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Field theory of nonlinear gyrofluid models

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Vollständiger Abdruck der von der Fakultät für Physik
der Technischen Universität München
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften (Dr.rer.nat.)
genehmigten Dissertation.

Vorsitzender: Univ.-Prof. Dr. R. Gross

Prüfer der Dissertation: 1. Hon.-Prof. Dr. R. Wilhelm

2. Univ.-Prof. Dr. M. Lindner

Die Dissertation wurde am 10.02.2005 bei der
Technischen Universität München eingereicht und
durch die Fakultät für Physik am 04.05.2005 angenommen.

To my parents,
Xaris and Frederica Strintzi

Abstract

The research presented in this thesis is part of the research area of controlled nuclear fusion, more specifically magnetic confinement fusion. It is motivated by observations in the experiment currently closest to an operating fusion reactor, the tokamak. In tokamaks large transport fluxes of particles and energy arise, which are thought to be the consequence of micro-instabilities, leading to a turbulent state. Although much progress in the understanding of turbulent tokamak transport has been made, the understanding is still incomplete. The gyrofluid model, which is further developed in this thesis, is one of the most commonly used models to describe the turbulent transport. It is a fluid like moment model able to treat finite-Larmor-radius effects to all orders.

Up to now, the gyrofluid equations were derived by taking moments of the Gyrokinetic equation (the kinetic equation in which an average over the fast gyromotion has been performed), using the so called strong drift ordering to partly linearise the final equation of evolution, and closing the system such that only a finite number of moments is retained. This method leads to equations that are homogeneous in the plasma parameters. Furthermore, the associated conservation laws are results of the construction of the equations, and it is therefore not clear how to generalise them.

In this thesis we present a method of deriving the gyrofluid equations that guarantees full non-linearity as well as an exact energy conservation law. Two sets of electrostatic gyrofluid equations are presented, in the first only the perpendicular temperature is kept, while the second retains both the parallel as well as the perpendicular temperature. In both cases a Lagrangian is used to derive the equations of evolution, such that the energy conservation law is automatically satisfied through Noether's theorem. The equations of evolution are calculated from the variation of the action integral with the help of constraints that in both cases represent conservation laws of the system. Two different variational methods are used, one in which the field variables are retained as dynamical variables (Lagrange multipliers method) and another in which the field variables are expressed as functions of only one dynamical variable (virtual displacement method). Both variational methods are equivalent, and yield the same equations of evolution. After the equations are obtained from the variation of the Lagrangian, diamagnetic cancellations are inserted manually in such a way that energy conservation is preserved. Comparison with previous models shows a very good agreement for the lowest order so called zero-Larmor-radius terms in both the sets of the gyrofluid equations. The final set of equations includes the evolution of the three or four first moments, as well as a Polarisation (Poisson) equation that determines the electric field, and an exact energy conservation law.

The new features of the models, namely the full-nonlinearity and the exact energy conservation law are important for understanding tokamak turbulent transport. Fully nonlinear equations are needed to study the highly nonlinear phenomena that occur especially in the edge of the tokamak, and furthermore, turbulent transport phenomena in general cover a spatial range involving significant variations of the background parameters. Moreover, a lack of an energy conservation law exact in all levels of nonlinearity can give rise to

unphysical drive or damping of the large scale fluxes and currents that participate in the turbulence. These considerations will help in the further study of turbulence in tokamaks.

Acknowledgements

I would like to thank many people that made this thesis possible. First of all, I thank my supervisor Bruce Scott for the time he spent with me and his advice in physics and non-physics related issues. His vast knowledge and intuition in physics was and is very helpful for me. Within these three years he has helped me understand and overcome many obstacles in my research, and I am grateful for all his help.

I would also like to thank Prof. Lackner, Prof. Weitzner, Prof. Brizard, Prof. Pfirsch and Dr. Correa-Resrepo for the many discussions on my subject. I thank especially Prof. Lackner for the really much time he spent with me and his advice in the two temperature model, and Dr. Correa-Restrepo for the many papers that he has given me - they have helped a lot. I also would like to thank Prof. Guenter and Prof. Wilhelm for reading my thesis and making useful corrections.

I would like to thank my professors in Greece who have helped me and inspired me through all the years of my physics studies. I thank Prof. Ioannou for everything that he has done for me, for the many discussions, and for giving me the possibility to travel abroad and see many aspects of physics. I also thank Prof. Moussas for the many hours and discussions in his office. Finally, I thank Dr. Tsois for interesting discussions in experimental plasma physics and also for introducing me at IPP.

Many other people have made my staying easy here. I thank all my colleagues in IPP, especially Philipp Lauber for correcting my German and discussing gyrokinetics with me, Frank Jenko for the discussions in gyrofluid and gyrokinetics and David Coster for solving all the computer problems I had and for the vast amount of his coffee that I have drunk these three years.

I would also like to thank my parents Xaris and Freerica Strintzi for their support and understanding, and also my friends, here and in Greece, for making my life more interesting and alive. Finally, I thank Arthur for being tolerant all these years and for making me very happy.

Contents

Chapter 1

Introduction

The research presented in this thesis is part of the larger research area of controlled nuclear fusion. This area of research has a practical application in a future energy source that has several advantages compared with the current sources of energy. Once achieved, nuclear fusion could provide the world with energy for a long time since there is no lack of fuel, also it would be CO_2 neutral, and would not produce large amounts of long lifetime nuclear waste [1]. However, the road to an operating fusion reactor is still long, partly due to the lack of the understanding of basic physics phenomena.

Nuclear fusion is more difficult to obtain than fission due to the large repulsive force between the nuclei. Only if the temperature (thermal velocity) of the fuel is large enough, the Coulomb barrier can be overcome in a collision between the nuclei and a fusion reaction can occur. The most favourable reaction in terms of minimum required thermal energy is that between Deuterium and Tritium generating an alpha particle and a neutron as well as releasing 14 MeV of energy

$$\mathrm{D} + \mathrm{T} \rightarrow {}^{4}\mathrm{He} + \mathrm{n} + 17.5\ \mathrm{MeV}. \tag{1.1}$$

Even for this reaction, though, the maximum fusion rate is obtained at energies around 10 keV, at which the nuclei are fully ionised, i.e. form a plasma. Since the thermal velocity of the ions is around 10^6 m/s at these energies they can move over the size of the device in a very short timescale, and a natural problem that occurs is that of their confinement for a sufficient long time such that enough fusion reactions have occurred. One can either use a very dense plasma for very short times, as is done in inertial fusion [2] , or confine the particles through an imposed magnetic field as is done in magnetic confinement fusion [3, 4]. This thesis deals with the latter area of research.

1.1 The tokamak

The concept closest to an operating reactor is the tokamak sketched in figure 1.1. Here we will outline the concept of a tokamak; a detailed description can be found in many

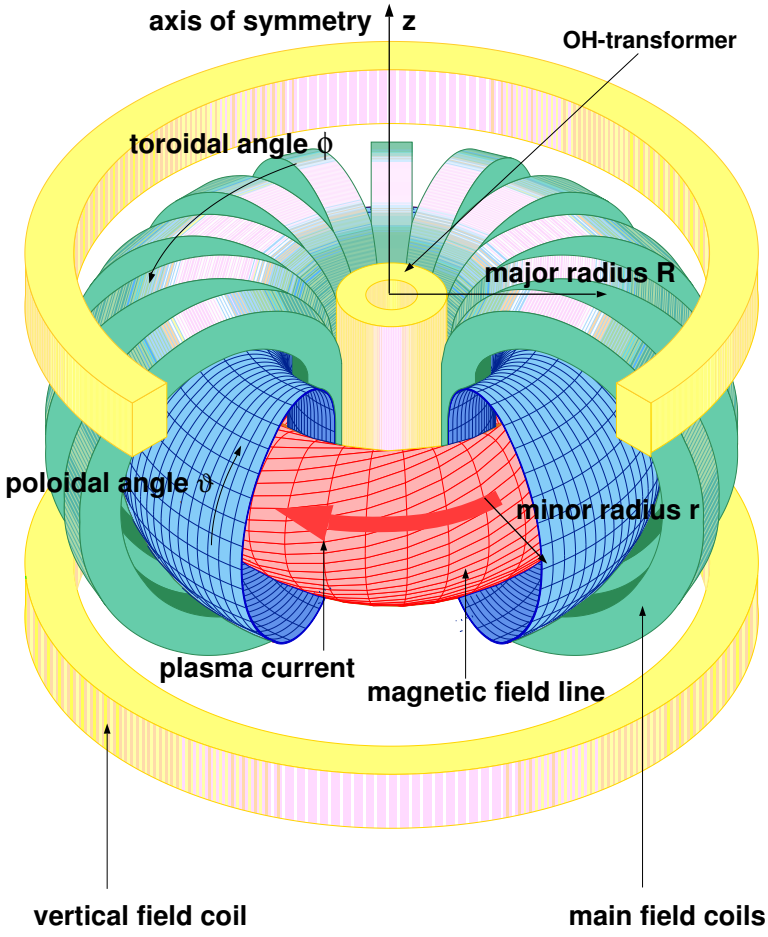

Figure 1.1: Schematic picture of the tokamak

textbooks, for example in [5]. In this device a toroidal magnetic field is imposed from the outside using the toroidal magnetic field coils. In an homogeneous magnetic field the charged particles move on spiral orbits, their motion perpendicular to the field being limited by the Lorentz force leading to a gyration around the field lines with a typical radius known as the Larmor radius, whereas the motion parallel to the field is force free. The magnetic field in the tokamak is strong (several Tesla) in the sense that the Larmor radius is much smaller than the size of the device. In a lowest order approximation, therefore, the motion is that of the particles in a homogeneous field determined by the local magnetic field at the particles position. The Lorentz force prevents the particles to move over large distances, whereas the topology of the field is such that the free motion of the particles along the field lines does not lead to their loss from the device. This picture suggests perfect particle confinement. Typical sizes for modern tokamaks are about one meter for the minor radius, a few meters for the major radius and about one millimetre for the ion Larmor radius.

The description in the above paragraph, however, is incomplete in the sense that higher

order effects cannot be neglected. Although small, the finite Larmor radius allows the particles to feel the inhomogeneity of the magnetic field. The latter is a direct consequence of the toroidal curvature. Because of this the gyration orbits are not perfectly closed, and the particles slowly drift away from the field lines. One can think of these drifts as being the consequence of a perpendicular (to the magnetic field) force acting on the particles, which is compensated by the Lorentz force leading to a velocity, the drift velocity. For stationary conditions these forces are due to the field inhomogeneity ($F = -\mu\nabla B$ with μ being the magnetic moment and B the magnetic field strength) the centrifugal force due to the motion along the toroidally curved field, and the possible perpendicular electric field ($\mathbf{E}$). Approximately the velocity ($\mathbf{v}$) of the centre of gyration can for the tokamak be written in the form

$$\mathbf{v} = v_{\|}\mathbf{b} + \frac{v_{\|}^2 + v_{\perp}^2/2}{\omega_c R}\mathbf{e}_z + \frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{1.2}$$

where $\mathbf{b}$ is the unit vector in the direction of the magnetic field and the z-axis is in the direction of the axis of symmetry, ω_c is the gyration frequency, R the distance to the axis of symmetry which is a measure of the toroidal curvature, and the indices parallel and perpendicular refer to the direction with respect to the magnetic field. The term due to $v_{\|}^2$ gives the curvature drift due to the magnetic field curvature $\mathbf{b}\cdot\nabla\mathbf{b}$. In a simple torus both $\mathbf{b}\cdot\nabla\mathbf{b}$ and $\nabla\log B$ are approximated by $-\nabla R/R$. Later in this thesis more accurate formulae are given, but for the present discussion the equation given above is sufficient.

The gyration frequency $\omega_c = eB/mc$, depends on the charge e and mass m of the particles. Therefore the vertical drift, the second term in equation (1.2), is in the opposite direction for the electrons and ions, and leads to a charge separation resulting in a vertical electric field. The latter will generate a drift radially outward through the last term of equation (1.2) leading to a rapid loss of the plasma. This is the reason why a tokamak cannot operate with only a toroidal field. A toroidal current is induced in the plasma using a transformer coil sketched in figure 1.1. This current generates a poloidal magnetic field, which makes that the field lines are helically wound mapping out magnetic flux surfaces. The helical field lines then connect top and bottom and short-circuit the electric field, preventing it from being formed and, therefore, preventing the outward motion of the plasma and the loss of confinement. A bulk toroidal hoop force in the outward direction of the major radius still exists, but is balanced by a vertical field (cf. vertical field coils in figure 1.1). Theoretically, it is sufficient to take the location of the outermost magnetic flux surface as given, and investigate the dynamics within. One can show that in a toroidally symmetric field, in which the poloidal magnetic field is of sufficient strength the particle orbits are perfectly confined inside the device. The poloidal magnetic field therefore provides the confinement through (partially) eliminating the effect of the drift. We will see however, that the drifts can play an important role in plasma instabilities leading to a turbulent state and radial transport of particles and energy.

In the discussion above the concept of charge separation has been used. Although, it did lead to an easy to understand physical picture of the dynamics, it is oversimplified. One of the main properties of a plasma is namely that of quasineutrality. For almost all

the phenomena of interest one can assume the electron and ion density to be equal. Only at very small length scales (below the so called Debye length) or high frequencies (above the plasma frequency) does charge separation play a role. These concepts (charge separation and quasineutrality) are reconciled by a phenomenon known as polarisation, by which corrections from the ion inertia in a time-dependent electric field effectively neutralise the total charge by maintaining a divergence-free total current. Polarisation will be treated in detail within the models discussed in this thesis.

To obtain fusion a sufficiently dense plasma of sufficiently high temperature (T) must be confined for a sufficiently long time for a large amount of fusion reactions to occur. The confinement is usually measured in the so called confinement time (τ_E) which is the ratio of the total stored energy and the power used to heat the plasma. The most favourable state to operate a reactor is when this heating is provided by the fusion products themselves. In the reaction (1.1) the heating must come from the alpha particles, since the neutrons are not confined by the magnetic field. The energy carried by the neutrons is then used to generate electricity. The state in which the fusion reactions are able to sustain the temperature without any external heating is referred to as an ignited plasma, and this condition is reached when $nT\tau_E > 5 \cdot 10^{21}$ keV s m^{-3}, where n is the particle density. A plasma that produces as much energy through fusion reactions as is being used from the outside to heat it is said to have reached breakeven. It is the latter state that can be reached in the largest tokamak experiments today, and the next step is projected to generate ten times more energy through fusion than is used from the outside for heating.

From the discussion above it is clear that confinement is a key issue in obtaining a burning reactor. Single particle orbits are perfectly confined in a tokamak, but two effects can make that the particles are lost from the device. Collisions between the charged particles can scatter them out of there orbits. The transport fluxes that are due to these processes are described by neo-classical theory. The prefix "neo" is used here to stress that the fluxes depend critically on the magnetic field topology exceeding the fluxes in an homogeneous magnetic field by one or two orders of magnitude. Nevertheless, the observed transport fluxes largely exceed the neo-classical transport and are referred to as anomalous. This anomalous transport is thought to be the consequence of micro-instabilities leading to a turbulent state. The understanding of these instabilities and the nonlinear dynamics is of vital importance for the understanding of the energy confinement, which in turn is a key element in the operation of a burning reactor.

1.2 Turbulent transport

The tokamak equilibrium, discussed in the previous section, is not necessarily stable. Several sources of free energy exist: the density and temperature gradients of the components as well as the current, which can lead to instabilities [6]. These are often split into large and small scale instabilities, due to the difference in theoretical description they require. Large scale instabilities can often be described within the framework of Magneto-Hydro-

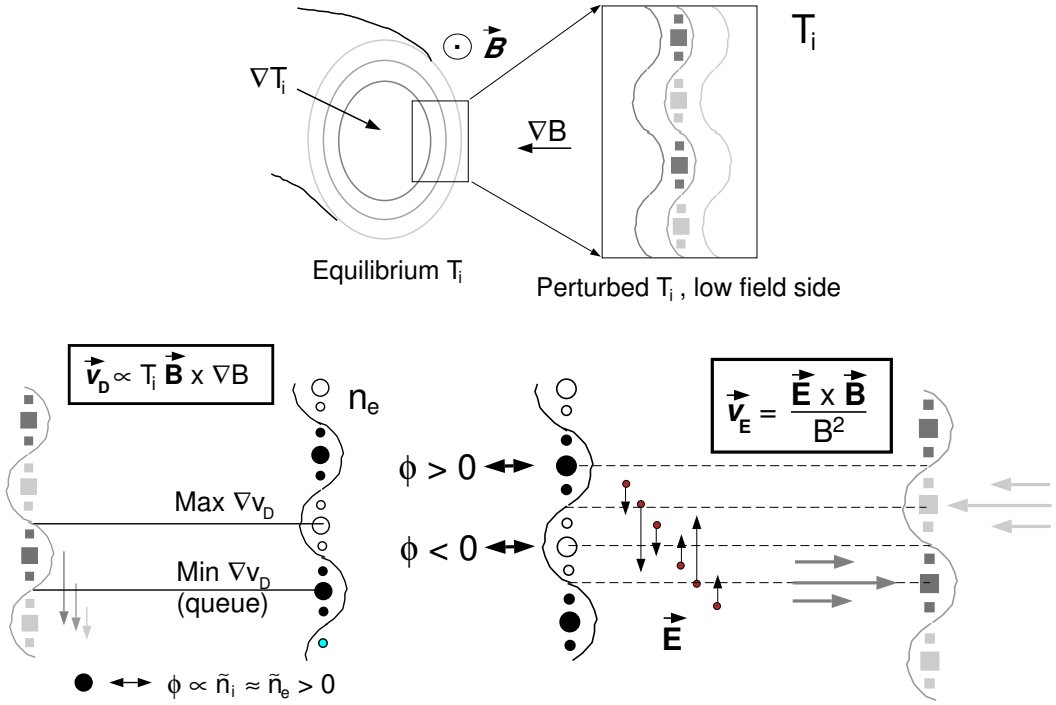

Figure 1.2: Schematic picture of the ITG instability.

Dynamics (MHD) which is essentially a one fluid theory, whereas smaller scale (known as micro-instabilities) require two (or more) fluid theories as well as careful treatment of the effects due to the finite Larmor radius (in general, temperature and parallel velocity fluctuations behave differently for the electrons, due to the smaller mass). Large scale instabilities are often (but not always) fatal for the plasma, leading to a rapid loss of confinement. They limit the operational space in terms of a maximum pressure that can be confined, and a maximum current that can be generated inside the plasma, both at a given toroidal magnetic field strength.

Micro-instabilities on the other hand have (perpendicular) length scales of the order of the Larmor radius. They are not fatal, although they do lead to turbulence and hence to a reduction of confinement. Because of there small scale they are often not directly observed (although they can be with special diagnostics) but lead to a radial flux of particles and energy as a function of the density and temperature gradients that drive them. It is commonly believed that they are the cause of the anomalous transport, and that they can hardly be avoided. That is, in the current reactor designs one expects the confinement time, of both energy as well as particles, to be set by these instabilities [7]. Some insight in the physics effects that has to be kept in order to study these instabilities can be obtained by discussing the physical picture of one of them, the toroidal ion temperature gradient mode (ITG). Consider an ion temperature perturbation in a magnetic surface in the form of a Fourier mode, as sketched in figures 1.2. It can be seen from equation (1.2) that the vertical drift velocity is proportional to the energy of the particles. It is, therefore, larger in the hot regions

compared to the cold regions. Associated with the temperature perturbation is a modulated velocity pattern that compresses the ions on the surface. The result is an ion density perturbation that is 90 degrees out of phase with the temperature perturbation. The compression of ions inside the magnetic surface then generates an electric field perturbation. The latter is a consequence of quasi-neutrality and the electron parallel force balance. Because of quasi-neutrality the electron and ion densities must be equal. The motion of the electrons over the surface, however, is much faster than the turbulent timescales, resulting in a force balance solution between the electron pressure perturbations due to the density fluctuations $T_e\nabla_{\|}n_e$ and the electric field $en\nabla_{\|}\phi$. The balance of these forces makes that the potential and the density perturbations are directly related (this response is known as the adiabatic electron approximation). The potential fluctuations lead to an $E \times B$ drift (last term of (1.2) that moves plasma from inside into the warm regions and from outside into the cold regions. On the outboard side of the magnetic surface, and for normal conditions in which the temperature gradient points inward, warm plasma is moved into the warm regions and cold plasma into the cold regions, therefore, enhancing the initial perturbation and causing an instability. On the inboard side cold plasma is moved into the warm regions and the mode is stable. This makes that these instabilities balloon on the outboard side of the flux surface. Inside and outside of the magnetic surface, however, are connected through the helical structure of the magnetic field. The drive of the mode has to be large enough to overcome the equilibrating along the magnetic field line. Since both the compression of the ions on the flux surface as well as the $E \times B$ velocity involve a spatial derivative along the magnetic surface the growth rate of the mode increases as the wave vector of the perturbation increases. However, for wave lengths approaching the Larmor radius, the finite size of the particle orbits makes that the particles start averaging the electric field potential leading to a reduction in the $E \times B$ velocity. This effect is known as a finite Larmor radius effect, and it reduces the growth rate of the mode. The maximum growth rate is therefore often obtained at $k\rho_i \approx 0.3$, where k is the wave vector and ρ_i is the Larmor radius. The small scale of the order of the Larmor radius has led to the terminology micro-instabilities. Turbulence dependent on the ITG physical mechanism has been found to describe the ion heat transport in the tokamak core reasonably well.

The description above shows that it is important to keep two fluid effects while the electrons are in parallel force balance, describe the finite ion Larmor radius effects accurately and, in a moment approach, keep at least the evolution of the temperature besides density and fluid velocity. The ITG is not the only instability that can occur in the plasma. Of the many others that can occur it is worth here to mention the drift wave. Different from the ITG which does not need any dissipation to be unstable, and which is therefore referred to as reactive, the drift wave is driven unstable due to the friction between electrons and ions breaking the adiabatic force balance discussed above. It also does not rely on the drift velocity due to the magnetic field inhomogeneity (although its character is altered by it), and therefore, can also appear in an homogeneous magnetic field. Opposite to the ITG, the electrons mostly provide the dynamics for the drift wave, and the ions the inertia. Because the growth rate of this instability is proportional to the friction, it is more prominent in

the edge of the plasma which is more collisional, whereas the ITG dominates more in the core. Both are fundamentally two-fluid mechanisms, and hence cannot be treated with a conventional MHD (magneto-hydro-dynamics) model.

The micro instabilities are not fatal to plasma operation and are always present in the normal plasma discharges. Obviously they are in a saturated state, and it is this state that determines the fluxes of particles and energy. The saturated state is a turbulent state with many modes (different wave numbers, stable and unstable) present at the same time. Only nonlinear simulations can accurately determine the fluxes.

1.3 This thesis

From the previous sections it is clear that non-linear simulations of the turbulent state are essential for the understanding of the confinement properties of a tokamak. Although much progress towards the prediction of turbulent tokamak transport [8, 9, 10] has been made, there are still many transport phenomena which are poorly understood. Different models can be employed for this purpose. A full kinetic description is the most accurate, and is indeed used in several codes. However such a model is a five dimensional problem (2 dimensions in velocity space since one can average over the fast gyro-motion, and 3 dimensions in configuration space) that has to be integrated in time, and is, therefore, very resource intensive in computations. Many physics effects can be well captured in fluid equations that can be obtained through momentum building of the kinetic equation. The gyrofluid model [11, 12, 13, 14, 15, 16] is one of the most commonly-used. It is much less computationally intensive than the analogous gyrokinetic [17, 19, 18] models, and although it is less accurate than the gyrokinetic one [20], experience shows it to retain most of the qualitative properties necessary for understanding turbulent transport [12, 13, 14, 15, 16].

Up to now, tokamak microturbulence has been treated using strong drift ordering, also called gyrokinetic ordering: because all gradient lengths (density, temperature) are assumed large compared to the Larmor radius, nonlinearities are discarded except in the processes of advection by the ExB velocity and the contributions to the dynamics parallel to the magnetic field due to field line disturbances, which are kept because the gradients across the magnetic field of the background and the disturbances are taken to be of similar size [21]. This yields equations which are homogeneous in their parameters, even though variations in the background profiles are treated self consistently. For example, parameters such as the ion gyroradius normalised to the plasma radius and the plasma beta (pressure energy over magnetic field energy) are treated as constants even though the density and temperature of each species is a dependent variable. Both electrostatic (treating perturbations of only the electric field) [12, 13, 14] and electromagnetic (treating perturbations both of the electric and the magnetic field) [11, 15, 16] gyrofluid models in the literature depend on this partially linearised derivation. They start by taking moments of the gyrokinetic equation under the ordering and close the hierarchy using approximate forms for higher moments not retained in the model. These approximations include a gyroaveraging procedure which

allows finite Larmor radius (FLR) effects to be treated to arbitrary order (cf. also [22]). The focus on the linear part of the dynamics is underscored by the lack of an energy theorem for the resulting equations. In computational work, nonlinear evolution subsequent to a linear growth phase often remains undiagnosed, merely assumed to follow the same physics as the linear instability itself.

At any level of nonlinearity in the equations, however, an energy conservation law is crucial for the various processes, which deal not only with the instability mechanism itself but also several conservative transfer pathways all of which can influence the overall statistical equilibrium of the turbulence. That is, although in the linear theory small errors in the conservation laws lead merely to small errors in the growth rate, in the non-linear theories any error in the conservation laws is a possible source of free energy. Thus in turbulence, an inexact energy conservation law can give rise to unphysical drive or damping of the quantities examined, which can be especially damaging when the conservative interaction of turbulence and large scale flows is considered [23, 24]. This last scale flows or zonal flows play an important role in controlling the turbulent fluctuation level. To describe magnetised plasma turbulence, a consistently derived model which includes an exact energy conservation law is desired.

Although the underlying gyrokinetic equation conserves energy exactly [18], existing gyrofluid models usually do not regard an energy theorem at all (cf. however [25]). Under drift ordering, conservation of a quadratic quantity related to energy (commonly called "free energy" or "entropy") should still be maintained, as is done in the more conventional two fluid treatments [26, 27, 28, 29]. It can be shown that the best developed model [14] maintains free energy conservation at zeroth order in the finite Larmor radius (FLR), but breaks it in the FLR terms due to the different techniques it uses in forming the FLR corrections to the moment equations and to the field potential equations which provide self consistency [30]. Beyond this, for such a robustly nonlinear situation as edge turbulence, which is not necessarily dependent upon linear microinstabilities [28, 31], it is desired to move beyond these local treatments and form a gyrofluid model which conserves global energy as well, as was recently done for the two fluid drift equations including polarisation currents [32]. While it is simple to set up the moment equations themselves in terms of total divergences [12, 14], thereby conserving thermal and kinetic energy, self consistent incorporation of the "drift energy" due to the electrostatic potential is more subtle. This step is required to put gyrofluid theory on the same level as gyrokinetic theory.

The first attempt of deriving a gyrofluid model that conserves global energy, was one neglecting FLR effects more or less based upon a magnetohydrodynamic (MHD) ordering in the sense that its set of assumptions were consistent with the ion pressure gradient being ordered small compared to the force density due to the electric field [33]. By contrast, the usual situation for gradient driven turbulence, for which the adiabatic response of the electrons is always important [26, 31], finds the perpendicular electric field and ion pressure gradient at comparable levels for the regime of comparable ion and electron temperatures [35]. In this first attempt, an equation for the fluid velocity is obtained from the variation of a system Lagrangian using constrained variations of a fluid spatial displacement field. The

energy conservation law then follows consistently from Noether's theorem [34]. The Lagrangian is not systematically derived, but constructed from the one for particles under the drift-kinetic ordering (all scale lengths are large compared to the Larmor radius). It includes the polarisation drift current but not the corresponding FLR corrections. Specifically, the "diamagnetic vorticity" (vorticity of the diamagnetic velocity, a fluid velocity perpendicular to the magnetic field that arrises from the balance of the perpendicular pressure gradient and the Lorentz force) is lost, and the model is restricted to the MHD limit. Another feature of the fluid models, the so called diamagnetic cancellation, is also not taken into account. In the fluid models advection terms due to the diamagnetic velocity are cancelled naturally by other terms in the fluid equations. This is because, although the diamagnetic velocity produces a current, there is no actual motion of the guiding centers of the particles, such that there is no advection of particles or energy. In this model advection by the diamagnetic velocity appears in the final equations, an incompatibility with pure fluid models in which these cancellations are made automatically [36, 37]. So we have to extend the model to capture these nominally FLR effects which are nevertheless important at large scale.

In this thesis two different models of gyrofluid equations are derived in a systematic way using methods from field theory already known from studies of fluid dynamics or MHD [38]. Rather than building moments of the kinetic equations, a Lagrangian is used here to derive the equations. This approach can guarantee energy conservation through the Noether theorem [34]. In this approach several technical problems appear, that will be pointed out, and solved to obtain a consistent set of equations. Among them, diamagnetic cancellations and diamagnetic vorticity are treated in detail. Furthermore, the equations derived do not only conserve energy but also are fully nonlocal and non-linear (i.e. no linearisation is applied anywhere in contrast to the previous derivations of the theory).

The thesis is structured as follows. Chapter 2 deals with Lagrangian field theory, and outlines the necessary tools to deal with the problem such as the Lagrangian multipliers and the virtual displacement. Chapter 3 then treats the gyro-kinetic and gyro-fluid theory. It reviews the state of gyro-fluid theory before the work presented in this thesis. Chapter 4 addresses a set of gyro-fluid equations that has all the properties of energy conservation, non-linearity and non-locality, but assumes an one temperature model dealing only with the thermal motion in the component perpendicular to the magnetic field (the use of only one temperature simplifies greatly the constraints that have to be imposed on the Lagrangian). Although not complete in the sense that it neglects the thermal motion along the field lines, it forms a system in which energy transfer mechanisms can be investigated. Chapter 5 then deals with the two temperature system keeping both the perpendicular as well as the parallel temperature. The constraints that have to be used in this case are much more complex, and a different method, the virtual displacement instead of the Lagrangian multipliers, is used to derive the equations of motion. Finally the last chapter gives a summary, conclusion, and outlook of the work presented.

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Chapter 2

Lagrangian field theory

In classical mechanics there exist a large amount of tools for dealing with dynamical systems. The purpose of this chapter is not to analyse in detail all these tools, but rather to describe some of the methods that will be used later in dealing with gyrokinetic and gyrofluid equations. Since a Lagrangian will be used in the formulation of these equations, this chapter will introduce some methods in describing both discrete as well as continuous systems and fields with the use of a Lagrangian formulation. Section 2.1 treats the Lagrangian of a discrete system. The Euler-Lagrange equations are derived and the Lie perturbation method is introduced. In section 2.2 the Lagrangian formulation of a continuous system and fields, for field variables independent of each other, is described, and the energy conservation law is derived from Noether's theorem. Finally, section 2.3 describes the Lagrangian formulation of a continuous system in which the field variables are dependent on each other. Two methods for solving such systems are presented through the derivation of the evolution equations of an ideal charged fluid together with the equations for the electromagnetic fields.

2.1 Lagrangian methods of discrete systems

Here, some properties of the Lagrangian of a discrete system are summarised. These can be found in any good text book on classical mechanics [1, 2, 3], as well as in many papers dealing with gyrokinetics [4, 5, 6, 7]. In general, the state of any system with N degrees of freedom requires the specification of $2N$ variables, which in Lagrangian mechanics are taken to be the generalised coordinates $\mathbf{q}$ and their time derivatives $\dot{\mathbf{q}}$. The equations of motion are obtained from the Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}})$ which is defined as the difference between the kinetic (E_{kin}) and potential (V) energy of the system

$$L = E_{kin} - V. \tag{2.1}$$

In Hamiltonian mechanics, on the other hand, the $2N$ variables describing the state of a

system are taken to be the generalised coordinates $\mathbf{q}$ and their momenta $\mathbf{p}$, where

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} \tag{2.2}$$

The evolution of the system is obtained from the Hamiltonian $H(\mathbf{q},\mathbf{p})$ which is connected with the Lagrangian through the Legendre transformation. For canonical coordinates this relation is

$$H = \mathbf{p}\cdot\dot{\mathbf{q}} - L, \tag{2.3}$$

and the evolution of the system are given by Hamilton's equations:

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}. \tag{2.4}$$

The Lagrangian can then (again for canonical coordinates) be written in the form

$$L = \mathbf{q}\cdot\dot{\mathbf{q}} - H_c(\mathbf{q},\mathbf{p},t), \tag{2.5}$$

where H_c is the canonical Hamiltonian, and $(\mathbf{q},\mathbf{p})$ the canonical coordinates.

For an arbitrary coordinate system $\mathbf{z} = \mathbf{z}(\mathbf{q},\mathbf{p},t)$ the Lagrangian transforms to

$$L = \gamma_i \dot{z}_i - H(\mathbf{z},t), \tag{2.6}$$

where

$$\gamma_i = \mathbf{p}\cdot\frac{\partial \mathbf{q}}{\partial z^i}, \; H = H_c - \mathbf{p}\cdot\frac{\partial \mathbf{q}}{\partial t}. \tag{2.7}$$

A more convenient form for the Lagrangian is then the notation of the fundamental one-form of Poincare-Cartan

$$\gamma = \gamma_\mu dz^\mu = \gamma_i dz^i - H\,dt \tag{2.8}$$

where $z^0 = t, \gamma_0 = -H$. The fundamental one-form has the advantage that it does not change its structure under a coordinate transformation $Z^\mu = Z^\mu(\mathbf{z},t)$:

$$\gamma = \gamma_\mu dz^\mu = \Gamma_\mu dZ^\mu, \; \Gamma_\mu = \gamma_\sigma \frac{\partial z^\sigma}{\partial Z^\mu} \tag{2.9}$$

With the Lagrangian given, the equations of motion can be derived from the variation of the action integral

$$S = \int_{t_1}^{t_2} L\,dt. \tag{2.10}$$

yielding the Euler-Lagrange equations

$$\hat{\omega}_{ij}\frac{dz^j}{dt} = \frac{\partial H}{\partial z^\mu} + \frac{\partial \gamma_i}{\partial t}, \tag{2.11}$$

where

$$\hat{\omega}_{ij} = \frac{\partial \gamma_j}{\partial z^i} - \frac{\partial \gamma_i}{\partial z^j} \tag{2.12}$$

are the components of the Lagrange tensor also called "'symplectic structure'. The Euler-Lagrange equations are invariant under any gauge transformation $\gamma \to \gamma + dS$, where S is any scalar function of the extended phase-space.

The equations above show the well known property that, once the Lagrangian is given, the equations of motion can be derived from it. Furthermore, properties of the Lagrangian are translated in properties of the dynamical system. Every symmetry of the Lagrangian, for instance, leads to a conservation law through Noether's theorem. But the Lagrangian can also be manipulated to obtain approximate equations in which a fast periodic motion is eliminated. This is the object of gyro-kinetics in which the fast gyration of the particles around the field line is eliminated, with the equations describing the motion of the centre of gyration. Below the procedure through which the new Lagrangian is derived is briefly outlined. Further reading on this topic can be found in the references [4, 5, 6, 7].

Consider a fundamental one-form that can be written in two separate parts, one equilibrium and one perturbed part:

$$\gamma = \gamma_0 + \varepsilon\gamma_1, \tag{2.13}$$

where ε is a small parameter. Moreover, consider that the first order fundamental one-form depends on a coordinate θ, which one wants to eliminate from the Euler-Lagrange equations. This is done using the following steps:
(1) considering only the equilibrium part of the fundamental one-form, one finds coordinates such that they include the coordinate θ and its canonical momentum, and transforms the Lagrangian in the new coordinates.
(2) Then the equilibrium and perturbed part are considered together, and another transformation is found that removes the coordinate θ from the perturbed part. This is done efficiently by the Lie transformation method.

A Lie coordinate transform can be written as:

$$Z^\mu = T z^\mu, \tag{2.14}$$

where T is in general a series of individual Lie transforms $T = ...T_3T_2T_1$, the nth- order transformation T_n eliminating the coordinate θ dependence in the n-th order component of the symplectic structure and Hamiltonian. An individual Lie transform T_n can be written as

$$T_n = \exp(\varepsilon^n L_n). \tag{2.15}$$

Scalars transform as $S = T^{-1}s$, and the Lie derivative L_n acts on scalars as

$$L_n f = g_n^\mu \frac{\partial f}{\partial z^\mu}, \quad g_n^\mu = \frac{\partial Z^\mu}{\partial \varepsilon^n}, \tag{2.16}$$

where g_n^μ is the generator of the Lie transformation T_n. The one-form transforms as

$$\Gamma = T^{-1}\gamma + dS, \quad \Gamma = \gamma_\mu dZ^\mu, \tag{2.17}$$

where S is a gauge transformation, and the Lie derivative L_n acts on one-forms as

$$(l_n\gamma)_\mu = g_n^\sigma \left(\frac{\partial \gamma_\mu}{\partial z^\sigma} - \frac{\partial \gamma_\sigma}{\partial z^\mu} \right). \tag{2.18}$$

Under the Lie transformation, the fundamental one-form to various orders in ε is given by

$$\Gamma_0 = \gamma_0 + dS_0, \tag{2.19}$$

$$\Gamma_1 = \gamma_1 - L_1\gamma_0 + dS_1, \tag{2.20}$$

$$\Gamma_2 = \gamma_2 - L_1\gamma_1 + \left(\frac{1}{2}L_1^2 - L_2\right) + dS_2. \tag{2.21}$$

The generators g_n^σ as well as the gauge functions dS_n can be chosen arbitrarily such that the final one-form and thus the final equations of motion do not depend on θ. The elimination of the coordinate θ is then done by the following rules: First the generators g_n^σ are chosen such that all explicitly time dependent parts appear only in the perturbed Hamiltonian, thereby eliminating the explicit time derivative from equation (2.11). Secondly, the gauge functions dS_n are chosen such that the θ dependence is removed from the transformed one-form. The Lie transform eliminates the coordinate θ from the Lagrangian without changing the symplectic structure or the form of the Lagrange tensor.

2.2 Lagrangian methods for continuous systems

The transition from a discrete to a continuous system is achieved by introducing the Lagrangian density ($\mathcal{L}$) as well as the field variables (η) of the system. The action integral of a continuous system then contains an integral over configuration space

$$S = \int dt \int d^3x\, \mathcal{L}[\eta_\alpha, \dot{\eta}_\alpha, \nabla\eta_\alpha, \mathbf{x}, t] \tag{2.22}$$

such that the Lagrangian of the system is given by the integral over the Lagrangian density

$$L[(\eta_\alpha), (\dot{\eta}_\alpha)] = \int d\mathbf{x}\, \mathcal{L}_\alpha[\eta_\alpha, \dot{\eta}_\alpha, \nabla\eta_\alpha, \mathbf{x}, t]. \tag{2.23}$$

The field variables $\eta_\alpha = \eta_\alpha(\mathbf{x}, t)$ specify the state of the system (e.g. density, pressure, velocity, electric and magnetic potential etc.) and α is a label that denotes a member of the set of fields. The Lagrangian density $\mathcal{L}$ depends on the field variables, their first derivatives towards space and time, and can also depend explicitely on space and time. For closed systems the Lagrangian density does not depend explicitly on space or time, similar to the situation of a closed discrete system where the Lagrangian does not depend explicitely on time. From now on we will investigate only closed systems, for which the Lagrangian can be written as

$$L[(\eta_\alpha), (\dot{\eta}_\alpha)] = \int d\mathbf{x}\, \mathcal{L}_\alpha[\eta_\alpha, \dot{\eta}_\alpha, \nabla\eta_\alpha]. \tag{2.24}$$

In the case of field variables η_α that are independent, the evolution equations of the system resemble the Euler-Lagrange equations of a discrete system, and can be found through the variation of the action integral

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\eta}_\alpha}\right) + \nabla\left(\frac{\partial\mathcal{L}}{\partial\nabla\eta_\alpha}\right) - \frac{\partial\mathcal{L}}{\partial\eta_\alpha} = 0. \tag{2.25}$$

The Euler-Lagrange equations are then as many as there are independent fields.

As mentioned before, the Lagrangian formulation provides the means of finding conservation laws that the system obeys, such as energy, momentum, or angular momentum conservation, through Noether's theorem. Conservation laws are related to symmetries. For every Lagrangian system, momentum conservation is directly related to space symmetry, and energy conservation is directly related to time symmetry. Here, the energy conservation law of a continuous system will be derived, since it will play an important role in the following chapters. Consider the infinitesimal transformation

$$\begin{aligned} &t \to t' = t + \delta t, \\ &\mathbf{x} \to \mathbf{x}' = \mathbf{x}, \\ &\eta_\alpha(\mathbf{x},t) \to \eta'_\alpha(\mathbf{x}',t') = \eta_\alpha(\mathbf{x},t+\delta t), \end{aligned} \tag{2.26}$$

such that

$$\delta\eta_\alpha(\mathbf{x},t) = \eta'_\alpha(\mathbf{x},t) - \eta_\alpha(\mathbf{x},t) = \delta t\,\dot{\eta}_\alpha. \tag{2.27}$$

This infinitesimal transformation changes the action integral

$$S \to S' = \int_{t'_1}^{t'_2} L'\,dt'. \tag{2.28}$$

Using the Euler -Lagrange equations, we can write the variation of the action integral as

$$\delta S = S' - S = -\int_{t_1}^{t_2} dt \left[\frac{dG}{dt} + \sum_\alpha \int d\mathbf{x}\nabla\cdot\mathbf{J}_\alpha\right], \tag{2.29}$$

where

$$G = \delta t \left(\sum_\alpha \int d\mathbf{x}\dot{\eta}_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial\dot{\eta}_\alpha} - L\right), \tag{2.30}$$

and

$$\mathbf{J}_\alpha = \delta t\,\dot{\eta}_\alpha \frac{\partial \mathcal{L}_\alpha}{\partial\nabla\eta_\alpha}. \tag{2.31}$$

The Lagrangian that we have considered has no explicit time dependence, i.e. its time dependence is only through the field variables $\eta_\alpha(\mathbf{x},t)$. Thus the action integral S is invariant under the infinitesimal transformation in equation (2.26). Then, it follows from equation (2.29) that

$$\frac{dG}{dt} + \sum_\alpha \int d\mathbf{x}\nabla\cdot\mathbf{J}_\alpha = 0, \tag{2.32}$$

since the boundaries t_1 and t_2 can be arbitrarily chosen. If the quantity $\mathbf{J}_\alpha$ vanishes on the boundaries of the integral $\int d\mathbf{x}$, then the quantity G is conserved

$$\frac{dG}{dt} = 0. \tag{2.33}$$

Thus, from equation (2.33), one immediately obtains the conservation law for the total energy

$$\mathcal{E}_{tot} = \sum_{\alpha} \int d\mathbf{x}\, \dot{\eta}_{\alpha} \frac{\partial \mathcal{L}_{\alpha}}{\partial \eta_{\alpha}} - L = const. \tag{2.34}$$

The energy conservation law arising from the time symmetry of the problem, is an example of Noether's theorem, that deals with the connection between symmetries and conservation properties of a Lagrangian system. The properties derived in this section will be used in subsequent chapters. Again they can be found in any good text book [1, 2, 3], to which the reader is also referred for further reading.

2.3 Ideal fluid equations and electrodynamics

An additional complication in the study of a continuous system with the use of a Lagrangian method, appears when the field variables are not independent. In this case, equation (2.25) is not valid. Two methods for finding the evolution equations for such systems are introduced here through the example of an ideal fluid of charged particles embedded in an electromagnetic field. This system contains five field variables: the density n, the velocity $\mathbf{u}$, the pressure p, the electric field $\mathbf{E}$, and the magnetic field $\mathbf{B}$. These are not independent of each other because the system is constrained by the conservation laws. Conservation of mass, for instance, connects the evolution of the density with the velocity field. And besides mass conservation, the system satisfies energy and entropy conservation.

The conservation laws of mass and entropy can be conveniently formulated in two distinct coordinate systems, either by considering a volume of fluid which is fixed in space (in which case one obtains the Eulerian form), or by considering a volume which is moving with the fluid (the Lagrangian form). The fixed volume system and the comoving system are related to each other by a transformation in the spatial coordinates

$$\begin{aligned} dt' &= dt & dt &= dt' \\ d\mathbf{x}' &= d\mathbf{x} - \mathbf{u}\,dt & d\mathbf{x} &= d\mathbf{x}' + \mathbf{u}dt, \end{aligned}$$

where the prime denotes the comoving system. The transformation of the derivatives from one frame to the other is

$$\begin{aligned} \nabla' &= \nabla \\ \frac{d}{dt} &= \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \end{aligned}$$

where d/dt denotes the time derivative in the comoving system (advective derivative) and $\partial/\partial t$ denotes the time derivative in the fixed volume system (for more information see [1])

The conservation laws of mass and entropy will be derived below using the co-moving frame, i.e. in the Lagrangian form. Let V be a specified volume that moves with the fluid at a velocity $\mathbf{u}$. After a time interval δt the volume has moved over a distance $\delta\mathbf{x}$, such that,

if $\delta t \to 0$, $\delta\mathbf{x}/\delta t = \mathbf{u}$. Since the volume moves with the fluid there is no flux of material across its boundaries, but the volume itself can be deformed. The latter leads to a change in the volume, which can be proven to satisfy

$$\delta V = V \nabla \cdot \delta\mathbf{x}. \tag{2.35}$$

The conservation of mass demands that the total particle number (N) inside the co-moving volume is constant

$$\delta N = \delta n V + n \delta V = 0. \tag{2.36}$$

Thus the change in the density (n) is

$$\delta n = -n\frac{\delta V}{V} = -n\delta t\, \nabla \cdot \mathbf{u}. \tag{2.37}$$

Letting $\delta t \to 0$, the mass conservation law is

$$\frac{dn}{dt} + n\nabla \cdot \mathbf{u} = 0. \tag{2.38}$$

Entropy conservation can be derived in a similar way. For the element moving with the fluid the change in entropy (s) is

$$\delta s = 0, \tag{2.39}$$

or

$$\frac{ds}{dt} = 0. \tag{2.40}$$

Both conservation laws can then be tranformed to the Eulerian form using the transformations of the derivatives from one frame to the other:

$$\begin{aligned} &\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \\ &\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = 0. \end{aligned} \tag{2.41}$$

The entropy conservation can also be written as

$$\frac{\partial ns}{\partial t} + \nabla \cdot (ns\mathbf{u}) = 0, \tag{2.42}$$

or, in terms of pressure

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0, \tag{2.43}$$

for $s = p/n^\gamma$ with $\gamma = 5/3$ for an ideal fluid. The first equation of (2.41) and the equation (2.43) form two constraints on the evolution of the fluid. They link density and pressure to the velocity field making them dependent field variables.

The Lagrangian of a fluid of charged particles with charge e and mass m in an electromagnetic field is given by [1, 3]

$$L = \int d\mathbf{x} \left[\frac{ne}{c}\mathbf{A}\cdot\mathbf{u} + \frac{nm}{2}u^2 - \frac{3}{2}p - ne\phi + \frac{1}{8\pi}(E^2 - B^2) \right], \tag{2.44}$$

where ϕ and $\mathbf{A}$ are the electric and magnetic potentials respectively, related to the electric and magnetic field through

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{2.45}$$

The above representations of the fields make that Faraday's law is automatically satisfied and that the magnetic field is divergence free.

In the Lagrangian, the term $(3/2)p$ appears as a potential energy. This term is in fact the internal energy of the system, and follows from the equipartition theorem for non-equilibrium gases. Consider a system of N particles that are not in equilibrium but are moving with an average (flow) velocity

$$\mathbf{u} = \frac{2}{Nm}\sum_i \mathbf{p}_i, \tag{2.46}$$

where $\sum_i$ denotes the summation over all particles and $\mathbf{p}_i = m\mathbf{v}_i$ is the canonical momentum of the particle i that moves with velocity $\mathbf{v}_i$. The equipartition theorem states that the kinetic temperature of the system is

$$\frac{3}{2}NT = \sum_i \frac{1}{2m}\left\langle \sum_i (\mathbf{p}_i - m\mathbf{u}_i)^2 \right\rangle. \tag{2.47}$$

If the energy of this system is given by kinetic energy only

$$E = \sum_i \frac{1}{2m}\mathbf{p}_i^2,$$

the corresponding fluid energy, and thus the Hamiltonian, will be

$$E = H_c = \int d\mathbf{x} \left(\frac{1}{2}m\mathbf{u}^2 + \frac{3}{2}p \right). \tag{2.48}$$

The Lagrangian of the fluid is then given by

$$L = \mathbf{p}\cdot\dot{\mathbf{q}} - H_c = \int d\mathbf{x} \left(\frac{1}{2}m\mathbf{u}^2 - \frac{3}{2}p \right). \tag{2.49}$$

The term $(E^2 - B^2)/8\pi$ is the Lagrangian for the electromagnetic field. The field variables in this expression are $(n, \mathbf{u}, p, \phi, \mathbf{A})$, and, as stated before, constraints in the form of the mass and entropy conservation laws must be used to specify the connection between the field variables.

There are two methods to insert the constraints into the Lagrangian, the Lagrange multiplier method (Euler-Lagrange), and the virtual displacement method (Euler-Poincare). Below we will examine each one separately.

2.3.1 Lagrange multipliers method

In this method every constraint is directly introduced in the action integral, however, multiplied with an arbitrary function of space and time, called the Lagrange multiplier. For fields the Lagrange multipliers depend on $(\mathbf{x},t)$ and therefore are themselves an additional set of fields. The action integral is then varied towards the fields as well as the multipliers, treating all as independent variables. The resulting equations contain the multipliers, which must then be eliminated to obtain the evolution equations. These equations can be proven to minimise the action, under the boundary condition that the evolution is limited by the constraints.

In our example the Lagrange multipliers $q_0(\mathbf{x},t)$ and $q_1(\mathbf{x},t)$ are introduced for the mass and entropy conservation, respectively. The action integral then is

$$S = \int dt \int d\mathbf{x} \left[\mathcal{L} + q_0 \left(\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) \right) + q_1 \left(\frac{\partial ns}{\partial t} + \nabla \cdot (ns\mathbf{u}) \right) \right], \tag{2.50}$$

where $\mathcal{L}$ is the Lagrangian density of the fluid

$$\mathcal{L} = \frac{ne}{c}\mathbf{A}\cdot\mathbf{u} + \frac{nm}{2}u^2 - \frac{3}{2}n^\gamma s - ne\phi + \frac{1}{8\pi}(E^2 - B^2) \tag{2.51}$$

and the constraints are written in a way that the calculations for finding the equation of evolution become easier. As before, the entropy is chosen to be that of an ideal fluid with three degrees of freedom

$$s = \frac{p}{n^\gamma}, \qquad \gamma = \frac{5}{3}, \tag{2.52}$$

The variation of the action towards the field variables and Lagrange multipliers leads to an equation where each varied quantity ($\delta n, \delta\mathbf{u}$, etc.) is multiplied with some coefficient that, in this case, is a differential equation. In each case the coefficients of the variations must independently vanish. This yields the dynamical equations of the system, which, for each varied quantity, are

$$\delta n: \quad \frac{\partial \mathcal{L}}{\partial n} = \frac{\partial q_0}{\partial t} + \mathbf{u}\cdot\nabla q_0 + s\left(\frac{\partial q_1}{\partial t} + \mathbf{u}\cdot\nabla q_1 \right), \tag{2.53}$$

$$\delta\mathbf{u}: \quad \frac{\partial \mathcal{L}}{\partial \mathbf{u}} = n\nabla q_0 + ns\nabla q_1, \tag{2.54}$$

$$\delta s: \quad \frac{\partial \mathcal{L}}{\partial s} = n\left(\frac{\partial q_1}{\partial t} + \mathbf{u}\cdot\nabla q_1 \right), \tag{2.55}$$

$$\delta q_0: \quad \frac{\partial n}{\partial t} = -\nabla\cdot(n\mathbf{u}), \tag{2.56}$$

$$\delta q_1: \quad \frac{\partial ns}{\partial t} = -\nabla\cdot(ns\mathbf{u}), \tag{2.57}$$

$$\delta\phi: \quad \frac{1}{4\pi}\left[\frac{1}{c}\nabla\cdot\frac{\partial \mathbf{A}}{\partial t} + \nabla(\nabla\phi) \right] = ne, \tag{2.58}$$

$$\delta\mathbf{A}: \quad -\frac{1}{4\pi c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi \right) = \frac{1}{4\pi}\nabla\times(\nabla\times\mathbf{A}) - \frac{1}{c}ne\mathbf{u}. \tag{2.59}$$

The fourth and fifth equation, due to the variation of the multipliers, are the original constraints, and the two last equations, resulting from the variation of the electric and magnetic potentials, are the Maxwell's equations

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \frac{4\pi}{c} ne\mathbf{u}, \qquad \nabla \cdot \mathbf{E} = 4\pi ne. \tag{2.60}$$

The final task is, therefore, to eliminate the multipliers from the first three equations to obtain the fluid velocity (the constraints already give the equations for the evolution of the density and pressure). From equation (2.54) the Clebsch representation of the fluid velocity [8, 9] is found

$$\frac{e}{c}\mathbf{A} + m\mathbf{u} = \nabla q_0 + s\nabla q_1. \tag{2.61}$$

Taking the curl of equation (2.61), one finds

$$\nabla \times \left(\frac{e}{c}\mathbf{A} + m\mathbf{u}\right) = \nabla s \times \nabla q_1. \tag{2.62}$$

This equation when $s = const$ will give that the vorticity is equal to

$$\nabla \times \mathbf{u} = -\frac{e}{mc}\mathbf{B}. \tag{2.63}$$

That is, the vorticity is then entirely specified by the magnetic field. To avoid specifying the vorticity, one can introduce one more constraint (Lin constraint) in the Lagrangian, which will then have the form

$$L = \int d\mathbf{x} \left[\mathcal{L} + q_0\left(\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u})\right) + q_1\left(\frac{\partial ns}{\partial t} + \nabla \cdot (ns\mathbf{u})\right) + q_2\left(\frac{\partial C}{\partial t} + \nabla \cdot (C\mathbf{u})\right)\right], \tag{2.64}$$

where C is an arbitrary variable (Lin variable) [9, 10, 11]. The Lin constraint gives for the Clebsch representation of the velocity

$$\frac{e}{c}\mathbf{A} + m\mathbf{u} = \nabla q_0 + s\nabla q_1 + \frac{C}{n}\nabla q_2 \tag{2.65}$$

such that the magnetic field and the vorticity have an arbitrary form. It does, however, not change any of the other results of the variation, so the rest of the calculations will be done with equation (2.61).

First q_0 is eliminated. This is done by adding two equations: the time derivative of equation (2.61), and the gradient of equation (2.61) contracted with $\mathbf{u}$. After some calculations, one arrives at

$$\frac{\partial}{\partial t}\left(\frac{e}{c}\mathbf{A} + m\mathbf{u}\right) + \nabla \cdot \left(\frac{m}{2}\mathbf{u}\right) = -\frac{1}{n}\nabla p - e\nabla\phi + \mathbf{u} \times (\nabla s \times \nabla q_1). \tag{2.66}$$

Observing then from equation (2.61) that

$$\mathbf{u} \times (\nabla s \times \nabla q_1) = \mathbf{u} \times [\nabla \times (\nabla q_0 + s\nabla q_1)] = \mathbf{u} \times \left[\nabla \times \left(\frac{e}{c}\mathbf{A} + m\mathbf{u}\right)\right], \tag{2.67}$$

one finds the equation for the evolution of the fluid velocity

$$nm\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=en\left(\mathbf{E}+\frac{\mathbf{u}\times\mathbf{B}}{c}\right)-\nabla p. \tag{2.68}$$

The energy conservation law can be found by Noether's theorem. From equation (2.34) the global energy is

$$\mathcal{E}=\int d\mathbf{x}\left[\frac{nm}{2}u^2+\frac{3}{2}p+\frac{1}{8\pi}(E^2+B^2)\right], \tag{2.69}$$

which constitutes of the kinetic energy, the thermal energy, and the energy in the electromagnetic field.

2.3.2 Virtual displacement method

The second method for introducing the constraints in the Lagrangian is the virtual displacement method. In this method a perturbation of the Lagrangian is expressed in terms of a virtual displacement (in space) as well as the perturbed electric and magnetic potentials. To obtain such an expression the perturbed density, velocity, as well as pressure must be expressed as a functional form of the virtual displacement and background quantities. These expressions are obtained directly from the conservation laws or, equivalently, from the evolution equations in Lagrangian form that describe these constraints.

Consider an Eulerian displacement field $\xi(\mathbf{x},t)$, where $\mathbf{x}(t)$ is the unperturbed position of a fluid element at time t. Here, ξ will be a virtual displacement of the fluid, that is, a displacement of fluid elements in the co-moving frame subject to conservation laws, which enter as constraints that determine the variation of the field variables (usually the fluid ones) after the arbitrary displacement field $\xi(\mathbf{x},t)$ is applied. Let then $\hat{\mathbf{x}}(t)$ be the new position of the fluid element after applying the virtual displacement ξ. The perturbed location at time t will be the sum of the unperturbed position and the virtual displacement ξ

$$\hat{\mathbf{x}}(t)=\mathbf{x}(t)+\xi[\mathbf{x}(t),t]. \tag{2.70}$$

The time derivative of $\hat{\mathbf{x}}$ is a new velocity at the new position

$$\frac{d\hat{\mathbf{x}}}{dt}=\hat{\mathbf{u}}(\hat{\mathbf{x}},t)=\frac{d\mathbf{x}}{dt}+\frac{\partial\xi}{\partial t}+\frac{d\mathbf{x}}{dt}\cdot\nabla\xi. \tag{2.71}$$

Since $d\mathbf{x}/dt=\mathbf{u}(\mathbf{x},t)$, one obtains

$$\hat{\mathbf{u}}(\hat{\mathbf{x}},t)=\mathbf{u}(\mathbf{x},t)+\frac{\partial\xi}{\partial t}+\mathbf{u}\cdot\nabla\xi. \tag{2.72}$$

In an Eulerian description the variation of a quantity is the value of the quantity after the displacement but at the original position minus the quantity before displacement at the original position

$$\delta\mathbf{u}=\hat{\mathbf{u}}(\mathbf{x},t)-\mathbf{u}(\mathbf{x},t), \tag{2.73}$$

where $\hat{\mathbf{u}}(\mathbf{x},t)$ can be found through an expansion of $\hat{\mathbf{u}}(\hat{\mathbf{x}},t)$ with respect to ξ, retaining terms up to the first order in ξ

$$\hat{\mathbf{u}}(\hat{\mathbf{x}},t) = \mathbf{u}(\hat{\mathbf{x}},t) + \xi \cdot \nabla \hat{\mathbf{u}}. \tag{2.74}$$

Thus the Eulerian variation of the velocity is

$$\delta \mathbf{u} = \frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi - \xi \cdot \nabla \mathbf{u}. \tag{2.75}$$

Below the density and pressure variation will be directly derived from the conservation laws. The density variation can be found from mass conservation. The mass of the perturbed volume element is equal to the mass of the unperturbed volume element

$$\hat{n}(\hat{\mathbf{x}},t) d\hat{\mathbf{x}} = n(\mathbf{x},t) d\mathbf{x}. \tag{2.76}$$

The volume element in the new coordinates is related to that of the old coordinates through the Jacobian $J(\mathbf{x},t)$ of the transformation

$$d\hat{\mathbf{x}} = J(\mathbf{x},t) d\mathbf{x}, \tag{2.77}$$

with the Jacobian satisfying

$$J(\mathbf{x},t) = \det \frac{\partial \hat{x}_i}{\partial x_k} = 1 + \nabla \cdot \xi \tag{2.78}$$

up to the first order in ξ. Expanding then $\hat{n}$ up to the first order in ξ

$$\hat{n}(\hat{\mathbf{x}},t) = \hat{n}(\mathbf{x},t) + \xi \cdot \nabla n, \tag{2.79}$$

mass conservation yields

$$\hat{n}(\mathbf{x},t) + \xi \cdot \nabla n = \frac{n(\mathbf{x},t)}{1 + \nabla \cdot \xi} = n(\mathbf{x},t)[1 - \nabla \cdot \xi], \tag{2.80}$$

and the density variation is

$$\delta n = \hat{n}(\mathbf{x},t) - n(\mathbf{x},t) = -\nabla \cdot (u\xi). \tag{2.81}$$

Proceeding in the same way, the constraint of entropy conservation can be written as

$$\frac{\hat{p}(\hat{\mathbf{x}},t)}{[\hat{n}(\hat{\mathbf{x}},t)]^{\gamma}} = \frac{p(\mathbf{x},t)}{[n(\mathbf{x},t)]^{\gamma}}. \tag{2.82}$$

Using the equation for the perturbed density yields

$$\hat{p}(\hat{\mathbf{x}},t) = p(\mathbf{x},t)[J(\mathbf{x},t)]^{-\gamma}, \tag{2.83}$$

and thus the equation for the variation of the pressure is

$$\delta p = \hat{p}(\mathbf{x},t) - p(\mathbf{x},t) = -\xi \cdot \nabla p - \gamma p \nabla \cdot \xi. \tag{2.84}$$

The variations of density and pressure can also be found from their equations of evolution, which will be shown below. The two approaches are equivalent, since the evolution equations can be directly derived from the conservation laws. Starting point are the equations

$$\frac{dn}{dt} + n\nabla \cdot \mathbf{u} = 0 \tag{2.85}$$

$$\frac{dp}{dt} + \gamma p \nabla \cdot \mathbf{u} = 0. \tag{2.86}$$

Here, one must differentiate between the Eulerian and Lagrangian variations. The Eulerian variation $\delta\eta$ of a field variable $\eta(\mathbf{x},t)$ is given by

$$\delta\eta = \hat{\eta}(\mathbf{x},t) - \eta(\mathbf{x},t), \tag{2.87}$$

whereas the Lagrangian variation $\hat{\delta}\eta$ of η is given by

$$\hat{\delta}\eta = \hat{\eta}(\hat{\mathbf{x}},t) - \eta(\mathbf{x},t). \tag{2.88}$$

The relation between the two variations can be found using the equation (2.70)

$$\delta\eta = \hat{\delta}\eta - \xi \cdot \nabla\eta, \tag{2.89}$$

which can be used to describe the Eulerian variation of a space integral for fixed volumes:

$$\begin{aligned} \delta \int dV n\eta &= \int dV[-\eta\nabla\cdot(n\xi) + n\delta\eta] = \\ &= \int dV[n\hat{\delta}\eta - \nabla\cdot(n\eta\xi)] = \int dV n\hat{\delta}\eta. \end{aligned} \tag{2.90}$$

Thus the Lagrangian variation of the fluid velocity is

$$\hat{\delta}\mathbf{u} = \frac{d\hat{\mathbf{x}}}{dt} - \frac{d\mathbf{x}}{dt} = \frac{d\xi}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\hat{\xi}. \tag{2.91}$$

The Lagrangian variation of the density is derived by considering the finite time-difference of equation (2.86)

$$\frac{\Delta n}{\Delta t} + n\nabla\cdot\frac{\Delta\mathbf{x}}{\Delta t} = 0. \tag{2.92}$$

In the limit $\Delta t \to 0$ and $\Delta\mathbf{x} \to \xi$ one finds for the Lagrangian variation of the density

$$\Delta n \to \hat{\delta} n = -n\nabla\cdot\xi. \tag{2.93}$$

Then the Eulerian variation of the density is

$$\delta n = -\nabla \cdot (n\xi). \tag{2.94}$$

Combining equation (2.94) with equation (2.75) the Eulerian variation of the momentum can be found:

$$\begin{aligned} \delta(n\mathbf{u}) &= \frac{\partial(n\xi)}{\partial t} + \nabla \cdot [n(\mathbf{u}\xi - \xi\mathbf{u})] - \xi\left(\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{u}\right) = \\ &= \frac{\partial(n\xi)}{\partial t} + \nabla \cdot [n(\mathbf{u}\xi - \xi\mathbf{u})]. \end{aligned} \tag{2.95}$$

In a similar way we can proceed in finding the variation of the pressure from equation (2.86). The Lagrangian variation is

$$\hat{\delta} p = -\gamma p \nabla \cdot \xi, \tag{2.96}$$

and the Eulerian variation is

$$\delta p = \hat{p}(\mathbf{x}, t) - p(\mathbf{x}, t) = -\xi \cdot \nabla p - \gamma p \nabla \cdot \xi. \tag{2.97}$$

After deriving the variations of the density, velocity and pressure in terms of the virtual displacement, the variation of the Lagrangian in Eulerian form can be expressed in terms of the displacement as well as the perturbed electric and magnetic potentials, through the replacement of $\delta n, \delta\mathbf{u}$ and δp with their equivalent equations that involve ξ. After doing some manipulations with the space and time derivatives, the variation of the Lagrangian can be written in the form

$$\begin{aligned} \delta\mathcal{L} &= \delta\phi\left(\frac{\nabla \cdot \mathbf{E}}{4\pi} - en\right) + \delta\mathbf{A} \cdot \left(\frac{1}{4\pi c}\frac{\partial \mathbf{E}}{\partial t} - \frac{\nabla \times \mathbf{B}}{4\pi} + \frac{e}{c} n\mathbf{u}\right) - \\ &\quad -\xi \cdot \left[n\frac{d}{dt}\left(m\mathbf{u} + \frac{e}{c}\mathbf{A}\right) + \nabla p + n\nabla\mathbf{u} \cdot \left(m\mathbf{u} + \frac{e}{c}\mathbf{A}\right) - \right. \\ &\quad \left. -n\nabla\left(\frac{m}{2}u^2 - e\phi + \frac{e}{c}\mathbf{u} \cdot \mathbf{A}\right)\right] + \frac{\partial}{\partial t}\left[n\xi \cdot \left(m\mathbf{u} + \frac{e}{c}\mathbf{A}\right) - \frac{\mathbf{E} \cdot \delta\mathbf{A}}{4\pi c}\right] + \\ &\quad +\nabla \cdot \left[\frac{5}{2}p\xi + n\mathbf{u}\left(m\mathbf{u} + \frac{e}{c}\mathbf{A}\right) \cdot \xi - \right. \\ &\quad \left. -n\xi\left(\frac{m}{2}u^2 - e\phi + \frac{e}{c}\mathbf{u} \cdot \mathbf{A}\right) - \left(\frac{\mathbf{E}\delta\phi}{4\pi} + \frac{\delta\mathbf{A} \times \mathbf{B}}{4\pi}\right)\right]. \end{aligned} \tag{2.98}$$

Then stationarity of the action integral $\int dt d\mathbf{x}$ with respect to the variations of ϕ and $\mathbf{A}$ will again yield the Maxwell equations

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \frac{4\pi}{c} ne\mathbf{u}, \qquad \nabla \cdot \mathbf{E} = 4\pi ne, \tag{2.99}$$

whereas stationarity of the action integral with respect to the virtual displacement yields the fluid momentum equation

$$nm\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=en\left(\mathbf{E}+\frac{\mathbf{u}\times\mathbf{B}}{c}\right)-\nabla p. \tag{2.100}$$

The rest of the terms gives the local and global energy conservation laws

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial}{\partial t}\left[n\xi\cdot\left(m\mathbf{u}+\frac{e}{c}\mathbf{A}\right)-\frac{\mathbf{E}\cdot\delta\mathbf{A}}{4\pi c}\right]+\nabla\cdot\left[\frac{5}{2}p\xi+n\mathbf{u}\left(m\mathbf{u}+\frac{e}{c}\mathbf{A}\right)\cdot\xi-\right.\\ &\left.-n\xi\left(\frac{m}{2}u^2-e\phi+\frac{e}{c}\mathbf{u}\cdot\mathbf{A}\right)-\left(\frac{\mathbf{E}\delta\phi}{4\pi}+\frac{\delta\mathbf{A}\times\mathbf{B}}{4\pi}\right)\right].\end{aligned} \tag{2.101}$$

Here, $\xi, \delta\mathbf{A}$ and $\delta\phi$ are taken as a result of the dynamical evolution of the system, and not as virtual variations, that is

$$\xi=-\mathbf{u}\delta t,\quad \delta\phi=-\frac{\partial\phi}{\partial t}\delta t,\quad \delta\mathbf{A}=-\frac{\partial\mathbf{A}}{\partial t}\delta t. \tag{2.102}$$

The variation of the Lagrangian density is

$$\delta\mathcal{L}=-\frac{\partial\mathcal{L}}{\partial t}\delta t. \tag{2.103}$$

Replacing the variations, and after some calculations the local energy conservation law is retrieved

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{nm}{2}u^2+\frac{3}{2}p+\frac{1}{8\pi}(E^2+B^2)\right]+\\ &+\nabla\cdot\left[\mathbf{u}\left(\frac{nm}{2}u^2+\frac{5}{2}p\right)+\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right]=0,\end{aligned} \tag{2.104}$$

and, integrating this equation over space gives the global energy conservation law. The global energy then is

$$\mathcal{E}=\int d\mathbf{x}\left[\frac{nm}{2}u^2+\frac{3}{2}p+\frac{1}{8\pi}(E^2+B^2)\right] \tag{2.105}$$

The difference in the two methods presented here, the Lagrange multiplier (Euler-Lagrange) method and the virtual displacement (Euler-Poincare) method, is that in the former the the dynamical fields are varied independently, while in the latter the variations are introduced through the displacement. Both methods are commonly used in many applications of classical mechanics, as well as quantum field theory and relativity, and are useful for studying systems under constraints. They are equivalent, yielding the same evolution equations as well as the same conservation laws[12]. They can be used alternatively,

depending on the problem studied. In general the mathematics connected with the virtual displacement method is somewhat easier compared with the method using Lagrange multipliers. The latter method, however, is more elegant and has the advantage that the equations of motion can be expressed in an alternative form if the multipliers are not (or are not all) eliminated.

In both the methods there is an arbitrariness that is introduced through the choice of the Lagrangian and its constraints. In fact, the Lagrangian and the constraints of a problem cannot be derived, and the validity of them is verified only by the validity of the final equations of evolution.

In the next chapters, we will use both methods to derive gyrofluid equations starting from a gyrofluid Lagrangian and imposing constraints for the fields. Thus on Chapter 4 we will use the Lagrange multiplier method to derive gyrofluid equations for only one temperature, and in Chapter 5 we will use the virtual displacement method to derive gyrofluid equations for two temperatures. The change in method from Chapter 4 to Chapter 5 was motivated by the far greater mathematical complexity of the two temperature system.

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Chapter 3

Gyrokinetic and gyrofluid theory

Two of the most commonly used descriptions in the study of low frequency plasma turbulence are the gyrokinetic and gyrofluid models. The latter describes the evolution of a few moments of the former. Although, gyrofluid is not as accurate a description as the kinetic description, it is computationally more economic, and retains most physics effects, needed for a qualitative as well as quantitative description of turbulence [1]. In this chapter both models are presented, and the derivation of the evolution equations is briefly outlined. In Section 3.1 the physics assumptions of the two models, giving the ordering and limitations, are discussed. In Section 3.2 the derivation of the gyrokinetic equations is sketched and discussed. And, finally, Section 3.3 discusses the derivation of the gyrofluid model prior to the work of this thesis.

3.1 Gyrokinetic and gyrofluid

Gyrokinetics deals with the motion of individual charged particles in an electromagnetic field. It is a reduced description that can be applied in the case of low frequency dynamics in a strong magnetic field, as will be explained below.

The trajectory of a charged particle in a magnetic field can be viewed as the sum of two separate motions: a circular motion around the magnetic field lines (gyromotion) with frequency $\Omega = eB/mc$ which is called gyrofrequency (cyclotron frequency), and a drifting motion (Fig.3.1). The combination of the two motions makes a spiral trajectory around a central line created by the centers of the circular motion, which are called the gyrocenters.

In many cases in plasma physics one has to deal with phenomena that evolve much slower than the gyromotion, and for which the electromagnetic field with space scale lengths of the order of the gyroradius can be treated as a small perturbation against a background field that has scale lengths much larger than this radius. One can then use a perturbation theory that essentially consists of an average over the fast gyromotion, and reduces the description to the motion of the gyrocenters rather than the particle position. This theory is known as gyrokinetics.

In electrostatic gyrokinetics the small parameter that is used to describe the perturbed

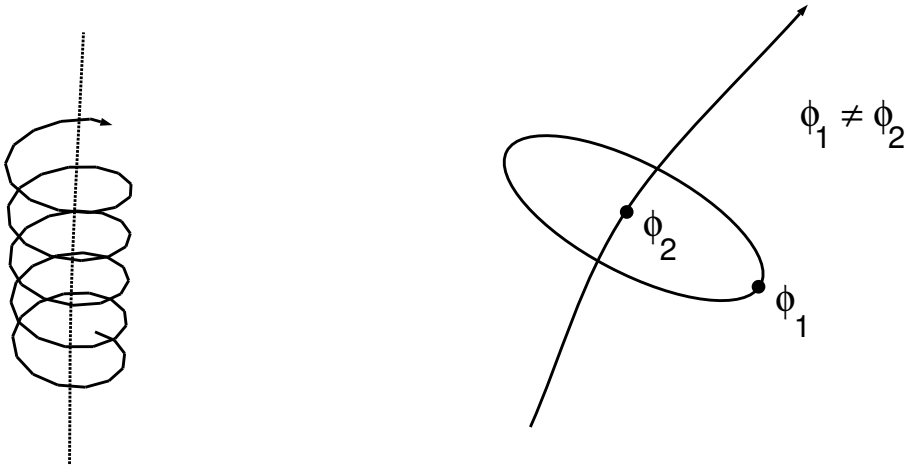

Figure 3.1: Left: Motion of a particle: gyration and motion of the gyrocenter. Right: The electric and magnetic fields in the position of the gyrocenter and the real position of the particle are not the same.

quantities is the relative amplitude of the electrostatic potential $e\phi/T_e$ where T_e is the electron temperature. The length scales associated with this small parameter however are arbitrary. These assumptions are in contrast with the complementary theory known as drift kinetics in which the electrostatic potential amplitude is arbitrary but the small parameter that describes the perturbations is the perpendicular wave length $k_\perp^2 \rho_i^2$.

In gyrokinetics care is to be taken in the evaluation of the electric and magnetic field, which change from the position of the real particle to that of the gyrocenter (Fig. 3.1). The gyroaveraging procedure must be applied to the fields, with the relevant field for the gyrocenter motion being given by the mean value of the fields on the particle gyro-orbit. However, also in calculating the electromagnetic field itself one has to consider the finite size of the orbit. This is shown in Fig. 3.2: the gyrocenter P contributes charge to all the points of the ring, and the gyrocenters that contribute charge to Q lie on the dashed ring.

The distribution function of the particles can be separated in a (small) part that depends on the gyroangle and a part that is gyroangle independent. The latter describes the gyrocenters, while the former is treated as an additional perturbative contribution to the density and formulates the polarisation term in the Poisson equation. The gyrokinetic polarisation density includes polarisation effects to all orders in the gyroradius. In contrast, the polarisation effects appears in the drift kinetic description not as a density perturbation, but in the form of the polarisation drift.

Since gyrokinetics is an approximation to the full particle motion, it has limited validity. It can describe phenomena that are slower than the gyrofrequency, and energies of the field perturbation smaller than the thermal energy. The gyrokinetic ordering for the electrostatic case is given by

$$\frac{\omega}{\Omega} \sim \frac{k_\parallel}{k_\perp} \sim \frac{e\phi}{T_e} \sim \frac{\rho_i}{L_n} = O(\varepsilon) \;\; and \;\; k_\perp \rho_i = O(1). \tag{3.1}$$

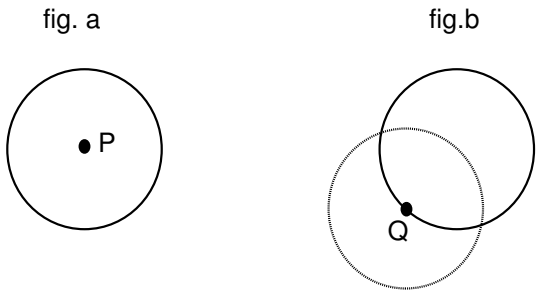

Figure 3.2: The gyrocenter P contributes charge to all the points of the ring and is sensitive to the field potential at the points on the ring. The gyrocenters that contribute charge to Q lie on the dashed ring.

Here ω and Ω_i are the characteristic fluctuation frequency and the ion gyrofrequency respectively, $k_{||}$ and $k_{\perp}$ are the components of the wave vector parallel and perpendicular to the magnetic field respectively, ρ_i is the ion gyroradius, $L_n = n/|\nabla n|$ is the density scale length of the background, ϕ is the fluctuating electrostatic potential and ε is a parameter indicating the smallness of ρ_i compared with L_n. Note that, although $e\phi/T_e = O(\varepsilon)$, $eL_n\nabla\phi/T_e = O(1)$ since small scales are allowed for the perturbed potential. If $\nabla n \approx 0$, then instead of the density scale length of the background L_n, the temperature scale length of the background $L_T = T/|\nabla T|$ is used, which is always defined since in general a confined plasma has a finite ∇p.

Because of the gyro-averaging performed in gyrokinetics, the final equations depend on five instead of six coordinates, and no longer contain the fast gyrofrequency. Thus, although gyrokinetics is mathematically more demanding than the full particle description, it is computationally more efficient since the numerical time integration does not have to resolve the very fast gyromotion, while, within its limits, giving qualitatively as well as quantitatively correct results. The gyrokinetic theory is a basic framework to describe microinstabilities, turbulence and anomalous transport observed in magnetically confined plasmas.

Gyrofluid deals with the motion of many gyrocenters (Fig. 3.3). It can essentially be viewed as a moment approach applied to the gyrokinetic equations. Therefore, the gyrokinetic ordering applies, and also gyrofluid contains the gyro-averaging procedure for the fields. Due to finite amount of moments kept in the gyrofluid theory it is less accurate than gyrokinetics, but also more computationally efficient.

3.2 Electrostatic gyrokinetic equations

The gyrokinetic theory describes the motion of charged particles in an electromagnetic field, no longer resolving the fast gyration, but retaining it only in an averaging procedure of the fields and in the space density corrections known as polarisation. Here, the derivation of the gyrokinetic equations is briefly outlined, keeping only perturbations in the electric

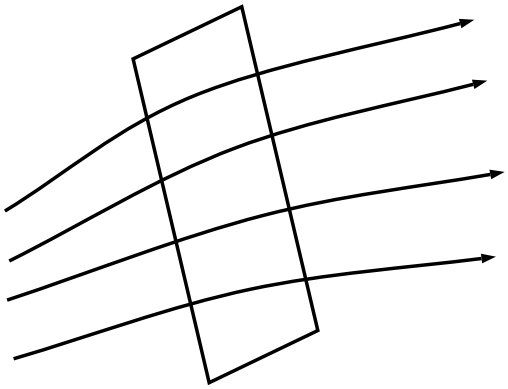

Figure 3.3: Gyrofluid follows many gyrocenters together, viewed as parts of a fluid.

field (generated by other particles or externally applied), i.e. in the electro-static limit. The derivation uses the phase-space Lagrangian variation method, and the gyromotion is eliminated using the Lie perturbation method. The formalism that is discussed briefly here can be found in the publications [2, 3, 4, 5].

The motion of charged particles in an electromagnetic field, as well as the evolution equation for the electromagnetic field, can be described by a Lagrangian that is the sum of the Lagrangian of the particles and the Lagrangian of the electromagnetic field (L_f) [5]

$$\begin{aligned} L &= L_f + \sum_\alpha \int d^3\mathbf{x}_0 d^3\mathbf{v}\, f_\alpha(\mathbf{x}_0,\mathbf{v}_0,t_0)\cdot \\ &\quad L_\alpha[\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0,t_0;t),\mathbf{v}(\mathbf{x}_0,\mathbf{v}_0,t_0;t),\dot{\mathbf{x}}(\mathbf{x}_0,\mathbf{v}_0,t_0;t)], \end{aligned} \tag{3.2}$$

where $f_\alpha(\mathbf{x}_0,\mathbf{v}_0,t_0)$ is the distribution function of species α at an arbitrary specified initial time t_0, and $\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0,t_0;t)$ and $\mathbf{v}(\mathbf{x}_0,\mathbf{v}_0,t_0;t)$ are the position and velocity of a particle at time t, which satisfy the initial condition

$$\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0,t_0;t_0) = \mathbf{x}_0, \quad \mathbf{v}(\mathbf{x}_0,\mathbf{v}_0,t_0;t_0) = \mathbf{v}_0. \tag{3.3}$$

In equation (3.2) the single-particle Lagrangian (L_α)

$$L_\alpha = \left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right)\cdot\dot{\mathbf{x}} - \left(\frac{1}{2}mv^2 + e\phi\right) \equiv \mathbf{p}\cdot\dot{\mathbf{x}} - H_c, \tag{3.4}$$

where $\mathbf{p}$ is the canonical momentum and H_c is the canonical Hamiltonian, is summed over all particle species α. The field Lagrangian L_f for the electro-static case reduces to

$$L_f = \frac{1}{8\pi}\int d^3\mathbf{x}|\nabla\phi|^2, \tag{3.5}$$

where only the perturbed electric potential is considered.

The Lagrangian in equation (3.2) is the starting point for the derivation of the equations for the single-particle motion, the gyrokinetic equation, as well as the Poisson equation and the energy conservation law of gyrokinetics.

3.2.1 Particle motion

To derive the equations for the particle motion, one only needs to manipulate the single-particle Lagrangian in equation (3.4).

In the gyrokinetic electrostatic system, the electric field and its scalar potential is assumed to consist of a static and a perturbed part

$$\mathbf{E} = \mathbf{E}_0(\mathbf{x}) + \varepsilon\mathbf{E}(\mathbf{x},t), \quad \phi = \phi_0(\mathbf{x}) + \varepsilon\phi(\mathbf{x},t). \tag{3.6}$$

However, assuming the static $E \times B$ drift velocity to be of the order $O(\varepsilon v_t)$, where v_t is the thermal velocity, one can treat both perturbation and static parts together. That is, one can put $\mathbf{E}_0 = \phi_0 = 0$, retaining the perturbations $\mathbf{E}$ and ϕ that now include also the static part.

The fundamental one-form corresponding to the single-particle Lagrangian is separated into two parts, the equilibrium (γ_0), and the perturbed part (γ_1) as in equation (2.13)

$$\gamma_0 = \left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) \cdot d\mathbf{x} - \frac{1}{2}mv^2 dt, \tag{3.7}$$

$$\gamma_1 = -e\phi dt. \tag{3.8}$$

Both parts depend on the gyromotion. To be able to proceed with the Lie perturbation method, two transformations are needed: one to retrieve the gyroangle coordinate and its canonical momentum, and one to remove the gyroangle dependence from the equilibrium part of the fundamental one-form [2]. The first transformation is (Fig. 3.4)

$$(\mathbf{x},\mathbf{v}) \to \mathbf{z} = (\mathbf{x}, v_\parallel, \mu, \theta), \tag{3.9}$$

where

$$v_\parallel = \mathbf{v}\cdot\mathbf{b}, \;\; \mu = \frac{mv_\perp^2}{2B}, \;\; \mathbf{v}_\perp = \mathbf{v} - v_\parallel\mathbf{b}, \;\; \theta = \arctan\frac{\mathbf{v}\cdot\hat{e}_1}{\mathbf{v}\cdot\hat{e}_2}, \tag{3.10}$$

and $(\hat{e}_1, \hat{e}_2, \mathbf{b} = \mathbf{B}/B)$ are unit vectors which form a right- handed orthogonal system at $\mathbf{x}$.

To follow the perpendicular part of the velocity $v_\perp$, one can also introduce another right-handed orthogonal system at $\mathbf{x}$, consisting of the vectors $(\hat{a}, \hat{c}, \mathbf{b})$ such that

$$\begin{aligned} \hat{a} &= \cos\theta\hat{e}_1 - \sin\theta\hat{e}_2 \\ \hat{c} &= -\sin\theta\hat{e}_1 - \cos\theta\hat{e}_2, \end{aligned} \tag{3.11}$$

with $\hat{a} = \mathbf{b} \times \hat{c}$ and the velocity given by

$$\mathbf{v} = v_\parallel\mathbf{b} + \mathbf{v}_\perp\hat{c}. \tag{3.12}$$

To remove the gyroangle θ dependence from the equilibrium part of the fundamental one-form, the guiding-center transformation is introduced (Fig. 3.5)

$$\mathbf{z} = (\mathbf{x}, v_\parallel, \mu, \theta) \to \mathbf{Z} = (\mathbf{X}, V_\parallel, \mu, \theta), \tag{3.13}$$

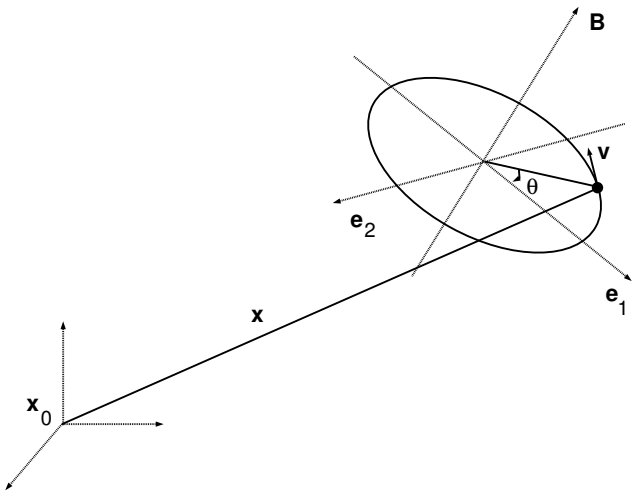

Figure 3.4: Coordinates $(\mathbf{x},\mathbf{v})$ and $(\mathbf{x},v_{\|},\mu,\theta)$

where

$$\mathbf{X}=\mathbf{x}+\varepsilon\rho+O(\varepsilon^2),\ V_{\|}=v_{\|}+O(\varepsilon),\ \mu=\mu+O(\varepsilon),\ \theta=\theta+O(\varepsilon), \tag{3.14}$$

and the gyroradius and the gyrofrequency are defined by

$$\rho=\frac{\mathbf{v}\times\mathbf{b}}{\Omega}=\frac{v_{\perp}\hat{a}}{B},\ \ \Omega=\frac{eB}{mc}. \tag{3.15}$$

In the last transformation the space coordinate introduced $\mathbf{X}$ follows the guiding center of the motion of the particle (in figure 3.5) rather than the actual particle (in figure 3.4), while the space coordinate θ and the momenta $V_{\|},\mu$ stay the same. In this way the space coordinate that indicates the position of the particle $\mathbf{x}$ can be divided in two parts, one $\mathbf{X}$ that does not depend on θ and another ρ that does. Thus the separation of the coordinates, and hence the removal of the gyroangle, becomes easier.

For the transformation of the fundamental one-form from the particle phase space coordinates to the guiding center coordinates, the value of the background potential is taken at the guiding center positions and then Taylor expanded in the actual particle positions retaining up to the first order in the small parameter ε. Then a gauge transformation is applied to arrive at a Lagrangian that does not depend on the guiding-phase angle.

In our case, the only part of the Lagrangian that changes is the symplectic part. Thus for the transformation, the magnetic potential is Taylor expanded as

$$\mathbf{A}(\mathbf{x})=\mathbf{A}(\mathbf{X}+\rho)\approx\mathbf{A}(\mathbf{X})+\rho\cdot\nabla\mathbf{A}(\mathbf{X}) \tag{3.16}$$

and the differential $d\mathbf{x}$ becomes

$$d\mathbf{x}=d\mathbf{X}+d\rho. \tag{3.17}$$

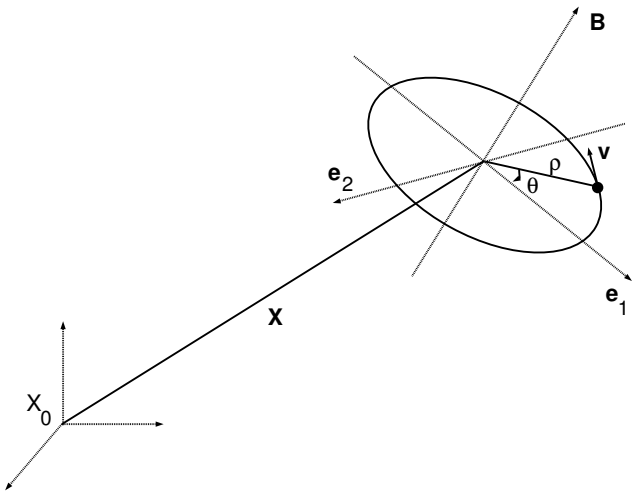

Figure 3.5: Guiding - center coordinates. The space coordinate **X** is now pointing to the gyrocenter, instead of the actual particle in figure 3.4. The space coordinate θ, as well as the momenta $\mu, V_{||}$ remain the same.

Inserting the Taylor expansion in the fundamental one-form of Eq. (3.7), new gauge-dependent terms appear. To remove these a gauge transformation is applied. In general, if one considers the Lagrangian $L(\mathbf{z}, \dot{\mathbf{z}}, t)$ and if $S = S(\mathbf{z}, t)$ is an arbitrary scalar, the equations of motion are invariant under the substitution $L \to L + dS/dt$, or

$$\gamma_i \to \gamma_i + \frac{\partial S}{\partial z_i}, \quad H \to H - \frac{\partial S}{\partial t}. \tag{3.18}$$

The gauge S that is chosen to simplify the Lagrangian is

$$S = -\frac{e v_\perp}{cB} \hat{a} \cdot \mathbf{A} - \varepsilon \frac{e v_\perp^2}{2cB} \hat{a} \cdot \nabla \mathbf{A} \cdot \hat{a}. \tag{3.19}$$

After performing the Taylor expansion and adding the gauge transformation, in terms of the guiding-center coordinates $(\mathbf{X}, V_{||}, \mu, \theta)$ the fundamental one-forms are

$$\gamma_0 = \left[\frac{e}{c}\mathbf{A}(\mathbf{X}) + m V_{||} \mathbf{b}(\mathbf{X})\right] \cdot d\mathbf{X} + \frac{mc}{e} \mu d\theta - \left(\mu B + \frac{m V_{||}^2}{2}\right) dt, \tag{3.20}$$

$$\gamma_1 = -e\phi(\mathbf{X} - \rho) dt. \tag{3.21}$$

Now the Lie perturbation method is applied. Since the equilibrium part γ_0 does not depend on the gyroangle, one can put the gauge $dS_0 = 0$, and

$$\Gamma_0 = \gamma_0. \tag{3.22}$$

since the transformations do not involve time, one can also put $g_n^t = 0$ for the generators of the Lie transform.

For the first order one-form, equation (2.20) gives

$$\begin{aligned}\Gamma_1 &= dS_1 - e\phi dt + g_1^\theta d\mu - g_1^\mu d\theta + m(g_1^{\mathbf{X}} \cdot \mathbf{b} dV_\parallel - g_1^{V_\parallel} \mathbf{b} \cdot d\mathbf{X}) + \\ &\quad + \frac{e}{c} g_1^{\mathbf{X}} \times \mathbf{B}^* \cdot d\mathbf{X} + (\Omega g_1^\mu + m V_\parallel g_1^{V_\parallel} + \mu g_1^{\mathbf{X}} \cdot \nabla\Omega) dt, \end{aligned} \tag{3.23}$$

where

$$\mathbf{B}^* = \nabla \times \mathbf{A} + \frac{mc}{e} V_\parallel \nabla \times \mathbf{b}. \tag{3.24}$$

In $\mathbf{B}^*$ the parallel velocity $V_\parallel$ commutes with the space derivative ∇, such that the generalised field $\mathbf{B}^*$ is divergence-free.

The Lie transform generators as well as the gauge scalars can be chosen such that the perturbations appear only in the Hamiltonian and Γ_1 has no θ- dependence. One then obtains

$$\Gamma_1 = -e < \phi > dt, \tag{3.25}$$

where

$$< \phi >= \frac{1}{2\pi} \int d\theta \phi(\mathbf{X} + \rho, t). \tag{3.26}$$

The generators of the transformation g_1 are calculated such that equations (3.23) and (3.25) are equal. After some calculations the generators are found to be:

$$\begin{aligned} g_1^{\mathbf{X}} &= -\frac{\varepsilon}{B} \left[\mathbf{b} \times \frac{e}{B} \nabla < \phi > + e\mathbf{b} \cdot \frac{\partial \tilde{\Phi}}{\partial V_\parallel} \right], \;\; g_1^{V_\parallel} = 0, \\ g_1^\mu &= \varepsilon \frac{e}{B} < \tilde{\phi} >, \;\; g_1^\theta = -\varepsilon \frac{e}{B} \frac{\partial \tilde{\Phi}}{\partial \mu}. \end{aligned} \tag{3.27}$$

We apply the same procedure for the second order one-form. For that we have that $\gamma_2 = 0$, such that

$$\Gamma_2 = dS_2 - \frac{1}{2} L_1 \gamma_1 - \frac{1}{2} L_1 \Gamma_1 - L_2 \gamma_0 \tag{3.28}$$

The gauge S_2 as well as the generators g_2 can be arbitrarily chosen, whereas the generators g_1 are the ones found in the evaluation of the first order one-form. In this case the gauge and generators are calculated such that only the Hamiltonian part of Γ_2 is non-zero. After extensive calculations and by applying the gyrokinetic ordering, the second order part is found to be

$$\Gamma_2 = \frac{e^2}{2\Omega} \left(\frac{\partial}{\partial \mu} < \tilde{\phi}^2 > + \frac{1}{\Omega} < \nabla \tilde{\Phi} \cdot \mathbf{b} \times \nabla \tilde{\phi} > \right) dt, \tag{3.29}$$

where

$$\tilde{\phi} = \phi - < \phi >, \;\; \tilde{\Phi} = \int^\theta \tilde{\phi} d\theta. \tag{3.30}$$

Although the Lie transform can be taken to higher order in ε, the combination of zeroth, first and second order is enough to capture the necessary physics completing the parts of the energy conservation law. Obviously, every next order is smaller by a factor ε and represents a smaller correction to the equations of motion. First order in this description, however, is not sufficient since it is the second order that yields the polarisation density in the Poisson equation and part of the $E \times B$ energy in the energy conservation law, effects that must be kept in an accurate description of the plasma dynamics [3].

The total fundamental one-form after the transformation is

$$\begin{aligned}\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 \quad = \quad & \left[\frac{e}{c}\mathbf{A}(\bar{\mathbf{X}}) + m\bar{V}_{\|}\mathbf{b}(\bar{\mathbf{X}})\right] \cdot d\bar{\mathbf{X}} + \bar{\mu} d\bar{\theta} - \\ & \left[m\frac{\bar{V}_{\|}^2}{2} + \bar{\mu}\Omega(\bar{\mathbf{X}}) + e\Psi\right] dt, \end{aligned} \tag{3.31}$$

where

$$\Psi = <\phi> - \frac{e^2}{2\Omega}\left(\frac{\partial}{\partial\bar{\mu}} <\tilde{\phi}^2> + \frac{1}{\Omega} <\nabla\tilde{\Phi}\cdot\mathbf{b}\times\nabla\tilde{\phi}>\right). \tag{3.32}$$

Together with the Lagrangian, the coordinates are transformed as well. Through the Lie transform, the coordinates transform from guiding-center to gyrocenter

$$\mathbf{Z} = (\mathbf{X}, V_{\|}, \mu, \theta) \rightarrow \bar{\mathbf{Z}} = (\bar{\mathbf{X}}, \bar{V}_{\|}, \bar{\mu}, \bar{\theta}), \tag{3.33}$$

where

$$\mathbf{Z} = \bar{\mathbf{Z}} - \bar{g}_1 + O(\varepsilon^2), \tag{3.34}$$

Substituting the transformation generators $\bar{g}_1$ found in equation 3.27, in equation 3.34, the Lie transformed coordinates are:

$$\begin{aligned}\mathbf{X} &= \bar{\mathbf{X}} + \frac{\varepsilon}{B}\left[\mathbf{b}\times\frac{e}{B}\nabla <\phi> + e\mathbf{b}\cdot\frac{\partial\tilde{\Phi}}{\partial V_{\|}}\right], \\ V_{\|} &= \bar{V}_{\|}, \\ \mu &= \bar{\mu} - \varepsilon\frac{e}{B} <\tilde{\phi}>, \\ \theta &= \bar{\theta} + \varepsilon\frac{e}{B}\frac{\partial\tilde{\Phi}}{\partial\mu}. \end{aligned} \tag{3.35}$$

The gyrokinetic one-form in equation (3.31) is written in the gyrokinetic coordinates. Finally, the nonvanishing components of the Lagrange tensor in equation (2.12) are

$$\begin{aligned}\hat{\omega}_{\bar{\mathbf{X}}_i\bar{\mathbf{X}}_j} &= \frac{e}{c}\varepsilon_{ijk}B_k^*, \\ \hat{\omega}_{t\bar{\mu}} &= \Omega + e\frac{\partial\Psi}{\partial\bar{\mu}}, \\ \hat{\omega}_{t\bar{\mathbf{X}}} &= \bar{\mu}\nabla\Omega + e\nabla\Psi, \end{aligned}$$

$$\begin{aligned}\hat{\omega}_{\bar{\mathbf{X}}\bar{V}_\parallel} &= -m\mathbf{b},\\ \hat{\omega}_{t\bar{V}_\parallel} &= m\bar{V}_\parallel,\\ \hat{\omega}_{\bar{\theta}\bar{\mu}} &= -1.\end{aligned} \tag{3.36}$$

From equation (2.11) the equations of motion for the single particle can be found

$$\frac{d\bar{\mu}}{dt} = 0, \tag{3.37}$$

$$\frac{d\bar{\theta}}{dt} = \Omega + e\frac{\partial\Psi}{\partial\bar{\mu}}, \tag{3.38}$$

$$\frac{d\bar{\mathbf{X}}}{dt} = \bar{V}_\parallel\mathbf{b} + \frac{1}{B^*_\parallel}\mathbf{b}\times\left(\frac{c}{e}\bar{\mu}\nabla\Omega + \frac{mc\bar{V}_\parallel^2}{e}\mathbf{b}\cdot\nabla\mathbf{b} + c\nabla\Psi\right), \tag{3.39}$$

$$m\frac{d\bar{V}_\parallel}{dt} = -\frac{1}{B^*_\parallel}\mathbf{B}^*\cdot\left(\bar{\mu}\nabla\Omega + e\nabla\Psi\right). \tag{3.40}$$

where $B^*_\parallel = \mathbf{B}^*\cdot\mathbf{b}$ such that $\mathbf{B}^* = B^*_\parallel\mathbf{b} + (mc\bar{V}_\parallel/e)\mathbf{b}\times\mathbf{b}\cdot\nabla\mathbf{b}$

Equations (3.37, 3.38,3.39, 3.40) describe the motion of the gyrocenter in an inhomogeneous magnetic field, under a perturbation due to the electric field. Equation (3.37) shows that there is an invariant of motion, the magnetic moment $\bar{\mu}$, and equation (3.38) shows that to zeroth order in ε, the time evolution of the gyroangle is given by the gyrofrequency Ω. It is also instructive to analyse the different contribution to the velocity given in equation (3.39), which were already mentioned in the introduction (a) a velocity parallel to the magnetic field $\bar{V}_\parallel\mathbf{b}$,
(b) a velocity perpendicular to the magnetic field due to the inhomogeneity of the magnetic field strength, the grad B drift

$$\frac{1}{B^*_\parallel}\mathbf{b}\times\frac{c}{e}\bar{\mu}\nabla\Omega, \tag{3.41}$$

(c) a velocity perpendicular to the magnetic field due to the parallel motion of the particle in a curved magnetic field, the curvature drift

$$\frac{1}{B^*_\parallel}\mathbf{b}\times\frac{mc\bar{V}_\parallel^2}{e}\mathbf{b}\cdot\nabla\mathbf{b}, \tag{3.42}$$

The curvature drift given above is derived from the equation of motion by splitting $\mathbf{B}^*$ in its components $\mathbf{B}^* = B^*_\parallel\mathbf{b} + (mc\bar{V}_\parallel/e)\mathbf{b}\times\mathbf{b}\cdot\nabla\mathbf{b}$.
(d) a velocity perpendicular to the magnetic field due to the electric field, the $E\times B$ velocity

$$\frac{1}{B^*_\parallel}\mathbf{b}\times c\nabla\Psi. \tag{3.43}$$

In equation (3.39) the polarisation drift due to the time dependence of the electric field does not appear. Technically, this is because the electric field appears only in the Hamiltonian and not the symplectic part of the Lagrangian. The physics effects due to this drift, however, are still contained in the model. Instead of a polarisation drift in the equation of motion, a polarisation term will appear in the Poisson equation, describing the same physics.

3.2.2 Gyrokinetic equation

The gyrokinetic equation is directly derived from the Liouville theorem of volume conservation in phase space, and represents the time evolution of the distribution function $f(\bar{\mathbf{Z}},t)$ of the particles [6].

The statistical state of any system is determined by a phase-space distribution function $f(\mathbf{q},\mathbf{p};t)$ where $(\mathbf{q},\mathbf{p})$ are the coordinates of the system. The distribution function satisfies

$$\int d\mathbf{Q}d\mathbf{P}|J(\mathbf{Q},\mathbf{P})|f(\mathbf{Q},\mathbf{P};t) = N, \tag{3.44}$$

where N is the total number of particles in the system. Upon any transformation $(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$, even if this is non-canonical, the distribution function [7] transforms as:

$$f(\mathbf{q},\mathbf{p};t) \to |J(\mathbf{Q},\mathbf{P})|f(\mathbf{Q},\mathbf{P};t), \tag{3.45}$$

where $|J(\mathbf{Q},\mathbf{P})|$ is the Jacobian of the transformation. Since the coordinates that we will use do not depend on time, the Jacobian of the transformation $(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$ will be assumed here time-independent.

For any coordinates $(\mathbf{Q},\mathbf{P})$, and for every variable $a = a(\mathbf{Q},\mathbf{P};t)$, Hamilton's equations (equations 2.4) can be written in the compact form

$$\dot{a} = [H,a]. \tag{3.46}$$

The distribution function then satisfies

$$\frac{\partial f}{\partial t} = [H,f], \tag{3.47}$$

or, more explicitely

$$\frac{\partial f}{\partial t} = [H,\mathbf{Q}]\frac{\partial f}{\partial \mathbf{Q}} + [H,\mathbf{P}]\frac{\partial f}{\partial \mathbf{P}}. \tag{3.48}$$

This is the Liouville equation, which states that for a non-interacting system, the distribution function obeys an equation of evolution that makes that the total volume of the system in phase space is conserved.

From Hamilton's equations, one can also prove that

$$\frac{\partial}{\partial Q_i}(|J|[H,Q_i]) + \frac{\partial}{\partial P_i}(|J|[H,P_i]) = 0. \tag{3.49}$$

Combining equation (3.48) with equation (3.49), it is easily derived that

$$\frac{\partial}{\partial t}(|J|f) = \frac{\partial}{\partial \mathbf{Q}}([H,\mathbf{Q}]|J|f) + \frac{\partial}{\partial \mathbf{P}}([H,\mathbf{P}]|J|f). \tag{3.50}$$

This is the basic evolution equation of the distribution function.

Returning to the gyrokinetic coordinates, and substituting the Lagrange tensor components equation (3.36) in equation (3.48), one finds the gyrokinetic equation

$$\frac{\partial f}{\partial t} + \frac{d\bar{V}_\parallel}{dt}\frac{\partial f}{\partial \bar{V}_\parallel} + \frac{d\bar{\mathbf{X}}}{dt}\cdot\frac{\partial f}{\partial \bar{\mathbf{X}}} = 0, \tag{3.51}$$

where $d\bar{V}_\parallel/dt$ and $d\bar{\mathbf{X}}/dt$ are given by equations (3.40) and (3.39), respectively. Moreover, from equation (3.50) one obtains

$$\frac{\partial}{\partial t}(B_\parallel^* f) + \frac{\partial}{\partial \bar{V}_\parallel}\left(\frac{d\bar{V}_\parallel}{dt}B_\parallel^* f\right) + \frac{\partial}{\partial \bar{\mathbf{X}}}\cdot\left(\frac{d\bar{\mathbf{X}}}{dt}B_\parallel^* f\right) = 0, \tag{3.52}$$

where $B_\parallel^*$ is the Jacobian of the transformation $(\mathbf{x},\mathbf{v}) \rightarrow (\bar{\mathbf{Z}})$.

3.2.3 Poisson equation and energy conservation law

To find the Poisson equation, and the energy conservation law, the full Lagrangian of particles and fields is needed [5]. This Lagrangian is obtained by substituting equation (3.31) into equation (3.2)

$$L = \sum_\alpha \int d^6\bar{\mathbf{Z}} B_\parallel^*(\bar{\mathbf{Z}}) f(\bar{\mathbf{Z}},t)\cdot L_\alpha[\bar{\mathbf{Z}},\dot{\bar{\mathbf{Z}}},t] + \frac{1}{8\pi}\int d^3\mathbf{x}|\nabla\phi|^2, \tag{3.53}$$

where

$$L_\alpha[\bar{\mathbf{Z}},\dot{\bar{\mathbf{Z}}},t] = \left(\frac{e}{c}\mathbf{A} + m\bar{V}_\parallel\mathbf{b}\right)\cdot\dot{\bar{\mathbf{X}}} + \bar{\mu}\dot{\bar{\theta}} - \left(m\frac{\bar{V}_\parallel^2}{2} + \bar{\mu}\Omega + e\Psi\right). \tag{3.54}$$

The gyrokinetic Poisson equation is derived from the variation of the action integral towards ϕ, $\delta S/\delta\phi = 0$:

$$\nabla^2\phi = -4\pi\sum_\alpha e_\alpha \int d^6\bar{\mathbf{Z}}\cdot B_\parallel^*\cdot\delta^3[\bar{\mathbf{X}} + \varepsilon\bar{\rho} - \mathbf{x}]\cdot\left[f + \frac{e}{\Omega}\left(\tilde{\phi}\frac{\partial f}{\partial\bar{\mu}} + \frac{1}{\Omega}\nabla\tilde{\Phi}\cdot\mathbf{b}\times\nabla f\right)\right]. \tag{3.55}$$

The polarisation arrises from the term that contains $(e/\Omega)\tilde{\phi}\partial f/\partial\mu$ in the Poisson equation. To see this, one has to go back to the second order Lagrangian, and manipulate the term $-(e^2/2\Omega)\,\partial < \tilde{\phi}^2 > /\partial\mu$. Taylor expanding the electric potential $\phi(\rho)$ in a series of ρ and retaining only the two first terms in the expansion yields

$$\phi = < \phi > + \rho\cdot\nabla\phi, \tag{3.56}$$

such that

$$< \tilde{\phi}^2 >= \frac{\rho^2}{2}(\nabla_\perp \phi)^2. \tag{3.57}$$

Then the term $-(e^2/2\Omega)\,\partial < \tilde{\phi}^2 > /\partial\mu$ in the Lagrangian can be proven to be equal to

$$-\frac{e^2}{2\Omega}\frac{\partial < \tilde{\phi}^2 >}{\partial\mu} = -\frac{1}{2}mu_E^2. \tag{3.58}$$

Using this approximation in the Lagrangian and taking the variation towards the potential ϕ one finds the new approximated Poisson equation to be

$$\nabla^2\phi = -4\pi\sum_\alpha e_\alpha \int d^6\bar{\mathbf{Z}}\cdot B_\parallel^* \cdot \delta^3[\bar{\mathbf{X}} + \varepsilon\bar{\rho} - \mathbf{x}]\cdot\left[f + \frac{e}{\Omega^2}\nabla\tilde{\Phi}\cdot\mathbf{b}\times\nabla f + \nabla\cdot\frac{fmc^2}{B^2}\nabla_\perp\phi\right]. \tag{3.59}$$

The last term in the Poisson equation is a more common representation of the polarisation term.

Equation (3.59) can be written in a somewhat simplified way as

$$n_i e = < n_i e > + \nabla\cdot\frac{n_i m_i c^2}{B^2}\nabla_\perp\phi = n_e e \tag{3.60}$$

where n_i, n_e are the ion and electron particle density respectively, and $< n_i e >$ is the contribution to the ion particle density from the gyrophase independent part of the distribution function. In Equation (3.60) we have neglected the space charge density since it is small compared to the polarisation piece, and have set the total space densities equal because of quasineutrality. The double requirement that the densities must be equal and that the electric field is electrostatic is made possible by the existence of the polarisation term. This necessity of the polarisation term in the Poisson equation then makes it necessary to keep the second order corrections in the Lagrangian.

The energy conservation law is found by applying Noether's theorem, equation (2.34), to the Lagrangian of equation (3.53). Using equation (3.40) the conserved total energy is given by

$$\begin{aligned}
E_{tot} &= \sum_\alpha \int d^6\bar{\mathbf{Z}} B_\parallel^* f(\bar{\mathbf{Z}},t)\cdot\left(\dot{\bar{\mathbf{Z}}}\frac{\partial L_\alpha}{\partial\dot{\bar{\mathbf{Z}}}}\right) - L = \\
&= \sum_\alpha \int d^6\bar{\mathbf{Z}} B_\parallel^* f(\bar{\mathbf{Z}},t) H(\bar{\mathbf{Z}},t) - L_f = \\
&= \sum_\alpha \int d^6\bar{\mathbf{Z}} B_\parallel^* f\cdot\left[\frac{m\bar{V}_\parallel^2}{2} + \bar{\mu}B + \frac{e^2}{2\Omega}\left(\frac{\partial}{\partial\bar{\mu}} < \tilde{\phi}^2 > + \right.\right. \\
&\quad \left.\left. + \frac{1}{\Omega} < \nabla\tilde{\Phi}\cdot\mathbf{b}\times\nabla\tilde{\phi} > \right)\right] + \frac{1}{8\pi}\int |\mathbf{E}|^2 d^3\mathbf{x}.
\end{aligned} \tag{3.61}$$

To see this derivation clearly, we use a more simplified representation of the Poisson equation

$$\sum_{\alpha}\left[\int d^3\mathbf{v}eJ_0 f+\nabla\cdot\frac{nmc^2}{B^2}\nabla_{\perp}\phi\right]=0, \tag{3.62}$$

where $n=\int d^3\mathbf{v}f$ the particle density and J_0 is a Hermitian operator that denotes the gyroaveraging procedure. Here we have kept only the polarisation and the electric potential term. Multiplying the Poisson equation (3.62) with ϕ and integrating over the whole volume one can find that

$$\sum_{\alpha}\int d^3\mathbf{x}\int d^3v eJ_0\phi f=\sum_{\alpha}\int d^3\mathbf{x}nmv_E^2, \tag{3.63}$$

since the operator J_0 is Hermitian. Thus we have that

$$\sum_{\alpha}\int d^3\mathbf{x}\left(\int d^3\mathbf{v}eJ_0\phi f-nm\frac{v_E^2}{2}\right)=\sum_{\alpha}\int d^3\mathbf{x}\,nm\frac{v_E^2}{2} \tag{3.64}$$

which means that the potential energy $e\langle\phi\rangle f$ together with $-mnv_E^2/2$ represent the the $E\times B$ energy.

The same approximations for the Lagrangian yields

$$L=\left(\frac{e}{c}\mathbf{A}+mv_{\parallel}\mathbf{b}\right)\dot{\mathbf{X}}+\mu\dot{\theta}-\left(m\frac{v_{\parallel}^2}{2}+\mu B+eJ_0\phi-\frac{nm}{2}v_E^2\right) \tag{3.65}$$

and then by using equation (3.64) the energy becomes

$$\begin{aligned}\mathcal{E} &= \sum_{\alpha}\int d^6\mathbf{Z}f\,H=\sum_{\alpha}\int d^6\mathbf{Z}\left(m\frac{v_{\parallel}^2}{2}+\mu B+eJ_0\phi-\frac{nm}{2}v_E^2\right)= \\ &\sum_{\alpha}\int d^6\mathbf{Z}\left(m\frac{v_{\parallel}^2}{2}+\mu B+\frac{nm}{2}v_E^2\right).\end{aligned} \tag{3.66}$$

Here one can easily distinguish the different parts of the total energy of the system, namely the thermal $(mv_{\parallel}^2/2+\mu B)$ and drift $(nmv_E^2)/2$ energy.

3.3 Gyrofluid electrostatic equations

The gyrofluid equations describe the time evolution of a few moments of the gyrokinetic equation. Here, the electrostatic model first derived by [8, 9], to which the reader is referred for a more detailed derivation, is presented. We will concentrate on a set of four moments: the gyrocenter density n, the parallel velocity $u_{\parallel}$, and the parallel $p_{\parallel}$ and perpendicular pressure $p_{\perp}$. In the work presented in references [8, 9] the equations of evolution are directly derived from the Gyrokinetic equation, and the moment hierarchy is closed by approximations of the higher order moments.

3.3.1 General approximations used for the model

In taking the moments of the gyrokinetic equation (3.52), many approximations [8, 9, 10, 11, 12] are introduced. In this section these approximations will be discussed, since they are important for understanding the limitations of the model.

Before taking the moments an approximation is introduced through a simplification of the gyrokinetic Lagrangian, where only the $O(\varepsilon)$ terms are kept. Thus, Ψ is approximated with $< \phi >= J_0\phi$, where J_0 is the linear operator that performs the gyroaveraging, which in Fourier space is the Bessel function $J_0(k_\perp v_\perp/\Omega)$. This approximation eliminates the polarisation term in the Poisson equation. it is then, however, put back by hand in linearised form.

The final gyrofluid equations are partly linearised, i.e. the usual gyrofluid models are delta-f models. Although the $E \times B$ advection nonlinearity is kept, all other terms are linearised according to the gyrokinetic ordering. The $E \times B$ advection nonlinearity is kept due to the fact that, although the electric field is of order epsilon $e\phi/T_e = O(\varepsilon)$, its gradient is of order 1 $eL_n\nabla\phi/T_e = O(1)$, and the perpendicular wave vector satisfies $k_\perp\, \rho_i = O(1)$. Then the term $\mathbf{u}_E \cdot \nabla$ (and hence $\nabla\phi \times \nabla$) is of the same order as the other terms in the equations. For all other terms of the form $(\partial f/\partial v_\parallel) \cdot g(\phi)$, where g is any operator acting on ϕ, the distribution function f is approximated by the Maxwellian.

Also the model uses the approximations $B^*_\parallel \approx B$ and $\mathbf{B}^* \approx \mathbf{B}$, after the ∇B and curvature drifts are fully developed in the equations of motion. Consequently, the Jacobian of the transformation from Cartesian to gyrokinetic coordinates is approximated by B.

The model is derived for low-beta plasmas in which $|\nabla \times \mathbf{B}| << |\nabla B|$. The ∇B and curvature drift can then be approximated to

$$\mathbf{v}_d = \frac{v_\parallel^2 + \mu B}{\Omega B^2}\mathbf{B} \times \nabla B. \tag{3.67}$$

The approximation in the curvature drift makes the model a zero-current model, since this approximation for the gyrokinetic equation means

$$\nabla \times \mathbf{b} \cdot \nabla f = \nabla\left(\frac{1}{B}\right) \times \mathbf{B} \cdot \nabla f + \frac{1}{B}\nabla \times \mathbf{B} \cdot \nabla f \approx \nabla\left(\frac{1}{B}\right) \times \mathbf{B} \cdot \nabla f, \tag{3.68}$$

i.e. it assumes that the term $\nabla \times \mathbf{B} \cdot \nabla f$ and thus the term $\mathbf{J} \cdot \nabla f$ is small.

Moreover, terms of the form $\nabla B \cdot \nabla \times \mathbf{B}$ and $\nabla\phi \times (\nabla \times \mathbf{B})$ are put to zero, since for low-beta plasmas the current $\mathbf{J} = (c/4\pi)\nabla \times \mathbf{B}$ is mostly parallel and the $\nabla\phi$ and ∇B mostly perpendicular.

A set of approximations is made for the gyro-averaging operator J_0. According to [12], J_0 and its derivatives are approximated by

$$\begin{aligned}
&< J_0 >= \Gamma_0^{1/2},\ < J_0 v_\parallel >= v_t\Gamma_0^{1/2},\ < J_0 v_\parallel^2 >= v_t^2\Gamma_0^{1/2},\\
&< J_0 v_\perp^2 >= v_t^2(2\Gamma_0^{1/2} + \hat{\nabla}_\perp^2),\ < J_0 v_\parallel^3 >= v_t^3\Gamma_0^{1/2},\\
&< J_0 v_\parallel v_\perp^2 >= v_t^3(2\Gamma_0^{1/2} + \hat{\nabla}_\perp^2),
\end{aligned} \tag{3.69}$$

with the modified Laplacian operators defined by

$$\frac{1}{2}\hat{\nabla}_{\perp}^{2}\Psi = b\frac{\partial\Gamma_0^{1/2}}{\partial b}\phi, \;\; \hat{\hat{\nabla}}_{\perp}^{2}\Psi = b\frac{\partial^2}{\partial b^2}(b\Gamma_0^{1/2})\phi, \tag{3.70}$$

where $\Psi = \Gamma_0^{1/2}\phi$, $b = (k_{\perp}^2 v_t^2/\Omega^2)$, and v_t is the thermal velocity $v_t = \sqrt{T_{\perp}/m}$. Derivatives of J_0 are approximated as

$$\begin{aligned} &\nabla J_0(\alpha) = J_1(\alpha)\frac{\alpha}{2B}\nabla B, \;\; \alpha = \frac{k_{\perp}v_{\perp}}{\Omega} \\ &< J_1\alpha > \approx \hat{\nabla}_{\perp}^2 - 2\hat{\hat{\nabla}}_{\perp}^2. \end{aligned} \tag{3.71}$$

The final gyrofluid equations are also normalised. For normalisation the following relations are used

$$\begin{aligned} &(t, k_{\parallel}, k_{\perp}) \rightarrow \left(\frac{tv_t}{L_n}, k_{\parallel}L_n, k_{\perp}\rho\right) \\ &(\phi, n, u, p) \rightarrow \frac{\rho}{L_n}\left(\frac{e\phi}{T_0}, \frac{n_1}{n_0}, \frac{u_1}{v_t}, \frac{p_{\perp}}{n_0 m v_t^2}\right), \end{aligned} \tag{3.72}$$

here the quantities with the subscript "1" are perturbed quantities and the quantities with the subscript "0" are background quantities, ρ is the Larmor radius and L_n is the equilibrium scale length of the density.

3.3.2 Gyrokinetic equation

The derivation of the gyrofluid equations starts from the gyrokinetic equation

$$\frac{\partial}{\partial t}(B_{\parallel}^{*}f) + \frac{\partial}{\partial \bar{V}_{\parallel}}\left(\frac{d\bar{V}_{\parallel}}{dt}B_{\parallel}^{*}f\right) + \frac{\partial}{\partial \bar{\mathbf{X}}}\cdot\left(\frac{d\bar{\mathbf{X}}}{dt}B_{\parallel}^{*}f\right) = 0. \tag{3.73}$$

For convenience, from now on, the gyrokinetic coordinates are written as $(\mathbf{x}, v_{\parallel}, \mu, \theta)$. The particle parallel velocity will be denoted by $v_{\parallel}$, and the fluid velocity by $u_{\parallel}$. Under the approximations discussed in the previous section, equation (3.73) can be written as

$$\begin{aligned} &\frac{\partial}{\partial t}(fB) + \nabla\cdot\left[fB(v_{\parallel}\mathbf{b} + \mathbf{v}_E + \mathbf{v}_d)\right] + \\ &+\frac{\partial}{\partial v_{\parallel}}\left[fB\left(-\frac{e}{m}\mathbf{b}\cdot\nabla < \phi > -\mu\mathbf{b}\cdot\nabla B + v_{\parallel}(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\mathbf{v}_E\right)\right] = 0, \end{aligned} \tag{3.74}$$

where $\mathbf{v}_E = (c/B)\mathbf{b}\times\nabla < \phi >$ is the $E\times B$ velocity, and $\mathbf{v}_d$ is given by equation (3.67).

Equation (3.74) is manipulated, grouping together all velocity dependent terms, to bring it in a more suitable form for the integration over velocity space. For more compact notation the following definition

$$i\omega_d = \frac{v_t^2}{\Omega B^2}\mathbf{B}\times\nabla B\cdot\nabla, \tag{3.75}$$

is introduced. Using the approximations of low-beta plasmas, and linearising the terms, after some algebra, the gyrokinetic equation can be written as

$$\frac{\partial}{\partial t}(fB)+B\nabla_{\|}\frac{fBv_{\|}}{B}+\mathbf{v}_E\cdot\nabla(fBJ_0)+2fBJ_0\cdot\frac{e}{T}i\omega_d\phi+$$
$$+\frac{e}{T}i\omega_d\left(fBJ_1\phi\frac{k_\perp v_\perp}{2\Omega}\right)+\frac{i\omega_d}{v_t^2}[fB(v_{\|}^2+\mu B)]-\frac{e}{m}\nabla_{\|}\left(J_0\phi B\frac{\partial f_0}{\partial v_{\|}}\right)+$$
$$+\frac{e}{m}J_0\phi B\frac{\partial f_0}{\partial v_{\|}}\left(\frac{\mu B}{v_t^2}-1\right)\nabla_{\|}\ln B-\mu B\frac{\partial}{\partial v_{\|}}(fB)\nabla_{\|}\ln B-$$
$$-\frac{\partial}{\partial v_{\|}}(fBJ_0v_{\|})\frac{e}{T}i\omega_d\phi=0, \tag{3.76}$$

where f_0 is the Maxwellian distribution function.

3.3.3 General gyrofluid equations

While deriving, from equation (3.76), the evolution equations for the four moments of interest, the following moments will appear in the equations

$$n=\int d^3v\,f,\ nu_{\|}=\int d^3v\,fv_{\|},\ nT_{\|}=p_{\|}=\int d^3v\,f(v_{\|}-v_{\|})^2,$$
$$nT_\perp=p_\perp=\frac{m}{2}\int d^3v\,fv_\perp^2,\ q_{\|}=m\int d^3v\,f(v_{\|}-v_{\|})^3,$$
$$q_\perp=\frac{m}{2}\int d^3v\,fv_\perp^2(v_{\|}-u_{\|}),\ r_{\|,\|}=\int d^3v\,f(v_{\|}-u_{\|})^4,$$
$$r_{\|,\perp}=\frac{m}{2}\int d^3v\,fv_\perp^2(v_{\|}-u_{\|})^2,\ r_{\perp,\perp}=\frac{m}{4}\int d^3v\,fv_\perp^4. \tag{3.77}$$

Using these expressions, the integrals of the form $\int dv_{\|}d\mu v_{\|}^j\mu^k$ of the equation (3.76) lead to the following four equations

$$\frac{\partial n}{\partial t}+B\nabla_{\|}\frac{nu_{\|}}{B}+\mathbf{v}_\phi\cdot\nabla(n<J_0>)+2n<J_0>\frac{e}{T}i\omega_d\phi+$$
$$+\frac{e}{T}i\omega_d\left(\phi n\frac{<J_1\alpha>}{2}\right)+\frac{1}{T}i\omega_d(p_{\|}+p_\perp+nmu_{\|}^2)=0, \tag{3.78}$$

$$\frac{\partial}{\partial t}nu_{\|}+B\nabla_{\|}\left(\frac{p_{\|}}{m}+nu_{\|}^2\right)/B+\mathbf{v}_\phi\cdot\nabla(n<J_0v_{\|}>)+2n<J_0v_{\|}>\frac{e}{T}i\omega_d\phi+$$
$$+\frac{e}{T}i\omega_d\left(\frac{\phi n<J_1v_{\|}\alpha>}{2}\right)+\frac{1}{T}i\omega_d(q_{\|}+q_\perp+3p_{\|}u_{\|}+p_\perp u_{\|}+nmu_{\|}^3)+$$
$$+\frac{e}{m}\nabla_{\|}n<J_0>\phi+\frac{e}{m}n<J_0(v_\perp^2/2v_t^2-1)>\phi\nabla_{\|}\ln B+\frac{p_\perp}{m}\nabla_{\|}\ln B+$$
$$+n<J_0v_{\|}>\frac{e}{T}i\omega_d\phi=0, \tag{3.79}$$

$$\frac{\partial}{\partial t}(p_\parallel + nmu_\parallel^2) + B\nabla_\parallel \frac{q_\parallel + 3p_\parallel u_\parallel + nmu_\parallel^2}{B} + \mathbf{v}_\phi \cdot \nabla(n < J_0 v_\parallel^2 >) +$$
$$+2n < j_0 v_\parallel^2 > \frac{e}{T} i\omega_d \phi + \frac{e}{T} i\omega_d \left(\frac{\phi n < J_1 v_\parallel^2 \alpha >}{2} \right) +$$
$$+\frac{1}{T} i\omega_d (r_{\parallel,\parallel} + r_{\parallel,\perp} + 4q_\parallel u_\parallel + 6p_\parallel u_\parallel^2 + p_\perp u_\parallel^2 + nmu_\parallel^4) +$$
$$+2\frac{e}{m}\nabla_\parallel v < J_0 v_\parallel > \phi + \frac{2e}{m} n < j_0 v_\parallel (v_\perp^2/2v_t - 1) > \phi \nabla_\parallel \ln B +$$
$$+2(q_\perp + p_\perp u_\parallel)\nabla_\parallel \ln B + 2n < j_0 v_\parallel^2 > \frac{e}{T} i\omega_d \phi = 0, \tag{3.80}$$

$$\frac{\partial}{\partial t}\frac{p_\perp}{B} + B\nabla_\parallel \frac{q_\perp + p_\perp u_\parallel}{B^2} + \mathbf{v}_\phi \cdot \nabla \frac{n < J_0 v_\perp^2 >}{2B} + 2\frac{n < J_0 v_\perp^2 >}{2B}\frac{e}{T} i\omega_d \phi +$$
$$+\frac{e}{T} i\omega_d \left(\frac{\phi n < J_1 v_\perp^2 \alpha >}{4B} \right) + \frac{1}{T} i\omega_d \frac{r_{\parallel,\perp} + r_{\perp,\perp} + q_\perp u_\parallel + p_\perp u_\parallel^2}{B} = 0, \tag{3.81}$$

where the notation $n < A >= \int d^3v\, fA = 2\pi \int dv_\parallel d\mu\, fBA$ has been used. These are the general evolution equations for the first four gyrofluid moments. Each equation for the time evolution of a moment contains terms that depend on higher moments. A closure approximation is needed for all those moments for which no evolution equation exists. In this case these are the heat fluxes $q_\parallel, q_\perp$ and $r_{\parallel,\parallel}, r_{\parallel,\perp}, r_{\perp,\perp}$. The closures are made empirically, by comparing gyrofluid with gyrokinetic results in the linear regime [9], and will not be discussed here.

3.3.4 Finite Larmor radius effects

The terms in the evolution equations that contain angle brackets are integrals over the gyroaverage operator, and are connected with the gyroaveraged potential. Since the finite size of the Larmor radius is explicitly introduced through such terms they are known as Finite Larmor Radius (FLR) terms. The terms have the form of a Bessel function (or its derivative) multiplied with parallel and/or perpendicular velocity components to a certain power: $< J_0 v_\parallel^i \mu^j v >$. These can be approximated with expressions given in Section (3.3.1), leading, after normalisation, to

$$\frac{\partial n}{\partial t} + B\nabla_\parallel \frac{u_\parallel}{B} - \left(1 + \frac{\eta_\perp}{2}\hat{\nabla}_\perp^2\right) i\omega_* \Psi + \left(2 + \frac{1}{2}\hat{\nabla}_\perp^2\right) i\omega_d \Psi + i\omega_d (p_\parallel + p_\perp) = 0, \tag{3.82}$$

$$\frac{\partial u_\parallel}{\partial t} + B\nabla_\parallel \frac{p_\parallel}{B} + \nabla_\parallel \Psi + \left(p_\perp + \frac{1}{2}\hat{\nabla}_\perp^2 \Psi\right) \nabla_\parallel \ln B + i\omega_d (q_\parallel + q_\perp + 4u_\parallel) = 0, \tag{3.83}$$

$$\frac{\partial p_\parallel}{\partial t} + B\nabla_\parallel \frac{q_\parallel + 3u_\parallel}{B} + 2(q_\perp + u_\parallel)\nabla_\parallel \ln B - \left(1 + \eta_\parallel + \frac{\eta_\perp}{2}\hat{\nabla}_\perp^2\right) i\omega_* \Psi + \\ + \left(4 + \frac{1}{2}\hat{\nabla}_\perp^2\right) i\omega_d \Psi + i\omega_d (r_{\parallel,\parallel} + r_{\parallel,\perp}) = 0, \tag{3.84}$$

$$\frac{\partial p_\perp}{\partial t} + B^2\nabla_\parallel \frac{q_\perp + u_\parallel}{B^2} - \left[1 + \frac{1}{2}\hat{\nabla}_\perp^2 + \eta_\perp \left(1 + \frac{1}{2}\hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2\right)\right] i\omega_* \Psi + \\ + \left(3 + \frac{3}{2}\hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2\right) i\omega_d \Psi + i\omega_d (r_{\parallel,\perp} + r_{\perp,\perp}) = 0. \tag{3.85}$$

where

$$i\omega_* = -\frac{cT}{eBn_0}\nabla n_0 \cdot \mathbf{b} \times \nabla \tag{3.86}$$

and $\eta_\parallel = L_n/L_{T_\parallel}$, $\eta_\perp = L_n/L_{T_\perp}$, with $L_{T_\parallel}$ and $L_{T_\perp}$ being the equilibrium scale lengths of parallel and perpendicular temperature.

These are the evolution equations of the first four gyrofluid moments in which all integrals over velocity space have been worked out. Only the closure would still have to be applied. For comparison of this model with our model, which is worked out in the following chapters, only the Maxwellian parts of the higher moments $r_{\parallel,\parallel}, r_{\parallel,\perp}, r_{\perp,\perp}$ are needed. These are

$$r_{\parallel,\parallel} = \frac{3p_\parallel^2}{n}, \;\; r_{\parallel,\perp} = \frac{p_\parallel p_\perp}{n}, \;\; r_{\perp,\perp} = \frac{2p_\perp^2}{n}. \tag{3.87}$$

The gyrofluid model of [8] presented here retains much of the physics of a confined plasma. However, it lacks two major features necessary for the correct description of turbulent transport especially in the edge of the tokamak, namely the full-nonlinearity and the energy conservation law.

The model described here is partly non-linear. Although the $E \times B$ nonlinearity is kept, all other nonlinearities are discarded following the gyrokinetic ordering, making the model a delta-f model. The consequence of the linearisation is that the nonlinear evolution of microinstabilities following the linear evolution is often undiagnosed.

Moreover, the gyrofluid model described here does not conserve any form of energy. Although in linear theory the impact of the lack of energy conservation to the transport computations is small, since a small deviation of energy conservation leads to a small change in the growth rate, in the nonlinear theory the impact can be very large, since then violation of the energy conservation can lead to a source of free energy and, hence, in large errors in the description of the turbulent state.

In this model equations (3.82, 3.83, 3.84 and 3.85) are conservation laws but only in the neglect of field energy and momentum. Thus equation (3.82) represents the conservation of mass, equation (3.83) represents the conservation of parallel momentum, and equations (3.84) and (3.85) represent the conservation of the internal energy at zero order in the Larmor radius. However, the equations together with the Poisson equation of the model do not conserve total energy, since the energy conservation breaks in the FLR terms because

of the different treatment used for the moment equations and the Poisson equation. Thus although the internal energy is conserved (at least to the zeroth order in the FLR terms) in this model, the internal energy together with the field energy is not conserved. Although the field energy is much smaller than the internal energy, because of the interaction between flows and turbulence which have comparable energy, and because the transfer dynamics is all of similar magnitude, we need to conserve field and thermal energy together.

From the above it is obvious that the model should be extended such that the main drawbacks of it are removed. A new model should be derived, that in fully nonlinear and conserves total energy. The new model presented in the next chapters, although more evolving algebraically, contains both of these features.

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Chapter 4

Gyrofluid theory: three-moment model

The gyrofluid equations (3.82, 3.83, 3.84, 3.85) derived in the previous chapter are partly linearised, and, although they maintain free energy conservation at zeroth order in the FLR, they break it in the FLR terms due to the different techniques the model uses in forming the FLR corrections to the moment equations and to the field potential equations which provide self consistency.

In this chapter we present a new method for a consistent derivation of the gyrofluid moment equations and the (field) polarisation equation starting from a gyrofluid Lagrangian. The equations are derived in a two part procedure. The first part is purely Hamiltonian, in which the equations are obtained from the principle of least action. The gyrofluid Lagrangian is constructed using the gyrokinetic one as a guide as in [23]. Particle and entropy conservation are used as constraints, and are introduced in the Lagrangian using the Lagrange multiplier method. The velocity is written in the "Clebsch" representation, which is introduced in Section 2.3.1, and previously used in hydrodynamic and MHD models [29, 30, 31]. The action involves the density, velocity and pressure, but not higher moments. Variation of the fluid variables and the Lagrange multipliers leads to the generalised drift velocity and the gyrofluid moment equations. Variation of the potential leads to the self consistent polarisation equation. The energy conservation law is then obtained in a consistent way from the Jacobi integral of the gyrofluid Lagrangian (Noether's theorem). The proper conservation law follows directly from the Lagrangian, and it is exactly satisfied by the equations of evolution.

In the second part, we account for the higher moments by incorporating the diamagnetic cancellation in each equation [26, 28], that is, insertion of perpendicular diamagnetic momentum and heat fluxes to cancel advection by the diamagnetic velocity (the part due to the pressure gradient) in a way not impacting the energy conservation law. Diamagnetic fluxes arise from the combination of first-order FLR corrections and general gradients in the kinetic theory [27]. This diamagnetic cancellation is automatically satisfied in the conventional gyrofluid derivation, which obtains the residual compressional effects arising from the diamagnetic fluxes in a different way [5]. However, this relies on the higher moments (eg. viscosity, heat flux) which cannot be treated in a Lagrangian unless they are kept as

dynamical variables, so in our method the diamagnetic fluxes have to be treated separately. The procedure however is systematic, as will be shown in chapter 4.4. The advantage of the Lagrangian method is its rigorous treatment of the exchange between fluid energy and ExB drift energy. The systematic construction of the diamagnetic fluxes — through pure divergences arranged to cancel the diamagnetic advection effects term by term — ensures that this exact energy conservation is not lost.

The model presented herein is electrostatic and keeps only the perpendicular temperature (a two degree of freedom model). The organisation of the rest of the Chapter is as follows: in Section 4.1 the construction of the gyrofluid Lagrangian from the gyrokinetic one is presented, and the constraints that will be used are given. In Section 4.2 the Clebsch forms for the magnetic field and velocity are presented and the gyrofluid moment and polarisation equations before the diamagnetic cancellation are found. In Section 4.3 the energy conservation law from the Noether's theorem, which must subsequently survive the diamagnetic cancellations is derived. In Section 4.4 the diamagnetic cancellations are described and the final equations for momentum and temperature are presented. In Section 4.5 the explicit form of the energy conservation law, determining the energy exchange terms is presented.

4.1 Lagrangian and constraints

In this Section, we construct the gyrofluid Lagrangian starting with the gyrokinetic one. Taking only the long wavelength form of gyroscreening, the gyrokinetic one-form equation (3.31) can be written as

$$\gamma = \left(\frac{e}{c}\mathbf{A} + mv_{\|}\mathbf{b}\right)\cdot d\mathbf{X} + \mu\frac{mc}{e}d\theta - \left(e<\phi>(\mu,\mathbf{X},t) + \mu B + \frac{mv_{\|}^2}{2} - \frac{mv_E^2}{2}\right)dt, \quad (4.1)$$

where the coefficient of dt is the Hamiltonian and the terms left of it represent the symplectic form. The velocity variables $v_{\|}$ and μ refer to the particle motion itself, while

$$\mathbf{v}_E = (c/B^2)\mathbf{B}\times\nabla\phi \quad (4.2)$$

is the ExB velocity. The $mv_E^2/2$ term represents the long wavelength form of gyroscreening, and is derived from the second order potential term in the Hamiltonian, and will lead to the field potential term in the polarisation equation [32]. It can be thought of as a form of field energy, coming from the interaction of the particles with the electromagnetic field. In combination with the electric potential energy $-e<\phi>$ and the electric field Lagrangian it gives the $E\times B$ energy in the energy conservation law (equation 4.36 for the gyrofluid energy).

Although we take the long wavelength limit for the gyroscreening, the potential is still gyroaveraged since this is necessary to obtain the finite pressure corrections to the polarisation equation, as will be clear later. The long wavelength limit for the gyroscreening is

taken for reasons of computational tractability. It is actually a useful approximation because although the turbulence extends down to ion gyroradius scales, the gyroscreening correction to the potential is significant only for large flow velocities, at long wavelength. Its contribution to polarisation is then modelled as a simple elliptic operator which is easily solved. Many computational models revert to this long wavelength form for polarisation; it does not affect results unless the "hyperfine" scale dynamics between the ion and electron gyroradii [33] are addressed. It is important to note that the different level of the treatment of gyroaveraging and gyroscreening does not affect consistency, as these effects enter the energetics separately, the gyroscreening through polarisation and by adding to the potential, and the gyroaveraging as an Hermitian operator in the consistency between ExB and thermal energy (cf. Section 4.5).

To construct the gyrofluid Lagrangian, we follow the "recipe" applied in [23], based on two steps:

(i) The terms that do not contribute to the internal energy of the fluid are constructed by simply taking the lowest moment of the respective terms of the gyrokinetic Lagrangian. For example, the term $(e/c)\mathbf{A}\cdot\mathbf{v}$ in the gyrokinetic Lagrangian becomes $n(e/c)\mathbf{A}\cdot\mathbf{u}$ in the gyrofluid one, where v is the particle velocity and u is the fluid velocity.

(ii) The terms that contribute to the internal energy of the fluid are replaced with the internal energy $nT_\perp$.

Then we can set the gyrofluid Lagrangian density to be

$$\mathcal{L}_f = nmu_\parallel \mathbf{b}\cdot\mathbf{u} - nm\frac{u_\parallel^2}{2} + n\frac{e}{c}\mathbf{A}\cdot\mathbf{u} - nT_\perp - ne||\phi||(\mathbf{x},t;T_\perp) + nm\frac{v_E^2}{2}, \tag{4.3}$$

where now $||...||$ denotes a gyrofluid gyroaveraging operator to be specified later, and $\mathbf{x}$ is the gyrofluid space coordinate. It is important to keep in mind that $||...||$ depends upon $T_\perp$ while taking variations and, later, time derivatives. Although the drift-fluid Lagrangian of [23] and our gyrofluid Lagrangian are similar, there exist three differences: the gyrofluid Lagrangian contains FLR corrections to ϕ, we treat $T_\perp$ rather than an isotropic T, and we arrange the terms such that the gyroscreening term appears only in the Hamiltonian (i.e., not in the "symplectic" part involving contraction with $\mathbf{u}$).

Here one should note that the Lagrangian is constructed, not derived, and the method that we followed here is not a mathematical derivation of the gyrofluid Lagrangian from the gyrokinetic one. Thus the validity of the Lagrangian follows from the correspondence of the resulting model equations to the other models in regimes where both are valid, as well as the energy conservation law.

Secondly, that here the $u_\parallel$ is not a priori equal to $\mathbf{u}\cdot\mathbf{b}$. This is found by varying the quantity $u_\parallel$, in the action integral arising from the gyrofluid Lagrangian (equation 4.3) which yields

$$u_\parallel = \mathbf{u}\cdot\mathbf{b}, \tag{4.4}$$

analogously to the extra velocity variable used by Littlejohn [34]. After this, we can replace the quantity $u_\parallel$ with $\mathbf{u}\cdot\mathbf{b}$ in the Lagrangian. Thus we can re-set the gyrofluid Lagrangian

density as

$$\mathcal{L}_f = n\frac{e}{c}\mathbf{A}\cdot\mathbf{u} + nm\frac{u_\parallel^2}{2} - nT_\perp - ne||\phi|| + nm\frac{v_E^2}{2}. \tag{4.5}$$

Together with $\mathcal{L}_f$, we introduce constraints for the system, arising from the conservation of particles and thermal entropy. To have a nonvanishing magnetic field in the absence of an entropy gradient, we also introduce a Lin variable, q_2, which however does not change the resulting moment equations (in compressible hydrodynamics this is done to allow an arbitrary fluid vorticity [29]). The constraints are

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{u}) = 0, \tag{4.6}$$

$$\frac{\partial T_\perp}{\partial t} + \nabla\cdot(T_\perp\mathbf{u}) = 0, \tag{4.7}$$

$$\frac{\partial q_2}{\partial t} + \nabla\cdot(q_2\mathbf{u}) = 0. \tag{4.8}$$

The first constraint (equation (4.6)) sets material continuity as the equation governing the density, and the second (equation (4.7)) is equivalent to constancy of perpendicular thermal entropy along flow streamlines: $ds_\perp/dt = 0$, where $s_\perp = p_\perp/n^2$, with $p_\perp = nT_\perp$ is the perpendicular pressure [35]. Finally, equation (4.8) inserts the Lin variable.

The general Lagrangian of this model contains the Lagrangian of the gyrofluid, that of the electric field, and the constraints. We introduce the Lagrange multipliers p_0, p_1, p_2 for the constraints, and insert the additional Lagrangian for the electric field. The final form for the Lagrangian is then:

$$\begin{aligned} L = \int d^3\mathbf{x} \Bigg[& n\frac{e}{c}\mathbf{A}\cdot\mathbf{u} + nm\frac{u_\parallel^2}{2} - nT_\perp - ne||\phi|| + nm\frac{v_E^2}{2} + \\ & + p_0\left(\frac{\partial n}{\partial t} + \nabla\cdot n\mathbf{u}\right) + p_1\left(\frac{\partial T_\perp}{\partial t} + \nabla\cdot T_\perp\mathbf{u}\right) + p_2\left(\frac{\partial q_2}{\partial t} + \nabla\cdot q_2\mathbf{u}\right) + \\ & + \frac{E^2}{8\pi}\Bigg], \end{aligned} \tag{4.9}$$

where $E = -\nabla\phi$ is the electric field. Variation of this with respect to the dependent variables and Lagrange multipliers will yield the system of equations describing the model.

4.2 Dynamical equations of the model

Herein we present the Clebsch form of the magnetic field and velocity, determined by the coefficients of the velocity variation, then derive the gyrofluid moment equations by eliminating the Lagrange multipliers, and then derive the polarisation equation by taking the variation of the electrostatic potential. The method that we will use is the method of the Lagrange multipliers, presented in Section 2.3.1 for a hydrodynamic model.

4.2.1 Clebsch forms

We proceed by varying the Lagrangian with respect to the independent variables $n, \mathbf{u}, T_\perp, p_1, p_2, p_3, q_2$ and ϕ. From the variation of the Lagrange multipliers we recover the constraint equations (4.6,4.7,4.8). From the variation of $\mathbf{u}$, remembering that $u_\parallel = \mathbf{u}\cdot\mathbf{b}$, we get the Clebsch form for the generalised magnetic potential:

$$\frac{e}{c}\mathbf{A} + mu_\parallel\mathbf{b} = \nabla p_0 + \frac{T_\perp}{n}\nabla p_1 + \frac{q_2}{n}\nabla p_2. \tag{4.10}$$

As shown in Section 2.3.1, in a hydrodynamic or MHD model one finds a form like this for the velocity [29, 30, 31], but here the term in $\mathcal{L}_f$ contracted with $\mathbf{u}$ is proportional to a generalised form of $\mathbf{A}$ rather than $\mathbf{u}$, so we find this form on the left hand side of equation (4.10) instead. One can see the necessity of the Lin variable q_2, and its Lagrange multiplier p_2, by taking the curl of equation (4.10):

$$\nabla\times\left(\frac{e}{c}\mathbf{A} + mu_\parallel\mathbf{b}\right) = \nabla\left(\frac{T_\perp}{n}\right)\times\nabla p_1 + \nabla\left(\frac{q_2}{n}\right)\times\nabla p_2. \tag{4.11}$$

The role of the Lin variable q_2 is to allow arbitrary forms in the magnetic field ($\nabla\times\mathbf{A}$) and the parallel velocity, in the absence of an entropy gradient (cf. reference [29]), noting that $\nabla s_\perp = \nabla(T_\perp/n)$.

From the variation of the density, we get:

$$\frac{\partial\mathcal{L}_f}{\partial n} - \frac{\partial p_0}{\partial t} - \mathbf{u}\cdot\nabla p_0 = 0, \tag{4.12}$$

and from the variation of the temperature, we get

$$\frac{\partial\mathcal{L}_f}{\partial T_\perp} - \left(\frac{\partial p_1}{\partial t} + \mathbf{u}\cdot\nabla p_1\right) = 0. \tag{4.13}$$

Finally, from the variation of the quantity q_2 we get

$$\frac{\partial}{\partial t}p_2 + \mathbf{u}\cdot\nabla p_2 = 0. \tag{4.14}$$

4.2.2 The gyrofluid moment equations

We can eliminate the Lagrange multipliers from equations (4.10, 4.12, 4.13) by first taking the time derivative of equation (4.10), then contracting equation (4.10) with $\mathbf{u}$ and taking its gradient, and then adding the two resulting equations. This re-introduces $\mathbf{u}$ to the model, and determines the form of the drift velocity across the magnetic field. After some calculations, we find

$$nm\frac{\partial u_\parallel}{\partial t}\mathbf{b} + n\nabla\left(m\frac{u_\parallel^2}{2}\right) + \nabla p_\perp +$$

$$+ n\nabla\left(e||\phi|| - m\frac{v_E^2}{2}\right) + T_\perp \nabla\left(ne\frac{\partial||\phi||}{\partial T_\perp}\right) =$$
$$= n\mathbf{u}\times\left[\nabla\left(\frac{T_\perp}{n}\right)\times\nabla p_1 + \nabla\left(\frac{q_2}{n}\right)\times\nabla p_2\right], \tag{4.15}$$

where once again $p_\perp = nT_\perp$ is the perpendicular pressure. The term involving the derivative with respect to $T_\perp$ arises from the dependence of $||...||$ upon $T_\perp$. We will see the role this has in energy conservation later.

The manipulations that follow become simpler if we define generalised potentials, given by

$$\psi = ||\phi|| - \frac{m}{e}\frac{v_E^2}{2} \qquad\qquad \Omega = \frac{\partial||\phi||}{\partial T_\perp}. \tag{4.16}$$

These can be regarded as the total potential and its first FLR correction, respectively.

The right hand side of equation (4.15) contains some of the Clebsch potentials from equation (4.10) in the brackets. Substituting in for these, we arrive at the general equation determining the gyrofluid velocity:

$$nm\frac{\partial u_\parallel}{\partial t}\mathbf{b} + nm(\mathbf{u}\cdot\nabla u_\parallel)\mathbf{b} - \frac{ne}{c}\mathbf{u}\times\mathbf{B}^* + \nabla p_\perp + n\nabla(e\psi) + T_\perp\nabla(ne\Omega) = 0, \tag{4.17}$$

where

$$\mathbf{B}^* = \nabla\times\mathbf{A} + \frac{mc}{e}u_\parallel\nabla\times\mathbf{b}. \tag{4.18}$$

To find the gyrofluid velocity itself we cross equation (4.17) with $\mathbf{b}$ to find

$$n\mathbf{u} = nu_\parallel\mathbf{b}^* + \frac{c}{eB_\parallel^*}\mathbf{b}\times\nabla p_\perp + \frac{nc}{eB_\parallel^*}\mathbf{b}\times\nabla(e\psi) + \frac{cT_\perp}{eB_\parallel^*}\mathbf{b}\times\nabla(ne\Omega), \tag{4.19}$$

where $\mathbf{b}^* = \mathbf{B}^*/B_\parallel^*$, with $B_\parallel^* = \mathbf{b}\cdot\mathbf{B}^*$. The gyrofluid velocity consists of the parallel velocity, diamagnetic velocity, and generalised ExB drift. It is important to note that $\mathbf{B}^*$ is not a divergence free field, in contrast to the gyrokinetic version where $v_\parallel$ (appearing instead of $u_\parallel$) is a separate velocity variable commuting with the gradient. Here, $u_\parallel$ is a field variable with spatial dependence, whose appearance in $\mathbf{B}^*$ arises from the fact that $\mathbf{B}^*$ is a fluid moment of the gyrokinetic version of the generalised magnetic field (cf. reference [9]). We could have constructed $\mathbf{B}^*$ as $\nabla\times(e/c)\mathbf{A}^*$, with $(e/c)\mathbf{A}^* = (e/c)\mathbf{A} + mu_\parallel\mathbf{b}$, but at the cost of less transparency in the role of advection by $u_\parallel$ in the equations.

From equation (4.19) one can see that the diamagnetic velocity still appears — it is the part due to $\nabla p_\perp$. Advection by this piece is what we will have to eventually cancel by adding diamagnetic fluxes in Section 4.4.

Contracting equation (4.17) with $\mathbf{b}^*$ we get the equation for parallel momentum

$$nm\left(\frac{\partial}{\partial t}u_\parallel + \mathbf{u}\cdot\nabla u_\parallel\right) + \mathbf{b}^*\cdot\nabla p_\perp + n\mathbf{b}^*\cdot\nabla(e\psi) + T_\perp\mathbf{b}^*\cdot\nabla(ne\Omega). \tag{4.20}$$

The moment equations for n and $T_\perp$ are already given in equations (4.6,4.7). These three equations, with the drift velocity given in equation (4.19), constitute the gyrofluid moment part of the model, prior to the diamagnetic cancellations, below.

Comparison to previous models, especially [5], requires expansion of the compact forms, especially in $\mathbf{u}$, in the three gyrofluid moment equations. We will delay this, however, until after the diamagnetic cancellations.

4.2.3 The polarisation equation

The equations presented so far describe independent evolution of each species of the plasma particles, through the gyrofluid moment variables, each with its own charge state and particle mass. Moreover, a specific equation for the gyroaveraged potential has not appeared, and an arbitrary form may be used for the gyroaveraging of ϕ denoted by $||\phi||$. We now specify the field equation by taking the functional derivative of the Lagrangian with respect to ϕ. Here, we may still have an arbitrary form for $||...||$ but now it must have a well defined Hermitian conjugate, denoted by $||...||^\dagger$. We also have to add up the contributions of all the species to the total system Lagrangian.

The part of the Lagrangian dependent upon ϕ is

$$L = \cdots + \int d^3x \left(\sum_i \left[nm\frac{v_E^2}{2} - ne||\phi|| \right] + \frac{E^2}{8\pi} \right), \tag{4.21}$$

where the subscript i is the species label, and the species dependent nature of e, m, n, and $||...||$, dependent upon $T_\perp$, is understood. Variation with respect to ϕ, noting that v_E^2 and E^2 are quadratic in the gradients of ϕ, integrating in each case one of these gradients by parts, and denoting the Hermitian conjugate of $||...||$ as $||...||^\dagger$, yields

$$\delta L = \cdots - \int d^3x \delta\phi \left(\sum_i \left[||ne||^\dagger + \nabla \cdot \frac{nmc^2}{B^2} \nabla_\perp \phi \right] + \frac{\nabla^2 \phi}{4\pi} \right). \tag{4.22}$$

The demand that this be stationary requires the quantity in parentheses to vanish, so that

$$\sum_i \left[||ne||^\dagger + \nabla \cdot \frac{nmc^2}{B^2} \nabla_\perp \phi \right] + \frac{\nabla^2 \phi}{4\pi} = 0. \tag{4.23}$$

This is the polarisation equation. The last term is the divergence of the electric field, and it comes from the electric field energy in the Lagrangian. The terms inside the square brackets give the total space charge density of species i. The two contributions to this are the gyrocenter density, the first term, reflecting the part of the gyrokinetic distribution function which is independent of the gyrophase, and the polarisation density, the second term, reflecting the gyrophase dependent part [9, 3].

If the electric field divergence is neglected, we recover the situation of quasineutrality: effects due to the actual space charge density are neglected even though the electric field

divergence is finite. This is a matter of assuming the space charge is overcome by polarisation, or in terms of parameters, that the Alfvén velocity is subrelativistic. One can see here why the gyroscreening term in the Hamiltonian is important, since it is what gives rise to the polarisation density in equation (4.23). Neglecting this would have made quasineutrality impossible to recover.

It is possible to use a Taylor expansion form for the gyroaveraging, which will lead to computationally tractable forms. The formal gyroaverage given a particular directed Larmor radius ρ is [10]

$$\oint d\theta\,\phi = \oint d\theta\,\phi(\mathbf{R}+\rho) = \oint d\theta \exp(\rho\cdot\nabla)\phi(\mathbf{R}), \tag{4.24}$$

with the integral taken over the gyrophase angle and the Taylor expansion compactly written as the exponential operator. The fluid moment of this over a Maxwellian is constructed, term by term, using the fact that for the Maxwellian all even moments can be expressed in terms of the temperature, yielding

$$||\phi|| = \exp\left(\frac{\rho_i^2}{2}\nabla_\perp^2\right)\phi(\mathbf{R}), \tag{4.25}$$

where $\nabla_\perp^2$ is the perpendicular Laplacian given by $\nabla\cdot\nabla_\perp$, and ρ_i is the thermal gyroradius for species i, given by $\rho_i^2 = m(c/eB)^2 T_\perp$. It is important to note that this operator is not self-adjoint, so that in the integrations by parts used to derive the polarisation equation the entire chain in the order of operations is reversed. The polarisation equation then becomes

$$\sum_i \left[\exp\left(\nabla_\perp^2\frac{\rho_i^2}{2}\right)(ne) + \nabla\cdot\frac{nmc^2}{B^2}\nabla_\perp\phi\right] + \frac{\nabla^2\phi}{4\pi} = 0, \tag{4.26}$$

in which each power of the combination of $\nabla_\perp^2$ and ρ_i^2 acts upon ne.

Using these forms we may also directly compute the FLR potential,

$$\Omega = \frac{\partial||\phi||}{\partial T_\perp} = \frac{\rho_i^2}{2T_\perp}\nabla_\perp^2||\phi||. \tag{4.27}$$

It is also possible to use a Padé approximant for the gyroaveraging operator, as in the standard local models [3, 5]. Therein, the gyroaveraging is done at the gyrokinetic level using the Bessel function J_0 in Fourier space, taking as argument the product of the perpendicular wavenumber $k_\perp$ and the particle velocity $v_\perp$. The closure therein uses the fact that the average of J_0^2 over a Maxwellian is the function Γ_0, taking as argument the product $k_\perp^2\rho_i^2$. So the model uses the approximation $||\phi|| = \Psi$, which is defined as

$$\Psi \equiv \Gamma_0^{1/2}(k_\perp^2\rho_i^2)\phi = \sum_{\mathbf{k}}\left[\Gamma_0(k_\perp^2\rho_i^2)\right]^{1/2}\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{R}), \tag{4.28}$$

wherein the gyroradius ρ_i is a constant parameter. In a fully nonlocal model it is not possible to use Fourier forms explicitly. Nevertheless, the Padé approximants given in

[3, 5] can be used generally if $k_\perp^2$ is replaced by $-\nabla_\perp^2$, as is necessary even in a local model if the boundary conditions preclude Fourier methods [6]. In this case we have

$$||\phi|| = \left(1 - \frac{\rho_i^2}{2}\nabla_\perp^2\right)^{-1}\phi, \tag{4.29}$$

and the polarisation equation becomes

$$\sum_i \left[\left(1 - \nabla_\perp^2 \frac{\rho_i^2}{2}\right)^{-1}(ne) + \nabla\cdot\frac{nmc^2}{B^2}\nabla_\perp\phi\right] + \frac{\nabla^2\phi}{4\pi} = 0, \tag{4.30}$$

where the operator in the first term is the Hermitian conjugate of the Padé approximant to the gyroaveraging operator acting upon the potential.

In models which do not involve scales below the gyroradius of any species it is possible to use these expansions keeping only the first term in each case. Whichever version is used, the principal consideration is consistency between $||\phi||$ and the polarisation equation, and the dependence of $||...||$ upon $T_\perp$.

The corrections to the density variable due to the finite gyroradius are what give rise to the diamagnetic vorticity. Assuming a single component, singly charged, quasineutral plasma in which the electron gyroradius is negligible, both this FLR effect and the polarisation itself come solely from the ions. Taking the lowest order FLR correction to the gyroaveraging (equations (4.25),(4.26)), we have

$$n_i e + \nabla_\perp^2 \frac{M_i c^2 n_i T_{i\perp}}{2eB^2} + \nabla\cdot\frac{n_i M_i c^2}{B^2}\nabla_\perp\phi = n_e e. \tag{4.31}$$

The second term on the left hand side, the FLR correction to $n_i e$, is the same size as the polarisation term if $n_i e\nabla_\perp^2\phi$ is comparable to $\nabla_\perp^2 p_{i\perp}$. The term due to ϕ reflects the ExB vorticity, and the term due to $p_{i\perp}$ reflects the diamagnetic vorticity. Since the electron parallel dynamics causes $n_e e\nabla\phi$ to be comparable to disturbances in ∇p_e generally, it follows that if the electron and ion temperatures are comparable, then the ExB and diamagnetic vorticities will be comparable. This is also known from fluid drift models and it is important in breaking the conservation of mean squared vorticity in small scale turbulence [25]. Even at arbitrarily large scale, therefore, we require retention of this lowest order FLR correction to the ions if the perpendicular electric field is not large compared to the ion pressure gradient; in other words, any time MHD ordering is not applicable. This is the usual situation of gradient driven turbulence [17, 21].

4.3 The energy integral

The energy conservation law can be obtained from the Lagrangian using the Noether's theorem [24], or Jacobi integral, as has been done for the gyrokinetic-Maxwell system [36, 37], and been presented here in Section 2.2 In general, if $\eta_\alpha(\mathbf{x},t)$ are the field variables,

and $\mathcal{L}_\alpha$ is the part of the Lagrangian density that contains η_α and $\dot{\eta}_\alpha \equiv \partial\eta_\alpha/\partial t$, the energy integral will be given by equation (2.34)

$$\frac{\partial \mathcal{E}}{\partial t} = 0 \qquad \text{with} \qquad \mathcal{E} = \sum_a \int d^3x \dot{\eta}_a \frac{\partial \mathcal{L}_a}{\partial \dot{\eta}_a} - L. \tag{4.32}$$

In the present case, the field variables η_a are $\{n, T_\perp, q_2\}$ for each species (the variables appearing explicitly with the time derivative). This form is also found using the Legendre transformation

$$\mathcal{E} = \int d^3x \left(\dot{n}\frac{\partial \mathcal{L}}{\partial \dot{n}} + \dot{T}_\perp \frac{\partial \mathcal{L}}{\partial \dot{T}_\perp} + \dot{q}_2 \frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \mathcal{L} \right), \tag{4.33}$$

where $\mathcal{L}$ is the Lagrangian density given by $L = \int d^3x \mathcal{L}$.

Each of the time derivatives appears in a quadratic form with a Lagrange multiplier, so that the latter drop out of the energy (Hamiltonian density). The energy is given by

$$\mathcal{E} = \int d^3x \left(\sum_i \left[nm\frac{u_\parallel^2}{2} + nT_\perp + ne||\phi|| - nm\frac{v_E^2}{2} \right] - \frac{E^2}{8\pi} \right), \tag{4.34}$$

noting the sum over species i. The terms in the square brackets are the parallel kinetic energy, thermal energy and the drift energy due to the generalised potential including the gyroscreening, respectively, and the last term is the proper electric field energy. The drift energy appears as a combination with a term that looks like a charge density, which is ultimately why the ExB drift energy and electric field energy appear with minus signs. The positive sign can be restored using the polarisation equation: multiplying equation (4.23) by ϕ and integrating over d^3x, we find

$$\int d^3x \left(\sum_i \left[nm\frac{v_E^2}{2} \right] + \frac{E^2}{8\pi} \right) = \int d^3x \left(\sum_i \left[ne||\phi|| - nm\frac{v_E^2}{2} \right] - \frac{E^2}{8\pi} \right). \tag{4.35}$$

Substituting into equation (4.34) for $ne||\phi||$, we then find the total energy in more familiar terms:

$$\mathcal{E} = \int d^3x \left(\sum_i \left[nm\frac{u_\parallel^2}{2} + nT_\perp + nm\frac{v_E^2}{2} \right] + \frac{E^2}{8\pi} \right). \tag{4.36}$$

Now we have just the three contributions: parallel kinetic energy, thermal energy, and potential energy, the latter split between the ExB drift energy of the plasma and the proper electric field energy. In the case of quasineutrality (cf. discussion after equation (4.23)), the neglect of the electric field divergence in equation (4.23) corresponds consistently to the neglect of $E^2/8\pi$ in equation (4.36)

A similar treatment is used to recast the time dependence of the terms in $\mathcal{E}$ involving ϕ in terms of the time derivatives of the density and temperature. We present this below, after treating the diamagnetic cancellation.

4.4 Diamagnetic cancellation

The equations of evolution of the parallel velocity and temperature as derived here contain terms reflecting advection by the diamagnetic velocity, $\mathbf{u}^* = (c/neB_\parallel^*)\mathbf{b}\times\nabla p_\perp$, as noted after equation (4.19). More specifically, the momentum equation contains the term $\mathbf{u}^*\cdot\nabla u_\parallel$, and the temperature equation contains the combination $\mathbf{u}^*\cdot\nabla T_\perp + T_\perp\nabla\cdot\mathbf{u}^*$ arising from advection by and divergence of $\mathbf{u}^*$. Since the diamagnetic velocity does not correspond to a displacement of the gyrocenters, and thus to an actual transport of the conserved quantities, these diamagnetic advection terms in the gyrofluid moment equations should be cancelled by introducing appropriate flux divergences corresponding to higher moments not appearing in the original Lagrangian. This procedure is also necessary to obtain the correct correspondence between fluid and gyrofluid models as explained elsewhere [38].

For fluid models this cancellation appears automatically through the higher moments. There too exists the diamagnetic velocity arising from the pressure gradient. In the temperature equation a diamagnetic heat flux arises from the temperature gradient, and in the momentum equation gyroviscosity arises from the velocity gradient. The diamagnetic advection terms arising from the diamagnetic velocity are then cancelled from the diamagnetic fluxes. For instance, $\mathbf{u}^*\cdot\nabla T_\perp$ is cancelled in the fluid temperature equation by the diamagnetic heat flux [26], and $\mathbf{u}^*\cdot\nabla u_\parallel$ is cancelled in the fluid momentum equation by the non-diagonal elements of the pressure tensor, equivalent to the diamagnetic momentum flux [28].

In a gyrofluid model this cancellation must be done manually, since the higher moments are not included in the Lagrangian. During this it is necessary to avoid changes to the energy integral arising from the application of Noether's theorem to the model. Thus the final equations of evolution do not arise in all detail directly from the Lagrangian, but they still conserve energy exactly.

In the velocity equation, the term added is

$$\nabla\cdot\blacksquare_{\perp\parallel} = \nabla\cdot\left(\frac{mc}{eB_\parallel^*}p_\perp\mathbf{b}\times\nabla u_\parallel\right), \tag{4.37}$$

and it resembles the divergence of the perp-parallel component of the pressure tensor that does the diamagnetic cancellation in the fluid models. Since this term is a total divergence, it conserves parallel momentum exactly. Further, since the contraction of the stress tensor itself (the term in brackets) with $\nabla u_\parallel$ vanishes, it also conserves energy exactly. This term is the same as the Braginskii gyroviscosity for the parallel flow [27].

To zeroth order in ϕ, the term that is added to the temperature equation is

$$\nabla\cdot\mathbf{q}_\perp^{(0)} = \nabla\cdot\left(2\frac{c}{eB_\parallel^*}p_\perp\mathbf{b}\times\nabla T_\perp\right), \tag{4.38}$$

which resembles the divergence of the perpendicular heat flux that works together with the diamagnetic velocity divergence to do the cancellation in the fluid models. This term is the same as the Braginskii diamagnetic heat flux for this model [27].

This zeroth order heat flux by itself breaks energy conservation (potential plus thermal), so this must be restored using appropriate FLR corrections. The heat flux affects the part of the energy evolution resulting from the time derivative of the temperature, according to

$$\frac{\partial \mathcal{E}}{\partial t} = \cdots + \int d^3x \sum_i \left[n \frac{\partial T_\perp}{\partial t} + ne\Omega \frac{\partial T_\perp}{\partial t} \right], \tag{4.39}$$

where the second term is the FLR correction to the evolution of the potential energy (cf. next Section), arising from the dependence of $||\phi||$ upon $T_\perp$ (this is also the origin of Ω). The appropriate corrections to cancel this piece are added to the heat flux. In addition, we must specify a further FLR correction, given by

$$\mathbf{w} = \frac{cT_\perp}{eB^*_\parallel} \mathbf{b} \times \nabla (e\Omega), \tag{4.40}$$

to the velocity advecting the temperature in order to maintain the energy conservation law — this will correspond the extra FLR nonlinearity present in the perpendicular temperature equation of [3, 5], as noted below. The origin of this quantity is the residual arising from $\mathbf{q}^{(0)}_\perp \cdot \nabla(e\Omega)$ which would be left uncancelled in the energy theorem if we did not further correct the temperature equation. The manipulations make use of the fact that $\mathbf{w} \cdot \nabla(e\Omega)$ and $\mathbf{q}^{(0)}_\perp \cdot \nabla(e\Omega) + 2n\mathbf{w} \cdot \nabla T_\perp$ both vanish. It is important to note that these forms occur with $\nabla(e\Omega)$ and not $\nabla(ne\Omega)$, in contrast to the term appearing in the gyrofluid velocity, equation (4.19).

The final correction to the temperature equation is

$$\begin{aligned} 2T_\perp \nabla \cdot (n\mathbf{w}) + \nabla \cdot (\mathbf{q}_\perp) = \\ = \nabla \cdot \left(2 \frac{c}{eB^*_\parallel} p_\perp \mathbf{b} \times \nabla T_\perp \right) - 2 \frac{c}{B^*_\parallel} p_\perp \mathbf{b} \times \nabla \frac{\partial ||\phi||}{\partial T_\perp} \cdot \nabla T_\perp + \\ + \nabla \cdot \left(2 \frac{c}{B^*_\parallel} \frac{\partial ||\phi||}{\partial T_\perp} p_\perp \mathbf{b} \times \nabla T_\perp \right) + \nabla \cdot \left(2T_\perp^2 n \frac{c}{B^*_\parallel} \mathbf{b} \times \nabla \frac{\partial ||\phi||}{\partial T_\perp} \right), \end{aligned} \tag{4.41}$$

where the FLR-corrected heat flux is given by

$$\mathbf{q}_\perp = (1 + e\Omega)\mathbf{q}^{(0)}_\perp = 2p_\perp (1 + e\Omega) \frac{c}{eB^*_\parallel} \mathbf{b} \times \nabla T_\perp. \tag{4.42}$$

Thus we cancel the diamagnetic advection by adding a heat flux term, and order by order we assure energy conservation by adding further FLR corrections to both the heat flux and the advecting velocity. The procedure remains systematic.

Including these diamagnetic flux terms in the gyrofluid moment equations, we find the density evolution unchanged

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{u} = 0, \tag{4.43}$$

and the parallel velocity equation becomes,

$$nm\left(\frac{\partial}{\partial t}u_{\parallel}+\mathbf{u}\cdot\nabla u_{\parallel}\right)+\nabla\cdot\blacksquare_{\perp\parallel}=-\mathbf{b}^{*}\cdot\left(\nabla p_{\perp}+ne\nabla\psi+T_{\perp}\nabla ne\Omega\right), \tag{4.44}$$

and the temperature equation becomes

$$n\left(\frac{\partial T_{\perp}}{\partial t}+\mathbf{u}\cdot\nabla T_{\perp}\right)+p_{\perp}\nabla\cdot\mathbf{u}+2T_{\perp}\nabla\cdot n\mathbf{w}+\nabla\cdot\mathbf{q}_{\perp}=0. \tag{4.45}$$

These three equations, plus specification of $\mathbf{u}$, $\blacksquare_{\perp\parallel}$, $\mathbf{w}$, and $\mathbf{q}_{\perp}$ in equations (4.19, 4.37, 4.40, 4.42), respectively, plus the polarisation equation, form the final equations of the model. The corresponding energy conservation law is given in equations (4.32, 4.36).

For comparison to previous models, we expand these forms by substituting equations (4.19, 4.37, 4.41), leaving forms in which only the dependent variables appear. This gives the expanded density equation:

$$\begin{aligned}\frac{\partial n}{\partial t}+\frac{c}{B_{\parallel}^{*}}\mathbf{b}\times\nabla\psi\cdot\nabla n+\nabla\cdot\left(nu_{\parallel}\mathbf{b}^{*}\right)-n\mathcal{K}(\psi)-\frac{1}{e}\mathcal{K}(p_{\perp})+\\+\frac{c}{B_{\parallel}^{*}}\mathbf{b}\times\nabla\left(n\Omega\right)\cdot\nabla T_{\perp}-T_{\perp}\mathcal{K}(n\Omega)=0,\end{aligned} \tag{4.46}$$

the expanded parallel velocity equation

$$\begin{aligned}nm\frac{\partial u_{\parallel}}{\partial t}+nm\frac{c}{B_{\parallel}^{*}}\mathbf{b}\times\nabla\psi\cdot\nabla u_{\parallel}+nmu_{\parallel}\mathbf{b}^{*}\cdot\nabla u_{\parallel}-\frac{m}{e}p_{\perp}\mathcal{K}(u_{\parallel})+\\+\mathbf{b}^{*}\cdot\nabla p_{\perp}+ne\mathbf{b}^{*}\cdot\nabla\psi+\\+T_{\perp}m\frac{c}{B_{\parallel}^{*}}\mathbf{b}\times\nabla\left(n\Omega\right)\cdot\nabla u_{\parallel}+T_{\perp}\mathbf{b}^{*}\cdot\nabla\left(ne\Omega\right)=0,\end{aligned} \tag{4.47}$$

and the expanded perpendicular temperature equation

$$\begin{aligned}n\frac{\partial T_{\perp}}{\partial t}+n\frac{c}{B_{\parallel}^{*}}\mathbf{b}\times\nabla\psi\cdot\nabla T_{\perp}+n\nabla\cdot\left(T_{\perp}u_{\parallel}\mathbf{b}^{*}\right)-\\-nT_{\perp}\mathcal{K}(\psi)-\frac{T_{\perp}}{e}\mathcal{K}(p_{\perp})-2\frac{p_{\perp}}{e}\mathcal{K}(T_{\perp})+\\+T_{\perp}^{2}\frac{c}{B_{\parallel}^{*}}\mathbf{b}\times\nabla\Omega\cdot\nabla n+2p_{\perp}\frac{c}{B_{\parallel}^{*}}\mathbf{b}\times\nabla\Omega\cdot\nabla T_{\perp}-\\-T_{\perp}^{2}\mathcal{K}(n\Omega)-2nT_{\perp}^{2}\mathcal{K}(\Omega)-2p_{\perp}\Omega\mathcal{K}(T_{\perp})=0,\end{aligned} \tag{4.48}$$

while the conserved energy is still that appearing in equations (4.34, 4.36). In these forms, the operator $\mathcal{K}$ denotes the compressibility of perpendicular drift fluxes in a toroidal magnetic field, defined as

$$\mathcal{K}(f)\equiv\nabla\cdot\left(\frac{c}{BB_{\parallel}^{2}}\mathbf{B}\times\nabla f\right), \tag{4.49}$$

for any scalar f. The equations of evolution of the gyrofluid moment variables resemble to a large extent equations (3.82, 3.83, 3.84, 3.85) by Beer and Hammett [5]. The differences that arise can be separated in three groups: (1) differences because of the use of only one temperature in this model (2) differences because of lack of higher order moments in the Lagrangian and (3) differences because of the energy conservation.

In the first group lie differences in the terms that include pressure or temperature, and arise because of the different models used. More specifically, in this paper we treat only the perpendicular temperature, for two degrees of freedom, rather than the separate set of parallel and perpendicular temperatures. Especially in the density equation, those differences can disappear if one replaces $p_\perp$ with $(p_\parallel + p_\perp)/2$ in the curvature terms (including the FLR curvature term).

In the second group lie differences in terms that come from the closures of higher order moments (e.g., heat flux). In this model we do not include higher order moments than the pressure in the Lagrangian, and the closures are done automatically, by choosing the gyro-averaged potential to be operated by the Bessel function (equation (4.23)). Thus for instance, Landau damping terms do not appear in our model.

In the third, and most important group, lie differences because of the energy conservation law. Especially in the FLR terms, previous models do not follow energy conservation. Beer and Hammett use three potentials, all arising from ϕ: the gyro-reduced potential, Ψ, defined in equation (4.28), an FLR potential arising from the first derivative with respect to temperature, $(1/2)\widehat{\nabla}_\perp^2\Psi$, and a second FLR potential arising from a further temperature derivative, $\widehat{\widehat{\nabla}}_\perp^2\Psi$. The first two of these correspond to our $||\phi||$ according to equation (4.28) and $\widehat{\Omega} = n\Omega$ (note the v_E^2 component in our ψ doesn't appear due to the drift ordering used in the local models), and we could also derive a second FLR potential $\widehat{\widehat{\Omega}}$ by taking a further temperature derivative of $\widehat{\Omega}$. As shown elsewhere [20] these FLR potentials correspond energetically with the second and higher moments in the polarisation energetics, e.g., $\widehat{\Omega}$ with $T_\perp$ and $\widehat{\widehat{\Omega}}$ with the non-Maxwellian fourth moment $r_\perp$ (where the Maxwellian part is in our case the combination $2p_\perp T_\perp$ appearing in the diamagnetic cancellations). If $r_\perp$ is not kept as a dynamical variable in its own right, there is no place for $\widehat{\widehat{\Omega}}$ in the energetics. Reference [20] repairs the situation by substituting $\widehat{\widehat{\Omega}}$ with $\widehat{\Omega}$ with the appropriate coefficients. In our model terms with $\widehat{\widehat{\Omega}}$ do not appear, and one can see the correlation between $T_\perp$ and $\widehat{\Omega}$ in the energetics. Thus in the density and temperature equations the curvature terms $\mathcal{K}(p_\perp)$ and $T_\perp\mathcal{K}(n\Omega)$ appear in combination. Moreover, one can see that our polarisation equation (equation (4.23)) reduces to the local one [5, 6], i.e., in agreement with the local model in the local limit.

4.5 Explicit form of the energy conservation law

Here we present the energy conservation law in terms of the evolution of each of the contributions to the total energy and the transfer mechanisms which exist between them. The total energy is given by

$$\mathcal{E} = \int d^3x \left(\sum_i \left[nm\frac{u_\parallel^2}{2} + nT_\perp + nm\frac{v_E^2}{2} \right] + \frac{E^2}{8\pi} \right), \tag{4.50}$$

representing the parallel kinetic energy, thermal energy, ExB drift energy, and electric energy, the latter two collectively making up the potential energy. Here and below, quantities appearing in square brackets are summed over species.

The time derivative of the potential energy involves contributions due to $\partial\phi/\partial t$, which are eliminated using the polarisation equation. We use equation (4.35) to recast these as

$$\begin{aligned} &\frac{\partial}{\partial t}\int d^3x \left(\sum_i \left[nm\frac{v_E^2}{2} \right] + \frac{E^2}{8\pi} \right) \\ &\quad = \int d^3x \sum_i \left[e\psi\frac{\partial n}{\partial t} + ne\Omega\frac{\partial T_\perp}{\partial t} \right] \\ &\quad + \int d^3x \left(\sum_i \left[||ne||^\dagger + \nabla\cdot\frac{nmc^2}{B^2}\nabla_\perp\phi \right] + \frac{\nabla^2\phi}{4\pi} \right)\frac{\partial\phi}{\partial t}, \end{aligned} \tag{4.51}$$

wherein it is evident that the last combination in parentheses vanishes, due to equation (4.23). The Hermitian property of the gyroaveraging operator has been used to have it act upon ne in this line. Together with the time derivatives of the thermal and parallel kinetic energies, we have

$$\frac{\partial}{\partial t}\int d^3x \left(\sum_i \left[nm\frac{v_E^2}{2} \right] + \frac{E^2}{8\pi} \right) = \int d^3x \sum_i \left[e\psi\frac{\partial n}{\partial t} + ne\Omega\frac{\partial T_\perp}{\partial t} \right] \tag{4.52}$$

$$\frac{\partial}{\partial t}\int d^3x \sum_i \left[nT_\perp \right] = \int d^3x \sum_i \left[T_\perp\frac{\partial n}{\partial t} + n\frac{\partial T_\perp}{\partial t} \right] \tag{4.53}$$

$$\frac{\partial}{\partial t}\int d^3x \sum_i \left[nm\frac{u_\parallel^2}{2} \right] = \int d^3x \sum_i \left[m\frac{u_\parallel^2}{2}\frac{\partial n}{\partial t} + nmu_\parallel\frac{\partial u_\parallel}{\partial t} \right], \tag{4.54}$$

for the evolution of the pieces of the total energy, cast in terms of the time derivatives solely of the gyrofluid moment variables multiplied by various combinations involving all the dependent variables.

Applying equations (4.46–4.48) to evaluate the time derivatives, we obtain equations that contain either total divergences — which conserve energy within each equation — or terms that exchange energy between the kinetic, thermal and potential contributions to the

energy. Use is made of the Poynting cancellation [22], whose simplest form in this context is

$$\nabla\cdot(p_\perp\mathbf{u}_E+e\psi\mathbf{u}^*)=-\mathcal{K}(p_\perp e\psi)\,, \tag{4.55}$$

where $\mathbf{u}_E$ is the main ExB drift given by $\mathbf{u}_E=(c/eB^*_\parallel)\mathbf{b}\times\nabla(e\psi)$. In the following, we also define the FLR correction to the ExB velocity as $\mathbf{w}_E=(cT_\perp/neB^*_\parallel)\mathbf{b}\times\nabla(ne\Omega)$, noting that due to the factors of n this differs slightly from $\mathbf{w}$. It is useful to do the manipulations involving only n, $u_\parallel$, $T_\perp$ or $p_\perp$, and the pieces of $\mathbf{u}$ first, and then add the corrections due to $\blacksquare_{\perp\parallel}$, $\mathbf{q}_\perp$, and $\mathbf{w}$.

The contribution of each species to the ExB energy equation is given by

$$\begin{aligned}&e\psi\frac{\partial n}{\partial t}+ne\Omega\frac{\partial T_\perp}{\partial t}+\nabla\cdot\left[(ne\psi+p_\perp e\Omega)\,\mathbf{u}+p\,(\mathbf{u}_E+\mathbf{w}_E)+e\Omega\,(\mathbf{q}_\perp+2p\,\mathbf{w})\right]\\&=u_\parallel\mathbf{b}^*\cdot[ne\nabla\psi+T_\perp\nabla(ne\Omega)]+p\nabla\cdot\mathbf{u}_E+p\nabla\cdot\mathbf{w}_E-2n\mathbf{w}\cdot\nabla T_\perp.\end{aligned} \tag{4.56}$$

The contribution of each species to the parallel kinetic energy equation is given by

$$\frac{\partial}{\partial t}\left(nM\frac{u_\parallel^2}{2}\right)+\nabla\cdot\left[nM\frac{u_\parallel}{2}\mathbf{u}+u_\parallel\blacksquare_*\right]=-u_\parallel\mathbf{b}^*\cdot[ne\nabla\psi+T_\perp\nabla(ne\Omega)+\nabla p]\,. \tag{4.57}$$

The contribution of each species to the thermal energy equation is given by

$$\begin{aligned}&\frac{\partial p_\perp}{\partial t}+\nabla\cdot\left[p\,(\mathbf{u}_E+\mathbf{w}_E)+2p\left(\mathbf{u}^*+u_\parallel\mathbf{b}^*+\mathbf{w}\right)+\mathbf{q}_\perp\right]\\&\qquad=u_\parallel\mathbf{b}^*\cdot\nabla p-p\nabla\cdot\mathbf{u}_E-p\nabla\cdot\mathbf{w}_E+2n\mathbf{w}\cdot\nabla T_\perp.\end{aligned} \tag{4.58}$$

Terms appearing under the total divergences give the main transport effects, of which the largest contributors are those due to $\mathbf{u}_E$ [22]. Pairs of identical terms appearing with opposite sign in two equations give the transfer processes. The Poynting cancellation is represented by the ExB transport entering with the single factor of p and direct transfer from ExB to thermal energy occurring through compression of $\mathbf{u}_E$ and $\mathbf{w}_E$. The transport effects conserve energy within each corresponding equation, with the nonlinear parts (dominantly, the ExB fluxes) exchanging energy within a given dependent variable but between various scales. The transfer processes are very useful in computations to decide the physical mechanism behind particular aspects of the dynamics [15, 21]. They are conservative, which is shown trivially by summing equations (4.56–4.58), upon which the transfer terms cancel mutually.

The transfer processes are all clearly identifiable. The ones directly between the ExB and thermal energy are the main and FLR ExB compression, and the FLR heat flux pieces arising from $\mathbf{w}\cdot\nabla T_\perp$, or equivalently $\mathbf{q}_\perp^{(0)}\cdot\nabla(e\Omega)$. The parallel dynamics also transfers energy between the ExB and thermal components, but mediated by the parallel kinetic energy. These include the Alfvén transfer (from $\nabla\psi$), its FLR correction (from $\nabla ne\Omega$), and the parallel compressibility (from ∇p). Note that upon summation the ion and electron charges

subtract, so that $\sum_i ne\psi$ is the generalised vorticity and $\sum_i neu_\parallel$ is the parallel current, $J_\parallel$. The parallel compressibility splits into two pieces given by sound waves ($u_\parallel \nabla_\parallel p$, with p the total pressure) and the adiabatic transfer between the pressure and the Alfvén dynamics ($J_\parallel \nabla_\parallel p_e$, with p_e the electron pressure).

The model derived in this chapter describes correctly the dynamics in the perpendicular plane. ExB velocity, diamagnetic compression and FLR terms are all retained. It does, however, not distinguish between the parallel and perpendicular temperature, and contains no dissipation due to collisions and Landau damping. The lack of dissipation makes that this model does not describe drift waves. Reactive instabilities like the ion temperature gradient (ITG) and electron temperature gradient (ETG) modes are retained. Although the principle dynamics of these modes occurs in the perpendicular plane non negligible corrections due to the different parallel temperature and landau damping occur. Our model can therefore be used for basic studies on the dynamics but is not expected to describe the instabilities with high accuracy. In the next chapter we will lift one of the restrictions of the model, and include the parallel temperature as a dynamical variable.

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Chapter 5

Gyrofluid theory: four-moment model

In Chapter 4, we derived a fully nonlinear set of electrostatic gyrofluid equations that satisfy an exact energy theorem. This set contains the evolution equations for density, momentum, and perpendicular temperature, as well as the polarisation equation for the electrostatic potential. The energy conservation law, which is guaranteed by the Noether method through the Lagrangian formulation of gyrofluid dynamics, is a unique and important feature of this model, which is thought to be especially important for the numerical simulation of turbulence. Although the single-temperature gyrofluid model is consistent, it does not contain the parallel-temperature effects necessary for a quantitatively accurate description of turbulence in magnetised plasmas. In the present Chapter, we extend our previous model to include these effects. Although the model is still electrostatic, it should give an accurate description of low β turbulent plasmas.

The extension to anisotropic temperatures ($T_{\parallel} \neq T_{\perp}$) turns out to be nontrivial. In the single-temperature case, the constraints used in the variation of the Lagrangian density can be chosen to be the particle and entropy conservation laws. In the two-temperature case, one needs an additional constraint and one can no longer rely on (only) conservation laws. The approach used here is to derive the constraints for the parallel and perpendicular pressure through the full Chew-Goldberger-Low (CGL) equations [8, 9]. Alternative constraints (despite being physically motivated) are later shown to fail in deriving valid gyrofluid moment equations. A further difference with the previous Chapter is the use of the virtual fluid displacement ξ in deriving Eulerian variations for the gyrofluid moments, instead of the Lagrangian-multiplier method (which turns out to be algebraically more involved).

Gyrofluid equations that include anisotropic parallel and perpendicular pressures have been derived previously through the derivation of moments of the gyrokinetic equation [1]-[6]. Although these equations (see Section 3.3.4) do not conserve energy, they agree to a large extent with our present gyrofluid model.

The remainder of this Chapter is organised as follows. In Section 5.1, the gyrofluid Lagrangian is presented and the constraints on the density and anisotropic pressures used in the variational principle are given. In Section 5.2, the variation of the gyrofluid action

functional is presented, and the gyrofluid equations of motion before the insertion of diamagnetic cancellations, as well as the polarisation equation are derived. In Section 5.3, the energy conservation law by applying the Noether method is derived. In Section 5.4, we describe how the diamagnetic cancellations are inserted into our gyrofluid equations and present the final equations for density, momentum, parallel and perpendicular pressures. Also the explicit form of the energy conservation law with each energy-exchange term clearly identified is presented. Lastly, in Section 5.5, the validity of other possible constraints is discussed and each resulting set of equations with previous models is compared.

5.1 Gyrofluid Lagrangian density and Lagrangian constraints

5.1.1 Gyrofluid Lagrangian density

The four-moment gyrofluid Lagrangian for the anisotropic-temperature model is constructed in the same way as that of the one-temperature model in Section 4.1, based on the work of Pfirsch and Correa-Restrepo [10]. Here, the gyrofluid Lagrangian density is defined as

$$\mathcal{L}_f = \frac{1}{2} nm \left(u_\parallel^2 + |\mathbf{u}_E|^2 \right) - \left(p_\perp + \frac{p_\parallel}{2} \right) + en \left(\mathbf{A} \cdot \frac{\mathbf{u}}{c} - \|\phi\| \right), \tag{5.1}$$

where $\mathbf{u}_E = (c/B)\,\mathbf{b} \times \nabla\phi$ is the $E \times B$ velocity, $u_\parallel \equiv \mathbf{u} \cdot \mathbf{b}$ denotes the gyrofluid velocity parallel to the magnetic field $\mathbf{B} \equiv \nabla \times \mathbf{A} = B\,\mathbf{b}$, and the nonlocal operator $\|...\|$ denotes a gyrofluid gyro-averaging operator. For better understanding of the calculations, we define the gyrofluid gyroaveraging operator as follows. First, we note that, in our four-moment model, the electrostatic gyro-potential $\|\phi\|$ depends on the perpendicular temperature $T_\perp \equiv p_\perp/n$ and is assumed not to depend on higher-order gyrofluid moments. Thus, a general expression for $\|\phi\|$ can be given as

$$\|\phi\| \equiv \mathcal{P}\left(\frac{\rho^2}{2}\, \nabla_\perp^2 \right) \phi = \sum_k \frac{\mathcal{P}^{(k)}(0)}{2^k\, k!}\, \rho^{2k} \nabla_\perp^{2k} \phi, \tag{5.2}$$

where $\rho^2 = (mc^2/e^2B^2)\, T_\perp$ is the squared Larmor radius and $\mathcal{P}(x)$ represents a smooth function such that $\mathcal{P}^0(0) \equiv 1$, i.e., we have $\|\phi\| \to \phi$ in the zero-Larmor-radius (ZLR) limit. Two choices for $\mathcal{P}(x)$ will be discussed below; here, we note that the operator $\nabla_\perp^2$ in the Taylor expansion (5.2) acts on ϕ alone. The Hermitian conjugate operator $\|...\|^\dagger$ is defined in terms of equation (5.2) as

$$\|\chi\|^\dagger \equiv \sum_k \frac{\mathcal{P}^{(k)}(0)}{2^k\, k!}\, \nabla_\perp^{2k} \left(\chi \rho^{2k} \right) = \mathcal{P}\left(\nabla_\perp^2\, \frac{\rho^2}{2} \right) \chi, \tag{5.3}$$

where χ is an arbitrary function and we made use of the identity

$$\begin{aligned} \chi \|\phi\| \;=\; & \phi\, \|\chi\|^\dagger + \nabla_\perp \cdot \left[\left(\chi_1 + \nabla_\perp^2 \chi_2 + \cdots \right) \nabla_\perp \phi - \phi\, \nabla_\perp \left(\chi_1 + \nabla_\perp^2 \chi_2 + \cdots \right) \right. \\ & \left. + \left(\chi_2 + \cdots \right) \nabla_\perp \nabla_\perp^2 \phi - \nabla_\perp^2 \phi\, \nabla_\perp \left(\chi_2 + \cdots \right) + \cdots \right], \end{aligned} \tag{5.4}$$

where

$$\chi_k \equiv \frac{\partial(\chi\|\phi\|)}{\partial(\nabla_\perp^{2k}\phi)} = \frac{\mathcal{P}^{(k)}(0)}{2^k\,k!}\,\chi\rho^{2k}.$$

Note that the operator $\nabla_\perp^2$ in equation (5.3) now acts on χ and powers of the Larmor radius ρ.

The $E\times B$ kinetic term $m|\mathbf{u}_E|^2/2$ in equation (5.1) is again the long-wavelength form of gyro-screening (i.e., the ZLR part of the second-order potential term in the gyrokinetic Hamiltonian [11]), which plays a major role in the derivation of the polarisation equation. Although the $E\times B$ kinetic term is only an approximation of the full second-order gyrokinetic Hamiltonian, it is useful since the gyro-screening correction to the potential is important only for large flow velocities and long wavelengths. Thus, the addition of full finite-Larmor-radius (FLR) effects for this quadratic term will not qualitatively alter any computational results. The fact that the gyro-screening term appears only in the Hamiltonian leads to the appearance of polarisation effects only in Poisson's equation for the scalar potential ϕ, which is desirable for computational reasons.

Although the drift-fluid Lagrangian of [10] and our gyrofluid Lagrangian are similar, there exist three differences: (a) the gyrofluid Lagrangian (5.1) contains FLR corrections to the scalar potential ϕ, (b) the anisotropic temperatures $T_\perp$ and $T_\parallel$ are treated separately rather than as a single isotropic temperature T, and (c) we have arranged the terms such that the gyro-screening term appears only in the Hamiltonian (i.e., not in the symplectic part involving contraction with $\mathbf{u}$; see [7] for further details). The total Lagrangian density for our four-moment gyrofluid model is the sum of the gyrofluid Lagrangian density (5.1) and the Lagrangian density of the electric field, expressed as

$$\mathcal{L} = \frac{1}{2}\,nm\left(u_\parallel^2 + |\mathbf{u}_E|^2\right) - n\left(T_\perp + \frac{T_\parallel}{2}\right) + en\left(\mathbf{A}\cdot\frac{\mathbf{u}}{c} - \|\phi\|\right) + \frac{|\mathbf{E}|^2}{8\pi}, \tag{5.5}$$

where the variational fields are the four gyrofluid moments $(n,\mathbf{u},p_\parallel,p_\perp)$ for each fluid species (here, summation over particle species is assumed wherever appropriate) and the electrostatic potential ϕ.

5.1.2 Lagrangian constraints

In order to proceed with our variational principle, we introduce constraints on the gyrofluid moments $(n,p_\parallel,p_\perp)$, based on the continuity (mass conservation) equation

$$\frac{dn}{dt} + n\,\nabla\cdot\mathbf{u} = 0, \tag{5.6}$$

where $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ denotes the total time derivative, and the modified CGL equations for the perpendicular and parallel pressures [8, 9]:

$$\frac{dp_\parallel}{dt} + p_\parallel\,\nabla\cdot\mathbf{u} + 2\,p_\parallel\,\mathbf{bb}:\nabla\mathbf{u} = -2\nabla\cdot\mathbf{q}_{\parallel\perp} + 4\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}), \tag{5.7}$$

$$\frac{dp_\perp}{dt} + p_\perp\,\nabla\cdot\mathbf{u} + p_\perp\,(\mathbf{I}-\mathbf{bb}):\nabla\mathbf{u} = -\nabla\cdot\mathbf{q}_{\perp\perp} - 2\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}), \tag{5.8}$$

where $\mathbf{q}_{\parallel\perp}$ and $\mathbf{q}_{\perp\perp}$ are the parallel and perpendicular heat fluxes, respectively. The pressure constraints, without the terms including the heat fluxes, can also be obtained by general conservation laws. If the heat fluxes are removed from the pressure equations (5.7)-(5.8), all gyrofluid quantities $\eta^a = (n, p_\parallel, p_\perp)$ are directly related to the velocity $\mathbf{u}$. Using these equations as constraints for the Lagrangian density (5.5), we can obtain the evolution equation for the gyrofluid velocity $\mathbf{u}$.

We obtain expressions for the Lagrangian variations $\Delta\eta^a$ in terms of the virtual fluid displacement ξ by taking the limits

$$\lim_{\Delta t\to 0}\left(\frac{d\eta^a}{dt}\,\Delta t\right) \equiv \Delta\eta^a \quad \text{and} \quad \lim_{\Delta t\to 0}(\mathbf{u}\,\Delta t) \equiv \xi,$$

so that the variations of the pressures from equations (5.7) and (5.8) become

$$\left.\begin{aligned} \Delta p_\parallel &= -p_\parallel\,\nabla\cdot\xi - 2\,p_\parallel\,\mathbf{bb}:\nabla\xi \\ \Delta p_\perp &= -p_\perp\,\nabla\cdot\xi - p_\perp\,(\mathbf{I}-\mathbf{bb}):\nabla\xi \end{aligned}\right\}. \tag{5.9}$$

The heat-flux terms in equations (5.7)-(5.8), which were not used in the Lagrangian variations (5.11) for $p_\parallel$ and $p_\perp$, are added to the pressure equations later on since they play an important role in the diamagnetic cancellations; these cancellations refer to the fact that the diamagnetic velocity does not lead to advection. The present Lagrangian formulation also requires one additional Lagrangian variation:

$$\Delta\mathbf{u} \equiv d\xi/dt = \partial\xi/\partial t + \mathbf{u}\cdot\nabla\xi \tag{5.10}$$

for the fluid velocity $\mathbf{u}$. For a correct treatment of FLR effects associated with the gyrofluid electrostatic potential $\|\phi\|$, instead of the pressure variations we will use the temperature variations, such that the set of the Lagrangian variations used will be:

$$\left.\begin{aligned} \Delta n &= -n\,\nabla\cdot\xi \\ \Delta\mathbf{u} &= \partial\xi/\partial t + \mathbf{u}\cdot\nabla\xi \\ \Delta T_\parallel &= -2T_\parallel\,\mathbf{bb}:\nabla\xi \\ \Delta T_\perp &= -T_\perp\,(\mathbf{I}-\mathbf{bb}):\nabla\xi \end{aligned}\right\}. \tag{5.11}$$

where Δn is obtained from equation (5.6). The variations used are the temperature ones rather than the pressure ones, since the FLR corrections of the gyrofluid electrostatic potential $||\phi||$ depend on $T_\perp$ rather than $p_\perp$. If the pressure constraints are used, after the Lagrangian variation terms in the form $\frac{\partial||\phi||}{\partial p_\perp}$ are introduced, which are more difficult to handle. The variations in equation (5.11) will be used to derive the equation of evolution for the system as well as the polarisation equation and the energy conservation law. For

better comparison though with previous models, the pressure constraints will be used to derive the gyrofluid equations and to make the diamagnetic cancellation, such that the final equations of evolution will be for the quantities $(n, u_\parallel, p_\parallel, p_\perp)$.

Lastly, one should note that the gyrofluid Lagrangian density (5.5) was constructed, not derived, and that the method adopted here is not a mathematical derivation of the gyrofluid Lagrangian from the gyrokinetic (single-particle) Lagrangian. The validity of the present gyrofluid Lagrangian follows from the correspondence of the resulting gyrofluid equations to those of the other models in regimes where both are valid, as well as the exactness of the energy conservation law. The same applies to the constraint equations, e.g., equations (5.7)-(5.8), which can be introduced arbitrarily, and their validity follows from the validity of the resulting evolution equations.

5.2 Gyrofluid dynamical equations

In this Section, we present a variational principle based on the action functional $S \equiv \int \mathcal{L}\, d^3x dt$ using the virtual-displacement method associated with the Eulerian variations $(\delta n, \delta \mathbf{u}, \delta T_\parallel, \delta T_\perp)$, as was done for the simple fluid case in Section 2.3.2. Here, each Eulerian variation $\delta\chi$ is defined in terms of its Lagrangian variation $\Delta\chi$ as $\delta\chi \equiv \Delta\chi - \xi\cdot\nabla\chi$, so that, using equation (5.11), we find

$$\left.\begin{aligned} \delta n &= -\nabla\cdot(n\,\xi) \\ \delta \mathbf{u} &= \partial_t \xi + (\mathbf{u}\cdot\nabla)\xi - (\xi\cdot\nabla)\mathbf{u} \\ \delta T_\parallel &= -\xi\cdot\nabla T_\parallel - 2T_\parallel\, \mathbf{bb}:\nabla\xi \\ \delta T_\perp &= -\nabla\cdot(T_\perp\,\xi) + T_\perp\, \mathbf{bb}:\nabla\xi \end{aligned}\right\}. \tag{5.12}$$

From the Eulerian variation of the gyrofluid Lagrangian (5.5), we can now derive the moment equation for the gyrofluid velocity $\mathbf{u}$, as well as the polarisation equation for the scalar potential ϕ. Note that the Eulerian variation $\delta T_\perp$ in equation (5.12) is used also in connection with the FLR dependence of $\|\phi\|$:

$$\delta\|\phi\|(\phi, T_\perp) \equiv \|\delta\phi\| + \delta T_\perp \frac{\partial\|\phi\|}{\partial T_\perp}, \tag{5.13}$$

where

$$\frac{\partial\|\phi\|}{\partial T_\perp} = \frac{mc^2}{2e^2B^2}\,\mathcal{P}'\left(\frac{\rho^2}{2}\nabla_\perp^2\right)\,\nabla_\perp^2\phi \;\equiv\; \Omega(\phi, T_\perp; B), \tag{5.14}$$

and $\mathcal{P}'(x)$ denotes the first derivative of $\mathcal{P}(x)$. As a result of the mass scaling in equation (5.14), the ion FLR correction Ω_i is much larger than the electron FLR correction Ω_e.

5.2.1 Variation of the Lagrangian density

It is straightforward to derive the total variation $\delta\mathcal{L}$ of the gyrofluid Lagrangian density (5.5), which depends on the variational fields $(n, \mathbf{u}, T_\parallel, T_\perp, \phi)$, so that we obtain

$$\delta\mathcal{L} = \left(\nabla\delta\phi \cdot \frac{\partial\mathcal{L}}{\partial(\nabla\phi)} - en\,\|\delta\phi\|\right) + \delta n\,\frac{\partial\mathcal{L}}{\partial n} + \delta\mathbf{u}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}} - \left[\frac{n}{2}\,\delta T_\parallel + (1+e\,\Omega)\,n\,\delta T_\perp\right]. \quad (5.15)$$

Here, from equation (5.5), we find

$$\frac{\partial\mathcal{L}}{\partial n} = \frac{m}{2}\,u_\parallel^2 + \frac{e}{c}\,\mathbf{A}\cdot\mathbf{u} - e\,\psi - \left(T_\perp + \frac{1}{2}\,T_\parallel\right) \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\mathbf{u}} = n\left(m\,u_\parallel\,\mathbf{b} + \frac{e}{c}\,\mathbf{A}\right), \quad (5.16)$$

where we introduced the definition

$$e\,\psi(\phi, T_\perp; B) \equiv e\,\|\phi\| - (mc^2/2B^2)\,|\nabla\phi|^2, \quad (5.17)$$

which combines the linear (with FLR corrections) and nonlinear (in the ZLR limit) electrostatic terms of the gyrokinetic Hamiltonian [11].

By inserting the Eulerian variations (5.12) into the gyrofluid Lagrangian variation (5.15), we obtain (after some algebra)

$$\begin{aligned}\delta\mathcal{L} \;=\; & -\xi\cdot\left\{\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\mathbf{u}} + \nabla\cdot\left(\mathbf{u}\,\frac{\partial\mathcal{L}}{\partial\mathbf{u}}\right) + \nabla\mathbf{u}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}}\right. \\ & \left. + \nabla\cdot\mathsf{P}^* - n\left[\nabla\left(\frac{\partial\mathcal{L}}{\partial n} + T_\perp + \frac{1}{2}\,T_\parallel\right) + e\,\Omega\,\nabla T_\perp\right]\right\} \\ & - \delta\phi\left(\nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\phi)} + \|en\|^\dagger\right) + \frac{\partial\Lambda}{\partial t} + \nabla\cdot\Gamma, \end{aligned} \quad (5.18)$$

where the tensor P^* denotes the FLR-corrected CGL pressure tensor

$$\mathsf{P}^* \;\equiv\; \mathsf{P}_{CGL} + e\,\Omega\,p_\perp\,(\mathbf{I}-\mathbf{bb}) = p_\parallel\,\mathbf{bb} + p_\perp\,(1+e\,\Omega)\,(\mathbf{I}-\mathbf{bb}), \quad (5.19)$$

the Hermitian conjugate operator $\|\cdots\|^\dagger$ is defined in equation (5.3), and we introduced the scalar field

$$\Lambda \equiv \xi\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}}, \quad (5.20)$$

and the vector field

$$\begin{aligned}\Gamma \;\equiv\; & \mathbf{u}\,\Lambda + \left(\mathsf{P}^* - n\,\frac{\partial\mathcal{L}}{\partial n}\,\mathbf{I}\right)\cdot\xi + \delta\phi\,\frac{\partial\mathcal{L}}{\partial(\nabla\phi)} \\ & + \left[\left(\delta\phi\,\nabla_\perp\frac{\partial(en\|\phi\|)}{\partial(\nabla_\perp^2\phi)} - \nabla_\perp\delta\phi\,\frac{\partial(en\|\phi\|)}{\partial(\nabla_\perp^2\phi)}\right) + \cdots\right]. \end{aligned} \quad (5.21)$$

Note that, while the last two terms $\partial_t\Lambda + \nabla\cdot\Gamma$ in equation (5.18) do not play a role in the variational principle $\int\delta\mathcal{L}\,d^3x\,dt = 0$, they play a crucial role in the derivation of exact conservation laws based on the Noether method.

5.2.2 Gyrofluid equations

Gyrofluid velocity

From the Eulerian variation of the Lagrangian density (5.18), the stationarity of the action functional with respect to a arbitrary virtual fluid displacement ξ yields the Euler-Poincaré equation

$$\begin{aligned} 0 \;=\; & \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \mathbf{u}} + \nabla\cdot\left(\mathbf{u}\,\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right) + \nabla\mathbf{u}\cdot\frac{\partial \mathcal{L}}{\partial \mathbf{u}} + \nabla\cdot\mathsf{P}^* \\ & -n\left[\nabla\left(\frac{\partial \mathcal{L}}{\partial n} + \frac{1}{2}T_{\|} + T_{\perp}\right) + e\,\Omega\,\nabla T_{\perp}\right], \end{aligned} \tag{5.22}$$

which describes the evolution of the gyrofluid velocity $\mathbf{u}$. Upon substituting the Lagrangian derivatives (5.16) into the Euler-Poincaré equation (5.22), and using the fact that the background magnetic field $\mathbf{B}$ is assumed to be a time-independent nonuniform vector field, we obtain

$$\begin{aligned} 0 \;=\; & mn\left(\frac{\partial u_{\|}}{\partial t} + \mathbf{u}\cdot\nabla u_{\|}\right)\mathbf{b} - \frac{en}{c}\,\mathbf{u}\times\mathbf{B}^* + en\,\nabla\psi \\ & +\nabla p_{\perp} + T_{\perp}\,\nabla\left(e\,n\,\Omega\right) + \nabla\left[\left(p_{\Delta} - e\,\Omega\,p_{\perp}\right)\mathbf{bb}\right], \end{aligned} \tag{5.23}$$

where $p_{\Delta} = p_{\|} - p_{\perp}$, and we have introduced the following definitions

$$\left.\begin{aligned} \mathbf{B}^* &\equiv \mathbf{B} + (mc/e)\,u_{\|}\,\nabla\times\mathbf{b} \\ B^*_{\|} &\equiv \mathbf{B}^*\cdot\mathbf{b} = B + (mc/e)\,u_{\|}\,\mathbf{b}\cdot\nabla\times\mathbf{b} \\ \mathbf{b}^* &\equiv \mathbf{B}^*/B^*_{\|} = \mathbf{b} + mu_{\|}\,\mathbf{C} \end{aligned}\right\}, \tag{5.24}$$

with the magnetic-curvature term $\mathbf{C}$ defined as $\mathbf{C} \equiv (c/eB^*_{\|})\,\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b})$.

Equation (5.23) can also be written in a more compact form by introducing the gradient of ψ evaluated at constant perpendicular temperature, $\nabla^{\top}\psi$, defined through equations (5.2) and (5.14) as

$$\nabla\|\phi\| = \|\nabla\phi\| + \frac{\partial\|\phi\|}{\partial B}\,\nabla B + \frac{\partial\|\phi\|}{\partial T_{\perp}}\,\nabla T_{\perp} \;\equiv\; \nabla^{\top}\|\phi\| + \Omega\,\nabla T_{\perp},$$

where $\partial\|\phi\|/\partial B \equiv -2\,(T_{\perp}/B)\,\Omega$. Then equation (5.23) can be written as:

$$0 \;=\; mn\left(\frac{\partial u_{\|}}{\partial t} + \mathbf{u}\cdot\nabla u_{\|}\right)\mathbf{b} - \frac{en}{c}\,\mathbf{u}\times\mathbf{B}^* + en\,\nabla^{\top}\psi + \nabla\cdot\mathsf{P}^*. \tag{5.25}$$

Note that the vector field $\mathbf{B}^*$ is again not a divergenceless field (since $\nabla u_{\|} \neq 0$), as is common to all guiding-center and gyrocenter Hamiltonian models (in which parallel gyrofluid velocity $u_{\|}$ is replaced with the parallel guiding-center velocity $v_{\|}$).

It is easy to see that equation (5.23) can be divided into two equations: one equation that expresses the gyrofluid velocity $\mathbf{u}$ in terms of the gyrofluid moments $(n, u_\parallel, p_\parallel, p_\perp)$ and the electrostatic potential ϕ, and one equation that describes the time evolution of the parallel gyrofluid velocity $u_\parallel$. The first equation is obtained by taking the cross-product of equation (5.23) with $\mathbf{b}$, which yields the following expression for the gyrofluid velocity

$$\mathbf{u} \equiv u_\parallel \mathbf{b}^* + \frac{c\mathbf{b}}{enB^*_\parallel} \times \left\{ en\nabla\psi + \nabla p_\perp + T_\perp \nabla(en\,\Omega) + \nabla[(p_\Delta - e\Omega\, p_\perp)\,\mathbf{bb}] \right\}, \quad (5.26)$$

Therefore, according to equation (5.26), the gyrofluid velocity $\mathbf{u}$, consists of the parallel velocity $u_\parallel \equiv \mathbf{u}\cdot\mathbf{b}$ and the following perpendicular gyrofluid velocities

$$\left.\begin{aligned} \mathbf{u}^* &\equiv (c/enB^*_\parallel)\,\mathbf{b}\times\nabla p_\perp \\ \mathbf{u}_\psi &\equiv (c/B^*_\parallel)\,\mathbf{b}\times\nabla\psi \\ \mathbf{u}_C &\equiv (p_\Delta/n)\,\mathbf{C} \end{aligned}\right\}, \quad (5.27)$$

which represent the diamagnetic velocity, the generalised $E\times B$ velocity, and the curvature-drift velocity, respectively, and the following gyro-potential FLR corrections

$$\mathbf{w}_\Omega \equiv \frac{cT_\perp \mathbf{b}}{nB^*_\parallel} \times \nabla(n\Omega) \quad \text{and} \quad \mathbf{w}_C \equiv -\,e\Omega T_\perp\,\mathbf{C} \quad (5.28)$$

to the $E\times B$ velocity and curvature-drift velocity, respectively.

Evolution equation for $u_\parallel$

The evolution equation for parallel gyrofluid velocity $u_\parallel$ can be derived by taking the dot-product of equation (5.23) with $\mathbf{b}^*$:

$$\begin{aligned} mn\left(\frac{\partial u_\parallel}{\partial t} + \mathbf{u}\cdot\nabla u_\parallel\right) &= -\ \mathbf{b}^*\cdot[\ en\,\nabla\psi + \nabla p_\perp + T_\perp\,\nabla(en\Omega)\ ] \\ &\quad -\ \nabla\cdot[\ (p_\Delta - e\Omega\, p_\perp)\,\mathbf{b}\ ], \end{aligned} \quad (5.29)$$

where we have used the identity (valid for any function f)

$$\nabla\cdot(f\,\mathbf{bb}) = [\,\nabla\cdot(f\,\mathbf{b})\,]\mathbf{b} + f\,(\mathbf{b}\cdot\nabla\mathbf{b}),$$

so that we find $\mathbf{b}^*\cdot\nabla\cdot(f\,\mathbf{bb}) = \nabla\cdot(f\,\mathbf{b})$. The gyrofluid equation (5.29) for $u_\parallel$ includes terms associated with the parallel electric field and its FLR corrections as well as parallel thermal forces.

The set of gyrofluid equations of motion for the four gyrofluid moments $(n, p_\parallel, p_\perp, u_\parallel)$ are, thus, given by equations (5.6), (5.7)-(5.8), and (5.29), respectively. Each of these gyrofluid equations involves the advection operator $\mathbf{u}\cdot\nabla$ and the divergence $\nabla\cdot\mathbf{u}$; it is the diamagnetic part of the advection operator that must be eliminated from equations (5.7)-(5.8) and (5.29) by adding suitable diamagnetic fluxes.

5.2.3 Polarisation equation

The polarisation equation for ϕ can be found by considering the terms in the variation (5.18) that involve the variation of the potential $\delta\phi$. Here, we must remember to add up the contributions of all the species to the total Lagrangian density.

Using the variation of the Lagrangian density (5.18), we obtain the Euler-Lagrange equation for ϕ:

$$0 = \sum_i \|(en)\|^\dagger + \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}, \tag{5.30}$$

where summation over particle species ($\sum_i$) is shown explicitly. Using equation (5.5), we find

$$\frac{\partial\mathcal{L}}{\partial(\nabla\phi)} = \frac{1}{4\pi}\nabla\phi + \sum_i \frac{nmc^2}{B^2}\nabla_\perp\phi, \tag{5.31}$$

so that the Euler-Lagrange (5.30) becomes the polarisation equation

$$\sum_i\left[\|(en)\|^\dagger + \nabla_\perp\cdot\left(\frac{nmc^2}{B^2}\nabla_\perp\phi\right)\right] + \frac{1}{4\pi}\nabla^2\phi = 0. \tag{5.32}$$

Comparing equation (5.32) with equation 4.23 it is clear that the polarisation equation is not affected by the introduction of the parallel temperature in the model. This arises from the fact that the terms that contain the gyrofluid electrostatic potential (5.2) do not depend on $p_\parallel$. Further details concerning equation (5.32) can be found in Section 4.2.3.

Lastly, we note that two versions for the function $\mathcal{P}(x)$, appearing in the definition of the gyrofluid scalar potential (5.2), are commonly used in gyrofluid applications: the function $\mathcal{P}(x) = e^x$ or its Padé approximant $\mathcal{P}(x) = (1-x)^{-1}$. It is important to note that, whatever form is adopted for the gyrofluid scalar potential $\|\phi\|$, it must be used consistently throughout the model in order to ensure energy conservation.

5.3 Energy conservation law

In this Section we present the local and global forms of the energy conservation law, as they arise from the application of the Noether method. For this purpose, we point out that, as a result of the variational principle $\int \delta\mathcal{L}\, d^3x\, dt = 0$, the only remaining terms in the Eulerian variation of the Lagrangian density (5.18) become the Noether equation

$$\delta\mathcal{L} = \frac{\partial\Lambda}{\partial t} + \nabla\cdot\Gamma, \tag{5.33}$$

where Λ and Γ are defined in equations (5.20)-(5.21). The energy and momentum conservation laws are derived from the Noether equation (5.33) by considering infinitesimal time and space translations, respectively. Here we focus our attention on the local and global energy conservation laws associated with our electrostatic gyrofluid model.

5.3.1 Local energy conservation law

We derive the local form of the energy conservation law from the Noether equation (5.33) by considering infinitesimal time translations $t \to t + \delta t$, from which we obtain the following expressions for the virtual fluid displacement ξ and the Eulerian variations $\delta\phi$ and $\delta\mathcal{L}$:

$$\xi \;=\; -\,\mathbf{u}\,\delta t, \quad \delta\phi \;=\; -\,\delta t\,\partial_t\phi, \quad \text{and} \quad \delta\mathcal{L} \;=\; -\,\delta t\,\partial_t\mathcal{L}. \tag{5.34}$$

Inserting these expressions into equations (5.20)-(5.21), we obtain

$$\Lambda \;\equiv\; -\,\delta t\,\left(\mathbf{u}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}}\right), \tag{5.35}$$

$$\begin{aligned}\Gamma \;\equiv\; & -\,\delta t\left[\,\mathbf{u}\left(\mathbf{u}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}} - n\,\frac{\partial\mathcal{L}}{\partial n}\right) + \mathsf{P}^*\cdot\mathbf{u} + \frac{\partial\phi}{\partial t}\,\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\right.\\ & \left. +\,\left(\frac{\partial\phi}{\partial t}\,\nabla_\perp\frac{\partial(en\|\phi\|)}{\partial(\nabla_\perp^2\phi)} - \nabla_\perp\frac{\partial\phi}{\partial t}\,\frac{\partial(en\|\phi\|)}{\partial(\nabla_\perp^2\phi)} + \cdots\right)\,\right],\end{aligned} \tag{5.36}$$

where summation over fluid species is implied wherever appropriate. By combining these expressions, we arrive at the primitive form of the local energy conservation law;

$$\frac{\partial\varepsilon'}{\partial t} + \nabla\cdot\mathbf{S}' \;=\; 0,$$

where the primitive energy density is

$$\varepsilon' \;\equiv\; \mathbf{u}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}} - \mathcal{L} \;=\; \frac{1}{2}nmu_\parallel^2 + p_\perp + \frac{p_\parallel}{2} + \left(en\psi - \frac{|\mathbf{E}|^2}{8\pi}\right), \tag{5.37}$$

and the primitive energy-density flux is

$$\begin{aligned}\mathbf{S}' \;\equiv\; & \mathbf{u}\left(\mathbf{u}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{u}} - n\,\frac{\partial\mathcal{L}}{\partial n}\right) + \mathsf{P}^*\cdot\mathbf{u} + \frac{\partial\phi}{\partial t}\,\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\\ & +\,\left(\frac{\partial\phi}{\partial t}\,\nabla_\perp\frac{\partial(en\|\phi\|)}{\partial(\nabla_\perp^2\phi)} - \nabla_\perp\frac{\partial\phi}{\partial t}\,\frac{\partial(en\|\phi\|)}{\partial(\nabla_\perp^2\phi)} + \cdots\right)\\ =\; & \mathbf{u}\left(\frac{1}{2}nmu_\parallel^2 + p_\perp + \frac{p_\parallel}{2} + en\psi\right) + \mathsf{P}^*\cdot\mathbf{u} + \frac{\partial\phi}{\partial t}\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\\ & +\,\left[\,\frac{\partial\phi}{\partial t}\,\nabla_\perp\left(\frac{\partial(en\,\|\phi\|)}{\partial(\nabla_\perp^2\phi)} + \cdots\right) - \nabla_\perp\frac{\partial\phi}{\partial t}\,\left(\frac{\partial(en\,\|\phi\|)}{\partial(\nabla_\perp^2\phi)} + \cdots\right)\,\right].\end{aligned} \tag{5.38}$$

In order to arrive at the final form of the energy conservation law, we need to rearrange terms in equation (5.37). By substituting the polarisation equation (5.30) into equation (5.4), we find

$$en\,\|\phi\| \;=\; \frac{|\mathbf{E}|^2}{4\pi} + mn\,|\mathbf{u}_E|^2 - \nabla\cdot\mathbf{D}, \tag{5.39}$$

where

$$\mathbf{D} \equiv \phi \frac{\partial \mathcal{L}}{\partial(\nabla\phi)} - \frac{\partial(en\,\|\phi\|)}{\partial(\nabla_\perp^2\phi)} \nabla_\perp\phi + \phi\,\nabla_\perp \left(\frac{\partial(en\,\|\phi\|)}{\partial(\nabla_\perp^2\phi)} \right) + \cdots \tag{5.40}$$

so that the last terms in equation (5.37) become

$$en\psi - \frac{|\mathbf{E}|^2}{8\pi} = \frac{1}{2} nm\,|\mathbf{u}_E|^2 + \frac{|\mathbf{E}|^2}{8\pi} - \nabla\cdot\mathbf{D}. \tag{5.41}$$

Hence, we express the primitive energy density (5.37) as $\varepsilon' \equiv \varepsilon - \nabla\cdot\mathbf{D}$, where the final form of the energy density is defined as

$$\varepsilon \equiv \frac{1}{2} nm \left(u_\parallel^2 + |\mathbf{u}_E|^2 \right) + p_\perp + \frac{p_\parallel}{2} + \frac{|\mathbf{E}|^2}{8\pi}, \tag{5.42}$$

and we obtain the local form of the energy conservation law

$$\frac{\partial\varepsilon}{\partial t} + \nabla\cdot\mathbf{S} = 0, \tag{5.43}$$

where the final form of the energy density flux is defined as

$$\mathbf{S} \equiv \mathbf{S}' - \frac{\partial\mathbf{D}}{\partial t}. \tag{5.44}$$

After some partial cancellations, the final form of the energy density flux is

$$\mathbf{S} = \mathbf{u}\left(\frac{1}{2} nm\,u_\parallel^2 + p_\perp + \frac{p_\parallel}{2} + en\psi \right) + \mathsf{P}^*\cdot\mathbf{u} + \mathbf{S}_\phi, \tag{5.45}$$

where $\mathsf{P}^*\cdot\mathbf{u} = p_\perp\,(1+e\Omega)\,\mathbf{u} + (p_\Delta - e\Omega\,p_\perp)\,u_\parallel\,\mathbf{b}$ and the electrostatic energy density flux is defined as

$$\mathbf{S}_\phi = -\,\phi\,\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)} + \nabla_\perp \frac{\partial(en\,\|\phi\|)}{\partial(\nabla_\perp^2\phi)} + \cdots \right) + \nabla_\perp\phi\,\frac{\partial}{\partial t}\left(\frac{\partial(en\,\|\phi\|)}{\partial(\nabla_\perp^2\phi)} + \cdots \right). \tag{5.46}$$

In the next Section, after heat fluxes are inserted back into the pressure evolution equations and diamagnetic cancellations are performed with the addition of terms in the gyrofluid equations, the local energy conservation law (5.43) is converted into a new energy equation $\partial_t\varepsilon + \nabla\cdot\mathbf{S}^* = 0$, in which heat fluxes and diamagnetic-cancellation terms result in a modified energy density flux $\mathbf{S}^*$. This new form ensures that the total energy $\mathcal{E} = \int \varepsilon\, d^3x$ satisfies the global energy conservation law $d\mathcal{E}/dt = 0$.

5.3.2 Global energy conservation law

The global energy conservation law $d\mathcal{E}/dt = 0$ can be derived from the local energy conservation law (5.43) by integrating it over space. Here, the global energy is defined as

$$\mathcal{E} = \int d^3x \left[\frac{1}{2} nm\,u_\parallel^2 + \left(p_\perp + \frac{p_\parallel}{2} \right) + \left(\frac{1}{2} nm|\mathbf{u}_E|^2 + \frac{|\mathbf{E}|^2}{8\pi} \right) \right]. \tag{5.47}$$

In this form, the parallel kinetic energy, the internal energy, and the electric field energy explicitly appear. In a later section, we will present the time evolution of each of the separate terms that constitute the energy conservation law in order to identify the energy-exchange processes that allow the transfer of energy between the three types of gyrofluid (parallel kinetic, internal, and field) energies.

5.4 Diamagnetic cancellations and energy conservation

The gyrofluid velocity (5.23) contains the diamagnetic velocity $\mathbf{u}^*$. Since the gyrofluid moment-equations are derived by inserting the gyrofluid velocity $\mathbf{u}$ in the Lagrangian constraints, diamagnetic advection terms appear in the equations of evolution for the parallel velocity and the two anisotropic pressures. More specifically, the momentum equation for $u_\parallel$ contains the term $\mathbf{u}^* \cdot \nabla u_\parallel$, and the parallel and perpendicular pressure equations contain a combination of the terms $\mathbf{u}^* \cdot \nabla p$ and $p \nabla \cdot \mathbf{u}^*$ (where here p is either $p_\parallel$ or $p_\perp$). These diamagnetic-advection terms should be cancelled in the gyrofluid evolution equations by the introduction of appropriate terms containing higher-order moments, as was done in Section 4.4 for the one temperature gyrofluid model. These correspond to the FLR corrections to perpendicular fluxes in conventional fluid models [16, 15], but arise naturally due to grad-B and curvature drifts in the moment-based derivation of local gyrofluid models [4]. Here, due to the presence of two temperatures instead of one in section 4.4, although the gyroviscosity and the zeroth order in FLR perpendicular heat flux added stay the same, the FLR corrections of the perpendicular heat flux change to include the parallel temperature. Moreover a parallel-perpendicular heat flux is introduced to make the diamagnetic cancellation in the parallel temperature equation.

Since the higher-order moment terms cannot be derived from the Lagrangian, the diamagnetic cancellations must be done manually. However, there exists a constraint in the addition of higher-order moment terms, namely, that the global energy conservation law should not be altered. Here, the terms added are derived from the Vlasov equation, and FLR corrections are then introduced to conserve energy. Thus, the final moment equations are not directly derived from the Lagrangian, but they still conserve energy exactly.

5.4.1 Parallel gyrofluid dynamics

The diamagnetic cancellation needed for the parallel momentum equation (5.29) involves the addition of the term $-\nabla \cdot \Pi^*_\parallel$, associated with the non-diagonal part of the pressure tensor [13], on the right side of equation (5.29). From Vlasov theory, the diamagnetic-cancellation term is found to be [14]:

$$-\nabla \cdot \Pi^*_\parallel = -\nabla \cdot \left(p_\perp \frac{mc\mathbf{b}}{eB^*_\parallel} \times \nabla u_\parallel \right) = mn\mathbf{u}^* \cdot \nabla u_\parallel + p_\perp \mathcal{K}(mu_\parallel), \tag{5.48}$$

where the magnetic differential operator $\mathcal{K}(\cdots)$ is defined by the identity

$$\mathcal{K}(f) = -\nabla\left(\frac{c}{eB_\parallel^*}\mathbf{b}\times\nabla f\right), \tag{5.49}$$

valid for arbitrary functions f. This term is the same as the one used in section 4.4 for the one-temperature model, since only the perpendicular pressure is involved in gyroviscosity, and no additional term is required to be added for the energy conservation law. In fact as written here, equation (5.48) conserves energy by itself, since $u_\parallel\ \nabla\cdot\Pi_\parallel^* = \nabla\cdot(u_\parallel\ \Pi_\parallel^*)$. We should note that the substitution of B to $B_\parallel^*$ is done here, and throughout this Section, to make the diamagnetic cancellation exact.

The time evolution of the parallel kinetic energy density is, therefore, expressed as

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{mn}{2}\,u_\parallel^2\right) = -\,\nabla\cdot\left(\frac{mn}{2}\,u_\parallel^2\mathbf{u} + u_\parallel\,\Pi_\parallel^*\right)\\ &-\mathbf{u}\cdot\left\{en\nabla\psi + \nabla p_\perp + T_\perp\nabla(en\Omega) + \nabla\left[(\,p_\Delta - e\Omega\,p_\perp\,)\,\mathbf{bb}\right]\right\},\end{aligned} \tag{5.50}$$

where we used the expression (5.26) for the gyrofluid velocity to write $u_\parallel\,\mathbf{b}^*\cdot\{\cdots\} = \mathbf{u}\cdot\{\cdots\}$ for the second term on the right side of equation (5.50).

5.4.2 Internal energy

To consider the time evolution of the internal energy $p_\perp + p_\parallel/2$, we rewrite the two pressure equations (5.7)-(5.8) in the form

$$\begin{aligned}&\frac{\partial p_\parallel}{\partial t} + \nabla\cdot(p_\parallel\mathbf{u}) + 2p_\parallel\mathbf{bb}:\nabla\mathbf{u} =\\ &= -\,2\nabla\cdot\mathbf{q}_{\parallel\perp} + 4\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) + 2\,Q_\parallel,\end{aligned} \tag{5.51}$$

$$\begin{aligned}&\frac{\partial p_\perp}{\partial t} + \nabla\cdot(p_\perp\mathbf{u}) + p_\perp(\mathbf{I} - \mathbf{bb}):\nabla\mathbf{u} =\\ &= -\nabla\cdot\mathbf{q}_{\perp\perp} - 2\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) + Q_\perp,\end{aligned} \tag{5.52}$$

where $Q_\parallel$ and $Q_\perp$ are additional terms (to be determined later) associated with FLR corrections to the electrostatic scalar field ϕ.

To zeroth order in the electrostatic potential ϕ, the heat fluxes are derived from the Vlasov equation directly and are found in [9] to be expressed as

$$\mathbf{q}_{\parallel\perp} = \frac{1}{2}\,\frac{cp_\perp\mathbf{b}}{eB_\parallel^*}\times\nabla T_\parallel + p_\parallel\,\mathbf{u}_C, \tag{5.53}$$

and

$$\mathbf{q}_{\perp\perp} = 2\,\frac{cp_\perp\mathbf{b}}{eB_\parallel^*}\times\nabla T_\perp. \tag{5.54}$$

Inserting these diamagnetic heat fluxes, using the definition (5.49), we find

$$\begin{aligned} -2\nabla\cdot\mathbf{q}_{\parallel\perp}+4\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) &= n\mathbf{u}^*\cdot\nabla T_\parallel + p_\perp\,\mathcal{K}(T_\parallel) \\ &\quad -\,2\,p_\perp\,\mathbf{C}\cdot\nabla T_\parallel - 2\,\nabla\cdot\left(T_\parallel\,p_\Delta\,\mathbf{C}\right) \end{aligned} \tag{5.55}$$

for the parallel pressure equation (5.7), and

$$-\nabla\cdot\mathbf{q}_{\perp\perp}-2\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) = 2\,n\mathbf{u}^*\cdot\nabla T_\perp + 2\,p_\perp\,\mathcal{K}(T_\perp) + p_\perp\,\mathbf{C}\cdot\nabla T_\parallel \tag{5.56}$$

for the perpendicular pressure equation (5.8).

Using equations (5.51) and (5.52), the time evolution of the internal energy can, therefore, be expressed as

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{p_\parallel}{2}+p_\perp\right) &= -\nabla\cdot\left[\left(\frac{p_\parallel}{2}+p_\perp\right)\mathbf{u}+\mathsf{P}_{CGL}\cdot\mathbf{u}+\left(\mathbf{q}_{\parallel\perp}+\mathbf{q}_{\perp\perp}\right)\right] \\ &\quad +\,\mathbf{u}\cdot\nabla\cdot\mathsf{P}_{CGL}+\left(Q_\parallel+Q_\perp\right), \end{aligned} \tag{5.57}$$

where suitable energy-conserving expressions for the FLR-corrected heat fluxes $Q_\parallel$ and $Q_\perp$ are now determined by considering the expression for the time evolution of the electrostatic field energy.

5.4.3 Electrostatic field energy

By making use of the electrostatic field energy equation (5.41), we write the following expression for the time evolution of the electrostatic field energy

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{mn}{2}\,|\mathbf{u}_E|^2+\frac{|\mathbf{E}|^2}{8\pi}\right) &= \nabla\cdot\frac{\partial\mathbf{D}}{\partial t}+\frac{\partial}{\partial t}\left(en\psi-\frac{|\mathbf{E}|^2}{8\pi}\right) \\ &= e\,(\psi-T_\perp\,\Omega)\,\frac{\partial n}{\partial t}+e\Omega\,\frac{\partial p_\perp}{\partial t}-\nabla\cdot\mathbf{S}_\phi, \end{aligned} \tag{5.58}$$

where we made use of the polarisation equation (5.32) and the definition (5.46) for $\mathbf{S}_\phi$. Here, using the gyrofluid continuity (5.6), we obtain

$$e\,(\psi-T_\perp\Omega)\,\frac{\partial n}{\partial t} = -\nabla\cdot\left[\,e\,(n\psi-p_\perp\,\Omega)\,\mathbf{u}\,\right]+en\,\mathbf{u}\cdot\left[\,\nabla\psi-\nabla(T_\perp\Omega)\,\right],$$

and, using the perpendicular pressure equation (5.52), we obtain

$$\begin{aligned} e\Omega\,\frac{\partial p_\perp}{\partial t} &= -\,e\Omega\,\nabla\cdot(p_\perp\,\mathbf{u})-\mathsf{P}_\Omega:\nabla\mathbf{u}-e\Omega\,\left[\,\nabla\cdot\mathbf{q}_{\perp\perp}+2\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b})-Q_\perp\,\right] \\ &= -\,\nabla\cdot\left(\,e\Omega\,p_\perp\,\mathbf{u}+\mathsf{P}_\Omega\cdot\mathbf{u}\,\right)+\mathbf{u}\cdot\left[\,p_\perp\,\nabla(e\Omega)+\nabla\cdot\mathsf{P}_\Omega\,\right] \\ &\quad -\,e\Omega\,\left[\,\nabla\cdot\mathbf{q}_{\perp\perp}+2\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b})-Q_\perp\,\right], \end{aligned}$$

where $\mathsf{P}_\Omega \equiv \mathsf{P}^* - \mathsf{P}_{CGL} = e\Omega\, p_\perp\, (\mathbf{I} - \mathbf{bb})$ denotes the FLR-correction to the CGL pressure tensor. Hence, the first two terms on the right side of equation (5.58) can be written as

$$\begin{aligned} e\,(\psi - T_\perp\,\Omega)\,\frac{\partial n}{\partial t} + e\Omega\,\frac{\partial p_\perp}{\partial t} \;&=\; -\,\nabla\cdot\left(\, e\,n\psi\,\mathbf{u} + \mathsf{P}_\Omega\cdot\mathbf{u}\,\right) \\ &\quad +\mathbf{u}\cdot\left(en\,\nabla\psi - en\,\Omega\nabla T_\perp + \nabla\cdot\mathsf{P}_\Omega\right) \\ &\quad -\,e\Omega\left[\nabla\cdot\mathbf{q}_{\perp\perp} + 2\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) - Q_\perp\right]. \end{aligned} \tag{5.59}$$

We now require that, in order for the last term $Q_\Omega \equiv Q_\parallel + Q_\perp$ appearing on the right side of equation (5.57) to cancel the last three terms on the right side of equation (5.59), the latter terms must be written up to an exact spatial divergence as

$$-\,e\Omega\,\nabla\cdot\mathbf{q}_{\perp\perp} - 2\,e\Omega\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) + e\Omega\,Q_\perp \;=\; -\,Q_\Omega - \nabla\cdot\mathbf{S}_\Omega.$$

Hence, we set the additional terms $Q_\parallel$ and $Q_\perp$ in equations (5.57) and (5.59) to be

$$\left.\begin{aligned} Q_\parallel \;&=\; 2\,e\Omega\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \\ Q_\perp \;&=\; -\,\mathbf{q}_{\perp\perp}\cdot\nabla(e\Omega) - \nabla\cdot\left(\mathbf{q}_{\perp\perp}\,e\Omega\right) \end{aligned}\right\}, \tag{5.60}$$

so that

$$Q_\Omega \equiv Q_\parallel + Q_\perp = n\mathbf{w}_C\cdot\nabla T_\parallel + 2\,e\Omega\,p_\perp\,\mathcal{K}(T_\perp) + 2n\,(2\,\mathbf{w}_\Omega - e\Omega\,\mathbf{u}^*)\cdot\nabla T_\perp, \tag{5.61}$$

and $\mathbf{S}_\Omega \equiv e\Omega\,(2 + e\Omega)\,\mathbf{q}_{\perp\perp}$, with $\mathbf{q}_{\perp\perp}$ defined in equation (5.54). Combining equations (5.59)-(5.61) into equation (5.58), the time evolution of the electrostatic field energy is, therefore, expressed as

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{mn}{2}|\mathbf{u}_E|^2 + \frac{|\mathbf{E}|^2}{8\pi}\right) \;&=\; -\nabla\cdot\left(\, e\,n\psi\,\mathbf{u} + \mathsf{P}_\Omega\cdot\mathbf{u} + \mathbf{S}_\phi + \mathbf{S}_\Omega\,\right) \\ &\quad +\mathbf{u}\cdot\left(en\,\nabla\psi - en\,\Omega\nabla T_\perp + \nabla\cdot\mathsf{P}_\Omega\right) - Q_\Omega. \end{aligned} \tag{5.62}$$

5.4.4 Explicit form of the energy conservation law

When diamagnetic cancellations and heat fluxes are introduced into the four-moment gyrofluid equations, the local energy conservation law (5.43) is modified. By combining the evolution equations for the parallel kinetic energy (5.50), the internal energy (5.57), and the electrostatic field energy (5.62), the local energy conservation law (5.43) becomes the local energy equation

$$\frac{\partial\varepsilon}{\partial t} + \nabla\cdot\mathbf{S} \;=\; -\,\nabla\cdot\left[\, u_\parallel\,\Pi_\parallel^* + \mathbf{q}_{\parallel\perp} + (1 + e\Omega)^2\,\mathbf{q}_{\perp\perp}\,\right], \tag{5.63}$$

which ensures that the total energy $\mathcal{E} = \int \varepsilon d^3x$ still satisfies the global energy conservation law $d\mathcal{E}/dt = 0$.

We now identify the energy-exchange processes that transfer energy between the three different types of gyrofluid energy. First, we write down expressions describing the time evolution of each type of energy (e.g., parallel kinetic energy, internal energy, and field energy). Thus, the contribution of each species to the integrated parallel kinetic energy (5.50) is

$$\frac{d}{dt}\int\left(\frac{mn}{2}\,u_{\parallel}^2\right)d^3x \;=\; -\int \mathbf{u}\cdot\left(en\,\nabla\psi - en\Omega\nabla T_{\perp} + \nabla\cdot\mathsf{P}^*\right)d^3x, \tag{5.64}$$

the contribution of each species to the integrated internal energy (5.57) is

$$\frac{d}{dt}\int\left(\frac{p_{\parallel}}{2}+p_{\perp}\right)d^3x \;=\; \int\left[\,\mathbf{u}\cdot\left(\nabla\cdot\mathsf{P}_{CGL}\right)+Q_{\Omega}\,\right]d^3x, \tag{5.65}$$

and the contribution of each species to the integrated electrostatic field energy (5.62) is

$$\frac{d}{dt}\int\left(\frac{mn}{2}\,|\mathbf{u}_E|^2+\frac{|\mathbf{E}|^2}{8\pi}\right)d^3x \;=\; \int\left[\,\mathbf{u}\cdot\left(en\,\nabla\psi - en\Omega\nabla T_{\perp}+\nabla\cdot\mathsf{P}_{\Omega}\right)-Q_{\Omega}\,\right]d^3x. \tag{5.66}$$

The terms on the right side of equations (5.64)-(5.66) appear in pairs with opposite sign, and give the energy-exchange processes. For example, the FLR-correction heat flux Q_{Ω} is involved in energy exchange between the electrostatic field energy and the internal energy, while the FLR-correction pressure tensor P_{Ω} is involved in energy exchange between the electrostatic field energy and the parallel kinetic energy. The contributions from $en\,\nabla\psi$, $en\Omega\nabla T_{\perp}$ and $\nabla\cdot\mathsf{P}_{CGL}$ in equations (5.64)-(5.66), on the other hand, describe standard energy-exchange processes.

5.4.5 Comparison with previous models

We now write explicit final expressions for the gyrofluid density n, the gyrofluid parallel velocity $u_{\parallel}$, and the parallel and perpendicular gyrofluid pressures $p_{\parallel}$ and $p_{\perp}$. The gyrofluid continuity is expressed in expanded form as

$$\begin{aligned}\frac{d_E n}{dt}+\nabla\cdot\left[n\left(u_{\parallel}\,\mathbf{b}^*+\mathbf{u}_C+\mathbf{w}_C\right)\right] \;&=\; \left[\,\mathcal{K}(p_{\perp})+n\,\mathcal{K}(e\psi)+T_{\perp}\,\mathcal{K}(en\Omega)\,\right]\\ &\quad+\;\mathbf{u}^*\cdot\nabla(en\Omega)+\mathbf{w}_{\Omega}\cdot\nabla n,\end{aligned} \tag{5.67}$$

where $d_E/dt \equiv \partial/\partial t + \mathbf{u}_{\psi}\cdot\nabla$. With the insertion of the diamagnetic-cancellation term (5.48), the evolution equation for the gyrofluid parallel velocity is expressed in expanded form as

$$\begin{aligned}mn\left(\frac{\partial u_{\parallel}}{\partial t}+\mathbf{u}'\cdot\nabla u_{\parallel}\right)-p_{\perp}\,\mathcal{K}(mu_{\parallel}) \;&=\; -\,\mathbf{b}^*\cdot\left[\,en\,\nabla\psi+\nabla p_{\perp}+T_{\perp}\,\nabla(en\Omega)\,\right]\\ &\quad-\;\nabla\cdot\left[\,(p_{\Delta}-e\Omega\,p_{\perp})\,\mathbf{b}\,\right],\end{aligned} \tag{5.68}$$

where $\mathbf{u}' \equiv \mathbf{u} - \mathbf{u}^*$ denotes the gyrofluid velocity without its diamagnetic contribution. Lastly, with the insertion of the diamagnetic-cancellation terms (5.55)-(5.56) and (5.60), the gyrofluid parallel and perpendicular pressure equations are

$$\begin{aligned} \frac{\partial p_\parallel}{\partial t} &= -\nabla\cdot\left(p_\parallel\,\mathbf{u} + 2\,p_\parallel\,u_\parallel\,\mathbf{b} + 2\,\mathbf{q}_{\parallel\perp}\right) + 2\,\mathbf{u}\cdot\nabla\cdot\left(p_\parallel\,\mathbf{bb}\right) \\ &\quad + 4\,(1+e\Omega)\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}), \end{aligned} \tag{5.69}$$

and

$$\begin{aligned} \frac{\partial p_\perp}{\partial t} &= -\nabla\cdot\left[\,p_\perp\,\mathbf{u} + p_\perp\,(\mathbf{I}-\mathbf{bb})\cdot\mathbf{u} + (1+e\Omega)\,\mathbf{q}_{\perp\perp}\,\right] + \mathbf{u}\cdot\nabla\cdot\left[\,p_\perp\,(\mathbf{I}-\mathbf{bb})\,\right] \\ &\quad - 2\,\mathbf{q}_{\parallel\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) - \mathbf{q}_{\perp\perp}\cdot\nabla(e\Omega), \end{aligned} \tag{5.70}$$

where the heat fluxes $\mathbf{q}_{\parallel\perp}$ and $\mathbf{q}_{\perp\perp}$ are defined in equations (5.53) and (5.54).

The results of the present two-temperature gyrofluid model can be compared with the one-temperature model presented in Chapter 4, where the gyrofluid equations are derived by including the perpendicular temperature only. Since the FLR-corrected CGL pressure tensor (5.19) includes the parallel pressure $p_\parallel$, the two gyrofluid models agree except for the terms that arise from the pressure anisotropy $p_\Delta = p_\parallel - p_\perp$.

Our results can also be compared with the gyrofluid model of Beer and Hammett [4] presented in Section 3.3.4, since this Beer-Hammett model is the most extended one, which contains all the previously developed gyrofluid models. The equations of evolution of the gyrofluid moments presented here are nearly identical to those of Beer and Hammett. The differences that arise can be separated in two categories: (1) differences in the non-FLR terms and (2) differences in the FLR terms.

The non-FLR terms coincide almost exactly in the two models, except terms that come from closures of higher-order moments, which our model is not able to retrieve. In our model, gyrofluid moments higher than pressure moments cannot be included in the gyrofluid Lagrangian, and the closures are done automatically when choosing the constraints for the variation. Thus, for instance, Landau damping is not included in our model, though it can be added by hand afterwards as long as the energy conservation remains exact. A special difference in the non-FLR terms between the two models - and the only one of the kind that appears - is a different term in the momentum equation (5.68): the magnetic term $\mathcal{K}(u_\parallel)$ in our model compared to $2\,\mathcal{K}(u_\parallel) \approx i\omega_d(4u_\parallel)$ in Section 3.3.4. This difference arises from the fact that the additional magnetic contribution originates from the parallel-parallel heat flux $\mathbf{q}_{\parallel\parallel}$, which we cannot fully retrieve.

The differences in the FLR terms between the two models arise from the higher-order moment closures, but also because of the energy conservation law. Especially in the FLR terms, previous models do not conserve energy. Although our polarization equation reduces to the local one [5] and so is in agreement with the local model in the local limit, we do not exactly retrieve the results of [5]. That is, although we find a strong correlation between n and $\|\phi\|$ and $p_\perp$ and Ω, there exists another FLR term that is not correlated with $p_\perp$. This difference arises because of the constraints introduced here, which lead to another closure than that of [5].

5.5 Other pressure constraints

As noted before, both the Lagrangian and the constraints can at first be chosen arbitrarily. Their validity follows from the validity of the resulting equations of evolution, and also from the energy conservation law. Although the choice for the density-constraint equation (5.6) is straightforward, the choice for the pressure-constraint equations is not, since there are many sets of pressure equations used in the literature. In the course of constructing a valid gyrofluid model, we examined many possible pressure constraints. As a test for their validity, we chose to compare the non-FLR terms of the evolution equations with those in Section 3.3.4.

Here, we will discuss two sets of pressure constraints: (1) constraints that are derived from the conservation of entropy for one and two degrees of freedom for $p_\parallel$ and $p_\perp$, respectively; and (2) constraints derived from the double adiabatic theory [8].

The first set of constraints include the conservation of particles as well as the conservation of the entropy $d(p/n^\gamma)/dt = 0$ for $p_\parallel$ ($\gamma = 3$) and $p_\perp$ ($\gamma = 2$), respectively, each written in conservative form as

$$\begin{aligned} &\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{u}) = 0 \\ &\frac{\partial}{\partial t}\left(\frac{p_\parallel}{n^2}\right) + \nabla\cdot\left(\frac{p_\parallel}{n^2}\,\mathbf{u}\right) = 0 \\ &\frac{\partial}{\partial t}\left(\frac{p_\perp}{n}\right) + \nabla\cdot\left(\frac{p_\perp}{n}\,\mathbf{u}\right) = 0 \end{aligned} \tag{5.71}$$

After doing the variation of the action as was done here, one can find the gyrofluid velocity excluding terms that contain Ω to be:

$$\mathbf{u} \;=\; u_\parallel\,\mathbf{b}^* + \frac{c\mathbf{b}}{enB^*_\parallel}\times\nabla(p_\perp + p_\parallel) + \mathbf{u}_\psi. \tag{5.72}$$

Inserting the gyrofluid velocity in the constraints equation (5.72), the gyrofluid equations of evolution are derived. From the gyrofuid velocity one can see already the differences between the two models. In the velocity derived with the entropy constraints the diamagnetic drift includes both pressures, whereas in the velocity derived with the CGL constraints the diamagnetic velocity depends only on $p_\perp$. Moreover, in the velocity equation (5.72), there exists no curvature drift $\mathbf{u}_C$, since it comes from the dependence of the pressure constraints on the magnetic field. These constraints lead to differences between this first model and previous results even in the density equation, and more specifically in the curvature terms, where this model finds twice the curvature terms than found in Section 3.3.4. Moreover, all terms that arise from the curvature drift do not appear in this first model. To eliminate these differences, another set of constraints that includes dependence on the magnetic field and that does not include the parallel pressure in the diamagnetic drift is needed.

The second set of constraints include the density conservation as well as the double

adiabatic equations [8] written in conservative form as

$$\begin{aligned}
&\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{u}) = 0 \\
&\frac{\partial}{\partial t}\left(\frac{p_\perp}{B}\right) + \nabla\cdot\left(\frac{p_\perp}{B}\mathbf{u}\right) = 0 \\
&\frac{\partial}{\partial t}\left(\frac{p_\parallel p_\perp^2}{n^4}\right) + \nabla\cdot\left(\frac{p_\parallel p_\perp^2}{n^4}\mathbf{u}\right) = 0
\end{aligned} \tag{5.73}$$

The two last constraints can be derived from the conservation of magnetic moment for the perpendicular pressure equation $d(p_\perp/nB)/dt = 0$, and the conservation of bounce action for the parallel pressure equation $d(p_\parallel B^2/n^3)/dt = 0$. Proceeding as we did before, we can find the gyrofluid velocity, which excluding terms that contain Ω is:

$$\mathbf{u} = u_\parallel\,\mathbf{b}^* + \frac{c\mathbf{b}}{enB_\parallel^*}\times\nabla p_\parallel + \mathbf{u}_\psi + \mathbf{u}_C. \tag{5.74}$$

Here, the curvature drift velocity appears, but the diamagnetic drift depends only on the parallel pressure and not on the perpendicular one. This is not consistent with the fluid theories, where the diamagnetic velocity depends only on the $p_\perp$. In this second model, the density equation coincides with equation (3.82) found by previous models. However, there are differences between the two models on the momentum and the pressure equations on the diamagnetic terms, even after diamagnetic cancellations are applied. Moreover, this second model also does not retrieve the correct curvature terms in the momentum equation.

To summarise, of the three sets of constraints presented in this chapter, only the total CGL constraints (5.6)-(5.8) give the expected from fluid theory gyrofluid velocity, and also equations of evolution that coincide to zeroth order in the FLRs with the previous models derived from the Gyrokinetic equation.

The model described in this chapter has similar limitations as the model described in the previous chapter with one important difference in that it keeps the parallel temperature as a dynamical variable. The timescales associated with the turbulence are often shorter than the collision time and, therefore, an isotropic temperature can not be maintained through collisions. The dynamics perpendicular and parallel to the field is largely (but not completely) decoupled and the temperature perturbations are determined by different equations. For an accurate description of the system one therefore needs to distinguish between the parallel and perpendicular temperatures and keep both as dynamic variables.

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Chapter 6

Conclusions

In this dissertation two sets of electrostatic gyrofluid equations that describe the evolution of the density, momentum and temperature have been derived, one for a single temperature and one for an anisotropic temperature. In each case, a polarisation equation derived from the same formalism allows for the calculation of the electric field evolution. Each model satisfies an exact energy conservation law that includes energy-exchange terms involving parallel kinetic energy, internal energy, and field energy. Together, they provide an accurate description of much of the physics relevant for microinstability driven turbulence.

Previous electrostatic gyrofluid models are local, that is nonlinearities are discarded except the $E \times B$ velocity advection. These models are delta-f models with equations which are homogeneous in the plasma parameters. Moreover, although previous models conserve free energy in the zeroth order FLR terms, they do not conserve energy in the FLR corrections because of the different approximations used in the moment equations and the field potential equation, and also because of the different closures.

Our model is the first fully inhomogeneous gyrofluid model as well as the first one that satisfies an exact energy conservation theorem. The fact that the model is inhomogeneous makes the set of equations especially suited for turbulence computations which intend to treat either large amplitude disturbances (relative fluctuation levels of order unity) or strong variations of parameters (e.g., collisionality, relative magnetic inductivity, temperature, etc.) across the spatial domain. Energy conservation is essential because even a small violation of energy conservation which would not strongly affect the turbulence dynamics *per se* becomes damaging when treating secondary (e.g., zonal flow) effects. These zonal flows are global flows which play an important role in controlling the level of turbulent fluctuations.

To guarantee energy conservation we have used the Lagrangian approach in which the conservation property is directly obtained via Noether's theorem. This method is new compared with the usual approach which uses moment building of the gyrokinetic equation. The Lagrangian is constructed to be analogous to the single particle Lagrangian, and a variation procedure yields all the equations of our set. The validity of the Lagrangian and the constraints introduced for the variational procedure was verified by the validity of

the final evolution equations and the energy conservation theorem. Diamagnetic cancellation is taken into account in a second step, through the addition of the non-diagonal part of the pressure tensor and the perpendicular heat flux in the momentum and temperature equations, respectively. This is done according to a systematic procedure chosen to maintain energy conservation. Thus all non-electrostatic diamagnetic-corrections terms were computed from the Vlasov equation, and the FLR-corrected terms associated with ϕ were chosen such that the energy conservation is still satisfied.

The constraints, used to derive the evolution equations, play an important role, since different constraints yield strongly different evolution equations. For the three-moment model the used constraints are density and entropy conservation. For the four-moment model in addition to density conservation, the Chew-Goldberger-Low equations are used to constrain the parallel and perpendicular pressures. Other equations often used in the literature are also examined, and the results for each were presented and compared.

To verify the validity of the Lagrangian and the constraints, a comparison with previous models was necessary. The agreement of the models can not be perfect because of the energy conservation law and the different ways used to derive the evolution equations. Thus, although our model should agree with the previously published to zeroth order in the FLR terms except those derived from higher order moments (eg. heat fluxes), the models should not have the same FLR terms since previous models do not conserve energy in these terms. Actual comparison with previous models shows that there is a very good agreement in the zero order FLR terms, but not in the FLR corrections. From the comparison we can deduce that the Lagrangian and the constraints used are the correct ones for the derivation of an electrostatic four moment gyrofluid model.

6.0.1 Future directions

In this thesis a consistent set of gyrofluid equations has been developed which allows for numerical computation of turbulent transport. However, in order to provide a more complete and accurate model of turbulent transport, the model presented here should be extended, and the approximations used to derive the gyrofluid equations should be relaxed. Here we will emphasise three of the most important approximations: the electrostatic approximation, the four-moment approach and the use of only passing electrons.

So far the model described uses the electrostatic approximation, assuming that turbulence can be described purely in terms of a fluctuating electrostatic potential, i.e. the magnetic fluctuations are neglected. This is equivalent to assuming that the electrodynamical beta $\beta_e \equiv c_s^2/v_A^2$ is smaller not only than unity, but also [1] the mass ratio $\beta_e << m_e/M_i$, which does not hold in modern tokamaks. Additionally, steep gradient phenomena [2] make drift wave turbulence itself electromagnetic when $\hat{\beta} = \beta_e(qR/L_T)^2 > 1$. The electrostatic approximation removes all electromagnetic modes such as shear Alfven waves, and their associated instabilities from the system. Hence simulations using the model presented here cannot describe such modes, which are predicted by linear theory to be unstable in the core as well as the edge of tokamak plasmas at sufficient high β. In addition, the electro-

static model cannot describe finite-β modifications to electrostatic modes such as the ITG instabilities, which are, however, substantial at the experimentally observed β values. Thus, future work should extend the current model to include electromagnetic effects. This can be done by including the parallel magnetic potential fluctuation in the Lagrangian, specifically in the symplectic part, in a similar way as is done for the gyrokinetic electromagnetic Lagrangian.

Another approximation that needs to be relaxed is the four-moment approximation. So far, we have included only the first four moments (density, parallel velocity, parallel and perpendicular pressure) as dynamical variables, and we have derived evolution equations for those four moments. A next step would be to include the next moments, namely the parallel heat fluxes, as dynamical variables, thus arriving to a six-moment model. The addition of the parallel heat fluxes is necessary to include physics such as Landau damping, which, in the case of the four moment model derived with the Lagrangian method, is not included. The addition of the heat fluxes is nontrivial, since the Lagrangian used so far does not depend on them. Since the heat fluxes are odd moments in the velocities (like the fluid velocity itself), the variation of the heat fluxes should in some way follow that of the velocity. Thus, it is possible that an additional displacement, namely a heat displacement, is needed together with the fluid (mass) displacement [3, 4, 5].

Another physical aspect of a tokamak plasma in the tokamak core that should be included in the model to correctly describe turbulent transport is the trapping of the electrons in the inhomogeneous magnetic field. There are instabilities associated with the trapped electrons like the one known as the Trapped Electron Mode (TEM) and, furthermore, trapped electrons also increase the growth rate of the ITG mode. For low frequency turbulence the trapped electron response could be described by bounce averaged equations, i.e. equations in which one has averaged over the trapped orbit in much the same way as one averages over the gyromotion. Trapped electrons equations have been developed in a local model [6], but as with previous versions the emphasis was only on the linear theory and there was no treatment of conservative energetics. The starting point for the derivation of global, fully nonlinear trapped electron equations with an energy conservation law would not be gyrokinetics, but a new theory should be developed in which the bounce averaging is treated. Despite this complication a complete transport model should be able to predict particle and electron transport, and thus, the nonadiabatic electron dynamics are a necessary ingredient.

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