmit power and for convenience of notation we will set $P_{\text{T}} = 1$ in the following.

We assume that $h_{k,i}[n]$ are complex random variables. The channels between the users are independent and we assume *uncorrelated fading* accross the transmit antennas. Hence, the entries of the channel vector $\mathbf{h}_k[n]$, i.e. $h_{k,i}[n] \ \forall\, i, k$, are independent and identically distributed (i.i.d). Furthermore, we assume *fast fading*, where at each time slot n we have a new independent realization of the MISO channel $\mathbf{h}_k[n] \ \forall\, k$.

With opportunistic beamforming, a beamforming vector $\mathbf{w}[n]$ is generated at random for every time slot n. With the beamforming vector at time slot n, the MISO channel becomes an equivalent *single-input single-output* (SISO) channel for user k given by $h_{k,\text{eq}}[n] = \mathbf{w}^{\text{T}}[n] \cdot \mathbf{h}_k[n] \in \mathbb{C}$ as shown in (1)-(2). For each random beam, each user computes their *signal-to-noise ratio* (SNR) or equivalently under our assumptions the square magnitude of their equivalent SISO channel, i.e. $|h_{k,\text{eq}}|^2$, and feeds it back to the base station. Considering no fairness, the base station maximizes the sum throughput by scheduling the user with the largest $|h_{k,\text{eq}}|^2$. As stated in [4], the distribution of the beamforming vector $\mathbf{w}[n]$ should be such that it matches the distribution of the channel vector $\mathbf{h}_k[n]$, i.e. the distribution of the small scale fading. One of the most common assumptions in the literature when there is no line of sight is to assume *Rayleigh fading* [5, 6]. In this case, in order to match the channel, the beamforming vectors $\mathbf{w}[n]$ have to be generated according to an isotropic distribution [7]. In the following, we drop the time index n for simplicity.

3. TRUNCATED RAYLEIGH FADING DISTRIBUTION

As stated before, it is common to use Rayleigh fading to model the small scale fading when there is no line of sight. In such a case the elements of the channel vector $\mathbf{h}_k[n]$, i.e. $h_{k,i}$ for $i = 1, \ldots, N$, are zero-mean unit-variance complex Gaussian random variables. Let us express $\mathbf{h}_{k,i} = x + \text{j} \cdot y = z \cdot \text{e}^{\text{j}\theta}$, where $z = |h_{k,i}|$ and $\theta = \arg h_{k,i}$ are the magnitude and phase of $h_{k,i}$, respectively. With Rayleigh fading, we have that x and y are independent zero-mean unit-variance Gaussian distributed. Through a transformation of variables, it is well known the magnitude of $h_{k,i}$ is Rayleigh distributed [8], with *probability density function* (pdf)

$$f_Z(z) = 2 \cdot z \cdot \text{e}^{-z^2} \qquad z > 0, \quad (3)$$

and the distribution of the phase θ is uniform over $[0, 2\pi[$.

Consider in the following a single-user SISO link with Rayleigh fading. With the Rayleigh distribution there exist a *non-zero probability* of having a finite *very large channel magnitude* $z_{\text{L}} < \infty$, i.e.

$$\forall\, z_{\text{L}} < \infty : \ \text{Prob}[z \geq z_L] > 0. \quad (4)$$

Nevertheless, in practice this is not possible since we have to abide by the *law of conservation of energy*, i.e. the received power cannot be arbitrarily large since the received power cannot exceed the power emitted by the transmitter[1]. With the Rayleigh fading model, there exist a *non-zero probability* that the received power is larger than the transmit power. Of course, the probability that the amplitude z is very large tends towards zero[2] as z increases and thus, by using the Rayleigh fading model it would be *highly unlikely* that the channel realization of a user has such a value.

However, consider now the following scenario: a broadcast channel with only $N = 1$ transmit antenna and K independent users with Rayleigh fading channels, where the base station schedules the user with the largest channel gain among the K users. The capacity achieved with this approach grows as $\ln \ln K$ [9], and hence the capacity is unbounded as $K \rightarrow \infty$. When the number of users is large, we have a large number of independent Rayleigh realizations and naturally the maximum among K such independent random variables increases without bound as $K \rightarrow \infty$. Nevertheless, as stated before this is not realistic. The un-physicality of the Rayleigh model is due to the right tail of the pdf, i.e. (4),

[1] or: actually the power emitted by the transmitter minus the propagation losses.

[2] Although as stated before it is only zero in the limit!

which when modelling the fading for a single user can be neglected, but cannot be ignored when modelling the maximum of K single user channels as $K \to \infty$.

To overcome this impracticality, we propose simply to use a *truncated* version of the Rayleigh distribution in order to abide by the law of physics. Let us denote the maximum possible magnitude of the truncated channel z_{tr} as A. In the present work, we will not address the problem of how to determine the value of A, which anyhow turns out to be unimportant for our main result. Truncating the Rayleigh distribution given in (3) at A gives us now that the channel magnitude z_{tr} has the following distribution

$$f_{Z_{\mathrm{tr}}}(z_{\mathrm{tr}}) = \begin{cases} \dfrac{2 \cdot z_{\mathrm{tr}} \cdot \mathrm{e}^{-z_{\mathrm{tr}}^2}}{1 - \mathrm{e}^{-A^2}} & 0 < z_{\mathrm{tr}} \leq A, \\ 0 & \text{else,} \end{cases} \quad (5)$$

which is basically the same distribution and formula as in (3), but with a limited domain and normalized by

$$F_{Z_{\mathrm{tr}}}(A) = (1 - \mathrm{e}^{-A^2}), \quad (6)$$

where $F_{Z_{\mathrm{tr}}}(A)$ is the *cumulative distribution function* (cdf) of (3) evaluated at A. This is required in order that

$$\int_0^\infty f_{Z_{\mathrm{tr}}}(z_{\mathrm{tr}}) \cdot \mathrm{d}z_{\mathrm{tr}} = 1. \quad (7)$$

The expected value of the square of z_{tr}, i.e. $\mathrm{E}\left[z_{\mathrm{tr}}^2\right]$, is given by

$$\mathrm{E}\left[z_{\mathrm{tr}}^2\right] = \frac{1 - (1 + A^2) \cdot \mathrm{e}^{-A^2}}{1 - \mathrm{e}^{-A^2}}. \quad (8)$$

When $A \to \infty$, then we have the Rayleigh fading distribution and hence $\mathrm{E}\left[z^2\right] = 1$. However, for $A = 2.5$ for instance, $\mathrm{E}\left[z_{\mathrm{tr}}^2\right] = 0.9879$.

In order to observe the difference between the *true* or *original* Rayleigh distribution and the *truncated* Rayleigh distribution, we depict in Fig. 2 the distribution for diferent values of A. Additionally, in Fig. 3 we have zoomed in onto the right tail of the fading distributions. Notice that there is practically no difference between the original Rayleigh distribution and the truncated one at $A = 2.5$. The probability that the channel magnitude is larger than A for the Rayleigh distribution is given by $1 - F_{Z_{\mathrm{tr}}}(A)$, which for $A = 1.5, 2.0$ and 2.5 is equal to 10.54%, 1.83% and 0.19% from the original Rayleigh distribution, respectively. Note that for $A \geq 2.5$ this percentage is practically negligible.

Now, we will assume that the $|h_{k,i}|$ for $i = 1, \ldots, N$, and $k = 1, \ldots, K$, are i.i.d. random variables with distribution $f_{Z_{\mathrm{tr}}}(z_{\mathrm{tr}})$ given by (5) with phase $\arg h_{k,i}$ being uniform distributed. We will refer to this type of fading as *truncated Rayleigh fading*. Under truncated Rayleigh fading, note that in the limit $A \to \infty$, the channels become zero-mean unit-variance complex Gaussian random variables, i.e. as $A \to \infty$ we have the original Rayleigh fading.

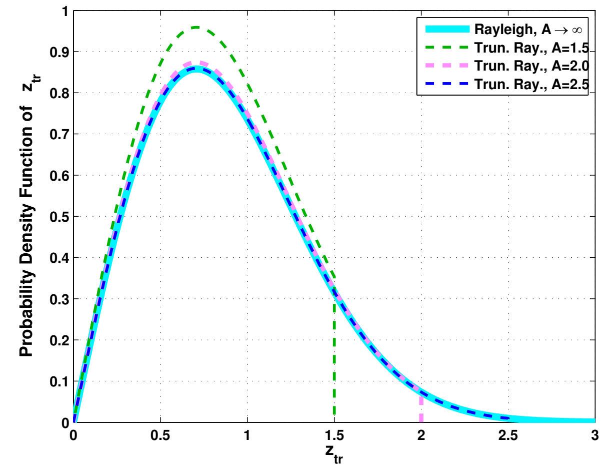

Fig. 2. Original and Truncated Rayleigh distributions for $A = 1.5, 2.0, 2.5$.

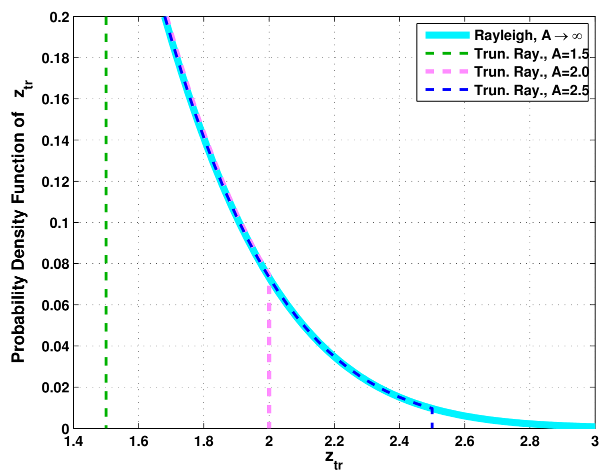

Fig. 3. Right Tail of Truncated Rayleigh distributions for $A = 1.5, 2.0, 2.5$.

3.1. Distribution Properties

Before continuing let us present two important properties of the truncated Rayleigh fading, that will be used further on. The pdf of the magnitude $z_{tr} = h_{k,i}$ and phase $\theta = \arg h_{k,i}$ of the channel $\forall\, k, i$ should have the following properties:

1. **Asymptotically-Achievable Finite Magnitude:**

$$\exists\, A > 0 : \forall\, \epsilon_A > 0 : \exists\, \epsilon_p > 0 :$$
$$\int_{z'_{\mathrm{tr}}}^\infty f_{Z_{\mathrm{tr}}}(z_{\mathrm{tr}})\, \mathrm{d}z_{\mathrm{tr}} = \begin{cases} \geq \epsilon_p & \text{for } z'_{\mathrm{tr}} = A - \epsilon_A, \\ = 0 & \text{for } z'_{\mathrm{tr}} > A. \end{cases}$$

This means that,

$$\Pr\left[A - \epsilon_A \leq z_{\mathrm{tr}} \leq A\right] \geq \epsilon_A \epsilon_p > 0,$$

for arbitrarily small $\epsilon_A > 0$, i.e., the probability is greater than zero, that $|h_{k,i}|\ \forall\, k, i$ has a value inside an arbitrarily small interval which is arbitrarily close to A. Also, we have that $|h_{k,i}|$ cannot exceed A

$$\Pr\left[z_{\text{tr}} > A\right] = 0.$$

2. **Asymptotically-Achievable Discrete Phase:**

$$\exists\, \phi \in \mathbb{R} : \forall\, \epsilon_\phi > 0 : \exists\, \epsilon_{p'} > 0 :$$
$$\int_{\phi-\epsilon_\phi}^{\phi} f_\Theta(\theta)\, \mathrm{d}\theta \geq \epsilon_{p'}.$$

This means that, there is at least one angle θ for which

$$\Pr\left[\phi - \epsilon_\phi \leq \theta \leq \phi\right] \geq \epsilon_\phi \epsilon_{p'} > 0,$$

for arbitrarily small $\epsilon_\phi > 0$. The phase of the channels inside an arbitrarily small interval which is arbitrarily close to θ occur with probability larger than zero.

4. BEAMFORMING GAIN WITH OPPORTUNISTIC BEAMFORMING

Let is now compute the beamforming gain obtained with opportunistic beamforming in a MU-MISO downlink with N transmit antennas where K users experience truncated Rayleigh fading for a given A. For a given A, number of users K and number of antennas N and a fixed beamforming vector $\mathbf{w} \in \mathbb{C}^N$, let us denote

$$\gamma(\mathbf{w}, \mathbf{h}_k, K, A) = \max_{k=\{1,\ldots,K\}} |h_{k,\text{eq}}|^2 = \max_{k=\{1,\ldots,K\}} |\mathbf{w}^{\text{H}} \cdot \mathbf{h}_k|^2, \tag{9}$$

i.e., the squared magnitude of the equivalent channel of the scheduled user. The beamforming gain is with respect to the single transmit antenna case. For $N = 1$, let us assume that we transmit only with the first transmit antenna by using $\mathbf{w} = \text{e}_1 = [1,\ 0,\ \ldots,\ 0]^{\text{T}}$. For each user k, the SISO channel $h'_{k,\text{eq}} = \mathbf{w}'^{,\text{H}} \cdot \mathbf{h}_k = h_{k,1}$ is just the first element of channel vector $\mathbf{h}_k$. For $N = 1$, (9) becomes

$$\gamma(\mathbf{w} = \text{e}_1, \mathbf{h}_k, K, A) = \max_{k=\{1,2,\ldots,K\}} |h_{k,1}|^2. \tag{10}$$

Let us now compute the beamforming gain Γ as $K \to \infty$ and $A \to \infty$. Since two limits are involved, then we can define the gain in two ways, depending on the order of the limits:

$$\Gamma_1(N, \mathbf{w}) = \lim_{K\to\infty} \left(\lim_{A\to\infty} \frac{\text{E}\left[\gamma(\mathbf{w}, \mathbf{h}_k, K, A)\right]}{\text{E}\left[\gamma(\mathbf{w} = \text{e}_1, \mathbf{h}_k, K, A)\right]} \right) \tag{11}$$

$$\Gamma_2(N, \mathbf{w}) = \lim_{A\to\infty} \left(\lim_{K\to\infty} \frac{\text{E}\left[\gamma(\mathbf{w}, \mathbf{h}_k, K, A)\right]}{\text{E}\left[\gamma(\mathbf{w} = \text{e}_1, \mathbf{h}_k, K, A)\right]} \right) \tag{12}$$

The order of the limits is important because each order has a different interpretation and leads to different results.

4.1. First Ordering: $\Gamma_1(N, \mathbf{w})$

Let us first consider the first ordering, i.e. (11), where the beamforming gain can be simplified as follows

$$\Gamma_1(N, \mathbf{w}) = \lim_{K\to\infty} \left(\lim_{A\to\infty} \frac{\text{E}\left[\gamma(\mathbf{w}, \mathbf{h}_k, K, A)\right]}{\text{E}\left[\gamma(\mathbf{w} = \text{e}_1, \mathbf{h}_k, K, A)\right]} \right) \tag{13}$$

$$= \lim_{K\to\infty} \left(\frac{\lim_{A\to\infty} \text{E}\left[\gamma(\mathbf{w}, \mathbf{h}_k, K, A)\right]}{\lim_{A\to\infty} \text{E}\left[\gamma(\mathbf{w} = \text{e}_1, \mathbf{h}_k, K, A)\right]} \right) \tag{14}$$

$$= \lim_{K\to\infty} \left(\frac{\text{E}\left[\lim_{A\to\infty} \gamma(\mathbf{w}, \mathbf{h}_k, K, A)\right]}{\text{E}\left[\lim_{A\to\infty} \gamma(\mathbf{w} = \text{e}_1, \mathbf{h}_k, K, A)\right]} \right) \tag{15}$$

Equalities (14) and (15) folow from

$$0 < \lim_{A\to\infty} \gamma(\mathbf{w}, \mathbf{h}_k, K, A) < \infty, \tag{16}$$

in case that $A < \infty$ and $N < \infty$, which also implies

$$0 < \lim_{A\to\infty} \text{E}\left[\gamma(\mathbf{w}, \mathbf{h}_k, K, A)\right] < \infty, \tag{17}$$

from which equality (13) and (14) follow.

Let us now compute $\Gamma_1(N, \mathbf{w})$ from (15). We consider first the limit $A \to \infty$. As stated before as $A \to \infty$, we have the original Rayleigh fading, i.e. the elements of $\mathbf{h}_k$ are i.i.d. zero-mean unit-variance circularly symmetric complex Gaussians random variables. In this case, $h_{k,\text{eq}}$ can be expressed as a weighted sum of independent complex Gaussians, where the weights are the elements of $\mathbf{w} = [w_1, \ldots, w_N]^{\text{T}}$. Hence,

$$h_{k,\text{eq}} = \mathbf{w}^{\text{H}} \cdot \mathbf{h}_k = \sum_{i=1}^{N} w_i \cdot h_{k,i}, \tag{18}$$

which due to the central limit theorem, is also zero-mean complex Gaussian distributed with variance

$$\sum_{i=1}^{N} |w_i|^2 = \|\mathbf{w}\|_2^2 = 1,$$

i.e., unit variance. Therefore, the distribution of $h_{k,\text{eq}}$ for $k = 1, \ldots, K$, is independent of $\mathbf{w}$ and of N. Additionally, the $h_{k,\text{eq}}$ for $k = 1, \ldots, K$, have the same distribution as that of the elements of $\mathbf{h}_k$, including $h_{k,1}$. Since the previous argument holds as $A \to \infty$, then the distribution of $\lim_{A\to\infty} \max_{k=\{1,2,\ldots,K\}} |h_{k,\text{eq}}|^2$ is the same as that of $\lim_{A\to\infty} \max_{k=\{1,2,\ldots,K\}} |h_{k,1}|^2$. Recalling (9) and (10), then

$$\frac{\text{E}\left[\lim_{A\to\infty} \gamma(\mathbf{w}, \mathbf{h}_k, K, A)\right]}{\text{E}\left[\lim_{A\to\infty} \gamma(\mathbf{w} = \text{e}_1, \mathbf{h}_k, K, A)\right]} = 1, \tag{19}$$

and (15) results in

$$\Gamma_1(N) = 1, \tag{20}$$

indicating that there is *no additional gain*, resulting from beamforming gain, when we deploy multiple transmit antennas with opportunistic beamforming. However, exchanging the order of the limits leads us to a different result.

4.2. Second Ordering: $\Gamma_2(N, \mathbf{w})$

For the following argument, assume without loss of generality that the random beamforming vector is fixed with an equal power allocation over the antennas, i.e.

$$\tilde{\mathbf{w}} = \sqrt{\frac{1}{N}} \cdot \left[\mathrm{e}^{-\mathrm{j}\theta_1}, \mathrm{e}^{-\mathrm{j}\theta_2}, \ldots, \mathrm{e}^{-\mathrm{j}\theta_N}\right]^{\mathrm{T}}, \tag{21}$$

where $\theta_1, \theta_2, \ldots, \theta_N$ are angles satisfying the asymptotic achievable discrete phase property described in Section 3.1.

Let us now consider taking the limits in the opposite sequence as described in the previous section with the previous described beamforming vector $\tilde{\mathbf{w}}$. Retaking it from (11) and following a similar development as (13)-(15), we have that

$$\begin{aligned}
\Gamma_2(N, \tilde{\mathbf{w}}) &= \lim_{A\to\infty} \left(\lim_{K\to\infty} \frac{\mathrm{E}\left[\gamma(\tilde{\mathbf{w}}, \mathbf{h}_k, K, A)\right]}{\mathrm{E}\left[\gamma(\mathbf{w} = \mathrm{e}_1, \mathbf{h}_k, K, A)\right]} \right) && (22)\\
&= \lim_{A\to\infty} \left(\frac{\lim_{K\to\infty} \mathrm{E}\left[\gamma(\tilde{\mathbf{w}}, \mathbf{h}_k, K, A)\right]}{\lim_{K\to\infty} \mathrm{E}\left[\gamma(\mathbf{w} = \mathrm{e}_1, \mathbf{h}_k, K, A)\right]} \right) && (23)\\
&= \lim_{A\to\infty} \left(\frac{\mathrm{E}\left[\lim_{K\to\infty} \gamma(\tilde{\mathbf{w}}, \mathbf{h}_k, K, A)\right]}{\mathrm{E}\left[\lim_{K\to\infty} \gamma(\mathbf{w} = \mathrm{e}_1, \mathbf{h}_k, K, A)\right]} \right) && (24)
\end{aligned}$$

Let us now consider a channel vector $\mathbf{h}_{k,*}$ with the following properties

$$\begin{aligned}
\forall\, i: \qquad & A - \epsilon_A \leq |h_{k,i,*}| \leq A, && (25)\\
& \phi_i - \epsilon_\phi \leq \arg h_{k,i,*} \leq \phi_i. && (26)
\end{aligned}$$

Since $\mathbf{h}_{k,*}$ is arbitrarily close to a channel matched to the beamforming vector $\tilde{\mathbf{w}}$, and furthermore its norm is arbitrarily close to the maximum possible norm, it follows that

$$\begin{aligned}
\forall\, \epsilon > 0 : |\tilde{\mathbf{w}}^{\mathrm{H}} \cdot \mathbf{h}_{k,*}|^2 &\geq \sup |\tilde{\mathbf{w}}^{\mathrm{H}} \cdot \mathbf{h}_{k,*}|^2 - \epsilon \\
&\geq NA^2 - \epsilon, && (27)
\end{aligned}$$

which is because of

$$\begin{aligned}
\sup_{\mathbf{w},\mathbf{h}_k} |\tilde{\mathbf{w}}^{\mathrm{H}} \cdot \mathbf{h}_{k,*}|^2 &= \sup_{\mathbf{w},\mathbf{h}_k} \gamma(\mathbf{w}, \mathbf{h}_k, K, A) \\
&= \left(\sum_{i=1}^{N} \frac{A}{\sqrt{N}} \right)^2 \\
&= NA^2. && (28)
\end{aligned}$$

Based on the properties described in Section 3.1, as $K \to \infty$, there will be, with probability

$$P \geq \lim_{K\to\infty} \left(1 - \left(1 - (\epsilon_p \cdot \epsilon_A \cdot \epsilon_\phi \cdot \epsilon_{p'})^N\right)^K \right) = 1, \tag{29}$$

such a vector $\mathbf{h}_{k,*}$. Hence, with probability $P = 1$, we have that,

$$\begin{aligned}
\forall\, \epsilon > 0 : NA^2 - \epsilon &\leq |\tilde{\mathbf{w}}^{\mathrm{H}} \cdot \mathbf{h}_{k,*}|^2 \\
&\leq \lim_{K\to\infty} \gamma(\tilde{\mathbf{w}}, \mathbf{h}_k, K, A) \\
&\leq NA^2, && (30)
\end{aligned}$$

and

$$\begin{aligned}
\forall\, \epsilon > 0 : NA^2 - \epsilon &\leq |\tilde{\mathbf{w}}^{\mathrm{H}} \cdot \mathbf{h}_{k,*}|^2 \\
&\leq \mathrm{E}\left[\lim_{K\to\infty} \gamma(\tilde{\mathbf{w}}, \mathbf{h}_k, K, A)\right] \\
&\leq NA^2, && (31)
\end{aligned}$$

where (31) follows from (30) because $\lim_{K\to\infty} \gamma(N, K, A)$ does not depend on K and is not a random variable anymore as shown in (28). Finally, the ratio

$$\frac{\mathrm{E}\left[\lim_{K\to\infty} \gamma(\mathbf{w}, \mathbf{h}_k, K, A) | \mathbf{w}\right]}{\mathrm{E}\left[\lim_{K\to\infty} \gamma(\mathbf{w} = \mathrm{e}_1, \mathbf{h}_k, K, A)\right]} = N, \tag{32}$$

and does not depend on A, so we do not need to take limit $A \to \infty$. Thus using (27), (31) and (32) we obtain that (24)

$$\forall\, \epsilon > 0 : N - \epsilon \leq \Gamma_2(N, \tilde{\mathbf{w}}) = \Gamma_2(N, \mathbf{w}) \leq N. \tag{33}$$

The asymptotically obtainable beamforming gain is arbitrarily close to the number of transmit antennas, which represents the maximum beamforming gain. Here we have assumed for simplicity that the beamforming vector is given by (21). Nevertheless, the result (33) also holds if the magnitudes of the beamforming vector $\mathbf{w}$ are also random. This result is in striking contrast to the result from (20), where no beamforming gain is concluded.

The first ordering of the limits leads to the well known result, result (20), that the multiple transmit antennas with opportunistic beamforming provide no additional benefit under fast Rayleigh fading as stated in [4]. However, we rule out this ordering because it conflicts with the laws of physics as we allow for $A \to \infty$ and as explained in Section 3 this is unfeasible. With the second ordering, we do not use $A \to \infty$ and by simulaton results we confirm its validity. Hence, the multiuser diversity is increased when deploying multiple transmit anntenas with opportunistic beamforming as a result of the beamforming or antenna gain as $K \to \infty$. In the next section we confirm that the second.

5. SIMULATION RESULTS

In order to confirm our results from the last section, we present simulation results for a MU-MISO downlink channel with truncated Rayleigh fading for different values of A, number of transmit antennas N and number of users K. In Fig. 4, we depict the cdf of $\Gamma_2(N)$ in dB for $N = 1, 2$ for $K = 10^6$ users and $A = 2.5$. For the case of $N = 1$, the cdf is almost a step-function at $10 \cdot \log_{10} A^2 \approx 7.96$ dB. This comes about, because the fairly large number of users ($K = 10^5$) guarantees that, at least one SISO channel realization from the K users has a magnitude which is very close to $A = 2.5$. In the case of $N = 2$, the cdf shows a larger spread since, in order to achieve close to maximum beamforming gain, both the magnitude and the angle of two channel coefficients have to closely match the beamforming vector $\mathbf{w}$, which is much less probable than in the case of $N = 1$. The average beamforming gain in dB is

$$10 \cdot \log_{10}\left(\frac{\mathrm{E}\left[\Gamma_2(N=2)\right]}{\mathrm{E}\left[\Gamma_2(N=1)\right]}\right) \approx 2.56 \quad \text{dB}. \qquad (34)$$

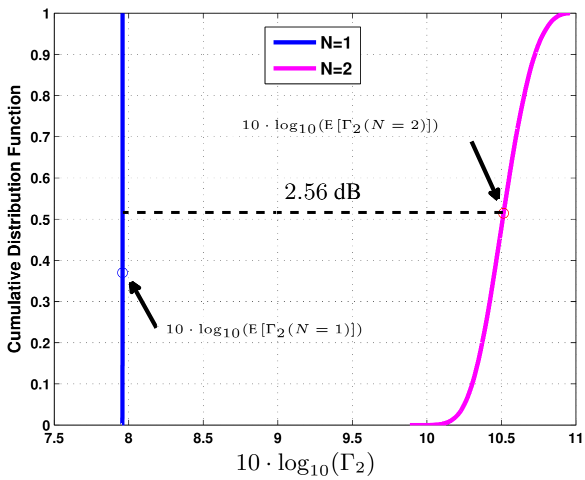

Fig. 4. CDF of $\Gamma_2(N)$ for $N = 1, 2$, $K = 10^6$ and $A = 2.5$.

Looking in detail in Fig. 4, it can be seen that in 1% of the cases, the beamforming gain is above 2.8 dB, while in 99% of the cases it is above 1.54 dB. Notice that for $K \to \infty$, the cdf for both $N = 1$ and $N = 2$ becomes a step-function and the asymptotic beamforming gain of $10 \cdot \log_{10}(2) \approx 3$ dB is reached. The asymptotic beamforming gain $\Gamma_2(N)$ is independent of the finite value A. As a consequence, A can be made arbitrarily large, while an asymptotic beamforming gain which is arbitrarily close to N is still achieved with probability $P = 1$ as $K \to \infty$.

In Fig. 5, we depict the average beamforming gain for different number of users for $N = 2$ and different values of A. As K increases the average beamforming gain increases because the probability of finding a user whose channel is closely matched to the beamforming vector increases. For the same reason as before is the increase larger for a smaller A. Notice that there is always a fraction of the maximum beamforming gain even for a moderate number of users.

In Fig. 6, we show the percentage of the maximum beamforming gain in dB achieved with different number of transmit antennas N for $K = 10^5$ and $A = 2.0$. The maximum beamforming gain increases as N increases. However, as the number of transmit antennas increase, it is less probable to find a user whose channel is closely matched to the beamforming vector. Thus, a larger number of users K would be required to achieve the same percentage of the maximum beamforming gain as N increases. Even though the beamforming gain increases with the number of transmit antennas,

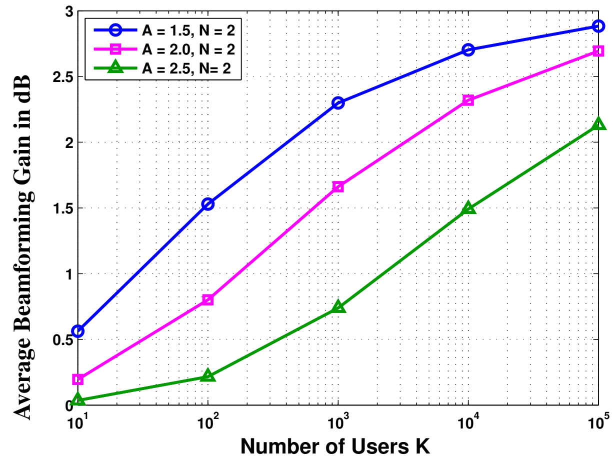

Fig. 5. Average Beamforming Gain in dB for Different Number of Users

a larger number of users is required to come closer to the maximum beamforming gain.

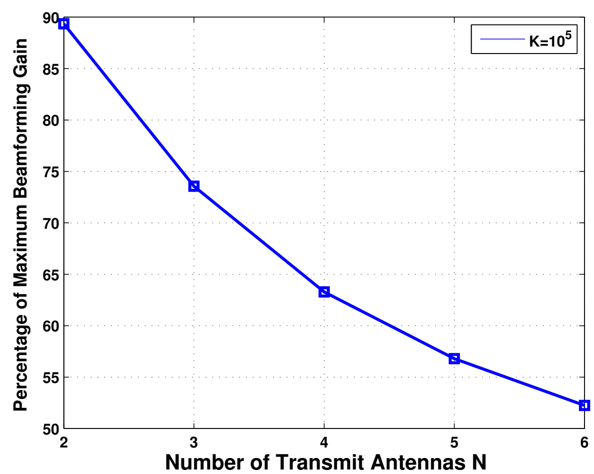

Fig. 6. Achieved Percentage of Maximum Beamforming Gain with $K = 10^5$

6. CONCLUSIONS

In this work we have shown that the deployment of multiple transmit antennas with opportunistic beamforming in a fast fading downlink scenario increases the degree of multiuser diversity as a result of having beamforming gain for a large number of users. The truncated Rayleigh distribution was proposed as a more realistic fading model to show our results. Even though the maximum beamforming gain is achieved when $K \to \infty$, there is always a fraction of the beamforming gain for finite number of users. For a small number of users, large number of transmit anntenas or large A, it is more difficult to achieve the maximum beamforming gain, since it is less probable of finding a user closely matched to the random beamforming vector.

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