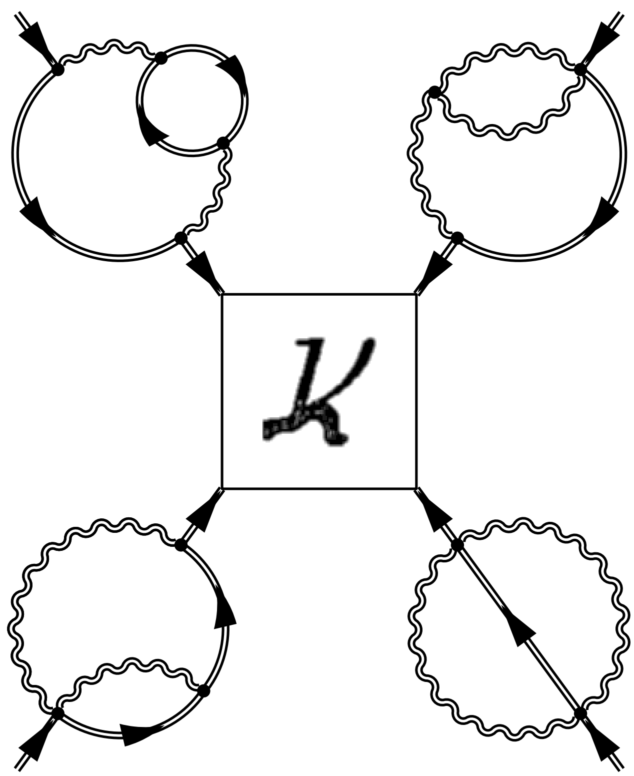
Running Neutrino Masses



Dissertation von Michael Ratz

Technische Universität München
Physik-Department
Institut für Theoretische Physik T30
Professor Dr. Manfred Lindner

Technische Universitt Mnchen
Physik Department
Institut für Theoretische Physik T30d
Univ.-Prof. Dr. M. Lindner

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Dipl.-Phys. Univ. Michael Ratz

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Summary

The renormalization group running of a seesaw mass matrix for the neutrinos in the Standard Model, a class of Two Higgs Models and in the Minimal Supersymmetric Standard Model is studied. A formalism for calculating β-functions for tensorial quantities in MS-like renormalizations schemes is presented. The corresponding Renormalization Group Equations for all three models are derived. Moreover, a general method of checking β-functions of effective theories is presented and applied in an example. Furthermore, a "construction kit" for computing β-functions of renormalizable and non-renormalizable operators in general $N=1$ supersymmetric theories is derived. Finally, the effects of non-degenerate see-saw scales on the renormalization group evolution of mixing angles are investigated.

Zusammenfassung

Es wird das Renormierungsgruppen-Laufen einer Seesaw-Massenmatrix für die Neutrinos im Standardmodell, in einer Klasse von Zwei-Higgs-Modellen und in der minimalen supersymmetrischen Erweiterung des Standardmodells untersucht. Ein Formalismus für die Berechnung von β-Funktionen für tensorielle Größen in MS-artigen Renormierungsschemata wird präsentiert. Die Renormierungsgruppen-Gleichungen für alle drei oben genannten Modelle werden hergeleitet. Darüberhinaus wird eine allgemeine Methode zur Überprüfung von β-Funktionen von effektiven Operatoren dargestellt und in einem Beispiel verwendet. Ausserdem wird ein "Baukasten-System" zur Berechnung von β-Funktionen für renormierbare und nicht-renormierbare Operatoren in allgemeinen $N = 1$ supersymmetrischen Theorien hergeleitet. Abschliessend werden die Effekte von nicht entarteten Seesaw-Skalen auf das Verhalten von Mischungswinkeln unter der Renormierungsgruppe untersucht.

Contents

1 Introduction

1.1 Motivation

The Standard Model (SM) yields an adequate and extremely precise description of high energy phenomena. Nevertheless, there are strong hints both from theory and experiment that it cannot be the ultimate theory. The measurements of atmospheric neutrinos performed by the Super-Kamiokande collaboration [1] have given clear indications of neutrino oscillations, and thus evidence for physics beyond the SM. Moreover, solar neutrino measurements strongly point towards neutrino oscillations, even on astrophysical scales. The recently published data of the Sudbury Neutrino Observatory [2,3] favors the Large Mixing Angle solution of the solar neutrino problem. From the theorist's point of view the SM suffers from some severe problems. It is certain that the SM is not the whole story since gravity is not incorporated. Consequently, one has to assume that the SM is only an effective theory which emerges from some underlying theory.

Therefore, the aim of many theorists is to elaborate a unified theory which reduces the unsatisfactory large number of SM parameters and which then can be regarded as (more) fundamental. Usually a typical energy scale, call it Λ, is associated with such a theory,, i.e. its predictions are expected to hold at this scale. Due to quantum effects, those predictions are altered if they are translated to an energy scale at which measurements can be made. This translation can be accomplished by solving the corresponding Renormalization Group Equations (RGE's), or, in other words, by running the predictions down to a lower scale. The gauge coupling evolution provides an impressive example of running quantities. In this respect, the Minimal Supersymmetric extension of the SM (MSSM) is especially interesting since it allows for gauge coupling unification without adding different representations or gauge groups to the model [4–6], thus leading to a so-called Grand Unified Theory (GUT). In this example, the unification scale is $M_{\mathrm{GUT}} = \mathscr{O}(10^{16}\,\mathrm{GeV})$, which is on the one hand much larger than the electroweak scale, $M_{\mathrm{EW}} = \mathscr{O}(10^{2}\,\mathrm{GeV})$, but on the other hand significantly smaller than the scale $\mathscr{O}(10^{18}\,\mathrm{GeV})$, at which gravity is expected to become important.

Moreover, from an underlying theory typically higher-dimensional, effective operators arise which are suppressed by some inverse powers of Λ. These operators are fixed at Λ and have to be run down to some lower scale, which is usually the electroweak scale or below, in order to be compared with the experiment. Although these operators are non-renormalizable by power-counting, their RGE's can be calculated in the Effective Field Theory (EFT) approach in which an expansion in inverse powers of Λ is performed. It turns out that the observed neutrino masses may well be described by an effective operator, the more so since the relative smallness of neutrino masses may find an attractive explanation in the see-saw mechanism [7,8]. Since there exist three generations of fundamental SM fermions, their mass parameters consist besides the mass eigenvalues of mixing angles and CP phases. It is quite remarkable that in the lepton sector one or more of the mixing angles are found to be large [9] whereas the corresponding quantities in the quark sector turn out to be small. Since in many unified models quark and lepton mixing have a common origin, it is interesting to study whether this discrepancy can be explained by a different RGE evolution of the corresponding mixing angles. It has to be stressed that studying the RGE's of the parameters is not an option but must be performed in any approach to elaborate a unified theory.

1.2 Goals of this Study

In this thesis, we will therefore address the question of calculating RGE's. In particular, we will concentrate on the derivation of β-functions for effective operators. Since neutrino masses and mixings are of great current interest, we will compute the β-functions for the effective neutrino mass operator in the SM and in its two most prominent extensions, the Two Higgs Doublet Models (2HDM's) and the MSSM. In the SM and the 2HDM's, these β-functions existed in the literature but were not correct. The correct β-functions were re-derived in collaboration with S. Antusch, M. Drees, J. Kersten and M. Lindner [10,11]. In this thesis, some unpublished details of calculation and formulae as well as a new method of checking β-functions of arbitrary effective theories are presented.

In the MSSM, the β-functions were only known at one-loop, and the two-loop β-function was computed in collaboration with S. Antusch [12]. While usually the calculations were performed in the component field approach, in which the non-renormalization theorem is not manifest, the two-loop result is obtained by using supergraphs. With this method, calculations are simplified considerably since due to the non-renormalization theorem only the wavefunction renormalization constants have to be taken into account for operators of the superpotential. Although the theorem was known to be applicable to renormalizable operators quite for a while, the fact that it holds for non-renormalizable operators as well was not clear until 1998 [13]. Therefore, the method of calculating β-functions for higher-dimensional operators with the supergraph method represents significant simplification of existing techniques for calculation. The relevant formalism will be presented in great detail.

Furthermore, the effects of non-degenerate see-saw scales on the evolution of mass parameters were discussed in collaboration with S. Antusch, J. Kersten and M. Lindner in [14]. The case where the SM and the MSSM are extended by an arbitrary number of heavy singlets which have explicit (Majorana) masses with a non-degenerate spectrum is considered and extended to the 2HDM's in this study. The RGE's that govern the evolution of the neutrino mass matrix in the effective theories arising between the mass thresholds are calculated. In particular, the evolution of the mixing angle is investigated in a 2×2 example in order to illustrate the differences in the evolution compared to a treatment where all heavy degrees of freedom are integrated out at a common scale.

1.3 Outline

This thesis is organized as follows: After a short review[1] of some basic concepts of renormalization, a general formalism is derived which allows for computing the β-functions for tensorial quantities in MS-like renormalization schemes directly from the counterterms, even if additive renormalization is imposed. The third chapter deals with the idea of effective theories. The procedure of integrating out heavy degrees of freedom is exemplified in a simple extension of the SM, and this way the lowest-dimensional, effective neutrino mass operator compatible with the gauge symmetries of the SM is introduced. In the fourth chapter, the renormalization of this operator in the SM is addressed. Besides a short review of the calculation of the corresponding β-function, a general method of checking β-functions for arbitrary effective operators is presented. This method is applied to verify the coefficients of the result mentioned before. Chapter five is dedicated to the calculation of neutrino mass operator RGE's in extensions of the SM. The β-functions for these operators in a class of 2HDM's is derived in great detail. Chapter six concerns with the calculation of RGE's in supersymmetric theories. A general construction kit is presented which allows for computing two-loop β-functions of arbitrary, in particular even higher-dimensional operators of the superpotential with only little effort. All β-functions for the couplings of the MSSM extended by singlet "neutrino" superfields are explicitly specified. Moreover, the formulae given in this chapter make it trivial to determine β-functions for any superpotential operator in the MSSM or the extension mentioned before. In chapter seven, the solutions of the RGE's are analyzed analytically and numerically, and the effect of multiple, non-degenerate mass thresholds is briefly discussed within the effective theories described before in an example with two generations. In the final chapter, conclusions are drawn and a short outlook concerning possible applications of this study is given.

[1] In this thesis, there appear some sections which are quite introductory. The intention of those parts is to fix the notation and conventions used in this study.

2 Renormalization

First we review some elementary formulae which are used througout this thesis, thus fixing the notation. At the end of this chapter, a general method for calculating β-functions from counterterms is derived.

2.1 Greens and Vertex Functions

2.1.1 Generating Functional

Consider a quantum field theory of some field φ, described by a Lagrangian $\mathscr{L}(\varphi, \partial_\mu \varphi)$. As usual, the generating functional is introduced by

$$\mathcal{Z}[J] = \mathscr{N} \int \mathcal{D}\varphi \, \exp\left\{ \mathrm{i} \int \mathrm{d}^4 x \, \mathscr{L}(\varphi, \partial_\mu \varphi) + J\varphi \right\} \ . \tag{2.1-1}$$

From this, one can calculate the Greens functions

$$G_N(x_1, \ldots x_N) = \langle - | \boldsymbol{T} \left\{ \boldsymbol{\varphi}(x_1) \cdots \boldsymbol{\varphi}(x_N) \right\} | - \rangle = \left. \frac{\delta^N \mathcal{Z}[J]}{\mathrm{i}\delta J(x_1) \cdots \mathrm{i}\delta J(x_N)} \right|_{J=0} \ , \tag{2.1-2}$$

where $\boldsymbol{T}$ denotes the time-ordering operator and $|-\rangle$ the vacuum of the theory. Here and throughout this thesis, operators are written boldface. Using perturbation theory, the Greens functions can be represented by Feynman diagrams. The Feynman rules depend on the specific shape of the Lagrangian.

2.1.2 Generating Functional for Connected Diagrams

It is useful to distinguish between Greens functions corresponding to topologically connected diagrams and Greens functions corresponding to disconnected diagrams. Examples are depicted in figures 2.1(a) and 2.1(b). One can show that the functional $\mathcal{W}[J]$, defined by

$$\mathcal{Z}[J] = \exp\left\{ \mathrm{i}\mathcal{W}[J] \right\} \ , \tag{2.1-3}$$

generates only connected N-point functions [15], i.e. the connected N-point functions are calculated by

$$\mathscr{G}_N(x_1, \ldots x_N) = (-\mathrm{i})^N \frac{\delta^N \mathcal{W}[J]}{\delta J(x_1) \cdots \delta J(x_N)} \ . \tag{2.1-4}$$

2.1.3 Generating Functional for OPI diagrams

Consider a diagram whose legs are amputated. Such a diagram is called **reducible** if it can be divided into two subdiagrams by cutting one inner line. Otherwise we speak of **One Particle Irreducible (OPI)** diagrams. Similarly to the connected diagrams, the OPI diagrams can be generated by a functional as well. However, for the definition of this functional we will need some preliminary considerations.

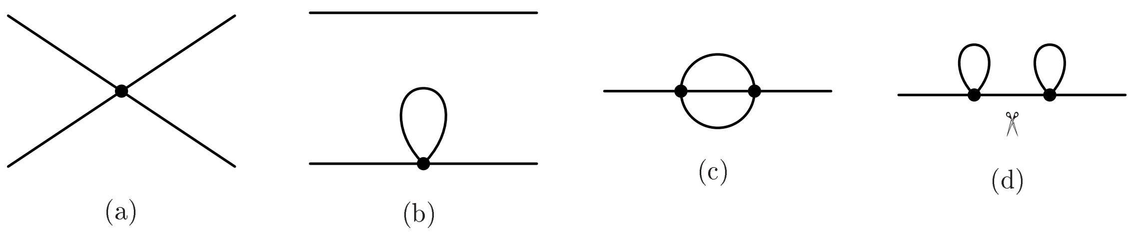


Figure 2.1: Diagram (a) is connected, diagram (b) is not, diagram (c) is OPI while diagram (d) is not since it can be subdivided into two diagrams by cutting the line marked by the scissors.

We define

$$\varphi_c(x) := \frac{\delta \mathcal{W}[J]}{\delta J(x)} \tag{2.1-5}$$

as the **classical field**. The Legendre transform of $\mathcal{W}$ w.r.t. φ_c,

$$\Gamma[\varphi_c] := \mathcal{W}[J] - \int \mathrm{d}^4x \, J(x)\varphi_c(x) \ , \tag{2.1-6}$$

is called **effective action**. As **vertex function** we define

$$\Gamma_N(x_1, \ldots x_N) := (-\mathrm{i})^N \frac{\delta^N \Gamma[\varphi_c]}{\delta\varphi_c(x_1)\cdots\delta\varphi_c(x_N)} \ . \tag{2.1-7}$$

Up to factors of i, these correspond to the OPI diagrams, or, more precisely, the effective action is the generating functional for the OPI vertex functions.

Equation (2.1-7) can be inverted,

$$\Gamma[\varphi_c] = \sum_{N=1}^{\infty} \frac{\mathrm{i}^N}{N!} \int \mathrm{d}^4x_1 \cdots \int \mathrm{d}^4x_N \, \Gamma_N(x_1, \ldots x_N) \, \varphi_c(x_1)\cdots\varphi_c(x_N) \ . \tag{2.1-8}$$

Furthermore, $\Gamma_2(x, y)$ and $\mathscr{G}_2(x, y)$ are inverse to each other,

$$\begin{aligned} \int \mathrm{d}^4y \, \mathscr{G}_2(x, y) \, \Gamma_2(y, z) &= -\mathrm{i} \int \mathrm{d}^4y \, \frac{\delta^2 \mathcal{W}[J]}{\delta J(x) \, \delta J(y)} \, \frac{\delta^2 \Gamma[\varphi_c]}{\delta\varphi_c(y) \, \delta\varphi_c(z)} \\ &= \mathrm{i} \int \mathrm{d}^4y \, \frac{\delta\varphi_c(x)}{\delta J(y)} \, \frac{\delta J(y)}{\delta\varphi_c(z)} = \mathrm{i}\delta^{(4)}(x - z) \ . \end{aligned} \tag{2.1-9}$$

Further differentiation yields

$$\begin{aligned}\frac{\delta^N \Gamma[\varphi_c]}{\delta\varphi_c(x_1)\cdots\delta\varphi_c(x_N)} &= \\ &= -\mathrm{i}^N \int \mathrm{d}y_1 \ldots \mathrm{d}y_N \, \Gamma_2(x_1,y_1)\cdots\Gamma_2(x_N,y_N)\, \frac{\delta^N \mathcal{W}[J]}{\delta J(y_1)\cdots\delta J(y_N)} \,. \end{aligned} \tag{2.1-10}$$

Thus, the N-point vertex functions correspond to the Greens functions divided by their external legs. Therefore we conclude that under scaling of the fields, $\varphi \to \zeta \cdot \varphi$, the connected Greens functions transform oppositely to the vertex functions,

$$\mathscr{G}_N(x_1,\ldots x_N) \quad \xrightarrow{\varphi\to\zeta\varphi} \quad \zeta^N \mathscr{G}_N(x_1,\ldots x_N)\,, \tag{2.1-11a}$$

$$\Gamma_N(x_1,\ldots x_N) \quad \xrightarrow{\varphi\to\zeta\varphi} \quad \zeta^{-N} \Gamma_N(x_1,\ldots x_N)\,. \tag{2.1-11b}$$

Equivalently, we may consider the vertex functions in momentum space, i.e. the Fourier transforms of the vertex functions in coordinate space. We will use the arguments $p, q, \ldots$ in order to signify the Fourier transform,

$$\mathscr{G}_N(p_1,\ldots p_N) := \int \frac{\mathrm{d}^4 x_1}{(2\pi)^4}\, e^{\mathrm{i}p_1\cdot x_1} \cdots \int \frac{\mathrm{d}^4 x_N}{(2\pi)^4}\, e^{\mathrm{i}p_N\cdot x_N}\, \mathscr{G}_N(x_1,\ldots x_N) \quad \text{etc.} \tag{2.1-12}$$

Since the vertex functions always contain energy-momentum conservation in the form of a δ function, $\Gamma_N(p_1,\ldots p_N) \sim \delta^{(4)}(p_1+\cdots p_N)$, it is convenient to define **proper vertex functions** $\overline{\Gamma}_N(p_1,\ldots p_{N-1})$ by

$$\Gamma_N(p_1,\ldots p_N) =: \overline{\Gamma}_N(p_1,\ldots p_{N-1}) \cdot \delta^{(4)}(p_1+\cdots p_N)\,. \tag{2.1-13}$$

2.2 Basic Concepts of Renormalization

We consider a theory of some real scalar fields φ with a mass parameter m and a coupling g involving N_g fields. The generalization to more parameters is not performed explicitly for the sake of keeping the expressions simple.

2.2.1 Dimensional Regularization

In this thesis, for non-supersymmetric calculations dimensional regularization is used since it has the following advantages:

(1) It manifestly preserves almost all symmetries.

(2) The regularized graphs are no harder calculated than the unregularized ones.

(3) Renormalization is particularly simple in conjunction with the MS scheme.

The procedure of dimensional regularization consists of calculating graphs formally in d dimensions. Useful formulae for this purpose can be found in appendix E. Divergences show up as poles in the deviation from 4 dimensions, $\epsilon := 4 - d$. In order to keep the mass dimension of the quantities in d dimensions the same as in 4, a parameter μ carrying mass dimension one is introduced. Then quantities Q can be written in d dimensions as $\mu^{D_Q \epsilon} Q$, where D_Q is chosen appropriately. However, the specific value of μ is completely arbitrary.

2.2.2 Bare and Renormalized Quantities

The theory is described by a **bare Lagrangian**

$$\mathscr{L}_{\mathrm{B}} = \mathscr{L}_{\mathrm{B}} \left(\varphi_{\mathrm{B}}, m_{\mathrm{B}}, g_{\mathrm{B}} \right) \ . \tag{2.2-1}$$

For definiteness we use

$$\mathscr{L}_{\mathrm{B}} = \frac{1}{2} \left(\partial_\mu \varphi_{\mathrm{B}} \right) \left(\partial^\mu \varphi_{\mathrm{B}} \right) - \frac{m_{\mathrm{B}}^2}{2} \varphi_{\mathrm{B}}^2 - \frac{g_{\mathrm{B}}}{N_g!} \varphi_{\mathrm{B}}^{N g} \ . \tag{2.2-2}$$

The bare quantities are related to the **renormalized** ones by

$$\begin{aligned} \varphi_{\mathrm{B}} &= Z_\varphi^{\frac{1}{2}} \, \varphi \ , & \text{(2.2-3a)} \\ m_{\mathrm{B}}^2 &= Z_\varphi^{-1} \left(m^2 + \delta m^2 \right) \quad \text{and} & \text{(2.2-3b)} \\ g_{\mathrm{B}} &= \mu^{D_g \epsilon} \, Z_\varphi^{-\frac{N_g}{2}} \left(g + \delta g \right) . & \text{(2.2-3c)} \end{aligned}$$

The **renormalization constants** Z_φ, δm^2 and δg are chosen such that, in a given order in perturbation theory, amplitudes calculated with the Lagrangian (2.2-1) are finite. δm^2 and δg are referred to as **counterterms**, and Z_φ as **wavefunction renormalization** constant. The bare Lagrangian is related to the renormalized one, $\mathscr{L}$, by

$$\mathscr{L}_{\mathrm{B}} \left(\varphi_{\mathrm{B}}, m_{\mathrm{B}}, g_{\mathrm{B}} \right) = \mathscr{L} \left(\varphi, m, g \right) + \mathscr{C} \left(\varphi, m, g \right) \tag{2.2-4}$$

with $\mathscr{C}$ being called **counterterm Lagrangian**. It is a general feature of a renormalizable theory that all $\mathscr{L}$, $\mathscr{L}_{\mathrm{B}}$ and $\mathscr{C}$ have the same shape except for a *finite* number of couplings that have to be included. The splitting (2.2-4) is completely arbitrary under the condition that the quantities appearing in the renormalized Lagrangian are finite.

2.2.3 MS-like Renormalization Schemes

However, we may fix a certain choice of the splitting (2.2-4) by demanding that the counterterm be proportional to ϵ^{-1}. This prescription is known as **Minimal Subtraction**

(MS). Consequently, for the MS-scheme equations (2.2-3) read

$$\varphi_{\mathrm{B}} = \left(1 + \sum_{k=1}^{\infty} \frac{\delta Z_{\varphi,k}(g,m,\mu)}{\epsilon^k}\right)^{\frac{1}{2}} \varphi \,, \tag{2.2-5a}$$

$$m_{\mathrm{B}} = Z_\varphi^{-1} \left(m + \sum_{k=1}^{\infty} \frac{\delta m_{,k}(g,m,\mu)}{\epsilon^k}\right) \,, \tag{2.2-5b}$$

$$g_{\mathrm{B}} = \mu^{D_g \epsilon} Z_\varphi^{-\frac{N_g}{2}} \left(g + \sum_{k=1}^{\infty} \frac{\delta g_{,k}(g,m,\mu)}{\epsilon^j}\right) \,. \tag{2.2-5c}$$

Stated differently, the renormalized quantities in equations (2.2-3) are functions of μ. Since μ is completely arbitrary, we can study the implications of rescaling μ.

2.2.4 Derivation of the Renormalization Group Equation

Starting from the bare Lagrangian $\mathscr{L}_{\mathrm{B}}$, we can determine the N-point functions

$$(G_N)_{\mathrm{B}}(x_1, \dots x_n) = \langle - | \boldsymbol{T}\{\boldsymbol{\varphi}_{\mathrm{B}}(x_1) \cdots \boldsymbol{\varphi}_{\mathrm{B}}(x_N)\} | - \rangle \,. \tag{2.2-6}$$

Using $\varphi = Z_\varphi^{-1/2} \varphi_{\mathrm{B}}$, we find for the vacuum expectation value of the time-ordered product of the *renormalized* fields

$$\langle - | \boldsymbol{T}\{\boldsymbol{\varphi}(x_1) \cdots \boldsymbol{\varphi}(x_N)\} | - \rangle = Z_\varphi^{-N/2} \langle - | \boldsymbol{T}\{\boldsymbol{\varphi}_{\mathrm{B}}(x_1) \cdots \boldsymbol{\varphi}_{\mathrm{B}}(x_N)\} | - \rangle \tag{2.2-7}$$

or

$$Z_\varphi^{N/2}(g,m,\mu)\, G_N(\{x_i\}, m, g, \mu) = (G_N)_{\mathrm{B}}(\{x_i\}, m_{\mathrm{B}}, g_{\mathrm{B}}) \,. \tag{2.2-8}$$

Since the right-hand side is independent of μ, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \left\{ Z_\varphi^{N/2}(g,m,\mu) G_N\Big(\{x_i\}, m, g, \mu\Big)\right\} \stackrel{!}{=} 0 \,. \tag{2.2-9}$$

An analogous relation holds for the connected Greens functions $\mathscr{G}_N$ and for the vertex functions Γ_N as well. Using the fact that the renormalized quantities are functions of μ and equation (2.2-3), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \left\{ Z_\varphi^{-N/2}(\mu) \overline{\Gamma}_N\Big(\{p_i\}, m(\mu), g(\mu), \mu\Big)\right\} = 0 \,, \tag{2.2-10}$$

where the proper vertex functions $\overline{\Gamma}_N$ were considered and equation (2.1-11) was used. Accordingly, one can substitute

$$\frac{\mathrm{d}}{\mathrm{d}\mu} = \frac{\partial}{\partial\mu} + \frac{\mathrm{d}g}{\mathrm{d}\mu}\frac{\partial}{\partial g} + \frac{\mathrm{d}m}{\mathrm{d}\mu}\frac{\partial}{\partial m} \,, \tag{2.2-11}$$

and use the definitions

$$\beta(g,m,\mu,\epsilon) \;:=\; \mu\frac{\mathrm{d}g}{\mathrm{d}\mu} \xrightarrow{\epsilon\to 0} \beta(g,m,\mu)\;, \tag{2.2-12a}$$

$$\gamma(g,m,\mu,\epsilon) \;:=\; \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_\varphi \xrightarrow{\epsilon\to 0} \gamma(g,m,\mu)\;, \tag{2.2-12b}$$

$$m^2\,\gamma_m(g,m,\mu,\epsilon) \;:=\; -\mu\frac{\mathrm{d}m}{\mathrm{d}\mu} \xrightarrow{\epsilon\to 0} m^2\,\gamma_m(g,m,\mu)\;. \tag{2.2-12c}$$

β is called **beta function** while γ_m and γ are referred to as **anomalous dimension** of the mass and the field, respectively. With these definitions inserted in (2.2-11), the **Renormalization Group Equation (RGE)** reads

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(g,m,\mu)\,\frac{\partial}{\partial g} - \frac{N}{2}\gamma(g,m,\mu) - m^2\,\gamma_m(g,m,\mu)\,\frac{\partial}{\partial m}\right\}\overline{\Gamma}_N(\{p_i\},g,m,\mu) = 0\;. \tag{2.2-13}$$

It is convenient to use the variable $t = \ln\left(\frac{\mu}{\mu_0}\right)$. Then the solution of the RGE has the property

$$\overline{\Gamma}_N\left(\{p_i\},g,m,\mu_0\right) = \overline{\Gamma}_N\left(\{p_i\},g(t),m(t),e^t\mu_0\right)\cdot\exp\left\{-\frac{N}{2}\int\limits_0^t \mathrm{d}\tau\,\gamma\big(g(\tau)\big)\right\}\;, \tag{2.2-14}$$

where $g(t)$ and $m(t)$ are the solutions of

$$\frac{\partial g}{\partial t}(t) \;=\; \beta(g(t),m(t),\mu)\;, \tag{2.2-15a}$$

$$\frac{\partial m}{\partial t}(t) \;=\; \gamma_m\big(g(t),m(t),\mu\big)\,m^2(t)\;, \tag{2.2-15b}$$

with the initial conditions $g(0) = g$ and $m(0) = m$.

2.2.5 The β-Function

In order to calculate β, we use the equation

$$\mu\frac{\mathrm{d}g_\mathrm{B}}{\mathrm{d}\mu} = 0 \tag{2.2-16}$$

and insert the expression for g_B. For example, let us consider the case that equation (2.2-3c) can be rewritten in the form

$$g_\mathrm{B} = \mu^{D_g\epsilon}\,Z_g\,g\;, \tag{2.2-17}$$

where

$$Z_g = 1 + \sum_{k=1}^{\infty}\frac{\delta Z_{g,k}}{\epsilon^k} = 1 + \delta Z_g \tag{2.2-18}$$

and where D_g is chosen as explained at the end of section 2.2.1. By differentiation we obtain

$$\begin{aligned} \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g_{\mathrm{B}} &= \beta(g,m,\mu,\epsilon)\,\mu^{D_g\epsilon}Z_g + g\,D_g\epsilon\,\mu^{D_g\epsilon}\,Z_g + g\,\mu^{D_g\epsilon}\frac{\partial\delta Z_g}{\partial g}\beta(g,m,\mu,\epsilon) \\ &= \mu^{D_g\epsilon}\left[\beta(g,m,\mu,\epsilon)\left(Z_g + g\frac{\partial\delta Z_g}{\partial g}\right) + g\,D_g\epsilon\,Z_g\right] . \end{aligned} \tag{2.2-19}$$

On the other hand, β-functions are finite as $\epsilon \to 0$. We can therefore make the ansatz

$$\beta(g,m,\mu,\epsilon) = \beta(g,m,\mu) + \epsilon\beta^{(1)}(g,m,\mu) + \cdots + \epsilon^n\beta^{(n)}(g,m,\mu)\ , \tag{2.2-20}$$

where n is an arbitrary integer. Note that in this case the power of ϵ is not related to the order of perturbation theory. We insert now equation (2.2-18) and the expansion (2.2-20) and obtain by comparing coefficients

$$\beta(g,m,\mu,\epsilon) = -\epsilon D_g g + \sum_{k=1}^{n} D_g\,g^2\frac{\partial\delta Z_{g,k}}{\partial g}\epsilon^{k-1}\ , \tag{2.2-21}$$

i.e.

$$\beta(g,m,\mu) = D_g\,g^2\frac{\partial\delta Z_{g,1}}{\partial g}\ . \tag{2.2-22}$$

This is the standard formula for calculating β-functions for scalar couplings in MS-like renormalization schemes where multiplicative renormalization is used. It will be generalized in section 2.3.

Remark 2.2.1. Dimensional analysis tells us that in mass-independent renormalization schemes

$$g(\mu') = G\left(g(\mu), \frac{\mu'}{\mu}\right) \tag{2.2-23}$$

with some appropriate function G. Differentiating w.r.t. μ' yields at $\mu' = \mu$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu) = \beta\left(g(\mu)\right) \tag{2.2-24}$$

where

$$\beta\left(g(\mu)\right) = \frac{\partial}{\partial\zeta}G\left(g(\mu),\zeta\right)\Big|_{\zeta=1}\ . \tag{2.2-25}$$

Thus, in mass-independent schemes, β does not depend on μ explicitly.

2.2.6 Scale Transformations

Now we study what happens if we rescale the external momenta, $\{p_i\} \to \{\zeta \cdot p_i\}$. We assign to each Greens function G_N a mass dimension which is equal N times the mass dimension of the field φ, $\dim[\varphi] = d_\varphi$, and find for example

$$\dim[G_N(x_1, \dots x_N)] = \dim[\langle - | \boldsymbol{T}\{\boldsymbol{\varphi}(x_1) \cdots \boldsymbol{\varphi}(x_N)\} | - \rangle] = N \cdot d_\varphi \,. \tag{2.2-26}$$

Analogously, we find

$$\dim\left[\overline{\Gamma}_N(p_1, \dots p_{N-1})\right] = 4 - N\, d_\varphi \,. \tag{2.2-27}$$

By rescaling all variables which carry mass dimension we obtain

$$\overline{\Gamma}_N\left(\{\zeta p_i\}, \zeta^{d_g} g(\mu), \zeta m(\mu), \zeta\mu\right) = \zeta^{d_{\overline{\Gamma}_N}} \overline{\Gamma}_N\left(\{p_i\}, g(\mu), m(\mu), \mu\right) \,, \tag{2.2-28}$$

where $d_g := \dim[g]$ and $d_{\overline{\Gamma}_N} = \dim\left[\overline{\Gamma}_N\right]$. Inserting the solution of the RGE, we obtain with $\zeta = e^t$

$$\begin{aligned}
&\overline{\Gamma}_N(\{e^t p_i\}, g, m, \mu) \overset{(2.2\text{-}14)}{=} \\
&\overset{(2.2\text{-}14)}{=} \quad \overline{\Gamma}_N(e^t\{p_i\}, g(t), m(t), e^t\mu) \exp\left\{ -\frac{N}{2}\int_0^t \mathrm{d}\tau\, \gamma\big(g(\tau)\big) \right\} \\
&\overset{(2.2\text{-}28)}{=} \quad \exp\left\{ d_{\overline{\Gamma}_N} t - \frac{N}{2}\int_0^t \mathrm{d}\tau\, \gamma\big(g(\tau)\big) \right\} \overline{\Gamma}_N\left(\{p_i\}, e^{-d_g t} g\,(t)\,, e^{-t} m\,(t)\,, \mu\right) \,,
\end{aligned} \tag{2.2-29}$$

i.e. rescaling the external momenta can be compensated by a multiplicative factor and a change in the renormalized mass m and the renormalized coupling g. This formula is crucial for understanding the meaning of RGE's and running quantities. It describes the behavior of physical quantities under the change of the energy scale. Note that rescaling on-shell external momenta is only possible for massless particles.

2.3 β-Functions in MS-like Renormalization Schemes

In this section we derive a formula which allows us to compute the β-function for a quantity Q directly from the counterterm δQ in the MS scheme. In particular, we generalize the usual formalism, equation (2.2-22), for calculating β-functions to include tensorial quantities as well as non-multiplicative renormalization.

As a generalization of (2.2-5c) we impose the relation

$$\begin{aligned}
Q_{\mathrm{B}} &= Z_{\phi_1}^{n_1} \cdots Z_{\phi_M}^{n_M} \, [Q + \delta Q] \mu^{D_Q \epsilon} \, Z_{\phi_{M+1}}^{n_{M+1}} \cdots Z_{\phi_N}^{n_N} \\
&= \left(\prod_{i \in I} Z_{\phi_i}^{n_i} \right) [Q + \delta Q] \mu^{D_Q \epsilon} \left(\prod_{j \in J} Z_{\phi_j}^{n_j} \right) \,,
\end{aligned} \tag{2.3-1}$$

where $I = \{1, \dots M\}$ and $J = \{M+1, \dots N\}$, and the n_i are integer or half-integer valued.

Remark 2.3.1. D_Q controls the dependence of Q on ϵ and should not be confused with its mass dimension. Consider for example a quantity κ, coupling to two fermions ψ and two bosons ϕ, and with $\dim[\kappa_{\rm B}] = 3 - d$. Then we obtain a relation

$$\kappa_{\rm B} = (Z_\phi^{-\frac{1}{2}})^T\,(Z_\psi^{-\frac{1}{2}})^T\,[\kappa + \delta\kappa]\,\mu^{D_\kappa \epsilon}\,Z_\phi^{-\frac{1}{2}}\,Z_\psi^{-\frac{1}{2}}\;, \tag{2.3-2}$$

where $\dim[\kappa] = -1$ and $D_\kappa = 1$ follows from $\dim[\psi] = \frac{d-1}{2}$ and $\dim[\phi] = \frac{d-2}{2}$.

δQ and the wavefunction renormalization constants can be expressed as follows,

$$\delta Q \;=\; \delta Q(\{V_A\})\;, \tag{2.3-3a}$$
$$Z_{\phi_i} \;=\; Z_{\phi_i}(\{V_A\}) \qquad (1 \leq i \leq N)\;, \tag{2.3-3b}$$

where $\{V_A\}$ denotes the set of variables of the theory including the one under consideration, Q, i.e. $\{V_A\} = \{Q, \dots\}$. Note that $V_A = V_A(\mu)$ are functions of the renormalization scale μ, but δQ and Z_{ϕ_i} do not depend explicitly on μ in an MS-like renormalization scheme.

Taking the derivative of equation (2.3-1) yields

$$\begin{aligned} 0 \;&\overset{!}{=}\; \mu^{-D_Q\epsilon}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Q_{\rm B} \\ &= \left(\prod_{i\in I} Z_{\phi_i}^{n_i}\right)\left[\beta_Q + \sum_A \left\langle \frac{\mathrm{d}\delta Q}{\mathrm{d}V_A}\middle|\beta_{V_A}\right\rangle + \epsilon D_Q(Q+\delta Q)\right]\left(\prod_{j\in J} Z_{\phi_j}^{n_j}\right) \\ &\quad + \left(\prod_{i\in I} Z_{\phi_i}^{n_i}\right)[Q+\delta Q]\,\times \\ &\qquad \times\left\{\sum_{j\in J}\left(\prod_{j'<j} Z_{\phi_{j'}}^{n_{j'}}\right)\left[\sum_A \left\langle \frac{\mathrm{d}Z_{\phi_j}^{n_j}}{\mathrm{d}V_A}\middle|\beta_{V_A}\right\rangle\right]\left(\prod_{j''>j} Z_{\phi_{j''}}^{n_{j''}}\right)\right\} \\ &\quad + \left\{\sum_{i\in I}\left(\prod_{i'<i} Z_{\phi_{i'}}^{n_{i'}}\right)\left[\sum_A \left\langle \frac{\mathrm{d}Z_{\phi_i}^{n_i}}{\mathrm{d}V_A}\middle|\beta_{V_A}\right\rangle\right]\left(\prod_{i''>i} Z_{\phi_{i''}}^{n_{i''}}\right)\right\}\times \\ &\qquad \times[Q+\delta Q]\left(\prod_{j\in J} Z_{\phi_j}^{n_j}\right)\;. \end{aligned} \tag{2.3-4}$$

Here we have introduced the notation

$$\left\langle\frac{\mathrm{d}F}{\mathrm{d}x}\middle|y\right\rangle := \begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}x}y & \text{for scalars } x, y \\ \displaystyle\sum_n \frac{\mathrm{d}F}{\mathrm{d}x_n}y_n & \text{for vectors } x = (x_n), y = (y_n) \\ \displaystyle\sum_{m,n} \frac{\mathrm{d}F}{\mathrm{d}x_{mn}}y_{mn} & \text{for matrices } x = (x_{mn}), y = (y_{mn}) \\ \dots & \text{etc. .} \end{cases} \tag{2.3-5}$$

We will solve equation (2.3-4) and the corresponding expressions for the other V_A by expanding all quantities in powers of ϵ. In the MS-scheme the quantities δV_A and Z_{ϕ_i} can be expanded as

$$\delta V_A = \sum_{k\geq 1} \frac{\delta V_{A,k}}{\epsilon^k} , \tag{2.3-6a}$$

$$Z_{\phi_i} = \mathbb{1} + \sum_{k\geq 1} \frac{\delta Z_{\phi_i,k}}{\epsilon^k} =: \mathbb{1} + \delta Z_{\phi_i} . \tag{2.3-6b}$$

From equations (2.2-20) and (2.3-6) we find that

$$\frac{\mathrm{d}Z_{\phi_i}^{n_i}}{\mathrm{d}V_A} = n_i\, Z_{\phi_i}^{n_i-1} \frac{\mathrm{d}Z_{\phi_i}}{\mathrm{d}V_A} = n_i \frac{\mathrm{d}\delta Z_{\phi_i}}{\mathrm{d}V_A} + \mathscr{O}\left(\frac{1}{\epsilon^2}\right) = \mathscr{O}\left(\frac{1}{\epsilon}\right) , \tag{2.3-7}$$

where the lowest possible power of $\frac{1}{\epsilon}$ appearing on the right side of (2.3-7) is 1. Our analysis of equation (2.3-4), starting with the inspection of the ϵ^n term, then shows that $\beta_{V_A}^{(n)}$ vanishes. Repeating this argument for successively smaller positive powers of ϵ implies that

$$\beta_{V_A}^{(k)}(\{V_A\}) = 0 \qquad \forall\, k \in \{2, \dots, n\} , \tag{2.3-8a}$$

$$\beta_{V_A}^{(1)}(\{V_A\}) = -D_{V_A} V_A . \tag{2.3-8b}$$

Note that these terms do not contribute to the β-function in 4 dimensions, i.e. for $\epsilon \to 0$, but they are necessary to read off $\beta_Q(\{V_A\})$ from equation (2.3-4), leading to the result

$$\begin{aligned} \beta_Q(\{V_A\}) = {} & D_{V_A} \left\langle \frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle - D_Q\, \delta Q_{,1} \\ & + Q \cdot n_j \left[D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_j,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \right] + n_i \left[D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \right] \cdot Q , \end{aligned} \tag{2.3-9}$$

where summation over A, i and j is implied. Note that for complex quantities V_A we have to treat the complex conjugates Q^* and V_A^* as additional independent variables. Consider for example the RGE for a complex quantity Q in a theory in which also the complex variable V appears. Then one can set $V_1 = Q$, $V_2 = Q^*$, $V_3 = V$ and $V_4 = V^*$ and apply equation (2.3-9).

Remark 2.3.2. If we impose multiplicative renormalization,

$$\delta Q = \delta Z_Q \cdot Q \qquad \text{with} \quad \delta Z_Q = \sum_{k=1}^{N} \frac{\delta Z_{Q,k}}{\epsilon^k} , \tag{2.3-10}$$

we find

$$\left\langle \frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}Q} \middle| Q \right\rangle = \left\langle \frac{\mathrm{d}\delta Z_{Q,1}}{\mathrm{d}Q} \middle| Q \right\rangle \cdot Q + \delta Q_{,1} , \tag{2.3-11a}$$

$$\left\langle \frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle = \left\langle \frac{\mathrm{d}\delta Z_{Q,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \cdot Q \qquad \forall\, V_A \neq Q , \tag{2.3-11b}$$

and therefore conclude

$$\begin{aligned}\beta_Q(\{V_A\}) &= -\epsilon\, D_Q\, Q + \sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{Q,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \cdot Q \\ &\quad + Q \cdot n_j \left[D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_j,1}}{\mathrm{d}V_A} V_A \right\rangle \right] + n_i \left[D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \right] \cdot Q \ , \qquad (2.3\text{-}12)\end{aligned}$$

where the second line remains unchanged compared to equation (2.3-9). The first line corresponds to equation (2.2-22), generalized for tensorial quantities.

In the following, we will make use of formula (2.3-9). In particular, it turns out that in most of the following problems, multiplicative renormalization as specified in equation (2.3-10) spoils the symmetry of the operators under consideration, so that it cannot be imposed. Therefore, equation (2.3-9) represents a crucial element for the calculation of the β-functions of this thesis.

3 Effective Field Theories

As already mentioned in the introduction, neutrino masses may well be described by an effective operator. In this chapter, this statement is substantiated. Moreover, we will describe a scenario in which a number of effective theories arise, corresponding to the number of non-degenerate mass eigenvalues of some heavy states.

3.1 Basic Ideas of an Effective Field Theory

3.1.1 Purpose of Effective Field Theories

Purpose of an Effective Field Theory (EFT) is to eliminate in a fundamental theory those degrees of freedom (d.o.f.) whose masses are far above the energy scales we are interested in. Consider for example a fundamental field theory with light d.o.f. ϕ (mass scale m), heavy d.o.f. Φ (mass scale $M \gg m$) and a Lagrangian $\mathscr{L}_{\mathrm{fundamental}}(\phi, \Phi)$. We are interested in an effective theory described by $\mathscr{L}_{\mathrm{eff}}(\phi)$, i.e. we look for a $\mathscr{L}_{\mathrm{eff}}(\phi)$ which fulfills

$$\begin{aligned} \exp(\mathrm{i}\mathcal{W}) &= \int \mathcal{D}\phi\,\mathcal{D}\Phi\,\exp\Big\{\mathrm{i}\underbrace{\int \mathrm{d}^4x\,\mathscr{L}_{\mathrm{fundamental}}(\phi,\Phi)}_{=S(\phi,\Phi)}\Big\} \\ &\overset{p_{\mathrm{E}}^2 \ll M^2}{\simeq} \int \mathcal{D}\phi\,\exp\Big\{\mathrm{i}\underbrace{\int \mathrm{d}^4x\,\mathscr{L}_{\mathrm{eff}}(\phi)}_{=S_{\mathrm{eff}}(\phi)}\Big\}\;, \end{aligned} \tag{3.1-1}$$

where p_{E}^2 denotes the Euclidean momentum squared, $p_{\mathrm{E}}^2 = E^2 + \vec{p}^{\,2}$.

3.1.2 Remark on the Terminology "Integrating Out"

Integrating out a real field Φ can be seen as the following sequence of steps:

(1) In a Lagrangian

$$-\mathscr{L}_{\mathrm{fundamental}} = -\frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi + \frac{M^2}{2}\,\Phi^2 + a(\phi)\,\Phi - \mathscr{L}_{\mathrm{l.d.o.f.}}(\phi) \tag{3.1-2}$$

neglect the kinetic term, leading to

$$-\mathscr{L}_{\mathrm{infrared}} = \frac{M^2}{2}\,\Phi^2 + a(\phi)\,\Phi - \mathscr{L}_{\mathrm{l.d.o.f.}}(\phi)\;. \tag{3.1-3}$$

(2) Complete the square

$$-\mathscr{L}_{\text{infrared}} + \mathscr{L}_{\text{l.d.o.f.}}(\phi) = \frac{M^2}{2}\,\Phi^2 + a(\phi)\,\Phi = \frac{M^2}{2}\left(\Phi + \frac{a(\phi)}{M^2}\right)^2 - \frac{a(\phi)^2}{2M^2}$$
$$=: \tilde{\Phi}^2 - \frac{a(\phi)^2}{2M^2}\,. \qquad (3.1\text{-}4)$$

(3) Perform the path integral over $\tilde{\Phi}$.

This leads to an effective Lagrangian

$$\mathscr{L}_{\text{eff}} = \frac{a(\phi)^2}{2M^2} + \mathscr{L}_{\text{l.d.o.f.}}(\phi)\;, \qquad (3.1\text{-}5)$$

where the field Φ has disappeared from the theory (and, of course, from the path integral). Deriving the field equations from $\mathscr{L}_{\text{infrared}}$, we find $\Phi = a(\phi)/2M^2$. Inserting this expression in $\mathscr{L}_{\text{infrared}}$ also leads to the effective Lagrangian (3.1-5).

Note that this procedure may also applied to complex fields Φ. For spinors the result is similar, only the way to obtain it differs a little bit. In particular, integrating out heavy fermions is equivalent to inserting the equations of motion in which the kinetic term is set to 0.

3.1.3 Decoupling Theorem

The Appelquist-Carrazone decoupling theorem [16] states that the low-energy effects of heavy particles, i.e. of particles with large direct mass terms, are either suppressed by inverse powers of the heavy masses, or they get absorbed into renormalizations of the couplings and fields of the EFT obtained by integrating out the heavy particles. Note that the theorem can in general not be applied to spontaneously broken gauge theories.

This theorem yields the main justification for considering EFT's which are valid in certain energy ranges. An application of the theorem is given in the next section.

3.2 Matching

We will explain the method of **matching** by an explicit example. In many mass models, left-handed neutrino masses are described by the effective operator

$$\mathscr{L}_\kappa = \frac{1}{4}\mu^\epsilon \sum_{f,g=1}^{n_F} \kappa_{gf}\, \overline{\ell^{c}_{\text{L}}}{}^{g}_{c}\phi_d\,(\varepsilon_{cd}\varepsilon_{ba} + \varepsilon_{ca}\varepsilon_{bd})\phi_a \ell^f_{\text{L}b}\tfrac{1}{2} + \text{h.c.}\;, \qquad (3.2\text{-}1)$$

where ℓ^f_{L}, $f \in \{1,\ldots,n_F\}$, are the $\text{SU}(2)_\text{L}$-doublets of SM leptons, ϕ is the Higgs doublet and $\ell^{\text{C}}_{\text{L}} := (\ell_\text{L})^{\text{C}}$ is the charge conjugate of the lepton doublet.

The operator (3.2-1) is symmetric under a permutation of the SU(2) indices, and furthermore it is assumed to be symmetric under a permutation of the generation indices f and g, i.e. κ is a symmetric matrix in generation space. We chose to write down the SU(2) index symmetry explicitly. We did this in order to read off the corresponding Feynman rule (C.9) directly. Therefore we can rewrite equation (3.2-1) in the form of [17,10],

$$\mathscr{L}_\kappa = \frac{1}{4}\mu^\epsilon \sum_{f,g=1}^{n_F} \kappa_{gf}\, \overline{\ell^c_{\mathrm{L}c}}^g \phi_d\, \phi_a \ell^f_{\mathrm{L}b} \varepsilon_{cd}\varepsilon_{ba} + \text{h.c.} = \frac{1}{4}\mu^\epsilon [\overline{\ell^c_\mathrm{L}}\varepsilon\phi]\, \kappa\, [\ell_\mathrm{L}\varepsilon\phi] + \text{h.c.}\ , \tag{3.2-2}$$

where we have also introduced a matrix representation of equation (3.2-1).

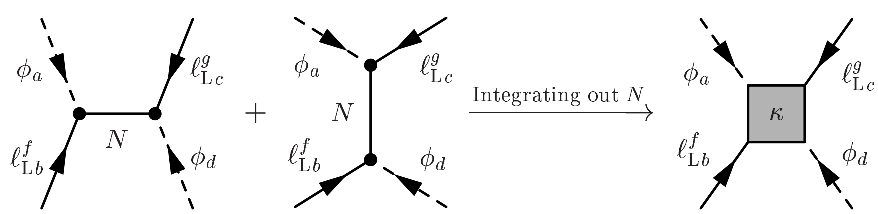
3.2.1 Tree-Level Matching

The operator (3.2-1) arises for example as an effective vertex of a possible full theory, described by a Lagrangian

$$\mathscr{L}_\mathrm{SM} + \frac{1}{2}\sum_{i=1}^{n_G} \overline{N}^i \mathrm{i}\partial\!\!\!/ N^i - \left(\sum_{i=1}^{n_G} \frac{M_i}{2} \overline{N}^i N^i + \mu^{\frac{\epsilon}{2}} \sum_{i=1}^{n_G}\sum_{f=1}^{n_F} (Y_\nu^\dagger)_{fi} \overline{\ell}^f_\mathrm{L} \widetilde{\phi} P_\mathrm{R} N^i + \text{h.c.} \right)\ , \tag{3.2-3}$$

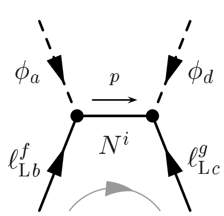
where Y_ν denotes the Yukawa coupling matrix and M the right-handed Majorana mass matrix. The Majorana spinor N, as defined in equation (B.14), was introduced.

Integrating out N is performed by the **tree-level matching**



, (3.2-4)

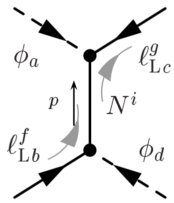
at the scale μ_R, the lowest eigenvalue of the mass matrix M. In other words, we replace e.g. the left diagram in the large M limit in the following way:



$$= [-\mathrm{i}(Y_\nu^T)_{gh}\varepsilon_{cd}\mu^{\frac{\epsilon}{2}}]\, P_\mathrm{L}\, \frac{p\!\!\!/ + M_i}{p^2 - M_i^2}\, [-\mathrm{i}(Y_\nu)_{hf}(\varepsilon^T)_{ab}\mu^{\frac{\epsilon}{2}}]\, P_\mathrm{L}$$

$$\xrightarrow[i=1,\dots n_G]{p \ll M_i} \mathrm{i}\mu^\epsilon (Y_\nu^T M^{-1} Y_\nu)_{gf} \varepsilon_{cd}\varepsilon_{ba} P_\mathrm{L}\ . \tag{3.2-5}$$

The gray arrow indicates the fermion flow, as defined in appendix C.1. Analogously, we get the same result for the second diagram except for a permutation of the indices of ε,



$$= [-\mathrm{i}(Y_\nu^T)_{gh}\varepsilon_{ca}\mu^{\frac{\epsilon}{2}}]\,P_\mathrm{L}\,\frac{\mathrm{i}(\not{p}+M_h)}{p^2-M_h^2}\,[-\mathrm{i}(Y_\nu)_{hf}\varepsilon_{bd}\mu^{\frac{\epsilon}{2}}]\,P_\mathrm{L}$$

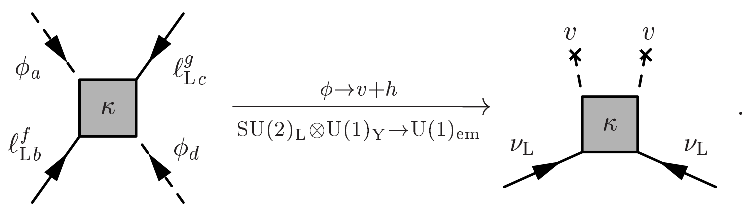
$$\xrightarrow[i=1,\ldots n_G]{p\ll M_i} \mathrm{i}\mu^\epsilon(Y_\nu^T M^{-1} Y_\nu)_{gf}\varepsilon_{ca}\varepsilon_{bd}P_\mathrm{L}\ , \tag{3.2-6}$$

where the permutation of SU(2) indices of equation (3.2-1) explicitly appears. Accordingly, we obtain

$$\kappa = 2Y_\nu^T M^{-1} Y_\nu \tag{3.2-7}$$

at the matching scale μ_R, which is commonly taken to be one of the eigenvalues of M. However, if the eigenvalue spectrum of M is non-degenerate, we obtain several effective theories and matching conditions. We will address this issue in a moment.

After spontaneous breaking of the electroweak symmetry, the Majorana mass term for the left-handed neutrino ν_L arises by inserting the vacuum expectation value (vev) of the Higgs field,



In the Lagrangian language, this corresponds to

$$\mathscr{L}_\kappa \xrightarrow[\mathrm{SU(2)_L\otimes U(1)_Y\to U(1)_{em}}]{\phi\to v+h} \mathscr{L}_{\nu\nu} = \frac{1}{2}(m_\nu)_{gf}\,\overline{\nu_\mathrm{L}^\mathrm{C}}^g\,\nu_\mathrm{L}^f + \text{h.c.}\ , \tag{3.2-8}$$

where $m_\nu = v^2\,Y_\nu^T \cdot M^{-1} \cdot Y_\nu$, thus justifying the terminology "effective neutrino mass operator".

For large eigenvalues of M, i.e. $M_i \gg v$, the effective left-handed neutrino masses are obviously suppressed. This is the well-known type I seesaw mechanism [7,8].

3.2.2 Matching at Multiple Thresholds

In analogy to the procedure described above, we may match a number of effective theories to each other. This is e.g. necessary if we consider more than one generation of right-handed neutrinos which are non-degenerate in mass. The ranges of the different EFT's are depicted in figure 3.1.

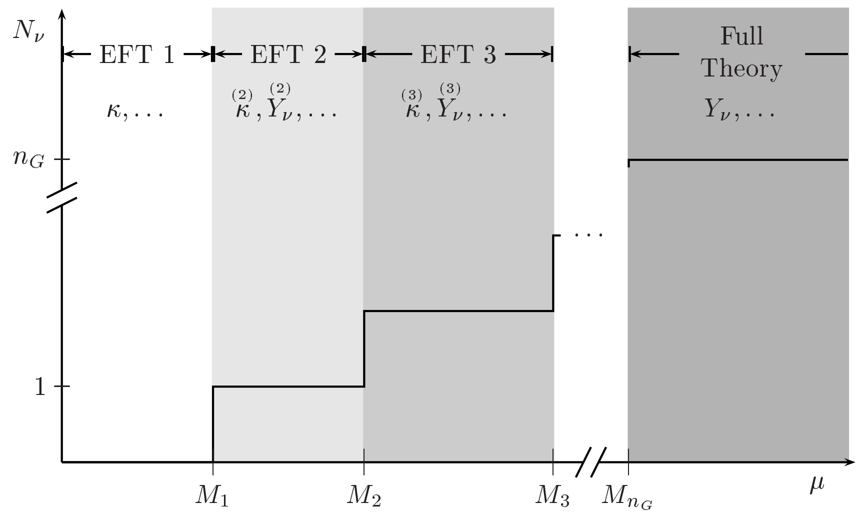


Figure 3.1: Illustration of the ranges of the different effective theories. Each theory corresponds to a number of heavy neutrinos which are integrated out whereas the remaining, lighter ones are treated as massless.

Consider the situation that the eigenvalues of the mass matrix M (i.e. the masses of the mass eigenstates $\{N^1, \ldots N^{n_G}\}$) have a certain hierarchy,

$$M_1 < M_2 < \cdots < M_{n_G} \, . \tag{3.2-9}$$

In the region where $n_G - n + 1$ heavy singlet fields are integrated out, the Yukawa coupling is a $(n-1) \times n_F$ matrix and will be referred to as $\overset{(n)}{Y}_\nu$,

$$Y_\nu \to \left(\begin{array}{ccc} (Y_\nu)_{1,1} & \cdots & (Y_\nu)_{1,n_F} \\ \vdots & & \vdots \\ (Y_\nu)_{n-1,1} & \cdots & (Y_\nu)_{n-1,n_F} \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array} \right) \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ b \\ c \end{array}}\right\} =: \overset{(n)}{Y}_\nu \, , \\ \left.\vphantom{\begin{array}{c} a \\ b \\ c \end{array}}\right\} \begin{array}{l} n_G - n + 1 \text{ heavy, sterile} \\ \text{neutrinos integrated out} \, . \end{array} \end{array} \tag{3.2-10}$$

Furthermore, it turns out to be useful to define $\overset{(n)}{M}$ as the $(n-1) \times (n-1)$ sub-matrix of the singlet mass matrix and $\overset{(n)}{\kappa}_{gf}$ as the effective vertex below the nth threshold. The tree-level matching condition at the nth threshold reads

$$\overset{(n)}{\kappa}_{gf}\Big|_{M_n} := \overset{(n+1)}{\kappa}_{gf}\Big|_{M_n} + 2 \left(\overset{(n+1)}{Y}{}_\nu^T \right)_{gn} M_n^{-1} \left(\overset{(n+1)}{Y}_\nu \right)_{nf}\Big|_{M_n} , \quad n = 1, \ldots n_G \, . \tag{3.2-11}$$

Note that $\overset{(1)}{\kappa}_{gf} \equiv \kappa_{gf}$, as defined previously in equation (3.2-7). We will see that similar definitions can be made in the 2HDM's and the MSSM extended by singlet superfields.

The next step consists in the derivation of the RGE's that govern the evolution of $\overset{(n)}{\kappa}_{gf}$, $\overset{(n)}{M}$ and $\overset{(n)}{Y}_\nu$ between the mass thresholds. This will be done in the chapters 4, 5 and 6 for the SM, a class of 2HDM's and the MSSM extended by right-handed neutrinos, respectively. In chapter 7, we will perform a numerical analysis of the predictions of the corresponding RGE's.

4 Standard Model Calculations

In this chapter, the calculation of the β function for the κ operator in the SM is reviewed, for which existed a result in the literature that was not entirely correct. In order to verify our result we develop a general method of checking β-functions for effective operators.

4.1 Renormalization of the Effective Neutrino Mass Operator in the SM

4.1.1 Extracting Poles from One-Loop Diagrams

It is clear from formula (2.3-9) that for the calculation of β-functions only the poles in ϵ of the counterterms and renormalization constants are needed. In order to extract the divergent parts of a given diagram, we use the following sequence of steps:

(1) Calculate the diagram in d dimensions.

(2) Express the result by Passarino-Veltman functions (cf. appendix F). This can be automatized by `FeynCalc` [18].

(3) Extract the poles in ϵ by the use of table F.1.

4.1.2 β_κ in the Standard Model

In this section we just summarize the main results of the calculation of β_κ since the details can be found in [19]. For all of our SM calculations we use the field content of table 4.1-1. As we have seen in section 3.2, the dimension 5 operator $\mathscr{L}_\kappa$ arises by integrating out the Majorana neutrinos N in a "full theory". We consider a Lagrangian, consisting of $\mathscr{L}_\kappa$, the SM Lagrangian $\mathscr{L}_{\mathrm{SM}}$ and proper counterterms $\mathscr{C}$,

$$\mathscr{L} = \mathscr{L}_\kappa + \mathscr{L}_{\mathrm{SM}} + \mathscr{C} \ . \tag{4.1-1}$$

The terms of $\mathscr{L}_{\mathrm{SM}}$ relevant for the calculation of the non-diagonal parts of the β-function are

$$\mathscr{L}_{\mathrm{kin}(\ell_\mathrm{L})} = \overline{\ell_\mathrm{L}}^f (\mathrm{i}\gamma^\mu \partial_\mu) \ell_\mathrm{L}^f \ , \tag{4.1-2a}$$

$$\mathscr{L}_{\mathrm{Higgs}} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \, \phi^\dagger \phi - \tfrac{1}{4}\lambda \, (\phi^\dagger \phi)^2 \ . \tag{4.1-2b}$$

Field	ϕ	q_{L}	d_{R}	u_{R}	ℓ_{L}	e_{R}	N
q_{Y}	$-\frac{1}{2}$	$+\frac{1}{6}$	$+\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$+1$	0
$\mathrm{SU}(2)_{\mathrm{L}}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{SU}(3)_{\mathrm{C}}$	$\mathbf{1}$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

Table 4.1-1: Quantum numbers of the SM fields and the additional Majorana neutrino.

The corresponding counterterms are defined by

$$\mathscr{C}_{\mathrm{kin}(\ell_{\mathrm{L}})} = \overline{\ell_{\mathrm{L}}}^{g}(\mathrm{i}\gamma^{\mu}\partial_{\mu})(\delta Z_{\ell_{\mathrm{L}}})_{gf}\ell_{\mathrm{L}}^{f} \,, \tag{4.1-3a}$$

$$\mathscr{C}_{\mathrm{Higgs}} = \delta Z_{\phi}(\partial_{\mu}\phi)^{\dagger}(\partial^{\mu}\phi) - \delta m^{2}\,\phi^{\dagger}\phi - \tfrac{1}{4}\,\delta\lambda(\phi^{\dagger}\phi)^{2} \,, \tag{4.1-3b}$$

$$\mathscr{C}_{\kappa} = \tfrac{1}{4}\delta\kappa_{gf}\,\overline{\ell^{\mathrm{C}}_{\mathrm{L}}}^{g}_{c}\varepsilon^{cd}\phi_{d}\;\ell^{f}_{\mathrm{L}b}\varepsilon^{ba}\phi_{a} + \mathrm{h.c.} \,, \tag{4.1-3c}$$

and evaluate to

$$\delta Z_{\ell_{\mathrm{L}},1} = -\frac{1}{16\pi^{2}}\left(Y_{e}^{\dagger}Y_{e} + \tfrac{1}{2}\xi_{B}g_{1}^{2} + \tfrac{3}{2}\xi_{W}g_{2}^{2}\right) \,, \tag{4.1-4a}$$

$$\delta Z_{\phi,1} = -\frac{1}{16\pi^{2}}\left[2\,\mathrm{Tr}\left(Y_{e}^{\dagger}Y_{e} + 3Y_{u}^{\dagger}Y_{u} + 3Y_{d}^{\dagger}Y_{d}\right) - \tfrac{1}{2}(3-\xi_{B})g_{B}^{2} - \tfrac{3}{2}(3-\xi_{W})g_{2}^{2}\right] \,, \tag{4.1-4b}$$

$$\delta\kappa_{,1} = -\frac{1}{16\pi^{2}}\left[2\kappa(Y_{e}^{\dagger}Y_{e}) + 2(Y_{e}^{\dagger}Y_{e})^{T}\kappa - \lambda\kappa - (\tfrac{3}{2}-\xi_{B})g_{1}^{2}\kappa - (\tfrac{3}{2}-3\xi_{W})g_{2}^{2}\kappa\right] \,, \tag{4.1-4c}$$

where ξ_B and ξ_W are the gauge fixing parameters used in R_ξ gauge. With these and formula (2.3-9), we obtain

$$16\pi^{2}\beta_{\kappa} = -\frac{3}{2}\left[\kappa\,\left(Y_{e}^{\dagger}Y_{e}\right) + \left(Y_{e}^{\dagger}Y_{e}\right)^{T}\,\kappa\right] + \lambda\,\kappa - 3g_{2}^{2}\,\kappa + 2\,\mathrm{Tr}\left(3Y_{u}^{\dagger}Y_{u} + 3Y_{d}^{\dagger}Y_{d} + Y_{e}^{\dagger}Y_{e}\right)\,\kappa \,. \tag{4.1-5}$$

This is the β-function that governs the evolution of the dimension 5 effective neutrino mass operator in the SM. If it turns out that the observed neutrino masses are described by this operator, this is the β-function that provides the link between physics at a high energy scale and the observations.

Between the thresholds, the counterterms look similar to (4.1-4). Additionally, the contributions from $\overset{(n)}{Y}_\nu$ have to be respected. We obtain

$$\delta\overset{(n)}{Z}_{\ell_\mathrm{L},1} = -\frac{1}{16\pi^2}\left[\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + Y_e^\dagger Y_e + \frac{1}{2}\xi_B g_1^2 + \frac{3}{2}\xi_W g_2^2\right] , \tag{4.1-6a}$$

$$\begin{aligned}\delta\overset{(n)}{Z}_{\phi,1} = -\frac{1}{16\pi^2}\Big[&2\ \mathrm{Tr}\left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right) + 2\ \mathrm{Tr}(Y_e^\dagger Y_e) + 6\ \mathrm{Tr}(Y_u^\dagger Y_u) + 6\ \mathrm{Tr}(Y_d^\dagger Y_d) \\ &+ \frac{1}{2}(\xi_B - 3)g_1^2 + \frac{3}{2}(\xi_W - 3)g_2^2\Big] ,\end{aligned} \tag{4.1-6b}$$

$$\delta\overset{(n)}{Z}_{N,1} = -\frac{1}{16\pi^2}\left[2\,\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger\right] . \tag{4.1-6c}$$

The vertex renormalizations are given by

$$\delta\overset{(n)}{Y}_{\nu,1} = -\frac{1}{16\pi^2}\left[2\,\overset{(n)}{Y}_\nu (Y_e^\dagger Y_e) + \frac{1}{2}\xi_B g_1^2\,\overset{(n)}{Y}_\nu + \frac{3}{2}\xi_W g_2^2\,\overset{(n)}{Y}_\nu\right] , \tag{4.1-7a}$$

$$\begin{aligned}\delta\overset{(n)}{\kappa}_{,1} = -\frac{1}{16\pi^2}\Big[&2\,(Y_e^\dagger Y_e)^T \overset{(n)}{\kappa} + 2\overset{(n)}{\kappa}(Y_e^\dagger Y_e) - \lambda\overset{(n)}{\kappa} \\ &+ \frac{1}{2}(2\xi_B - 3)g_1^2\overset{(n)}{\kappa} + \frac{3}{2}(2\xi_W - 1)g_2^2\overset{(n)}{\kappa}\Big] ,\end{aligned} \tag{4.1-7b}$$

$$\delta\overset{(n)}{M} = 0 . \tag{4.1-7c}$$

From these, we obtain the β-function below the nth threshold,

$$\begin{aligned}16\pi^2\overset{(n)}{\beta}_\kappa = &-\frac{3}{2}(Y_e^\dagger Y_e)^T\overset{(n)}{\kappa} - \frac{3}{2}\overset{(n)}{\kappa}(Y_e^\dagger Y_e) + \frac{1}{2}\left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right)^T\overset{(n)}{\kappa} + \frac{1}{2}\overset{(n)}{\kappa}\left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right) \\ &+ 2\ \mathrm{Tr}(Y_e^\dagger Y_e)\overset{(n)}{\kappa} + 2\ \mathrm{Tr}\left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right)\overset{(n)}{\kappa} + 6\ \mathrm{Tr}(Y_u^\dagger Y_u)\overset{(n)}{\kappa} \\ &+ 6\ \mathrm{Tr}(Y_d^\dagger Y_d)\overset{(n)}{\kappa} - 3g_2^2\overset{(n)}{\kappa} + \lambda\overset{(n)}{\kappa} .\end{aligned} \tag{4.1-8}$$

In addition, we will need the β-functions for $\overset{(n)}{Y}_\nu$ and $\overset{(n)}{M}$, which can be calculated from the counterterms

$$\begin{aligned}16\pi^2\overset{(n)}{\beta}_{Y_\nu} = \overset{(n)}{Y}_\nu\Big[&\frac{3}{2}\left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right) - \frac{3}{2}(Y_e^\dagger Y_e) + \mathrm{Tr}\left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right) + \mathrm{Tr}(Y_e^\dagger Y_e) \\ &+ 3\ \mathrm{Tr}(Y_u^\dagger Y_u) + 3\ \mathrm{Tr}(Y_d^\dagger Y_d) - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\Big]\end{aligned} \tag{4.1-9}$$

and

$$16\pi^2\overset{(n)}{\gamma}_M = \left(\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger\right)\overset{(n)}{M} + \overset{(n)}{M}\left(\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger\right)^T . \tag{4.1-10}$$

The remaining β-functions are found to be the same as in [20], if one substitutes $Y_\nu \to \overset{(n)}{Y}_\nu$. They can be inferred from some more general formulae at the end of section 5.2.5 where they arise as a special case. Numerical investigations of the corresponding system of coupled differential equations will be performed in chapter 7.

4.2 Checking β-Functions in Effective Theories

In this section we present a method for checking β-functions in effective theories and apply it to our result, equation (4.1-5). We compare some amplitudes $\overline{\Gamma}_N$ in a full theory with predictions from the RGE. In particular, we use the fact that calculating $\overline{\Gamma}_N$ in the effective theory should be equivalent to calculating $\overline{\Gamma}_N$ in the full theory and taking the limit $|p_i| \ll M$, as illustrated in figure 4.1. Considering the finite variations of amplitudes in the full theory and comparing them with the predictions from the β-function of the effective theory makes it possible to verify the correctness of the latter.

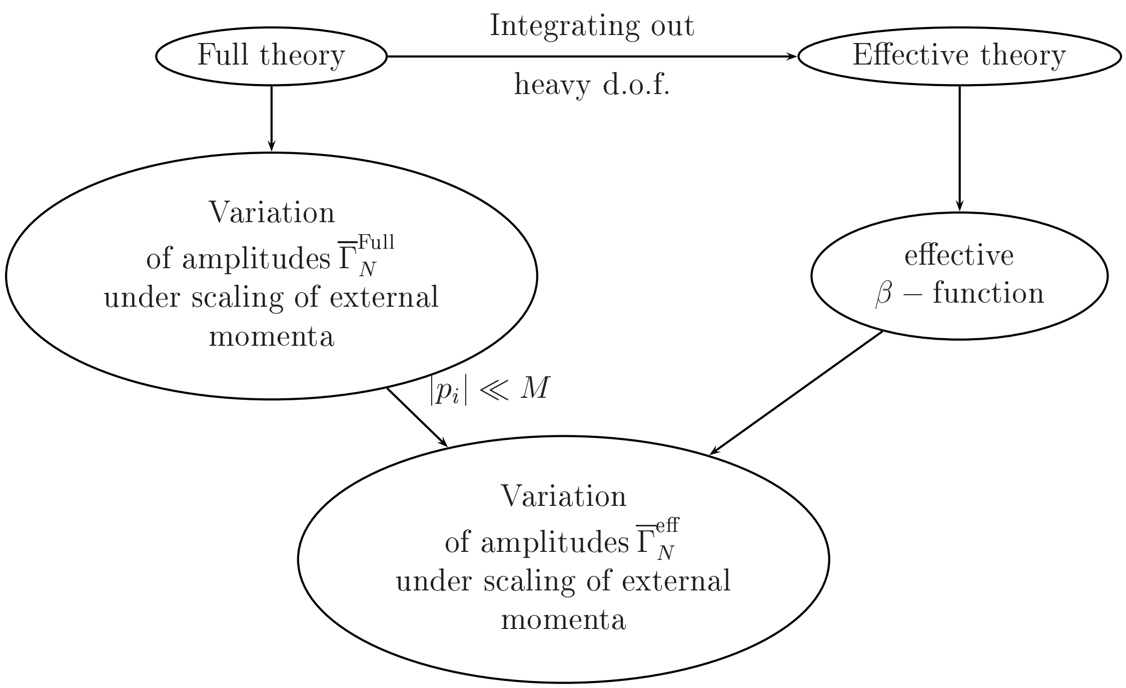


Figure 4.1: General strategy for checking β-functions.

We consider for $p_i \ll M$ an effective theory with dimensionless couplings $\{g_k\}$ and an effective coupling K of mass dimension d_K. Possible mass terms (except for M in the full theory) are ignored as they do not affect the MS β-functions. Consider an amplitude which is linear in K,

$$\overline{\Gamma}_N^{\text{eff}}(\{p_i\},\{g_k\},K,\mu_0) = a(\{p_i\},\{g_k\},\mu_0)\cdot K \ . \tag{4.2-1}$$

Then equation (2.2-29) tells us that

$$\begin{aligned}\overline{\Gamma}_N^{\text{eff}}(\{e^t p_i\},\{g_k\},K,\mu_0) \quad &= \quad \exp\left\{(d_{\overline{\Gamma}_N} - d_K)t - \frac{1}{2}\sum_j \int\limits_0^t \mathrm{d}\tau\, \gamma_{\phi_j}\big(\{g_k(\tau)\}\big)\right\}\times \\ &\qquad \times a(\{p_i\},\{g_k(t)\},\mu_0)\ K(t) \ . \end{aligned} \tag{4.2-2}$$

Accordingly, we obtain

$$K(t) \;=\; \exp\left\{-(d_{\overline{\Gamma}_N}-d_K)\,t+\frac{1}{2}\sum_j\int\limits_0^t \mathrm{d}\tau\,\gamma_{\phi_j}\big(\{g_k(\tau)\}\big)\right\}\times \\ \times\frac{\overline{\Gamma}_N^{\mathrm{eff}}\left(\{e^t p_i\},\{g_k\},K,\mu_0\right)}{a(\{p_i\},\{g_k(t)\},\mu_0)}\;. \qquad (4.2\text{-}3)$$

Then differentiating w.r.t. t yields for $t=0$

$$\beta_K \;=\; \left\{-d_{\overline{\Gamma}_N}+d_K+\frac{1}{2}\sum_j\gamma_{\phi_j}\big(\{g_k(\mu_0)\}\big)+ \right.\\ \left.+\frac{1}{\overline{\Gamma}_N^{\mathrm{eff}}\left(\{p_i\},\{g_k\},K,\mu_0\right)}\left.\frac{\mathrm{d}\overline{\Gamma}_N^{\mathrm{eff}}\left(\{e^t p_i\},\{g_k\},K,\mu_0\right)}{\mathrm{d}t}\right|_{t=0}\right\}\times K \\ -\frac{\overline{\Gamma}_N^{\mathrm{eff}}\left(\{p_i\},\{g_k\},K,\mu_0\right)}{a^2\left(\{p_i\},\{g_k\},\mu_0\right)}\sum_\ell\frac{\partial a\left(\{p_i\},\{g_k\},\mu_0\right)}{\partial g_\ell}\,\beta_{g_\ell}\;. \qquad (4.2\text{-}4)$$

The crucial step is now to replace $\overline{\Gamma}_N^{\mathrm{eff}}$ by the corresponding amplitude in the full theory, $\overline{\Gamma}_N^{\mathrm{Full}}$, and to perform the transition to the effective theory limit after calculating the right-hand side of (4.2-4),

$$\beta_K \;=\; \lim_{p_i\ll M}\left\{-d_{\overline{\Gamma}_N}+d_K+\frac{1}{2}\sum_j\gamma_{\phi_j}\big(\{g_k(\mu_0)\}\big)+\right.\\ +\frac{1}{\overline{\Gamma}_N^{\mathrm{Full}}\left(\{p_i\},\{g_{k'}\},M,\mu_0\right)}\left.\frac{\mathrm{d}\overline{\Gamma}_N^{\mathrm{Full}}\left(\{e^t p_i\},\{g_{k'}\},M,\mu_0\right)}{\mathrm{d}t}\right|_{t=0} \\ \left.-\frac{1}{a\left(\{p_i\},\{g_k\},\mu_0\right)}\sum_\ell\frac{\partial a\left(\{p_i\},\{g_k\},\mu_0\right)}{\partial g_\ell}\,\beta_{g_\ell}\right\}\times K\;, \qquad (4.2\text{-}5)$$

where $\{g_{k'}\}$ denotes the set of couplings of the full theory. On the other hand, β_K is linear in K,

$$\beta_K = b(\{g_k\})\cdot K\;, \qquad (4.2\text{-}6)$$

since K corresponds to an effective operator and is therefore suppressed by some inverse powers of M, i.e. higher powers of K have to be neglected in the EFT approach. A K-

independent term is not possible because the g_k are dimensionless. Accordingly, we obtain

$$\begin{aligned} b(\{g_k\}) \quad = \quad \lim_{p_i \ll M} \Bigg\{ & -d_{\overline{\Gamma}_N} + d_K + \frac{1}{2}\sum_j \gamma_{\phi_j}(\mu_0) \\ & + \frac{1}{\overline{\Gamma}_N^{\text{Full}}\left(\{p_i\},\{g_{k'}\},M,\mu_0\right)} \left.\frac{\mathrm{d}\overline{\Gamma}_N^{\text{Full}}\left(\{e^t p_i\},\{g_{k'}\},M,\mu_0\right)}{\mathrm{d}t}\right|_{t=0} \\ & - \frac{1}{a\left(\{p_i\},\{g_k\},\mu_0\right)} \sum_\ell \frac{\partial a\left(\{p_i\},\{g_k\},\mu_0\right)}{\partial g_\ell}\,\beta_{g_\ell} \Bigg\} \;. \end{aligned} \tag{4.2-7}$$

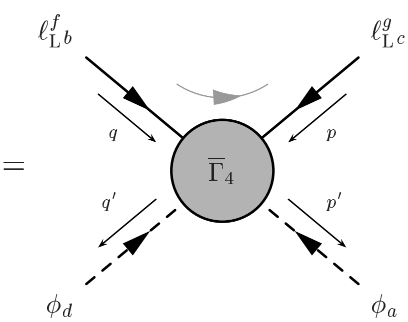
This formula is a general result and can be applied to the β-function of any effective operator. It may easily be generalized to tensorial operators K. In the following, this is illustrated by two examples.

4.2.1 The λ Correction

To demonstrate the method, we define the abbreviation

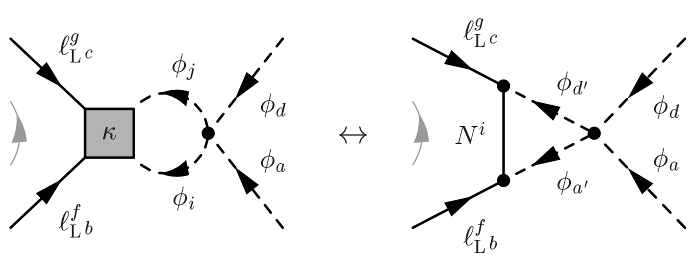
$$\overline{\Gamma}_4 = \Gamma\Big(\ell^f_{\mathrm{L}b}(p) + \ell^g_{\mathrm{L}c}(q) \to \phi_d(p') + \phi_a(q')\Big)$$

$$= \quad \Big\langle \overline{\boldsymbol{\ell}^{\mathrm{C}\,g}_{\mathrm{L}c}}(p)\,\boldsymbol{\phi}^\dagger_a(-q')\,\boldsymbol{\phi}^\dagger_d(-p')\,\boldsymbol{\ell}^f_{\mathrm{L}b}(q)\Big\rangle\Big|_{\text{amp}} = \tag{4.2-8}$$



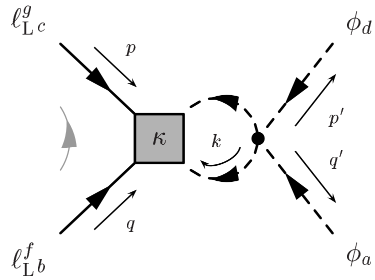
.

In order to verify the coefficient of the β-function coming from the Higgs self-interaction λ we investigate the relation



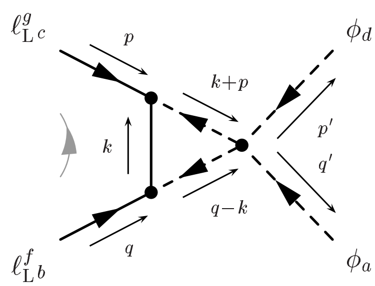
.

For simplicity we consider only one heavy neutrino, i.e. $N^i = N^1$. Furthermore, we set all anomalous dimensions γ_{φ_i} equal to zero because all wavefunction renormalization factors proportional to λ are equal in the full and the effective theory. The effective theory diagram evaluates to [19]



$$= \quad -\frac{\mathrm{i}}{16\pi^2}\mu^\epsilon \lambda \kappa_{gf} \tfrac{1}{2}(\varepsilon_{ba}\varepsilon_{cd} + \varepsilon_{ca}\varepsilon_{bd}) P_{\mathrm{L}} \frac{1}{\epsilon} + \text{UV finite}\,, \tag{4.2-9}$$

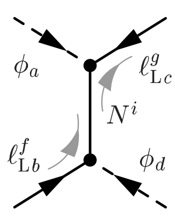
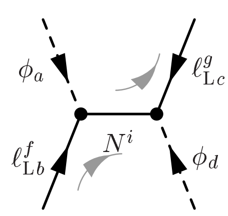
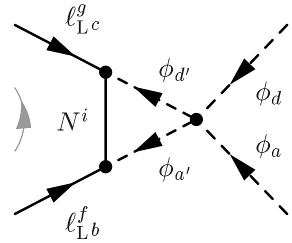
where K of the previous section is here specialized to κ. For the full theory diagram we obtain



$$\begin{aligned} = \quad & \frac{\mathrm{i}}{16\pi^2}\mu^\epsilon (Y_\nu^T)_{gi}(Y_\nu)_{if}\lambda \tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ab} + \varepsilon_{ca}\varepsilon_{db}) \\ & \times M_i\, C_0(p^2, q^2, (p+q)^2, m^2, M_i^2, m^2)\, P_{\mathrm{L}} \end{aligned} \tag{4.2-10}$$

We define

$$\overline{\Gamma}_4^{\mathrm{Full}}(s) = \sum_{\substack{\text{tree-level} \\ +\,\text{one-loop} \\ \text{OPI-diagrams}}} \Bigg|_{\substack{p^2=q^2=0 \\ (p+q)^2=s}} \tag{4.2-11}$$

$=$  $+$  $+$  $\Bigg|_{\substack{p^2=q^2=0 \\ (p+q)^2=s}}$

and use equation (4.2-10) in order to obtain

$$\begin{aligned}\Delta\overline{\Gamma}_4^{\text{Full}}(t,s_0) &= \overline{\Gamma}_4^{\text{Full}}(e^{2t}s_0) - \overline{\Gamma}_4^{\text{Full}}(s_0) \\ &= \frac{\mathrm{i}}{16\pi^2}\lambda\,\mu^\epsilon\,\underbrace{(Y_\nu^T)_{gi}(Y_\nu)_{if}\frac{1}{2}(\varepsilon_{cd}\varepsilon_{ab}+\varepsilon_{ca}\varepsilon_{db})}_{=\frac{1}{2}M_i\,\kappa_{gf}}\,M_i\,\times \\ &\quad\times\left[C_0(e^{2t}0,e^{2t}0,e^{2t}s_0,m^2,M_i^2,0) - C_0(0,0,s_0,m^2,M_i^2,0)\right]\,P_\mathrm{L}\;,\end{aligned} \tag{4.2-12}$$

where the on-shell momentum configuration

$$\left.\begin{aligned} p &= (\mu_0,\mu_0,0,0) \\ q &= (\mu_0,-\mu_0,0,0)\end{aligned}\right\} \quad\Rightarrow\quad \left\{\begin{aligned} p^2=q^2 &= 0\;, \\ (p+q)^2 &= 4\mu_0^2 =: s_0\;,\end{aligned}\right. \tag{4.2-13}$$

has been used for convenience.

Consequently, with $s=e^{2t}s_0$ we obtain

$$\begin{aligned}\overline{\Gamma}_4^{\text{Full}}(s) &= \kappa(4M^2)\cdot\left[1+\frac{1}{32\pi^2}\lambda\,M_i^2\,C_0(0,0,s,m^2,M_i^2,0)\right]+\mathscr{O}(\lambda^2) \\ &\approx \kappa(4M^2)\cdot\left\{\begin{array}{ll}\left[1+\frac{1}{32\pi^2}\lambda\,\ln\left(e^{2t}\right)\right]+\mathscr{O}(\lambda^2)\;, & s\ll M^2\;, \\ 1+\mathscr{O}(\lambda^2)\;, & s\gg M^2\;,\end{array}\right.\end{aligned} \tag{4.2-14}$$

where (F.16) was used.

From this we derive according to (4.2-7) with $d_{\overline{\Gamma}_4}=d_\kappa$

$$\beta_\kappa = \frac{1}{16\pi^2}\lambda\,\kappa + \mathscr{O}(\lambda^2)\;, \tag{4.2-15}$$

where due to $\beta_\lambda\sim\lambda^2$ the β_λ term was neglected. This confirms the corresponding term in (4.1-5). Hence the prediction for the evolution of $\overline{\Gamma}_4$ from the renormalization group is given by

$$\overline{\Gamma}_4^{\text{RGE}}(t,M,s_0) := \left\{\begin{array}{ll}\overline{\Gamma}_4^{\text{Full}}(s_0)+\frac{\lambda}{16\pi^2}\kappa\,\ln e^t\;, & e^{2t}s_0<M^2\;, \\ \overline{\Gamma}_4^{\text{Full}}(s_0)\;, & e^{2t}s_0>M^2\;.\end{array}\right. \tag{4.2-16}$$

Of course, this can only be regarded as the prediction from the RGE if λ itself does not run, which is true at the order λ. Going to λ^2, we would have to solve the corresponding set of coupled differential equations.

In order to verify (F.16) and therefore (4.2-14), we compare C_0 with its approximation which is identical to $\overline{\Gamma}_4^{\text{RGE}}$. This is illustrated in figure 4.2.

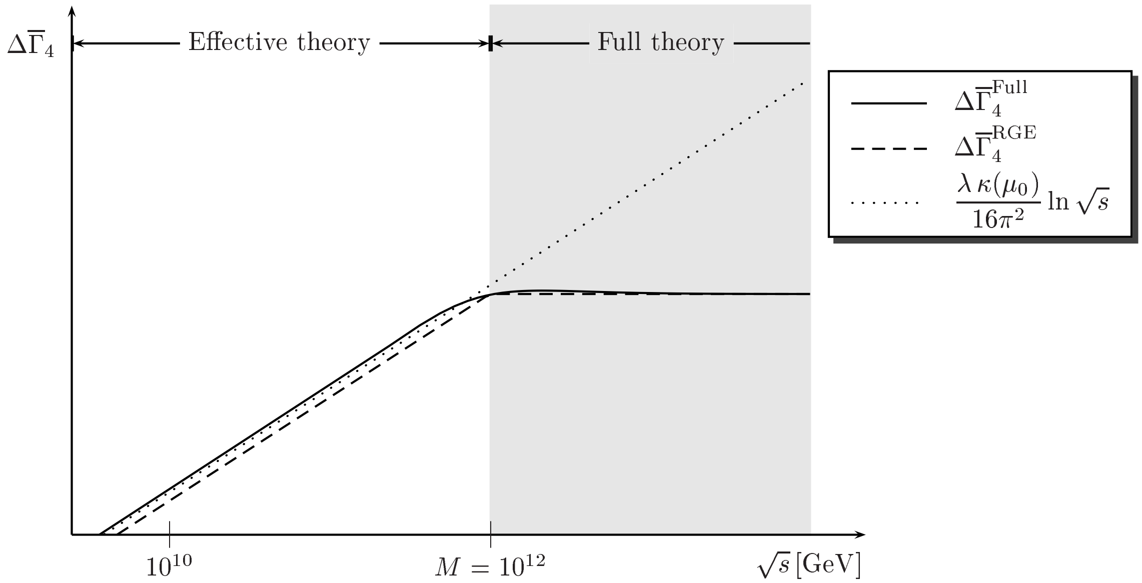


Figure 4.2: Comparison between $\Delta\overline{\Gamma}_4^{\mathrm{Full}}$ and $\Delta\overline{\Gamma}_4^{\mathrm{RGE}}$.

4.2.2 Checking the Non-Diagonal Part of β_κ

Now we extend the above analysis to the non-diagonal part of β_κ since for this there exists a discrepancy with the literature [17]. We consider the relation between the "effective diagrams" shown in figure 4.3 on the one hand and the "triangle" and "box diagrams" of figure 4.4 and figure 4.5, on the other hand. Again, we do not include the wavefunction renormalization factors in our analysis because these lead in the full theory to a β-function for the Y_e Yukawa matrix which is in accordance with the literature [21–23]. However, we include the case of an arbitrary number of heavy singlets ($i = 1, \dots n_G$) in our analysis. The divergence structure of the "triangle diagrams" coincides with the divergence structure of

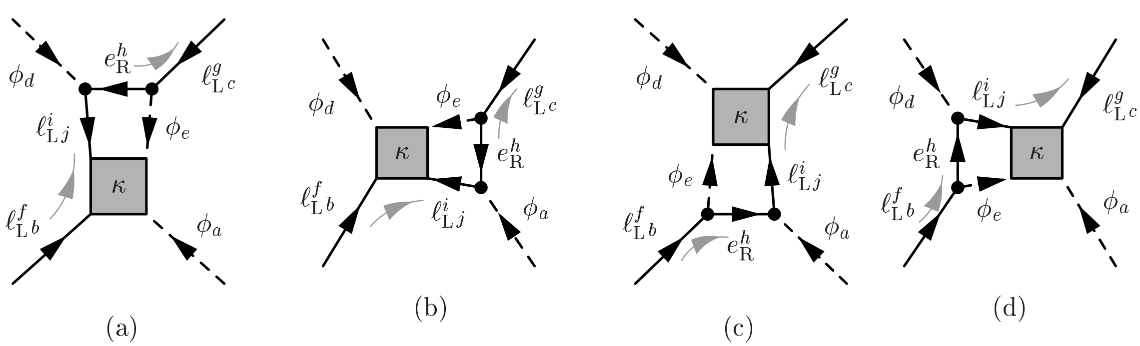


Figure 4.3: The vertex corrections proportional to Y_e in the effective theory.

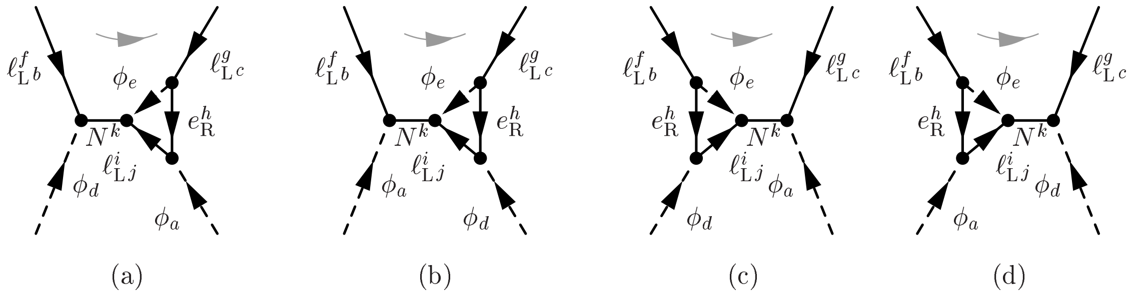


Figure 4.4: The "triangle diagrams" of the full theory.

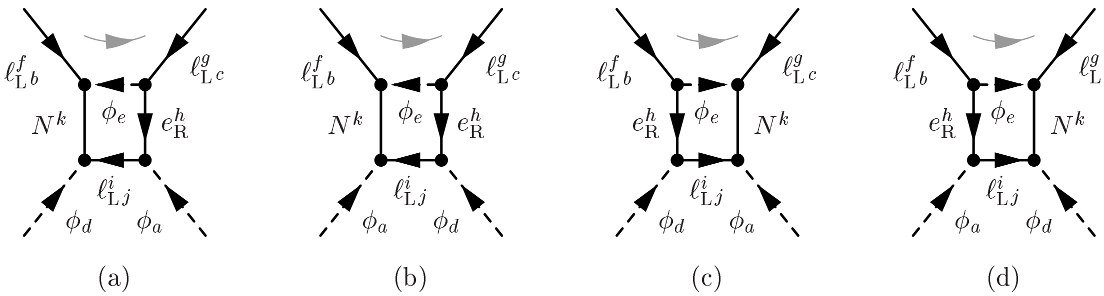
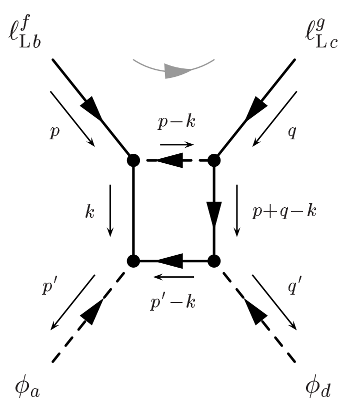


Figure 4.5: The "box diagrams" of the full theory.

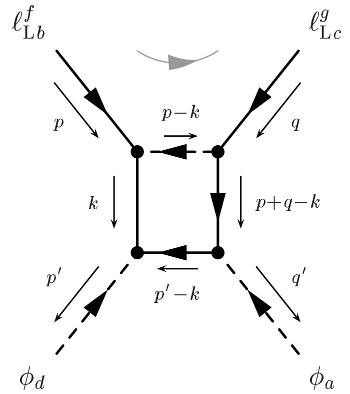
the "effective diagrams", whereas the "box diagrams" have no counterpart in the effective theory. We compute for example



$$= \qquad (4.2\text{-}17)$$

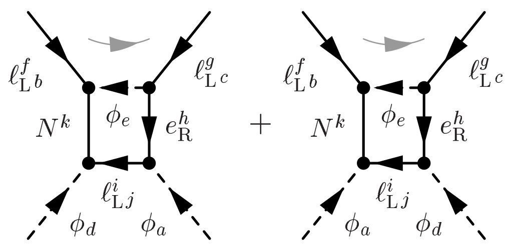
$$= \mu^{2\epsilon}(Y_e^T Y_e^*)_{gh}(Y_\nu^T)_{hk}(Y_\nu)_{kf}\varepsilon_{da}\varepsilon_{bc} \times$$
$$\times \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(\not{p} + \not{q} - \not{k})(\not{p}' - \not{k})\,(\not{k} + M_k)}{(p+q-k)^2\,(p'-k)^2\,(k^2 - M_k^2)\;[(k-p)^2 - m^2]}\, P_\mathrm{L}\;. \qquad (4.2\text{-}18)$$

Note that the SU(2)-structure of the diagram does not coincide with the structure we found for κ. Furthermore, we can interchange the SU(2)-indices of the Higgs fields and obtain for the second box diagram



$$
\begin{aligned}
&= \\
&= \mu^{2\epsilon}(Y_e^T Y_e^*)_{gh}(Y_\nu^T)_{hk}(Y_\nu)_{kf}\varepsilon_{ad}\varepsilon_{bc}\times \\
&\quad \times\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{(\not p+\not q-\not k)\,(\not q'-\not k)\,(\not k+M_k)}{(p+q-k)^2\,(q'-k)^2\,(k^2-M_k^2)\,[(k-p)^2-m^2]}\,P_{\mathrm{L}}\ , \qquad (4.2\text{-}19)
\end{aligned}
$$

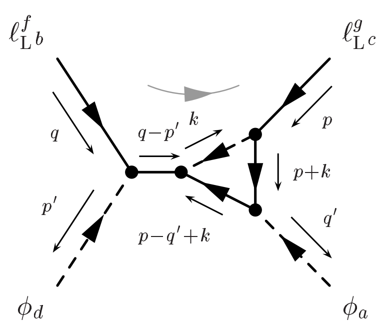
i.e. the negative of equation (4.2-18) with p' and q' interchanged. In other words,



$= 0 \quad \text{for } p' = q'$.

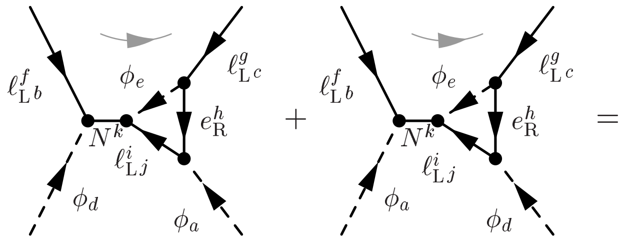
This also holds for the other two box diagrams, therefore we conclude that *for* $p' = q'$ *the box diagrams do not contribute to the finite corrections of the amplitude* $\overline{\Gamma}_4$ *in the full theory.*

Now we compute the triangle diagrams. The first one yields



$$
\begin{aligned}
&= \\
&= \frac{\mathrm{i}}{16\pi^2}(Y_e^T Y_e^* Y_\nu^T)_{gk}\underbrace{\frac{M_k+\not q-\not p'}{M_k^2-(q-p')^2}}_{\xrightarrow{q,p'\ll M_k}\frac{1}{M_k}}(Y_\nu)_{kf}\mu^\epsilon\varepsilon_{ca}\varepsilon_{bd}\times \\
&\quad \times\frac{\mu^\epsilon}{\mathrm{i}\pi^2}\int\mathrm{d}^d k\,\frac{\not p+\not k}{(p+k)^2}\,\frac{\not p-\not q'+\not k}{(p-q'+k)^2}\,\frac{1}{k^2-m^2}\,P_{\mathrm{L}}\ . \qquad (4.2\text{-}20)
\end{aligned}
$$

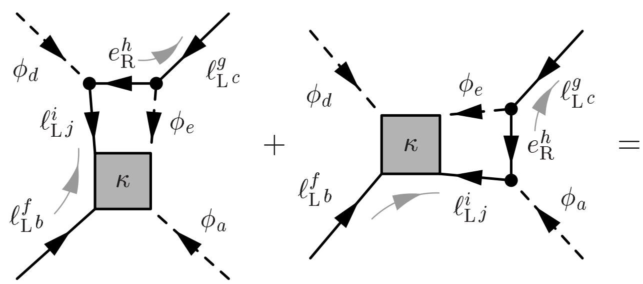
The second diagram corresponds to the same expression with p' and q' and the SU(2)-indices a and d interchanged. Therefore we find



$$= \frac{\mathrm{i}}{16\pi^2}(Y_e^T Y_e^*)_{gh} \underbrace{2(Y_\nu^T)_{hk}\frac{M_k}{M_k^2-(q-p')^2}(Y_\nu)_{kf}}_{\xrightarrow{q,p'\ll M_k}\kappa_{hf}} \mu^\epsilon \tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd}) \times$$

$$\times \frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^d k \, \frac{\not{p}+\not{k}}{(p+k)^2}\frac{\not{p}-\not{q}'+\not{k}}{(p-q'+k)^2}\frac{1}{k^2-m^2}P_{\mathrm{L}} \qquad (4.2\text{-}21)$$

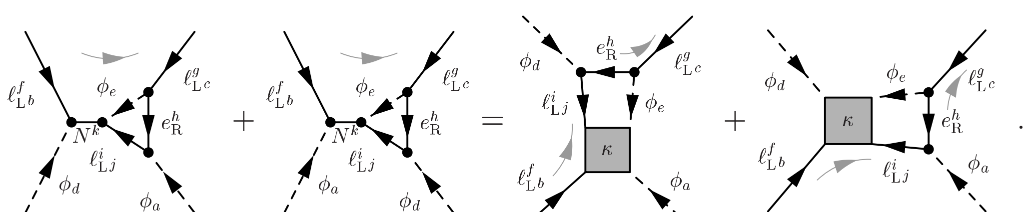
for $p'=q'$. On the other hand, if we set $p'=q'$, we find with the same labeling of momenta



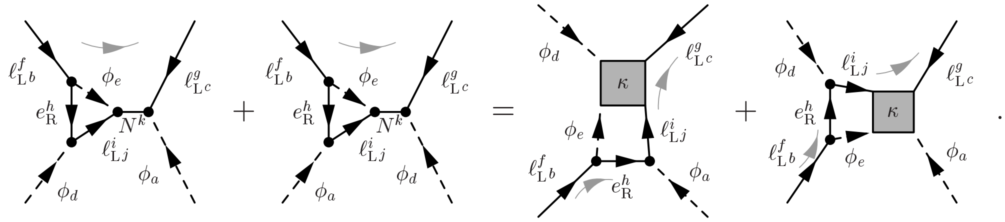
$$= \frac{\mathrm{i}}{16\pi^2}(Y_e^T Y_e^* \kappa)_{gf}\mu^\epsilon \tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd}) \times$$

$$\times \frac{\mu^\epsilon}{\mathrm{i}\pi^2}\int \mathrm{d}^d k \, \frac{\not{p}+\not{k}}{(p+k)^2}\frac{\not{p}-\not{q}'+\not{k}}{(p-q'+k)^2}\frac{1}{k^2-m^2}P_{\mathrm{L}} \, . \qquad (4.2\text{-}22)$$

Thus, in the effective theory limit $p,q \ll M_k$, we obtain for $p'=q'$



Repeating the arguments below equation (4.2-20) we find in the effective theory limit $p,q \ll M_k$ for $p'=q'$



We found so far that the box diagrams cancel out and do not give a contribute to $\overline{\Gamma}_4$ when we scale the momenta. Furthermore, the contributions of the effective diagrams and the triangle diagrams coincide in the limit $p, q \ll M_k$ and $p' = q'$. Thus we have shown that the finite changes of $\overline{\Gamma}_4$ are equal in the effective theory limit.

Hence, we may use the full theory for the remainder of our check, which consists of comparing the change of two diagrams under a rescaling of the external momenta with the prediction from the RGE. First we evaluate the integral in equation (4.2-22), using Passarino-Veltman functions and the `FeynCalc` [18] package.

$$\begin{aligned}
\mathcal{I} &:= \frac{\mu^\epsilon}{i\pi^2}\int \mathrm{d}^d k \, \frac{\not{p}+\not{k}}{(p+k)^2}\frac{\not{p}-\not{p}'+\not{k}}{(p-p'+k)^2}\frac{1}{k^2-m^2} \\
&= \frac{1}{2\left[(p\cdot p')^2-p^2p'^2\right]}\Big\{\left[-p^2p'^2+p\cdot p'\,\not{p}\not{p}'\right]B_0\big(p^2,0,m^2\big)+ \\
&\quad +\left[2\,(p\cdot p')^2-p'^2\,p\cdot p'-p^2p'^2+(p'^2-p\cdot p')\,\not{p}\not{p}'\right]B_0\big((p-p')^2,0,m^2\big)+ \\
&\quad +\left[p'^2\,p\cdot p'-p'^2\,\not{p}\not{p}'\right]B_0\big(p'^2,0,0\big)- \\
&\quad -p'^2\left[(p^2-m^2)\,p\cdot p'-p^2p'^2+(m^2-p^2+p\cdot p')\,\not{p}\not{p}'\right]\times \\
&\quad \times C_0\big(p^2,p'^2,(p-p')^2,m^2,0,0\big)\Big\}
\end{aligned} \tag{4.2-23}$$

In analogy to equation (4.2-11), we define $\overline{\Gamma}_4(t)$ as the sum of all OPI diagrams with the external momenta rescaled according to $(p-p')^2 \propto s_0\, e^{2t}$, and introduce the difference by $\Delta\overline{\Gamma}_4^{\text{triangle}}(t) := \overline{\Gamma}_4^{\text{triangle}}(s_0e^{2t}) - \overline{\Gamma}_4^{\text{triangle}}(s_0)$. The latter is proportional to the corresponding change of $\mathcal{I}$. For $e^t\sqrt{s_0} \gg m^2$, we can use the approximation

$$B_0\big(e^{2t}s_0,0,m^2\big)-B_0\big(s_0,0,m^2\big)\approx -2\ln\left(e^t\right)\ . \tag{4.2-24}$$

Besides, due to equation (F.16), the contribution from the C_0 term can be neglected.

Using all these simplifications, we find

$$\Delta\overline{\Gamma}_4^{\text{triangle}} = -\frac{1}{16\pi^2}(Y_e^\dagger Y_e)^T\kappa(s_0)\,\ln\left(e^{2t}\right)\ . \tag{4.2-25}$$

Thus, the terms with a non-trivial Dirac structure have dropped out. Moreover, differentiating this result w.r.t. t verifies our result for the counterterm, equation (4.1-4c), and the corresponding term of β_κ.

The calculation for the remaining two triangle diagrams can be done in an analogous way. Alternatively, one can avoid it by arguing that their contribution to the β-function has to be the transpose of that from the diagrams we considered because of the symmetry of κ. This completes the check of those parts of the β-function that are due to triangle diagrams.

4.2.3 Summary

We have illustrated our method of checking β-functions by two explicit calculations, thus verifying our result equation (4.1-5). Note that the latter has also been confirmed independently by [24]. It has to be stressed that this method may be applied to any β-function of an effective theory. It may be extended to some softly broken supersymmetric theories in which, due to the non-renormalization theorem, the calculation of the full, i.e. supersymmetric, theory β-functions requires only little effort, as shown in chapter 6.

5 Extensions of the Standard Model

The SM is in an excellent shape experimentally. Nevertheless, as was already argued before, it is most probably an effective theory. It may be easily extended without spoiling the good agreement with the experimental data. A very minimalistic extension consists in the enlargment of the Higgs sector, especially in a model with two Higgs doublets. As we will argue in section 5.1, such models arise effectively in many unified theories in an intermediate stage. Also the MSSM requires two Higgs doublets.

5.1 The Left-Right-Symmetric Extension of the Standard Model

The left-right-symmetrical model is a gauge theory which contains in addition to the gauge groups of the SM a "right-handed" $SU(2)_R$,

$$G_{LR} = SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \, . \tag{5.1-1}$$

5.1.1 The Scalar Sector

In order to obtain spontaneous symmetry breaking (SSB) $G_{LR} \to G_{SM}$, one utilizes a scalar sector which differs from that of the SM. In the early days of the development of the LR models, people used the two doublets. For an overview see e.g. [25].

However, it is not possible to construct a Majorana mass term for the right-handed neutrino with those scalars. Therefore the tendency is to use two triplets Δ_L and Δ_R, whose Yukawa couplings and vev structure lead to the desired Majorana masses. Since one does not have to extend the fermionic content of the model in comparison to the SM and since one can "explain" the smallness of the neutrino masses via the see-saw mechanism, this model is often referred to as *minimal left-right-symmetric model*. More details can be found in [26].

Furthermore, a scalar field is needed which plays the role of the Higgs field, i.e. its Yukawa couplings to the SM fermions should generate their masses. The most appealing extension of the scalar sector which fulfills this requirement consists of the introduction of a bi-doublet Φ. As we will see in section 5.1.4, after spontaneous parity breaking this leads effectively to a Two-Higgs-Doublet Model (2HDM).

5.1.2 Transformation Properties of the Fundamental Fields

As one can see from table 5.1-1, the fermions fit quite well into the representation space of the gauge groups, i.e. they transform either trivially or under the fundamental representation of each component.

	SU(3)	SU(2)$_\text{L}$	SU(2)$_\text{R}$	$U(1)_{B-L}$
$q_\text{L} = \begin{pmatrix} u_\text{L} \\ d_\text{L} \end{pmatrix}$	**3**	**2**	**1**	$\frac{1}{3}$
$q_\text{R} = \begin{pmatrix} u_\text{R} \\ d_\text{R} \end{pmatrix}$	**3**	**1**	**2**	$\frac{1}{3}$
$\ell_\text{L} = \begin{pmatrix} \nu_\text{L} \\ e_\text{L} \end{pmatrix}$	**1**	**2**	**1**	-1
$\ell_\text{R} = \begin{pmatrix} \nu_\text{R} \\ e_\text{R} \end{pmatrix}$	**1**	**1**	**2**	-1
$\Phi = \begin{pmatrix} \phi_1^0 & \phi_2^+ \\ \phi_1^- & \phi_2^0 \end{pmatrix}$	**1**	**2**	**2**	0
$\Delta_\text{L} = \begin{pmatrix} \Delta_\text{L}^+ & \Delta_\text{L}^{++} \\ \Delta_\text{L}^0 & -\Delta_\text{L}^+ \end{pmatrix}$	**1**	**3**	**1**	-1
$\Delta_\text{R} = \begin{pmatrix} \Delta_\text{R}^+ & \Delta_\text{R}^{++} \\ \Delta_\text{R}^0 & -\Delta_\text{R}^+ \end{pmatrix}$	**1**	**1**	**3**	-1

Table 5.1-1: The fermions of the SM and the scalars of the minimal left-right-symmetric model and their transformation properties in left-right-symmetric gauge theories.

5.1.3 Parity Breaking

The breaking of parity occurs spontaneously, i.e. the vev's of the scalar fields do not respect parity symmetry. In particular, the vacuum expectation values of the triplets are given by

$$\langle \Delta_\text{L} \rangle = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \langle \Delta_\text{R} \rangle = \begin{pmatrix} 0 & 0 \\ v_\text{R} & 0 \end{pmatrix} \ . \tag{5.1-2}$$

5.1.4 Fermion Masses

The fermion masses arise from the Yukawa couplings of the bi-doublet Φ to the fermions

$$\begin{aligned} \mathscr{L}^\Phi_\text{Yukawa} \quad = \quad & -\Big\{ \overline{q}^g_\text{L} \left[(Y_1)_{gf}\Phi + (Y_2)_{gf}\widehat{\Phi} \right] q^f_\text{R} + \\ & +\overline{\ell}^g_\text{L} \left[(Y_3)_{gf}\Phi + (Y_4)_{gf}\widehat{\Phi} \right] \ell^f_\text{R} + \text{h.c.} \Big\} \ , \end{aligned} \tag{5.1-3}$$

and from

$$\mathscr{L}_{\text{Yukawa}}^{\Delta} = -\left\{ (Y_5)_{gf}\,\overline{\ell_{\text{L}}^{\text{C}g}}\,\Delta_{\text{L}}\,\ell_{\text{L}}^{f} + (Y_5)_{gf}\,\overline{\ell_{\text{R}}^{\text{C}g}}\,\Delta_{\text{R}}\,\ell_{\text{R}}^{f} + \text{h.c.} \right\} . \tag{5.1-4}$$

Inserting the triplet vev's (5.1-2), one finds

$$-\mathscr{L}_{\text{Majorana}} = (Y_5)_{gf}\,\overline{\nu_{\text{R}}^{\text{C}g}}\nu_{\text{R}}^{f} v_{\text{R}}^{2} + \text{h.c.} =: \frac{1}{2} M_{gf}\,\overline{\nu_{\text{R}}^{\text{C}g}}\nu_{\text{R}}^{f} + \text{h.c.} . \tag{5.1-5}$$

In particular, the Majorana mass term is proportional to a Yukawa-matrix. Therefore it is well-motivated to assume that the eigenvalue spectrum is non-degenerate.

After the breaking of $\text{SU}(2)_{\text{R}}$, the scalar field Φ transforms as

$$\Phi = \left(\phi^{(1)}, \widehat{\phi}^{(2)}\right) \to \left(U_{\text{L}}\phi^{(1)}, U_{\text{L}}\widehat{\phi}^{(2)}\right) \tag{5.1-6}$$

with

$$\phi^{(1)} = \begin{pmatrix} \phi^{(1)\,0} \\ \phi^{(1)\,-} \end{pmatrix} , \quad \widehat{\phi}^{(2)} = \begin{pmatrix} (\phi^{(2)\,-})^{*} \\ -(\phi^{(2)\,0})^{*} \end{pmatrix} \quad \text{and} \quad \phi^{(2)} = \begin{pmatrix} \phi^{(2)\,0} \\ \phi^{(2)\,-} \end{pmatrix} , \tag{5.1-7}$$

thus leading to to a model with two Higgs doublet of the SM type.

5.2 Neutrino Mass Operator RGE in a Class of Two-Higgs Doublet Models

As we have seen in the previous section, the Higgs sector of the SM can well be enlarged by some additional $\text{SU}(2)_{\text{L}}$ doublet scalar fields $\phi^{(i)}$ $(1 \leq i \leq n_{\text{H}})$. These can in general couple to the SM fermions via the Yukawa couplings

$$\begin{aligned} -\mathscr{L}_{\text{Yukawa}}^{(i)} &= (Y_e^{(i)})_{gf}\overline{e_{\text{R}}^{g}}\phi_a^{(i)\dagger}\delta^{ab}\ell_{\text{L}b}^{f} \\ &\quad + (Y_d^{(i)})_{gf}\overline{d_{\text{R}}^{g}}\phi_a^{(i)\dagger}\delta^{ab}Q_{\text{L}b}^{f} + (Y_u^{(i)})_{gf}\overline{u_{\text{R}}^{g}}Q_{\text{L}b}^{f}\varepsilon^{ba}\phi_a^{(i)} + \text{h.c.} . \end{aligned} \tag{5.2-1}$$

The notation is chosen in such a way that all $\phi^{(i)}$ transform as $(\mathbf{2}, \frac{1}{2})$ under $\text{SU}(2)_{\text{L}} \otimes \text{U}(1)_{\text{Y}}$. In particular, for $n_{\text{H}} = 1$ we obtain the SM. This is also true for the Feynman rules (see appendix C).

Note that there are tight phenomenological constraints on Yukawa couplings. As pointed out in [27–29], it is very hard to construct viable models in which one type of SM fermions e, d and u couples to two or more Higgs bosons, since this in general leads to tree-level flavor-changing neutral currents (FCNC's). To avoid them it is convenient to restrict the discussion to models in which the fermions couple to at most one Higgs. As a consequence, the suffix "(i)" on the Yukawa couplings in equation (5.2-1) is redundant and will be omitted in the following.

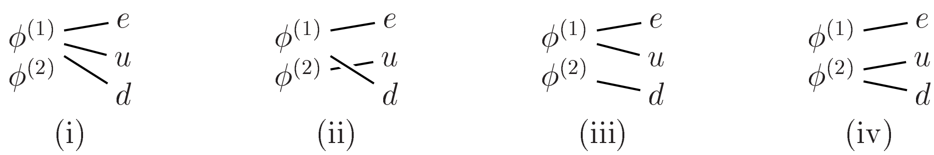


Table 5.2-1: Classification of the 2-Higgs models with natural FCNC suppression and tree level mass terms for all SM fermion types. Note that model (i) is usually referred to as "type I" and (ii) as "type II" in the literature.

5.2.1 Classification of 2HDM's

We concentrate on models with two Higgs doublets for simplicity, i.e. $n_{\mathrm{H}} = 2$, and consider only schemes in which each of the right-handed SM fermions couples to exactly one Higgs boson. All inequivalent possibilities are classified in table 5.2-1. By convention, the scalar which couples to e_{R} is defined to be $\phi^{(1)}$.

Furthermore, we allow for an extension of the models by right-handed neutrinos in order to give rise to effective neutrino mass operators, defined analogously to (3.2-1).

For our calculations we introduce coefficients $z_\psi^{(i)}$ which are defined to be 1 if the fermion ψ couples to the Higgs boson $\phi^{(i)}$ and 0 otherwise. For the models classified in table 5.2-1 they are given by table 5.2-2. Since in table 5.2-2 only the effective theories are classified, the coefficients $z_\nu^{(i)}$ do not appear there. In order to avoid FCNC's, we impose the $\mathbb{Z}_2$

	(i)	(ii)	(iii)	(iv)
$z_u^{(1)}$	1	0	1	0
$z_u^{(2)}$	0	1	0	1
$z_d^{(1)}$	1	1	0	0
$z_d^{(2)}$	0	0	1	1

Table 5.2-2: The coefficients $z_\psi^{(i)}$ for the Two Higgs Doublet Models classified in table 5.2-1. The coefficients $z_\nu^{(i)}$ are not shown as they do not specify the model.

symmetry

$$\phi^{(1)} \to \phi^{(1)}\;, \quad \phi^{(2)} \to -\phi^{(2)} \quad \text{and} \quad \psi \to (-1)^{z_\psi^{(2)}}\,\psi\;, \quad (\psi \in \{u,d\})\;. \tag{5.2-2}$$

For example, in scheme (ii) all fields transform trivially except for

$$\begin{pmatrix} \phi^{(2)} \\ u \end{pmatrix} \to -\begin{pmatrix} \phi^{(2)} \\ u \end{pmatrix}\;. \tag{5.2-3}$$

The most general Higgs self-interaction Lagrangian is then

$$\begin{aligned} \mathscr{L}_{\text{2Higgs}} \quad = \quad & -\frac{\lambda_1}{4}\left(\phi^{(1)\dagger}\phi^{(1)}\right)^2 - \frac{\lambda_2}{4}\left(\phi^{(2)\dagger}\phi^{(2)}\right)^2 \\ & -\lambda_3\left(\phi^{(1)\dagger}\phi^{(1)}\right)\left(\phi^{(2)\dagger}\phi^{(2)}\right) - \lambda_4\left(\phi^{(1)\dagger}\phi^{(2)}\right)\left(\phi^{(2)\dagger}\phi^{(1)}\right) \\ & -\left[\frac{\lambda_5}{4}\left(\phi^{(1)\dagger}\phi^{(2)}\right)^2 + \text{h.c.}\right]\;. \end{aligned} \tag{5.2-4}$$

5.2.2 Lagrangian of a Possible Full Theory

In section 5.1 we have seen that a 2HDM with additional right-handed neutrinos may emerge from a broken left-right symmetry. Allowing for a more general coupling of the Higgs doublets to the fermions, we arrive at a Lagrangian of a possible "full" theory which is given by

$$\mathscr{L} = \sum_{\psi\in\{q_\mathrm{L},d_\mathrm{R},u_\mathrm{R},\ell_\mathrm{L},e_\mathrm{R}\}} \mathscr{L}_\mathrm{kin}^{\psi} + \mathscr{L}_\nu + \sum_{i=1,2} \mathscr{L}_\mathrm{kin}^{\phi^{(i)}} + \mathscr{L}_\mathrm{2Higgs} + \mathscr{L}_\mathrm{Yukawa} + \mathscr{L}_\mathrm{Gauge} \;, \tag{5.2-5}$$

where

$$\mathscr{L}_\mathrm{kin}^{\psi} = \overline{\psi}^f \,\mathrm{i}\not{D}^{(\psi)}\, \psi^f \;, \tag{5.2-6a}$$

$$\mathscr{L}_\nu = \frac{1}{2}\overline{N}^i \,\mathrm{i}\not{\partial}\, N^i - \left(\frac{1}{2}\overline{N}^i M_{ij} N^j + \sum_{i=1,2} z_\nu^{(i)}\, N^j\, (Y_\nu)_{jf}\, \ell_{\mathrm{L}b}^f \varepsilon^{ba} \phi_a^{(i)} + \mathrm{h.c.}\right) \;, \tag{5.2-6b}$$

$$\mathscr{L}_\mathrm{kin}^{\phi^{(i)}} = \left(D_\mu^{(\phi)}\phi^{(i)}\right)^\dagger \left(D^{(\phi)\,\mu}\phi^{(i)}\right) - m_i^2\, \phi^{(i)\,\dagger}\phi^{(i)} \;, \tag{5.2-6c}$$

and $\mathscr{L}_\mathrm{Gauge}$ is defined appropriately. By integrating out the heavy degrees of freedom, we obtain an effective neutrino mass operator of the form

$$\mathscr{L}_\kappa^{(ii)} = \frac{1}{4}\kappa_{gf}^{(ii)}\, \overline{\ell_{\mathrm{L}}^{\mathrm{C}}}{}^g_c \varepsilon^{cd} \phi_d^{(i)}\, \ell_{\mathrm{L}b}^f \varepsilon^{ba} \phi_a^{(i)} + \mathrm{h.c.} \qquad (i = 1 \text{ or } 2) \;, \tag{5.2-7}$$

where $\kappa^{(ii)} = 2 z_\nu^{(i)}\, Y_\nu^T \cdot M^{-1} \cdot Y_\nu$ at the matching scale. In analogy to the SM case, $\kappa_{gf}^{(ii)}$ are symmetric matrices with respect to the generation indices g and f.

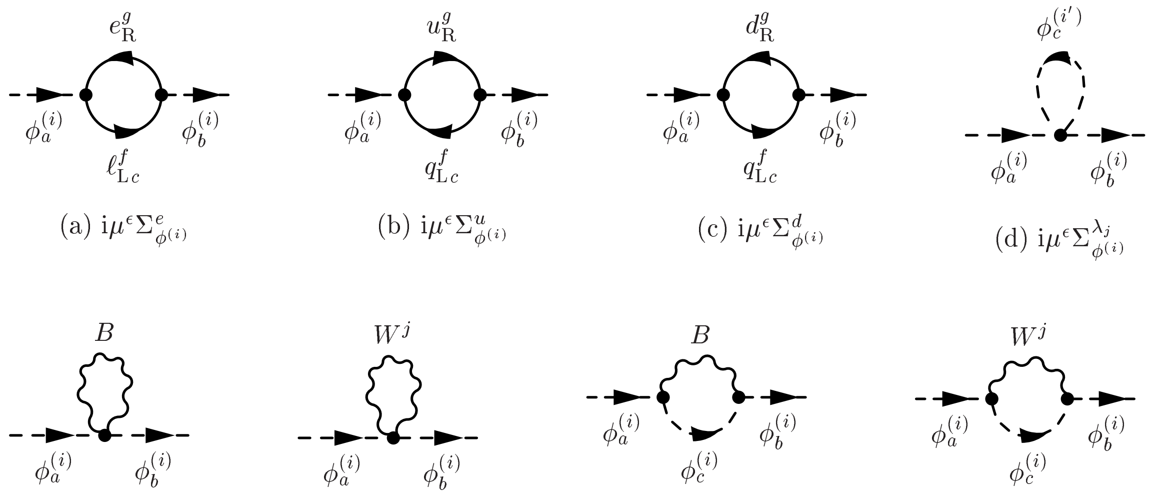
5.2.3 Effective Neutrino Mass Operators

The lowest dimensional effective neutrino mass operators compatible with the symmetry (5.2-2) are given by the sum of operators of the type (5.2-7),

$$\mathscr{L}_\kappa = \mathscr{L}_\kappa^{(11)} + \mathscr{L}_\kappa^{(22)} \;. \tag{5.2-8}$$

As demonstrated before, it is possible that only one of these operators, e.g. $\mathscr{L}_\kappa^{(22)}$, arises from integrating out heavy degrees of freedom in a specific model. For example, in the full theory of the previous section, a non-zero $\kappa^{(ii)}$ arises at the matching scale only if $z_\nu^{(i)}$ is 1. However, as we shall see, both operators mix due to the renormalization group evolution and therefore have to be taken into account simultaneously.

As long as the symmetry (5.2-2) is valid, $\mathscr{L}_\kappa^{(11)}$ and $\mathscr{L}_\kappa^{(22)}$ represent the only possible dimension 5 operators containing two ℓ_L fields. If this symmetry was broken, further couplings would appear in the Higgs interaction Lagrangian (5.2-4), and the natural FCNC suppression would be spoiled. In this case, more terms in the β-functions would arise besides the additional effective operators appearing in the Lagrangian.



(a) $\mathrm{i}\mu^\epsilon\Sigma^{e}_{\phi^{(i)}}$ (b) $\mathrm{i}\mu^\epsilon\Sigma^{u}_{\phi^{(i)}}$ (c) $\mathrm{i}\mu^\epsilon\Sigma^{d}_{\phi^{(i)}}$ (d) $\mathrm{i}\mu^\epsilon\Sigma^{\lambda_j}_{\phi^{(i)}}$

(e) $\mathrm{i}\mu^\epsilon\Sigma^{B,1}_{\phi^{(i)}}$ (f) $\mathrm{i}\mu^\epsilon\Sigma^{W^j,1}_{\phi^{(i)}}$ (g) $\mathrm{i}\mu^\epsilon\Sigma^{B,2}_{\phi^{(i)}}$ (h) $\mathrm{i}\mu^\epsilon\Sigma^{W^j,2}_{\phi^{(i)}}$

Figure 5.1: Diagrams contributing to the self-energy of the field $\phi^{(i)}$.

5.2.4 Calculation of the β-Functions

The wavefunction renormalization constants are defined by

$$\psi^f_{\mathrm{B}} = (Z_\psi^{\frac{1}{2}})_{fg}\psi^g \tag{5.2-9}$$

for the SM fermions $\psi \in \{q_{\mathrm{L}}, u_{\mathrm{R}}, d_{\mathrm{R}}, \ell_{\mathrm{L}}, e_{\mathrm{R}}\}$,

$$\phi^{(i)}_{\mathrm{B}} = Z^{\frac{1}{2}}_{\phi^{(i)}}\phi^{(i)} \tag{5.2-10}$$

for the Higgs doublets and similarly for the remaining fields. For the Higgs fields $\phi^{(i)}$ we obtain the self-energy diagrams of figure 5.1, which yield

$$\mathrm{i}\left(\Sigma^{e}_{\phi^{(i)}}\right)_{ab}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{8\pi^2}\,\mathrm{Tr}\left(Y_e^\dagger Y_e\right)\,\delta_{ab}\,p^2\frac{1}{\epsilon}\ , \tag{5.2-11a}$$

$$\mathrm{i}\left(\Sigma^{u}_{\phi^{(i)}}\right)_{ab}\Big|_{\mathrm{div}} = \frac{3\mathrm{i}}{8\pi^2}\,\mathrm{Tr}\left(Y_u^\dagger Y_u\right)\,\delta_{ab}\,p^2\frac{1}{\epsilon}\ , \tag{5.2-11b}$$

$$\mathrm{i}\left(\Sigma^{d}_{\phi^{(i)}}\right)_{ab}\Big|_{\mathrm{div}} = \frac{3\mathrm{i}}{8\pi^2}\,\mathrm{Tr}\left(Y_d^\dagger Y_d\right)\,\delta_{ab}\,p^2\frac{1}{\epsilon}\ , \tag{5.2-11c}$$

$$\mathrm{i}\left(\Sigma^{\lambda_j}_{\phi^{(i)}}\right)_{ab}\Big|_{\mathrm{div}} = \frac{3\mathrm{i}}{16\pi^2}\lambda_j\,m_i^2\frac{1}{\epsilon}\ , \tag{5.2-11d}$$

$$\mathrm{i}\left(\Sigma^{B,1}_{\phi^{(i)}}\right)_{ab}\Big|_{\mathrm{div}} = 0\ , \tag{5.2-11e}$$

$$\mathrm{i}\sum_j\left(\Sigma^{W^j,1}_{\phi^{(i)}}\right)_{ab}\Big|_{\mathrm{div}} = 0\ , \tag{5.2-11f}$$

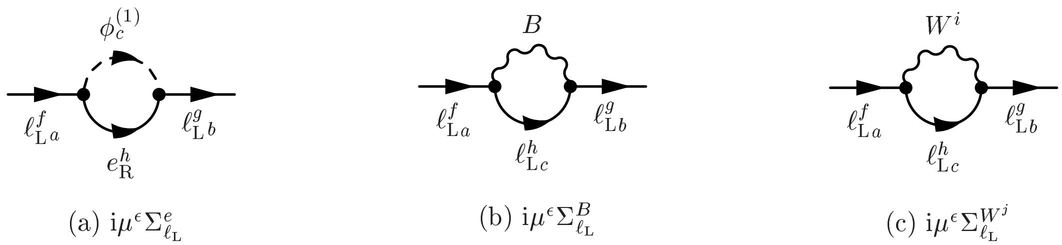


Figure 5.2: Diagrams relevant for the self-energy of the ℓ_L field.

$$\mathrm{i}\left(\Sigma^{B,2}_{\phi^{(i)}}\right)_{ab}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{16\pi^2}\left[2\xi_B(p^2-m_i^2)-6p^2\right]\, y^2_{\phi^{(i)}}\, g_1^2\, \delta_{ab}\,(p^2-m^2)\frac{1}{\epsilon}\,, \quad (5.2\text{-}11g)$$

$$\mathrm{i}\sum_j\left(\Sigma^{W^j,2}_{\phi^{(i)}}\right)_{ab}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{16\pi^2}\left[2\xi_W(p^2-m_i^2)-6p^2\right]\, \frac{3}{4}\, g_2^2\, \delta_{ab}\,(p^2-m^2)\frac{1}{\epsilon}\,, \quad (5.2\text{-}11h)$$

where the subscript "div" indicates the projection onto the divergent part of the corresponding expression. We demand

$$\Sigma_{\phi^{(i)}} + \delta Z_{\phi^{(i)}}\, p^2 + \delta m^2_{\phi^{(i)}} \stackrel{!}{=} \text{UV finite}\,, \quad (5.2\text{-}12)$$

and therefore obtain for the wavefunction renormalization constants

$$\begin{aligned}\delta Z_{\phi^{(i)}} &= \sum_{\psi\in\{e,u,d\}} \delta Z^{\psi}_{\phi^{(i)}} + \delta Z^{\lambda_i}_{\phi^{(i)}} + \delta Z^{\lambda_3}_{\phi^{(i)}} + \delta Z^{\lambda_4}_{\phi^{(i)}} + \sum_{k=1}^{2}\left[\delta Z^{B,k}_{\phi^{(i)}} + \sum_j \delta Z^{W^j,k}_{\phi^{(i)}}\right] \\ &= -\frac{1}{16\pi^2}\Big[\delta_{i1}\, 2\,\mathrm{Tr}(Y_e^\dagger Y_e) + z_u^{(i)}\, 6\,\mathrm{Tr}(Y_u^\dagger Y_u) + z_d^{(i)}\, 6\,\mathrm{Tr}(Y_d^\dagger Y_d) \\ &\qquad + \frac{1}{2}(\xi_B-3)\, g_1^2 + \frac{3}{2}(\xi_W-3)\, g_2^2\Big]\,\frac{1}{\epsilon}\,. \end{aligned} \quad (5.2\text{-}13)$$

The self-energy diagrams of the leptonic doublet (see figure 5.2) yield

$$\mathrm{i}\left(\Sigma^e_{\ell_\mathrm{L}}\right)^{gf}_{ab}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{16\pi^2}(Y_e^\dagger Y_e)^{gf}\,\delta_{ab}\,\not{p}\, P_\mathrm{L}\frac{1}{\epsilon}\,, \quad (5.2\text{-}14a)$$

$$\mathrm{i}\left(\Sigma^B_{\ell_\mathrm{L}}\right)_{ab}\Big|_{\mathrm{div}} = \frac{2\mathrm{i}}{16\pi^2} g_1^2\, y_\ell^2\, \xi_B\, \delta_{ab}\,\not{p}\, P_\mathrm{L}\frac{1}{\epsilon}\,, \quad (5.2\text{-}14b)$$

$$\mathrm{i}\sum_j\left(\Sigma^{W^j}_{\ell_\mathrm{L}}\right)_{ab}\Big|_{\mathrm{div}} = \frac{2\mathrm{i}}{16\pi^2} g_2^2\,\frac{3}{4}\,\xi_W\, \delta_{ab}\,\not{p}\, P_\mathrm{L}\frac{1}{\epsilon}\,. \quad (5.2\text{-}14c)$$

The condition

$$\Sigma_{\ell_\mathrm{L}} + \delta Z_{\ell_\mathrm{L}}\,\not{p} + \delta m_{\ell_\mathrm{L}} \stackrel{!}{=} \text{UV finite} \quad (5.2\text{-}15)$$

leads to the following wavefunction renormalization for ℓ_L:

$$\begin{aligned}\delta Z_{\ell_\mathrm{L}} &= \delta Z^e_{\ell_\mathrm{L}} + \delta Z^B_{\ell_\mathrm{L}} + \sum_j \delta Z^{W^j}_{\ell_\mathrm{L}} \\ &= -\frac{1}{16\pi^2}\left[Y_e^\dagger Y_e - \tfrac{1}{2}\xi_B g_1^2 - \tfrac{3}{2}\xi_W g_2^2\right]\frac{1}{\epsilon}\,. \end{aligned} \quad (5.2\text{-}16)$$

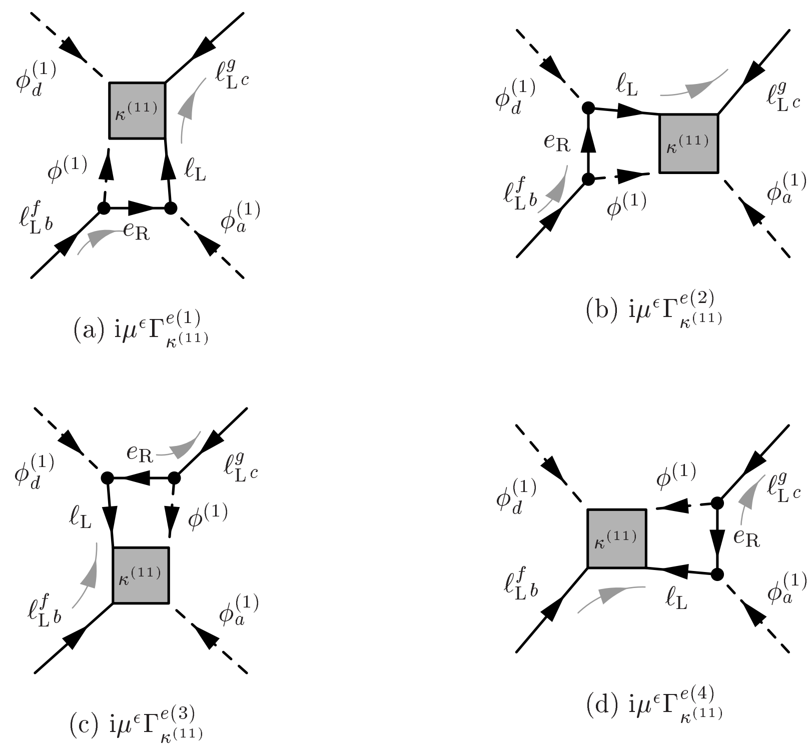


Figure 5.3: The one-loop vertex corrections that arise due to the Yukawa coupling Y_e. In the class of 2HDM's under consideration these affect only the renormalization of $\kappa^{(11)}$.

We find for the vertex corrections arising due to the Y_e Yukawa coupling, which are depicted in figure 5.3,

$$\mathrm{i}\left(\Gamma^{e(1)}_{\kappa^{(11)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{8\pi^2}\left(\kappa^{(11)}Y_e^\dagger Y_e\right)_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{cb}\varepsilon_{da})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \tag{5.2-17a}$$

$$\mathrm{i}\left(\Gamma^{e(2)}_{\kappa^{(11)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{8\pi^2}(\kappa^{(11)}Y_e^\dagger Y_e)_{gf}\,\tfrac{1}{2}(\varepsilon_{ca}\varepsilon_{bd}-\varepsilon_{cb}\varepsilon_{da})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \tag{5.2-17b}$$

$$\mathrm{i}\left(\Gamma^{e(3)}_{\kappa^{(11)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{8\pi^2}(Y_e^T Y_e^*\kappa^{(11)})_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{cb}\varepsilon_{da})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \tag{5.2-17c}$$

$$\mathrm{i}\left(\Gamma^{e(4)}_{\kappa^{(11)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{8\pi^2}(Y_e^T Y_e^*\kappa^{(11)})_{gf}\,\tfrac{1}{2}(\varepsilon_{ca}\varepsilon_{bd}-\varepsilon_{cb}\varepsilon_{da})\,P_\mathrm{L}\frac{1}{\epsilon}\;. \tag{5.2-17d}$$

For the corrections from the B gauge interactions, as shown in figure 5.4, we obtain

$$\mathrm{i}\left(\Gamma^{B(1)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{32\pi^2}\xi_B\,\mu^\epsilon\,g_1^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \tag{5.2-18a}$$

$$\mathrm{i}\left(\Gamma^{B(2)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{32\pi^2}\xi_B\,\mu^\epsilon\,g_1^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \tag{5.2-18b}$$

$$\mathrm{i}\left(\Gamma^{B(3)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{32\pi^2}\xi_B\,\mu^\epsilon g_1^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \tag{5.2-18c}$$

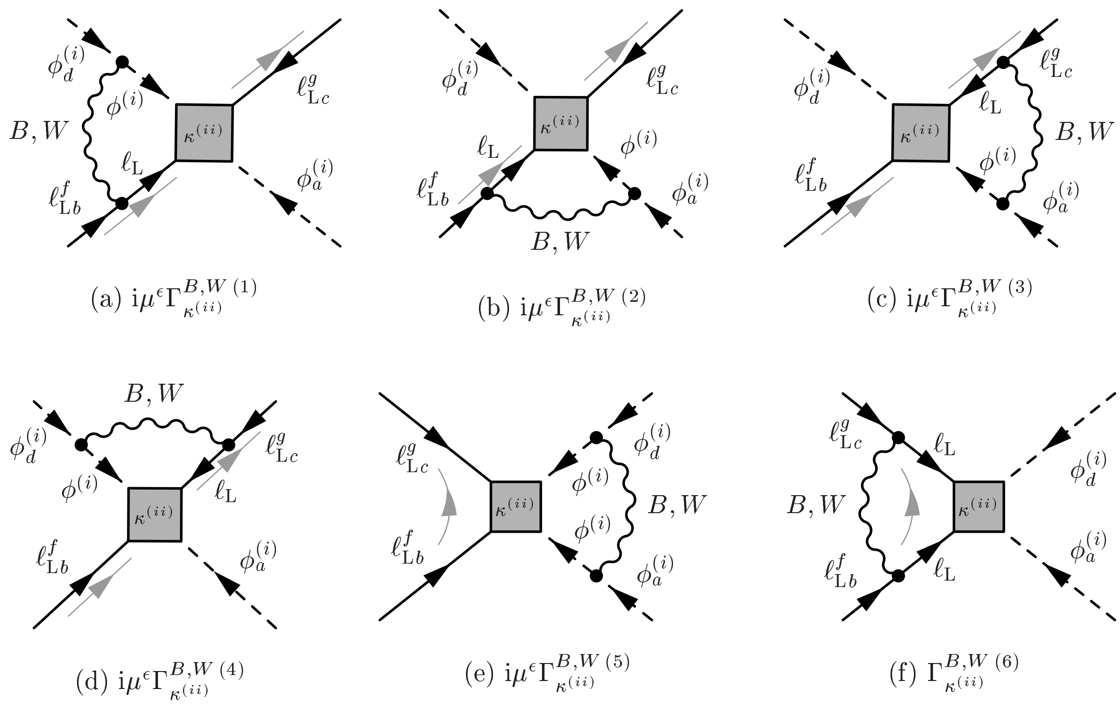


Figure 5.4: One-loop vertex corrections that arise due to gauge interactions. All these diagrams have a counterpart in the SM.

$$\mathrm{i}\left(\Gamma^{B(4)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{32\pi^2}\xi_B\,\mu^\epsilon\,g_1^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_{\mathrm{L}}\frac{1}{\epsilon}\,, \tag{5.2-18d}$$

$$\mathrm{i}\left(\Gamma^{B(5)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{-\mathrm{i}}{32\pi^2}\xi_B\,\mu^\epsilon\,g_1^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_{\mathrm{L}}\frac{1}{\epsilon}\,, \tag{5.2-18e}$$

$$\mathrm{i}\left(\Gamma^{B(5)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{-\mathrm{i}}{32\pi^2}(3+\xi_B)\,\mu^\epsilon\,g_1^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_{\mathrm{L}}\frac{1}{\epsilon}\,, \tag{5.2-18f}$$

while the W diagrams yield

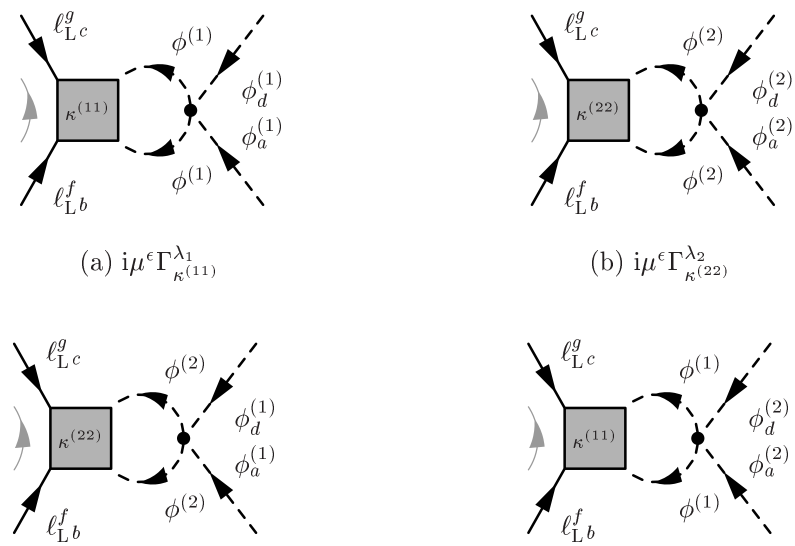
$$\mathrm{i}\left(\Gamma^{W(1)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{32\pi^2}\xi_W\,g_2^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+3\varepsilon_{ca}\varepsilon_{bd}-2\varepsilon_{cb}\varepsilon_{da})\,P_{\mathrm{L}}\frac{1}{\epsilon}\,, \tag{5.2-19a}$$

$$\mathrm{i}\left(\Gamma^{W(2)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{32\pi^2}\xi_W\,g_2^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(3\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd}-2\varepsilon_{cb}\varepsilon_{ad})\,P_{\mathrm{L}}\frac{1}{\epsilon}\,, \tag{5.2-19b}$$

$$\mathrm{i}\left(\Gamma^{W(3)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{32\pi^2}\xi_W\,g_2^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+3\varepsilon_{ca}\varepsilon_{bd}-2\varepsilon_{ad}\varepsilon_{bc})\,P_{\mathrm{L}}\frac{1}{\epsilon}\,, \tag{5.2-19c}$$

$$\mathrm{i}\left(\Gamma^{W(4)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{32\pi^2}\xi_W\,g_2^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(3\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd}-2\varepsilon_{da}\varepsilon_{bc})\,P_{\mathrm{L}}\frac{1}{\epsilon}\,, \tag{5.2-19d}$$

$$\mathrm{i}\left(\Gamma^{W(5)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\Big|_{\mathrm{div}} = \frac{-\mathrm{i}}{32\pi^2}\xi_W\,g_2^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_{\mathrm{L}}\frac{1}{\epsilon}\,, \tag{5.2-19e}$$



(a) $\mathrm{i}\mu^\epsilon\Gamma^{\lambda_1}_{\kappa^{(11)}}$ (b) $\mathrm{i}\mu^\epsilon\Gamma^{\lambda_2}_{\kappa^{(22)}}$

(c) $\mathrm{i}\mu^\epsilon\Gamma^{\lambda_5}_{\kappa^{(11)}}$ (d) $\mathrm{i}\mu^\epsilon\Gamma^{\lambda_5}_{\kappa^{(22)}}$

Figure 5.5: The diagrams coming from the Higgs interaction Lagrangian. While the diagrams (a) and (b) have a counterpart in the SM, the diagrams (c) and (d) appear only in the 2HDM's and lead to a mixing between the operators $\mathscr{L}_\kappa^{(11)}$ and $\mathscr{L}_\kappa^{(22)}$.

$$\mathrm{i}\left(\Gamma^{W(6)}_{\kappa^{(ii)}}\right)^{abcd}_{gf}\bigg|_{\mathrm{div}} \;=\; \frac{-\mathrm{i}}{32\pi^2}(3+\xi_W)\,g_2^2\,\kappa^{(ii)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})P_\mathrm{L}\,\frac{1}{\epsilon}\;. \qquad (5.2\text{-}19f)$$

The diagrams due to the Higgs self-interactions, see figure 5.5, evaluate to

$$\mathrm{i}\left(\Gamma^{\lambda_1}_{\kappa^{(11)}}\right)^{gf}_{abcd}\bigg|_{\mathrm{div}} \;=\; \frac{-\mathrm{i}}{16\pi^2}\lambda_1\,\kappa^{(11)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \qquad (5.2\text{-}20a)$$

$$\mathrm{i}\left(\Gamma^{\lambda_2}_{\kappa^{(22)}}\right)^{gf}_{abcd}\bigg|_{\mathrm{div}} \;=\; \frac{-\mathrm{i}}{16\pi^2}\lambda_2\,\kappa^{(22)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \qquad (5.2\text{-}20b)$$

$$\mathrm{i}\left(\Gamma^{\lambda_5}_{\kappa^{(11)}}\right)^{gf}_{abcd}\bigg|_{\mathrm{div}} \;=\; \frac{-\mathrm{i}}{16\pi^2}\lambda_5^*\,\kappa^{(22)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_\mathrm{L}\frac{1}{\epsilon}\;, \qquad (5.2\text{-}20c)$$

$$\mathrm{i}\left(\Gamma^{\lambda_5}_{\kappa^{(22)}}\right)^{gf}_{abcd}\bigg|_{\mathrm{div}} \;=\; \frac{-\mathrm{i}}{16\pi^2}\lambda_5\,\kappa^{(11)}_{gf}\,\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})\,P_\mathrm{L}\frac{1}{\epsilon}\;. \qquad (5.2\text{-}20d)$$

With the counterterms for the effective vertices defined by

$$\begin{aligned}\mathscr{C}_\kappa \;=\;& \frac{1}{4}\delta\kappa^{(11)}_{gf}\,\overline{\ell^{\mathrm{C}}_{\mathrm{L}}}^g_c\varepsilon^{cd}\phi^{(1)}_d\,\ell^f_{\mathrm{L}b}\varepsilon^{ba}\phi^{(1)}_a\\ &+\frac{1}{4}\delta\kappa^{(22)}_{gf}\,\overline{\ell^{\mathrm{C}}_{\mathrm{L}}}^g_c\varepsilon^{cd}\phi^{(2)}_d\,\ell^f_{\mathrm{L}b}\varepsilon^{ba}\phi^{(2)}_a+\mathrm{h.c.}\;,\end{aligned} \qquad (5.2\text{-}21)$$

and the condition that

$$0 \stackrel{!}{=} \delta\kappa^{(11)} + \left[\sum_{j=1}^{4} \Gamma^{e(j)}_{\kappa^{(11)}} + \sum_{j=1}^{6} \left(\Gamma^{B(j)}_{\kappa^{(11)}} + \Gamma^{W(j)}_{\kappa^{(11)}}\right) + \Gamma^{\lambda_1}_{\kappa^{(11)}} + \Gamma^{\lambda_5}_{\kappa^{(11)}}\right]_{\text{div}} , \tag{5.2-22a}$$

$$0 \stackrel{!}{=} \delta\kappa^{(22)} + \left[\sum_{j=1}^{6} \left(\Gamma^{B(j)}_{\kappa^{(22)}} + \Gamma^{W(j)}_{\kappa^{(22)}}\right) + \Gamma^{\lambda_2}_{\kappa^{(22)}} + \Gamma^{\lambda_5}_{\kappa^{(22)}}\right]_{\text{div}} \tag{5.2-22b}$$

in the MS scheme, we find

$$\begin{aligned}\delta\kappa^{(ii)} = -\frac{1}{16\pi^2} \big[&\delta_{i1}\, 2\kappa^{(ii)}(Y_e^\dagger Y_e) + \delta_{i1}\, 2(Y_e^\dagger Y_e)^T \kappa^{(ii)} \\ &- \lambda_i \kappa^{(ii)} - \delta_{i1}\lambda_5^* \kappa^{(22)} - \delta_{i2}\lambda_5 \kappa^{(11)} \\ &+ \left(\xi_B - \tfrac{3}{2}\right) g_1^2 \kappa^{(ii)} + \left(3\xi_W - \tfrac{3}{2}\right) g_2^2 \kappa^{(ii)}\big] \frac{1}{\epsilon} .\end{aligned} \tag{5.2-23}$$

Note that due to the appearance of the λ_5 terms even in the 1×1 case, i.e. even if the tensorial structure may be neglected, multiplicative renormalization cannot be imposed.

Using the technique described in section 2.3, the β-functions for $\kappa^{(ii)}$ are derived as

$$\begin{aligned}16\pi^2 \beta_{\kappa^{(ii)}} = &\left(\tfrac{1}{2} - 2\delta_{i1}\right) \left[\kappa^{(ii)}(Y_e^\dagger Y_e) + (Y_e^\dagger Y_e)^T \kappa^{(ii)}\right] \\ &+ \left[\delta_{i1}\, 2 \,\mathrm{Tr}(Y_e^\dagger Y_e) + z_u^{(i)}\, 6 \,\mathrm{Tr}(Y_u^\dagger Y_u) + z_d^{(i)}\, 6 \,\mathrm{Tr}(Y_d^\dagger Y_d)\right] \kappa^{(ii)} \\ &+ \lambda_i \kappa^{(ii)} + \delta_{i1}\lambda_5^* \kappa^{(22)} + \delta_{i2}\lambda_5 \kappa^{(11)} - 3 g_2^2 \kappa^{(ii)} .\end{aligned} \tag{5.2-24}$$

The terms proportional to λ_5 are responsible for the mixing of the effective operators mentioned before. The result for $\beta_{\kappa^{(11)}}$ differs from the one in [17] by a factor of 3 because of the term $\frac{1}{2} - 2\delta_{i1}$ in the first line. This discrepancy is analogous to the one encountered in the SM.

Note that in 2HDM's running effects are in general larger than in the SM due to the fact that the Yukawa couplings are enhanced, e.g. $(Y_e)_{\text{2HDM}} = (Y_e)_{\text{SM}}/\cos\beta$, where $\tan\beta = v^{(2)}/v^{(1)}$ with $v^{(i)}$ being the vev of the Higgs field $\phi^{(i)}$.

5.2.5 Running Between Thresholds

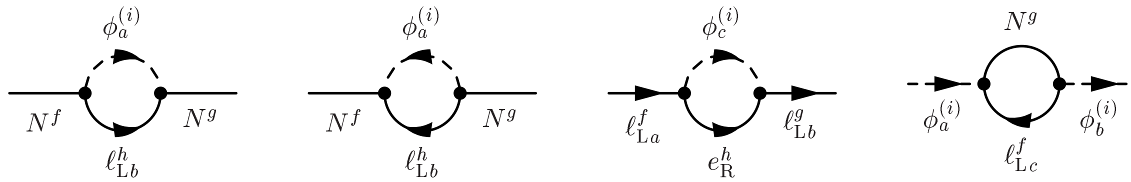
In the effective theories between the possible full theory described in section 5.2.2 and the 2HDM's, we find two contributions to the self-energy, coming from the diagrams 5.6(a) and 5.6(b),

$$\mathrm{i}(\Sigma_{N(\mathrm{R})})_{gf}\Big|_{\text{div}} = \frac{\mathrm{i}}{8\pi^2} \left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right)_{gf} \not{p}\, P_{\mathrm{R}} \frac{1}{\epsilon} , \tag{5.2-25a}$$

$$\mathrm{i}(\Sigma_{N(\mathrm{L})})_{gf}\Big|_{\text{div}} = \frac{\mathrm{i}}{8\pi^2} \left(\overset{(n)}{Y}{}_\nu^* \overset{(n)}{Y}{}_\nu^T\right)_{gf} \not{p} P_{\mathrm{L}} \frac{1}{\epsilon} . \tag{5.2-25b}$$

Therefore we calculate the following wavefunction renormalization constants for the neutrino:

$$\delta \overset{(n)}{Z}_N = -\frac{1}{8\pi^2} \left[\left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right)_{gf} P_{\mathrm{R}} + \left(\overset{(n)}{Y}{}_\nu^* \overset{(n)}{Y}{}_\nu^T\right)_{gf} P_{\mathrm{L}}\right] \frac{1}{\epsilon} . \tag{5.2-26}$$



(a) $\mathrm{i}\left(\Sigma_{N(\mathrm{R})}\right)_{gf}$ (b) $\mathrm{i}\left(\Sigma_{N(\mathrm{L})}\right)_{gf}$ (c) $\mathrm{i}(\Sigma^N_{\ell_\mathrm{L}})^{ba}_{gf}$ (d) $\mathrm{i}(\Sigma^N_{\phi^{(i)}})_{ba}$

Figure 5.6: Additional diagrams in the full theory. In this figure, $\phi^{(i)}$ denotes the Higgs doublet which couples to N.

Furthermore, the self-energy diagrams of the leptonic doublet and the Higgs which couples to N acquire additional contributions due to the diagrams shown in figures 5.6(c) and 5.6(d),

$$\mathrm{i}(\Sigma^N_{\ell_\mathrm{L}})^{ba}_{gf}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{16\pi^2}\left(\overset{(n)}{Y}{}^\dagger_\nu \overset{(n)}{Y}_\nu\right)_{gf}\delta_{ba}\,\not{p}\,P_\mathrm{L}\,\frac{1}{\epsilon}\,, \tag{5.2-27a}$$

$$\mathrm{i}(\Sigma^N_{\phi})_{ba}\Big|_{\mathrm{div}} = \frac{\mathrm{i}}{8\pi^2}\left(\overset{(n)}{Y}{}^\dagger_\nu\right)_{fg}\left(\overset{(n)}{Y}_\nu\right)_{gf}\delta_{ba}\,(p^2-2M_g^2)\,\frac{1}{\epsilon}\,, \tag{5.2-27b}$$

and correspondingly, we obtain the following additional wavefunction renormalization constants:

$$\delta\overset{(n)}{Z}_{\ell_\mathrm{L}} = -\frac{1}{16\pi^2}\left(\overset{(n)}{Y}{}^\dagger_\nu \overset{(n)}{Y}_\nu\right)_{gf} P_\mathrm{L}\,\frac{1}{\epsilon}\,, \tag{5.2-28a}$$

$$\delta\overset{(n)}{Z}_{\phi^{(i)}} = -\frac{1}{8\pi^2}\,\mathrm{Tr}\left(\overset{(n)}{Y}{}^\dagger_\nu \overset{(n)}{Y}_\nu\right)\frac{1}{\epsilon}\,. \tag{5.2-28b}$$

From these relations, the β-function for $\overset{(n)}{Y}_\nu$ is evaluated to

$$\begin{aligned}16\pi^2\,\overset{(n)}{\beta}_{Y_\nu} = \overset{(n)}{Y}_\nu\Bigg\{&\frac{3}{2}\overset{(n)}{Y}{}^\dagger_\nu \overset{(n)}{Y}_\nu - \frac{3}{2}Y_e^\dagger Y_e - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2 \\ &+\sum_{i=1}^{2} z_\nu^{(i)}\;\mathrm{Tr}\left[\delta_{i1}\,Y_e^\dagger Y_e + \overset{(n)}{Y}{}^\dagger_\nu \overset{(n)}{Y}_\nu + 3z_d^{(i)}\,Y_d^\dagger Y_d + 3z_u^{(i)}\,Y_u^\dagger Y_u\right]\Bigg\}\,.\end{aligned} \tag{5.2-29}$$

The β-functions of the remaining Yukawa couplings can be calculated analogously. They are quite similar to those of [30,20,31] with two exceptions: First of all, one has to replace $Y_\nu \to \overset{(n)}{Y}_\nu$, and secondly, in the trace terms one has to take into account that – unlike in the

SM – the Higgs doublets do not couple to all fermions. Proceeding like this, we obtain

$$16\pi^2 \overset{(n)}{\beta}_{Y_e} = Y_e \left\{ \frac{3}{2} Y_e^\dagger Y_e - \frac{3}{2} \overset{(n)}{Y}_\nu^\dagger \overset{(n)}{Y}_\nu - \frac{15}{4} g_1^2 - \frac{9}{4} g_2^2 \right. \tag{5.2-30a}$$

$$\left. + \operatorname{Tr}\left[Y_e^\dagger Y_e + z_\nu^{(1)} \overset{(n)}{Y}_\nu^\dagger \overset{(n)}{Y}_\nu + 3 z_d^{(1)} Y_d^\dagger Y_d + 3 z_u^{(1)} Y_u^\dagger Y_u \right] \right\} ,$$

$$16\pi^2 \overset{(n)}{\beta}_{Y_d} = Y_d \left\{ \frac{3}{2} Y_d^\dagger Y_d - \frac{3}{2} Y_u^\dagger Y_u - \frac{5}{12} g_1^2 - \frac{9}{4} g_2^2 - 8\, g_3^2 \right. \tag{5.2-30b}$$

$$\left. + \sum_{i=1}^{2} z_d^{(i)} \operatorname{Tr}\left[\delta_{i1}\, Y_e^\dagger Y_e + z_\nu^{(i)} \overset{(n)}{Y}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\, Y_d^\dagger Y_d + 3 z_u^{(i)} Y_u^\dagger Y_u \right] \right\} ,$$

$$16\pi^2 \overset{(n)}{\beta}_{Y_u} = Y_u \left\{ \frac{3}{2} Y_u^\dagger Y_u - \frac{3}{2} Y_d^\dagger Y_d - \frac{17}{12} g_1^2 - \frac{9}{4} g_2^2 - 8\, g_3^2 \right. \tag{5.2-30c}$$

$$\left. + \sum_{i=1}^{2} z_u^{(i)} \operatorname{Tr}\left[\delta_{i1}\, Y_e^\dagger Y_e + z_\nu^{(i)} \overset{(n)}{Y}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\, Y_d^\dagger Y_d + 3 z_u^{(i)} Y_u^\dagger Y_u \right] \right\} .$$

The RGE's for the gauge couplings can be taken from [31], since the introduction of the gauge singlets N^i does not change them at one-loop. They read

$$16\pi^2 \beta_{g_1} = \left(\frac{20}{9} n_F + \frac{1}{6} n_H \right) g_1^3 , \tag{5.2-31a}$$

$$16\pi^2 \beta_{g_2} = - \left(\frac{22}{3} - \frac{4}{3} n_F - \frac{1}{6} n_H \right) g_2^3 , \tag{5.2-31b}$$

$$16\pi^2 \beta_{g_3} = - \left(11 - \frac{4}{3} n_F \right) g_3^3 , \tag{5.2-31c}$$

where g_1 was not taken in GUT charge normalization. n_F and n_H appear in these formulae for the sake of generality. In particular, for $n_H = 1$ we obtain the SM evolution of the gauge couplings. Furthermore, the β-functions for the parameters of the Higgs interaction Lagrangian have to be specified. Up to terms arising due to the neutrino Yukawa couplings, these can also be inferred from [31]. Taking into account $\overset{(n)}{Y}_\nu$ yields

$$16\pi^2 \overset{(n)}{\beta}_{\lambda_1} = 6\, \lambda_1^2 + 8\, \lambda_3^2 + 6\, \lambda_3\, \lambda_4 + \lambda_5^2 - 3\, \lambda_1 \left(3 g_2^2 + g_1^2\right) + 3\, g_2^4 + \frac{3}{2} \left(g_1^2 + g_2^2\right)^2$$

$$+ 4\, \lambda_1 \operatorname{Tr}\left(Y_e^\dagger Y_e + z_\nu^{(1)} \overset{(n)}{Y}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\, z_d^{(1)} Y_d^\dagger Y_d + 3\, z_u^{(1)} Y_u^\dagger Y_u \right)$$

$$- 8 \operatorname{Tr}\left(Y_e^\dagger Y_e\, Y_e^\dagger Y_e + z_\nu^{(1)} \overset{(n)}{Y}_\nu^\dagger \overset{(n)}{Y}_\nu\, \overset{(n)}{Y}_\nu^\dagger \overset{(n)}{Y}_\nu \right.$$

$$\left. + 3\, z_d^{(1)} Y_d^\dagger Y_d\, Y_d^\dagger Y_d + 3\, z_u^{(1)} Y_u^\dagger Y_u\, Y_u^\dagger Y_u \right) , \tag{5.2-32a}$$

$$
\begin{aligned}
16\pi^2 \overset{(n)}{\beta}_{\lambda_2} \; = \; & 6\,\lambda_2^2 + 8\,\lambda_3^2 + 6\,\lambda_3\,\lambda_4 + \lambda_5^2 - 3\,\lambda_2\,(3g_2^2 + g_1^2) + 3\,g_2^4 + \frac{3}{2}\,(g_1^2 + g_2^2)^2 \\
& + 4\,\lambda_2\;\mathrm{Tr}\left(z_\nu^{(2)}\,\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\,z_d^{(2)}\,Y_d^\dagger Y_d + 3\,z_u^{(2)}\,Y_u^\dagger Y_u \right) \\
& - 8\,\mathrm{Tr}\left(z_\nu^{(2)}\,\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\,\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\,z_d^{(2)}\,Y_d^\dagger Y_d\,Y_d^\dagger Y_d + 3\,z_u^{(2)}\,Y_u^\dagger Y_u\,Y_u^\dagger Y_u \right) \;,
\end{aligned} \tag{5.2-32b}
$$

$$
\begin{aligned}
16\pi^2 \overset{(n)}{\beta}_{\lambda_3} \; = \; & (\lambda_1 + \lambda_2)\,(3\,\lambda_3 + \lambda_4) + 4\,\lambda_3^2 + 2\,\lambda_4^2 + \frac{1}{2}\lambda_5^2 - 3\,\lambda_3\,(3g_2^2 + g_1^2) \\
& + \frac{9}{4}\,g_2^4 + \frac{3}{4}\,g_1^4 - \frac{3}{2}\,g_1^2\,g_2^2 \\
& + 4\,\lambda_3\,\mathrm{Tr}\left(Y_e^\dagger Y_e + \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\,Y_d^\dagger Y_d + 3\,Y_u^\dagger Y_u \right) \\
& - 4\;\mathrm{Tr}\left(z_\nu^{(2)}\,Y_e^\dagger Y_e\,\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\;\left(z_d^{(1)}\,z_u^{(2)} + z_d^{(2)}\,z_u^{(1)} \right)\;Y_d^\dagger Y_d\,Y_u^\dagger Y_u \right) \;,
\end{aligned} \tag{5.2-32c}
$$

$$
\begin{aligned}
16\pi^2 \overset{(n)}{\beta}_{\lambda_4} \; = \; & 2\,(\lambda_1 + \lambda_2)\,\lambda_4 + 4\,(2\,\lambda_3 + \lambda_4)\,\lambda_4 + 8\lambda_5^2 - 3\,\lambda_1\,(3g_2^2 + g_1^2) + 3\,g_1^2\,g_2^2 \\
& + 4\,\lambda_4\,\mathrm{Tr}\left(Y_e^\dagger Y_e + \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\,Y_d^\dagger Y_d + 3\,Y_u^\dagger Y_u \right) \\
& + 4\;\mathrm{Tr}\left(z_\nu^{(2)} Y_e^\dagger Y_e\,\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\;\left(z_d^{(1)}\,z_u^{(2)} + z_d^{(2)}\,z_u^{(1)} \right)\;Y_d^\dagger Y_d\,Y_u^\dagger Y_u \right) \;,
\end{aligned} \tag{5.2-32d}
$$

$$
\begin{aligned}
16\pi^2 \overset{(n)}{\beta}_{\lambda_5} \; = \; & \lambda_5\;\Big[\lambda_1 + \lambda_2 + 8\,\lambda_3 + 12\,\lambda_4 - 6(g_1^2 + 3\,g_2^2) \\
& \quad + 2\;\mathrm{Tr}\left(Y_e^\dagger Y_e + \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + 3\,Y_d^\dagger Y_d + 3\,Y_u^\dagger Y_u \right) \Big] \;,
\end{aligned} \tag{5.2-32e}
$$

where we use different conventions for the couplings λ_1, λ_2 and λ_5 compared to [31]. Note that the terms which are quartic in the Yukawa couplings do not correspond to two-loop diagrams but stem from box diagrams of the type of figure 5.7. The extended SM β-function for $\lambda \equiv \lambda_1$ is obtained by setting $\lambda_2 = \cdots = \lambda_5 = 0$ and $z_\psi^{(1)} = 1$ for $\psi \in \{d, u, \nu\}$.

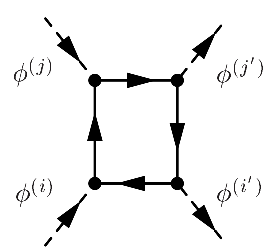


Figure 5.7: Typical one-loop box diagram leading to a correction for the Higgs self-interaction parameters which is quartic in the Yukawa couplings. The solid lines correspond to the fermions of the theory.

6 The β-Function Construction Kit

Supersymmetry is thought to play a role in solving many problems beyond the SM. For instance, GUT predictions for the unification of gauge couplings work best if the effects of relatively light supersymmetric particles are included [4–6]. Also, the hierarchy of mass scales, and particularly the fact that m_{EW} is much less than the scale at which gravity becomes important, appears to require relatively light supersymmetric particles, $M_{\mathrm{SUSY}} \lesssim 1\,\mathrm{TeV}$, for its stabilization.

As one can see from the gauge coupling evolution, the renormalization group has a great impact on the predictions for physical parameters because of the large hierarchy between M_{GUT}, which is predicted to be $\mathscr{O}(10^{16})\,\mathrm{GeV}$, and M_{SUSY}. Therefore it is interesting to study the renormalization group evolution of neutrino mass parameters in the MSSM. On the way of calculating the corresponding β-functions, we will develop a method, based on supergraph techniques, which allows for computing the RGE's for any, in particular even higher-dimensional operator of the superpotential with only little effort.

6.1 Preliminaries and Notation

Supersymmetry is an extension of the Poincaré symmetry of space-time. According to the Coleman-Mandula theorem [32], which holds under some very general assumptions, such an extension cannot be generated by bosonic operators. However, as is shown by Haag, Łopuszański and Sohnius [33], the Poincaré symmetry can be extended by fermionic generators $\{\boldsymbol{Q}^i\}_{i=1}^N$ and $\{\bar{\boldsymbol{Q}}^i\}_{i=1}^N$, which are referred to as SUSY generators. In this study, we consider only the case $N=1$.

6.1.1 Supernumbers and Superspace

Usually one introduces the **infinite dimensional Grassmann algebra** Λ_∞ whose generators $\{\zeta_i\}_{i=1}^\infty$ fulfill the anticommutation relations

$$\{\zeta_i,\zeta_j\} := \zeta_i\zeta_j + \zeta_j\zeta_i = 0 \qquad \forall\, i,j = 1,\dots\,. \tag{6.1-1}$$

Elements $z \in \Lambda_\infty$ are called **supernumbers**. They can be written as a sum

$$z = z_{\mathrm{B}} + z_{\mathrm{S}}\,, \tag{6.1-2}$$

where $z_{\mathrm{B}} \in \mathbb{C}$ and

$$z_{\mathrm{S}} = \sum_{n=1}^{\infty} \frac{1}{n!} z_{i_1 \cdots i_n} \zeta^{i_n} \ldots \zeta^{i_1} \qquad (z_{i_1 \cdots i_n} \in \mathbb{C}) \tag{6.1-3}$$

holds. Any supernumber z can be decomposed into an even and an odd part,

$$z = u + v \,, \tag{6.1-4a}$$

$$u = z_{\mathrm{B}} + \sum_{n=1}^{\infty} \frac{1}{(2n)!} z_{i_1 \cdots i_{2n}} \zeta^{i_{2n}} \ldots \zeta^{i_1} \,, \tag{6.1-4b}$$

$$v = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} z_{i_1 \cdots i_{2n+1}} \zeta^{i_{2n+1}} \ldots \zeta^{i_1} \,. \tag{6.1-4c}$$

Purely odd supernumbers are referred to as **a-numbers** and anticommute among each other. Purely even supernumbers are called **c-numbers** and commute with all supernumbers. The set of a-numbers $\mathbb{C}_{\mathrm{a}}$ is no sub-algebra of Λ_∞, whereas the set $\mathbb{C}_{\mathrm{c}}$ of c-numbers is.

For 4-dimensional supersymmetry, one usually introduces the subset $\mathbb{R}_c \subset \mathbb{C}_c$ of real supernumbers, and the superspace $\mathbb{S}$, which given by

$$\mathbb{S} = \left\{ z = (x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}});\ x^\mu \in \mathbb{R}_{\mathrm{c}} \wedge \theta_\alpha \in \mathbb{C}_{\mathrm{a}} \right\} \qquad (0 \leq \mu \leq 3,\ \alpha = 1, 2) \,. \tag{6.1-5}$$

The bar over the second θ means complex conjugation, the dot over the spinor index α is explained in appendix B.

6.1.2 Superspace Integration

We denote the integration over superspace coordinates by

$$\int \mathrm{d}^8 z := \int \mathrm{d}^4 x \int \mathrm{d}^4 \theta := \int \mathrm{d}^4 x \int \mathrm{d}^2 \theta \, \mathrm{d}^2 \bar{\theta} \,, \tag{6.1-6}$$

where

$$\mathrm{d}^2\theta = -\frac{1}{4} \mathrm{d}\theta^\alpha \, \mathrm{d}\theta^\beta \, \varepsilon_{\alpha\beta} = \frac{1}{4} \varepsilon^{\alpha\beta} \, \mathrm{d}\theta_\alpha \, \mathrm{d}\theta_\beta \,, \tag{6.1-7a}$$

$$\mathrm{d}^2\bar{\theta} = -\frac{1}{4} \mathrm{d}\bar{\theta}_{\dot{\alpha}} \, \mathrm{d}\bar{\theta}_{\dot{\beta}} \, \varepsilon^{\dot{\alpha}\dot{\beta}} = \frac{1}{4} \varepsilon_{\dot{\alpha}\dot{\beta}} \, \mathrm{d}\bar{\theta}^{\dot{\alpha}} \, \mathrm{d}\bar{\theta}^{\dot{\beta}} \,. \tag{6.1-7b}$$

The integrals over a-numbers are defined as

$$\int \mathrm{d}\theta_\alpha \, \theta^\beta = \delta_\alpha{}^\beta \qquad \text{and} \qquad \int \mathrm{d}\bar{\theta}^{\dot{\alpha}} \, \bar{\theta}_{\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{6.1-8}$$

so that

$$\int \mathrm{d}^2\theta \, \theta^2 = 1 \qquad \text{and} \qquad \int \mathrm{d}^2\bar{\theta} \, \bar{\theta}^2 = 1 \tag{6.1-9}$$

holds. As a consequence, we can define δ-functions with a-numbers as arguments,

$$\delta^2(\theta) = \theta^2 \,, \quad \delta^2(\bar\theta) = \bar\theta^2 \quad \text{and} \quad \delta^4(\theta) = \theta^2\,\bar\theta^2 \,. \tag{6.1-10}$$

They possess the standard properties

$$\int \mathrm{d}^2\theta\,\delta^2(\theta)\,f(\theta) = f(0)\,, \tag{6.1-11a}$$

$$\int \mathrm{d}^2\bar\theta\,\delta^2(\bar\theta)\,f(\bar\theta) = f(0)\,, \tag{6.1-11b}$$

$$\int \mathrm{d}^4\theta\,\delta^4(\theta)\,f(\theta,\bar\theta) = f(0)\,, \tag{6.1-11c}$$

$$\delta(\theta-\theta')\,f(\theta') = f(\theta)\,\delta(\theta-\theta')\,, \tag{6.1-11d}$$

as well as the unusual property

$$\delta(\theta)\,\delta(\theta) = 0\,. \tag{6.1-12}$$

6.1.3 Superfields

A **superfield** Φ, in the following denoted by a double-stroke letter, is a superanalytic mapping

$$\Phi :\ \mathbb{S} \to \mathbb{C}_c\,. \tag{6.1-13}$$

Due to the anticommuting properties of the θ coordinates, any superfield Φ can be expanded in so-called component fields,

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= C(x) + \sqrt{2}\,\theta\xi(x) + \sqrt{2}\,\bar\theta\bar\xi(x) + \theta\theta\,M(x) + \bar\theta\bar\theta\,M^*(x)\\ &\quad + \theta\sigma^\mu\bar\theta\,A_\mu(x) + \theta\theta\,\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\,\theta\lambda(x) + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\,D(x)\,,\end{aligned} \tag{6.1-14}$$

where C, M and D denote scalar fields, ξ and λ Weyl spinors and A a vector field. This expansion shows that the θ coordinates possess mass dimension $\dim[\theta] = -\frac{1}{2}$, and one can infer from equation (6.1-8) that $\mathrm{d}\theta_\alpha$ has the opposite mass dimension, $\dim[\mathrm{d}\theta_\alpha] = \frac{1}{2}$.

6.1.4 Vector-Superfields

A superfield $\mathbb{V}$ is called **vector-superfield** if it fulfills

$$\mathbb{V}^\dagger := \bar{\mathbb{V}} = \mathbb{V}\,. \tag{6.1-15}$$

In the component field expansion

$$\begin{aligned}\mathbb{V}(x,\theta,\bar\theta) &= C(x) + \sqrt{2}\theta\xi(x) + \sqrt{2}\bar\theta\bar\xi(x) + \theta\theta M(x) + \bar\theta\bar\theta M^*(x)\\ &\quad + \theta\sigma^\mu\bar\theta A_\mu(x) + \theta\theta\,\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\,\theta\lambda(x) + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta D(x)\,,\end{aligned} \tag{6.1-16}$$

C and D are real scalar fields, M is a complex scalar, A a real vector field, and ξ and λ complex spinors. Under an infinitesimal SUSY variation

$$\delta_{\text{SUSY}} = \mathrm{i}\left(\varepsilon^{\alpha}\boldsymbol{Q}_{\alpha} - \bar{\varepsilon}^{\dot{\alpha}}\bar{\boldsymbol{Q}}_{\dot{\alpha}}\right) , \tag{6.1-17}$$

the component fields of $\mathbb{V}$ behave as follows:

$$\begin{aligned}
\delta C &= \sqrt{2}\left(\varepsilon\xi + \bar{\varepsilon}\bar{\xi}\right) , && (6.1\text{-}18a)\\
\delta\xi_{\alpha} &= \sqrt{2}\,\varepsilon_{\alpha}M + \frac{1}{\sqrt{2}}\left(\sigma^{\mu}\bar{\varepsilon}\right)_{\alpha}\left(A_{\mu} - \mathrm{i}\partial_{\mu}C\right) , && (6.1\text{-}18b)\\
\delta M &= \bar{\varepsilon}\bar{\lambda} + \frac{\mathrm{i}}{\sqrt{2}}\partial_{\mu}\left(\xi\sigma^{\mu}\bar{\varepsilon}\right) , && (6.1\text{-}18c)\\
\delta A_{\mu} &= \varepsilon\sigma_{\mu}\bar{\lambda} + \lambda\sigma_{\mu}\bar{\varepsilon} - \frac{\mathrm{i}}{\sqrt{2}}\varepsilon\partial_{\mu}\xi + \frac{\mathrm{i}}{\sqrt{2}}\partial_{\mu}\bar{\xi}\bar{\varepsilon} && \\
&\quad + \sqrt{2}\varepsilon\sigma_{\mu\nu}\partial^{\nu}\xi - \sqrt{2}\bar{\varepsilon}\bar{\sigma}_{\mu\nu}\pi^{\nu}\bar{\xi} , && (6.1\text{-}18d)\\
\delta\bar{\lambda}^{\dot{\alpha}} &= \bar{\varepsilon}^{\dot{\alpha}}D - \frac{i}{2}\bar{\varepsilon}^{\dot{\alpha}}\partial_{\mu}A^{\mu} - \mathrm{i}\left(\bar{\sigma}^{\mu}\bar{\varepsilon}\right)^{\dot{\alpha}}\partial_{\mu}A_{\nu} , && (6.1\text{-}18e)\\
\delta D &= \mathrm{i}\partial_{\mu}\left(\lambda\sigma^{\mu}\bar{\varepsilon} + \bar{\lambda}\bar{\sigma}^{\mu}\varepsilon\right) . && (6.1\text{-}18f)
\end{aligned}$$

6.1.5 SUSY-Covariant Derivatives

The **SUSY-covariant derivative** is usually introduced by

$$\begin{aligned}
\boldsymbol{D}_{\alpha} &:= \partial_{\alpha} + \mathrm{i}\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu} , && (6.1\text{-}19a)\\
\bar{\boldsymbol{D}}_{\dot{\alpha}} &:= -\bar{\partial}_{\dot{\alpha}} - \mathrm{i}\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} , && (6.1\text{-}19b)
\end{aligned}$$

where ∂_{α} and $\bar{\partial}_{\dot{\alpha}}$ denote the derivative w.r.t. the a-number θ_{α} and $\bar{\theta}_{\dot{\alpha}}$, respectively. These derivatives obey the anticommutation rules

$$\begin{aligned}
\{\boldsymbol{D}_{\alpha}, \boldsymbol{D}_{\beta}\} &= \left\{\bar{\boldsymbol{D}}_{\dot{\alpha}}, \bar{\boldsymbol{D}}_{\dot{\beta}}\right\} = 0 , && (6.1\text{-}20a)\\
\left\{\boldsymbol{D}_{\alpha}, \bar{\boldsymbol{D}}_{\dot{\alpha}}\right\} &= 2\sigma^{\mu}_{\alpha\dot{\alpha}}\boldsymbol{P}_{\mu} . && (6.1\text{-}20b)
\end{aligned}$$

Furthermore, the following useful identities hold:

$$\begin{aligned}
\boldsymbol{D}^{\alpha}\boldsymbol{D}^{\beta} &= -\frac{1}{2}\varepsilon^{\alpha\beta}\boldsymbol{D}^{2} , && (6.1\text{-}21a)\\
\bar{\boldsymbol{D}}^{\dot{\alpha}}\bar{\boldsymbol{D}}^{\dot{\beta}} &= \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\boldsymbol{D}}^{2} , && (6.1\text{-}21b)\\
\boldsymbol{D}^{\alpha}\,\bar{\boldsymbol{D}}^{2}\boldsymbol{D}_{\alpha} &= \bar{\boldsymbol{D}}_{\dot{\alpha}}\boldsymbol{D}^{2}\,\bar{\boldsymbol{D}}^{\dot{\alpha}} , && (6.1\text{-}21c)\\
\bar{\boldsymbol{D}}^{2}\boldsymbol{D}^{2}\bar{\boldsymbol{D}}^{2} &= 16\,\Box\bar{\boldsymbol{D}}^{2} , && (6.1\text{-}21d)\\
\boldsymbol{D}^{2}\,\bar{\boldsymbol{D}}^{2}\boldsymbol{D}^{2} &= 16\,\Box\boldsymbol{D}^{2} . && (6.1\text{-}21e)
\end{aligned}$$

In Fourier space, we will have to specify the momentum,

$$\begin{aligned}
\boldsymbol{D}_{\alpha}(p) &:= \partial_{\alpha} + \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}p_{\mu} , && (6.1\text{-}22a)\\
\bar{\boldsymbol{D}}_{\dot{\alpha}}(p) &:= -\bar{\partial}_{\dot{\alpha}} - \theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}p_{\mu} . && (6.1\text{-}22b)
\end{aligned}$$

It is easily checked that the "transfer rule"

$$\delta^4(\theta-\theta')\,\overleftarrow{\boldsymbol{D}}'_\alpha(p) = -\overrightarrow{\boldsymbol{D}}_\alpha(-p)\,\delta^4(\theta-\theta') \tag{6.1-23}$$

holds; as a consequence, we find the useful relation

$$\delta^4(\theta-\theta')\,\overleftarrow{\boldsymbol{D}}'^2(p) = \overrightarrow{\boldsymbol{D}}^2(-p)\,\delta^4(\theta-\theta')\ . \tag{6.1-24}$$

Furthermore, by inserting $\delta^4(\theta-\theta') = (\theta-\theta')^2\,(\bar\theta-\bar\theta')^2$, we obtain the equation

$$\delta^4(\theta-\theta')\,\frac{1}{16}\boldsymbol{D}^2\,\bar{\boldsymbol{D}}^2\,\delta^4(\theta-\theta') = \delta^4(\theta-\theta')\ , \tag{6.1-25}$$

while

$$\delta(\theta-\theta')\times(\text{lower product of }\boldsymbol{D}'\text{s and }\bar{\boldsymbol{D}}'\text{s})\times\delta(\theta-\theta') = 0\ , \tag{6.1-26}$$

which will enable us to reduce $\boldsymbol{D}$'s and $\bar{\boldsymbol{D}}$'s in supergraph expressions.

6.1.6 Chiral Superfields

A superfield $\Phi_{\mathrm{L}} : \mathbb{S} \to \mathbb{C}_c$ or $\Phi_{\mathrm{R}} : \mathbb{S} \to \mathbb{C}_c$ is called **left-chiral** or **right-chiral**, if it fulfills

$$\bar{\boldsymbol{D}}_{\dot\alpha}\Phi_{\mathrm{L}}(x,\theta,\bar\theta) = 0 \quad\text{or}\quad \boldsymbol{D}_\alpha\Phi_{\mathrm{R}}(x,\theta,\bar\theta) = 0\ , \tag{6.1-27}$$

respectively.

Consider for example a left-chiral field. One usually changes coordinates according to

$$x^\mu \to y^\mu := x^\mu + \mathrm{i}\theta\sigma^\mu\bar\theta\ . \tag{6.1-28}$$

y^μ is chosen such that

$$\bar{\boldsymbol{D}}^{\dot\alpha}y^\mu = \left(-\bar\partial^{\dot\alpha} - \mathrm{i}\bar\sigma^{\mu\,\dot\alpha\beta}\theta_\beta\partial_\mu\right)\left(x^\mu + \mathrm{i}\theta\sigma^\mu\bar\theta\right) = 0$$

holds. In these new y-coordinates the covariant derivatives

$$\boldsymbol{D}^{(y)}_\alpha = \partial_\alpha + 2\mathrm{i}\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \quad\text{and}\quad \bar{\boldsymbol{D}}^{(y)}_{\dot\alpha} = \bar\partial_{\dot\alpha} \tag{6.1-29}$$

and the SUSY generators are given by

$$\boldsymbol{Q}^{(y)}_\alpha = -\mathrm{i}\partial_\alpha \quad\text{and}\quad \bar{\boldsymbol{Q}}^{(y)}_{\dot\alpha} = \mathrm{i}\bar\partial_{\dot\alpha} - 2\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial^{(y)}_\mu\ . \tag{6.1-30}$$

In particular, the left-chiral superfield in y-coordinates is independent of $\bar\theta$; therefore it can be expanded in powers of θ,

$$\Phi_{\mathrm{L}}(y,\theta) = \phi(y) + \sqrt{2}\,\theta\psi(y) + \theta\theta\,F(y)\ . \tag{6.1-31}$$

It can be shown that the equations of motion do not provide a derivative term for F. Therefore, this field is usually referred to as auxiliary field. We can go back to the original x-coordinates and find

$$\begin{aligned}\Phi_{\mathrm{L}}(x,\theta,\bar\theta) &= \phi(x) + \mathrm{i}\theta\sigma^\mu\bar\theta\,\partial_\mu\phi(x) + \frac{1}{4}\theta^2\,\bar\theta^2\,\Box\phi(x) \\ &\quad +\sqrt{2}\theta\psi(x) + \frac{\mathrm{i}}{\sqrt{2}}\,\theta^2\,\partial_\mu\psi(x)\sigma^\mu\bar\theta + \theta^2\,F(x)\ . \end{aligned} \tag{6.1-32}$$

One can show that the component fields transform under an infinitesimal SUSY transformation as

$$\delta\phi = \sqrt{2}\,\varepsilon^\alpha\psi_\alpha \tag{6.1-33a}$$

$$\delta\psi_\alpha = \sqrt{2}\varepsilon_\alpha F - \sqrt{2}\mathrm{i}\,(\sigma^\mu\bar\varepsilon)_\alpha\,\partial_\mu\phi \tag{6.1-33b}$$

$$\delta F = \sqrt{2}\mathrm{i}\,\partial_\mu(\psi\sigma^\mu\bar\varepsilon)\ . \tag{6.1-33c}$$

Since any left-chiral superfield can be written as

$$\Phi_{\mathrm{L}} = \bar{\boldsymbol{D}}^2\,\Psi \tag{6.1-34}$$

with a suitable, non-chiral superfield Ψ, we obtain by equation (6.1-21d)

$$\frac{1}{16}\,\Box^{-1}\,\bar{\boldsymbol{D}}^2\,\boldsymbol{D}^2\,\Phi_{\mathrm{L}} = \Phi_{\mathrm{L}} \tag{6.1-35}$$

With the aid of

$$\int \mathrm{d}^4x \int \mathrm{d}^4\theta = -\frac{1}{4}\int \mathrm{d}^4x \int \mathrm{d}^2\theta\,\bar{\boldsymbol{D}}^2\ , \tag{6.1-36}$$

we derive the useful identity

$$\int \mathrm{d}^4x \int \mathrm{d}^2\theta\, F\,G = -\frac{1}{4}\int \mathrm{d}^4x \int \mathrm{d}^4\theta\, F\,\Box^{-1}\,\boldsymbol{D}^2\,G\ , \tag{6.1-37}$$

where F and G are arbitrary chiral expressions.

In supersymmetric theories, matter fields are conventionally described by left-chiral superfields while gauge fields are contained in vector-superfields. From now on, we will denote-vector superfields by $\mathbb{V}$, $\mathbb{W}$ etc. All the chiral superfields will be assumed to be left-chiral, consequently the index "L" is redundant and will be omitted, i.e. in the following Φ will denote a left-chiral superfield.

6.1.7 Gauge Interactions

In SUSY, gauge transformations of a chiral field $\Phi^{(i)}$ in a given gauge group G are defined by

$$\Phi^{(i)} \to \sum_j \left[\exp\left(-2\mathrm{i}\,g\,\Lambda^A\mathsf{T}_A\right)\right]_{ij}\Phi^{(j)}\ , \tag{6.1-38a}$$

$$\bar\Phi^{(i)} \to \sum_j \left[\exp\left(2\mathrm{i}\,g\,\bar\Lambda^A\mathsf{T}_A\right)\right]_{ij}\bar\Phi^{(j)}\ , \tag{6.1-38b}$$

where $\{\mathbb{\Lambda}^A\}_{A=1}^{\dim G}$ are chiral superfields. For U(1) gauge theories, T has to be replaced by the charge q. The usual "kinetic" term $\sum_i \bar{\Phi}^{(i)}\Phi^{(i)}$ is generalized to

$$\sum_{i,j} \bar{\Phi}^{(i)} \, [\exp(2\,g\mathbb{V})]_{ij}] \; \Phi^{(j)} \tag{6.1-39}$$

with $\mathbb{V} := \mathbb{V}^A\mathsf{T}_A$ and $\mathbb{V}^A := \mathrm{i}\left(\bar{\mathbb{\Lambda}}^A - \mathbb{\Lambda}^A\right)$. By construction, $\mathbb{V}^A$ are vector-superfields. In order to guarantee the invariance of (6.1-39) under (6.1-38), $\mathbb{V}$ must transform as

$$\mathbb{V} \to \mathbb{V}' = \mathbb{V} + \mathrm{i}\left(\mathbb{\Lambda} - \bar{\mathbb{\Lambda}}\right) \; . \tag{6.1-40}$$

With the definitions of the field strength superfields

$$\mathbb{W}_\alpha \;=\; -\frac{1}{4}\bar{\boldsymbol{D}}\bar{\boldsymbol{D}} \, \exp(-\mathbb{V}) \, \boldsymbol{D}_\alpha \exp(\mathbb{V}) \quad \text{and} \tag{6.1-41a}$$

$$\bar{\mathbb{W}}_{\dot\alpha} \;=\; -\frac{1}{4}\boldsymbol{D}\boldsymbol{D} \, \exp(-\mathbb{V})\bar{\boldsymbol{D}}_{\dot\alpha} \exp(\mathbb{V}) \; , \tag{6.1-41b}$$

the gauge kinetic Lagrangian reads

$$\mathscr{L}_{\mathrm{Gauge}} = \frac{1}{4}\left(\int \mathrm{d}^2\theta \, \mathbb{W}^\alpha \mathbb{W}_\alpha + \int \mathrm{d}^2\bar\theta \, \bar{\mathbb{W}}_{\dot\alpha}\bar{\mathbb{W}}^{\dot\alpha}\right) + \mathscr{L}_{\mathrm{Gauge-Fixing}} + \mathscr{L}_{\mathrm{Ghost}} \; , \tag{6.1-42}$$

where the gauge-fixing Lagrangian is given by

$$\mathscr{L}_{\mathrm{Gauge-Fixing}} = -\frac{1}{\alpha} \, \mathrm{Tr} \int \mathrm{d}^4\theta \, \left(\boldsymbol{D}^2\mathbb{V}\right) \, \left(\bar{\boldsymbol{D}}^2\mathbb{V}\right) \; . \tag{6.1-43}$$

The gauge $\alpha=1$ is called **Fermi-Feynman gauge** and has the advantage that the propagator for a massless $\mathbb{V}$ is given by $1/p^2$, if it is fixed. Therefore we will use it in this thesis. Furthermore, the ghost Lagrangian $\mathscr{L}_{\mathrm{Ghost}}$ has to be included. However, since its explicit form is not needed in the following, it is not specified here.

6.1.8 Supersymmetric Lagrangians

In a supersymmetric theory, the gauge-invariant kinetic term is contained in

$$S_{\mathrm{Gauge-Matter}} = \sum_{i,j} \int \mathrm{d}^8 z \, \bar{\Phi}^{(i)} \, \left[\exp\left(2g\mathbb{V}\right)\right]_{ij} \; \Phi^{(j)} \; . \tag{6.1-44}$$

Yukawa-type interactions and mass terms are described by the superpotential, which is by definition an analytic function of the left-chiral fields. The requirement of renormalizability forbids couplings involving more than three powers of chiral superfields.

Consider now a general superpotential $\mathscr{W}(\Phi^{(1)}, \ldots \Phi^{(n)})$, where the superfields $\Phi^{(i)}$ have the following expansion in component fields:

$$\Phi^{(i)}(y,\theta) = \phi^{(i)}(y) + \sqrt{2}\,\theta\psi^{(i)}(y) + \theta\theta\, F^{(i)}(y) \; . \tag{6.1-45}$$

Since the superpotential is an analytic function of the chiral superfields, it is itself a chiral field. Besides, equation (6.1-33) shows that under SUSY variations the F component of any chiral field remains unchanged up to a total derivative. Hence the F component is a potential candidate for a SUSY Lagrangian. In particular, the F component of the superpotential is referred to as F-**term**.

In order to extract component field vertices out of the superpotential, it is useful to expand it in the fermionic coordinates, which yields the result

$$\begin{aligned} -\mathscr{L}_F &= \mathscr{W}(\Phi^{(1)},\dots\Phi^{(n)})\big|_{\theta\theta} + \text{h.c.} \\ &= -\left(\frac{1}{2}\sum_{i,j}\left.\frac{\partial^2\mathscr{W}}{\partial\Phi^{(i)}\,\partial\Phi^{(j)}}\right|_{\theta=0}\psi^{(i)}\psi^{(j)} - \sum_i \left.\frac{\partial\mathscr{W}}{\partial\Phi^{(i)}}\right|_{\theta=0} F^{(i)} + \text{h.c.}\right) . \end{aligned} \tag{6.1-46}$$

The same considerations we have made for the F component of a chiral expression apply to the D component of a vector-superfield, see equation (6.1-18f). Analogously, the $\theta\theta\,\bar\theta\bar\theta$ projection of a superfield expression $\mathscr{K}(\{\Phi^{(i)}\},\{\bar\Phi^{(i)}\})$ which fulfills $\mathscr{K}^\dagger = \mathscr{K}$, serves as a potential SUSY Lagrangian and is referred to as D-**term**. In general, it can be calculated via

$$\begin{aligned} \mathscr{L}_D &= \mathscr{K}\left(\{\Phi^{(i)}\},\{\bar\Phi^{(i)}\}\right)\big|_{\theta\theta\,\bar\theta\bar\theta} = \\ &= -\mathscr{G}_{k\ell}\left[\psi^{(\ell)}\sigma^\mu\partial_\mu\psi^{(k)} + F^{(k)}F^{(\ell)*} - \partial_\mu\phi^{(k)}\,\partial^\mu\phi^{(\ell)*} + \text{h.c.}\right] \\ &\quad -\left\{\left.\frac{\partial^3\mathscr{K}}{\partial\Phi^{(k)}\,\partial\Phi^{(\ell)}\,\partial\bar\Phi^{(m)}}\right|_{\theta=0}\left[\psi^{(k)}\psi^{(\ell)}\,F^{(\ell)*} + \left(\psi^{(\ell)}\sigma^\mu\bar\psi^{(m)}\right)\,\partial_\mu\phi^{(\ell)}\right] + \text{h.c.}\right\} \\ &\quad +\frac{1}{4}\left[\left.\frac{\partial^4\mathscr{K}}{\partial\Phi^{(k)}\,\partial\Phi^{(\ell)}\,\partial\bar\Phi^{(m)}\,\partial\bar\Phi^{(n)}}\right|_{\theta=0}\left(\psi^{(k)}\psi^{(\ell)}\right)\,\left(\bar\psi^{(m)}\bar\psi^{(n)}\right) + \text{h.c.}\right] , \end{aligned} \tag{6.1-47}$$

where we have introduced the so-called **Kähler-metric**, defined by

$$\mathscr{G}_{k\ell} = \left.\frac{\partial^2\mathscr{K}}{\partial\Phi^{(k)}\,\partial\bar\Phi^{(\ell)}}\right|_{\theta=0} . \tag{6.1-48}$$

The on-shell Lagrangian can be obtained by inserting the (algebraic) equations of motion for the auxiliary fields $F^{(i)}$. In particular, for a super-Yang-Mills action consisting of (6.1-42), (6.1-44) and a general superpotential $\mathscr{W}$, one derives the following F-term contribution to the on-shell Lagrangian:

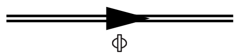
$$\mathscr{L}_F = -\frac{1}{2}\left(\sum_{i,j}\left.\frac{\partial^2\mathscr{W}}{\partial\Phi^{(i)}\,\partial\Phi^{(j)}}\right|_{\theta=0}\psi^{(i)}\psi^{(j)} + \text{h.c.}\right) - \sum_i\left|\left.\frac{\partial\mathscr{W}}{\partial\Phi^{(i)}}\right|_{\theta=0}\right|^2 . \tag{6.1-49}$$

6.2 Supergraphs

The supergraph method was invented in 1975 [34–36] and improved in 1979 [37]. It allows to represent expressions involving superfields by Feynman diagrams. Furthermore, its use makes it possible to keep the non-renormalization theorem manifest. Moreover, it has the advantage that the number of independent diagrams is clearly reduced compared to the component field calculations.

6.2.1 Supergraph Rules

We will signify chiral superfields in Feynman diagrams by straight double lines while vector-superfields are indicated by wiggly double lines,

Φ : 

and $\mathbb{V}$: 

.

The generating functional for chiral superfields is given by [38]

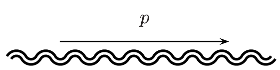
$$\mathcal{Z}\left[\mathbb{J},\bar{\mathbb{J}}\right]=\exp\left\{-\frac{\mathrm{i}}{2}\int \mathrm{d}^8 z\,\mathrm{d}^8 z'\left(\mathbb{J}(z),\bar{\mathbb{J}}(z)\right)\Delta_{\mathrm{GRS}}(z,z')\begin{pmatrix}\mathbb{J}(z')\\ \bar{\mathbb{J}}(z')\end{pmatrix}\right\}\,, \tag{6.2-1}$$

where Δ_{GRS} is the superfield propagator introduced by Grisaru, Roček and Siegel [37],

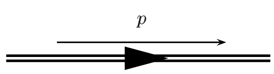
$$\Delta_{\mathrm{GRS}}(z,z')=\begin{pmatrix}-\frac{m}{4}\frac{\boldsymbol{D}^2}{\Box} & 1\\ 1 & -\frac{m}{4}\frac{\bar{\boldsymbol{D}}^2}{\Box}\end{pmatrix}\delta(z-z')\,. \tag{6.2-2}$$

Therefore, for each chiral superfield there exist three propagators, namely $\langle\bar{\Phi}\Phi\rangle$, $\langle\Phi\Phi\rangle$ and $\langle\bar{\Phi}\bar{\Phi}\rangle$. We use the convention that the arrow always points from $\bar{\Phi}$ to Φ.

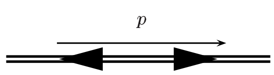
Supergraph rule 1 (Propagator Rules).

$\mathbb{V}(p,\theta)$ 

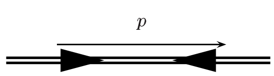
$\mathbb{V}(p,\theta')$: $-\frac{\mathrm{i}}{p^2}\delta^4(\theta-\theta')$ (6.2-3a)

$\bar{\Phi}(p,\theta)$ 

$\Phi(p,\theta')$: $\frac{\mathrm{i}}{p^2-m^2}\delta^4(\theta-\theta')$ (6.2-3b)

$\Phi(p,\theta)$ 

$\Phi(p,\theta')$: $\frac{\mathrm{i}m}{p^2(p^2-m^2)}\left(\frac{1}{4}\boldsymbol{D}^2(p)\right)\delta^4(\theta-\theta')$ (6.2-3c)

$\bar{\Phi}(p,\theta)$ 

$\bar{\Phi}(p,\theta')$: $\frac{\mathrm{i}m}{p^2(p^2-m^2)}\left(\frac{1}{4}\bar{\boldsymbol{D}}^2(p)\right)\delta^4(\theta-\theta')$ (6.2-3d)

For the first propagator, Fermi-Feynman gauge was applied.

Furthermore, for the vertices there arise products of SUSY-covariant derivatives:

Supergraph rule 2 (Vertex Rules).

- ✓ *Integrate each vertex over* $\mathrm{d}^4\theta$.
- ✓ *Integrate over loop momenta* $\int\frac{\mathrm{d}^4 k}{(2\pi)^4}$.
- ✓ *For each (anti-)chiral vertex with* n *internal lines let* $\bar{\boldsymbol{D}}^2(q_i)$ $(\boldsymbol{D}^2(q_i))$ *act on* $n-1$ *propagators associated with the internal lines.* q_i *is the momentum of flowing along the internal line away from the vertex.*

✓ *For each gauge vertex with n internal chiral lines let $\boldsymbol{D}^2(q_i)$ ($\bar{\boldsymbol{D}}^2(q_i)$) act on n propagators associated with the internal chiral lines if the chiral line points away from (towards) the vertex. q_i is again the momentum flowing along the internal line away from the vertex.*

For the analog of wavefunction renormalization we compute the effective action. It contains the kinetic term for the component fields in

$$\int \mathrm{d}^4p \int \mathrm{d}^2\theta' \, \mathrm{d}^2\bar{\theta}' \, \bar{\Phi}(-p,\theta) \, \Phi(p,\theta') \, \delta^4(\theta-\theta') \; .$$

Corrections to this can be written as

$$\int \mathrm{d}^4p \int \mathrm{d}^2\theta' \, \mathrm{d}^2\bar{\theta}' \, \bar{\Phi}(-p,\theta) \, \delta Z_\Phi \, \Phi(p,\theta') \, \delta^4(\theta-\theta') \; ,$$

thus leading to a Z_Φ factor.

Supergraph rule 3 (Effective Action). *In order to obtain the effective action, for each external (anti-)chiral leg multiply with a factor*

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4} \, \Phi(p,\theta) \quad \textit{or} \quad \int \frac{\mathrm{d}^4p}{(2\pi)^4} \, \bar{\Phi}(p,\theta) \; ,$$

respectively. External vector superfields lead to

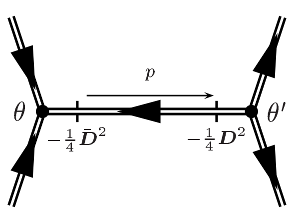
$$\int \frac{\mathrm{d}^4p}{(2\pi)^4} \, \mathbb{V}(p,\theta) \; .$$

p always denotes the outgoing momentum. Furthermore, multiply the whole expression with

$$(2\pi)^4 \, \delta\left(\sum p_{\mathrm{in}} - \sum p_{\mathrm{out}}\right) \; ,$$

where the sums are over incoming and outgoing momenta.

The covariant derivatives acting on the vertices are also denoted in the diagrams. This can be illustrated by an example:



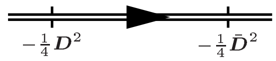
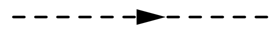
$$\sim \delta^4(\theta-\theta') \left(-\frac{1}{4}\boldsymbol{D}'^2\right) \frac{\delta^4(\theta-\theta')}{p^2-m^2} \left(-\frac{1}{4}\overleftarrow{\bar{\boldsymbol{D}}}^2(-p)\right) \; ,$$

where the arrow over the $\bar{\boldsymbol{D}}^2$ indicates that the derivative acts onto the expression left of it. The prime signifies that the corresponding $\boldsymbol{D}'$ is a derivative w.r.t. θ'. Using equation (6.1-24), we can simplify the expression to

$$\delta^4(\theta-\theta') \, \frac{1}{16} \, \boldsymbol{D}'^2 \, \bar{\boldsymbol{D}}^2(p) \, \frac{\delta^4(\theta-\theta')}{p^2-m^2} \overset{(6.1\text{-}25)}{=} \frac{\delta^4(\theta-\theta')}{p^2-m^2} \; .$$

Thus, whenever a chiral line with both $\boldsymbol{D}^2$ and $\bar{\boldsymbol{D}}^2$ acting on it appears in conjunction with $\delta^4(\theta-\theta')$, we can simply replace it by the ordinary Feynman propagator for scalar fields, $\mathrm{i}/(p^2-m^2)$.

Supergraph rule 4 ($\boldsymbol{D}$-Algebra). *Simplify the diagrams using*

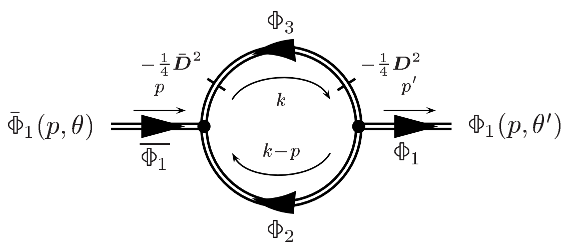
$\delta^4(\theta-\theta')\times$  $\to \delta^4(\theta-\theta')\times$  . (6.2-4)

6.2.2 Sample Calculations

Let us repeat the steps leading to supergraph rule 4 in an explicit example first. Consider a general Yukawa-type superpotential

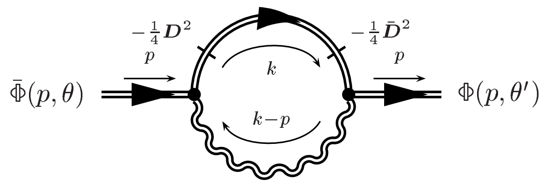
$$\mathscr{W}_{\text{Yukawa}} = y\,\Phi_1\Phi_3\Phi_2\ , \tag{6.2-5}$$

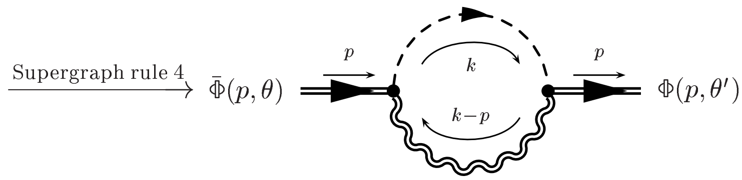
where each of the superfields transforms under some gauge groups. In this case, the Feynman rules (D.1) are easily derived. There is a non-vanishing supergraph for $\langle\bar{\Phi}_1\Phi_1\rangle$, namely



$$\begin{aligned}
&= (\mathrm{i}y)\,(\mathrm{i}y^*)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\bar{\Phi}(-p,\theta)\Phi(p',\theta)\,(2\pi)^4\,\delta^4(p-p')\times\\
&\quad\times\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left(-\frac{1}{4}\bar{\boldsymbol{D}}^2(k)\right)\frac{\mathrm{i}\delta^4(\theta-\theta')}{k^2}\left(-\frac{1}{4}\overleftarrow{\boldsymbol{D}}'^2(-k)\right)\frac{\mathrm{i}\delta^4(\theta-\theta')}{(k-p)^2}\ .\\
&\overset{(6.1\text{-}24)}{=} y^*y\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\bar{\Phi}(-p,\theta)\Phi(p,\theta)\\
&\quad\times\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{16}\bar{\boldsymbol{D}}^2(k)\,\boldsymbol{D}^2(k)\,\delta^4(\theta-\theta')\,\frac{\delta^4(\theta-\theta')}{k^2(k-p)^2}\\
&\overset{(6.1\text{-}25)}{=} y^*y\int\mathrm{d}^4p\,\bar{\Phi}(-p,\theta)\Phi(p,\theta)\,\delta^4(\theta-\theta')\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2(k-p)^2}\\
&= \int\mathrm{d}^4p\,\frac{\mathrm{i}}{16\pi^2}\,y^*y\,B_0(p^2,0,m^2)\int\mathrm{d}^4\theta'\,\bar{\Phi}(-p,\theta)\,\Phi(p,\theta')\delta^4(\theta-\theta')\ .
\end{aligned} \tag{6.2-6}$$

Now we use supergraph rule 4 in order to calculate gauge contributions to the $\langle\bar{\Phi}\Phi\rangle$ propagator. Consider a chiral superfield Φ which transforms under a representation R of a gauge group G. The wavefunction renormalization of a given field due to gauge interactions at one-loop is represented by the following diagram:





$$= \; -\int \mathrm{d}^4 p \, \frac{\mathrm{i}}{16\pi^2} \, 4g^2 \, c_2(R) \, B_0(p^2, 0, m^2) \int \mathrm{d}^4\theta' \, \bar{\Phi}(-p,\theta) \, \Phi(p,\theta') \, \delta^4(\theta-\theta') \; , \quad (6.2\text{-}7)$$

where $c_2(R)$ denotes the quadratic Casimir invariant of the irrep R, as explained in appendix A, and g the gauge coupling.

6.3 Renormalization of Supersymmetric Theories

6.3.1 Dimensional Reduction

It is well known that dimensional regularization, as described in section 2.2.1, violates SUSY explicitly since it introduces a mismatch between the number of gauge bosons and gaugino degrees of freedom. Hence we are obliged to choose a modified regularization prescription. Usually Dimensional Regularization via Dimensional Reduction (DRED) [39,40] is used. The main differences compared to dimensional regularization are:

(1) Only continuation to lower dimensions of space-time is allowed.

(2) The ranges of all Lorentz indices are kept the same, as if they were internal symmetry indices.

As a consequence, an N-extended supersymmetry in d dimensions must be reinterpreted as an N'-extended supersymmetry in $d' < d$ dimensions, where $N' > N$.

6.3.2 The Non-Renormalization Theorem

The Non-Renormalization-Theorem [41,42] is one of the major "miracles" which occur in conjunction with a supersymmetric theory. Consider a Lagrangian of N_Φ superfields,

$$\begin{aligned} \mathscr{L}_\mathrm{B} \;=\; & \left[\bar{\Phi}_\mathrm{B}^{(i)} \, \exp(-2\, g \cdot \mathbb{V}_\mathrm{B})_{ij} \, \Phi_\mathrm{B}^{(j)} \right]_{\theta\theta\,\bar{\theta}\bar{\theta}} + \left[\mathscr{W}\left(\{\Phi_\mathrm{B}^{(i)}\}\right) \Big|_{\theta\theta} + \mathrm{h.c.} \right] \\ & + \left[\sum_{n=1}^{S} \varepsilon_{\alpha\beta} \, (\mathbb{W}_\mathrm{B})_A^{n\,\alpha} \, (\mathbb{W}_\mathrm{B})^{n\,A\,\beta} \Big|_{\theta\theta} + \mathrm{h.c.} \right] \; , \qquad (6.3\text{-}1) \end{aligned}$$

where the superpotential can be written as

$$\mathscr{W} = \left\{ \sum_A \left[\prod_{i\in I} \left(\phi^{(i)}\right)^{n_i^A} \right] V_A \left[\prod_{j\in j} \left(\phi^{(j)}\right)^{n_j^A} \right] \right\} . \tag{6.3-2}$$

We have introduced the abbreviations

$$\mathbb{W}_\alpha^n = \frac{1}{8g_n} \overline{\boldsymbol{D}}^2 \left[\exp(2g_n \mathbb{V}^n)\, \boldsymbol{D}_\alpha \exp(-2g_n \mathbb{V}^n)\right] , \tag{6.3-3a}$$

$$g \cdot \mathbb{V} := \sum_{n=1}^{S} g_n \mathbb{V}^n \quad \text{and} \quad \mathbb{V}^n = \sum_{A=1}^{\dim G_n} \mathbb{V}_n^A \mathsf{T}_n^A , \tag{6.3-3b}$$

which also hold analogously for the bare fields. Then the non-renormalization theorem states the following: The counterterm for the operators of the superpotential can be chosen to vanish,

$$\delta V_A = 0 \quad \forall A . \tag{6.3-4}$$

In other words, only wavefunction renormalization constants have to be taken into account. It can be shown that this remarkable theorem also holds in non-renormalizable theories [13]. In particular, if only the superpotential contains higher-dimensional, non-renormalizable operators, the theorem applies as well. This means that one does not need any counterterms for the operators of the superpotential.

6.3.3 Wavefunction Renormalization Constants

The wavefunction renormalization constants of the matter superfields are defined as in the non-supersymmetric case,

$$Z_{ij} = \mathbb{1}_{ij} + \delta Z_{ij} , \tag{6.3-5}$$

i.e. they relate the bare $\phi_{\mathrm{B}}^{(i)}$ and the renormalized superfield, $\phi^{(i)}$, via

$$\phi_{\mathrm{B}}^{(i)} = \sum_{j=1}^{N_\phi} Z_{ij}^{\frac{1}{2}} \phi^{(j)} . \tag{6.3-6}$$

The renormalizable part of the superpotential is assumed to be

$$\mathscr{W}_{\mathrm{ren}} = \frac{1}{2} m_{(ij)} \, \phi^{(i)} \phi^{(j)} + \frac{1}{6} \lambda_{(ijk)} \, \phi^{(i)} \phi^{(j)} \phi^{(k)} . \tag{6.3-7}$$

The brackets indicate symmetrization of the indices of m and λ. Mass terms in equation (6.3-7) are ignored for the calculations as they do not affect the β-functions of the model. In addition to the terms of equation (6.3-7), higher dimensional operators may appear in the superpotential of an effective theory. These operators are generally suppressed by inverse powers of a large mass scale Λ. Although these operators are non-renormalizable by power counting, in the effective field theory approach one can renormalize the theory in an expansion in inverse powers of Λ. In the leading order of this expansion, the higher dimensional operators do not contribute to the wavefunction renormalization.

Using equations (6.2-6) and (6.2-7) as well as DRED, we calculate at the one-loop level

$$-\delta Z_{ij}^{(1)} = \frac{1}{(4\pi)^2}\frac{1}{\epsilon}\left[\sum_{k,\ell=1}^{N_\Phi} \lambda_{ik\ell}^{*}\lambda_{jk\ell} - 4\sum_{n=1}^{S} g_n^2\, c_2\left(R_n^{(i)}\right)\,\delta_{ij}\right] , \tag{6.3-8}$$

while at two-loop δZ_{ij} is given by [43]

$$\begin{aligned} -\delta Z_{ij}^{(2)} \;=\; \frac{-2+\epsilon}{(4\pi)^4\,\epsilon^2} \Bigg\{ & 4 \sum_{n,m=1}^{S} g_n^2\, c_2\big(R_n^{(i)}\big)\; g_m^2\, c_2\big(R_m^{(j)}\big)\;\delta_{ij} \\ & + 2\sum_{n=1}^{S} g_n^4\, c_2\big(R_n^{(i)}\big)\;\left(\overline{c}_{2,n} - 3\,c_1(G_n)\right)\,\delta_{ij} \\ & + \sum_{n=1}^{S}\sum_{k,\ell=1}^{N_\Phi} g_n^2\left[-\,c_2\big(R_n^{(i)}\big) + 2\,c_2\big(R_n^{(\ell)}\big)\right]\;\lambda_{ik\ell}^{*}\lambda_{jk\ell} \\ & - \frac{1}{2}\sum_{k,\ell,r,s,t=1}^{N_\Phi} \lambda_{ik\ell}^{*}\lambda_{\ell st}\,\lambda_{rst}^{*}\lambda_{jkr} \Bigg\} . \end{aligned} \tag{6.3-9}$$

Using the group-theoretical notation of appendix A, we introduced

$$\overline{c}_{2,n} := \sum_{i=1}^{N_\Phi} \ell\left(R_n^{(i)}\right)\cdot\prod_{m\neq n}\dim\left(R_m^{(i)}\right) , \tag{6.3-10}$$

where multiplication with (generalized) color-factors, as explained in the following, is implied. The diagrams leading to equation (6.3-9) are shown in figure 6.1.

6.3.4 Remarks on Generalized Color-Factors

In general, one demands that the trilinear terms in the superpotential,

$$\mathscr{W}_{\text{Yukawa}} = \sum_{i,j,k} \lambda_{(ijk)}\,\Phi^{(i)}\,\Phi^{(i)}\,\Phi^{(i)} \tag{6.3-11}$$

are singlets under gauge transformations. In the following, we will stick to couplings in which one field transforms only as singlet whereas the others transform opposite to each other, for this is the only configuration we will encounter in praxis. For instance in a SU(N) gauge theory, a typical trilinear term looks like

$$\mathbf{1}\cdot\boldsymbol{N}\cdot\overline{\boldsymbol{N}}\,. \tag{6.3-12}$$

In an one-loop diagram, as shown in figure 6.2(a), either the incoming and outgoing fields transform as singlets, in which case a color-factor has to be respected, or the incoming

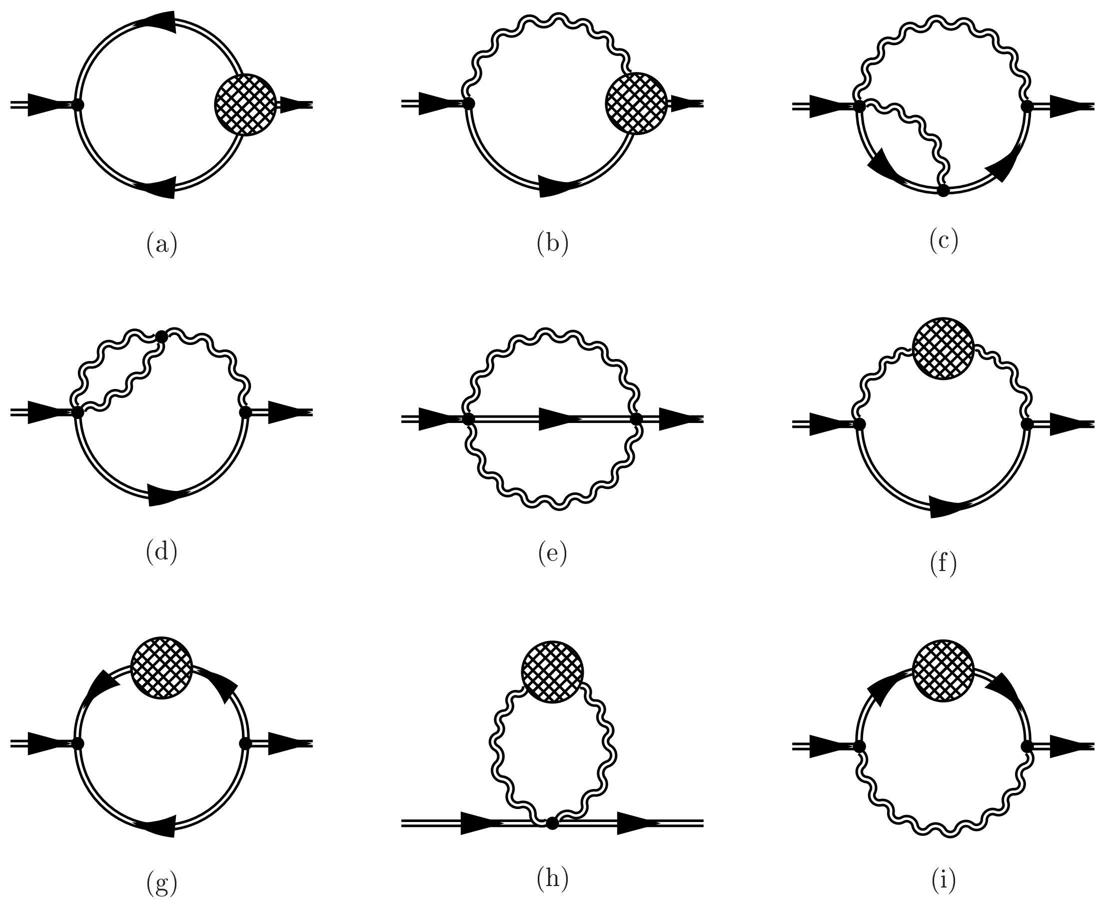


Figure 6.1: Two-loop supergraphs which contribute to the $\overline{\Phi}\Phi$ propagator. A blob denotes the relevant one-particle irreducible graph including any one-loop counterterm that may be required [43].

field transforms non-trivially, in which case there is no color-factor. Thus, we define the color-factor for a group factor as a function of the "internal" indices k and ℓ by

$$F_n(k,\ell) := \begin{cases} \dim R_n^{(k)} \;, & \dim R_n^{(k)} = \dim R_n^{(\ell)} \;, \\ 1 \;, & \text{otherwise} \end{cases} \tag{6.3-13}$$

and the color-factor for the whole gauge group

$$F(k,\ell) := \prod_{n=1}^{S} F_n(k,\ell) \;. \tag{6.3-14}$$

With these color-factors, equation (6.3-8) reads

$$\delta Z_{ij}^{(1)} = \frac{1}{(4\pi)^2}\frac{1}{\epsilon}\left[\sum_{k,\ell=1}^{N_\Phi} F(k,\ell)\,\lambda^*_{ik\ell}\lambda_{jk\ell} - 4\sum_{n=1}^{S} g_n^2\, c_2\left(R_n^{(i)}\right)\,\delta_{ij}\right] \;. \tag{6.3-15}$$

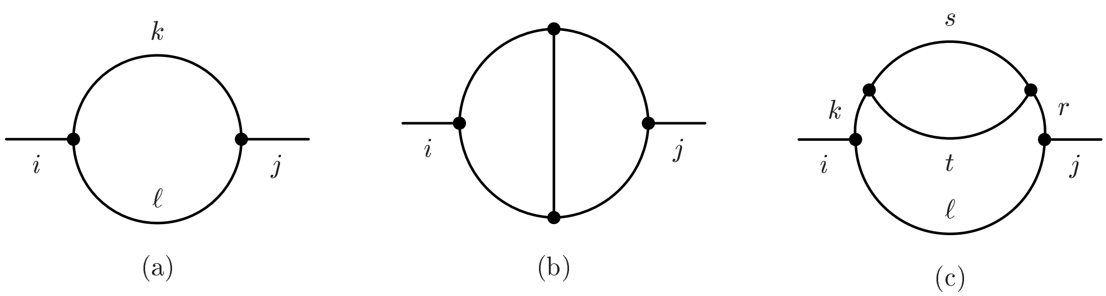


Figure 6.2: Topologies of the diagrams contributing to the one- and two-loop wavefunction renormalization in a theory with only trilinear couplings.

For the $\lambda^*\lambda\lambda^*\lambda$-term in principle we should take into account two topologies. However, the topology of figure 6.2(b) does not contribute to this term since it cannot be realized with massless chiral superfields. For the topology of figure 6.2(c), we only need to include the product of two color-factors so that the modified equation (6.3-9) reads

$$\begin{aligned}\delta Z^{(2)}_{ij} \;=\; \frac{-2+\epsilon}{(4\pi)^4\,\epsilon^2}\Bigg\{ & 4\sum_{n,m=1}^{S} g_n^2\, c_2\big(R_n^{(i)}\big)\; g_m^2\, c_2\big(R_m^{(j)}\big)\;\delta_{ij} \\ & + 2\sum_{n=1}^{S} g_n^4\, c_2\big(R_n^{(i)}\big)\;\big[\overline{\ell}_n - 3\,c_1(G_n)\big]\;\delta_{ij} \\ & + \sum_{n=1}^{S}\sum_{k,\ell=1}^{N_\Phi} F(k,\ell)\, g_n^2\;\big[-c_2\big(R_n^{(i)}\big) + 2\,c_2\big(R_n^{(\ell)}\big)\big]\;\lambda^*_{ik\ell}\lambda_{jk\ell} \\ & - \frac{1}{2}\sum_{k,\ell,r,s,t=1}^{N_\Phi} F(k,\ell)\,F(s,t)\,\lambda^*_{ik\ell}\lambda_{\ell st}\,\lambda^*_{rst}\lambda_{jkr}\Bigg\}\;. \end{aligned} \tag{6.3-16}$$

$\overline{\ell}_n$ equals $\overline{c}_{2,n}$ up to generation factors that have to be respected.

6.3.5 Calculating β-Functions

Consider a term of a general superpotential

$$\left[\prod_{i\in I}\big(\Phi^{(i)}\big)^{n_i}\right]\;Q\;\left[\prod_{j\in J}\big(\Phi^{(j)}\big)^{n_j}\right]\;, \tag{6.3-17}$$

where $I=\{1,\ldots M\}$ and $J=\{M+1,\ldots N\}$. Due to the non-renormalization theorem, the following relation between a bare quantity, Q_{B}, and the renormalized one, Q, holds:

$$Q_{\mathrm{B}} = \left(\prod_{i\in I} Z^{n_i}_{\Phi^{(i)}}\right)\;Q\;\left(\prod_{j\in J} Z^{n_j}_{\Phi^{(j)}}\right)\;. \tag{6.3-18}$$

Using DRED instead of dimensional regularization and following the same steps as in section 2.3, we find

$$\begin{aligned}\beta_Q(\{V_A\}) &= Q \cdot \sum_{j\in J} n_j \left[\sum_A D_{V_A} \left\langle \frac{\mathrm{d}Z_{\phi_j,1}}{\mathrm{d}V_A}\middle| V_A \right\rangle\right] \\ &\quad + \sum_{i\in I} n_i \left[\sum_A D_{V_A} \left\langle \frac{\mathrm{d}Z_{\phi_i,1}}{\mathrm{d}V_A}\middle| V_A \right\rangle\right] \cdot Q \,, \end{aligned} \tag{6.3-19}$$

where again $\{V_A\}$ denotes the set of all variables of the theory, including the one under consideration, Q. This relation in conjunction with the formulae for δZ, equations (6.3-15) and (6.3-16), is an important result, since it makes it possible to calculate β-functions in general $N=1$ supersymmetric theories with only little effort. It might therefore be of great interest for supersymmetric model building.

6.4 The νMSSM

The Minimal Supersymmetric Standard Model extended by neutrinos (νMSSM) contains the fields of the Minimal Supersymmetric Standard Model (MSSM) and additionally the singlet neutrino superfield which we will denote by ν^{C}. This minimal extension is particularly interesting since adding gauge singlets does not spoil the unification of the gauge couplings in the MSSM. With this extension, both Dirac masses and see-saw suppressed Majorana masses for the neutrinos are possible.

6.4.1 Field Content

We enumerate the νMSSM superfields as shown in table 6.4-1, in which the used symbols as well as the quantum numbers are specified. Note that we use GUT charge normalization for the $\mathrm{U}(1)_{\mathrm{Y}}$ charge.

Field	1	2	3	4	5	6	7	8
Symbol	$\phi^{(1)}$	$\phi^{(2)}$	q	d^{C}	u^{C}	l	e^{C}	ν^{C}
$\sqrt{\frac{5}{3}}q_{\mathrm{Y}}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{6}$	$+\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$+1$	0
$\mathrm{SU}(2)_{\mathrm{L}}$	**2**	**2**	**2**	**1**	**1**	**2**	**1**	**1**
$\mathrm{SU}(3)_{\mathrm{C}}$	**1**	**1**	**3**	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	**1**	**1**	**1**

Table 6.4-1: Quantum numbers of the superfields. q_{Y} denotes the $\mathrm{U}(1)_{\mathrm{Y}}$ charge in GUT normalization.

Their expansions in component fields read

$$
\begin{aligned}
\mathbb{h}_a^{(1)} &= \phi^{(1)} + \sqrt{2}\theta\tilde{\phi}^{(1)} + \theta\theta\, F_{h^{(1)}} \ , &(6.4\text{-}1a)\\
\mathbb{h}^{(2)} &= \phi^{(2)} + \sqrt{2}\theta\tilde{\phi}^{(2)} + \theta\theta\, F_{h^{(2)}} \ , &(6.4\text{-}1b)\\
\mathbb{q}^f &= \widetilde{q}^f + \sqrt{2}\theta q^f + \theta\theta\ F_q^f \ , &(6.4\text{-}1c)\\
\mathbb{u}^{\mathrm{C}\,g} &= \widetilde{u}^{\mathrm{C}\,g} + \sqrt{2}\theta u^{\mathrm{C}g} + \theta\theta\, F_u^g \ , &(6.4\text{-}1d)\\
\mathbb{d}^{\mathrm{C}\,g} &= \widetilde{d}^{\mathrm{C}\,g} + \sqrt{2}\theta d^{\mathrm{C}\,g} + \theta\theta\, F_d^g \ , &(6.4\text{-}1e)\\
\mathbb{l}^f &= \widetilde{\ell}^f + \sqrt{2}\theta\ell^f + \theta\theta\, F_\ell^f \ , &(6.4\text{-}1f)\\
\mathbb{e}^{\mathrm{C}\,g} &= \widetilde{e}^{\mathrm{C}\,g} + \sqrt{2}\theta e^{\mathrm{C}\,g} + \theta\theta\, F_e^g \ , &(6.4\text{-}1g)\\
\mathbb{v}^{\mathrm{C}\,g} &= \widetilde{\nu}^{\mathrm{C}\,g} + \sqrt{2}\theta\nu^{\mathrm{C}\,g} + \theta\theta\, F_\nu^g \ , &(6.4\text{-}1h)
\end{aligned}
$$

where the R-parity odd component fields are furnished with a tilde.

6.4.2 Superpotential

We consider the superpotential

$$
\begin{aligned}
\mathcal{W}_{\nu\mathrm{MSSM}} &= (Y_d)_{gf}\mathbb{d}^{\mathrm{C}g}\mathbb{h}_a^{(1)}\varepsilon^{ab}\mathbb{q}_b^f + (Y_u)_{gf}\mathbb{u}^{\mathrm{C}g}\mathbb{h}_a^{(2)}(\varepsilon^T)^{ab}\mathbb{q}_b^f \\
&\quad + (Y_e)_{gf}\mathbb{e}^{\mathrm{C}g}\mathbb{h}_a^{(1)}\varepsilon^{ab}\mathbb{l}_b^f + (Y_\nu)_{gf}\mathbb{v}^{\mathrm{C}g}\mathbb{h}_a^{(2)}\varepsilon^{ab}\mathbb{l}_b^f \\
&\quad + \frac{1}{2}M_{ij}\,\mathbb{v}^{\mathrm{C}i}\,\mathbb{v}^{\mathrm{C}j} \ . \qquad (6.4\text{-}2)
\end{aligned}
$$

It contains the couplings of the MSSM superpotential as well as the “Yukawa” coupling of the neutrino superfields and a direct mass term for the latter.

6.4.3 Integrating Out $\mathbb{v}^{\mathrm{C}}$

In this section, the procedure of integrating out heavy fields in supersymmetric theories is studied in an explicit example. For simplicity, we restrict ourselves to one generation. Consider the action

$$
\begin{aligned}
S \ = \ \int \mathrm{d}^4x \,\Bigg\{ \int \mathrm{d}^4\theta \,\Big[&\bar{\mathbb{v}}^{\mathrm{C}}\mathbb{v}^{\mathrm{C}} + \bar{\mathbb{h}}^{(2)} \exp\Big(2g_1\,\mathbb{B}q_{\mathrm{Y}}^{\phi^{(2)}} + g_2\,\mathbb{W}_i\sigma^i\Big)\,\mathbb{h}^{(2)} \\
&+ \bar{\mathbb{l}}\,\exp\big(2g_1\,\mathbb{B}q_{\mathrm{Y}}^{\ell} + g_2\,\mathbb{W}_i\sigma^i\big)\,\mathbb{l}\Big] \\
&+ \Big[\int \mathrm{d}^2\theta\, \mathscr{W}(\mathbb{v}^{\mathrm{C}},\mathbb{h}^{(2)},\mathbb{l}) + \text{h.c.}\Big]\Bigg\} \qquad (6.4\text{-}3)
\end{aligned}
$$

where we need to take into account only the following part of the νMSSM superpotential:

$$
\mathscr{W}_\nu = \frac{M}{2}\mathbb{v}^{\mathrm{C}}\mathbb{v}^{\mathrm{C}} + Y_\nu\,\mathbb{v}^{\mathrm{C}}\,\mathbb{h}_a^{(2)}\,(\varepsilon^T)^{ab}\,\mathbb{l}_b \ . \qquad (6.4\text{-}4)
$$

The procedure of integrating out $\mathbb{v}^{\mathrm{C}}$ can in principle be performed in two ways: Firstly, one can switch to a component field description of the theory and do it the usual way. Secondly, one can use supergraph techniques in order to calculate the desired effective action.

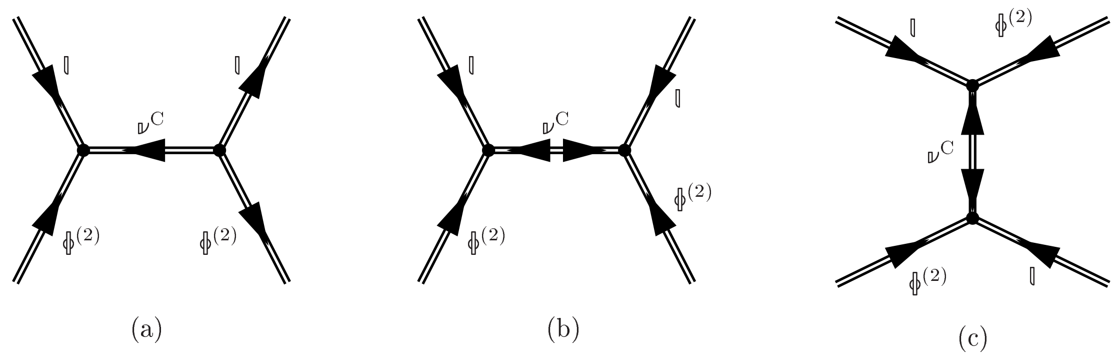


Figure 6.3: Tree-level supergraphs leading to an effective action where the $\mathbb{\nu}^{\mathrm{C}}$ fields are integrated out.

In the supergraph approach, we need to consider the three diagrams shown in figure 6.3 and those which arise by reverting the directions of all arrows. The first diagram 6.3(a) yields

$$-Y_\nu^\dagger Y_\nu \int \mathrm{d}^4x \int \mathrm{d}^4\theta \int \mathrm{d}^4\theta' \, \bar{\mathbb{\phi}}^{(2)}(x,\theta)\, \bar{\mathbb{l}}(x,\theta)\, \frac{\mathrm{i}\,\delta^4(\theta-\theta')}{-\Box - M^2}\, \mathbb{\phi}^{(2)}(x,\theta')\, \mathbb{l}(x,\theta')\;, \tag{6.4-5}$$

where we imply correct SU(2) index contractions. After neglecting the $\Box$ against the M^2 term, this leads to the contribution

$$\mathrm{i} \int \mathrm{d}^4x \int \mathrm{d}^4\theta \, \frac{Y_\nu^\dagger Y_\nu}{M^2}\, \bar{\mathbb{\phi}}^{(2)}(x,\theta)\, \bar{\mathbb{l}}(x,\theta)\, \mathbb{\phi}^{(2)}(x,\theta)\, \mathbb{l}(x,\theta) \tag{6.4-6}$$

to the effective action, which in turn gives rise to an additional D-term.

The second diagram 6.3(b) gives

$$-Y_\nu^2 \int \mathrm{d}^4x \int \mathrm{d}^4\theta \int \mathrm{d}^4\theta' \\ \mathbb{\phi}^{(2)}(x,\theta)\, \mathbb{l}(x,\theta)\, \frac{\mathrm{i}\,M\,\delta^4(\theta-\theta')}{-\Box\,(-\Box - M^2)}\, \frac{1}{4}\boldsymbol{D}^2\, \mathbb{\phi}^{(2)}(x,\theta')\, \mathbb{l}(x,\theta')\;. \tag{6.4-7}$$

Neglecting the second $\Box$ against M^2 and using equation (6.1-37), we find a contribution

$$\mathrm{i} \int \mathrm{d}^4x \int \mathrm{d}^2\theta \, \frac{Y_\nu^2}{M}\, \mathbb{\phi}^{(2)}(x,\theta)\, \mathbb{l}(x,\theta)\, \mathbb{\phi}^{(2)}(x,\theta)\, \mathbb{l}(x,\theta)\;, \tag{6.4-8}$$

which turns out to be a part of the effective superpotential.

When we include diagram 6.3(c) as well as the diagrams with reversed arrows, we obtain an effective action of the following form:

$$S_{\mathrm{eff}} = \int \mathrm{d}^4x \left\{ \int \mathrm{d}^4\theta \, \frac{4Y_\nu^\dagger Y^\nu}{M^2}\, \bar{\mathbb{\phi}}^{(2)}(x,\theta)\, \mathbb{l}(x,\theta)\, \mathbb{\phi}^{(2)}(x,\theta)\, \bar{\mathbb{l}}(x,\theta) \\ + \left[\int \mathrm{d}^2\theta\, 2\frac{Y_\nu^2}{M} \mathbb{\phi}^{(2)}\, \mathbb{l}\, \mathbb{\phi}^{(2)}\, \mathbb{l} + \mathrm{h.c.} \right] \right\}\;. \tag{6.4-9}$$

Hence, in leading order of the expansion in inverse powers of the presumably heavy scale M the effective action is given by

$$S_{\text{EFT}} = \int \mathrm{d}^4x \, \left[\int \mathrm{d}^2\theta \, 2 \frac{Y_\nu^2}{M} \phi^{(2)} \, \mathbb{l} \, \phi^{(2)} \, \mathbb{l} + \text{h.c.} \right] \,, \tag{6.4-10}$$

i.e. the D-term has dropped out. Consequently, we have justified the effective neutrino mass operator of the MSSM

$$\mathscr{W}_\kappa^{\text{MSSM}} = -\frac{1}{4} \kappa_{gf} \, \mathbb{l}_c^g \varepsilon^{cd} \phi_d^{(2)} \; \mathbb{l}_b^f \varepsilon^{ba} \phi_a^{(2)} + \text{h.c.} \,, \tag{6.4-11}$$

where the SU(2) and generation indices are introduced again and a combinatoric factor was respected. As in equation (3.2-7), κ is defined by

$$\kappa_{gf} = 2 \, (Y_\nu^T \, M^{-1} Y_\nu)_{gf} \tag{6.4-12}$$

at the matching scale.

6.4.4 Yukawa RGE's

Using equation (6.3-8), we find for the $1/\epsilon$-coefficients of the wavefunction renormalization constants

$$-(4\pi)^2 \, Z^{(1)}_{\phi^{(1)},1} \;=\; 6 \; \text{Tr}(Y_d^\dagger \cdot Y_d) + 2 \; \text{Tr}(Y_e^\dagger \cdot Y_e) - \frac{3}{5} \, g_1^2 - 3 \, g_2^2 \,, \tag{6.4-13a}$$

$$-(4\pi)^2 \, Z^{(1)}_{\phi^{(2)},1} \;=\; 6 \; \text{Tr}(Y_u^\dagger \cdot Y_u) + 2 \; \text{Tr}(Y_\nu^\dagger \cdot Y_\nu) - \frac{3}{5} \, g_1^2 - 3 \, g_2^2 \,, \tag{6.4-13b}$$

$$-(4\pi)^2 \, Z^{(1)}_{\mathbb{q},1} \;=\; 2 \, Y_d^\dagger \cdot Y_d + 2 \, Y_u^\dagger \cdot Y_u - \frac{1}{15} \, g_1^2 - 3 \, g_2^2 - \frac{16}{3} \, g_3^2 \,, \tag{6.4-13c}$$

$$-(4\pi)^2 \, Z^{(1)}_{\mathbb{d}^C,1} \;=\; 4 \, Y_d^* \cdot Y_d^T - \frac{4}{15} \, g_1^2 - \frac{16}{3} \, g_3^2 \,, \tag{6.4-13d}$$

$$-(4\pi)^2 \, Z^{(1)}_{\mathbb{u}^C,1} \;=\; 4 \, Y_u^* \cdot Y_u^T - \frac{16}{15} \, g_1^2 - \frac{16}{3} \, g_3^2 \,, \tag{6.4-13e}$$

$$-(4\pi)^2 \, Z^{(1)}_{\mathbb{l},1} \;=\; 2 \, Y_e^\dagger \cdot Y_e + 2 \, Y_\nu^\dagger \cdot Y_\nu - \frac{3}{5} \, g_1^2 - 3 \, g_2^2 \,, \tag{6.4-13f}$$

$$-(4\pi)^2 \, Z^{(1)}_{\mathbb{e}^C,1} \;=\; 4 \, Y_e^* \cdot Y_e^T - \frac{12}{5} \, g_1^2 \,, \tag{6.4-13g}$$

$$-(4\pi)^2 \, Z^{(1)}_{\nu^C,1} \;=\; 4 \, Y_\nu^* \cdot Y_\nu^T \,, \tag{6.4-13h}$$

where the the last six Z-factors are matrices in flavour space. From the two-loop diagrams we obtain

$$\begin{aligned} -(4\pi)^4 \, Z^{(2)}_{\phi^{(1)},1} \;=\; & - 9 \; \text{Tr}(Y_d^\dagger \cdot Y_d \cdot Y_d^\dagger \cdot Y_d) - 3 \; \text{Tr}(Y_d^\dagger \cdot Y_u \cdot Y_u^\dagger \cdot Y_d) \\ & - 3 \; \text{Tr}(Y_e^\dagger \cdot Y_e \cdot Y_e^\dagger \cdot Y_e) - \text{Tr}(Y_e^\dagger \cdot Y_\nu \cdot Y_\nu^\dagger \cdot Y_e) \\ & - \frac{2}{5} \, g_1^2 \; \text{Tr}(Y_d^\dagger \cdot Y_d) + \frac{6}{5} \, g_1^2 \; \text{Tr}(Y_e^\dagger \cdot Y_e) + 16 \, g_3^2 \; \text{Tr}(Y_d^\dagger \cdot Y_d) \\ & + \frac{207}{100} \, g_1^4 + \frac{9}{10} \, g_1^2 \, g_2^2 + \frac{15}{4} \, g_2^4 \,, \end{aligned} \tag{6.4-14a}$$

$$
\begin{aligned}
-(4\pi)^4\, Z^{(2)}_{\phi^{(2)},1} \;=\; & -3\ \mathrm{Tr}(Y_u^\dagger\cdot Y_d\cdot Y_d^\dagger\cdot Y_u) - 9\ \mathrm{Tr}(Y_u^\dagger\cdot Y_u\cdot Y_u^\dagger\cdot Y_u) \\
& - \mathrm{Tr}(Y_\nu^\dagger\cdot Y_e\cdot Y_e^\dagger\cdot Y_\nu) - 3\ \mathrm{Tr}(Y_\nu^\dagger\cdot Y_\nu\cdot Y_\nu^\dagger\cdot Y_\nu) \\
& + \frac{4}{5}\, g_1^2\ \mathrm{Tr}(Y_u^\dagger\cdot Y_u) + 16\, g_3^2\ \mathrm{Tr}(Y_u^\dagger\cdot Y_u) \\
& + \frac{207}{100}\, g_1^4 + \frac{9}{10}\, g_1^2\, g_2^2 + \frac{15}{4}\, g_2^4\,, \qquad (6.4\text{-}14b)
\end{aligned}
$$

$$
\begin{aligned}
-(4\pi)^4\, Z^{(2)}_{q,1} \;=\; & -2\, Y_d^\dagger\cdot Y_d\cdot Y_d^\dagger\cdot Y_d - 2\, Y_u^\dagger\cdot Y_u\cdot Y_u^\dagger\cdot Y_u \\
& -3\, Y_d^\dagger\cdot Y_d\ \mathrm{Tr}(Y_d\cdot Y_d^\dagger) - 3\, Y_u^\dagger\cdot Y_u\ \mathrm{Tr}(Y_u\cdot Y_u^\dagger) \\
& - Y_d^\dagger\cdot Y_d\ \mathrm{Tr}(Y_e\cdot Y_e^\dagger) - Y_u^\dagger\cdot Y_u\ \mathrm{Tr}(Y_\nu\cdot Y_\nu^\dagger) \\
& + \frac{2}{5}\, g_1^2\, Y_d^\dagger\cdot Y_d + \frac{4}{5}\, g_1^2\, Y_u^\dagger\cdot Y_u \\
& + \frac{199}{900}\, g_1^4 + \frac{1}{10}\, g_1^2\, g_2^2 + \frac{15}{4}\, g_2^4 \\
& + \frac{8}{45}\, g_1^2\, g_3^2 + 8\, g_2^2\, g_3^2 - \frac{8}{9}\, g_3^4\,, \qquad (6.4\text{-}14c)
\end{aligned}
$$

$$
\begin{aligned}
-(4\pi)^4\, Z^{(2)}_{d^C,1} \;=\; & -2\, Y_d^*\cdot Y_d^T\cdot Y_d^*\cdot Y_d^T - 2\, Y_d^*\cdot Y_u^T\cdot Y_u^*\cdot Y_d^T \\
& -6\, Y_d^*\cdot Y_d^T\ \mathrm{Tr}(Y_d\cdot Y_d^\dagger) - 2\, Y_d^*\cdot Y_d^T\ \mathrm{Tr}(Y_e\cdot Y_e^\dagger) \\
& + \frac{2}{5}\, g_1^2\, Y_d^*\cdot Y_d^T + 6\, g_2^2\, Y_d^*\cdot Y_d^T \\
& + \frac{202}{225}\, g_1^4 + \frac{32}{45}\, g_1^2\, g_3^2 - \frac{8}{9}\, g_3^4\,, \qquad (6.4\text{-}14d)
\end{aligned}
$$

$$
\begin{aligned}
-(4\pi)^4\, Z^{(2)}_{u^C,1} \;=\; & -2\, Y_u^*\cdot Y_d^T\cdot Y_d^*\cdot Y_u^T - 2\, Y_u^*\cdot Y_u^T\cdot Y_u^*\cdot Y_u^T \\
& -2\, Y_u^*\cdot Y_u^T\ \mathrm{Tr}(Y_\nu\cdot Y_\nu^\dagger) - 6\, Y_u^*\cdot Y_u^T\ \mathrm{Tr}(Y_u\cdot Y_u^\dagger) \\
& - \frac{2}{5}\, g_1^2\, Y_u^*\cdot Y_u^T + 6\, g_2^2\, Y_u^*\cdot Y_u^T \\
& + \frac{856}{225}\, g_1^4 + \frac{128}{45}\, g_1^2\, g_3^2 - \frac{8}{9}\, g_3^4\,, \qquad (6.4\text{-}14e)
\end{aligned}
$$

$$
\begin{aligned}
-(4\pi)^4\, Z^{(2)}_{l,1} \;=\; & -2\, Y_e^\dagger\cdot Y_e\cdot Y_e^\dagger\cdot Y_e - 2\, Y_\nu^\dagger\cdot Y_\nu\cdot Y_\nu^\dagger\cdot Y_\nu \\
& -3\, Y_e^\dagger\cdot Y_e\ \mathrm{Tr}(Y_d\cdot Y_d^\dagger) - Y_e^\dagger\cdot Y_e\ \mathrm{Tr}(Y_e\cdot Y_e^\dagger) \\
& -3\, Y_\nu^\dagger\cdot Y_\nu\ \mathrm{Tr}(Y_u\cdot Y_u^\dagger) - Y_\nu^\dagger\cdot Y_\nu\ \mathrm{Tr}(Y_\nu\cdot Y_\nu^\dagger) \\
& + \frac{6}{5}\, g_1^2\, Y_e^\dagger\cdot Y_e + \frac{207}{100}\, g_1^4 + \frac{9}{10}\, g_1^2\, g_2^2 + \frac{15}{4}\, g_2^4\,, \qquad (6.4\text{-}14f)
\end{aligned}
$$

$$
\begin{aligned}
-(4\pi)^4\, Z^{(2)}_{e^C,1} \;=\; & -2\, Y_e^*\cdot Y_e^T\cdot Y_e^*\cdot Y_e^T - 2\, Y_e^*\cdot Y_\nu^T\cdot Y_\nu^*\cdot Y_e^T \\
& -6\, Y_e^*\cdot Y_e^T\ \mathrm{Tr}(Y_d\cdot Y_d^\dagger) - 2\, Y_e^*\cdot Y_e^T\ \mathrm{Tr}(Y_e\cdot Y_e^\dagger) \\
& - \frac{6}{5}\, g_1^2\, Y_e^*\cdot Y_e^T + 6\, g_2^2\, Y_e^*\cdot Y_e^T + \frac{234}{25}\, g_1^4\,, \qquad (6.4\text{-}14g)
\end{aligned}
$$

$$-(4\pi)^4 \, Z^{(2)}_{\nu^{\mathrm{C}},1} = -\,2\,Y_\nu^* \cdot Y_e^T \cdot Y_e^* \cdot Y_\nu^T - 2\,Y_\nu^* \cdot Y_\nu^T \cdot Y_\nu^* \cdot Y_\nu^T - 6\,Y_\nu^* \cdot Y_\nu^T \;\mathrm{Tr}(Y_u \cdot Y_u^\dagger) - 2\,Y_\nu^* \cdot Y_\nu^T \;\mathrm{Tr}(Y_\nu \cdot Y_\nu^\dagger) + \frac{6}{5}\,g_1^2\,Y_\nu^* \cdot Y_\nu^T + 6\,g_2^2\,Y_\nu^* \cdot Y_\nu^T \,, \tag{6.4-14h}$$

respectively. From these, the two-loop Yukawa RGE's are derived,

$$\mu \frac{\mathrm{d}Y_x}{\mathrm{d}\mu} = \frac{1}{(4\pi)^2}\beta^{(1)}_{Y_x} + \frac{1}{(4\pi)^4}\beta^{(2)}_{Y_x} \,, \tag{6.4-15}$$

where $x \in \{d, u, e, \nu\}$. Using equation (6.3-19), the one-loop contributions to the β-functions are given by

$$\beta^{(1)}_{Y_d} = Y_d \cdot \left\{ 3\,Y_d^\dagger \cdot Y_d + Y_u^\dagger \cdot Y_u + 3\;\mathrm{Tr}(Y_d^\dagger \cdot Y_d) + \mathrm{Tr}(Y_e^\dagger \cdot Y_e) - \frac{7}{15}\,g_1^2 - 3\,g_2^2 - \frac{16}{3}\,g_3^2 \right\} \,, \tag{6.4-16a}$$

$$\beta^{(1)}_{Y_u} = Y_u \cdot \left\{ Y_d^\dagger \cdot Y_d + 3\,Y_u^\dagger \cdot Y_u + \mathrm{Tr}(Y_\nu^\dagger \cdot Y_\nu) + 3\;\mathrm{Tr}(Y_u^\dagger \cdot Y_u) - \frac{13}{15}\,g_1^2 - 3\,g_2^2 - \frac{16}{3}\,g_3^2 \right\} \tag{6.4-16b}$$

$$\beta^{(1)}_{Y_e} = Y_e \cdot \left\{ 3\,Y_e^\dagger \cdot Y_e + Y_\nu^\dagger \cdot Y_\nu + 3\;\mathrm{Tr}(Y_d^\dagger \cdot Y_d) + \mathrm{Tr}(Y_e^\dagger \cdot Y_e) - \frac{9}{5}\,g_1^2 - 3\,g_2^2 \right\} \,, \tag{6.4-16c}$$

$$\beta^{(1)}_{Y_\nu} = Y_\nu \cdot \left\{ Y_e^\dagger \cdot Y_e + 3\,Y_\nu^\dagger \cdot Y_\nu + 3\;\mathrm{Tr}(Y_u^\dagger \cdot Y_u) + \mathrm{Tr}(Y_\nu^\dagger \cdot Y_\nu) - \frac{3}{5}\,g_1^2 - 3\,g_2^2 \right\} \,, \tag{6.4-16d}$$

and the two-loop contributions are

$$
\begin{aligned}
\beta_{Y_d}^{(2)} \;=\;& Y_d \cdot \Big\{ -4\, Y_d^\dagger \cdot Y_d \cdot Y_d^\dagger \cdot Y_d - 2\, Y_u^\dagger \cdot Y_u \cdot Y_d^\dagger \cdot Y_d - 2\, Y_u^\dagger \cdot Y_u \cdot Y_u^\dagger \cdot Y_u \\
& - 9\ \mathrm{Tr}(Y_d^\dagger \cdot Y_d \cdot Y_d^\dagger \cdot Y_d) - 3\ \mathrm{Tr}(Y_d^\dagger \cdot Y_u \cdot Y_u^\dagger \cdot Y_d) \\
& - 3\ \mathrm{Tr}(Y_e^\dagger \cdot Y_e \cdot Y_e^\dagger \cdot Y_e) - \mathrm{Tr}(Y_e^\dagger \cdot Y_\nu \cdot Y_\nu^\dagger \cdot Y_e) \\
& - 9\, Y_d^\dagger \cdot Y_d\ \mathrm{Tr}(Y_d \cdot Y_d^\dagger) - 3\, Y_d^\dagger \cdot Y_d\ \mathrm{Tr}(Y_e \cdot Y_e^\dagger) \\
& - Y_u^\dagger \cdot Y_u\ \mathrm{Tr}(Y_\nu \cdot Y_\nu^\dagger) - 3\, Y_u^\dagger \cdot Y_u\ \mathrm{Tr}(Y_u \cdot Y_u^\dagger) \\
& + 6\, g_2^2\, Y_d^\dagger \cdot Y_d + \frac{4}{5}\, g_1^2\, Y_d^\dagger \cdot Y_d + \frac{4}{5}\, g_1^2\, Y_u^\dagger \cdot Y_u \\
& - \frac{2}{5}\, g_1^2\ \mathrm{Tr}(Y_d^\dagger \cdot Y_d) + \frac{6}{5}\, g_1^2\ \mathrm{Tr}(Y_e^\dagger \cdot Y_e) + 16\, g_3^2\ \mathrm{Tr}(Y_d^\dagger \cdot Y_d) \\
& + \frac{287}{90}\, g_1^4 + g_1^2\, g_2^2 + \frac{15}{2}\, g_2^4 + \frac{8}{9}\, g_1^2\, g_3^2 + 8\, g_2^2\, g_3^2 - \frac{16}{9}\, g_3^4 \Big\}\ , \qquad (6.4\text{-}17a)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{Y_u}^{(2)} \;=\;& Y_u \cdot \Big\{ -2\, Y_d^\dagger \cdot Y_d \cdot Y_d^\dagger \cdot Y_d - 2\, Y_d^\dagger \cdot Y_d \cdot Y_u^\dagger \cdot Y_u - 4\, Y_u^\dagger \cdot Y_u \cdot Y_u^\dagger \cdot Y_u \\
& - 3\, Y_d^\dagger \cdot Y_d\ \mathrm{Tr}(Y_d \cdot Y_d^\dagger) - Y_d^\dagger \cdot Y_d\ \mathrm{Tr}(Y_e \cdot Y_e^\dagger) \\
& - 9\, Y_u^\dagger \cdot Y_u\ \mathrm{Tr}(Y_u \cdot Y_u^\dagger) - 3\, Y_u^\dagger \cdot Y_u\ \mathrm{Tr}(Y_\nu \cdot Y_\nu^\dagger) \\
& - 3\ \mathrm{Tr}(Y_u^\dagger \cdot Y_d \cdot Y_d^\dagger \cdot Y_u) - 9\ \mathrm{Tr}(Y_u^\dagger \cdot Y_u \cdot Y_u^\dagger \cdot Y_u) \\
& - \mathrm{Tr}(Y_\nu^\dagger \cdot Y_e \cdot Y_e^\dagger \cdot Y_\nu) - 3\ \mathrm{Tr}(Y_\nu^\dagger \cdot Y_\nu \cdot Y_\nu^\dagger \cdot Y_\nu) \\
& + \frac{2}{5}\, g_1^2\, Y_d^\dagger \cdot Y_d + \frac{2}{5}\, g_1^2\, Y_u^\dagger \cdot Y_u + 6\, g_2^2\, Y_u^\dagger \cdot Y_u \\
& + \frac{4}{5}\, g_1^2\ \mathrm{Tr}(Y_u^\dagger \cdot Y_u) + 16\, g_3^2\ \mathrm{Tr}(Y_u^\dagger \cdot Y_u) + \frac{2743}{450}\, g_1^4 \\
& + g_1^2\, g_2^2 + \frac{15}{2}\, g_2^4 + \frac{136}{45}\, g_1^2\, g_3^2 + 8\, g_2^2\, g_3^2 - \frac{16}{9}\, g_3^4 \Big\}\ , \qquad (6.4\text{-}17b)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{Y_e}^{(2)} \;=\;& Y_e \cdot \Big\{ -4\, Y_e^\dagger \cdot Y_e \cdot Y_e^\dagger \cdot Y_e - 2\, Y_\nu^\dagger \cdot Y_\nu \cdot Y_e^\dagger \cdot Y_e - 2\, Y_\nu^\dagger \cdot Y_\nu \cdot Y_\nu^\dagger \cdot Y_\nu \\
& - 9\, Y_e^\dagger \cdot Y_e\ \mathrm{Tr}(Y_d \cdot Y_d^\dagger) - 3\, Y_e^\dagger \cdot Y_e\ \mathrm{Tr}(Y_e \cdot Y_e^\dagger) \\
& - Y_\nu^\dagger \cdot Y_\nu\ \mathrm{Tr}(Y_\nu \cdot Y_\nu^\dagger) - 3\, Y_\nu^\dagger \cdot Y_\nu\ \mathrm{Tr}(Y_u \cdot Y_u^\dagger) \\
& - 9\ \mathrm{Tr}(Y_d^\dagger \cdot Y_d \cdot Y_d^\dagger \cdot Y_d) - 3\ \mathrm{Tr}(Y_d^\dagger \cdot Y_u \cdot Y_u^\dagger \cdot Y_d) \\
& - 3\ \mathrm{Tr}(Y_e^\dagger \cdot Y_e \cdot Y_e^\dagger \cdot Y_e) - \mathrm{Tr}(Y_e^\dagger \cdot Y_\nu \cdot Y_\nu^\dagger \cdot Y_e) + \frac{6}{5}\, g_1^2\ \mathrm{Tr}(Y_e^\dagger \cdot Y_e) \\
& + 6\, g_2^2\, Y_e^\dagger \cdot Y_e - \frac{2}{5}\, g_1^2\ \mathrm{Tr}(Y_d^\dagger \cdot Y_d) + 16\, g_3^2\ \mathrm{Tr}(Y_d^\dagger \cdot Y_d) \\
& + \frac{27}{2}\, g_1^4 + \frac{9}{5}\, g_1^2\, g_2^2 + \frac{15}{2}\, g_2^4 \Big\}\ , \qquad (6.4\text{-}17c)
\end{aligned}
$$

$$\begin{aligned}\beta^{(2)}_{Y_\nu} \;=\; Y_\nu \cdot \Big\{ & -2\,Y_e^\dagger \cdot Y_e \cdot Y_e^\dagger \cdot Y_e - 2\,Y_e^\dagger \cdot Y_e \cdot Y_\nu^\dagger \cdot Y_\nu - 4\,Y_\nu^\dagger \cdot Y_\nu \cdot Y_\nu^\dagger \cdot Y_\nu \\ & - 3\,Y_e^\dagger \cdot Y_e\ \mathrm{Tr}(Y_d \cdot Y_d^\dagger) - Y_e^\dagger \cdot Y_e\ \mathrm{Tr}(Y_e \cdot Y_e^\dagger) \\ & - 3\,Y_\nu^\dagger \cdot Y_\nu\ \mathrm{Tr}(Y_\nu \cdot Y_\nu^\dagger) - 9\,Y_\nu^\dagger \cdot Y_\nu\ \mathrm{Tr}(Y_u \cdot Y_u^\dagger) \\ & - \mathrm{Tr}(Y_\nu^\dagger \cdot Y_e \cdot Y_e^\dagger \cdot Y_\nu) - 3\ \mathrm{Tr}(Y_\nu^\dagger \cdot Y_\nu \cdot Y_\nu^\dagger \cdot Y_\nu) \\ & - 3\ \mathrm{Tr}(Y_u^\dagger \cdot Y_d \cdot Y_d^\dagger \cdot Y_u) - 9\ \mathrm{Tr}(Y_u^\dagger \cdot Y_u \cdot Y_u^\dagger \cdot Y_u) \\ & + \frac{6}{5}\,g_1^2\,Y_e^\dagger \cdot Y_e + \frac{6}{5}\,g_1^2\,Y_\nu^\dagger \cdot Y_\nu + 6\,g_2^2\,Y_\nu^\dagger \cdot Y_\nu \\ & + \frac{4}{5}\,g_1^2\ \mathrm{Tr}(Y_u^\dagger \cdot Y_u) + 16\,g_3^2\ \mathrm{Tr}(Y_u^\dagger \cdot Y_u) \\ & + \frac{207}{50}\,g_1^4 + \frac{9}{5}\,g_1^2\,g_2^2 + \frac{15}{2}\,g_2^4 \Big\} \;. \end{aligned} \tag{6.4-17d}$$

Note that the two-loop MSSM RGE's for Y_d, Y_u and Y_e are easily obtained by setting $Y_\nu = 0$. The effort is clearly reduced compared to component field calculations [21–23].

6.4.5 Two-Loop β-Function for the Mass of the Singlet Superfield

From the wavefunction renormalization constants of the νMSSM, the β-function for the bilinear coupling of equation (6.4-2) can easily be computed using equation (6.3-19). At one-loop, we find

$$(4\pi)^2 \beta_M^{(1)} = 2\,M \cdot Y_\nu^* \cdot Y_\nu^T + 2\,Y_\nu \cdot Y_\nu^\dagger \cdot M \tag{6.4-18}$$

and the two-loop part of the β-function is given by

$$\begin{aligned}(4\pi)^4 \beta_M^{(2)} \;=\; M \cdot \Big[& -2\,Y_\nu^* \cdot Y_e^T \cdot Y_e^* \cdot Y_\nu^T - 2\,Y_\nu^* \cdot Y_\nu^T \cdot Y_\nu^* \cdot Y_\nu^T \\ & - 6\,Y_\nu^* \cdot Y_\nu^T\ \mathrm{Tr}(Y_u \cdot Y_u^\dagger) - 2\,Y_\nu^* \cdot Y_\nu^T\ \mathrm{Tr}(Y_\nu \cdot Y_\nu^\dagger) \\ & + \frac{6}{5}\,g_1^2\,Y_\nu^* \cdot Y_\nu^T + 6\,g_2^2\,Y_\nu^* \cdot Y_\nu^T \Big] \\ + \Big[& -2\,Y_\nu \cdot Y_e^\dagger \cdot Y_e \cdot Y_\nu^\dagger - 2\,Y_\nu \cdot Y_\nu^\dagger \cdot Y_\nu \cdot Y_\nu^\dagger \\ & - 6\,Y_\nu \cdot Y_\nu^\dagger\ \mathrm{Tr}(Y_u \cdot Y_u^\dagger) - 2\,Y_\nu \cdot Y_\nu^\dagger\ \mathrm{Tr}(Y_\nu \cdot Y_\nu^\dagger) \\ & + \frac{6}{5}\,g_1^2\,Y_\nu \cdot Y_\nu^\dagger + 6\,g_2^2\,Y_\nu \cdot Y_\nu^\dagger \Big] \cdot M \;. \end{aligned} \tag{6.4-19}$$

From this we derive between the thresholds the following β-function at one-loop

$$16\pi^2 \overset{(n)}{\beta}_M \;=\; 2\left(\overset{(n)}{Y}_\nu \overset{(n)}{Y_\nu^\dagger}\right) \overset{(n)}{M} + 2\,\overset{(n)}{M} \left(\overset{(n)}{Y}_\nu \overset{(n)}{Y_\nu^\dagger}\right)^T \;; \tag{6.4-20}$$

the two-loop result consists of equation (6.4-19) where Y_ν is replaced by $\overset{(n)}{Y}_\nu$

6.4.6 Two-Loop β-Function for the Effective Neutrino Mass Operator

We now apply the supergraph method to calculate the β-function for the lowest dimensional effective neutrino mass operator (6.4-11). The β-function can easily be computed using our method. Substituting $D_{g_i} = D_{Y_x} = \frac{1}{2}$ with $i \in \{1,2,3\}$ and $x \in \{u,d,e\}$, we obtain from equation (6.3-19)

$$\beta_\kappa = -Z_{\phi^{(2)},1} \cdot \kappa - \frac{1}{2}\, Z_{\mathbb{l},1}^T \cdot \kappa - \frac{1}{2}\, \kappa \cdot Z_{\mathbb{l},1} \; . \tag{6.4-21}$$

We can thus write the β-function for κ in the form

$$\beta_\kappa = X^T \cdot \kappa + \kappa \cdot X + \alpha\, \kappa \; , \tag{6.4-22}$$

where the complete flavour diagonal part is contained in α. We further split $X = X^{(1)} + X^{(2)}$ and $\alpha = \alpha^{(1)} + \alpha^{(2)}$ into their one- and two-loop part. Plugging in the wavefunction renormalization constants of equation (6.4-13b) and (6.4-13f) and setting $Y_\nu = 0$, our method reproduces the one-loop results of [44,17,11]

$$(4\pi)^2\, X^{(1)} = Y_e^\dagger \cdot Y_e \; , \tag{6.4-23a}$$

$$(4\pi)^2\, \alpha^{(1)} = -\frac{6}{5}\, g_1^2 - 6\, g_2^2 + 6\, \mathrm{Tr}(Y_u^\dagger \cdot Y_u) \; . \tag{6.4-23b}$$

Note that for $\mathrm{U}(1)_\mathrm{Y}$, we use GUT charge normalization as specified in table 6.4-1. At two-loop, with the wavefunction renormalization constants given in equations (6.4-14b) and (6.4-14f), we obtain

$$\begin{aligned}(4\pi)^4\, X^{(2)} = & -2\, Y_e^\dagger \cdot Y_e \cdot Y_e^\dagger \cdot Y_e \\ & + \left(\frac{6}{5}\, g_1^2 - \mathrm{Tr}(Y_e \cdot Y_e^\dagger) - 3\, \mathrm{Tr}(Y_d \cdot Y_d^\dagger)\right) Y_e^\dagger \cdot Y_e\end{aligned} \tag{6.4-24}$$

and

$$\begin{aligned}(4\pi)^4\, \alpha^{(2)} = & -6\, \mathrm{Tr}(Y_u^\dagger \cdot Y_d \cdot Y_d^\dagger \cdot Y_u) - 18\, \mathrm{Tr}(Y_u^\dagger \cdot Y_u \cdot Y_u^\dagger \cdot Y_u) \\ & + \frac{8}{5}\, g_1^2\, \mathrm{Tr}(Y_u^\dagger \cdot Y_u) + 32\, g_3^2\, \mathrm{Tr}(Y_u^\dagger \cdot Y_u) \\ & + \frac{207}{25}\, g_1^4 + \frac{18}{5}\, g_1^2\, g_2^2 + 15\, g_2^4 \; .\end{aligned} \tag{6.4-25}$$

At one-loop we obtain between the thresholds

$$\begin{aligned}16\pi^2 \overset{(n)}{\beta}_\kappa = & (Y_e^\dagger Y_e)^T \overset{(n)}{\kappa} + \overset{(n)}{\kappa} (Y_e^\dagger Y_e) + \left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right)^T \overset{(n)}{\kappa} + \overset{(n)}{\kappa} \left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right) \\ & + 2\, \mathrm{Tr}\left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right) \overset{(n)}{\kappa} + 6\, \mathrm{Tr}(Y_u^\dagger Y_u) \overset{(n)}{\kappa} - \frac{6}{5}\, g_1^2 \overset{(n)}{\kappa} - 6 g_2^2 \overset{(n)}{\kappa} \; ;\end{aligned} \tag{6.4-26}$$

and again the two-loop result is obtained analogously.

7 Applications

We have calculated the β-functions that govern the evolution of the neutrino mass parameters in the SM, the 2HDM's and the MSSM. It is therefore interesting to study whether there can be some generic predictions derived from the renormalization group equations. Furthermore, we can investigate the impact of a non-degenerate mass spectrum since we have also calculated the β-functions between the mass thresholds.

7.1 Neutrino Masses and Mixings

Usually in the quark sector the mixing angles are read off from the CKM matrix

$$V_{\mathrm{CKM}} = (U_{\mathrm{L}}^{(u)})^\dagger \cdot U_{\mathrm{L}}^{(d)} \,, \tag{7.1-1}$$

where

$$U_{\mathrm{R}}^{(u)\dagger} \cdot Y_u \cdot Y_u^\dagger \cdot U_{\mathrm{R}}^{(u)} = U_{\mathrm{L}}^{(u)\dagger} \cdot Y_u^\dagger \cdot Y_u \cdot U_{\mathrm{L}}^{(u)} = \mathrm{diag}(y_u^2, \dots) \,, \tag{7.1-2a}$$
$$U_{\mathrm{R}}^{(d)\dagger} \cdot Y_d \cdot Y_d^\dagger \cdot U_{\mathrm{R}}^{(d)} = U_{\mathrm{L}}^{(d)\dagger} \cdot Y_d^\dagger \cdot Y_d \cdot U_{\mathrm{L}}^{(d)} = \mathrm{diag}(y_d^2, \dots) \,. \tag{7.1-2b}$$

In the 2×2-case and for real entries of the Yukawa couplings, V_{CKM} can be chosen to be orthogonal and therefore parameterized by

$$V_{\mathrm{CKM}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \,, \tag{7.1-3}$$

thus defining the mixing angle θ. For Dirac masses, the MNS matrix is obtained by

$$V_{\mathrm{MNS}} = (U_{\mathrm{L}}^{(\nu)})^\dagger \cdot U_{\mathrm{L}}^{(e)} \,, \tag{7.1-4}$$

where $U_{\mathrm{L}}^{(\nu)}$ and $U_{\mathrm{L}}^{(e)}$ are defined analogously to (7.1-2). However, in the case of Majorana masses, $U_{\mathrm{L}}^{(\nu)}$ is defined by the relation

$$(U_{\mathrm{L}}^{(\nu)})^T \, m_\nu \, U_{\mathrm{L}}^{(\nu)} = \mathrm{diag}(m_1, \dots) \tag{7.1-5}$$

with m_ν being the left-handed neutrino mass matrix.

7.2 Maximal Mixing as Attractive Fixed Point?

Recent neutrino experiments give a clear evidence for large mixing angles in the lepton sector [9]. Since the corresponding angles in the quark sector are comparably small, it is interesting to study whether large mixing angles can be explained by renormalization effects.

Let us analyze the solution of the RGE's below the lowest threshold. We rewrite equation (4.1-5) in the form

$$16\pi^2 \frac{\mathrm{d}\kappa}{\mathrm{d}t} = \alpha\,\kappa - \frac{3}{2}\left[\kappa(Y_e^\dagger Y_e) + (Y_e^\dagger Y_e)^T \kappa\right] \,, \tag{7.2-1}$$

where α is a flavor diagonal term and Y_e can be chosen diagonal, if all Majorana neutrinos are integrated out. Especially in the two-flavor-case we obtain

$$Y_e^\dagger Y_e = \begin{pmatrix} y_1^2 & 0 \\ 0 & y_2^2 \end{pmatrix} \,. \tag{7.2-2}$$

This way we get originally four, but since $\kappa_{12} = \kappa_{21}$, three independent equations,

$$16\pi^2\,\dot\kappa_{11} = [\alpha - 3y_1^2]\cdot\kappa_{11} \,, \tag{7.2-3a}$$
$$16\pi^2\,\dot\kappa_{22} = [\alpha - 3y_2^2]\cdot\kappa_{22} \,, \tag{7.2-3b}$$
$$16\pi^2\,\dot\kappa_{12} = [\alpha - \tfrac{3}{2}(y_1^2 + y_2^2)]\cdot\kappa_{12} \,. \tag{7.2-3c}$$

Case 1: $\kappa_{12}(t_0) = 0$

If κ_{12} vanishes for a certain t_0, equation (7.2-3c) implies that $\kappa_{12} = 0$ holds for all t. The matrix κ is therefore diagonal and the mixing angle θ vanishes for all scales.

Case 2: $\kappa_{12}(t_0) \neq 0$

By forming quotients we can eliminate α in the equations,

$$16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa_{11}}{\kappa_{12}}\right) = \tfrac{3}{2}(y_1^2 - y_2^2)\frac{\kappa_{11}}{\kappa_{12}} \,, \tag{7.2-4a}$$
$$16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa_{22}}{\kappa_{12}}\right) = \tfrac{3}{2}(y_2^2 - y_1^2)\frac{\kappa_{22}}{\kappa_{12}} \,. \tag{7.2-4b}$$

This implies

$$\frac{\kappa_{11}}{\kappa_{12}}(t) = \frac{\kappa_{11}}{\kappa_{12}}(t_0)\cdot\exp\left[\frac{3}{32\pi^2}\int_0^t \mathrm{d}\tau\, y_1^2(\tau) - y_2^2(\tau)\right] \,, \tag{7.2-5a}$$

$$\frac{\kappa_{22}}{\kappa_{12}}(t) = \frac{\kappa_{22}}{\kappa_{12}}(t_0)\cdot\exp\left[\frac{3}{32\pi^2}\int_0^t \mathrm{d}\tau\, y_2^2(\tau) - y_1^2(\tau)\right] \,. \tag{7.2-5b}$$

Remark 7.2.1. Note that an initial value of $\kappa_{11} = \kappa_{22} = 0$ for $t = t_0$ implies that κ_{11} and κ_{22} vanish for all t and the mixing would stay maximal.

From the previous section it is clear that the mixing angle θ can be obtained from the relation

$$\cot 2\theta = \frac{1}{2}\left(\frac{\kappa_{22}}{\kappa_{12}}(t) - \frac{\kappa_{11}}{\kappa_{12}}(t)\right) . \tag{7.2-6}$$

Therefore, using the assumptions

(1) $y_2^2(\tau) > y_1^2(\tau)$ for $\tau < t_0$ and

(2) $\kappa_{11}(t_0) \neq 0$,

we obtain the relation

$$\lim_{t\to-\infty} \cot 2\theta \neq \frac{\pi}{4} \quad \text{in general} , \tag{7.2-7}$$

which implies in general a mixing angle $\theta \neq \frac{\pi}{4}$ in the infrared limit. In other words, we do not expect maximal mixing as a result of running effects.

Let us assume now that the entries of the Yukawa matrix Y_e do not run. Then we obtain from eq. (7.2-5)

$$\frac{\kappa_{11}}{\kappa_{12}}(t) = \frac{\kappa_{11}}{\kappa_{12}}(t_0) \cdot \exp\left[-\omega t\right] , \tag{7.2-8a}$$

$$\frac{\kappa_{22}}{\kappa_{12}}(t) = \frac{\kappa_{22}}{\kappa_{12}}(t_0) \cdot \exp\left[\omega t\right] \tag{7.2-8b}$$

with

$$\omega = \frac{3}{32\pi^2}(y_2^2 - y_1^2) . \tag{7.2-9}$$

If we insert for y_2 the τ Yukawa coupling $y_\tau = 1.777\,\mathrm{GeV}/v = \mathscr{O}(10^{-2})$ and for y_1 the electron Yukawa coupling $y_e = 511\,\mathrm{keV}/v = \mathscr{O}(10^{-6})$, we see that ω is tiny and θ stays approximately constant. Altogether we have found so far that maximal mixing is not favored by the renormalization group, and that running effects in the SM are in general relatively small below the lowest threshold.

7.3 Effects of Non-Degenerate Thresholds

7.3.1 Running of the Mixing Angle in an Example with Two Generations

We consider now in a 2×2-example the effects of thresholds. According to section 3.2.2, we have three energy regions in which different effective theories emerge: Above the largest

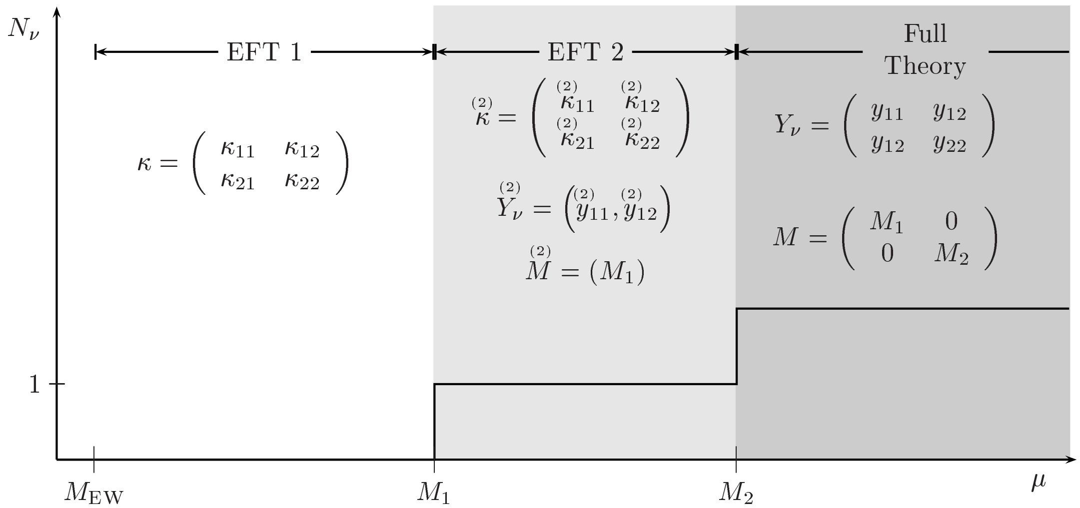


Figure 7.1: Illustration of the ranges of different effective theories in the SM with two generations and an additional effective neutrino mass operator.

mass eigenvalue M_2 of the Majorana mass matrix M a "full" theory, between both eigenvalues M_1 and M_2 the effective theory 2 and below M_1 the effective theory 1 which consists of the SM, a 2HDM or the MSSM with two generations and an additional effective neutrino mass operator.

In the energy region where heavy neutrinos are present, it is useful to introduce fictitious mixing angles which are defined as the mixing angles of (7.1-4) for the effective Majorana mass matrix of the non-sterile neutrinos, which is given by $\overset{(n)}{\kappa} + 2\overset{(n)}{Y}{}_\nu^T \overset{(n)}{M}{}^{-1} \overset{(n)}{Y}_\nu$. These angles can be seen as characteristics of the evolution of the mass parameters. Especially in the 2×2 case at hand, there is only one mixing angle.

Note that the matching condition (3.2-11) is non-trivial since the entries of the right-handed mass matrix run by themselves.

7.3.2 Numerical Results

Numerical results for the RG evolution of the mixing angle θ in a generic example with two generations of lepton doublets and two singlets are shown as solid lines in figure 7.2 for the SM, in figure 7.3 for the type (iv) 2HDM (cf. table 5.2-1), and in figure 7.4 for the MSSM. For all the examples, we chose

$$M = \begin{pmatrix} 10^8 & 0 \\ 0 & 10^{12} \end{pmatrix} \quad \text{and} \quad Y_\nu = \frac{1}{8} \begin{pmatrix} 6 & 2.5 \\ 0.2 & 8 \end{pmatrix} \tag{7.3-1}$$

at the GUT scale, for which we insert 10^{16} GeV. The electron Yukawa coupling is parametrized by

$$Y_e = \mathrm{diag}(0.0005, 0.01) \tag{7.3-2}$$

in the SM, and in the two other models $\tan\beta$ is taken into account.

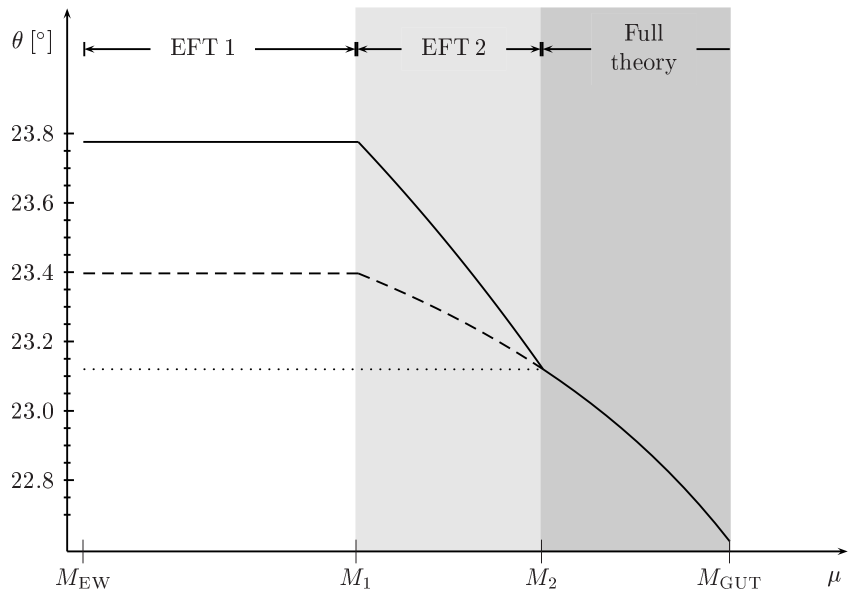


Figure 7.2: RG evolution of the mixing angle θ in the extended SM with 2 generations of lepton doublets and 2 singlets. We used $M_{\mathrm{GUT}} = 10^{16}$ GeV and the initial conditions $M_1(M_{\mathrm{GUT}}) = 10^8$ GeV, $M_2(M_{\mathrm{GUT}}) = 10^{12}$ GeV for the Majorana masses of the heavy neutrinos at this scale. Besides, we chose the initial values of the Yukawa coupling matrices $Y_\nu(M_{\mathrm{GUT}})$ to be real with (untuned) entries between 0.025 and 1. Further explanations are given in the text.

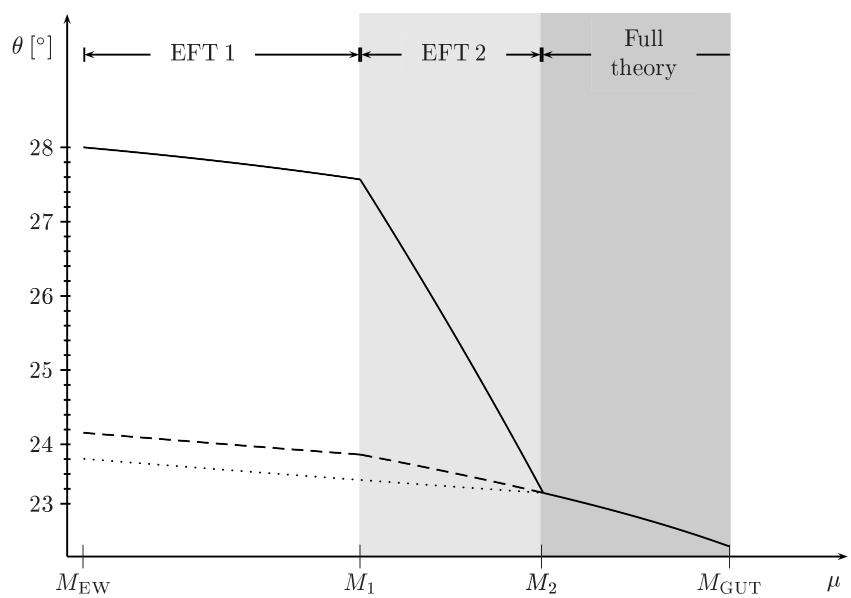


Figure 7.3: RG evolution of the mixing angle θ in the 2HDM of type (iv) in our classification scheme (cf. table 5.2-1) with 2 generations of lepton doublets, 2 singlets and $\tan\beta = 35$. The other parameters are the same as in the SM case (cf. figure 7.2). The running is enlarged due to the large $\tan\beta$. The second effective operator tends to decrease the running a little bit.

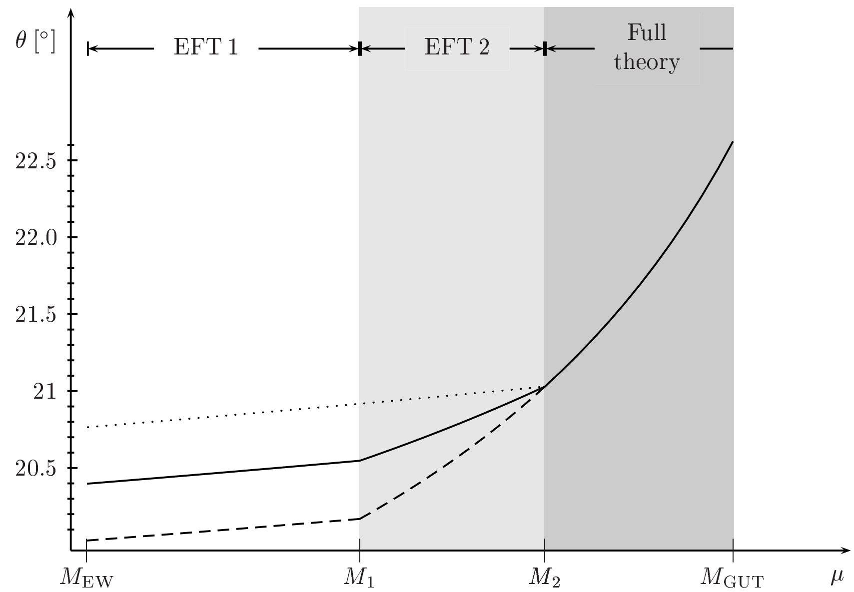


Figure 7.4: RG evolution of the mixing angle θ in the extended MSSM with 2 generations of lepton doublets, 2 singlets and $\tan\beta = 35$ as well as $M_{\mathrm{SUSY}} \approx M_{\mathrm{EW}}$ for simplicity. (A moderate change of the SUSY breaking scale M_{SUSY} does not change the qualitative picture.) The other parameters are the same as in the SM case (cf. figure 7.2).

The transitions to the various effective theories at the mass thresholds lead to pronounced kinks in the evolution. For comparison, the dotted and dashed lines in figures 7.2 and 7.4 show the results when both heavy neutrinos are integrated out at the higher or the lower threshold, respectively. Obviously, this produces large deviations from the true evolution, and the correct result need not even lie between the two extreme cases. Although this is only shown for the SM and for the 2HDM in our example, the same happens in the MSSM, if suitable initial values for the Yukawa couplings are chosen. Consequently, the correct running of the mixing angle cannot be reproduced by integrating out all heavy neutrinos at some intermediate mass scale $M_{\mathrm{int}} \in [M_1, M_2]$ in general. Therefore, studying the evolution of the renormalization group running between the thresholds is essential in any analysis in which non-degenerate mass scales are involved.

8 Summary and Outlook

8.1 Summary

In this thesis, the issue of calculating β-functions for non-renormalizable operators was addressed. A general formalism for computing β-functions for quantities with tensorial structure was derived, which also works if additive renormalization is imposed. It was applied to the calculation of the renormalization group equations for the lowest-dimensional effective neutrino mass operator in the SM and a class of 2HDM's. Furthermore, a general method of checking β-functions for effective operators by the aid of full theory amplitudes was developed. It was used to verify the result of the SM calculations for which existed a discrepancy with the literature. Due to the check and due to a second, independent confirmation, the result of our study can be regarded as the correct one. Furthermore, we presented the general formalism which enables one to deal with a number of effective theories arising from multiple heavy mass scales. We also argued that a scenario with non-degenerate mass scales is quite natural in the neutrino sector.

The analysis was extended to a class of 2HDM's in which two effective neutrino mass operators arise. It was shown that these operators mix with each other. All β-functions which govern the evolution of the neutrino mass parameters in the class of 2HDM's, in their extension by heavy singlets and in the regions between the mass thresholds and above were calculated.

A very important result of this thesis, however, basically stems from two observations. Firstly, the non-renormalization theorem of supersymmetry applies to any, even non-renormalizable operators of the superpotential, and secondly, these operators do not affect the coefficients of the β-functions in a rigid expansion in the sense of effective field theories. Hence, we were able to present a simple construction kit which enables one to calculate two-loop β-functions for any desired operator of the superpotential with only little effort. This method is based on supergraph techniques, therefore the amount of independent diagrams is clearly reduced compared to component field calculations, and SUSY is kept manifest so that the non-renormalization theorem can be applied directly. In an expansion in the sense of effective field theories, only the anomalous dimensions of the fields have to be computed, and the latter are only influenced by the renormalizable part of the superpotential and can therefore be taken from the literature. Moreover, our method has the advantage of being applicable to operators with tensorial structure, and allows for symbolical calculation as the color-factors were specified explicitly. In particular, with the results of chapter six one is able to calculate the β-function for any higher-dimensional operator of the superpotential within the framework of the MSSM immediately.

As a numerical application, the effects of non-degenerate see-saw scales on the evolution of the (fictitious) mixing angle was studied. The main message is that, in contrast to a naive estimate, the resulting running substantially differs from integrating out all heavy neutrinos at a common scale. In particular, running effects are sizeable above the lowest mass threshold. Besides, it turned out that in the Standard Model running effects below the lowest threshold are generically not large. This statement holds – depending on $\tan\beta$ – also in some regions of the parameter space in the 2HDM's and in the MSSM. We argued that maximal mixing is not a prediction of the renormalization group in the latter models by itself, i.e. it does not correspond to an attractive fixed point.

Altogether we have presented the foundations for the computation of the renormalization group evolution of neutrino mass parameters in the type I seesaw scenario for the SM, a class of 2HDM's and the MSSM. Moreover, the methods presented are quite general so that they can find an application in any renormalization group analysis.

8.2 Outlook

In this thesis, special attention was paid to methods for deriving β-functions. Consequently, an immediate application consists in the numerical analysis of the equations obtained in the calculations in more detail. In order to get touch with the experiments, it is worthwhile to consider the case of 3×3 matrices. In particular, predictions from specific models may be run down to the electroweak scale in order to be compared with the experimental data. Moreover, the methods described in this study are very general so that they might be of great interest for model building of any kind. For instance, the construction kit of chapter 6 makes it possible to design β-functions, i.e. the relations between their coefficients and the charactteristics of a given theory such as field content, gauge groups and interaction Lagrangian, become obvious immediately. Moreover, the method of chapter 6 may be extended to three-loop with the aid of [45,46].

It has to be stressed that in the neutrino sector, due to the absence of large hadronic uncertainties, in principle precision measurements of large accuracy are possible. Therefore, the analysis of two-loop β-functions, as were presented in chapter 6, as well as a closer investigation of threshold effects which were briefly discussed in chapter 7, are highly desirable.

Consequently, both more experimental data and more detailed numerical analyses are necessary in order to obtain better insight into physics beyond the SM. On the experimental side, running and planned as well as proposed projects, such as e.g. the KamLAND experiment [47] and e.g. neutrino factories as are very extensively discussed by M. Freund [48], will certainly provide some new and interesting, presumably even surprising results in the near and the more distant future. On the theoretical side, numerical analyses are presently performed [49] and will give new insights very soon. Hence, we can look forward to another very exciting period of neutrino physics.

Appendix

A Group-Theoretical Notation

A.1 General Notation

In this thesis we consider gauge groups G which consist of U(1) and simple group factors, i.e.

$$G = G_1 \otimes \cdots \otimes G_S \;, \tag{A.1}$$

where the G_i are either simple or U(1). The matrix representations of the generators of an arbitrary simple group G_i corresponding to the irrep (irreducible representation) R are denoted by $\{\mathsf{T}_A\}_{A=1}^{\dim G}$ and the structure constants by $f^A{}_{BC}$. The latter are defined by the relation

$$[\mathsf{T}_B, \mathsf{T}_C] = \mathrm{i}\, f^A{}_{BC}\, \mathsf{T}_A \;. \tag{A.2}$$

Furthermore, we use the group-theoretical constants

$$c_1(G)\, \delta^{AB} \;:=\; \sum_{C,D} f^{ACD} f^B{}_{CD} \;, \tag{A.3a}$$

$$c_2(R)\, \delta_{ab} \;:=\; \sum_{A} (\mathsf{T}^A \mathsf{T}^A)_{ab} \;, \tag{A.3b}$$

$$\ell(R)\, \delta^{AB} \;:=\; \mathrm{Tr}(\mathsf{T}^A \mathsf{T}^B) \;, \tag{A.3c}$$

where $\ell(R)$ is known as Dynkin index of the irrep R and $c_2(R)$ as the quadratic Casimir. The latter are related by

$$c_2(R) = \frac{\dim G_i}{\dim R}\, \ell(R) \;, \tag{A.4}$$

with $\dim G_i$ and $\dim R$ being the dimension of the simple group G_i and the irrep R, respectively. We use the convention that the generators of the irrep $\boldsymbol{N}$ of SU(N) are normalized such that $\ell(\boldsymbol{N}) = \frac{1}{2}$ holds. c_2 can then be obtained via $c_2(\boldsymbol{N}) = \frac{N^2-1}{2N}$ while for a U(1) theory both $\ell(R)$ and $c_2(R)$ are replaced by q^2 where q is the U(1) charge of the corresponding field. For any non-trivial irrep R of SU(N), the invariant $c_1(R)$ is given by N.

A.2 GUT Charge Normalization

Given an irrep R of a simple gauge group G, it is convenient to choose the generator matrices T such that

$$\mathrm{Tr}\,(\mathsf{T}_A\,\mathsf{T}_B) = \mathscr{N}(R)\,\delta_{AB}\;, \qquad A,B = 1,\dots \dim G \tag{A.5}$$

holds, where $\mathscr{N}(R)$ is a normalization constant. Calculation of the traces for the SM particles and the SM gauge group factors yields

$$\mathrm{Tr}\left[\left(\tfrac{1}{2}g_1\,y\right)^2\right] = \frac{10}{3}\,n_F\,g_1^2\;, \tag{A.6a}$$

$$\mathrm{Tr}\left[\left(\tfrac{1}{2}g_2\,\sigma_i\right)^2\right] = 2\,n_F\,g_2^2\;, \tag{A.6b}$$

$$\mathrm{Tr}\left[\left(\tfrac{1}{2}g_3\,\lambda_A\right)^2\right] = 2\,n_F\,g_3^2\;, \tag{A.6c}$$

where we chose for $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(3)_\mathrm{C}$ the Pauli-matrices σ_i and Gell-Mann-matrices λ_A, respectively, as representations of the generators. Thus, if all factors are embedded in some larger GUT gauge group, we are obliged to normalize y by a factor $\sqrt{\frac{3}{5}}$ in order to satisfy equation (A.5). Accordingly, g_1 has to be rescaled by a factor of $\sqrt{\frac{5}{3}}$.

B Spinorial Notation

B.1 Weyl Spinors

The spinor representation of the Lorentz group $\mathcal{L}_+^\uparrow$ can be decomposed as

$$\mathcal{L}_+^\uparrow = \mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})}\;, \tag{B.1}$$

where we can choose to represent $\mathrm{D}^{(\frac{1}{2},0)}$ by $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{D}^{(0,\frac{1}{2})}$ by the non-equivalent complex conjugate representation. Therefore, the transformation properties of $\mathrm{D}^{(\frac{1}{2},0)}$-spinors differ from those of $\mathrm{D}^{(0,\frac{1}{2})}$. Since there exist objects which transform under both representations simultaneously, we denote $\mathrm{D}^{(\frac{1}{2},0)}$ spinor indices by lower-case Greek letters α, β etc., whereas $\mathrm{D}^{(0,\frac{1}{2})}$ spinor indices are denoted by dotted lower-case Greek letters $\dot\alpha$, $\dot\beta$ etc.

Spinor indices are raised and lowered with the ε symbol,

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta\;, \tag{B.2}$$

where

$$\varepsilon = (\varepsilon^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \varepsilon^T = (\varepsilon_{\alpha\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\;. \tag{B.3}$$

Consequently, the scalar product of two spinors is symmetric, i.e.

$$\xi\cdot\eta := \xi^\alpha\varepsilon_{\alpha\beta}\eta^\beta = \xi^\alpha\eta_\alpha = -\xi_\alpha\eta^\alpha = \eta\cdot\xi\;, \tag{B.4}$$

where we used in the last step that the spinor components are a-numbers. Note that we also use the ε matrix for SU(2) contractions since $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$. The "square" of a spinor is defined[1] $\psi^2 = \psi \cdot \psi$. In the thesis, the dot "·" will be omitted. With these definitions, it is easy to check the relations

$$\theta\phi\,\theta\psi = -\frac{1}{2}\phi\psi\,\theta\theta\,, \tag{B.5a}$$

$$\bar\theta\bar\phi\,\bar\theta\bar\psi = -\frac{1}{2}\bar\phi\bar\psi\,\bar\theta\bar\theta\,. \tag{B.5b}$$

Furthermore, the σ**-matrices**, $\sigma^0_{\alpha\dot\alpha} := \mathbb{1}_2$,

$$\sigma^1_{\alpha\dot\alpha} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma^2_{\alpha\dot\alpha} := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\,, \quad \sigma^3_{\alpha\dot\alpha} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{B.6}$$

and

$$\bar\sigma^\mu := \begin{cases} \sigma^0\,, & \mu = 0 \\ -\sigma^\mu\,, & 1 \le \mu \le 3 \end{cases} \qquad \text{(in components)} \tag{B.7}$$

are used. With these, the following useful relations can be formulated:

$$(\sigma^\mu)_{\alpha\dot\alpha}\,(\sigma_\mu)_{\beta\dot\beta} = (\bar\sigma^\mu)_{\dot\alpha\alpha}\,(\bar\sigma_\mu)_{\dot\beta\beta} = 2\,\varepsilon_{\alpha\beta}\,\epsilon^{\dot\alpha\dot\beta}\,, \tag{B.8}$$

$$\theta\sigma^\mu\bar\theta\,\theta\sigma^\nu\bar\theta := \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,\theta^\beta\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta} = -\frac{1}{2}\theta\theta\,\bar\theta\bar\theta\,\eta^{\mu\nu}\,. \tag{B.9}$$

B.2 Dirac Spinors

Two Weyl spinors ξ and η can be combined to a **Dirac spinor**

$$\Psi := \begin{pmatrix} \xi \\ \bar\eta \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ \bar\eta^{\dot\alpha} \end{pmatrix}\,. \tag{B.10}$$

Here and throughout the thesis, we work in the Weyl basis, where γ_5 is diagonal, i.e.

$$\gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\,, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\,. \tag{B.11}$$

Using Dirac spinors turns out to be very convenient for calculating diagrams, the more so `FeynCalc` [18] is able to manage the resulting expressions. Therefore, the Weyl spinors of the SM are combined to Dirac spinors, and by the aid of the usual projectors $P_{\mathrm{L/R}} = (\mathbb{1} \mp \gamma_5)/2$ on left- and right-handed spinors, the Feynman rules for a chiral theory can be formulated. For example, the u-quark field is described by

$$\Psi_u = \begin{pmatrix} u_\alpha \\ (\bar u^{\mathrm{C}})^{\dot\alpha} \end{pmatrix}\,, \quad u_{\mathrm{R}} = P_{\mathrm{R}}\,\Psi_u = \begin{pmatrix} 0 \\ (\bar u^{\mathrm{C}})^{\dot\alpha} \end{pmatrix}\,. \tag{B.12}$$

For all other fermionic fields of the SM and the 2HDM's besides the singlet neutrino, analogous conventions are used.

[1] Note that in [38] a different convention is used!

B.3 Majorana Spinors

The charge conjugate of a Dirac spinor is defined by

$$\Psi^{\mathrm{C}} = \begin{pmatrix} \eta_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} = \mathsf{C}\,\overline{\Psi}^T \qquad \text{where} \quad \mathsf{C} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{B.13}$$

in the Weyl basis. Spinors which fulfill the condition $\Psi^{\mathrm{C}}=\Psi$ are called **Majorana spinors**. In the thesis, we use

$$N = \begin{pmatrix} {\nu^{\mathrm{C}}}_\alpha \\ \bar{\nu}^{\mathrm{C}\,\dot{\alpha}} \end{pmatrix} . \tag{B.14}$$

C Feynman Rules

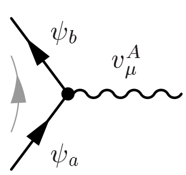
C.1 Feynman Rules in Fermion Number Violating Theories

According to [50], a fermion flow, indicated by a gray arrow, is introduced. It can be parallel or anti-parallel to the fermion number flow, which is in general not conserved in theories with Majorana fermions. The Feynman rules are then the common ones with two exceptions: Firstly, one has to read the diagrams reverse to the direction of the fermion flow and to write down the analytic expressions from left to right. Secondly, the propagator rules change as for example shown in (C.6). If fermion number is conserved (at least perturbatively), the fermion flow coincides with the fermion number flow. This is the case for the quark sector of the models we discuss in this thesis. Furthermore, the presented rules apply to four component spinors.

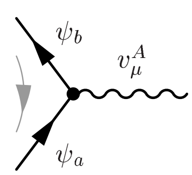
C.2 General SU(N) Gauge Theory

Gauge Boson - Fermion Interactions

The field ψ is assumed to transform under $\boldsymbol{N}$ of SU(N).



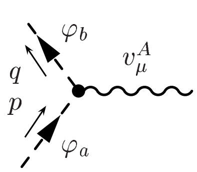
$: \; -\mathrm{i}\mu^{\frac{\epsilon}{2}} g \mathsf{T}^A_{ba}\gamma_\mu P_{\mathrm{L}}$



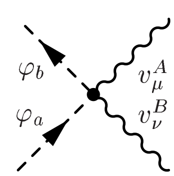
$: \; \mathrm{i}\mu^{\frac{\epsilon}{2}} g (\mathsf{T}^A)_{ba}\gamma_\mu P_{\mathrm{R}}$ (C.1)

Gauge Boson - Higgs Interactions

The field φ is assumed to transform under $\overline{\boldsymbol{N}}$ of SU(N).



$: \; \mathrm{i}\mu^{\frac{\epsilon}{2}} g (p_\mu + q_\mu)\mathsf{T}^A_{ab}$

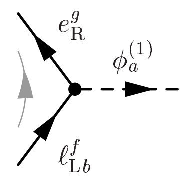


$: \; \mathrm{i}\mu^{\epsilon} g^2 \eta_{\mu\nu}[\mathsf{T}^A, \mathsf{T}^B]$ (C.2)

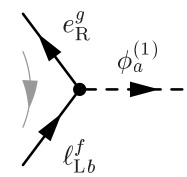
C.3 Rules for the Two Higgs Models of the Main Part

We formulate the Feynman rules for the relevant vertices and propagators of the model defined in section 5.2. Note that the corresponding rules for the SM are obtained by setting $\phi^{(1)} \equiv \phi$, $z_u^{(1)} = z_d^{(1)} = 1$ and omitting all vertices in which $\phi^{(2)}$ is involved.

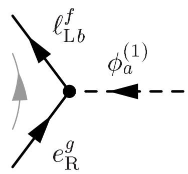
Yukawa Interactions



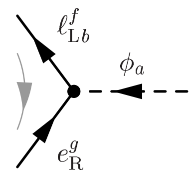
$: \; -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_e)_{gf}\delta_{ab}P_{\mathrm{L}}$



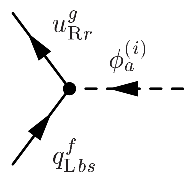
$: \; -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_e^T)_{fg}\delta_{ab}P_{\mathrm{L}}$ (C.3a)



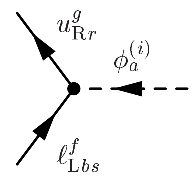
$: \; -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_e^\dagger)_{fg}\delta_{ab}P_{\mathrm{R}}$



$: \; -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_e^*)_{gf}\delta_{ab}P_{\mathrm{R}}$ (C.3b)

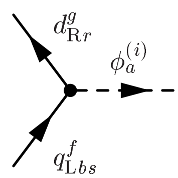


$: \; -\mathrm{i}z_u^{(i)}\mu^{\frac{\epsilon}{2}}(Y_u)_{gf}(\varepsilon^T)_{ab}P_{\mathrm{L}}$

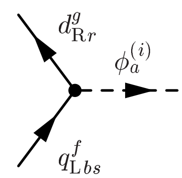


$: \; -\mathrm{i}z_u^{(i)}\mu^{\frac{\epsilon}{2}}(Y_u^T)_{fg}\varepsilon_{ba}\delta_{rs}P_{\mathrm{L}}$

(C.3c)



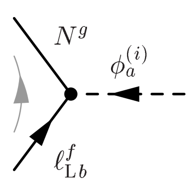
$: \; -\mathrm{i}z_d^{(i)}\mu^{\frac{\epsilon}{2}}(Y_d^\dagger)_{fg}\varepsilon_{ba}P_{\mathrm{R}}$



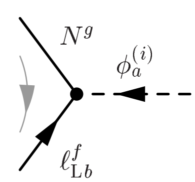
$: \; -\mathrm{i}z_d^{(i)}\mu^{\frac{\epsilon}{2}}(Y_d^*)_{gf}(\varepsilon^T)_{ab}P_{\mathrm{R}}$

(C.3d)

Additional Rules for the Possible Extension

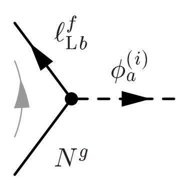


$: \; -\mathrm{i}z_\nu^{(i)}\mu^{\frac{\epsilon}{2}}(Y_\nu)_{gf}(\varepsilon^T)_{ab}P_{\mathrm{L}}$

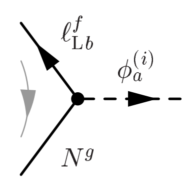


$: \; -\mathrm{i}z_\nu^{(i)}\mu^{\frac{\epsilon}{2}}(Y_\nu^T)_{fg}\varepsilon_{ba}P_{\mathrm{L}}$

(C.4a)



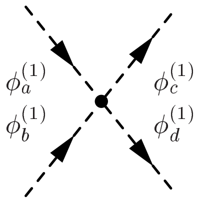
$: \; -\mathrm{i}z_\nu^{(i)}\mu^{\frac{\epsilon}{2}}(Y_\nu^\dagger)_{fg}\varepsilon_{ba}P_{\mathrm{R}}$



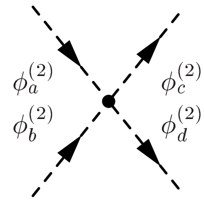
$: \; -\mathrm{i}z_\nu^{(i)}\mu^{\frac{\epsilon}{2}}(Y_\nu^*)_{gf}(\varepsilon^T)_{ab}P_{\mathrm{R}}$

(C.4b)

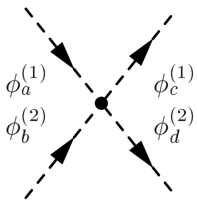
Feynman Rules for the Higgs Sector



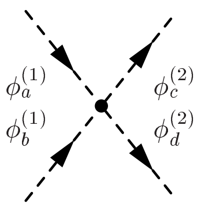
$$: \; -\mathrm{i}\mu^\epsilon \lambda_1 \frac{1}{2} \left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)$$



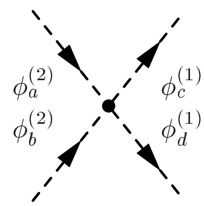
$$: \; -\mathrm{i}\mu^\epsilon \lambda_2 \frac{1}{2} \left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \tag{C.5a}$$



$$: \; -\mathrm{i}\mu^\epsilon \left(\lambda_3 \delta_{ac}\delta_{bd} + \lambda_4 \delta_{ad}\delta_{bc}\right) \tag{C.5b}$$



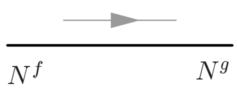
$$: \; -\mathrm{i}\mu^\epsilon \lambda_5^* \frac{1}{2} \left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)$$



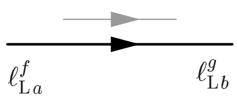
$$: \; -\mathrm{i}\mu^\epsilon \lambda_5 \frac{1}{2} \left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \tag{C.5c}$$

Propagators

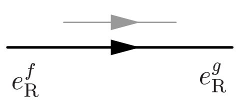
The momentum p used in the following formulae flows from left to right.



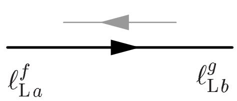
$$: \quad \mathrm{i}S_N(p) = \frac{\mathrm{i}(\not{p} + M_f)}{p^2 - M_f^2 + \mathrm{i}\varepsilon}\delta_{gf} \tag{C.6a}$$



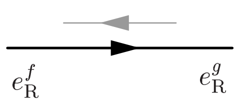
$$: \quad \mathrm{i}S_{\ell_\mathrm{L}}(p) = \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}\varepsilon}\delta_{gf}\delta_{ba} \tag{C.6b}$$



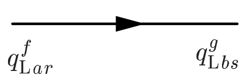
$$: \quad \mathrm{i}S_{e_\mathrm{R}}(p) = \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}\varepsilon}\delta_{gf} \tag{C.6c}$$



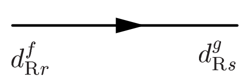
$$: \quad \mathrm{i}S_{\ell_\mathrm{L}}(-p) = \frac{-\mathrm{i}\not{p}}{p^2 + \mathrm{i}\varepsilon}\delta_{gf}\delta_{ba} \tag{C.6d}$$



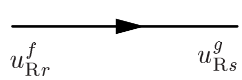
$$: \quad \mathrm{i}S_{e_\mathrm{R}}(-p) = \frac{-\mathrm{i}\not{p}}{p^2 + \mathrm{i}\varepsilon}\delta_{gf} \tag{C.6e}$$



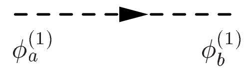
$$: \quad \mathrm{i}S_{q_\mathrm{L}}(p) = \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}\varepsilon}\delta_{gf}\delta_{ab}\delta_{rs} \tag{C.6f}$$



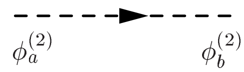
$$: \quad \mathrm{i}S_{d_\mathrm{L}}(p) = \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}\varepsilon}\delta_{gf}\delta_{rs} \tag{C.6g}$$



$$: \quad \mathrm{i}S_{u_\mathrm{R}}(p) = \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}\varepsilon}\delta_{gf}\delta_{rs} \tag{C.6h}$$



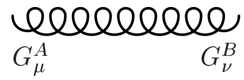
$$: \quad \mathrm{i}S_{\phi^{(1)}}(p) = \frac{\mathrm{i}}{p^2 - m_1^2 + \mathrm{i}\varepsilon}\delta_{ba} \tag{C.6i}$$



$$: \quad \mathrm{i}S_{\phi^{(2)}}(p) = \frac{\mathrm{i}}{p^2 - m_2^2 + \mathrm{i}\varepsilon}\delta_{ba} \tag{C.6j}$$

$$: \quad \mathrm{i}D_B^{\mu\nu}(p) = \mathrm{i}\frac{-\eta^{\mu\nu} + (1-\xi_B)\frac{p^\mu p^\nu}{p^2}}{p^2 + \mathrm{i}\varepsilon} \tag{C.6k}$$

$$: \quad \mathrm{i}D_{W^i}^{\mu\nu}(p) = \mathrm{i}\frac{-\eta^{\mu\nu} + (1-\xi_W)\frac{p^\mu p^\nu}{p^2}}{p^2 + \mathrm{i}\varepsilon}\delta_{ij} \tag{C.6l}$$

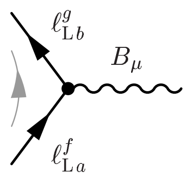


$$: \quad \mathrm{i}D_{G^A}^{\mu\nu}(p) = \mathrm{i}\frac{-\eta^{\mu\nu} + (1-\xi_G)\frac{p^\mu p^\nu}{p^2}}{p^2 + \mathrm{i}\varepsilon}\delta^{AB} \tag{C.6m}$$

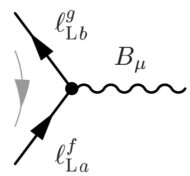
Note that the ghost rules are not listed since they are not needed for our calculations.

Gauge Boson - Fermion Interactions

In the following rules, GUT charge normalization for $\mathrm{U}(1)_\mathrm{Y}$ is not used.

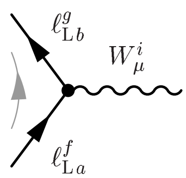


$: \ \frac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}} g_1 \delta_{gf}\delta_{ba}\gamma_\mu P_\mathrm{L}$

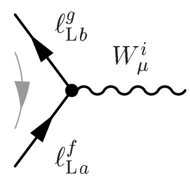


$: \ -\frac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}} g_1 \delta_{fg}\delta_{ab}\gamma_\mu P_\mathrm{R}$

(C.7a)

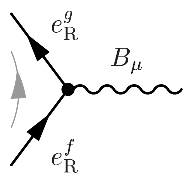


$: \ -\frac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}} g_2 \delta_{gf}\sigma^i_{ba}\gamma_\mu P_\mathrm{L}$

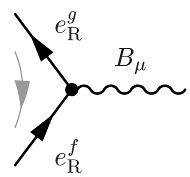


$: \ \frac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}} g_2 \delta_{fg}({\sigma^i}^T)_{ab}\gamma_\mu P_\mathrm{R}$

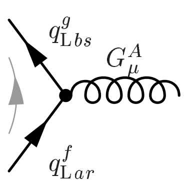
(C.7b)



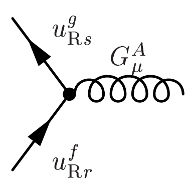
$: \ \mathrm{i}\mu^{\frac{\epsilon}{2}} g_1 \delta_{gf}\gamma_\mu P_\mathrm{R}$



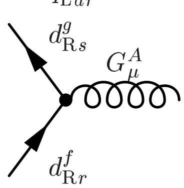
$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}} g_1 \delta_{fg}\gamma_\mu P_\mathrm{L}$ (C.7c)



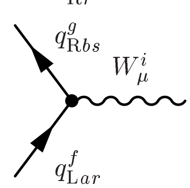
$: \ -\frac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}} g_3 \delta_{gf}\lambda^A_{sr}\delta_{ab}\gamma_\mu P_\mathrm{L}$



$: \ \frac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}} g_3 \delta_{fg}\lambda^A_{sr} P_\mathrm{R}$ (C.7d)

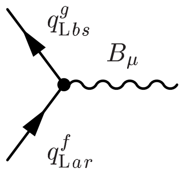


$: \ \frac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}} g_3 \delta_{fg}\lambda^A_{sr} P_\mathrm{R}$

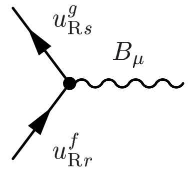


$: \ \frac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}} g_2 \delta_{fg}\sigma^i_{ba}\delta_{rs}\gamma_\mu P_\mathrm{L}$

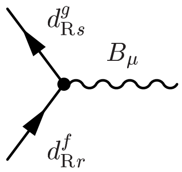
(C.7e)



$$: \; -\tfrac{\mathrm{i}}{6}\mu^{\frac{\epsilon}{2}}g_1\delta_{gf}\delta_{rs}\delta_{ab}\gamma_\mu P_{\mathrm{L}}$$

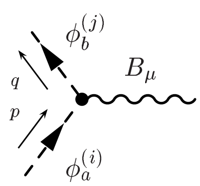


$$: \; \tfrac{2\mathrm{i}}{3}\mu^{\frac{\epsilon}{2}}g_1\delta_{fg}\delta_{rs}\gamma_\mu P_{\mathrm{R}} \qquad \text{(C.7f)}$$

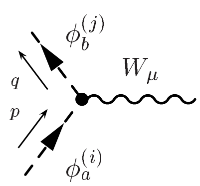


$$: \; -\tfrac{\mathrm{i}}{3}\mu^{\frac{\epsilon}{6}}g_1\delta_{gf}\delta_{rs}\delta_{rs}\delta_{ab}\gamma_\mu P_{\mathrm{R}} \qquad \text{(C.7g)}$$

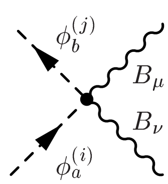
Gauge Boson - Higgs Interactions



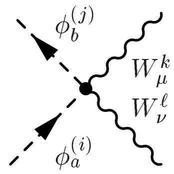
$$: \quad -\tfrac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}}g_1(p_\mu + q_\mu)\delta_{ba}\delta_{ij} \qquad \text{(C.8a)}$$



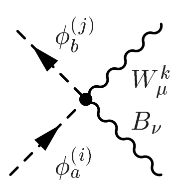
$$: \quad -\tfrac{\mathrm{i}}{2}\mu^{\frac{\epsilon}{2}}g_2(p_\mu + q_\mu)\sigma^i_{ba}\delta_{ij} \qquad \text{(C.8b)}$$



$$: \quad \tfrac{\mathrm{i}}{2}\mu^{\epsilon}g_1^2\eta_{\mu\nu}\delta_{ba}\delta_{ij} \qquad \text{(C.8c)}$$

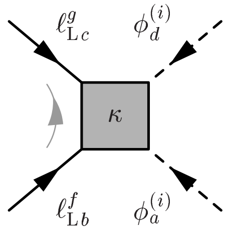


$$: \quad \tfrac{\mathrm{i}}{2}\mu^{\epsilon}g_2^2\eta_{\mu\nu}\delta_{k\ell}\delta_{ba}\delta_{ij} \qquad \text{(C.8d)}$$



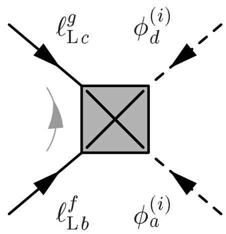
$$: \quad \tfrac{\mathrm{i}}{2}\mu^{\epsilon}g_1g_2\eta_{\mu\nu}\sigma^k_{ba}\delta_{ij} \qquad \text{(C.8e)}$$

Effective Vertices



$$: \ \mathrm{i}\mu^{\epsilon}\kappa_{gf}^{(ii)}\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})P_{\mathrm{L}} \qquad \text{(C.9)}$$

Effective Vertex Counterterms

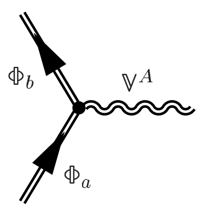


$$: \ \mathrm{i}\mu^{\epsilon}\delta\kappa_{gf}^{(ii)}\tfrac{1}{2}(\varepsilon_{cd}\varepsilon_{ba}+\varepsilon_{ca}\varepsilon_{bd})P_{\mathrm{L}} \qquad \text{(C.10)}$$

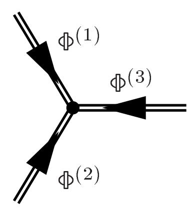
D Supergraph Rules

D.1 General Rules

The rules for the sample calculations of section 6.2.2 read:

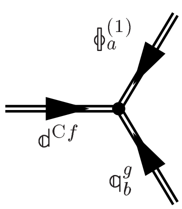


$$: \ 2\mathrm{i}\, g\, \mu^{\frac{\epsilon}{2}} \mathsf{T}_{ab}^{A}$$

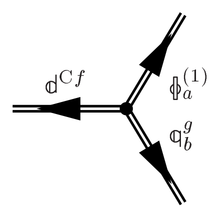


$$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}\lambda_{ijk} \qquad \text{(D.1)}$$

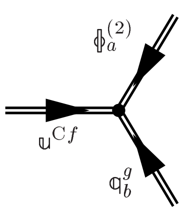
D.2 Rules for the MSSM Superpotential



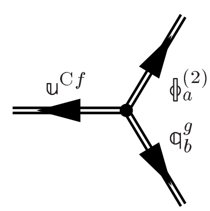
$$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_d)_{gf}\delta_{ab}$$



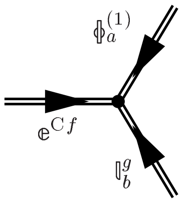
$$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_d^{\dagger})_{gf}\delta_{ab} \qquad \text{(D.2a)}$$



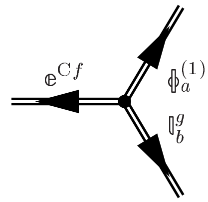
$$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_u)_{gf}\delta_{ab}$$



$$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_u^{\dagger})_{gf}\delta_{ab} \qquad \text{(D.2b)}$$



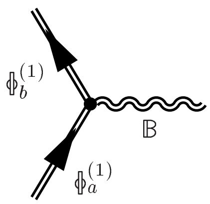
$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_e)_{gf}\delta_{ab}$



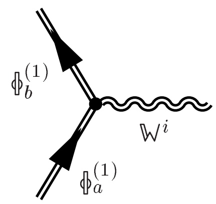
$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_e^\dagger)_{gf}\delta_{ab}$ (D.2c)

D.3 Rules for the Gauge Interactions

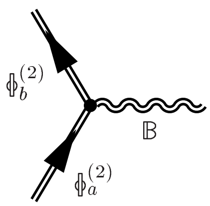
In the following rules, GUT charge normalization for $\mathrm{U}(1)_\mathrm{Y}$ is used.



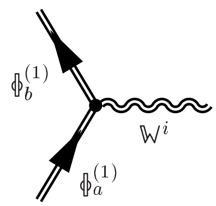
$: \ \sqrt{\frac{3}{5}}\mu^{\frac{\epsilon}{2}}g_1\delta_{ab}\delta_{gf}$



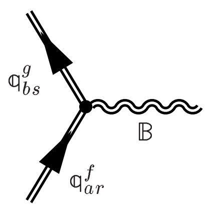
$: \ \mu^{\frac{\epsilon}{2}}g_2\sigma^i_{ba}\delta_{gf}$ (D.3a)



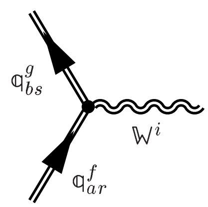
$: \ -\sqrt{\frac{3}{5}}\mu^{\frac{\epsilon}{2}}g_1\delta_{ab}\delta_{gf}$



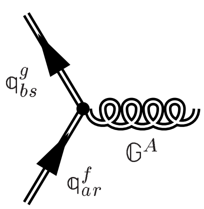
$: \ \mu^{\frac{\epsilon}{2}}g_2\sigma^i_{ba}\delta_{gf}$ (D.3b)



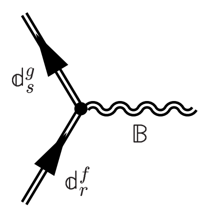
$: \ \sqrt{\frac{3}{5}}\frac{1}{3}\mu^{\frac{\epsilon}{2}}g_1\delta_{ab}\delta_{gf}\delta_{rs}$



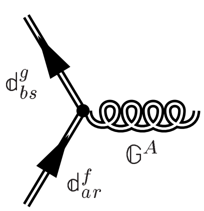
$: \ \mu^{\frac{\epsilon}{2}}g_2\sigma^i_{ba}\delta_{rs}\delta_{gf}$ (D.3c)



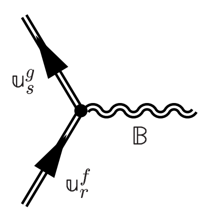
$: \ \mu^{\frac{\epsilon}{2}}g_3\delta_{ab}\lambda^A_{sr}\delta_{gf}$



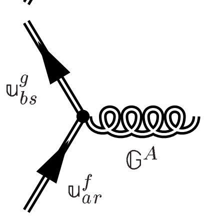
$: \ -\frac{2}{3}\sqrt{\frac{3}{5}}\mu^{\frac{\epsilon}{2}}g_1\delta_{gf}\delta_{rs}$ (D.3d)



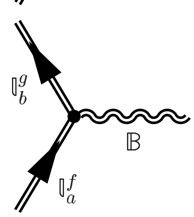
$: \ \mu^{\frac{\epsilon}{2}}g_3\lambda^A_{sr}\delta_{gf}$



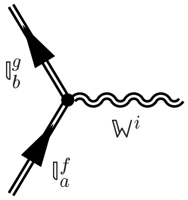
$: \ \frac{4}{3}\sqrt{\frac{3}{5}}\mu^{\frac{\epsilon}{2}}g_1\delta_{gf}\delta_{rs}$ (D.3e)



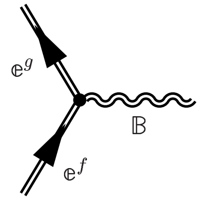
$: \ \mu^{\frac{\epsilon}{2}}g_3\lambda^A_{sr}\delta_{gf}$



$: \ -\sqrt{\frac{3}{5}}\mu^{\frac{\epsilon}{2}}g_1\delta_{ab}$ (D.3f)

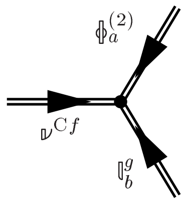


$: \ \mu^{\frac{\epsilon}{2}} g_2 \sigma^i_{ab}$

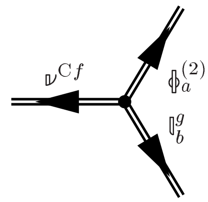


$: \ -2\sqrt{\frac{3}{5}}\mu^{\frac{\epsilon}{2}} g_1 \delta_{gf}$ (D.3g)

D.4 Additional Rules for the νMSSM

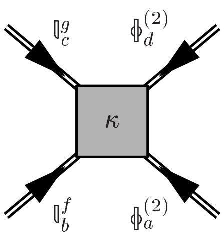


$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_\nu)_{gf}\delta_{ab}$

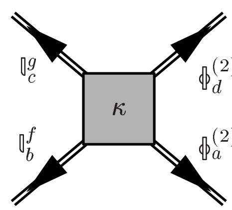


$: \ -\mathrm{i}\mu^{\frac{\epsilon}{2}}(Y_\nu^\dagger)_{gf}\delta_{ab}$ (D.4)

D.5 Rules for the Effective Neutrino Mass Operator



$: \ \mathrm{i}\mu^\epsilon \kappa_{gf}\frac{1}{2}\left(\varepsilon_{cd}\varepsilon_{ba} + \varepsilon_{ca}\varepsilon_{bd}\right)$



$: \ \mathrm{i}\mu^\epsilon \kappa_{gf}\frac{1}{2}\left(\varepsilon_{cd}\varepsilon_{ba} + \varepsilon_{ca}\varepsilon_{bd}\right)$

(D.5)

E Integrals in d Dimensions

E.1 Feynman Parameterization

Products of denominators can be rewritten in the following way:

$$\frac{1}{A_1\,A_2\cdots A_n} = \int_0^1 \mathrm{d}x_1\cdots\mathrm{d}x_n\,\delta\left(\sum x_i - 1\right)\frac{(n-1)!}{\left[x_1\,A_1 + x_2\,A_2 + \cdots x_n\,A_n\right]^n}\,, \tag{E.1a}$$

$$\frac{1}{A\,B} = \int_0^1 \mathrm{d}x\,\frac{1}{\left[x\,A + (1-x)\,B\right]^2}\,, \tag{E.1b}$$

$$\frac{1}{A^\alpha\,B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^1 \mathrm{d}x\,\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\left[x\,A + (1-x)\,B\right]^{\alpha+\beta}}\,, \tag{E.1c}$$

$$\frac{1}{A\,B\,C} = 2\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\,\frac{y}{\left[x\,y\,A + (1-x)\,y\,B + (1-y)\,C\right]^3}\,. \tag{E.1d}$$

E.2 d-Dimensional Integrals in Minkowski Space

Useful Integrals

After employing the Feynman parameterization it is often possible to transform the resulting integrals into one of the following by a change of variables.

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^n} = \mathrm{i}\frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2}, \tag{E.2a}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2-\Delta)^n} = -\mathrm{i}\frac{d}{2}\frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma\left(n-1-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-1-d/2}, \tag{E.2b}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2-\Delta)^n} = -\frac{\mathrm{i}}{2}\eta^{\mu\nu}\frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma\left(n-1-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-1-d/2}, \tag{E.2c}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\left(k^2\right)^2}{(k^2-\Delta)^n} = \mathrm{i}\frac{d(d+2)}{4}\frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma\left(n-2-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-2-d/2}, \tag{E.2d}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2-\Delta)^n} = \mathrm{i}\frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma\left(n-2-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-2-d/2} \times$$

$$\times\frac{1}{4}\left(\eta^{\mu\nu}\eta^{\rho\sigma}+\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}\right) . \tag{E.2e}$$

By symmetry, the integral over odd powers of k^μ in the numerator vanishes,

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} k^\mu f\left(k^2\right) = 0 . \tag{E.3a}$$

For products of the form $k^\mu k^\nu$ the following formula can be used:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} k^\mu k^\nu f\left(k^2\right) = \frac{1}{d}\eta^{\mu\nu} \int \frac{\mathrm{d}^d k}{(2\pi)^d} k^2 f\left(k^2\right) . \tag{E.3b}$$

Powers of Δ

$$\Delta^\epsilon = 1 + \epsilon \ln\Delta + \mathscr{O}\left(\epsilon^2\right). \tag{E.4a}$$

In particular:

$$\left(\frac{1}{\Delta}\right)^{n-d/2} = \left(\frac{1}{\Delta}\right)^{n-2} \cdot \left[1-\left(2-\tfrac{d}{2}\right)\ln\Delta + \mathscr{O}\left(\left(2-\tfrac{d}{2}\right)^2\right)\right] . \tag{E.4b}$$

Γ-Function

The Γ-function is defined as

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, t^{z-1} e^{-t} \tag{E.5}$$

and satisfies

$$\Gamma(x+1) = x\,\Gamma(x) \; , \tag{E.6}$$

i.e. it can be regarded as the continuous generalization of the faculty, $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. $\Gamma(z)$ diverges at 0 and negative integers. Around these poles it can be expanded in the following manner:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} + \psi(1) + \mathcal{O}(\epsilon) \; , \tag{E.7a}$$

$$\Gamma\left(-n + \frac{\epsilon}{2}\right) = \frac{(-1)^n}{n!}\left[\frac{2}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right] \; , \tag{E.7b}$$

where ϵ has to be positive and where ψ satisfies

$$\psi(z+1) = \psi(z) + \frac{1}{z} \; , \tag{E.8a}$$

$$\psi(1) = -\gamma_{\mathrm{E}} \; . \tag{E.8b}$$

$\gamma_{\mathrm{E}} = -0.5772\ldots$ is Euler's constant. Furthermore,

$$\Gamma(1-\epsilon) = 1 + \mathcal{O}(\epsilon^2) \; , \tag{E.9a}$$

$$\Gamma(2-\epsilon) = 1 - \epsilon + \mathcal{O}(\epsilon^2) \; . \tag{E.9b}$$

Beta-Function

Euler's B-function is defined by

$$B(\alpha+1, \beta+1) := \int_0^1 \mathrm{d}x\, x^\alpha (1-x)^\beta \; , \tag{E.10}$$

and can be expressed by Γ-functions,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \; . \tag{E.11}$$

Some useful expansions are given by

$$B(1-\epsilon, 1-\epsilon) = 1 + 2\epsilon + \mathcal{O}(\epsilon^2) \; , \tag{E.12a}$$

$$B(2-\epsilon, 1-\epsilon) = \tfrac{1}{2}(1 + 2\epsilon) + \mathcal{O}(\epsilon^2) \; , \tag{E.12b}$$

$$B(2-\epsilon, 2-\epsilon) = \tfrac{1}{6}(1 + \tfrac{5}{3}\epsilon) + \mathcal{O}(\epsilon^2) \; , \tag{E.12c}$$

$$B(3-\epsilon, 1-\epsilon) = \tfrac{1}{3}(1 + \tfrac{13}{6}\epsilon) + \mathcal{O}(\epsilon^2) \; . \tag{E.12d}$$

A Useful Formula

By combining the equations (E.4b) and (E.7a), we obtain

$$\Gamma\left(2 - \frac{d}{2}\right)\left(\frac{4\pi\,\mu^2}{\Delta}\right)^{2-d/2} = \frac{2}{\epsilon} - \gamma_{\mathrm{E}} - \ln\left(\frac{\Delta}{4\pi\,\mu^2}\right) + \mathcal{O}(\epsilon) \; . \tag{E.13}$$

F Passarino-Veltman Functions

In this appendix we introduce some standard integrals, known as **Passarino-Veltman functions** in the literature, since these standard integrals were first discussed by Passarino and Veltman [51]. The use of these functions has the advantage that the reduction of loop integrals can be automatized by using the `Mathematica` package `FeynCalc` [18]. Moreover, the functions are implemented in the `LoopTool` [52] package.

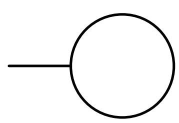
F.1 $\Delta_{\overline{\mathrm{MS}}}$ and the One-Point Integral A

First we define the quantity $\Delta_{\overline{\mathrm{MS}}}$ which represents the infinity in $d=4$ dimensions appearing in dimensional regularization,

$$\Delta_{\overline{\mathrm{MS}}} := \frac{2}{4-d} - \gamma_{\mathrm{E}} + \ln(4\pi) \; . \tag{F.1}$$

This is the part which is usually removed in Modified Minimal Subtraction ($\overline{\mathrm{MS}}$) [53]. Now we introduce the one-point function by

$$A(m^2) \;:=\; -\mathrm{i}\, 2^d\, \pi^{d-2}\, \mu^\epsilon \times$$



$$m \;=\; \frac{\mu^\epsilon}{\mathrm{i}\pi^2} \int \mathrm{d}^d q \, \frac{1}{q^2 - m^2} \; . \tag{F.2}$$

$$= \; m^2 \left(\frac{2}{\epsilon} - \ln \pi - \gamma_{\mathrm{E}} + 1 + \ln \frac{\mu^2}{m^2} \right) + \mathscr{O}(\epsilon) \; . \tag{F.3}$$

The `LoopTool` package [52] returns

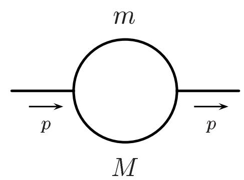
$$A(m^2) \xrightarrow{\texttt{LoopTools}} A_0(m^2) = m^2 \cdot (1 - \ln m^2) \; , \tag{F.4}$$

i.e. the diagram is calculated in the $\overline{\mathrm{MS}}$ scheme and μ is set to 1.

F.2 The Two-Point Integrals B

The two-point integrals are defined by

$$B_0(p^2,m^2,M^2) \;:=\; -i2^d\pi^{d-2}\mu^{4-d}\times$$



$$\begin{aligned}
&= \frac{\mu^\epsilon}{\mathrm{i}\pi^2}\int \mathrm{d}^d q\, \frac{1}{(q^2-m^2)\left((q+p)^2-M^2\right)}\;, && \text{(F.5a)}\\
B_\mu(p^2,m^2,M^2) &:= \frac{\mu^\epsilon}{\mathrm{i}\pi^2}\int \mathrm{d}^d q\, \frac{q_\mu}{(q^2-m^2)\left((q+p)^2-M^2\right)} && \text{(F.5b)}\\
&=: p_\mu B_1(p^2,m^2,M^2)\;, && \text{(F.5c)}\\
B_{\mu\nu}(p^2,m^2,M^2) &:= \frac{\mu^\epsilon}{\mathrm{i}\pi^2}\int \mathrm{d}^d q\, \frac{q_\mu q_\nu}{(q^2-m^2)\left((q+p)^2-M^2\right)} && \\
&=: p_\mu p_\nu B_{11}(p,m,M)+\eta_{\mu\nu}B_{00}(p,m,M)\;, && \text{(F.5d)}
\end{aligned}$$

where we have introduced the functions B_1, B_{00} and B_{11}.

These functions are related algebraically,

$$\begin{aligned}
B_1(p^2,m^2,M^2) &= \frac{1}{2p^2}\big[A(m^2)-A(M^2) \\
&\qquad + (M^2-m^2-p^2)B_0(p^2,m^2,M^2)\big]\;, && \text{(F.6a)}\\
B_{00}(p^2,m^2,M^2) &= \frac{1}{3p^2}\Bigg[A(M^2)-m^2B_0(p^2,m^2,M^2)-2(p^2+m^2-M^2)\times \\
&\qquad \times B_1(p^2,m^2,M^2)-\frac{1}{2}\left(m^2+M^2-\frac{1}{3}p^2\right)\Bigg]\;, && \text{(F.6b)}\\
B_{11}(p^2,m^2,M^2) &= \frac{1}{6}\Bigg[A(M^2)+2m^2B_0(p^2,m^2,M^2)+ \\
&\qquad +(p^2+m^2-M^2)B_1(p^2,m^2,M^2)m^2+M^2-\frac{1}{3}p^2\Bigg]\;, && \text{(F.6c)}
\end{aligned}$$

and for $d\to 4$ we obtain

$$\begin{aligned}
B_0(p^2,0,0) &= \Delta_{\overline{\mathrm{MS}}}-\ln\left(\frac{|p^2|}{\mu^2}\right)+2+\mathrm{i}\pi\,\theta(p^2)\;, && \text{(F.7a)}\\
B_0(0,m^2,0) &= B_0(0,0,m^2)=\Delta_{\overline{\mathrm{MS}}}-\ln\left(\frac{m^2}{\mu^2}\right)+1=\frac{1}{m^2}A(m^2)\;, && \text{(F.7b)}\\
B_0(p^2,0,M^2) &= \Delta_{\overline{\mathrm{MS}}}+2+\frac{M^2}{p^2}\ln\left(1-\frac{p^2}{M^2}\right)+\ln\left(\frac{M^2-p^2}{\mu^2}\right)\;. && \text{(F.7c)}
\end{aligned}$$

If one is interested in the finite part of B_0, the formula

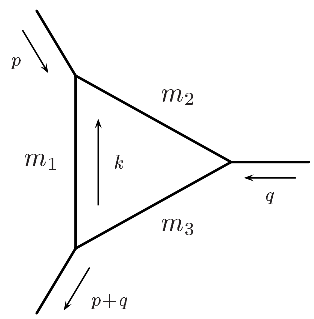
$$B_0(p^2, m^2, M^2) = \Delta_{\overline{\mathrm{MS}}} - \\ - \int_0^1 \mathrm{d}x \, \ln\left(\frac{x^2 p^2 - x(p^2 + m^2 - M^2) + m^2 - i\varepsilon}{\mu^2}\right) + \mathscr{O}(\epsilon) \quad \text{(F.8)}$$

may be useful. This implies that the infinity, that is the pole in ϵ, is proportional to $\Delta_{\overline{\mathrm{MS}}}$. Further poles are listed in table F.1.

F.3 The Three-Point Integrals C

The three-point integrals are defined by

$$C_0 := -\mathrm{i}2^d \pi^{d-2} \mu^\epsilon \times$$



$$= \frac{\mu^\epsilon}{\mathrm{i}\pi^2} \int \mathrm{d}^d k \, \frac{1}{(k^2 - m_1^2)\left((k+p)^2 - m_2^2\right)\left((k+p+q)^2 - m_3^2\right)} \, , \quad \text{(F.9a)}$$

$$C_\mu := \frac{\mu^\epsilon}{\mathrm{i}\pi^2} \int \mathrm{d}^d k \, \frac{k_\mu}{(k^2 - m_1^2)\left((k+p)^2 - m_2^2\right)\left((k+p+q)^2 - m_3^2\right)} \, , \quad \text{(F.9b)}$$

$$C_{\mu\nu} := \frac{\mu^\epsilon}{\mathrm{i}\pi^2} \int \mathrm{d}^d k \, \frac{k_\mu k_\nu}{(k^2 - m_1^2)\left((k+p)^2 - m_2^2\right)\left((k+p+q)^2 - m_3^2\right)} \, , \quad \text{(F.9c)}$$

$$C_{\mu\nu\rho} := \frac{\mu^\epsilon}{\mathrm{i}\pi^2} \int \mathrm{d}^d k \, \frac{k_\mu k_\nu k_\rho}{(k^2 - m_1^2)\left((k+p)^2 - m_2^2\right)\left((k+p+q)^2 - m_3^2\right)} \, , \quad \text{(F.9d)}$$

where $C_0 = C_0(p^2, q^2, (p+q)^2, m_1^2, m_2^2, m_3^2)$ etc.

C_0 can be expressed by dilogarithms [54,55],

$$C_0(s_1, s_2, s_3, m_1^2, m_2^2, m_3^2) = \\ \frac{1}{\sqrt{\lambda(s_1, s_2, s_3)}} \sum_{i=1}^{3} \sum_{j=1}^{2} \mathrm{Li}_2\left(\frac{x_i - 1}{x_i - z_i^j}\right) - \mathrm{Li}_2\left(\frac{x_i}{x_i - z_i^j}\right) \, , \quad \text{(F.10)}$$

where

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2\,(x\,y + y\,z + z\,x) \quad \text{(F.11)}$$

and

$$
\begin{aligned}
x_1 &= \frac{1}{2}\left(1+\frac{s_1-s_2-s_3}{\sqrt{\lambda(s_1,s_2,s_3)}}\right)+\frac{m_3^2}{\sqrt{\lambda(s_1,s_2,s_3)}}+ \\
&\quad +\frac{1}{2s_1}\left[-m_1^2\left(1+\frac{s_1+s_2-s_3}{\sqrt{\lambda(s_1,s_2,s_3)}}\right)+m_2^2\left(1-\frac{s_1-s_2+s_3}{\sqrt{\lambda(s_1,s_2,s_3)}}\right)\right]\ , &\text{(F.12a)} \\
z_1^{1/2} &= \frac{1}{2}\left(1-\frac{m_1^2-m_2^2}{s_1}\pm\sqrt{1-\frac{m_1^2-m_2^2}{s_1}-\frac{4\ (m_2^2-\mathrm{i}\,\varepsilon)}{s_1}}\right) &\text{(F.12b)}
\end{aligned}
$$

et cycl. For a real $a \leq 1$, the dilogarithm has the properties

$$
\begin{aligned}
\mathrm{Li}_2(a\pm\mathrm{i}\varepsilon) &= -\,\mathrm{Li}_2\left(\tfrac{1}{a}\right)+\frac{\pi^2}{3}-\frac{1}{2}(\ln a)^2\pm\mathrm{i}\ln a\ , &\text{(F.13a)} \\
\mathrm{Li}_2(1-a) &= -\,\mathrm{Li}_2(a)-\ln(a)\ \ln(1-a)+\zeta(2)\ , &\text{(F.13b)} \\
\mathrm{Li}_2(a) &= a+\frac{a^2}{4}+\frac{a^3}{9}+\mathscr{O}(a^4)\ . &\text{(F.13c)}
\end{aligned}
$$

Some other useful properties of the dilogarithm Li_2 can be found in [56].

Furthermore, if only one mass scale is involved, the relations

$$
C_0(0,0,s,M^2,0,0)=\frac{1}{s}\,\mathrm{Li}_2\left(\frac{s}{M^2}\right) \tag{F.14}
$$

and

$$
\begin{aligned}
\mathrm{Re}\,C_0(0,0,s,0,M^2,0) &= \frac{1}{s}\left\{\mathrm{Re}\left[\ln\left(\frac{M^2}{M^2+s}\right)\ln\left(\frac{M^2+s}{s}\right)-\mathrm{Li}_2\left(\frac{2\,M^2}{M^2+s}\right)\right.\right. \\
&\quad \left.\left.-\,\mathrm{Li}_2\left(\frac{s}{M^2+s}\right)\right]+\frac{\pi^2}{4}\right\} &\text{(F.15)}
\end{aligned}
$$

hold. From the second relation, we infer the approximation

$$
\mathrm{Re}\,C_0(0,0,s,0,M^2,0)\approx\begin{cases}\dfrac{1}{M^2}\ln\left(\dfrac{s}{M^2}\right)\ , & s\ll M^2\ , \\ 0\ , & s\gg M^2\ .\end{cases} \tag{F.16}
$$

Formula (F.14) is derived as follows: By equation (E.1a) we obtain

$$
\begin{aligned}
\mathscr{I} &:= \mu^\epsilon\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{1}{(k^2-M^2)\,(k+p)^2\,(k+p+q)^2} &\text{(F.17)} \\
&= \mu^\epsilon\int_0^1\mathrm{d}x\int_0^x\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{2}{\left[x(k^2-M^2)+(1-x-y)\,(k+p)^2+y\,(k+p+q)^2\right]^3}\ .
\end{aligned}
$$

Using the on-shell momenta $p^2 = q^2 = 0$ and $(p+q)^2 = s$, we can rewrite the term in brackets $[\dots] = \ell^2 - \Delta$, where $\ell = k + (1-x)\,p + y\,q$ and $\Delta = x\,M^2 - xy\,s$. By equation (E.2), we obtain

$$\begin{aligned}
\mathscr{I} &= \mathrm{i}\mu^\epsilon \frac{-1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(3-\frac{d}{2}\right)}{2} \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \, \frac{2}{\left[x\,M^2 - xy\,s\right]^{3-\frac{d}{2}}} \\
&= \frac{-\mathrm{i}}{(4\pi)^{\frac{d}{2}}} \,\mu^\epsilon \,\frac{\epsilon}{2}\,\Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 \mathrm{d}x \left[\frac{-1}{x\,s}\,\frac{-1}{2-\frac{d}{2}}\,\frac{1}{\left[x\,M^2 - xy\,s\right]^{2-\frac{d}{2}}}\right]_{y=0}^{y=x} \\
&= \frac{-\mathrm{i}}{(4\pi)^{\frac{d}{2}}} \,\mu^\epsilon \,\Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 \mathrm{d}x \,\frac{1}{x\,s} \left\{\frac{1}{\left[x\,M^2 - x^2\,s\right]^{\frac{\epsilon}{2}}} - \frac{1}{\left[x\,M^2\right]^{\frac{\epsilon}{2}}}\right\} \\
&\overset{(E.13)}{=} \frac{\mathrm{i}\pi^2}{(2\pi)^d}\,\frac{1}{s} \int_0^1 \mathrm{d}x \,\frac{1}{x} \ln\left(\frac{M^2}{M^2 - x\,s}\right) + \mathscr{O}(\epsilon) \\
&= \frac{\mathrm{i}\pi^2}{(2\pi)^d}\,\frac{1}{s}\,\mathrm{Li}_2\left(\frac{s}{M^2}\right) + \mathscr{O}(\epsilon)\;. \qquad \text{(F.18)}
\end{aligned}$$

For formula (F.15) we consider

$$\begin{aligned}
\mathscr{J} &= \mu^\epsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \,\frac{1}{k^2\,\left[(k+p)^2 - M^2\right]\,(k+p+q)^2} \qquad \text{(F.19)} \\
&= \mu^\epsilon \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int \frac{\mathrm{d}^d k}{(2\pi)^d} \,\frac{2}{\left\{(1-x-y)\,k^2 + y\left[(k+p)^2 - M^2\right] + x\,(k+p+q)^2\right\}^3}
\end{aligned}$$

By using $p^2 = q^2 = 0$, $(p+q)^2 = s$ and rewriting $\{\dots\} = \ell^2 - \Delta$ with $\ell = k + x(p+q) + yp$ and $\Delta = x(x+y-1)\,s + y\,M^2$, we obtain

$$\begin{aligned}
\mathscr{J} &= \mathrm{i}\mu^\epsilon \frac{-1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(3-\frac{d}{2}\right)}{2} \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \, \frac{2}{\left[x(x+y-1)\,s + y\,M^2\right]^{3-\frac{d}{2}}} \\
&= \frac{\mathrm{i}}{(2\pi)^d}\,\mu^\epsilon\,\Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 \mathrm{d}x \,\frac{-1}{x\,s - M^2} \left\{\frac{1}{\left[x(2x-1)\,s + x\,M^2\right]^{\frac{\epsilon}{2}}} - \frac{1}{\left[x(x-1)\,s\right]^{\frac{\epsilon}{2}}}\right\} \\
&= \frac{\mathrm{i}}{(2\pi)^d} \int_0^1 \mathrm{d}x \,\frac{1}{x\,s - M^2}\,\ln\left(\frac{(x-1)\,\frac{s}{M^2}}{(2x-1)\,\frac{s}{M^2} + 1}\right) + \mathscr{O}(\epsilon) \\
&= \frac{1}{s}\left\{\mathrm{Re}\left[\ln\left(\frac{M^2}{M^2+s}\right)\,\ln\left(\frac{M^2+s}{s}\right) - \mathrm{Li}_2\left(\frac{2\,M^2}{M^2+s}\right) - \mathrm{Li}_2\left(\frac{s}{M^2+s}\right)\right]\right. \\
&\qquad \left. + \frac{\pi^2}{4}\right\} + \mathscr{O}(\epsilon)\;. \qquad \text{(F.20)}
\end{aligned}$$

Conventionally the tensor integrals are decomposed in the following way [57],

$$C_\mu = p_\mu C_1 + q_\mu C_2 \,, \tag{F.21a}$$
$$C_{\mu\nu} = \eta_{\mu\nu} C_{00} + p_\mu p_\nu C_{11} + q_\mu q_\nu C_{22} + (pq)_{(\mu\nu)} C_{12} \,, \tag{F.21b}$$
$$\begin{aligned} C_{\mu\nu\rho} = \; & (p\eta)_{(\mu\nu\rho)} C_{001} + (q\eta)_{(\mu\nu\rho)} C_{002} + \\ & + p_\mu p_\nu p_\rho C_{111} + q_\mu q_\nu q_{2\rho} C_{222} + (qpp)_{(\mu\nu\rho)} C_{112} + (pqq)_{(\mu\nu\rho)} C_{122} \,, \end{aligned} \tag{F.21c}$$

where we have used the following abbreviations

$$(pq)_{(\mu\nu)} := p_\mu q_\nu + p_\nu q_\mu \,, \tag{F.22a}$$
$$(pqq)_{(\mu\nu\rho)} := p_\mu q_\nu q_\rho + q_\mu p_\nu q_\rho + q_\mu q_\nu p_\rho \,, \tag{F.22b}$$
$$(p\eta)_{(\mu\nu\rho)} := p_\mu \eta_{\nu\rho} + p_\nu \eta_{\mu\rho} + p_\rho \eta_{\mu\nu} \,. \tag{F.22c}$$

F.4 Divergent Parts of the Passarino-Veltman Functions

The non-vanishing poles of the of the Passarino-Veltman one-, two- and three-point functions are summarized in table F.1.

Integral	Divergent part
$A(m^2)$	$\frac{2}{\epsilon} m^2$
$B_0(p^2, m^2, M^2)$	$\frac{2}{\epsilon}$
$B_1(p^2, m^2, M^2)$	$-\frac{1}{\epsilon}$
$B_{11}(p^2, m^2, M^2)$	$\frac{2}{3\epsilon}$
$B_{00}(p^2, m^2, M^2)$	$-\frac{1}{6\epsilon}(p^2 - 3m^2 - 3M^2)$
$C_{00}(p^2, q^2, (p+q)^2, m_1^2, m_2^2, m_3^2)$	$\frac{1}{2\epsilon}$
$C_{00i}(p^2, q^2, (p+q)^2, m_1^2, m_2^2, m_3^2)$	$-\frac{1}{6\epsilon}$

Table F.1: The divergent parts of the one-, two- and three-point Passarino-Veltman functions.

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Nomenclature

$A(m^2)$ Passarino-Veltman one-point function, see equation (F.3), page 102

B_0, B_1, B_{21}, ... Passarino-Veltman two-point function, see equation (F.5), page 103

β β-function, see equation (2.2-12), page 16

$\tan\beta$ Ratio of the Higgs vev's in 2HDM's or the MSSM, page 52

$B(\alpha,\beta)$ Euler's B-function, see equation (E.10), page 101

C_0, C_μ, $C_{\mu\nu}$, $C_{\mu\nu\rho}$ Passarino-Veltman three-point function, see equation (F.9), page 104

$c_1(R)$ Group-theoretical invariant, see equation (A.3), page 89

$c_2(R)$ Quadratic Casimir invariant of the irrep R, see equation (A.3), page 89

$\mathbb{C}_\mathrm{a}$ Set of (complex) a-numbers, page 57

$\mathbb{C}_\mathrm{c}$ Set of (complex) c-number, page 57

$\boldsymbol{D}_\alpha$ SUSY-covariant derivative, see equation (6.1-19), page 59

$\Delta_{\overline{\mathrm{MS}}}$ Passarino-Veltman Δ, see equation (F.1), page 102

ϵ $\epsilon = 4 - d$, page 14

$F(k,\ell)$ Color-factor of G, see equation (6.3-14), page 70

$f^A{}_{BC}$ Structure constants, see equation (A.2), page 89

$F_n(k,\ell)$ Color-factor of G_n, see equation (6.3-13), page 70

γ Anomlaous dimension of the field, see equation (2.2-12), page 16

$\Gamma(z)$ Γ-function, see equation (E.5), page 101

$\Gamma[\varphi_c]$ Effective action, see equation (2.1-6), page 12

$\gamma_\mathrm{E} = -0.5772\ldots$ Euler's constant, see equation (E.8), page 101

γ_m Anomalous mass dimension, see equation (2.2-12), page 16

$\Gamma_N(x_1,\dots x_n)$ Vertex functions in coordinate space, see equation (2.1-7), page 12

$\overline{\Gamma}_N$ Proper vertex function, see equation (2.1-13), page 13

$\mathscr{G}_N$ Connected N-point function, see equation (2.1-4), page 11

irrep Irreducible representation, page 89

κ Coupling of the effective neutrino mass operator, see equation (3.2-1), page 23

Λ Scale of an embedding theory (see also M_{GUT}).

Λ_∞ Infinite dimensional Grassmann algebra, see equation (6.1-1), page 56

$\ell(R)$ Dynkin index of the irrep R, see equation (A.3), page 89

M_{GUT} Scale of an unified theory (see also Λ).

$\overline{\mathrm{MS}}$ Modified minimal subtraction

MS Minimal Subtraction.

p_{E}^2 Euclidean momentum squared, page 22

$\varphi_c(x)$ Classical field, see equation (2.1-5), page 12

$P_{\mathrm{L}}, P_{\mathrm{R}}$ Left- and right-handed projector, page 91

$\boldsymbol{Q}, \bar{\boldsymbol{Q}}$ SUSY generators , page 56

$\mathbb{S}$ Superspace, see equation (6.1-5), page 57

σ^μ, $\bar{\sigma}^\mu$ σ matrices, see equation (B.6), page 91

T_A Matrices of the generators of a gauge group, see equation (A.2), page 89

$\mathbb{W}_\alpha$, $\bar{\mathbb{W}}_{\dot\alpha}$ Field strength superfields, see equation (6.1-41), page 62

$\mathcal{W}[J]$ Generating functional for connected Greens functions, see equation (2.1-3), page 11

ξ_B, ξ_W, ξ_G Gauge fixing parameters for the SM gauge groups.

$\mathcal{Z}[J]$ Generating functional, see equation (2.1-1), page 11

Bibliography

[1] SuperKamiokande, Y. Fukuda et al., *Measurement of the flux and zenith-angle distribution of upward through-going muons by super-kamiokande*, Phys. Rev. Lett. **82** (1999), 2644–2648, hep-ex/9812014.

[2] SNO, Q. R. Ahmad et al., *Direct evidence for neutrino flavor transformation from neutral-current interactions in the sudbury neutrino observatory*, (2002), nucl-ex/0204008.

[3] SNO, Q. R. Ahmad et al., *Measurement of day and night neutrino energy spectra at sno and constraints on neutrino mixing parameters*, (2002), nucl-ex/0204009.

[4] J. R. Ellis, S. Kelley, and D. V. Nanopoulos, *Probing the desert using gauge coupling unification*, Phys. Lett. **B260** (1991), 131–137.

[5] U. Amaldi, W. de Boer, and H. Fürstenau, *Comparison of grand unified theories with electroweak and strong coupling constants measured at LEP*, Phys. Lett. **B260** (1991), 447–455.

[6] P. Langacker and M.-x. Luo, *Implications of precision electroweak experiments for $M(t)$, $\rho(0)$, $\sin^2\theta_{\rm W}$ and grand unification*, Phys. Rev. **D44** (1991), 817–822.

[7] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity* (P. van Nieuwenhuizen and D. Z. Freedman, eds.), North Holland, Amsterdam, 1979, p. 315.

[8] T. Yanagida, in *Proceedings of the Workshop on the Unified Theory and the Baryon Number in the Universe* (O. Sawada and A. Sugamoto, eds.), KEK, Tsukuba, Japan, 1979, p. 95.

[9] J. N. Bahcall, M. C. Gonzalez-Garcia, and C. Pena-Garay, *Before and after: How has the sno nc measurement changed things?*, (2002), hep-ph/0204314.

[10] S. Antusch, M. Drees, J. Kersten, M. Lindner, and M. Ratz, *Neutrino mass operator renormalization revisited*, Phys. Lett. **B519** (2001), 238–242, hep-ph/0108005.

[11] S. Antusch, M. Drees, J. Kersten, M. Lindner, and M. Ratz, *Neutrino mass operator renormalization in two Higgs doublet models and the MSSM*, Phys. Lett. **B525** (2002), 130–134, hep-ph/0110366.

[12] S. Antusch and M. Ratz, *Supergraph techniques and two-loop beta-functions for renormalizable and non-renormalizable operators*, (2002), hep-ph/0203027.

[13] S. Weinberg, *Non-renormalization theorems in non-renormalizable theories*, Phys. Rev. Lett. **80** (1998), 3702–3705, hep-th/9803099.

[14] S. Antusch, J. Kersten, M. Lindner, and M. Ratz, *Neutrino mass matrix running for non-degenerate see-saw scales*, (2002), hep-ph/0203233.

[15] S. Weinberg, *The quantum theory of fields. vol. 2: Modern applications*, Cambridge University Press, 1996, 489 p.

[16] T. Appelquist and J. Carazzone, *Infrared singularities and massive fields*, Phys. Rev. **D11** (1975), 2856.

[17] K. S. Babu, C. N. Leung, and J. Pantaleone, *Renormalization of the neutrino mass operator*, Phys. Lett. **B319** (1993), 191–198, hep-ph/9309223.

[18] R. Mertig, M. Böhm, and A. Denner, *FEYN CALC: Computer algebraic calculation of Feynman amplitudes*, Comput. Phys. Commun. **64** (1991), 345–359.

[19] J. Kersten, *Renormalization group evolution of neutrino masses*, Diploma thesis, T.U. München, 2001.

[20] P. H. Chankowski and S. Pokorski, *Quantum corrections to neutrino masses and mixing angles*, Int. J. Mod. Phys. **A17** (2002), 575–614, hep-ph/0110249.

[21] M. E. Machacek and M. T. Vaughn, *Two loop renormalization group equations in a general quantum field theory. 1. Wave function renormalization*, Nucl. Phys. **B222** (1983), 83.

[22] M. E. Machacek and M. T. Vaughn, *Two loop renormalization group equations in a general quantum field theory. 2. Yukawa couplings*, Nucl. Phys. **B236** (1984), 221.

[23] M. E. Machacek and M. T. Vaughn, *Two loop renormalization group equations in a general quantum field theory. 3. Scalar quartic couplings*, Nucl. Phys. **B249** (1985), 70.

[24] P. H. Chankowski and P. Wasowicz, *Low energy threshold corrections to neutrino masses and mixing angles*, hep-ph/0110237 (2001).

[25] G. Senjanović, *Spontaneous breakdown of parity in a class of gauge theories*, Nucl. Phys. **B153** (1979), 334.

[26] N. G. Deshpande, J. F. Gunion, B. Kayser, and F. Olness, *Left-right symmetric electroweak models with triplet Higgs*, Phys. Rev. **D44** (1991), 837–858.

[27] S. Weinberg, *Gauge theory of CP violation*, Phys. Rev. Lett. **37** (1976), 657.

[28] S. L. Glashow and S. Weinberg, *Natural conservation laws for neutral currents*, Phys. Rev. **D15** (1977), 1958.

[29] E. A. Paschos, *Diagonal neutral currents*, Phys. Rev. **D15** (1977), 1966.

[30] B. Grzadkowski and M. Lindner, *Nonlinear evolution of Yukawa couplings*, Phys. Lett. **B193** (1987), 71.

[31] C. T. Hill, C. N. Leung, and S. Rao, *Renormalization group fixed points and the higgs boson spectrum*, Nucl. Phys. **B262** (1985), 517.

[32] S. R. Coleman and J. Mandula, *All possible symmetries of the S matrix*, Phys. Rev. **159** (1967), 1251–1256.

[33] R. Haag, J. T. Łopuszański, and M. Sohnius, *All possible generators of supersymmetries of the S matrix*, Nucl. Phys. **B88** (1975), 257.

[34] R. Delbourgo, *Superfield perturbation theory and renormalization*, Nuovo Cim. **A25** (1975), 646.

[35] A. Salam and J. Strathdee, *Feynman rules for superfields*, Nucl. Phys. **B86** (1975), 142–152.

[36] K. Fujikawa and W. Lang, *Perturbation calculations for the scalar multiplet in a superfield formulation*, Nucl. Phys. **B88** (1975), 61.

[37] M. T. Grisaru, W. Siegel, and M. Roček, *Improved methods for supergraphs*, Nucl. Phys. **B159** (1979), 429.

[38] S. J. Gates, M. T. Grisaru, M. Roček, and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, Front. Phys. **58** (1983), 1–548, `hep-th/0108200`.

[39] W. Siegel, *Supersymmetric dimensional regularization via dimensional reduction*, Phys. Lett. **B84** (1979), 193.

[40] D. M. Capper, D. R. T. Jones, and P. van Nieuwenhuizen, *Regularization by dimensional reduction of supersymmetric and nonsupersymmetric gauge theories*, Nucl. Phys. **B167** (1980), 479.

[41] J. Wess and B. Zumino, *A lagrangian model invariant under supergauge transformations*, Phys. Lett. **B49** (1974), 52.

[42] J. Iliopoulos and B. Zumino, *Broken supergauge symmetry and renormalization*, Nucl. Phys. **B76** (1974), 310.

[43] P. West, *The Yukawa beta function in $N = 1$ rigid supersymmetric theories*, Phys. Lett. **B137** (1984), 371.

[44] P. H. Chankowski and Z. Pluciennik, *Renormalization group equations for seesaw neutrino masses*, Phys. Lett. **B316** (1993), 312–317, hep-ph/9306333.

[45] I. Jack, D. R. T. Jones, and C. G. North, *N = 1 supersymmetry and the three loop gauge beta function*, Phys. Lett. **B386** (1996), 138–140, hep-ph/9606323.

[46] I. Jack, D. R. T. Jones, and C. G. North, $n = 1$ *supersymmetry and the three loop anomalous dimension for the chiral superfield*, Nucl. Phys. **B473** (1996), 308–322, hep-ph/9603386.

[47] KamLAND, S. A. Dazeley, *An update on progress at KamLAND*, (2002), hep-ex/0205041.

[48] M. Freund, *Messung der fundamentalen Neutrinoparameter mittels Long-Baseline-Oszillationsexperimenten an zukünftigen Neutrinofabriken*, Ph.D. thesis, T.U. München, 2002.

[49] S. Antusch, J. Kersten, M. Lindner, and M. Ratz, *LMA solution from bi-maximal mixing by renormalization group running*, (2002), in preparation.

[50] A. Denner, H. Eck, O. Hahn, and J. Küblbeck, *Feynman rules for fermion number violating interactions*, Nucl. Phys. **B387** (1992), 467–484.

[51] G. Passarino and M. Veltman, *One loop corrections for* $e^+ e^-$ *annihilation into* $\mu^+ \mu^-$ *in the Weinberg model*, Nucl. Phys. **B160** (1979), 151.

[52] T. Hahn and M. Perez-Victoria, *Automatized one-loop calculations in four and d dimensions*, Comput. Phys. Commun. **118** (1999), 153–165, hep-ph/9807565.

[53] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, *Deep inelastic scattering beyond the leading order in asymptotically free gauge theories*, Phys. Rev. **D18** (1978), 3998.

[54] G. 't Hooft and M. Veltman, *Scalar one loop integrals*, Nucl. Phys. **B153** (1979), 365–401.

[55] B. A. Kniehl, *Radiative corrections for* $H \to Z\,Z$ *in the standard model*, Nucl. Phys. **B352** (1991), 1–26.

[56] A. Devoto and D. W. Duke, *Table of integrals and formulae for Feynman diagram calculations*, Riv. Nuovo Cim. **7N6** (1984), 1–39.

[57] A. Denner, *Techniques for calculation of electroweak radiative corrections at the one loop level and results for W physics at LEP200*, Fortschr. Phys. **41** (1993), 307–420.