# Iterative Detection Based on Reduced-Rank Equalization

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***Abstract*— In this paper, we consider an *iterative* or *turbo receiver* with linear detection using the *Wiener Filter* (WF), i. e. the optimal linear filter based on the *Mean Square Error* (MSE) criterion. Multiple antennas at the receiver increase the dimension of the observation vector which results in computational intense detectors. We extend an optimal but computational efficient algorithm, originally derived for a single receive antenna, to *Single-Input Multiple-Output* (SIMO) channels. To further reduce computational complexity, we apply the suboptimal low-rank *Multi-Stage Wiener Filter* (MSWF) and approximate additionally second order statistics of non-stationary random processes by their time-invariant averages. Complexity investigations reveal the enormous capability of the proposed algorithms to decrease computational effort. Moreover, simulation results show that the reduced-rank MSWF behave near optimum although the rank is drastically reduced to two or even one.**

## I. Introduction

Compensating *intersymbol interference* caused by frequency-selective channels is a fundamental task of practical communication systems. To meet this requirement, we investigate an iterative receiver structure performing joint symbol detection and decoding. This so-called *turbo equalizer* was introduced by Douillard et al. [1] and consists of an optimal *Maximum A Posteriori* (MAP) detector and a MAP decoder exchanging soft information in an iterative process. Simulation results have shown that this procedure eliminates intersymbol interference after several iteration steps such that the *Bit Error Rate* (BER) of coded transmission over the corresponding *Additive White Gaussian Noise* (AWGN) channel can be reached. Note that the proposed turbo system can be interpreted as an iterative decoding scheme for serial concatenated codes [2] where the inner code is the channel and the inner decoder is the detector. Unfortunately, the computational effort of this receiver is very high. Therefore, Wang and Poor [3] approximated the optimal non-linear detector using the *Wiener Filter* (WF), i. e. the linear filter with the *Minimum Mean Square Error* (MMSE), to reduce the complexity in a multi-user coded CDMA system.

In this paper, we consider a single-user uplink scenario for mobile communications with multiple antennas at the receiver. The resulting *Single-Input Multiple-Output* (SIMO) channel leads to a high computational complexity due to the high-dimensional observation vector. In a first step, we reduce the high effort for the computation of the WF weights [3] by extending the reduced-complexity algorithm introduced by Tüchler et al. [4] to the proposed SIMO channel. This method exploits the time-dependency of the auto-covariance matrix of the observation vector to decrease the computational complexity of its inversion by one order but is still optimal in the MMSE sense, i. e. it achieves the performance of the turbo equalizer proposed by [3].

To further reduce computational complexity, we introduce suboptimal solutions using reduced-rank equalization. The *Multi-Stage Wiener Filter* (MSWF) developed by Goldstein et al. [5] is a computationally cheap approximation of the WF. It has been shown by Honig et al. [6], [7] that the application of the MSWF is equivalent to Wiener filtering in the *Krylov subspace* of the auto-covariance matrix of the observation vector and the cross-covariance vector between the observation and the desired signal. Therefore, the efficient *Lanczos algorithm* [8], [9], [7] can be used to compute the reduced-rank filter weights.

Moreover, the approximation of second order statistics of non-stationary random processes by their time-invariant averages yields implementations with further reduced computational effort. This idea was developed for full-rank WFs [4] but holds also for the derived reduced-rank MSWF. A comparison of the needed *FLoating point OPerations* (FLOPs) as measure for the computational complexity for calculating the filter coefficients gives an impression for the capability of the proposed linear detectors.

Finally, simulation results show close to optimum behavior of the proposed solutions despite of their tremendously reduced computational complexity. Especially, the rank-one MSWF which is an easy to implement normalized *Matched Filter* (MF) followed by a scalar WF, turns out to have only a slightly smaller performance than the optimal WF. Besides, the MSWF outperforms the *Principal Component* (PC) [10] and the *Cross-Spectral* (CS) [11] method which are both based on the approximation of the WF in an eigen subspace of the auto-covariance matrix of the observation.

Throughout the paper, vectors and matrices are denoted by lower case bold and capital bold letters, respectively. The matrix $\mathbf{1}_n$ is the $n \times n$ identity matrix, $\boldsymbol{e}_\nu$ its $\nu$-th column, $\mathbf{0}_{m\times n}$ the $m \times n$ zero matrix, and $\mathbf{0}_n$ the $n$-dimensional zero vector. The operation '$\otimes$' denotes the Kronecker product, $\mathrm{E}\{\cdot\}$ expectation, $(\cdot)^*$ conjugate complex, $(\cdot)^{\mathrm{T}}$ transpose, $(\cdot)^{\mathrm{H}}$ Hermitian, i. e. conjugate transpose, $\|\cdot\|_2$ the Euclidian norm, and $\mathrm{O}(\cdot)$ the Landau symbol.

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## II. System Model

We consider the SIMO transmission model depicted in Fig. 1. The binary data block $\mathsf{b} \in \{0,1\}^B$ is encoded with rate $r$. The coded data block $\mathsf{c} \in \{0,1\}^{SQ}$ is interleaved, i. e. $\mathsf{c}' = \boldsymbol{\Pi}\mathsf{c} \in \{0,1\}^{SQ}$, and mapped to the complex symbol block $\mathsf{s} \in \mathbb{M}^S$ using the modulation alphabet $\mathbb{M}$ whose cardinality is $2^Q$, i. e. $B = SQr$. Afterwards, the symbol sequence $\mathsf{s}[k] = \mathsf{s}^{\mathrm{T}} \boldsymbol{e}_{k+1} \in \mathbb{M}$, $k \in \{0,1,\dots,S-1\}$, is transmitted over the SIMO channel with the coefficients $\boldsymbol{h}_\ell \in \mathbb{C}^R$, $\ell \in \{0,1,\dots,L-1\}$, received by $R$ antennas, and perturbed by stationary AWGN $\tilde{\mathsf{n}}[k] \in \mathbb{C}^R$ with the circular complex normal distribution $\mathcal{N}_c\left(\mathbf{0}_R, \sigma_{\mathsf{n}}^2 \mathbf{1}_R\right)$ whose variance is $\sigma_{\mathsf{n}}^2$. The received signal vector $\mathsf{r}[k] \in \mathbb{C}^R$ can finally be written as

$$\mathsf{r}[k] = \sum_{\ell=0}^{L-1} \boldsymbol{h}_\ell \mathsf{s}[k-\ell] + \tilde{\mathsf{n}}[k]. \tag{1}$$

For the derivation of the optimal linear equalizer filter with $K$ taps in Section III, we introduce the matrix-vector model

$$\mathsf{y}[k] = \boldsymbol{H}\mathsf{s}[k] + \mathsf{n}[k], \tag{2}$$

with the observation vector $\mathsf{y}[k] \in \mathbb{C}^N$, $N = RK$, the noise vector $\mathsf{n}[k] \in \mathbb{C}^N$, the symbol vector $\mathsf{s}[k] \in \mathbb{C}^{K+L-1}$ and the time-invariant channel convolutional matrix $\boldsymbol{H} \in \mathbb{C}^{N\times(K+L-1)}$ defined as follows:

$$\mathsf{y}[k] = \left[\mathsf{r}^{\mathrm{T}}[k], \mathsf{r}^{\mathrm{T}}[k-1], \dots, \mathsf{r}^{\mathrm{T}}[k-K+1]\right]^{\mathrm{T}}, \tag{3}$$

$$\mathsf{n}[k] = \left[\tilde{\mathsf{n}}^{\mathrm{T}}[k], \tilde{\mathsf{n}}^{\mathrm{T}}[k-1], \dots, \tilde{\mathsf{n}}^{\mathrm{T}}[k-K+1]\right]^{\mathrm{T}}, \tag{4}$$

$$\mathsf{s}[k] = \left[\mathsf{s}[k], \mathsf{s}[k-1], \dots, \mathsf{s}[k-K-L+2]\right]^{\mathrm{T}}, \tag{5}$$

$$\boldsymbol{H} = \sum_{\ell=0}^{L-1} \left[\mathbf{0}_{K\times\ell}, \mathbf{1}_K, \mathbf{0}_{K\times(L-1-\ell)}\right] \otimes \boldsymbol{h}_\ell. \tag{6}$$

Note that $\mathsf{y}[k]$ represents an $N$-dimensional subblock of the received symbol block $\mathsf{y} \in \mathbb{C}^{R(S+L-1)}$.

On receiver side, linear detector and decoder exchange soft information about the coded bits in an iterative process. In the following, soft information of a random variable $\mathsf{x}$ is represented by the well-known *Log-Likelihood Ratio* (LLR) [12] $l(\mathsf{x}) = \ln\left(\mathrm{P}(\mathsf{x}=0)/\mathrm{P}(\mathsf{x}=1)\right)$. The linear detector calculates the extrinsic information $\boldsymbol{l}_{\text{ext}}^{(\text{LD})}(\mathsf{c}') \in \mathbb{R}^{SQ}$ using the observation signal block $\mathsf{y}$ and the *a priori* information $\boldsymbol{l}_{\text{apr}}^{(\text{LD})}(\mathsf{c}') \in \mathbb{R}^{SQ}$ about the interleaved coded bits delivered by the decoder at the previous iteration step (cf. Section III). Further, the MAP decoder computes the extrinsic information $\boldsymbol{l}_{\text{ext}}^{(\text{D})}(\mathsf{c}) \in \mathbb{R}^{SQ}$ from the *a posteriori* LLRs (cf. e. g. [3]). Besides, the MAP decoder provides the decoded data bits

$$\tilde{\mathsf{b}}^{\mathrm{T}} \boldsymbol{e}_i = \underset{b\in\{0,1\}}{\operatorname{argmax}} \mathrm{P}\left(\mathsf{b}^{\mathrm{T}}\boldsymbol{e}_i = b | \boldsymbol{l}_{\text{apr}}^{(\text{D})}(\mathsf{c})\right), \tag{7}$$

$i \in \{1,2,\dots,B\}$, after the last iteration.

## III. Linear Detection

The estimate of the transmitted symbol sequence $\mathsf{s}[k]$ is obtained by linear equalization, i. e.

$$\hat{\mathsf{s}}[k] = \boldsymbol{w}^{\mathrm{H}}[k]\mathsf{y}[k] + a[k], \tag{8}$$

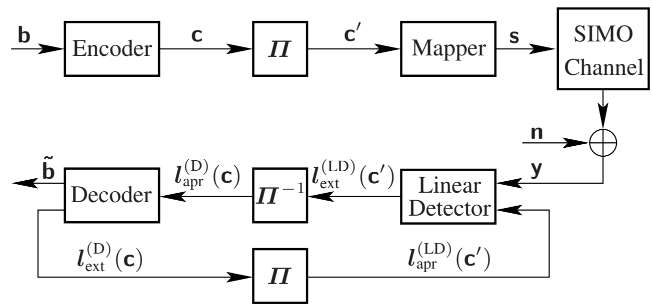


Fig. 1. System Model

where $\boldsymbol{w}[k]$ and $a[k]$ are linear *Time-Variant* (TV) filter coefficients. Note that we do not assume the symbol sequence $\mathsf{s}[k]$ to be stationary and zero mean. The scalar $a[k]$ is necessary due to the mean, i. e. the available *a priori* knowledge. The WF [13], [3] is the solution of the optimization

$$\{\boldsymbol{w}[k], a[k]\} = \underset{\{\boldsymbol{w},a\}}{\operatorname{argmin}}\, \xi_k(\boldsymbol{w}, a), \tag{9}$$

where $\xi_k(\boldsymbol{w}, a) = \mathrm{E}\{|\mathsf{s}[k-\kappa] - \hat{\mathsf{s}}[k]|^2\}$ is the *Mean Square Error* (MSE) at time index $k$ between the symbol $\mathsf{s}[k-\kappa]$ and its estimate $\hat{\mathsf{s}}[k]$ , i. e.

$$\begin{aligned}\xi_k(\boldsymbol{w}, a) = {} & 2\,\mathrm{Re}\left\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{m}_{\mathsf{y}}[k]a^* - \boldsymbol{w}^{\mathrm{H}}\boldsymbol{r}_{\mathsf{y},\mathsf{s}}[k] - a^* m_{\mathsf{s}}[k-\kappa]\right\} \\ & + \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{\mathsf{y}}[k]\boldsymbol{w} + \sigma_{\mathsf{s}}^2[k-\kappa] + |a|^2,\end{aligned} \tag{10}$$

with the latency time $\kappa$, the mean $\boldsymbol{m}_{\mathsf{y}}[k] = \mathrm{E}\{\mathsf{y}[k]\} \in \mathbb{C}^N$ and the auto-correlation matrix $\boldsymbol{R}_{\mathsf{y}}[k] = \mathrm{E}\{\mathsf{y}[k]\mathsf{y}^{\mathrm{H}}[k]\} \in \mathbb{C}^{N\times N}$ of the non-stationary observation vector random process $\mathsf{y}[k]$, and the mean $m_{\mathsf{s}}[k] = \mathrm{E}\{\mathsf{s}[k]\} \in \mathbb{C}$ and the power $\sigma_{\mathsf{s}}^2[k] = \mathrm{E}\{|\mathsf{s}[k]|^2\} \in \mathbb{R}_{0,+}$ of the scalar random process $\mathsf{s}[k]$. The cross-correlation vector between $\mathsf{y}[k]$ and $\mathsf{s}[k-\kappa]$ is given by $\boldsymbol{r}_{\mathsf{y},\mathsf{s}}[k] = \mathrm{E}\{\mathsf{y}[k]\mathsf{s}^*[k-\kappa]\} \in \mathbb{C}^N$. With the definition of the auto-covariance matrix $\boldsymbol{C}_{\mathsf{y}}[k] = \boldsymbol{R}_{\mathsf{y}}[k] - \boldsymbol{m}_{\mathsf{y}}[k]\boldsymbol{m}_{\mathsf{y}}^{\mathrm{H}}[k]$ and the cross-covariance vector $\boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k] = \boldsymbol{r}_{\mathsf{y},\mathsf{s}}[k] - \boldsymbol{m}_{\mathsf{y}}[k]m_{\mathsf{s}}^*[k]$, the solution of the optimization given in Eq. (9) computes as

$$\boldsymbol{w}[k] = \boldsymbol{C}_{\mathsf{y}}^{-1}[k]\boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k], \tag{11}$$

$$a[k] = m_{\mathsf{s}}[k-\kappa] - \boldsymbol{w}^{\mathrm{H}}[k]\boldsymbol{m}_{\mathsf{y}}[k]. \tag{12}$$

In order to ensure the adherence of the *turbo principle* [14], we choose the filter coefficients such that they do not depend on $m_{\mathsf{s}}[k-\kappa]$, i. e. we assume $m_{\mathsf{s}}[k-\kappa] = 0$. If we recall additionally the transmission model defined in Eq. (2), Eqs. (11) and (12) may be rewritten as

$$\boldsymbol{\omega}[k] = \left(\boldsymbol{H}\boldsymbol{\Gamma}_{\mathsf{s}}[k]\boldsymbol{H}^{\mathrm{H}} + \sigma_{\mathsf{n}}^2\mathbf{I}_N\right)^{-1}\boldsymbol{H}\boldsymbol{\Gamma}_{\mathsf{s}}[k]\boldsymbol{e}_{\kappa+1}, \tag{13}$$

$$\alpha[k] = -\boldsymbol{\omega}^{\mathrm{H}}[k]\boldsymbol{H}\boldsymbol{\mu}_{\mathsf{s}}[k], \tag{14}$$

where $\boldsymbol{\Gamma}_{\mathsf{s}}[k] = \mathbf{diag}\{c_{\mathsf{s}}[k], \dots, c_{\mathsf{s}}[k-\kappa+1], \sigma_{\mathsf{s}}^2[k-\kappa], c_{\mathsf{s}}[k-\kappa-1], \dots, c_{\mathsf{s}}[k-K-L+2]\}$ is a $(K+L-1)\times(K+L-1)$ diagonal matrix composed by the auto-covariances $c_{\mathsf{s}}[i] = \sigma_{\mathsf{s}}^2[i] - |m_{\mathsf{s}}[i]|^2 \in \mathbb{R}_{0,+}$ of the transmitted symbols and the mean $\boldsymbol{\mu}_{\mathsf{s}}[k] = [m_{\mathsf{s}}[k], \dots, m_{\mathsf{s}}[k-\kappa+1], 0, m_{\mathsf{s}}[k-\kappa-1], \dots, m_{\mathsf{s}}[k-K-L+2]]^{\mathrm{T}} \in \mathbb{C}^{K+L-1}$. Both, $\boldsymbol{\Gamma}_{\mathsf{s}}[k]$ and

$\boldsymbol{\mu}_{\mathsf{s}}[k]$ are computed using the *a priori* information $\boldsymbol{l}_{\text{apr}}^{(\text{LD})}(\mathbf{c}')$. In the sequel of this section, we consider QPSK modulation with $\sigma_{\mathsf{s}}^2[k] = 1$ for all $k$, as used in the simulations of Section IV. We define the vector of LLRs $\boldsymbol{l}^{(\text{LD})}(\mathbf{c}') = [l^{(\text{LD})}(\mathsf{c}'_{1,1}), l^{(\text{LD})}(\mathsf{c}'_{1,2}), \ldots, l^{(\text{LD})}(\mathsf{c}'_{S,1}), l^{(\text{LD})}(\mathsf{c}'_{S,2})]^{\text{T}}$ where $\boldsymbol{l}^{(\text{LD})}(\mathbf{c}') \in \mathbb{R}^{2S}$ can be either *a priori* or extrinsic information. It holds [4]

$$m_{\mathsf{s}}[i] = \frac{1}{\sqrt{2}} \left( \tanh \frac{l_{\text{apr}}^{(\text{LD})}(\mathsf{c}'_{i,1})}{2} + \mathrm{j} \tanh \frac{l_{\text{apr}}^{(\text{LD})}(\mathsf{c}'_{i,2})}{2} \right). \quad (15)$$

The relation between the *a priori* information and the mean of symbol sequences using other modulation alphabets can be found in [4]. Finally, the extrinsic information $\boldsymbol{l}_{\text{ext}}^{(\text{LD})}(\mathbf{c}')$ is generated by *soft demapping* using a Gaussian approach. It results for QPSK modulation [4]

$$l_{\text{ext}}^{(\text{LD})}(\mathsf{c}'_{k,1}) + \mathrm{j}\, l_{\text{ext}}^{(\text{LD})}(\mathsf{c}'_{k,2}) = \frac{\sqrt{8}}{1 - \boldsymbol{\omega}^{\text{H}}[k]\boldsymbol{H}\boldsymbol{e}_{\kappa+1}} \hat{\mathsf{s}}[k]. \quad (16)$$

Due to the inversion of the $N \times N$ matrix $\boldsymbol{C}_{\mathsf{y}}$ in Eq. (11), the computational complexity of the filter calculation is of $\mathrm{O}\left(N^3\right)$, $N = RK$, at each time index $k$. This results in computational intense algorithms if the number of antennas $R$, the number of filter taps $K$ to cover the channel of length $L$, and/or the symbol block length $S$ is high. Therefore, in the following, we consider different strategies to reduce the computational complexity of the linear detector.

### A. Reduced-Complexity Implementation

At each time index $k$, the computational complexity can be reduced by one order if we exploit the diagonal structure of $\boldsymbol{\Gamma}_{\mathsf{s}}[k]$ in Eq. (13). First, we partition the auto-covariance matrices at time indices $k$ and $k-1$

$$\boldsymbol{\Gamma}_{\mathsf{y}}[i] = \boldsymbol{H}\boldsymbol{\Gamma}_{\mathsf{s}}[i]\boldsymbol{H}^{\text{H}} + \sigma_{\mathsf{n}}^2 \mathbf{I}_N = \begin{bmatrix} \boldsymbol{D}[i] & \boldsymbol{E}[i] \\ \boldsymbol{E}^{\text{H}}[i] & \boldsymbol{F}[i] \end{bmatrix}, \quad (17)$$

and their inverses

$$\boldsymbol{\Gamma}_{\mathsf{y}}^{-1}[i] = \begin{bmatrix} \boldsymbol{D}_{-1}[i] & \boldsymbol{E}_{-1}[i] \\ \boldsymbol{E}_{-1}^{\text{H}}[i] & \boldsymbol{F}_{-1}[i] \end{bmatrix}, \quad i \in \{k, k-1\}, \quad (18)$$

such that $\boldsymbol{D}[k]$, $\boldsymbol{F}[k-1]$, $\boldsymbol{D}_{-1}[k]$, and $\boldsymbol{F}_{-1}[k-1]$ are $(N-R) \times (N-R)$ matrices, $\boldsymbol{E}[k]$, $\boldsymbol{E}^{\text{H}}[k-1]$, $\boldsymbol{E}_{-1}[k]$, and $\boldsymbol{E}_{-1}^{\text{H}}[k-1]$ are $(N-R) \times R$ matrices, and $\boldsymbol{F}[k]$, $\boldsymbol{D}[k-1]$, $\boldsymbol{F}_{-1}[k]$, and $\boldsymbol{D}_{-1}[k-1]$ are $R \times R$ matrices.

The fundamental idea of the reduced-complexity implementation [4] is to exploit the fact that $\boldsymbol{D}[k] = \boldsymbol{F}[k-1]$ in order to compute $\boldsymbol{\Gamma}_{\mathsf{y}}^{-1}[k]$, viz. $\boldsymbol{F}_{-1}[k]$, $\boldsymbol{E}_{-1}[k]$, and $\boldsymbol{D}_{-1}[k]$, using the block elements of $\boldsymbol{\Gamma}_{\mathsf{y}}[k]$ and the blocks of the already computed inverse $\boldsymbol{\Gamma}_{\mathsf{y}}^{-1}[k-1]$ from the previous time index. Finally, using the inversion lemma for partitioned matrices [13] and the matrix inversion lemma [13] yields the solution

$$\boldsymbol{F}_{-1}[k] = \left(\boldsymbol{F}[k] - \boldsymbol{E}^{\text{H}}[k]\boldsymbol{D}^{-1}[k]\boldsymbol{E}[k]\right)^{-1}, \quad (19)$$

$$\boldsymbol{E}_{-1}[k] = -\boldsymbol{D}^{-1}[k]\boldsymbol{E}[k]\boldsymbol{F}_{-1}[k], \quad (20)$$

$$\boldsymbol{D}_{-1}[k] = \boldsymbol{D}^{-1}[k] - \boldsymbol{E}_{-1}[k]\boldsymbol{E}^{\text{H}}[k]\boldsymbol{D}^{-1}[k], \quad (21)$$

where the inverse of $\boldsymbol{D}[k]$ can be expressed by the *Schur complement* [13] of $\boldsymbol{\Gamma}_{\mathsf{y}}^{-1}[k-1]$, i. e.

$$\boldsymbol{D}^{-1}[k] = \boldsymbol{F}_{-1}[k-1] - \boldsymbol{E}_{-1}^{\text{H}}[k-1]\boldsymbol{D}_{-1}^{-1}[k-1]\boldsymbol{E}_{-1}[k-1].$$

The enormous reduction in computational complexity is detailed investigated in Section III-C. Note that the proposed reduced-complexity algorithm is still optimum in the MMSE sense.

### B. Multi-Stage Wiener Filter (MSWF)

The reduced-rank WF $\boldsymbol{w}_D[k] \in \mathbb{C}^N$ denotes a rank $D$ approximation of the WF $\boldsymbol{w}[k]$ in a subspace spanned by the $D$ columns of the prefilter matrix $\boldsymbol{T}_D[k] \in \mathbb{C}^{N \times D}$, i. e. $\boldsymbol{w}_D[k] = \boldsymbol{T}_D[k]\boldsymbol{g}_D[k]$ where $\boldsymbol{g}_D[k]$ denotes a reduced-dimension WF obtained from the optimization

$$\{\boldsymbol{g}_D[k], a_D[k]\} = \underset{\{\boldsymbol{g},a\}}{\operatorname{argmin}}\, \xi_k\left(\boldsymbol{T}_D[k]\boldsymbol{g}, a\right), \quad (22)$$

with the MSE $\xi_k$ from Eq. (10). The solution easily computes as

$$\boldsymbol{w}_D[k] = \boldsymbol{T}_D[k]\left(\boldsymbol{T}_D^{\text{H}}[k]\boldsymbol{C}_{\mathsf{y}}[k]\boldsymbol{T}_D[k]\right)^{-1}\boldsymbol{T}_D^{\text{H}}[k]\boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k], \quad (23)$$

$$a_D[k] = m_{\mathsf{s}}[k-\kappa] - \boldsymbol{w}_D^{\text{H}}[k]\boldsymbol{m}_{\mathsf{y}}[k]. \quad (24)$$

The MSWF introduced by Goldstein et al. [5] chooses the first column $\boldsymbol{t}_1[k] \in \mathbb{C}^N$ of the prefilter matrix $\boldsymbol{T}_D[k] = [\boldsymbol{t}_1[k], \boldsymbol{t}_2[k], \ldots, \boldsymbol{t}_D[k]]$ as the normalized MF $\boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k]/\left\|\boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k]\right\|_2$. Thus, its output $\mathsf{s}_1[k]$ is maximum correlated to the desired signal $\mathsf{s}[k-\kappa]$. The $i$-th column $\boldsymbol{t}_i[k] \in \mathbb{C}^N$, $i \in \{2, 3, \ldots, D\}$, maximizes the real part of the correlation between its output $\mathsf{s}_i[k]$ and the output $\mathsf{s}_{i-1}[k]$ of the previous prefilter vector $\boldsymbol{t}_{i-1}[k]$, i. e. it fulfills the optimization

$$\begin{aligned} \boldsymbol{t}_i[k] &= \underset{\boldsymbol{t}}{\operatorname{argmax}}\, \mathrm{E}\left\{\mathrm{Re}\left\{\mathsf{s}_i[k]\mathsf{s}_{i-1}^*[k]\right\}\right\} \\ &\text{s. t.:} \quad \boldsymbol{t}^{\text{H}}\boldsymbol{t} = 1 \quad \text{and} \quad \boldsymbol{t}^{\text{H}}\boldsymbol{t}_j[k] = 0, \quad 1 \le j < i, \end{aligned} \quad (25)$$

if we restrict the prefilter vectors $\boldsymbol{t}_i[k]$ to be orthonormal. The solution is identical to the well-known *Lanczos algorithm* [8], [9], [7]

$$\boldsymbol{t}_i[k] = \frac{\boldsymbol{P}_{i-1}[k]\boldsymbol{P}_{i-2}[k]\boldsymbol{C}_{\mathsf{y}}[k]\boldsymbol{t}_{i-1}[k]}{\left\|\boldsymbol{P}_{i-1}[k]\boldsymbol{P}_{i-2}[k]\boldsymbol{C}_{\mathsf{y}}[k]\boldsymbol{t}_{i-1}[k]\right\|_2}, \quad (26)$$

with the projectors $\boldsymbol{P}_j[k] = \mathbf{1}_N - \boldsymbol{t}_j[k]\boldsymbol{t}_j^{\text{H}}[k]$ projecting onto the subspace orthogonal to $\boldsymbol{t}_j[k]$. Note that the auto-covariance matrix of the transformed observation vector $[\mathsf{s}_1[k], \mathsf{s}_2[k], \ldots, \mathsf{s}_D[k]]^{\text{T}}$ is tridiagonal due the Lanczos based prefilter matrix. With the knowledge that the prefilter vectors can be generated using the Lanczos procedure, it was shown [6], [7] that the columns of the prefilter matrix $\boldsymbol{T}_D[k]$ are basis vectors of the $D$-dimensional *Krylov subspace* [9]

$$\begin{aligned} \mathcal{K}^{(D)}\left(\boldsymbol{C}_{\mathsf{y}}[k], \boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k]\right) = \operatorname{span}\{&\boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k], \boldsymbol{C}_{\mathsf{y}}[k]\boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k], \\ &\ldots, \boldsymbol{C}_{\mathsf{y}}^{D-1}[k]\boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k]\}. \end{aligned} \quad (27)$$

Therefore, the MSWF can be seen as an MSE optimal approximation of the WF in the subspace $\mathcal{K}^{(D)}\left(\boldsymbol{C}_{\mathsf{y}}[k], \boldsymbol{c}_{\mathsf{y},\mathsf{s}}[k]\right)$.

Again, the filters $\boldsymbol{\omega}_D[k]$ and $\alpha_D[k]$, finally used in the linear detector of Fig. 1, are obtained from the solution given in Eq. (23) and (24), respectively, by assuming $m_{\mathsf{s}}[k-\kappa] = 0$.

### C. Time-Invariant Approach and Complexity Investigations

To avoid the calculation of the TV filter coefficients at each time index $k$, they can be set *Time-Invariant* (TI) by approximating the TV auto-covariance matrix $\boldsymbol{\Gamma}_{\mathsf{s}}[k]$ by its time average [4]

$$\bar{\boldsymbol{\Gamma}}_{\mathsf{s}} = \frac{1}{S}\sum_{i=1}^{S} \boldsymbol{\Gamma}_{\mathsf{s}}[i] \tag{28}$$

for each symbol block of length $S$. This method can be applied to either the optimal WF implementation or the suboptimal MSWF of Section III-B.

Table I shows the number of real-valued FLOPs needed for the filter computations of one iteration step where only the terms with $N^3$ and $N^2$ are considered. Up to the optimal RC approach, the computational complexity for the computation of the TV filters is $S$ times larger than the one for the TI methods. This is due to the fact that the TI filter coefficients have to be computed only once per each block instead of $S$ times. Thus, the TI approaches achieve an additional tremendous reduction in computational complexity, especially if $S \gg N$.

TABLE I
NUMBER OF FLOPS FOR FILTER COMPUTATIONS

| | TV | TI |
|---|---|---|
| Optimal | $S(\frac{4}{3}N^3 + 9N^2)$ | $\frac{4}{3}N^3 + 9N^2$ |
| Optimal, RC | $4N^3 + 8(S-1)(2R+1)N^2 + 9N^2$ | — |
| MSWF (D=2) | $16SN^2$ | $16N^2$ |
| MSWF (D=1) | $8SN^2$ | $8N^2$ |

The optimal method is based on a *Cholesky factorization* together with a *forward* and *backward recursion* [15] whereas the RC approach applies the ideas of Section III-A. Note that for each block, the latter one needs one inversion of a $N \times N$ matrix whose highest order term is three times larger than the highest order term of solving the corresponding equation system. Nevertheless, this inversion has to be performed only once per block whereas the optimal approach solves the equation system $S$ times at each time index $k$. Thus, its computational complexity can be neglected if $S \gg N$ which results in an order reduction.

Fig. 2 shows the exact number of FLOPs of the proposed TV filters for $R = 4$ receive antennas and a symbol block length of $S = 2048$. Note that only for $N > 45$, the RC approach has less FLOPs than the optimal one due to the high prefactor of the second order term. Besides, Fig. 2 demonstrates the computational efficiency of the suboptimal MSWFs with ranks $D \in \{1, 2\}$ compared to the optimal solutions. The MSWFs with ranks $D > 2$ are not considered here since the simulation results of Section IV will show that the rank $D = 2$ MSWF achieves already near optimal performance in the given turbo scenario.

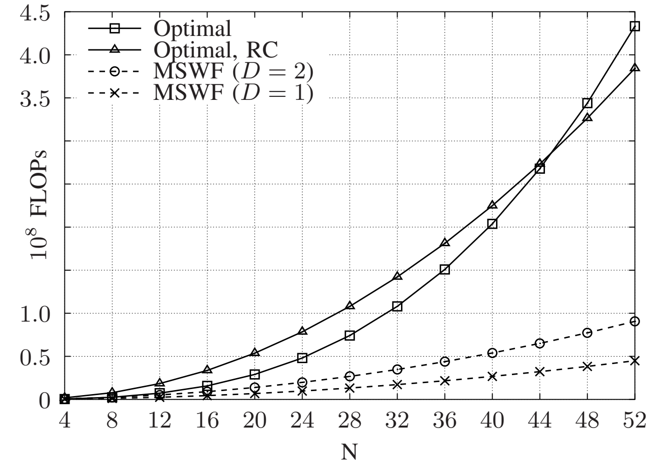


Fig. 2. FLOPs for Computation of TV Filter Coefficients

## IV. SIMULATION RESULTS

The $B = 2048$ bits of each data block $\mathbf{b}$ are encoded with a $(7,5)$-convolutional code, i.e. $r = 0.5$, and random interleaved. Then, the code bits are QPSK modulated and transmitted over the SIMO channel with an impulse response of length $L = 5$, each coefficient taken from a circular complex normal distribution with variance $1/L$. The channel is assumed to be known at the receiver and the noise variance $\sigma_{\mathsf{n}}^2$ is determined according to the receive *Signal to Noise Ratio* (SNR). $R = 4$ antennas at the receiver and a filter length of $K = 7$ for each antenna results in an observation vector $\mathbf{y}[k]$ with dimension $N = 28$. The decoder consists of the BCJR algorithm [16]. The latency time is chosen fixed to $\kappa = 5$.

Fig. 3 and 4 show the BER for TV and TI filter coefficients, respectively, averaged over several channel realizations. In Fig. 3 the lower bound is given by interference free transmission (coded AWGN). The optimal linear equalizer achieves no further improvement after one iteration. The $D = 1$ MSWF solution with one iteration and a significant reduced complexity is only 0.5 dB away from the optimal curve. Recall that the MSWF with $D = 1$ is equal to a normalized MF followed by a scalar WF. Thus, its computational complexity is reduced drastically compared to the full-rank WF. Note that contrary to the proposed rank-one MSWF which is used in any turbo iteration, the MF approaches introduced in [17] or [14] have a higher computational complexity since they are applied in a *hybrid manner*, i.e. the first iteration still applies a full-rank WF whereas only the following iterations are based on MF approximations. The $D = 2$ MSWF solution is not plotted since it has already the same performance as the optimal solution. In Fig. 4, it can be seen that the performance loss by TI filter coefficients is negligible despite of their enormous computational efficiency.

Finally, Fig. 5 depicts a BER comparison of the Krylov subspace based MSWF to eigen subspace based methods. The PC [10] algorithm chooses the columns of the prefilter matrix $\boldsymbol{T}_D[k]$ to be the $D$ principal eigenvectors of the auto-covariance matrix $\boldsymbol{\Gamma}_{\mathsf{y}}[k]$ corresponding to the largest eigenvalues. Goldstein et al. [11] introduced more recently the CS method which selects the $D$ eigenvectors of $\boldsymbol{\Gamma}_{\mathsf{y}}[k]$ achieving the smallest MSE. Thus, it is the optimal eigen subspace

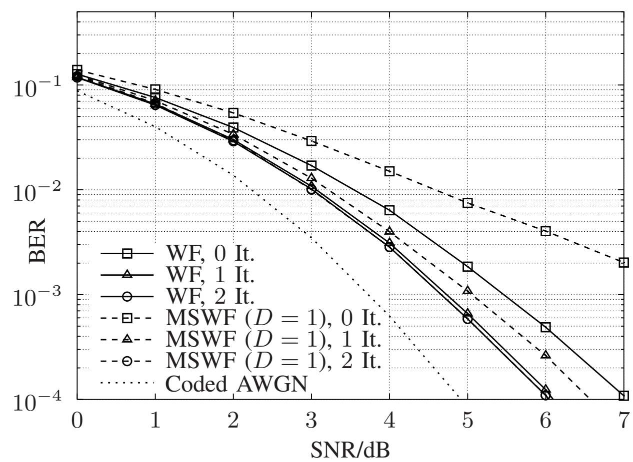


Fig. 3. BER for Time-Variant Filter Coefficients

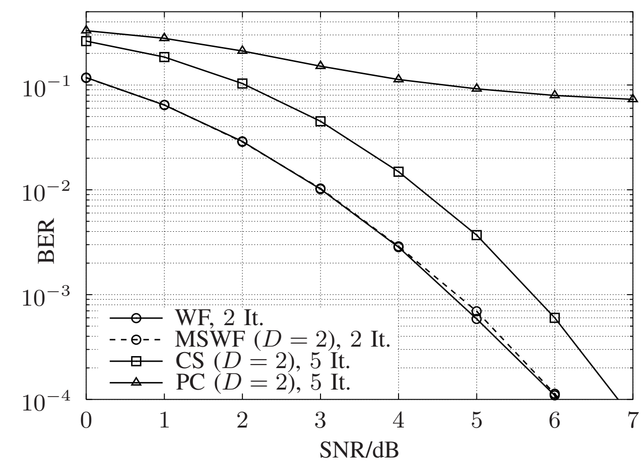


Fig. 5. Comparison to Eigen Subspace Based Methods

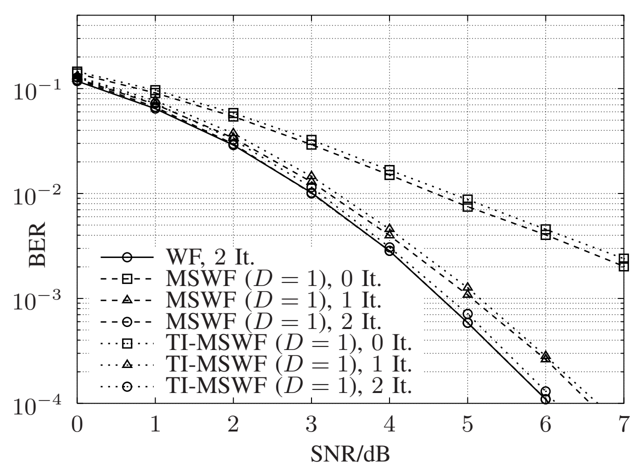


Fig. 4. BER for Time-Invariant Filter Coefficients

based method. Nevertheless, Fig. 5 shows the superiority of the MSWF compared to the PC and CS method although the latter ones were applied with more turbo iterations. Moreover, it can be seen that the MSWF with rank $D = 2$ has the same performance as the full-rank WF.

## V. Conclusions

In this paper, we have applied the reduced-rank MSWF to the linear detector of an iterative receiver. Simulation results of the coded transmission over a SIMO channel have shown that the reduced-rank MSWF achieves near optimal performance despite of a tremendous reduction in computational complexity compared to optimal reduced-complexity implementations. Even if the rank of the MSWF is reduced to one which results in an easy to compute normalized MF followed by a scalar WF, the performance of the turbo receiver is close to optimum.

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