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On a Problem of Erdős in Combinatorial Geometry

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Abstract

In this thesis, we solve a long-standing open problem in combinatorial geometry known as the *empty-hexagon problem* or *6-hole problem*. Erdős asked in 1977 whether every sufficiently large set of points in general position in the plane contains six points that form a convex hexagon without any points from the set in its interior. Such a configuration is called an *empty convex hexagon*. We answer the question in the affirmative. We show that every set that contains the vertex set of a convex 9-gon also contains an empty convex hexagon. The result is sharp in the sense that there exist sets that contain the vertex set of a convex 8-gon and do not contain an empty convex hexagon. It is known that every 1717-set of points in general position does contain the vertex set of a convex 9-gon and it is an open conjecture that 1717 can be replaced by 129.

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CHAPTER 1

Introduction

Combinatorial geometry deals with combinatorial questions regarding collections of geometric objects such as points, lines, balls, polytopes, hyperplanes, etc. The questions might concern, for example, the complexity of arrangements of objects of the above type or the occurrence of certain substructures in such arrangements; see [**HDK64, Ede87, EP95, PA95, Mat02, GO04, BMP05**] and references therein. The importance of the subject – apart from its apparent beauty and usefulness for educational purposes – lies in its close relationship to problems in such diverse fields as number theory, graph theory, combinatorial optimization or computational geometry.

In this thesis, we consider finite point sets in general position in the Euclidean plane. Here, *general position* means that no three points are collinear. We are interested in regular substructures that appear whenever a point set is sufficiently large. A classical theorem of Erdős and Szekeres states that we can find the vertex set of a convex n-gon in every point set of a suitable size (depending on n). Up to now it was an open problem raised by Erdős to decide, whether one could find the vertex set of an empty convex hexagon in every sufficiently large planar point set in general position. Here, an n-gon is called *empty* if no other points of the set lie in its interior.

1.1. Foundations from Combinatorial Geometry

We presume that the reader is familiar with basic concepts in convex geometry; see, for example, [**Mat02**]. In 1935, Erdős and Szekeres [**ES35**] proved the following theorem.

THEOREM 1 (**Erdős–Szekeres theorem**). *For each positive integer n there exists a smallest positive integer $g(n)$ such that every planar set of at least $g(n)$ points in general position contains n points that are the vertices of a convex n-gon.*

The standard proof of Theorem 1 combines two fundamental results, Ramsey's theorem [**Ram30**] from combinatorics and Carathéodory's theorem (see [**Eck93**]) from convex geometry.

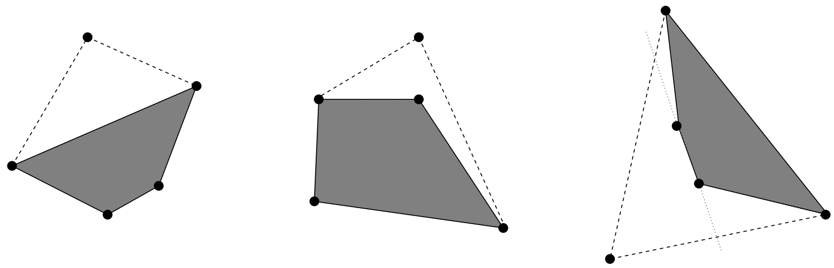

FIGURE 1.1. Klein's observation: Every set of five points in general position in the plane contains the vertex set of a convex quadrilateral.

Let X be a finite set and let $\binom{X}{p}$ denote the set $\{Y : Y \subseteq X \text{ and } |Y| = p\}$. An *$r$-coloring* of a set X is a partition $X = \cup_{i=1}^{r} X_i$ of X into pairwise disjoint sets X_i. If $x \in X_i$, x is said to have *color i*. Finally, a k-set is a set with exactly k elements.

THEOREM 2 (**Ramsey's theorem**). *For every choice of natural numbers p, r, n, there exists a natural number N such that whenever X is an N-element set and $c : \binom{X}{p} \to \{1, 2, \ldots, r\}$ is an arbitrary coloring of the system of all p-element subsets of X by r colors, then there is an n-element subset $Y \subseteq X$ such that all the p-tuples in $\binom{Y}{p}$ have the same color.*

Ramsey's theorem is proved by induction on p; we refer to [**Neš95**] for a proof and generalizations.

THEOREM 3 (**Carathéodory's theorem**). *Let X be a set in $\mathbb{R}^d$ and p a point in the convex hull of X. Then there is a subset Y of X consisting of $d+1$ or fewer points such that p lies in the convex hull of Y.*

This theorem can be proved by induction on the dimension of the space or it can be deduced either from Helly's theorem or Radon's theorem; see [**Eck93**] and references therein. Note that in the planar setting ($d = 2$), Carathéodory's theorem implies that n points are the vertices of a convex n-gon if every four of them are in convex position.

The following observation is attributed to E. Klein [**ES35**]. It can be derived from Figure 1.1.

OBSERVATION 1 (Klein). *Every set of five points in general position in the plane contains the vertex set of a convex quadrilateral.*

We can now give a proof of the Erdős-Szekeres theorem.

PROOF OF THEOREM 1 [**ES35**]. Consider a point set X in general position in the plane. Color a 4-tuple $T \subset X$ red if its four points form a convex quadrilateral and blue otherwise. If $|X|$ is sufficiently large, Ramsey's theorem provides an n-point subset $Y \subset X$ such that all 4-tuples from Y have the same color. For $n \geq 5$ this color cannot be blue, because any five points determine at least one red 4-tuple by Klein's observation. Therefore, Carathéodory's theorem implies that Y consists of the vertex set of a convex n-gon, since every four of its points are in convex position. □

The best known bounds for $g(n)$ are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1. \tag{1.1}$$

The upper bound in (1.1) was established recently by Tóth and Valtr [**TV05**] based on geometric modifications of a Ramsey-type argument in the context of *cups* and *caps*. Given a point set X in general position in the plane and a suitable coordinate system, where no two points of X have the same x-coordinate, a k-subset Y of X is called a *k-cup* if the points of Y lie on the graph of a convex function. Similarly, an *l-cap* is a set of l points that lie on the graph of a concave function. The authors refine an argument of Erdős and Szekeres [**ES35**] concerning the size of sets that contain an n-cup or an n-cap (and therefore an n-gon).

The lower bound in (1.1) is due to Erdős and Szekeres [**ES61**] who gave a suitable construction of point sets of cardinality 2^{n-2} that do not contain an n-gon. The rough idea is to properly arrange $n-1$ sets $X_0, \dots, X_{n-2}$, where X_i is a set of maximum cardinality that does not contain an $(n-i)$-cup or an $(i+2)$-cap. It is known that $|X_i| = \binom{n-2}{i}$ and it follows that $\sum_{i=0}^{n-2} \binom{n-2}{i} = 2^{n-2}$. The lower bound is known to be sharp for $n \leq 5$ [**ES35, KKS70, Bon74**] and is conjectured to be sharp for all n by Erdős and Szekeres.

CONJECTURE 1 (**Erdős-Szekeres conjecture** [**ES35, ES61**]).

$$g(n) = 2^{n-2} + 1. \tag{1.2}$$

The determination of the exact value of $g(n)$ constitutes a major open problem in combinatorial geometry and was one of Erdős's favourite problems [**Erd97**]. (His last contribution to the problem seems to be [**ETV96**], written some sixty years after the original paper [**ES35**].) Erdős offered \$500 for a proof of this conjecture, thereby indicating his estimation for the difficulty of finding such a proof [**Erd97**].

1.2. Problem Statement and Main Result

In 1977, Erdős [**Erd78, Erd81**] modified the original question of [**ES35**] about the occurrence of convex polygons in sufficiently large point sets in general position by adding an additional constraint. He now posed the problem of determining the smallest positive integer $h(n)$, if it exists, such that every set X of at least $h(n)$ points in general position in the plane contains n points that are the vertices of an *empty* convex polygon; that is, a convex n-gon whose interior does not contain any point of X. Trivially, $h(n) = n$ for $n \leq 3$. Klein's observation (Observation 1) also implies that $h(4) = 5$.

In 1978, Ehrenfeucht (unpublished; see [**Erd78, Erd79, Erd81**]) established the existence of $h(5)$. Furthermore, Harborth [**Har78**] and independently Morris (unpublished; see [**Erd81**]) determined the exact value $h(5) = 10$. Harborth's proof consists of a case analysis based on the existence of a (possibly non-empty) convex pentagon in every set of 10 points in general position (since $g(5) = 9$) together with a construction showing that $h(5) \geq 10$. See also [**Mat02**] for a presentation of a simpler alternative proof (based essentially on Ehrenfeucht's and Harborth's ideas) giving the worse bound $h(5) \leq g(6)$. Here, the starting point for a case analysis is a convex hexagon (guaranteed to exist in any $g(6)$-set) with a minimum number of points in its interior. (This approach inspired our main proof in Chapter 2.)

Surprisingly, in 1983 Horton [**Hor83**] showed that for all $n \geq 7$, $h(n)$ does not exist. He gave a construction for point sets of arbitrary size that do not contain an empty convex heptagon. Again, see [**Mat02**] for a streamlined presentation based on an analysis of Horton's construction by Valtr [**Val92**].

The problem of determining the existence of $h(6)$ remained unsettled, though it was popularized by Erdős and others; see, for example, [**Ede87**] (p. 43), [**CFG91**] (problem F8, pp. 155–156), [**KW91**] (ch. 1.5), [**Sch93**] (pp. 455–456), [**EP95**] (p. 862), [**Neš95**] (p. 1336), [**CG98**] (p. 8), [**Mat02**] (ch. 3), [**Pac04**] (p. 7), [**BMP05**] (ch. 8.2).

Problem 1 (**Empty-hexagon problem / 6-hole problem**). *Does $h(6)$ exist; that is, does every sufficiently large set of points in general position in the plane contain an empty convex hexagon?*

Based on an extensive computerized search, Overmars [**Ove03**] showed that $h(6) \geq 30$ (if it exists). Furthermore, in this paper Overmars speculates that bounds on the size of the convex hull or on the number of convex layers of a point set might be a starting point for an investigation of the empty-hexagon problem. Several authors expressed their belief that $h(6)$ does exist [**Hor83, BK01, Dum05**] and several conjectures were raised that if correct would imply this result [**Val97, Dum05**]. (See also Section 1.3.) See [**BK01**] for some problems that are directly linked to the empty-hexagon problem.

In this thesis, we prove the following theorem which implies that indeed *every sufficiently large planar point set in general position contains the vertex set of an empty convex hexagon.*

THEOREM 4. *$h(6) \leq g(9)$.*

More precisely, we show that every set that contains the vertex set of a convex 9-gon also contains an empty convex hexagon. The proof of Theorem 4 is given in Chapter 2. The upper bound for $g(n)$ in (1.1) immediately yields the following corollary.

COROLLARY 1. *$h(6) \leq 1717$.*

PROOF. $1717 = \binom{2\cdot 9-5}{9-2} + 1$. $\square$

Note that a proof of the Erdős-Szekeres conjecture would imply that $g(9) = 129$ and therefore, $h(6) \leq 129$. Note furthermore that there exist sets of points without empty convex hexagons that have eight points on the convex hull [**Ove03**]. In this sense, Theorem 4 is tight. (See also the discussion in Chapter 3.)

Addendum. After we had submitted our paper [**Ger**], the existence of empty convex hexagons in sufficiently large point sets in general position was established independently in 2005 by C. Nicolás [**Nica, Nicb**]. Here, the worse bound $h(6) \leq g(25)$ is proven by similar methods as we employ in Chapter 2.

1.3. Related Work

The Erdős-Szekeres paper [**ES35**] has inspired a great deal of research concerning generalizations of the original theorem and attempts at the settlement of the Erdős-Szekeres conjecture or the empty-hexagon problem. (Furthermore, it helped to popularize Ramsey's theorem.) As excellent recent surveys on the Erdős–Szekeres theorem and its related topics exist [**MS00, BK01, TV05, BMP05**], we only indicate the nature of the questions studied. All relevant research articles can be traced from the above surveys. Besides the above mentioned problems, the following directions of research were initiated:

Generalizations to higher dimensions. In $\mathbb{R}^d$, a point set is in *general position* if it does not contain $d+1$ points lying in a common hyperplane. Define $g_d(n)$ to be the smallest positive integer such that every set of $g_d(n)$ points in general position in $\mathbb{R}^d$ contains n points in convex position. (Note that $g_2(n) = g(n)$.) The existence of $g_d(n)$ can be established in analogy to the planar case. The problem of proving tight bounds for $g_d(n)$ is again open. The best known bounds are established via projection arguments.
Accordingly, one can ask for the existence of empty convex polytopes with a prescribed number of vertices in a sufficiently large point set X in general position in $\mathbb{R}^d$. A polytope is *empty* if its interior does not contain any point of X. It is possible to extend the ideas of Horton's construction to higher dimensions.

Generalization to planar convex sets. The Erdős-Szekeres theorem has a generalization for the case of convex bodies in the plane. Here, a family of pairwise disjoint convex sets is in *general position* if no set is contained in the convex hull of the union of two other sets of the family. It is in *convex position* if none of its members is contained in the convex hull of the others. Under certain conditions the constraint of disjointness can be relaxed. This problem also has a higher-dimensional variant.

Restricted point sets. For $\alpha > 0$, an n-point $X \subset \mathbb{R}^2$ is called *α-dense* if the ratio of the maximum and minimum distances occurring in X is at most $\alpha \cdot \sqrt{n}$. It is known that α-dense sets contain convex polygons with $\Omega(n^{\frac{1}{3}})$ vertices. (Compare this to the $\Omega(\log n)$ bound that follows from the Erdős-Szekeres theorem for general point sets.)

Counting (empty) convex polygons. Once the existence of (empty) convex polygons in a point set has been established, the next natural combinatorial question concerns the number of occurrences of such polygons.

Homogenous version. It has been established that for every N-set X of points in general position in the plane, there are n subsets, $Y_1, \ldots, Y_n \subset X$ each of size at least $c \cdot N$ such that each of their transversals forms a convex n-gon. In this context, a *transversal* of the sets $Y_1, \ldots, Y_n$ is a sequence of elements $y_1, \ldots y_n$ where $y_i \in Y_i$ for every i. Here, c denotes a positive constant depending on n.

Partitional variant. Here, the focus lies on the partition of a given point set in general position in the plane into vertex disjoint n-gons (possibly with a remainder).

Modular version. This direction is concerned with the existence of convex polygons, where the number of interior points is divisible by a given $q \in \mathbb{N}$.

Dual version. Here, a dual counterpart to the Erdős-Szekeres theorem in terms of arrangements of lines is studied. Further generalizations concern the replacement of lines by pseudolines.

Generalized Convexity. Investigations in this direction aim at those combinatorial properties of the plane that are essential to establish an Erdős-Szekeres type of result.

Algorithms. Here, the aim is to develop efficient algorithms for testing a given set for the size of the largest convex polygon contained in it or the size of the largest empty convex polygon contained in it. Particular emphasis has been laid on the automated search for point sets that do not contain an empty convex hexagon.

Others. Further research in relation to the above mentioned problems concerns point sets with a limitation on the maximum number of points in the interior of a triangle, colored versions, where a color is assigned to each point and the focus of attention lies on empty monochromatic n-gons, graph-theoretic variants (existence of complete subgraphs with a maximum possible number of crossings in any drawing) and purely combinatorial or algebraic variants.

1.4. Acknowledgements

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Furthermore, the author would like to thank Pavel Valtr and David Wood as well as the anonymous referees of *Discrete and Computational Geometry* for their comments on preliminary versions of his paper [**Ger**].

CHAPTER 2

Proof of Theorem 4

In this chapter we prove Theorem 4. We show that every set that contains the vertex set of a convex 9-gon also contains an empty convex hexagon. The proof will appear in [**Ger**].

2.1. Overview of the Proof

Proof. In the following, let X be a finite planar set of points in general position that contains the vertex set of a convex 9-gon. By the Erdős–Szekeres theorem [**ES35**] this is always the case if $|X| \geq g(9)$. Let $H \subseteq X$ be the vertex set of a convex 9-gon in X with the *minimum* $|X \cap \text{conv}(H)|$, where $\text{conv}(M)$ denotes the convex hull of the set M. Let $I := \text{conv}(H) \cap (X \setminus H)$ be the set of points of X inside the convex hull of H. Note that $\text{conv}(I)$ is a convex polygon and denote by ∂I its vertex set. If $|I| > 2$, let $J := \text{conv}(I) \cap (X \setminus \partial I)$ be the set of points of X inside the convex hull of ∂I. Note that $\text{conv}(J)$ is again a convex polygon and denote by ∂J its vertex set; see Figure 2.1. Let $i := |\partial I|$ and $j := |\partial J|$. Note that $0 \leq i, j \leq 8$ as otherwise there would be a 9-gon H' with smaller $|X \cap \text{conv}(H')|$. This leaves the 57 cases $0 \leq i \leq 2$ and $(i, j) \in \{3, \dots, 8\} \times \{0, \dots, 8\}$. We argue that in each case either an empty convex u-gon can be found ($u \geq 6$) or a convex 9-gon H' with smaller $|X \cap \text{conv}(H')|$ is present which contradicts the minimality condition imposed on H. (More precisely, the vertex set of an empty convex u-gon can be found. In the following, we do not make this distinction when the meaning is clear from the context.)

2.1.1. Notation.

We use (i, j) to denote a specific case, where i and j are defined as above. Sometimes we use the notation (i, j, k), where k refers to the number of points of X inside the convex hull of J; that is, the cardinality of $K := \text{conv}(J) \cap (X \setminus \partial J)$. The notation $\geq x$ indicates that x is a lower bound for i, j or k. Refer to Table 1 for locating the proof of a specific case.

2.1.2. Definitions.

Given three points in general position, P, Q, R, define the halfplane $H_{PQ}(R)$ as the open halfplane defined by the line $\overline{PQ}$ that contains R. A *convex chain* is a set of consecutive vertices of a convex polygon. Given a

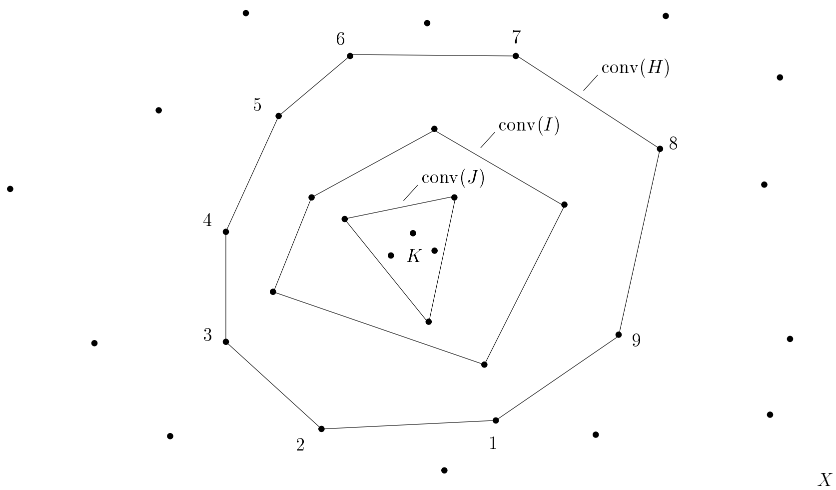

FIGURE 2.1. Basic notation

i/j	0	1	2	3	4	5	6	7	8
0	2.2	–	–	–	–	–	–	–	–
1	2.2	–	–	–	–	–	–	–	–
2	2.2	–	–	–	–	–	–	–	–
3	2.3	2.3	2.3	2.3	2.3	2.3	2.3	2.3	2.3
4	2.4	2.4	2.4	2.4	2.4	2.4	2.4	2.4	2.4
5	2.5	2.6.1	2.7	2.7	2.7	2.7	2.7	2.7	2.7
6	2.2	2.6.2	2.6.3	2.3/2.4	2.8	2.8	2.8	2.8	2.8
7	2.2	2.6.3	2.2	2.3/2.4	2.4	2.5/2.9	2.2/2.9	2.2/2.9	2.2/2.9
8	2.2	2.2	2.2	2.3	2.4	2.5/2.9	2.2/2.9	2.2/2.9	2.2/2.9

TABLE 1. Overview of the proof structure: for example, the proof for the case $(8, 5)$ is given in Sections 2.5 (special case) and 2.9 (general case).

convex chain of three points, $\overline{ABC}$, the *3-sector* specified by this chain is defined as

$$(ABC) := [H_{AB}(C) \cap H_{BC}(A)] \setminus \operatorname{conv}(\{A, B, C\}).$$

Note that three points in general position, S, T, U, lying in (ABC) can be used to construct a convex hexagon if $A, B, C \in (STU)$; see Figure 2.2a.

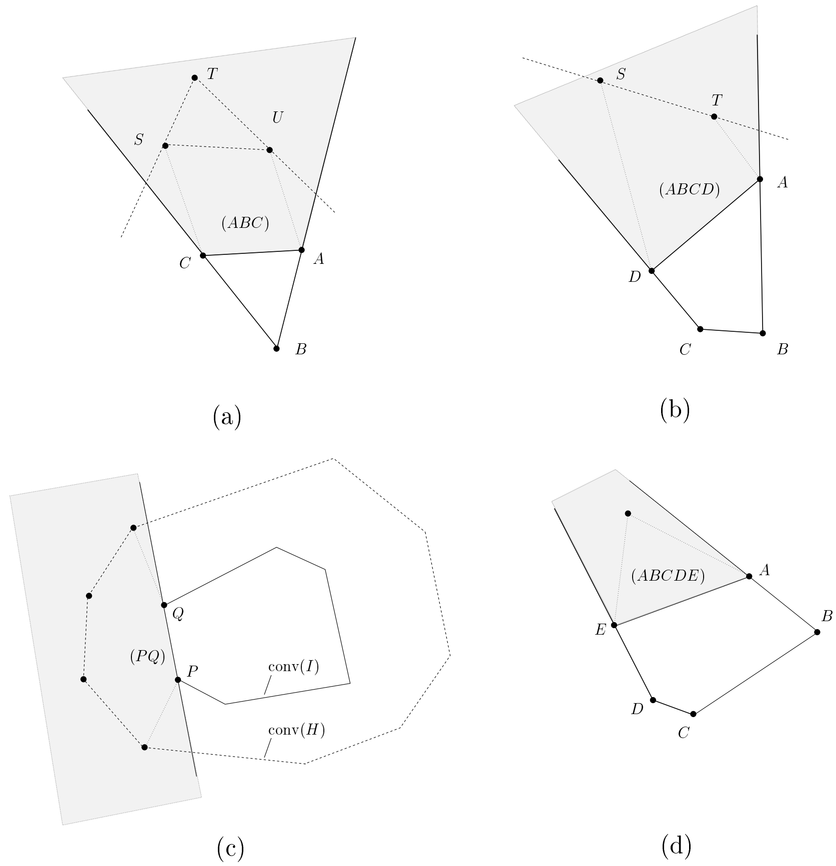

FIGURE 2.2. Definition: Sector

Given a convex chain of four points, $\overline{ABCD}$, the corresponding *4-sector* is defined as

$$(ABCD) := [(ABC) \cap (BCD)] \setminus \mathrm{conv}(\{A, B, C, D\}).$$

Note that two points, S, T, lying in $(ABCD)$ can be used to construct a convex hexagon if the line $\overline{ST}$ does not intersect $\mathrm{conv}(\{A, B, C, D\})$; see Figure 2.2b. This means that by construction, given an edge PQ of $\mathrm{conv}(I)$ (respectively $\mathrm{conv}(J)$), at most three vertices of $\mathrm{conv}(H)$ (respectively $\mathrm{conv}(I)$) can lie in an open halfplane that is defined by the line $\overline{PQ}$ and does not include any other point of I (respectively J) if no empty convex hexagon is to occur. In the following figures, we use the notation (PQ) to hint to this fact; see Figure 2.2c.

Finally, given a convex chain of five points, $\overline{ABCDE}$, the corresponding *5-sector* is defined as

$$(ABCDE) := [(ABCD) \cap (BCDE)] \setminus \mathrm{conv}(\{A, B, C, D, E\}).$$

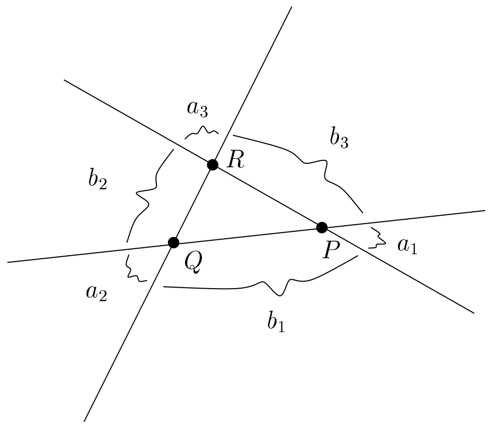

FIGURE 2.3. Notation for the cases $(3, \geq 0)$ and $(\geq 6, 3)$

Note that a single point lying in $(ABCDE)$ can be used to construct a convex 6-gon; see Figure 2.2d.

2.2. Elementary Cases

Note that the cases $\mathbf{(0,0)}$, $\mathbf{(\geq 6,0)}$ and $\mathbf{(\geq 3, \geq 6,0)}$ are trivial as an empty convex hexagon is present. The cases $\mathbf{(1,0)}$ and $\mathbf{(8,1)}$ can be dealt with by considering a line through the single interior point and one of the vertices of the convex 9- respectively 8-gon. Due to the general position, on one side of this line a convex chain of four vertices must be present which together with the two preselected points can be used to construct an empty convex hexagon. A similar argument settles the cases $\mathbf{(2,0)}$, $\mathbf{(8,2)}$ and $\mathbf{(7,2)}$.

2.3. The Cases $(3, \geq 0)$ and $(\geq 6, 3)$

We approach the cases $(3, \geq 0)$ and $(\geq 6, 3)$ in two batches:

2.3.1. The Cases $\mathbf{(3, \geq 0)}$ and $\mathbf{(8, 3)}$. Follow the notation as indicated in Figure 2.3. The variables stand for the number of vertices of the convex 9-respectively 8-gon in each sector. Assume that no empty convex hexagon is present. Note that

$$1 \leq a_1 + b_1 + a_2 \leq 3 \tag{2.3}$$

$$1 \leq a_2 + b_2 + a_3 \leq 3 \tag{2.4}$$

$$1 \leq a_3 + b_3 + a_1 \leq 3 \tag{2.5}$$

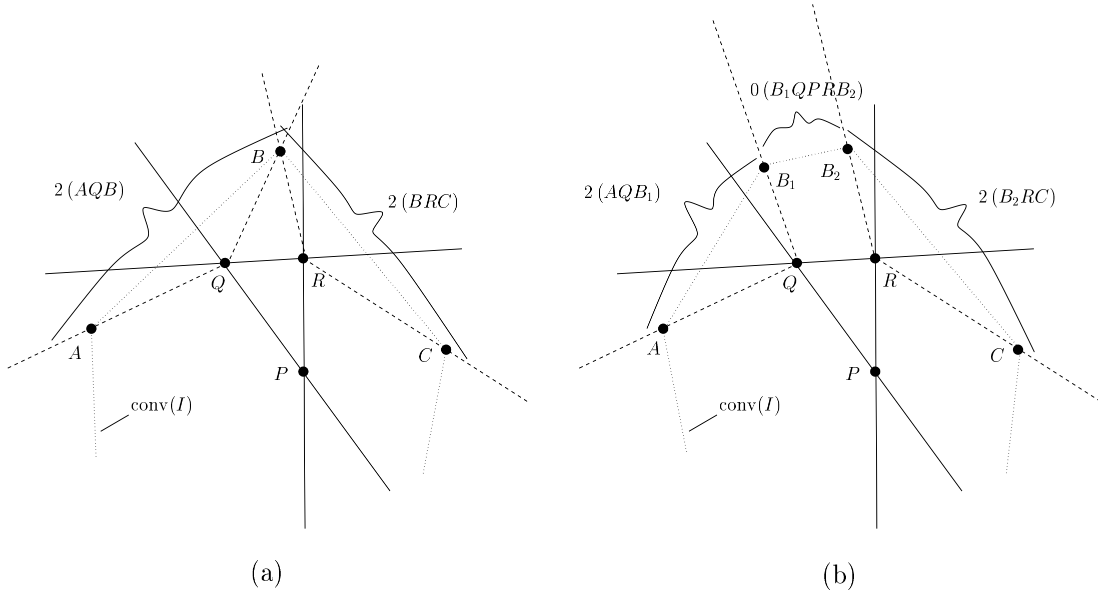

FIGURE 2.4. $(x, 3)$: degenerate cases with $(a_i, b_i, a_{i+1}) = (0, 1, 0)$ and $(a_i, b_i, a_{i+1}) = (0, 2, 0)$ respectively. Numbers indicate the number of vertices of the 9-gon that can lie in each sector without forming an empty convex hexagon.

by construction and as otherwise a convex chain of four vertices together with two vertices of the triangle could be used to form an empty convex hexagon. Also,

$$0 \leq b_i \leq 2 \quad (1 \leq i \leq 3) \tag{2.6}$$

as otherwise a convex chain of three points together with two vertices of the triangle and either the third vertex of the triangle or (if existent) one of its interior points can be used to form an empty convex hexagon. Summing up the upper bounds in (2.3) — (2.6) yields

$$2 \cdot \sum_{i=1}^{3} (a_i + b_i) \leq 15. \tag{2.7}$$

Therefore, at most seven vertices can be placed around the triangle and in the two cases at hand an empty convex hexagon is present.

2.3.2. The Cases $(6, 3)$ and $(7, 3)$. The cases $\mathbf{(6, 3, 0)}$ and $\mathbf{(7, 3, 0)}$ can be settled by a careful investigation of the $(a_1, b_1, a_2, b_2, a_3, b_3)$-tuples that are feasible for the set of constraints (2.3) — (2.6). Note that tuples (a_i, b_i, a_{i+1}) with $a_i = a_{i+1} = 0$ are not feasible, as a convex 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$

a_1	a_2	a_3	b_1	b_2	b_3	Solution
3	0	0	*	*	*	infeasible
2	1	1	0	(≤ 1)	0	$\sum_{i=1}^{3}(a_i+b_i) \leq 5$
2	1	0	0	(≤ 2)	(≤ 1)	(2,0,1,2,0,1)
2	0	0	*	*	*	infeasible
1	1	1	(≤ 1)	(≤ 1)	(≤ 1)	(1,1,1,1,1,1)
1	1	0	1	2	2	(1,1,1,2,0,2)
1	1	0	1	2	1	(1,1,1,2,0,1)
1	1	0	1	2	0	$\sum_{i=1}^{3}(a_i+b_i) = 5$
1	1	0	1	(≤ 1)	(≤ 1)	$\sum_{i=1}^{3}(a_i+b_i) \leq 5$
1	1	0	0	2	2	(1,0,1,2,0,2)
1	1	0	0	(≤ 2)	(≤ 1)	$\sum_{i=1}^{3}(a_i+b_i) \leq 5$
1	0	0	*	*	*	infeasible
0	0	0	*	*	*	infeasible

TABLE 2. Cases $(6,3,0)$ and $(7,3,0)$: Combinatorial subcases under the assumptions i) $a_1 \geq a_2 \geq a_3$ and ii) $b_2 \geq b_3$ for fixed (a_1, b_1, a_2). (* marks an arbitrary entry.) Note that constraint (2.3) implies $a_1 + a_2 \leq 3$.

could be constructed; see Figure 2.4. In Figure 2.4a, replace the vertices of the 9-gon lying in the union of sectors (AQB) and (BRC) (at least one by construction and at most four in total if no empty convex hexagon is present) by points from the convex chain $\overline{AQRC}$ of length four. In Figure 2.4b, accordingly replace the at most four vertices of the 9-gon lying in the union of sectors (AQB_1), (B_1QPRB_2) and (B_2RC).

Now assume without loss of generality that i) $a_1 \geq a_2 \geq a_3$ and ii) $b_2 \geq b_3$ for fixed (a_1, b_1, a_2); see Figure 2.3. Then the only solutions to the above set of constraints (modulo rotations and reflections) are $(2,0,1,2,0,1)$, $(1,1,1,1,1,1)$, $(1,1,1,2,0,2)$, $(1,1,1,2,0,1)$ and $(1,0,1,2,0,2)$; see Table 2. These can be treated individually as follows:

- The subcase $(2,0,1,2,0,1)$ can be treated as indicated in Figure 2.5. Here and in the following, numbers indicate the number of vertices of the outer polygon that can lie in each sector without forming an empty convex hexagon. As the union of sectors allows for at most eight points in convex position in the outmost layer, due to the presence of a convex 9-gon an empty convex hexagon must occur.

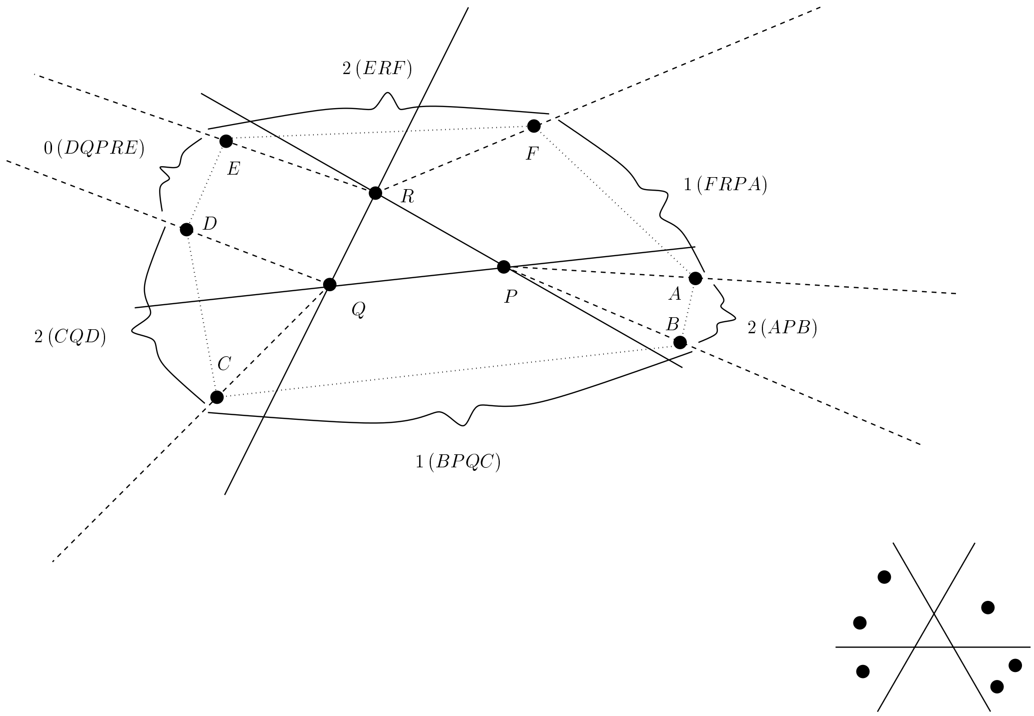

FIGURE 2.5. The case $(6, 3, 0)$ with $(2, 0, 1, 2, 0, 1)$

- Figure 2.6 indicates how to settle the subcase $(1, 1, 1, 1, 1, 1)$, provided the vertex Q of triangle PQR lies inside the triangle BDF. In that case the quadrilateral $BQDC$ exists. Similarly we can treat the case that some other of the points P, Q lies inside the triangle BDF. If none of the points P, Q, R lies inside the triangle BDF, the empty convex hexagon $PBQDRF$ occurs.
- Figure 2.7 indicates how to settle the subcase $(1, 1, 1, 2, 0, 2)$, provided that the point Q lies outside the triangle BCD. In that case the quadrilateral $CBQD$ exists. If Q lies inside the triangle BCD, the empty convex hexagon $BQDERP$ occurs.
- The subcase $(1, 1, 1, 2, 0, 1)$ can be treated as indicated in Figure 2.8.
- Figure 2.9 indicates how to settle the subcase $(1, 0, 1, 2, 0, 2)$.

The proof for the cases $\mathbf{(6, 3, \geq 1)}$ and $\mathbf{(7, 3, \geq 1)}$ is given in the following Section 2.4.

2.4. The Cases $(4, \geq 0)$ and $(\geq 7, 4)$

The cases $\mathbf{(4, \geq 0)}$ and $\mathbf{(\geq 7, 4)}$ can be dealt with simultaneously in three steps:

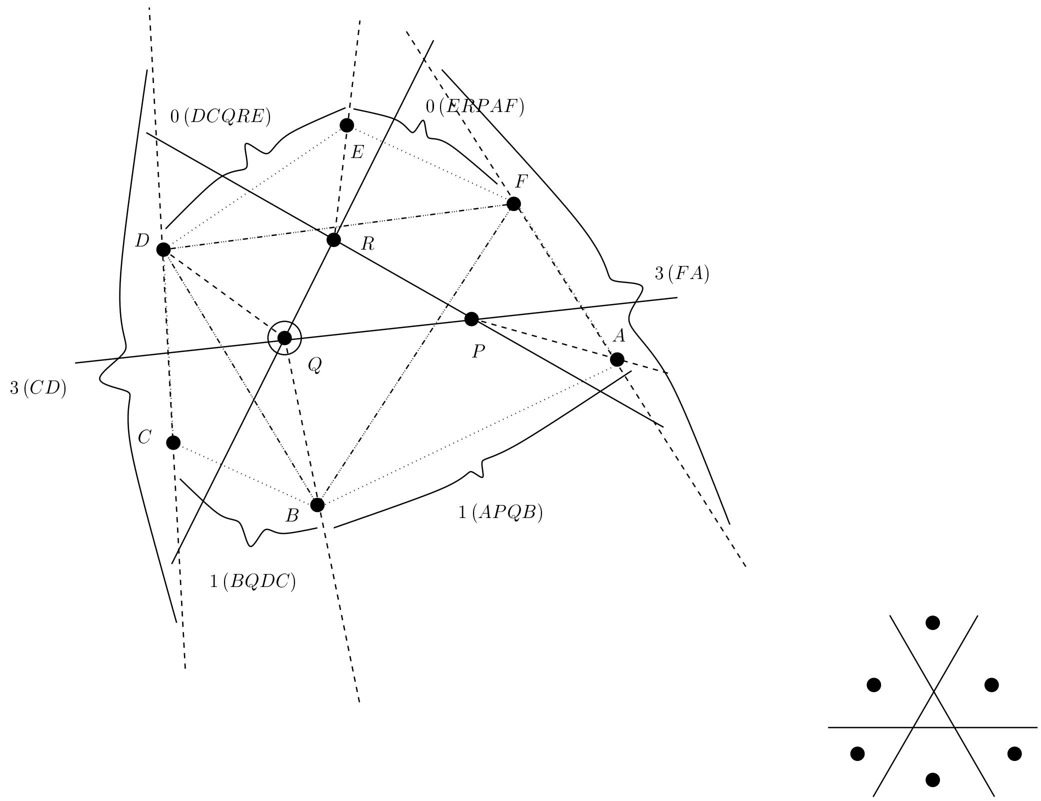

FIGURE 2.6. The case $(6, 3, 0)$ with $(1, 1, 1, 1, 1, 1)$. It is assumed that $Q \in \triangle BDF$.

2.4.1. Step 1a. First, consider the cases $\mathbf{(4, 0)}$ and $\mathbf{(8, 4, 0)}$. We use the same type of approach as in Section 2.3. Following the notation as indicated in Figure 2.10, where variables again refer to the number of vertices of the 9- respectively 8-gon lying in each sector, we arrive at the set of inequalities

$$1 \leq a_1 + b_1 + a_2 \leq 3 \tag{2.1}$$

$$1 \leq a_3 + b_3 + a_4 \leq 3 \tag{2.2}$$

if no empty convex hexagon is to occur. (Vertices lying in more than one sector are assigned arbitrarily to one particular sector they lie in and therefore only counted once.) If no empty convex hexagon is to be present, the constraint

$$0 \leq b_2 + b_4 \leq 1 \tag{2.3}$$

must also hold. By summing up the upper bounds in (2.1) — (2.3), it follows that at most seven vertices can be placed around the 4-gon, a contradiction in these two cases.

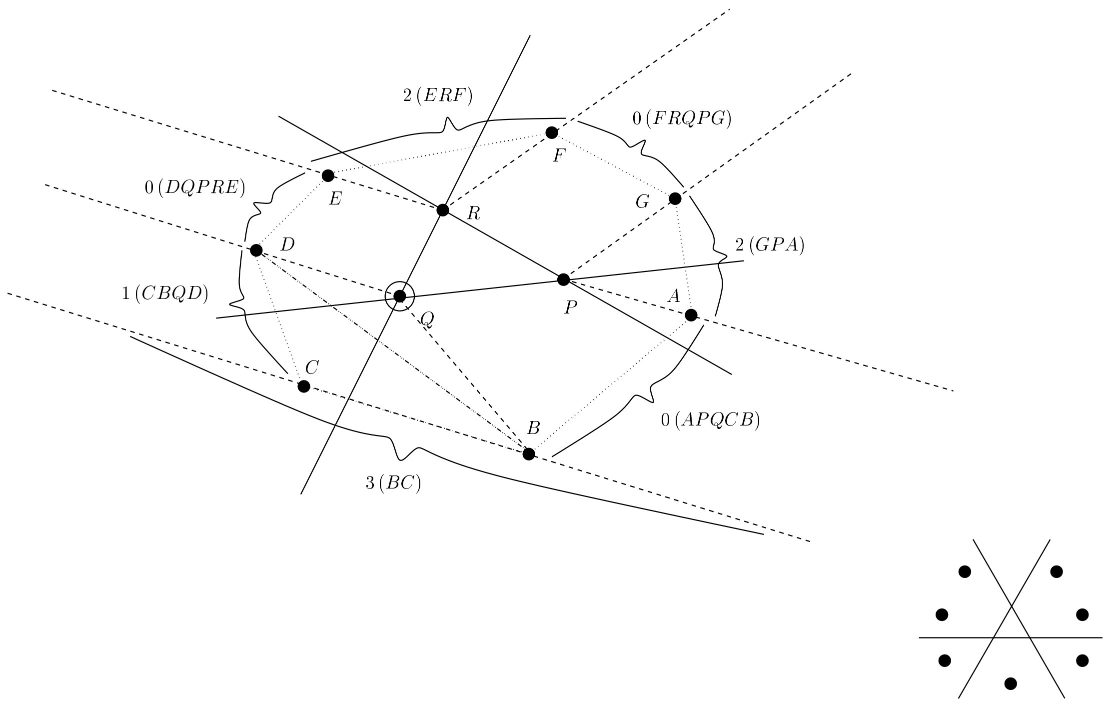

FIGURE 2.7. The case $(7,3,0)$ with $(1,1,1,2,0,2)$. It is assumed that $Q \notin \triangle BCD$.

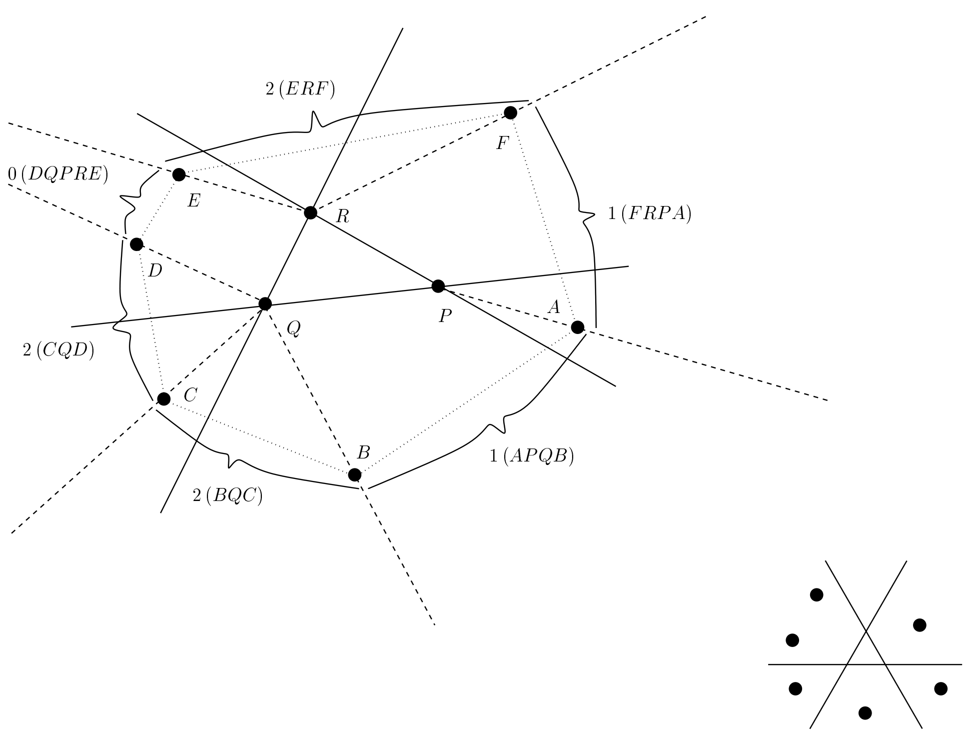

FIGURE 2.8. The case $(6,3,0)$ with $(1,1,1,2,0,1)$

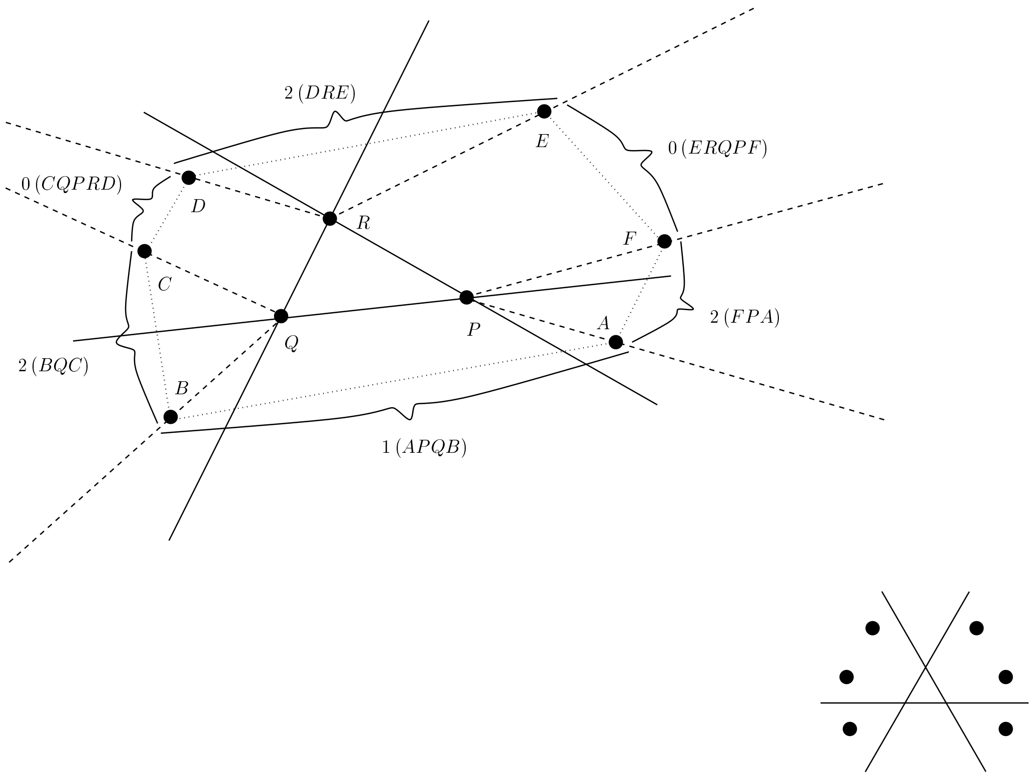

FIGURE 2.9. The case $(6, 3, 0)$ with $(1, 0, 1, 2, 0, 2)$

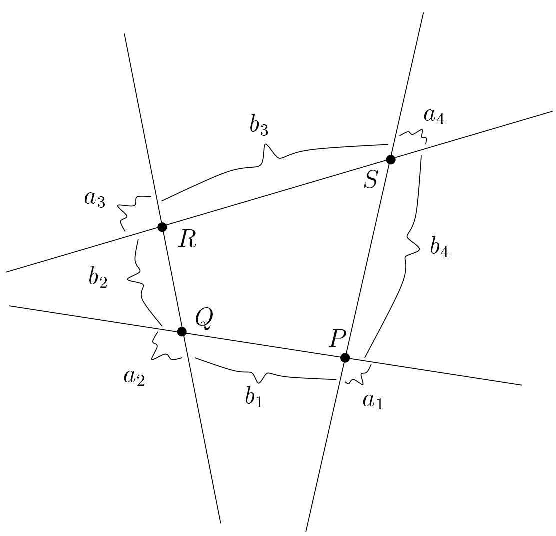

FIGURE 2.10. Notation for the cases $(4, \geq 0)$ and $(\geq 7, 4)$

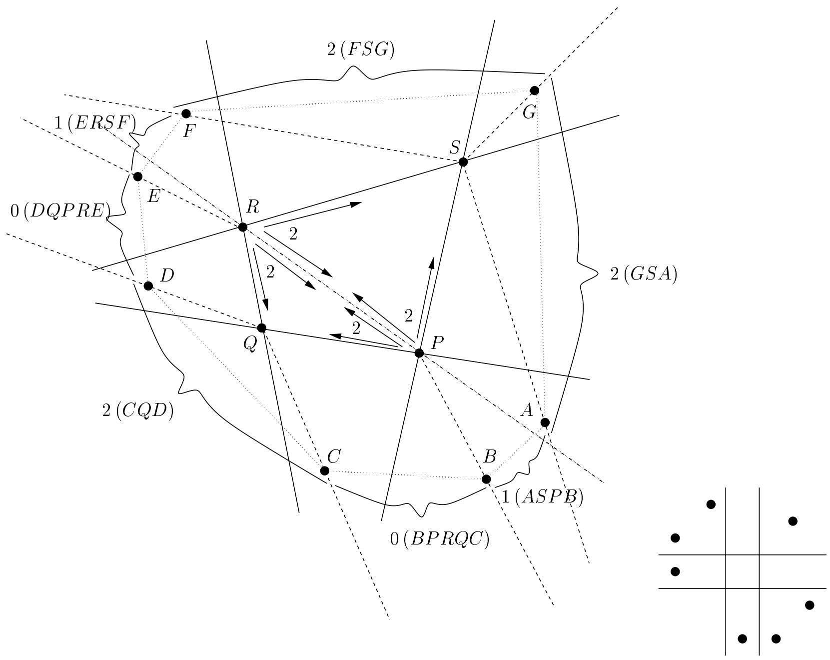

FIGURE 2.11. The case $(7, 4, 0)$ with $(2, 1, 0, 1, 2, 0, 1, 0)$

2.4.2. Step 1b. We next consider the case $\mathbf{(7, 4, 0)}$ and evaluate the feasible solutions to the set of constraints (2.1) — (2.3). By symmetry, any feasible $(a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4)$-tuple must also satisfy the following set of inequalities:

$$1 \leq a_1 + b_4 + a_4 \leq 3 \tag{2.4}$$

$$1 \leq a_2 + b_2 + a_3 \leq 3 \tag{2.5}$$

$$0 \leq b_1 + b_3 \leq 1. \tag{2.6}$$

It follows directly from (2.3) and (2.6) that $\sum_{i=1}^{4} b_i \leq 2$. Furthermore, it follows from (2.1) and (2.2) (respectively (2.4) and (2.5)) that if $b_2 = b_4 = 0$ or $b_1 = b_3 = 0$, at most six vertices can be placed around the 4-gon. Therefore, $(b_1, b_2, b_3, b_4) = (1, 1, 0, 0)$ without loss of generality. By choosing $a_1 \in \{0, 1, 2\}$, it follows that only the following $(a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4)$-tuples are feasible: $(2, 1, 0, 1, 2, 0, 1, 0)$, $(1, 1, 1, 1, 1, 0, 2, 0)$ and $(0, 1, 2, 1, 0, 0, 3, 0)$ (modulo rotations and reflections). These can be treated individually as follows:

- The subcase $(2, 1, 0, 1, 2, 0, 1, 0)$ can be treated as indicated in Figure 2.11. Note that at most two of the points D, E, F can lie in one of the sectors (QPR) and (RPS) without the occurrence of an empty convex hexagon. The same holds for A, B, C and the sectors (QRP) and (PRS). This is indicated by the arrows. Note that one 4- and one 5-sector arise.

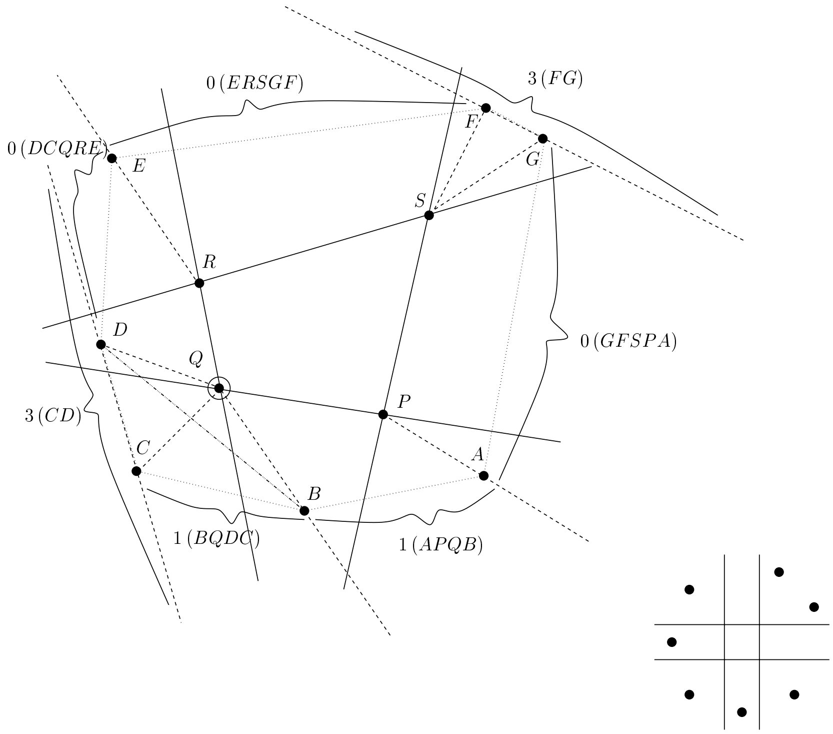

FIGURE 2.12. The case $(7, 4, 0)$ with $(1, 1, 1, 1, 1, 0, 2, 0)$. It is assumed that $Q \notin \triangle BCD$.

- Figure 2.12 indicates how to settle the subcase $(1, 1, 1, 1, 1, 0, 2, 0)$, provided that the vertex Q of the quadrilateral $PQRS$ lies outside the triangle BCD. In that case, the quadrilateral $BQDC$ exists. Note that if Q lies inside the triangle BCD, there exists an empty convex hexagon $BQDRSP$.
- The subcase $(0, 1, 2, 1, 0, 0, 3, 0)$ can be treated as indicated in Figure 2.13. Note that if B and C both lie in (PSQ) or both lie in (QSR), an empty convex hexagon occurs ($ABCQSP$ and $BCDRSQ$ respectively). Again, this is indicated by the arrows.

2.4.3. Step 2. Now we investigate the cases $\mathbf{(4, 1)}$ and $\mathbf{(8, 4, 1)}$. Consider the sectors occuring when rays emanate from the single point in J (respectively K) through the vertices of the convex 4-gon. Each of the four sectors can only contain two vertices of the convex 9- respectively 8-gon as otherwise an empty convex hexagon could be constructed. Since $4 \cdot 2 < 9$, in the case of the 9-gon an empty convex hexagon must occur. The case of the 8-gon is settled with a similar sector argument on the next level as indicated in Figure 2.14.

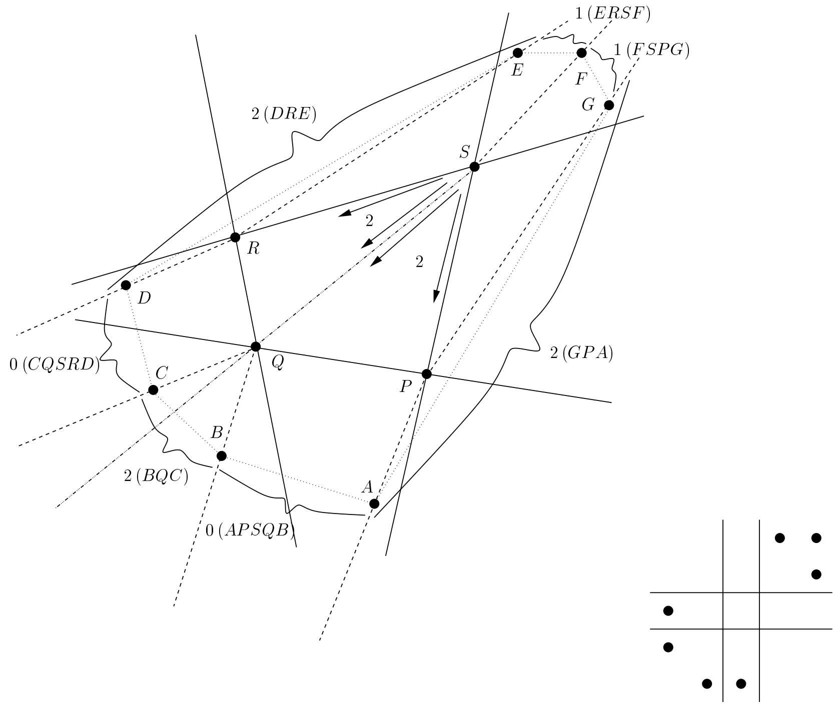

FIGURE 2.13. The case $(7, 4, 0)$ with $(0, 1, 2, 1, 0, 0, 3, 0)$

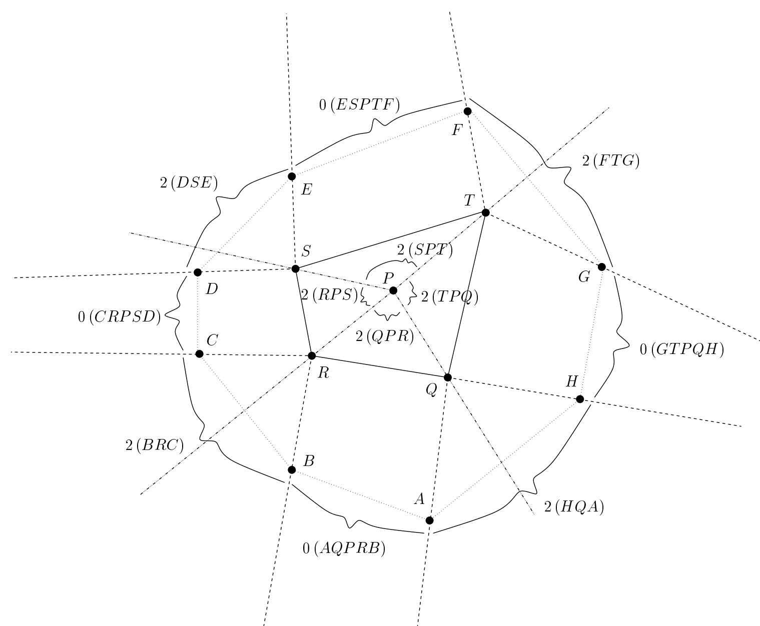

FIGURE 2.14. The case $(8, 4, 1)$

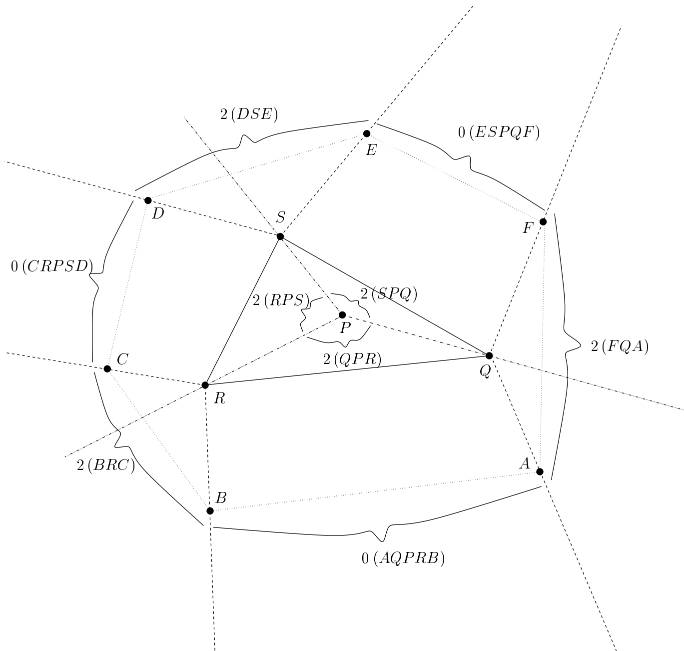

FIGURE 2.15. The case $(6, 3, 1)$

2.4.4. Step 3. In dealing with the cases $\mathbf{(4, \geq 2)}$, fix a point $P \in J$, construct the sectors as in Step 2 and afterwards replace P with an appropriate point from J in each sector (if necessary). Now argue as in Step 2. Proceed accordingly in the cases $\mathbf{(8, 4, \geq 2)}$ by choosing an arbitrary $P \in K$.

2.4.5. Remark. The approach of Sections 2.4.3 and 2.4.4 also works straightforward in the cases $\mathbf{(6, 3, \geq 1)}$ (as indicated in Figure 2.15), $\mathbf{(7, 3, \geq 1)}$ and $\mathbf{(7, 4, \geq 1)}$ (as indicated in Figure 2.16). Again, the idea is to fix a point $P \in K$ and to create sectors from rays emanating from P that pass through the vertices of the j-gon. Argue that each of these sectors can only contain at most two vertices of the i-gon without the occurrence of an empty convex hexagon. This remains true if other points of K should lie in some of the sectors. Now create another set of sectors such that their union covers the complete region outside of $\operatorname{conv}(I)$ as indicated in the figures. This approach is extended in Section 2.9 dealing with the cases $(\geq 7, \geq 5, \geq 1)$.

2.5. The Cases $(5, 0)$ and $(\geq 7, 5, 0)$

2.5.1. The Cases $\mathbf{(5, 0)}$ and $\mathbf{(8, 5, 0)}$. We use the same basic approach as in Sections 2.3 and 2.4, extending the concept and notation of Figures 2.3 and

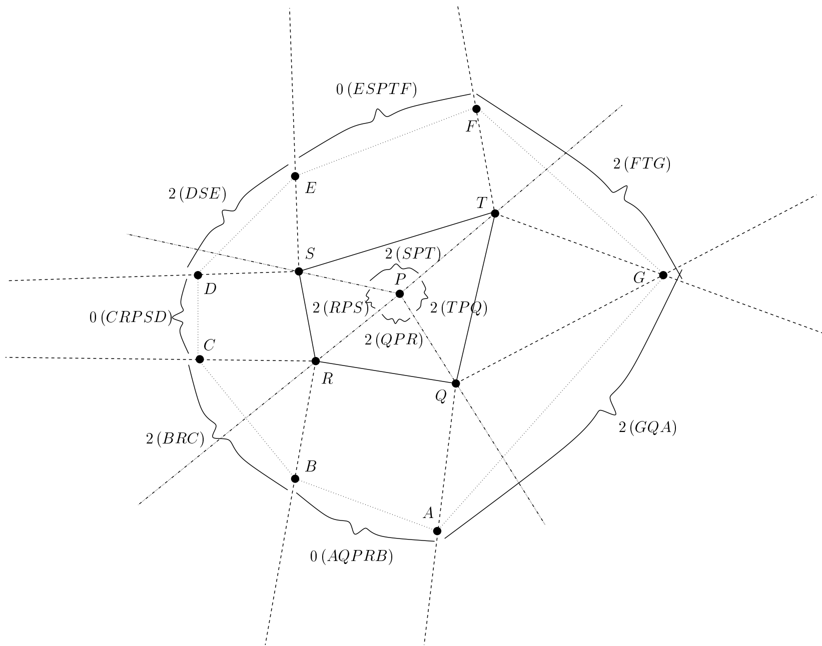

FIGURE 2.16. The case $(7, 4, 1)$

2.10 in the natural way. We arrive at the set of inequalities

$$b_i = 0 \quad (1 \leq i \leq 5) \quad \text{and} \tag{2.1}$$

$$1 \leq a_i + a_{i+1} \leq 3 \quad (1 \leq i \leq 5, a_6 := a_1) \tag{2.2}$$

if no empty convex hexagon is to be present (again counting vertices lying in more than one sector only once). This set of inequalities yields

$$2 \cdot \sum_{i=1}^{5} (a_i + b_i) \leq 15, \tag{2.3}$$

which implies the desired contradiction that an outer convex polygon with at most seven vertices can be present.

2.5.2. The Case $(7, 5, 0)$. A closer investigation of the constraints (2.1) — (2.3) shows that in this case the only feasible $(a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5)$-tuple (modulo rotation) is $(2, 0, 1, 0, 2, 0, 1, 0, 1, 0)$. This case can be settled as indicated in Figure 2.17.

2.6. Individual Cases

2.6.1. The Case $(5, 1)$. This case can be dealt with as indicated in Figure 2.18. Observe that P must lie in one of the triangles $\triangle ABD$, $\triangle BCE$, $\triangle CDA$,

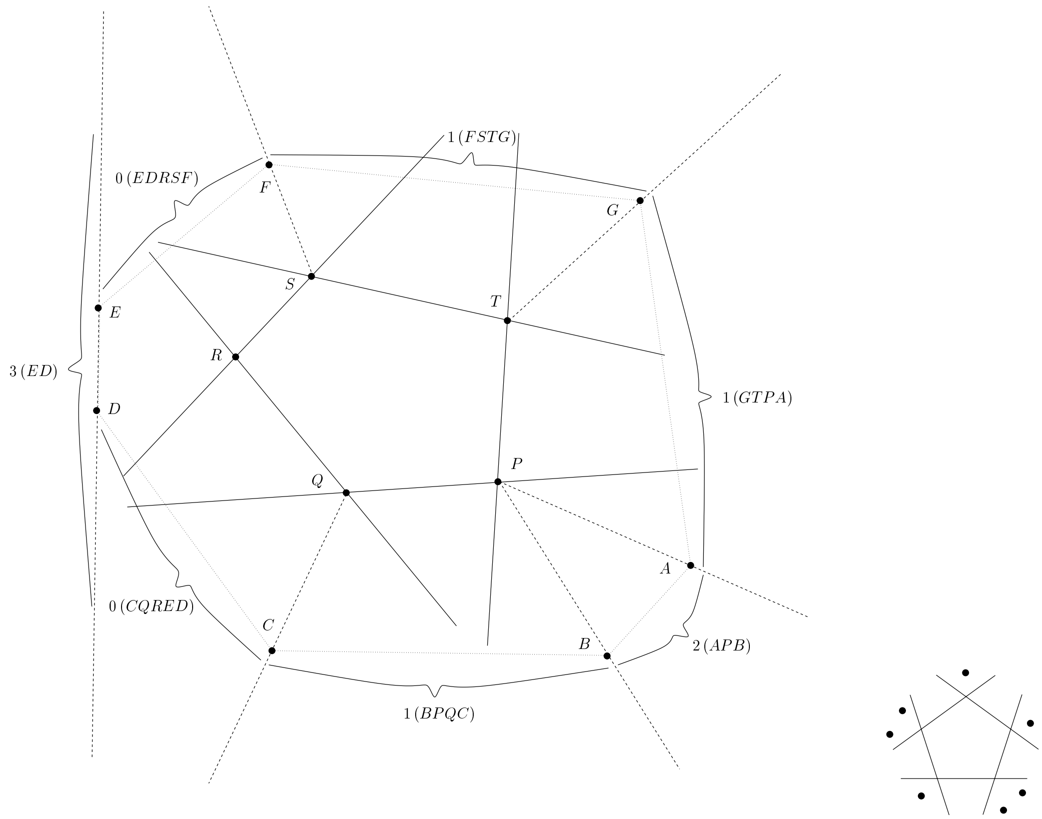

FIGURE 2.17. The case (7,5,0)

$\triangle DEB$ or $\triangle EAC$ (as these cover the convex 5-gon). Without loss of generality P is inside the triangle ABD (as in the figure). The line PD cuts the 5-gon into the two quadrilaterals $AEDP$ and $PDCB$ (and one triangle). It follows that $m_1+m_2 \leq 1$ and $n_1+n_2 \leq 1$ if no empty convex hexagon is to be present. (As in previous sections, variables refer to the number of vertices of the 9-gon lying in the corresponding sectors.) This leads to at most eight points that can be placed in convex position around the 5-gon without creating an empty convex hexagon.

2.6.2. The Case $(6,1)$. This case can be dealt with as indicated in Figure 2.19. Note that P must lie in one of the 4-gons $ADEF$ or $ABCD$ (as in the figure). Note furthermore that if in the latter case, $P \in \triangle ABC$ or $P \in \triangle BCD$, an empty convex hexagon occurs ($APCDEF$ respectively $BPDEFA$). Therefore, assume that the convex 4-gons $APCB$ and $CBPD$ exist and argue as indicated in the figure.

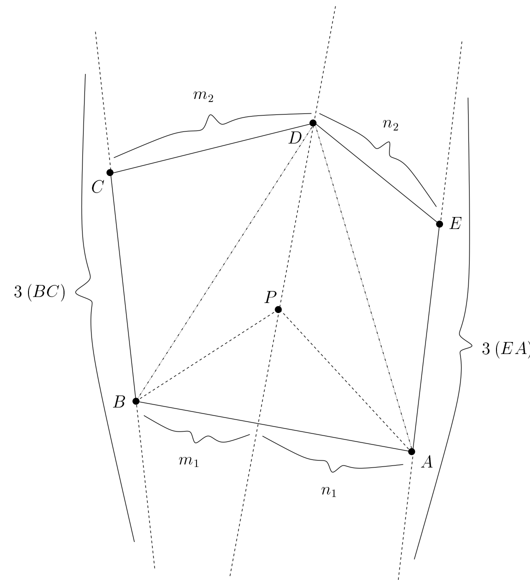

FIGURE 2.18. The case $(5, 1)$

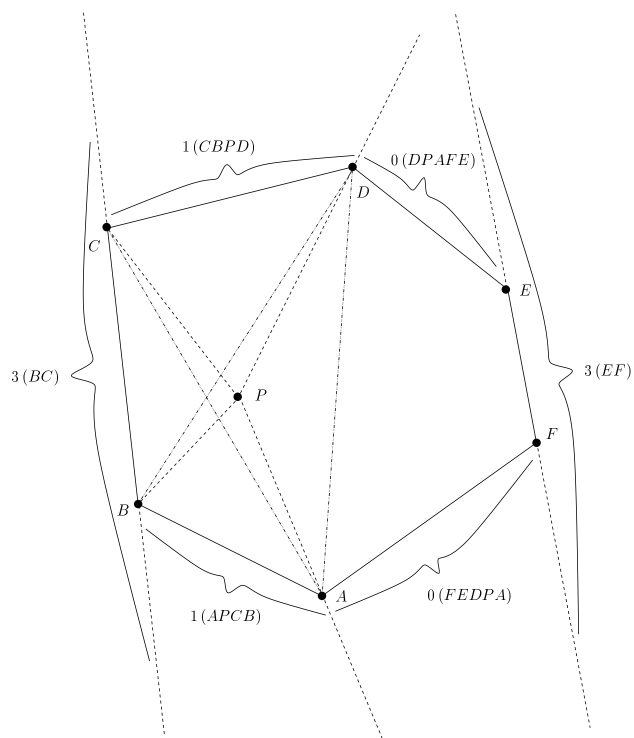

FIGURE 2.19. The case $(6, 1)$. See Section 2.6.2 for details.

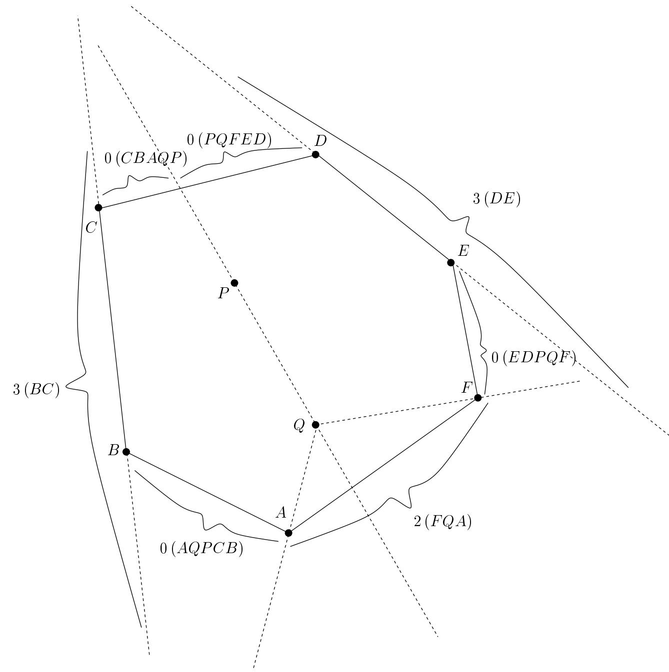

FIGURE 2.20. The case $(6,2)$. See Section 2.6.3 for details.

2.6.3. The Cases $(\mathbf{6,2})$ and $(\mathbf{7,1})$. The case $(\mathbf{6,2})$ can be dealt with as indicated in Figure 2.20. Note that if four vertices of the 6-gon lie on one side of the line $\overline{PQ}$, an empty convex hexagon can be constructed. The case $(\mathbf{7,1})$ is treated similarly. Here, one of the vertices of the convex 7-gon takes the role of P.

2.7. The Cases $(5, \geq 2)$

2.7.1. A key observation. The following observation is needed in later sections.

OBSERVATION 2. *Suppose that $j > 2$ and let $2 \leq t \leq \min\{i-1, j\}$. Consider a sequence of t consecutive vertices $V_1, V_2, \ldots, V_t$ of $\mathrm{conv}(J)$. Denote by T_n the set of vertices of the i-gon $\mathrm{conv}(I)$ lying in the halfplane that is defined by the line $\overline{V_n V_{n+1}}$ and that does not contain any other points of J. If $|\bigcup_{n=1}^{t-1} T_n| < t$, a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$ can be constructed.*

PROOF. We prove by induction over t. We use U_l $(l \in \mathbb{N}_0)$ to denote vertices of $\mathrm{conv}(I)$. Note that $|T_n| > 0$ for all n by the definition of J.

Let $t = 2$. Assume that $T_1 =: \{U_1\}$; see Figure 2.21. We claim that at most four vertices of the 9-gon can lie in the union of the 3-sectors $(U_0 V_1 U_1)$ and

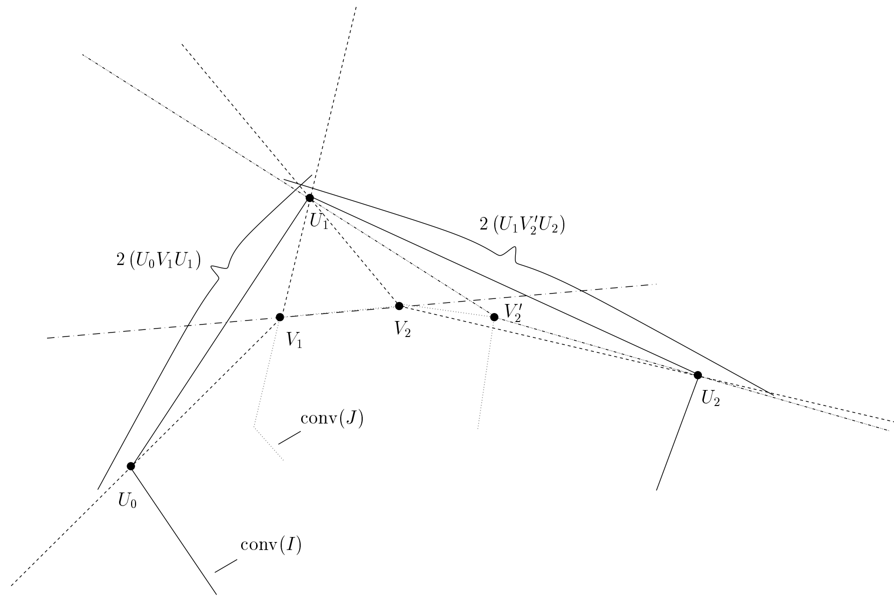

FIGURE 2.21. Observation 2: $t = 2$

$(U_1V_2U_2)$, where U_0 and U_2 are the vertices of conv(I) preceding and succeeding U_1. (Note that $U_0 \neq U_2$ as we presume $t < i$.) The bound follows directly if no other point of J lies within the triangles $\triangle U_0V_1U_1$ respectively $\triangle U_1V_2U_2$. Otherwise replace V_1 (respectively V_2) by appropriate $V_1' \in J \cap \triangle U_0V_1U_1$ and $V_2' \in J \cap \triangle U_1V_2U_2$ to obtain new 3-sectors $(U_0V_1'U_1)$ and $(U_1V_2'U_2)$ such that the corresponding triangles $\triangle U_0V_1'U_1$ and $\triangle U_1V_2'U_2$ do not contain any points of J. Note that these 3-sectors cover the region outside of $\text{conv}(I)$ that was originally covered by $(U_0V_1U_1)$ and $(U_1V_2U_2)$. (In fact, they cover a larger region.) Each of them allows for at most two vertices of the 9-gon without the occurrence of an empty convex hexagon and the claim follows. Replacing these vertices by points from the convex chain $\overline{U_0V_1V_2U_1}$ of length four yields a 9-gon H' with smaller $|X \cap \text{conv}(H')|$. (A similar argument was used in Section 2.3.2.)

Now let $t > 2$. We have to prove that if $|\bigcup_{n=1}^{t-1} T_n| < t$, a 9-gon H' with smaller $|X \cap \text{conv}(H')|$ can be constructed. If $|\bigcup_{n=1}^{t-1} T_n| < t - 1$, we are done by the induction hypothesis as $|\bigcup_{n=1}^{t-2} T_n| \leq |\bigcup_{n=1}^{t-1} T_n|$. Therefore, assume that $|\bigcup_{n=1}^{t-1} T_n| = t-1$. Label the consecutive vertices of $\text{conv}(I)$ as U_l $(l \in \mathbb{N}_0)$ in such a way that $U_1 \in T_1$ and $U_0 \notin T_1$. By the induction hypothesis this implies $U_2 \in T_1$ as otherwise $|T_1| = 1$. Now construct sectors as follows: start with the 3-sector $(U_0V_1U_1)$ that can hold at most two vertices of the 9-gon without the occurrence of an empty convex hexagon. (As above, replace V_1 by V_1' if necessary.) Next construct the 4-sectors $(U_1V_1V_2U_2), (U_2V_2V_3U_3), \ldots, (U_{t-2}V_{t-2}V_{t-1}U_{t-1})$ that can

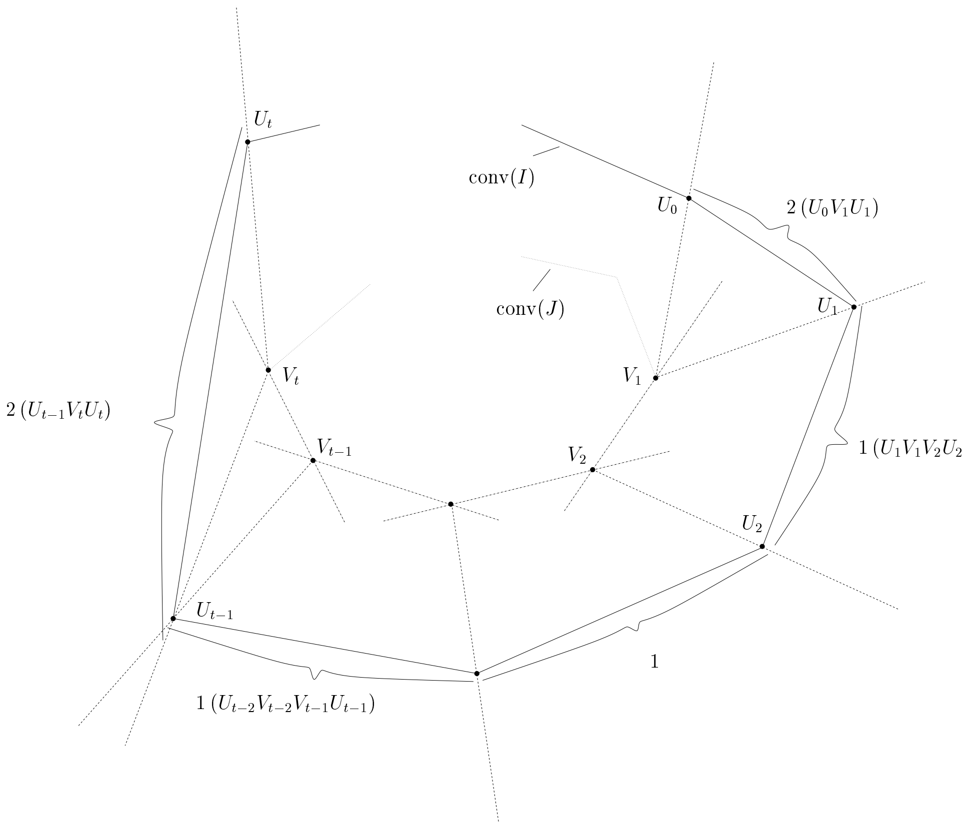

FIGURE 2.22. Observation 2: $t > 2$

hold at most one vertex of the 9-gon each if no empty convex hexagon is to occur; see Figure 2.22.

Note that at each step the construction is well-defined by the induction hypothesis. We can construct the 4-sector $(U_1V_1V_2U_2)$ as $U_1, U_2 \in T_1$. Assume there exists a smallest $p \in \mathbb{N}$ such that $U_pV_pV_{p+1}U_{p+1}$ is not a convex quadrilateral. This means that $U_p \in \left(\bigcup_{m=1}^{p-1} T_m\right) \setminus T_p$ or $U_{p+1} \in \left(\bigcup_{m=p+1}^{t-1} T_m\right) \setminus T_p$. In the first case, this implies $|\bigcup_{m=p}^{t-1} T_m| \leq (t-1)-p$. In the second case, it follows that $|\bigcup_{m=1}^{p} T_m| \leq p$. In both cases, the induction hypothesis implies that a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$ can be constructed.

Therefore, the 4-sectors can be constructed as described. Finally construct the 3-sector $(U_{t-1}V_tU_t)$ that can hold at most two vertices of the 9-gon without the occurrence of an empty convex hexagon. (As above, replace V_t by V_t' if necessary.) Note that $U_t \notin T_{t-1}$ as we presume $|\bigcup_{n=1}^{t-1} T_n| = t-1$. It follows that at most $2 \cdot 2 + (t-2) \cdot 1 = t+2$ vertices of the 9-gon can lie in the union of sectors

$$(U_0 V_1 U_1) \cup \bigcup_{l=1}^{t-2} (U_l V_l V_{l+1} U_{l+1}) \cup (U_{t-1} V_t U_t).$$

Replacing these vertices by points from the convex chain $\overline{U_0 V_1 V_2 \ldots V_t U_t}$ of length $t+2$ yields a 9-gon H' with smaller $|X \cap \operatorname{conv}(H')|$. $\square$

2.7.2. The Cases $(5, \geq 2)$. Consider the line through two consecutive vertices of $\operatorname{conv}(J)$, say P and Q, and let T_{PQ} be the set of vertices of the convex 5-gon lying in a halfplane that is defined by the line $\overline{PQ}$ and that does not contain any other points of J. (This halfplane is unique if $|J| > 2$.) Consider possible values for $|T_{PQ}|$:

- $|T_{PQ}| = 0$: This case is not possible by the definition of J.
- $|T_{PQ}| = 1$: In this case, a 9-gon H' with smaller $|X \cap \operatorname{conv}(H')|$ can be constructed. Set $t = 2$ in Observation 2.
- $2 \leq |T_{PQ}| \leq 3$: This is the assumption for our subsequent considerations.
- $|T_{PQ}| > 3$: In this case, an empty convex hexagon can be constructed by using a convex chain of four vertices of the 5-gon together with P and Q.

Therefore, assume that

$$2 \leq |T_{PQ}| \leq 3. \tag{2.1}$$

Let R be the next vertex on the convex hull of J after passing through P and Q (if $|J| = 2$ then $R = P$). Define the set T_{QR} accordingly (take the other halfplane if $|J| = 2$). For the same reasons as above assume that

$$2 \leq |T_{QR}| \leq 3 \tag{2.2}$$

and consider the following three possibilities:

$|T_{PQ} \cup T_{QR}| \geq 4$. In this case we can choose consecutive vertices A, B, C, D of the 5-gon such that $A, B \in T_{PQ}$ and $C, D \in T_{QR}$. Label the remaining vertex of the 5-gon E. Construct the two 4-sectors $(APQB)$ and $(CQRD)$ that can hold at most one vertex of the 9-gon each without the occurrence of an empty convex hexagon. Next construct the 3-sector (BQC) that can hold at most two vertices of the 9-gon if no empty convex hexagon is to occur. Construct furthermore the two 3-sectors (DRE) and (EPA). Note that the union of these five sectors covers the complete region outside of $\operatorname{conv}(I)$; see also Figure 2.23. Each of the two latter 3-sectors can hold at most two vertices of the 9-gon without the occurrence of an empty convex hexagon. (If necessary, replace R (respectively P) by appropriate $R' \in J \cap \triangle DRE$ and $P' \in J \cap \triangle EPA$ to obtain new 3-sectors

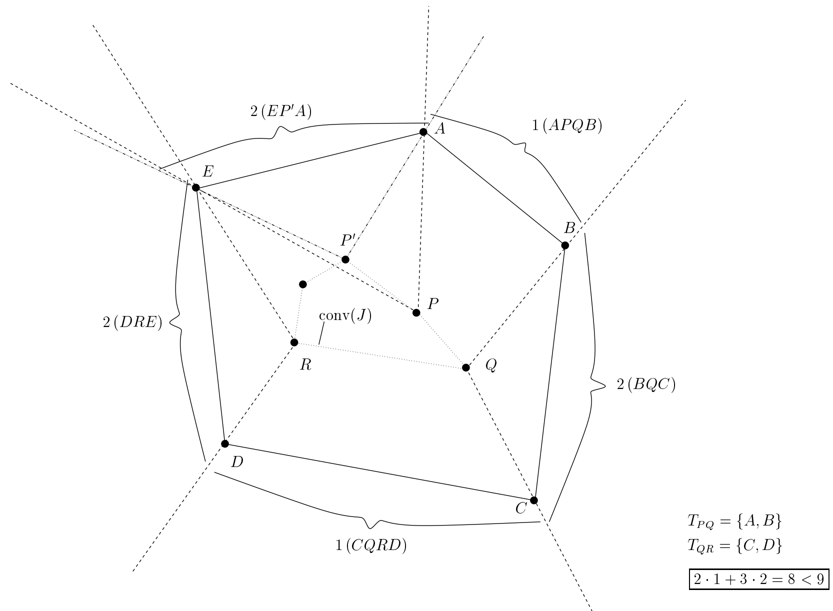

FIGURE 2.23. The cases $(5, \geq 2)$: Example with $|T_{PQ} \cup T_{QR}| \geq 4$.

$(DR'E)$ and $(EP'A)$ such that the corresponding triangles $\triangle DR'E$ and $\triangle EP'A$ do not contain any points of J as in the proof of Observation 2.) It follows that at most $2 \cdot 1 + 3 \cdot 2 = 8$ vertices of the 9-gon can be placed around the 5-gon without the occurrence of an empty convex hexagon. Note in particular that the case $\mathbf{(5,2)}$ is covered by the argument in this subsection.

$|T_{PQ} \cup T_{QR}| = 3$. The case $|T_{PQ} \cup T_{QR}| = 3$ can be treated by the same approach as in the previous subsection. Choose consecutive vertices A, B, C of the 5-gon such that $A, B \in T_{PQ}$ and $B, C \in T_{QR}$. Label the remaining vertices of the 5-gon D and E such that the vertices C, D, E are consecutive. Construct the two 4-sectors $(APQB)$ and $(BQRC)$. Next construct the 3-sectors (CRD), (DRE) and (EPA). As above, replace the points R and P by appropriate points in J and modify the 3-sectors if necessary. Again, we arrive at the contradiction that at most $2 \cdot 1 + 3 \cdot 2 = 8$ vertices of the 9-gon can be placed around the 5-gon without the occurrence of an empty convex hexagon.

$|T_{PQ} \cup T_{QR}| \leq 2$. This case leaves the possibility of constructing a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$. Set $t = 3$ in Observation 2.

2.8. The Cases $(6, \geq 4)$

The approach is similar to the one in Section 2.7. The key idea is to partition the region outside of $\mathrm{conv}(I)$ into two 3-sectors and four 4-sectors. Each 3-sector is defined by two consecutive vertices of the 6-gon and one vertex of conv(J). It can hold at most two vertices of the 9-gon if no empty convex hexagon is to occur. Each 4-sector is defined by two consecutive vertices of the 6-gon and two consecutive vertices of conv(J). It can hold at most one vertex of the 9-gon without the occurrence of an empty convex hexagon. It follows that a total of $2 \cdot 2 + 4 \cdot 1 = 8$ vertices of the 9-gon can be placed around the 6-gon without the occurrence of an empty convex hexagon.

Consider a chain of consecutive vertices of conv(J), $\overline{VWXYZ}$, where $V = Z$ if $j = 4$. Define the sets T_{VW},T_{WX},T_{XY} and T_{YZ} as in Section 2.7 (that is, T_{VW} is the set of vertices of the convex 6-gon lying in the halfplane defined by the line $\overline{VW}$ that does not contain any other points of J, etc.). As in Section 2.7, we assume that

$$(2.1) \qquad 2 \leq |T_{KL}| \leq 3 \quad ((K, L) \in \{(V, W), (W, X), (X, Y), (Y, Z)\}).$$

By setting $t = 3, 4, 5$ in Observation 2 (Section 2.7), it follows that we may also assume that

$$|T_{KL} \cup T_{LM}| \geq 3 \tag{2.2}$$

$$|T_{KL} \cup T_{LM} \cup T_{MN}| \geq 4 \tag{2.3}$$

$$|T_{VW} \cup T_{WX} \cup T_{XY} \cup T_{YZ}| \geq 5 \tag{2.4}$$

with $(K, L, M) \in \{(V, W, X), (W, X, Y), (X, Y, Z)\}$ (in (2.2)) and $(K, L, M, N) \in \{(V, W, X, Y), (W, X, Y, Z)\}$ (in (2.3)). Note that (2.4) also holds in the case $\mathbf{(6, 4)}$, where Observation 2 does not apply (since $t > j$). Note furthermore that by construction it is not possible that there is a $P \in T_{KL} \cap T_{MN}$ with $P \notin T_{LM}$ $((K, L, M, N) \in \{(V, W, X, Y), (W, X, Y, Z)\})$. We now give an explicit construction for the two 3-sectors and the four 4-sectors. A concrete example can be found in Figure 2.24. The combinatorial subcases are depicted in Figure 2.25.

2.8.1. $T_{WX} \cap T_{XY} \neq \emptyset$**.** Label the consecutive vertices of the 6-gon A, B, C, D, E, F such that $B \in T_{WX}$, $C \in T_{WX} \cap T_{XY}$ and $D \in T_{XY}$. Note that $F \notin T_{WX}$ and $F \notin T_{XY}$ as otherwise $|T_{WX}| > 3$ or $|T_{XY}| > 3$. Consider the following possibilities:

1. $A \notin (T_{VW} \cup T_{WX})$; see Figure 2.25a. It follows from (2.1) and (2.2) that $B, C \in T_{VW}$ and $D \in T_{WX}$. (2.3) implies $E \in T_{XY}$. Construct the three 4-sectors $(BVWC)$, $(CWXD)$ and $(DXYE)$. Next, construct the 3-sector (AVB). (Replace V by an appropriate $V' \in J \cap \triangle AVB$ if necessary.)

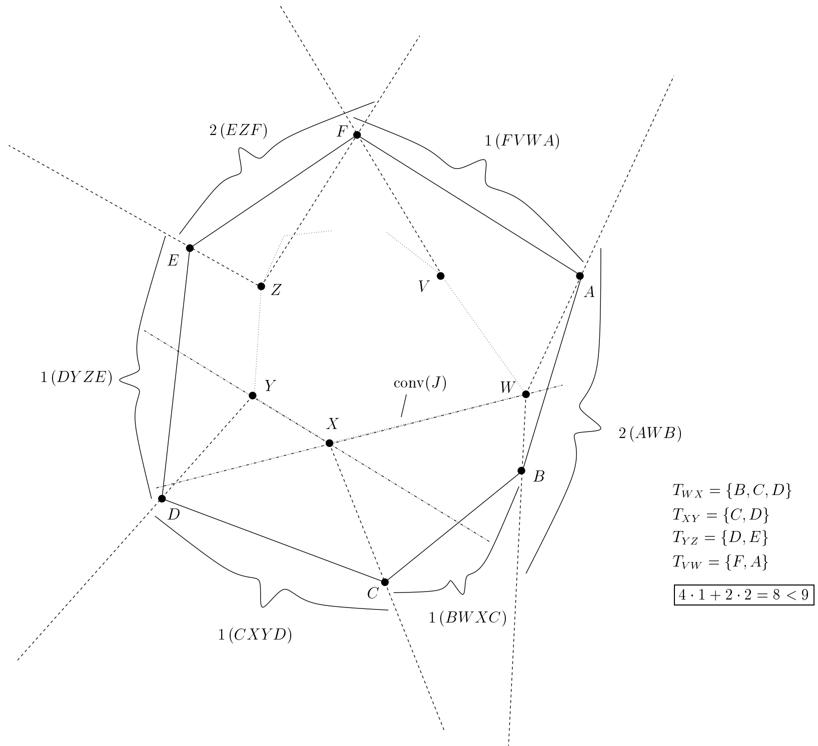

FIGURE 2.24. The cases $(6, \geq 4)$: Example with $T_{WX} \cap T_{XY} \neq \emptyset$.

- If $E \in T_{YZ}$ then (2.4) implies $F \in T_{YZ}$. Construct the 4-sector $(EYZF)$ and the 3-sector (FZA). (Again, replace Z by Z' if necessary.)
- If $E \notin T_{YZ}$ then it follows from (2.1) that $A, F \in T_{YZ}$. ($F \notin T_{XY}$ implies in particular $F \notin T_{XY} \setminus T_{YZ}$.) In this case construct the 3-sector (EYF) together with the 4-sector $(FYZA)$.

In both cases we arrive at a set of four 4-sectors and two 3-sectors as claimed. *In the following cases, assume that* $A \in (T_{VW} \cup T_{WX})$.

2. $E \notin (T_{XY} \cup T_{YZ})$. This case is symmetric to the previous one. Therefore, *in the following assume that* $E \in (T_{XY} \cup T_{YZ})$.
3. $A \in T_{WX} \setminus T_{VW}$; see Figure 2.25b. It follows from (2.1) that $E, F \in T_{VW}$. ($F \notin T_{WX}$ implies in particular that $F \notin T_{WX} \setminus T_{VW}$.) Construct the 3-sectors (FWA) and (BXC) together with the 4-sectors $(EVWF)$, $(AWXB)$ and $(CXYD)$. It follows that $D \in T_{YZ}$ as otherwise $|T_{YZ} \cup T_{VW}| = |\{E, F\}| < 3$. Note that $(E \in T_{VW}) \wedge (E \in (T_{XY} \cup T_{YZ}))$ implies $E \in T_{YZ}$. Therefore, we can construct the 4-sector $(DYZE)$. Again the

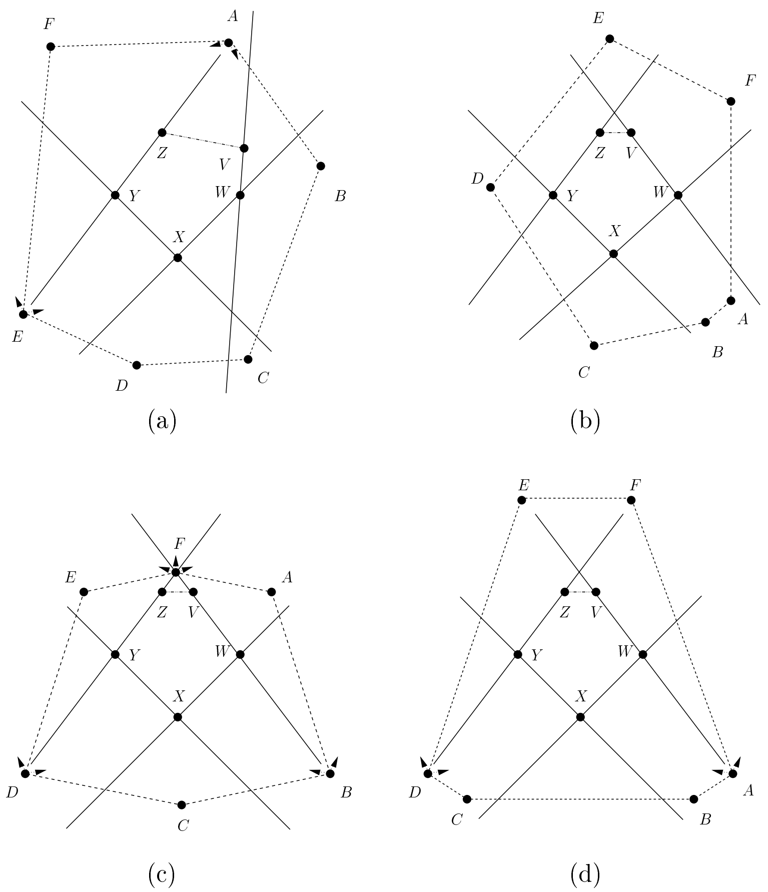

FIGURE 2.25. The cases $(6, \geq 4)$: Combinatorial subcases. The assumption in (a) — (c) is that $B \in T_{WX}$, $C \in T_{WX} \cap T_{XY}$ and $D \in T_{XY}$. In (a), $A \notin (T_{VW} \cup T_{WX})$. In (b), $A \in T_{WX} \setminus T_{VW}$ and $E \in (T_{XY} \cup T_{YZ})$. In (c), $A \in T_{VW}$ and $E \in T_{YZ}$. In (d), it is assumed that $A, B \in T_{WX} \setminus T_{XY}$ and $C, D \in T_{XY} \setminus T_{WX}$. Only those point positions that are essential for the construction of the sectors are indicated.

six sectors can be constructed as claimed. *In the following assume that* $A \in T_{VW}$.

4. $E \in T_{XY} \setminus T_{YZ}$. This case is symmetric to the previous one. Therefore, *in the following assume that* $E \in T_{YZ}$.
5. $A \in T_{VW}$ and $E \in T_{YZ}$; see Figure 2.25c. Construct the 4-sectors $(BWXC)$ and $(CXYD)$. Consider the following four possibilities:
   - $B \in T_{VW} \wedge D \in T_{YZ}$. Construct the 4-sector $(AVWB)$ together with the 3-sector (FVA). (Replace V by V' if necessary.) Accordingly,

construct the 4-sector $(DYZE)$ together with the 3-sector (EZF). (Replace Z by Z' if necessary.)

- $B \notin T_{VW} \wedge D \in T_{YZ}$. Construct the 4-sector $(DYZE)$ together with the 3-sector $(EZ'F)$ as in the previous subcase. If $B \notin T_{VW}$, it follows from (2.1) that $F \in T_{VW}$. In this case, construct the 3-sector (AWB) together with the 4-sector $(FVWA)$.
- $B \in T_{VW} \wedge D \notin T_{YZ}$. This subcase is symmetric to the previous one.
- $B \notin T_{VW} \wedge D \notin T_{YZ}$. It follows that $F \in T_{VW}$ and $F \in T_{YZ}$. Accordingly, construct the 4-sectors $(FVWA)$ and $(EYZF)$ together with the 3-sectors (AWB) and (DYE).

In each case, we arrive at a set of four 4-sectors and two 3-sectors that cover the complete region outside of $\mathrm{conv}(I)$ as claimed.

2.8.2. $T_{WX} \cap T_{XY} = \emptyset$. See Figure 2.25d. Then by construction, there exist consecutive vertices A, B, C, D of conv(I) such that $A, B \in T_{WX} \setminus T_{XY}$ and $C, D \in T_{XY} \setminus T_{WX}$. Construct the 4-sectors $(AWXB)$ and $(CXYD)$ as well as the 3-sector (BXC). Label the remaining vertices of the 6-gon E, F such that D, E, F are consecutive. Now distinguish four possibilities:

- $D \in T_{YZ} \wedge A \in T_{VW}$. It follows that $E \in T_{YZ}$ as otherwise $|T_{XY} \cup T_{YZ}| < 3$. Accordingly, $F \in T_{VW}$ as otherwise $|T_{VW} \cup T_{WX}| < 3$. Construct the 4-sectors $(DYZE)$ and $(FVWA)$ together with the 3-sector (EZF). (Replace Z by an appropriate Z' if necessary.)
- $D \notin T_{YZ} \wedge A \in T_{VW}$. As in the previous case, construct the 4-sector $(FVWA)$. If $E \in T_{XY} \setminus T_{YZ}$ it follows that $|T_{YZ} \cup T_{VW} \cup T_{WX}| \leq |\{A, B, F\}| < 4$. Therefore, assume that $E \in T_{YZ}$. It follows that $F \in T_{YZ}$ as otherwise $|T_{YZ}| < 2$. Construct the 3-sector (DYE) and the 4-sector $(EYZF)$.
- $D \in T_{YZ} \wedge A \notin T_{VW}$. This case is symmetric to the previous one.
- $D \notin T_{YZ} \wedge A \notin T_{VW}$. Note that this case is not feasible as it would imply $|T_{YZ} \cup T_{VW}| \leq |\{E, F\}| < 3$.

In each feasible case, we can construct the six sectors as claimed above.

2.9. The Cases $(\geq 7, \geq 5, \geq 1)$

Up to this point, we have settled all cases except for $(\boldsymbol{\geq 7, \geq 5, \geq 1})$. These cases, except for three special cases (see below), can all be settled via the same set of arguments. As above, let $K := \mathrm{conv}(J) \cap (X \setminus \partial J)$. Fix a point $P \in K$. Consider rays emanating from P through each vertex of the convex j-gon $\mathrm{conv}(J)$.

$(\mathbf{7,5,\geq 1})$	$(\mathbf{8,5,\geq 1})$
$\Pi(2,2,2,1,0)$	$\Pi(2,2,2,2,0)$
$\Pi(2,2,1,1,1)$	$\Pi(2,2,2,1,1)$

$(\mathbf{7,6,\geq 1})$	$(\mathbf{8,6,\geq 1})$
$\Pi(2,2,2,1,0,0)$	$\Pi(2,2,2,2,0,0)$
$\Pi(2,2,1,1,1,0)$	$\Pi(2,2,2,1,1,0)$
$(2,1,1,1,1,1)$	$\Pi(2,2,1,1,1,1)$

$(\mathbf{7,7,\geq 1})$	$(\mathbf{8,7,\geq 1})$
$\Pi(2,2,2,1,0,0,0)$	$\Pi(2,2,2,2,0,0,0)$
$\Pi(2,2,1,1,1,0,0)$	$\Pi(2,2,2,1,1,0,0)$
$\Pi(2,1,1,1,1,1,0)$	$\Pi(2,2,1,1,1,1,0)$
$(1,1,1,1,1,1,1)$	$(2,1,1,1,1,1,1)$

$(\mathbf{7,8,\geq 1})$	$(\mathbf{8,8,\geq 1})$
$\Pi(2,2,2,1,0,0,0,0)$	$\Pi(2,2,2,2,0,0,0,0)$
$\Pi(2,2,1,1,1,0,0,0)$	$\Pi(2,2,2,1,1,0,0,0)$
$\Pi(2,1,1,1,1,1,0,0)$	$\Pi(2,2,1,1,1,1,0,0)$
$(1,1,1,1,1,1,1,0)$	$\Pi(2,1,1,1,1,1,1,0)$
	$(1,1,1,1,1,1,1,1)$

TABLE 3. The cases $(\geq 7, \geq 5, \geq 1)$: Combinatorial subcases. (Π indicates possible permutations.)

This divides the region outside the j-gon into j sectors and in each sector at most two vertices of $\operatorname{conv}(I)$ can lie without forming an empty convex hexagon. (To see this, construct 3-sectors and replace P by an appropriate $P' \in K$ where needed.) Consider all possible vertex distributions. (These are summarized in Table 3.) We want to partition the region outside the convex i-gon $\operatorname{conv}(I)$ into sectors and to show that in each case at most eight vertices of the 9-gon can be placed inside the union of these sectors without creating an empty convex hexagon. The following three simple rules are sufficient to prove this:

2.9.1. The first rule. The first rule deals with two vertices of $\operatorname{conv}(I)$ lying in the same sector.

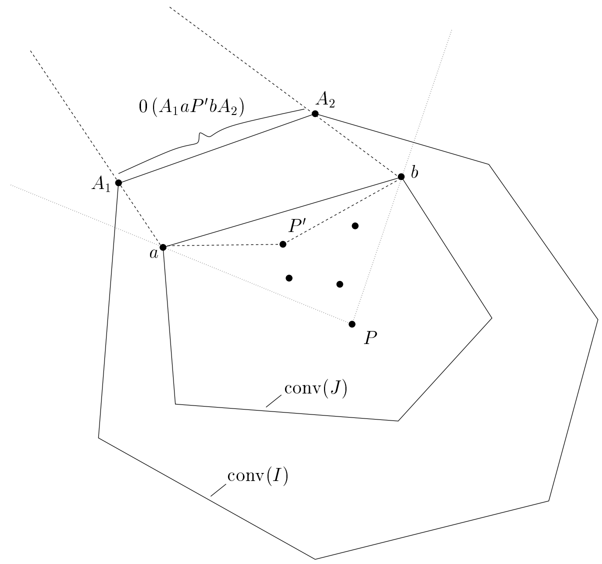

FIGURE 2.26. Rule 1

RULE 1. *Let A_1, A_2 denote two consecutive vertices of* conv(I) *lying in the same sector (aPb), where a and b are consecutive vertices of* conv(J). *Then no vertex of the 9-gon can lie in the sector (A_1abA_2) without the occurrence of an empty convex hexagon.*

PROOF. The claim follows directly from the presence of an empty convex 5-gon $A_1aP'bA_2$, where $P' \in K \cap \triangle aPb$ is chosen appropriately; see Figure 2.26. □

2.9.2. The second rule. The second rule gives an upper bound on the number of vertices of the 9-gon that can lie between two non-empty sectors.

RULE 2. *Let A_1, A_2 denote two consecutive vertices of* conv(I) *lying in distinct sectors (a_1Pb_1) and (a_2Pb_2), where a_1 and b_1 respectively a_2 and b_2 are consecutive vertices of* conv(J). *Suppose that a_1, b_1, a_2, b_2 are part of a chain of consecutive vertices of* conv(J). *Let $\mathcal{S} := (A_1b_1a_2A_2)$ if $A_1b_1a_2A_2$ is a convex quadrilateral and $\mathcal{S} := (A_1b_1A_2) \cup (A_1a_2A_2)$ otherwise. Then at most two vertices of the 9-gon can lie within $\mathcal{S}$.*

Remark: It is possible in Rule 2 that $b_1 = a_2$.

PROOF. A 3-sector that does not contain any points of J and covers the region of the sector $\mathcal{S}$ can be constructed by choosing A_1, A_2 and an appropriate a_r among the consecutive vertices of conv(J) between b_1 and a_2 (inclusively); see Figure 2.27. □

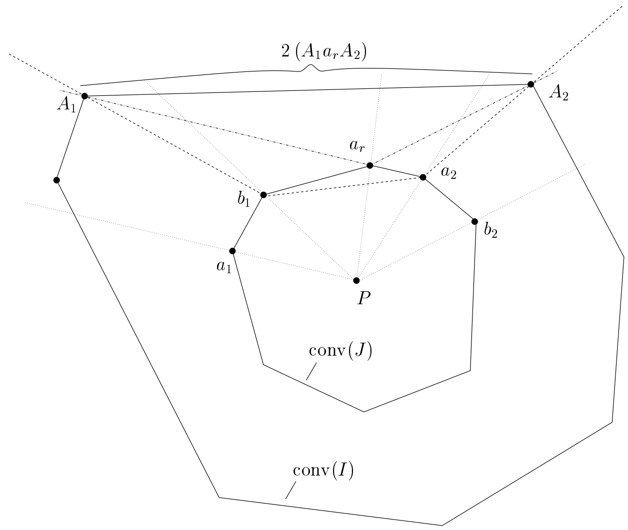

FIGURE 2.27. Rule 2

2.9.3. Application of Rules 1 and 2. The first two rules are already sufficient to settle the cases $(7,5,\geq 1)$ with distributions $\Pi(2,2,2,1,0)$, $(7,6,\geq 1)$ with distributions $\Pi(2,2,2,1,0,0)$, $(7,7,\geq 1)$ with distributions $\Pi(2,2,2,1,0,0,0)$, $(7,8,\geq 1)$ with distributions $\Pi(2,2,2,1,0,0,0,0)$, $(8,5,\geq 1)$ with distributions $\Pi(2,2,2,2,0)$, $(8,6,\geq 1)$ with distributions $\Pi(2,2,2,2,0,0)$, $(8,7,\geq 1)$ with distributions $\Pi(2,2,2,2,0,0,0)$ and $(8,8,\geq 1)$ with distributions $\Pi(2,2,2,2,0,0,0,0)$. To see this, apply Rule 1 whenever two consecutive vertices of $\operatorname{conv}(I)$ lie in the same sector. Note that two such vertices correspond to a 2 in the underlying distribution. For consecutive vertices of $\operatorname{conv}(I)$ lying in distinct sectors, apply Rule 2. Note that in the cases at hand, Rule 2 needs to be applied exactly four times as there are always exactly four non-zero entries in the corresponding distribution sequences. It follows that at most $4 \cdot 2 = 8$ vertices of the 9-gon can be placed without the occurrence of an empty convex hexagon. An example is given in Figure 2.28.

2.9.4. The third rule. The third rule deals with a sequence of sectors, where each sector contains at most one vertex of $\operatorname{conv}(I)$. See also Figure 2.29.

RULE 3. *Let $1 \leq n \leq i-2$. Consider a sequence $A_0, A_1, \ldots, A_{n+1}$ of consecutive vertices of* $\operatorname{conv}(I)$. *For $1 \leq l \leq n+1$, let $A_l \in (a_l P b_l)$, where a_l and b_l are consecutive vertices of* $\operatorname{conv}(J)$. *Suppose that for $1 \leq l \leq n$, each sector $(a_l P b_l)$ contains exactly one vertex of* $\operatorname{conv}(I)$ *and that $a_1, b_1, a_2, b_2, \ldots, a_{n+1}, b_{n+1}$ are*

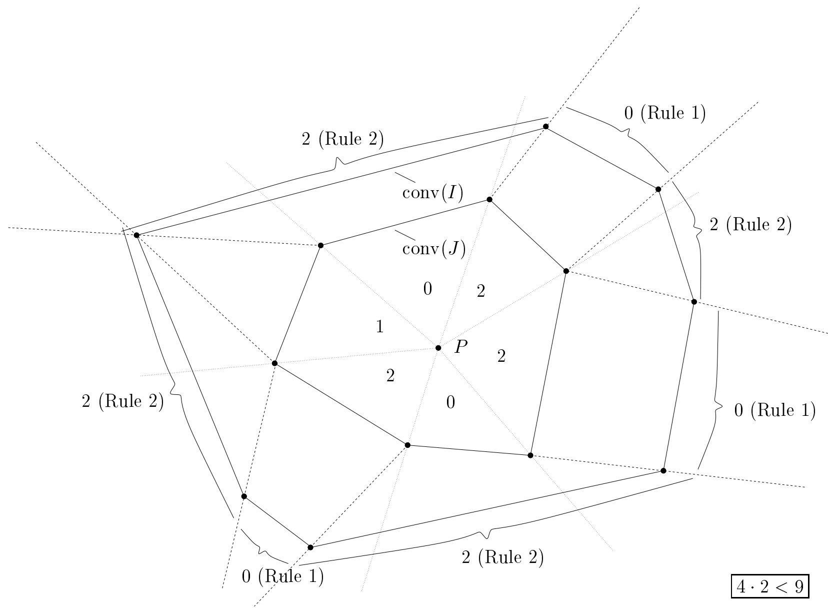

FIGURE 2.28. Application of Rules 1 and 2: Example for the case $(7, 6, \geq 1)$ with distribution $(2, 0, 2, 1, 0, 2)$.

part of a chain of consecutive vertices of $\operatorname{conv}(J)$*. Then at most* $n+2$ *vertices of the 9-gon lie in the union of sectors* $\bigcup_{l=1}^{n+1}(A_{l-1}a_lA_l)$*.*

Remark: It is possible in Rule 3 that $b_l = a_{l+1}$ $(1 \leq l \leq n)$ or $b_{n+1} = a_1$. Furthermore, it is possible that A_0 and A_{n+1} both lie in $(a_{n+1}Pb_{n+1})$.

PROOF. We prove by induction over n.

If $n = 1$, we can argue that it is not possible that A_1 lies above the line $\overline{a_1b_1}$ while A_0 and A_2 lie below it, where *lying below* refers to lying in the halfplane defined by $\overline{a_1b_1}$ that includes P. Otherwise, $|T_{a_1b_1}| = 1$ and a 9-gon H' with smaller $|X \cap \operatorname{conv}(H')|$ could be constructed. (Set $t = 2$ in Observation 2 in Section 2.7.) Assume that A_0 also lies above $\overline{a_1b_1}$. (The case that only A_1 and A_2 lie above the line is almost symmetric.) Construct the 4-sector $(A_0a_1b_1A_1)$ together with the 3-sector $(A_1a_2A_2)$. If necessary, replace a_2 by an appropriate $a_2' \in \triangle A_1a_2A_2$ to obtain a new 3-sector $(A_1a_2'A_2)$ with no points of J lying in $\triangle A_1a_2'A_2$. Together, the 4- and the 3-sector cover (at least) the region of $(A_0a_1A_1) \cup (A_1a_2A_2)$. This is clear for points lying in $(A_1a_2A_2)$ since $(A_1a_2'A_2)$ covers (at least) this region. Note that there cannot be a point $Q \in \big((A_0a_1A_1) \setminus (A_1a_2'A_2)\big) \setminus (A_0a_1b_1A_1)$. Such a point would have to lie in the shaded region in Figure 2.30. If a_2 lies to the right of $\overline{b_1A_1}$ (or $a_2 = b_1$) then $Q \in (A_1a_2'A_2)$. Otherwise $b_1 \in (A_1a_2A_2)$ and we could have chosen $a_2' := b_1$. The 4- and the 3-

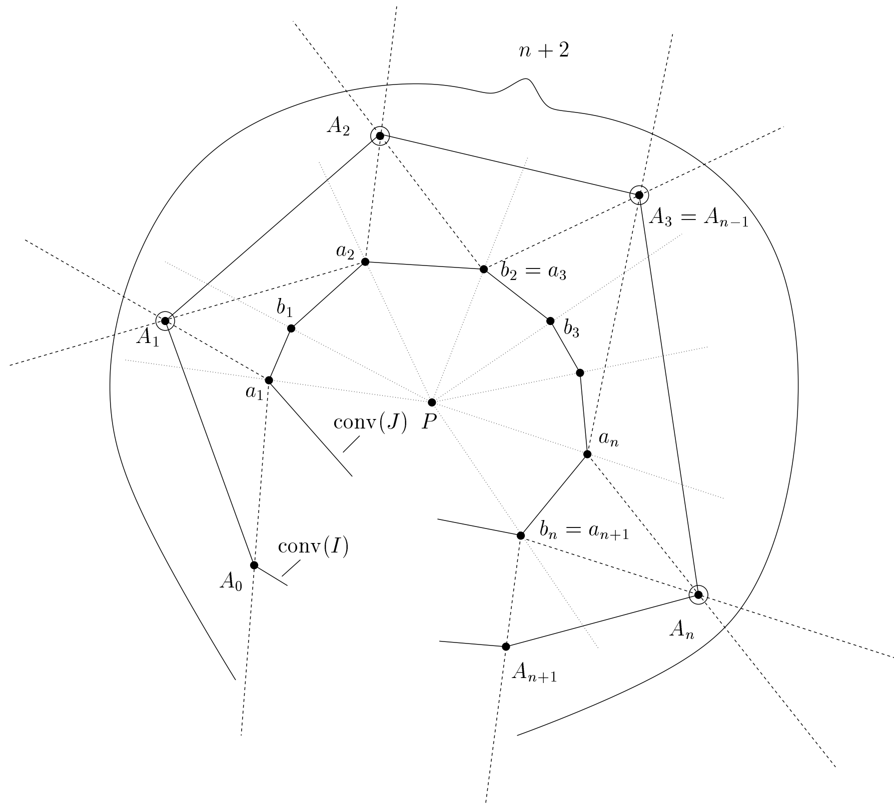

FIGURE 2.29. Rule 3

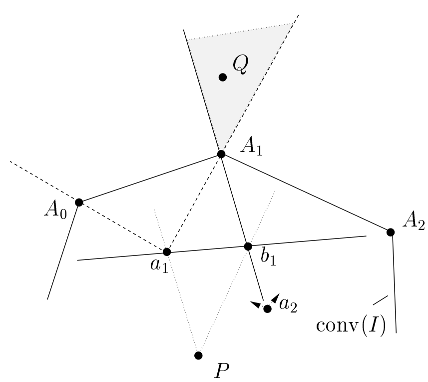

FIGURE 2.30. Proof of Rule 3: $n = 1$.

sector allow for at most $1+2=3$ vertices of the 9-gon without the occurrence of an empty convex hexagon.

For the induction step, assume that the claim is true for $1, 2, \ldots, n-1$. By the induction hypothesis, we know that at most $(n-1)+2$ vertices of the 9-gon can lie in the union of sectors $\bigcup_{l=1}^{n}(A_{l-1}a_lA_l)$. At most two additional vertices of the

9-gon can lie in the sector $(A_n a_n A_{n+1}) \setminus \bigcup_{l=1}^{n} (A_{l-1} a_l A_l)$ without the occurrence of an empty convex hexagon as it is part of the 3-sector $(A_n a_n A_{n+1})$. Therefore, the number of vertices of the 9-gon that can lie in the union of sectors $\bigcup_{l=1}^{n+1} (A_{l-1} a_l A_l)$ is at most $(n - 1 + 2) + 2 = n + 3$ if no empty convex hexagon is to occur. It also follows from the induction hypothesis that at most $(n-1)+2$ vertices of the 9-gon can lie in the union of sectors $\bigcup_{l=2}^{n+1} (A_{l-1} a_l A_l)$ without the occurrence of an empty convex hexagon. Accordingly, at most two additional vertices of the 9-gon can lie in the sector $(A_0 a_1 A_1) \setminus \bigcup_{l=2}^{n+1} (A_{l-1} a_l A_l)$ if no empty convex hexagon is to occur. Therefore, the above bound is sharp if and only if exactly two vertices of the 9-gon lie in the sectors $(A_0 a_1 A_1)$ and $(A_n a_{n+1} A_{n+1})$ respectively.

It follows that A_0 must lie below the line $\overline{a_1 b_1}$ and A_{n+1} must lie below the line $\overline{a_n b_n}$ as otherwise one could again replace one of the 3-sectors $(A_0 a_1 A_1)$ and $(A_n a_{n+1} A_{n+1})$ by the 4-sector $(A_0 a_1 b_1 A_1)$ respectively $(A_n a_n b_n A_{n+1})$ as above. This sector could hold only one vertex of the 9-gon (without the occurrence of an empty convex hexagon) and the union of all sectors would still cover the same region.

This implies $|\bigcup_{l=1}^{n} T_{a_l b_l}| = n < n + 1$, though, and a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$ can be constructed by Observation 2. To see this, note that $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ are part of a chain of consecutive vertices of $\mathrm{conv}(J)$ of length $L \geq n + 1$. Therefore, the claim follows. $\square$

2.9.5. Application of Rules 1–3. Based on the three rules we can now settle all the remaining subcases of $(\geq 7, \geq 5, \geq 1)$ with the exception of $(7, 7, \geq 1)$ with distribution $(1, 1, 1, 1, 1, 1, 1)$, $(7, 8, \geq 1)$ with distribution $(1, 1, 1, 1, 1, 1, 1, 0)$ and $(8, 8, \geq 1)$ with distribution $(1, 1, 1, 1, 1, 1, 1, 1)$. (These cases do not allow for a direct application of Rule 3. They are treated individually in the following subsections.) In the other cases, at least one 2 appears in the distribution sequence. We can argue as follows:

Whenever two consecutive vertices of $\mathrm{conv}(I)$ lie within the same sector, apply Rule 1. Note that two such vertices correspond to a 2 in the underlying distribution. No vertices of the 9-gon can lie in the corresponding sectors.

Now take maximal series of consecutive sectors containing at most one vertex of $\mathrm{conv}(I)$ each and apply Rule 3 (respectively Rule 2 if none of them contains a vertex). The number of vertices of the 9-gon that can lie in the union of all corresponding sectors is equal to $q + s \cdot 2$, where q is the total number of 1's in the underlying distribution and s is the number of distinct series. Note that s is equal to the number of gaps between two occurrences of a 2 in the distribution sequence. As this number is equal to the number of 2's in the sequence, it follows

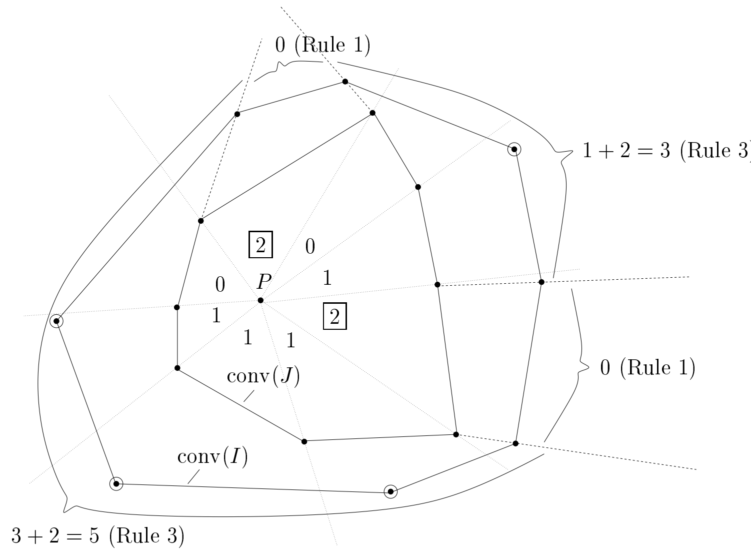

FIGURE 2.31. The case $(8, 8, \geq 1)$ with $(2, 1, 1, 1, 0, 2, 0, 1)$

that $q + s \cdot 2$ is equal to the sum of the elements of the distribution sequence. It can easily be verified that this sum is always smaller than 9. Therefore, in all these cases an empty convex hexagon occurs. (An example is given in Figure 2.31.)

2.9.6. The Case $(1, 1, 1, 1, 1, 1, 1)$**.** This case can be dealt with by applying Rule 3 with $n = 5$ seven times with each vertex of $\mathrm{conv}(I)$ as a starting point. Each 3-sector $(A_{r-1}a_rA_r)$ is left out exactly once. Therefore, in the union of all sectors at most $(7 \cdot (5 + 2))/6 < 9$ vertices of the 9-gon can lie without the occurrence of an empty convex hexagon.

2.9.7. The Case $(1, 1, 1, 1, 1, 1, 1, 0)$**.** For the case $(1, 1, 1, 1, 1, 1, 1, 0)$, label the vertices of the polygon $\mathrm{conv}(J)$ in clockwise order a_l $(1 \leq l \leq 8)$ and assume that the sector (a_6Pa_7) is the one that does not contain a vertex of $\mathrm{conv}(I)$. Applying Rule 3 with $n = 5$, we can conclude that at most $5 + 2 = 7$ vertices of the 9-gon can lie in the union of sectors $\bigcup_{l=1}^{6}(A_{l-1}a_lA_l)$; see Figure 2.32. Consider A_6. Note that it is not possible that A_6 lies below the line $\overline{a_1a_8}$ *and* below the line $\overline{a_6a_7}$ (where *below* refers to the halfplane that includes P), as otherwise a 9-gon $H' := (a_8a_1a_2 \ldots a_7A_6)$ with smaller $|X \cap \mathrm{conv}(H')|$ is present.

If A_6 lies above the line $\overline{a_1a_8}$, only one vertex of the 9-gon can lie in the then existing 4-sector $(A_6a_8a_1A_0)$ and therefore, without the occurrence of an empty

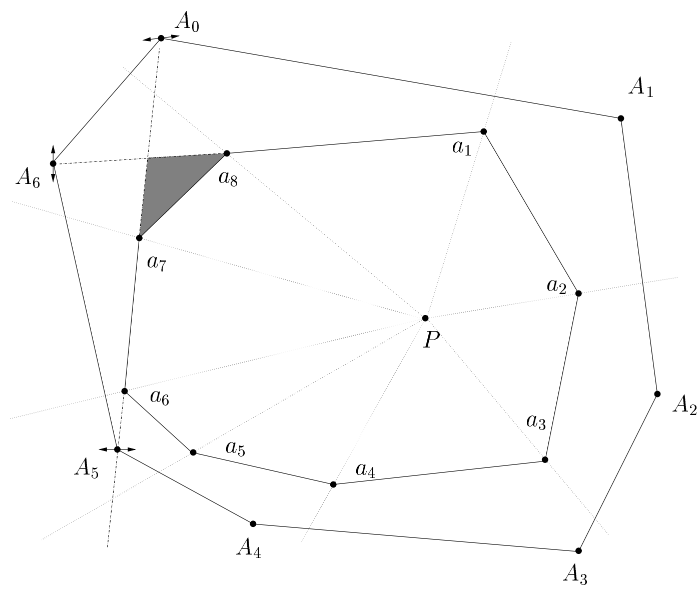

FIGURE 2.32. The case $(7, 8, \geq 1)$ with $(1, 1, 1, 1, 1, 1, 1, 1, 0)$

convex hexagon, at most eight vertices of the 9-gon can lie in the union of sectors

$$\bigcup_{l=1}^{6} (A_{l-1} a_l A_l) \cup (A_6 a_8 a_1 A_0),$$

which by construction covers the complete region outside of $\text{conv}(I)$.

Similarly, if A_0 lies above the line $\overline{a_6 a_7}$ (and therefore also A_6 by construction), at most eight vertices of the 9-gon can lie in the union of sectors

$$\bigcup_{l=1}^{6} (A_{l-1} a_l A_l) \cup (A_6 a_6 a_7 A_0)$$

(which by construction covers the complete region outside of $\text{conv}(I)$) without the occurrence of an empty convex hexagon.

Finally, if A_0 lies below the line $\overline{a_6 a_7}$ and A_6 lies above it, we know that A_5 must also lie above the line $\overline{a_6 a_7}$ as otherwise $|T_{a_6 a_7}| < 2$ and a 9-gon H' with smaller $|X \cap \text{conv}(H')|$ could be constructed by Observation 2. Therefore, the 4-sector $(A_5 a_6 a_7 A_6)$ exists, which can only hold one vertex of the 9-gon without the occurrence of an empty convex hexagon. Now, applying Rule 3 with $n = 5$ yields that at most seven vertices of the 9-gon can lie in the union of sectors

$$(A_6 a_8 A_0) \cup \bigcup_{l=1}^{5} (A_{l-1} a_l A_l)$$

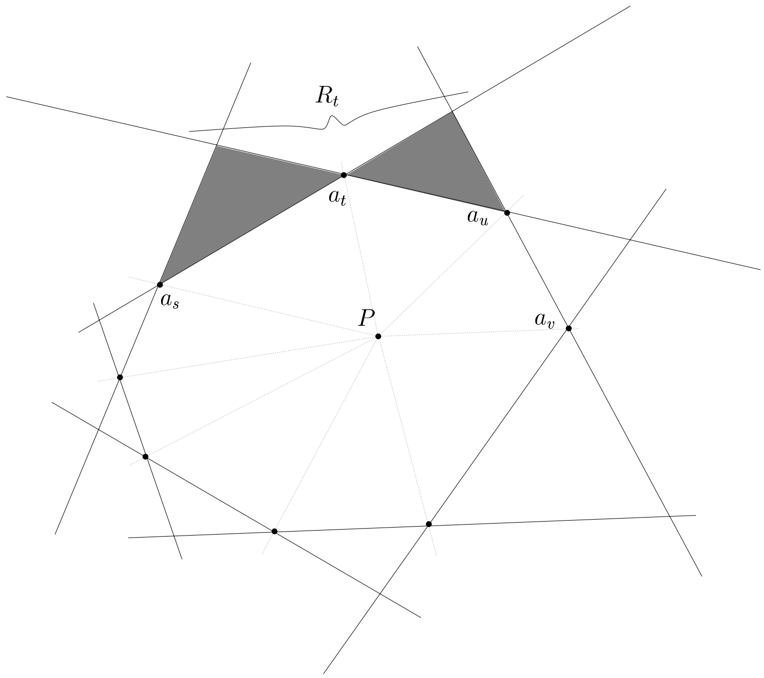

FIGURE 2.33. The case $(8, 8, \geq 1)$ with $(1, 1, 1, 1, 1, 1, 1, 1)$: Definition R_t.

without the occurrence of an empty convex hexagon. Therefore, without the occurrence of an empty convex hexagon, at most eight vertices of the 9-gon can lie in the union of sectors

$$(A_6 a_8 A_0) \cup \bigcup_{l=1}^{5} (A_{l-1} a_l A_l) \cup (A_5 a_6 a_7 A_6)$$

which by construction covers the complete region outside of $\operatorname{conv}(I)$.

2.9.8. The Case $(1, 1, 1, 1, 1, 1, 1, 1)$. Note that in the case $(1, 1, 1, 1, 1, 1, 1, 1)$, applying the induction argument with $n = 6$ eight times with each vertex of $\operatorname{conv}(J)$ as a starting point (in analogy to our approach to the case $(7, 7, \geq 1)$ with distribution $(1, 1, 1, 1, 1, 1, 1)$ in Section 2.9.6) only gives us an estimate of a total of $(8 \cdot (6 + 2))/7 > 9$ vertices of the 9-gon that can lie in the union of all sectors. Therefore, a different approach for this subcase is required.

Label the vertices of $\operatorname{conv}(J)$ in clockwise order as a_r $(1 \leq r \leq 8)$. Consider four consecutive vertices of the convex 8-gon $\operatorname{conv}(J)$, a_s, a_t, a_u and a_v. Note that no vertex of $\operatorname{conv}(I)$ can lie below the line $\overline{a_s a_t}$ *and* below the line $\overline{a_u a_v}$ (where *below* refers to the halfplance that includes P) as otherwise we could use such a

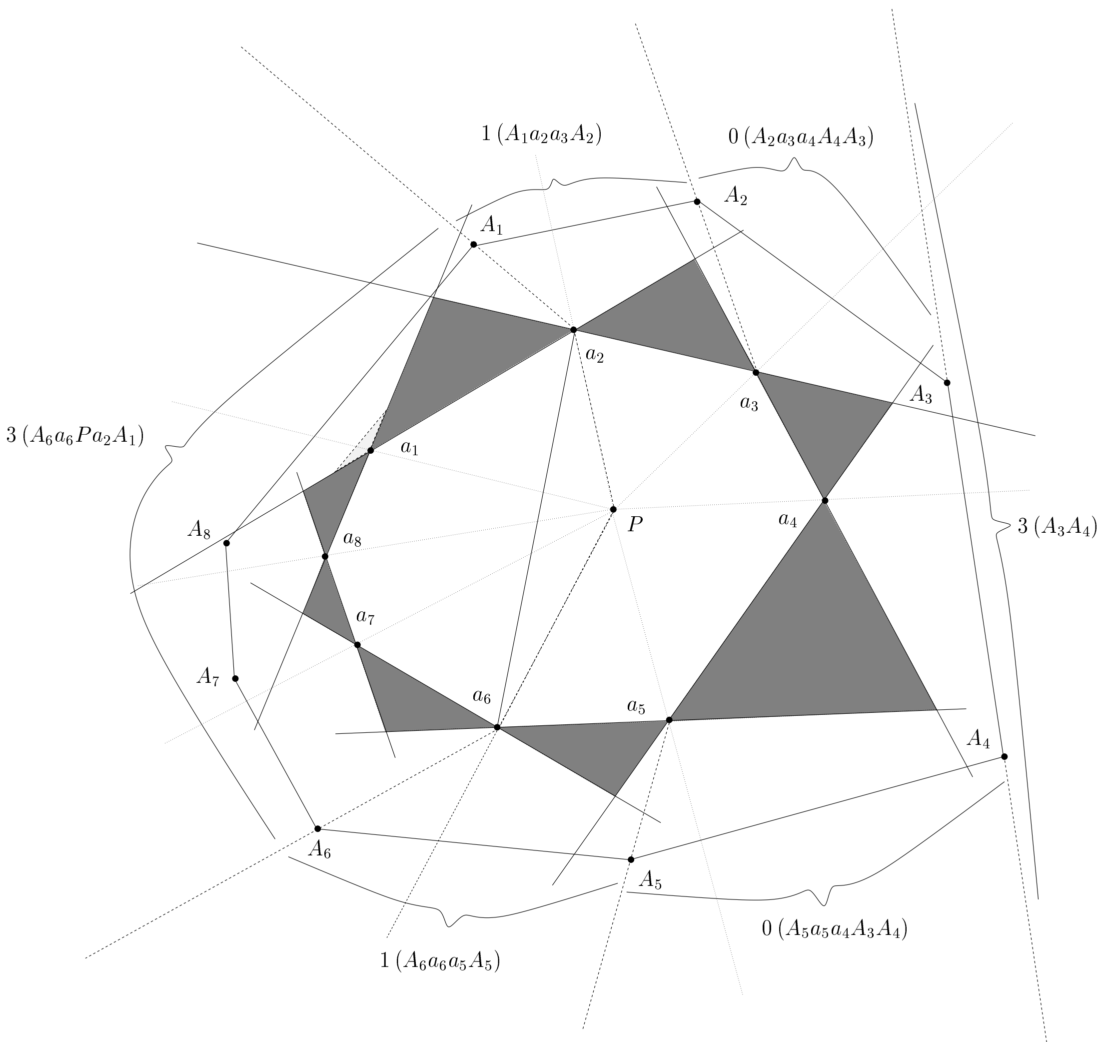

FIGURE 2.34. The case $(8, 8, \geq 1)$ with $(1, 1, 1, 1, 1, 1, 1, 1)$: R_1 contains no A_m.

vertex to construct a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$. Denote by R_t the region above both lines $\overline{a_s a_t}$ and $\overline{a_t a_u}$; see Figure 2.33. The union of all regions R_r $(1 \leq r \leq 8)$ defines the feasible region for vertices of $\mathrm{conv}(I)$. Label the vertices of $\mathrm{conv}(I)$ as A_m $(A_m \in (a_m P a_{m+1}),\ 1 \leq m \leq 8,\ a_9 := a_1)$. Note that A_m lies in R_m or R_{m+1} (or both) $(1 \leq m \leq 8,\ R_9 := R_1)$. Consider the following three possibilites:

There exists a region R_w with no A_m lying in it. There can be at most one such region as otherwise one could construct a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$. To see this, eliminate successively the possibilities that the next R_z with this property is R_{w+1}, R_{w+2}, R_{w+3} or R_{w+4}. In the first case, $|T_{a_w\, a_{w+1}}| < 2$ and in the other three cases one can replace one to three vertices of the 8-gon $a_1a_2\dots a_8$ by two to four points A_m in such a way that a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$ appears.

Let a_1 be the vertex associated with the region that does not contain any A_m. Since the existence of such a region is independent of the choice of P, we may assume that P lies in the pentagon $a_6a_2a_3a_4a_5$. Such a P must exist for otherwise an empty convex hexagon appears; see also Figure 2.34. (A different choice of P might result in a different distribution sequence. If this is the case, we arrive at a subcase that has already been settled.) As a consequence, at most three vertices of the 9-gon can lie in the sector $(A_6a_6Pa_2A_1)$ as otherwise a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$ could be constructed (as $5 + 4 = 9$).

We claim that $A_m \in R_m \cap R_{m+1}$ for $2 \leq m \leq 7$; that is, each A_m lies above both lines $\overline{a_{m-1}a_m}$ and $\overline{a_{m+1}a_{m+2}}$ $(2 \leq m \leq 7,\ a_9 := a_1)$. To see this, start from the line $\overline{a_1a_2}$ and work clockwise to prove that $A_m \in R_m$ $(2 \leq m \leq 7)$. Note that configurations, where $|\bigcup_{n=1}^{l-1} T_{a_n\, a_{n+1}}| < l$ $(2 \leq l \leq 8)$ yield a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$ by Observation 2 (Section 2.7). Now start from the line $\overline{a_1a_8}$ and work counter clockwise to prove that $A_m \in R_{m+1}$ $(2 \leq m \leq 7)$.

Finally, as we are assuming that no A_r lies in R_1, it follows that $A_1 \in R_2$ and $A_8 \in R_8$. Therefore, this case can be settled as indicated in Figure 2.34.

(At least) one A_m lies in each region R_r $(1 \leq r \leq 8)$ and, say, $A_1 \in R_1 \setminus R_2$. We claim that this implies that $A_u \in R_u$ $(3 \leq u \leq 8)$ as otherwise a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$ appears. To see this, first consider the point A_8. If $A_8 \in R_1 \setminus R_8$, the 9-gon $H' := A_1a_2a_3\dots a_8A_8$ with smaller $|X \cap \mathrm{conv}(H')|$ occurs. Therefore, $A_8 \in R_8$. Next, consider A_7, then A_6 and so on. Finally, as we are assuming that at least one A_m lies in each region R_r $(1 \leq r \leq 8)$ it follows that $A_2 \in R_2$; see Figure 2.35. In this case, the region outside of $\mathrm{conv}(I)$ can be partitioned into eight 4-sectors $(A_la_la_{l+1}A_{l+1})$ $(1 \leq l \leq 8,\ a_9 := a_1,\ A_9 := A_1)$ that together allow at most eight vertices of the 9-gon without the occurrence of an empty convex hexagon.

Each A_m lies in both R_m and R_{m+1}. Again, the region outside of $\mathrm{conv}(I)$ can be partitioned into eight 4-sectors $(A_la_la_{l+1}A_{l+1})$ $(1 \leq l \leq 8,\ a_9 := a_1,\ A_9 := A_1)$ that together allow at most eight vertices of the 9-gon without the occurrence of an empty convex hexagon. □

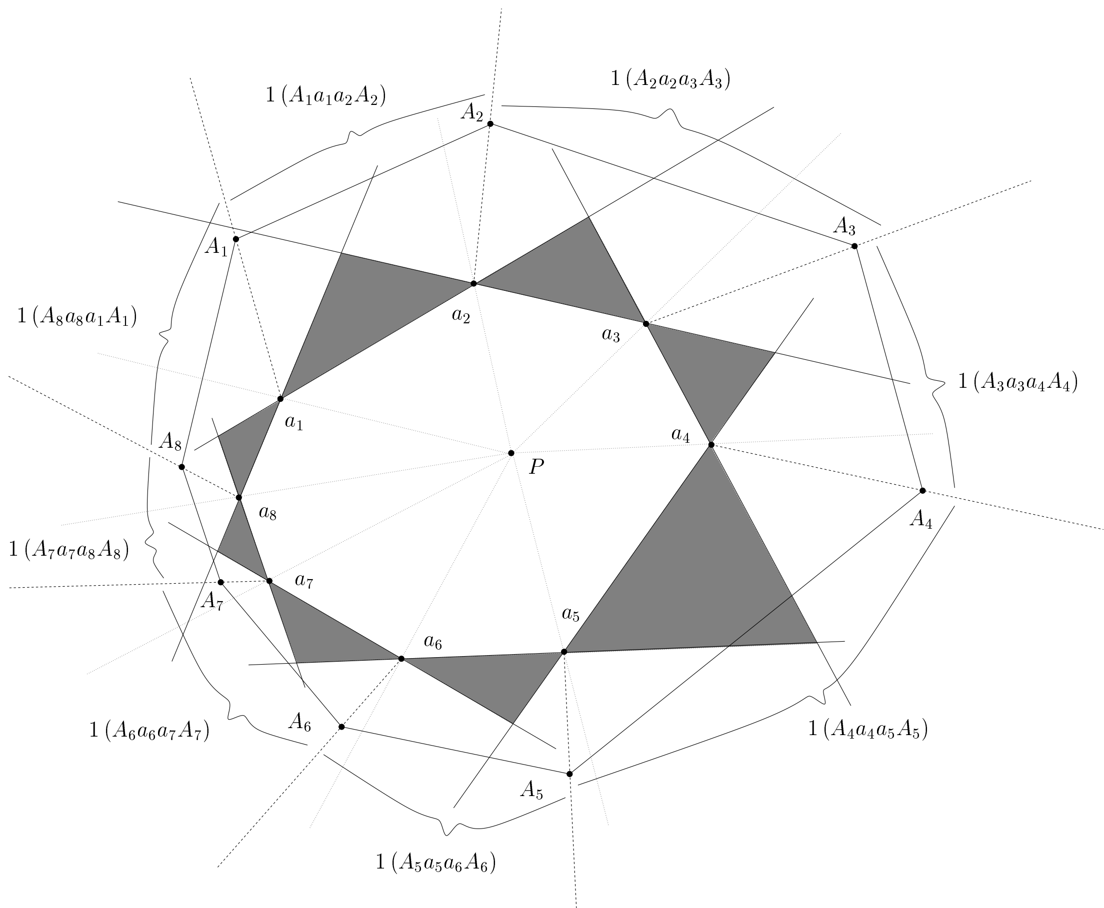

FIGURE 2.35. The case $(8, 8, \geq 1)$ with $(1, 1, 1, 1, 1, 1, 1, 1)$: $A_1 \in R_1 \setminus R_2$.

CHAPTER 3

Discussion

We have proven that every set that contains the vertex set of a convex 9-gon also contains an empty convex hexagon. As we have mentioned in Chapter 1, this result is tight in the sense that there are sets of points without empty convex hexagons that have eight points on the convex hull [**Ove03**]. In fact, we constructed such a set by hand when investigating the case $(5, 1)$ with an 8-gon instead of a 9-gon on the outmost layer; see Figure 3.1. Note that no 8-gon with less points in its interior can be constructed in the figure.

It is only natural to ask for the exact bound for $h(6)$. It seems likely that further case analysis with similar arguments as we have applied here will lead towards this goal. A first step in this direction might be achieved by a solution to the following problem.

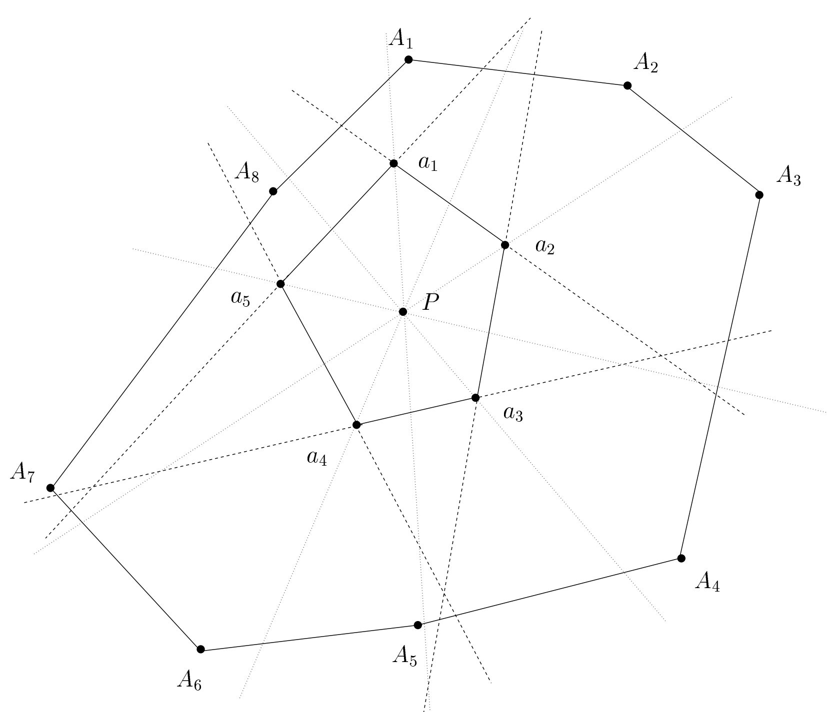

FIGURE 3.1. Example of a set that has eight points on the convex hull and no empty convex hexagon.

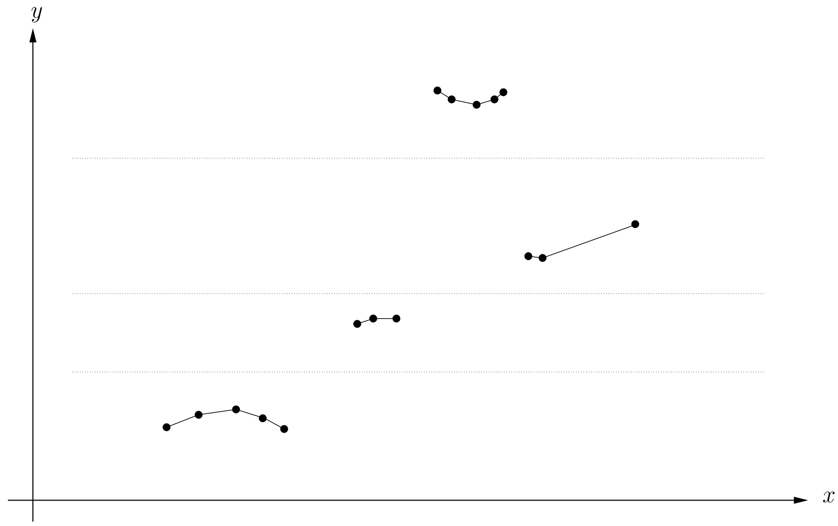

FIGURE 3.2. Example of a 16-point set with no convex hexagon.

PROBLEM 2. *Characterize all point sets in general position in the plane with eight points on the convex hull and no empty convex hexagon.*

It might also be interesting to investigate, where exactly the strategy of our proof fails if one tries to establish the existence of empty convex heptagons in every sufficiently large planar point set in general position (as opposed to Horton's result [**Hor83**]). It is possible that constructions of point sets without empty convex heptagons can be derived that differ from Horton's original one.

Some of the difficulties in the settlement of the Erdős-Szekeres conjecture arise from the fact that optimal sets are determined only up to geometric transformations that preserve convexity (such as rotations, reflections or certain projections). We make the following conjecture on the structure of optimal point sets for the first open case of this problem.

CONJECTURE 2. $g(6) = 17$. *Furthermore, after a suitable rotation of the coordinate system, every 16-point set that does not contain a convex hexagon can be decomposed into a 5-cap, a 3-cap, a 3-cup and a 5-cup (in ascending order of y-coordinates) as indicated in Figure 3.2.*

Furthermore, we express our belief that more general structural patterns will be observed in all extremal sets for the Erdős-Szekeres problem.

It is also possible to test the conjecture that $g(6) = 17$ based on an analysis of the convex layers of a 17-point set as already noticed in [**Bon74**]. Theoretically, this type of approach works for every fixed value of n, but already in the case $n = 6$ a large number of subcases needs to be treated individually. One exemplary case is treated in [**Fre06**]. Frei also shows that the original construction of Erdős and Szekeres [**ES61**] (as presented by Lovász [**Lov93**]) for a 16-point set that does not contain a convex hexagon is tight in the sense that the addition of a single point leads to the occurrence of a convex hexagon. The analysis is simplified by the fact that some of the conditions concerning the placement of the sets X_i (see Chapter 1) in the construction are sufficient but not necessary for the non-occurrence of a convex hexagon.

Under the assumption that the Erdős-Szekeres conjecture is true, the task of determining $h(6)$ reduces to the evaluation of point sets with at most $2^{9-2} + 1 = 129$ points. A computerized search might therefore be directed towards the following problem.

PROBLEM 3. *Investigate point sets in general position in the plane with at most* 129 *points and no convex 9-gon.*

Our proof of Theorem 4 has been analyzed by Valtr [**Val**]. Valtr shows that it can be shortened at the cost of a worse bound for $h(6)$ and concludes that "we do not see how to achieve Gerken's constant ... without an extensive case analysis as in [**Ger**]". Note the similarities between Valtr's analysis and the independent proof of Nicolás [**Nicb**].

Addendum. Very recently, Szekeres and Peters [**SP**] have announced a computer-based proof of $g(6) = 17$.

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