Discrete and Continuous Wavelet Transformations on the Heisenberg Group

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Contents

Chapter 1

Preface

Wavelet analysis is a still developing area in the mathematical sciences. Already early in the development of the wavelets both the discrete and the continuous transformation were examined.

The main aim of the theory of wavelet analysis is to find convenient ways to decompose a given function into elementary building blocks. Historically, the Haar basis, constructed in 1910 long before the term “wavelet” was created, was the first orthonormal wavelet basis in $L^2(\mathbb{R})$. But it was only recently discovered that the construction works because of an underlying multiresolution analysis structure. In the early 80’s, Strömberg [50] discovered the first continuous orthogonal wavelets. His wavelets have exponential decay and were in C^k, k arbitrary but finite. The next construction, independent of Strömberg, was the Meyer wavelet [40]. The images of the Meyer wavelets under the Fourier transform were compactly supported and were in C^k (k may be ∞). With the notion of multiresolution analysis, introduced by Mallat [38] and Meyer [41], a systematic framework for understanding these orthogonal expansions was developed, see for example [38] and [41] for details. This framework gave a satisfactory explanation for all these constructions, and provided a tool for the construction of other bases. Thus, multiresolution analysis is an important mathematical tool to understand and construct a wavelet basis of $L^2(\mathbb{R})$, i.e., a basis that consists of the scaled and integer translated versions of a finite number of functions.

In recent years, multiresolution analysis for the Euclidean group $\mathbb{R}$ has received extensive investigation. Also, various extensions and generalizations were considered. There are literally hundreds of sources dealing with this connection. In [37] multiresolution analysis for $\mathbb{R}^n$ whose scaling functions are characteristic functions are considered. Dahlke [8] extended multiresolution analysis to abelian locally compact groups. Baggett, et al. [2] considered the existence of wavelets in general Hilbert space based on the formulation of multiresolution analysis by using an abstract approach.

An alternative construction, imposing less restrictions on the wavelet functions, is the continuous wavelet transform. The continuous wavelet transformation can be interpreted as a phase space representation. Their filters and approximation characteristics have been examined. The group-theoretical approach allows a simple generalization for instance of wavelet transformation to high-dimensions Euclidean space (see [18]) or more general situations. Wavelet transformation in several dimension , exactly as in one dimension, may be derived from the similitude group of $\mathbb{R}^n$ $(n > 1)$, consisting of dilations, rotations and translations. Of course, the most interesting case of applications is $n = 2$, where wavelets have become a useful tool in image processing.

The construction of generalized continuous wavelet transform is investigated in the framework of irreducible, square-integrable representations of locally compact groups. The square integrability of representations guarantees the existence of a so-called admissible vector and an inverse wavelet transform [3], [28]. General existence theorems for square-integrable representation can be found in [12]. The existence of admissible vectors can also be considered when the irreducibility requirement can be dropped, as for example in [19], using a connection between generalized wavelet transforms and Plancherel theory.

Introduction to the Wavelets on $\mathbb{R}$

The wavelet transform of a function on $\mathbb{R}$, a signal so-called, depends on two variables:

scale and time. Suppose ψ is fixed. For $a \neq 0$ define

$$D_a\psi(x) := |a|^{-1/2}\,\psi(\frac{x}{a}).$$

Afterward $D_a\psi$ is translated by $b \in \mathbb{R}$. Thus one gets the functions

$$L_bD_a\psi(x) := D_a\psi(x-b) = |a|^{-1/2}\,\psi(\frac{x-b}{a}). \tag{1.1}$$

The functions $L_bD_a\psi$ are called *wavelets*; the function ψ is sometimes called *mother wavelet*; the system $\{L_bD_a\psi\}_{(b,a)\in\mathbb{R}\times\mathbb{R}^*}$ is called *wavelet system*. The *wavelet transform* of a signal $f \in L^2(\mathbb{R})$ is given by the scalar products of f with the wavelet system.
There exist two different types of wavelet systems, both referring to the basic form (1.1):

1. The continuous wavelet system: Here the dilation and translations parameters (b, a) vary over all of $\mathbb{R} \times \mathbb{R}^*$, and

2. The discrete wavelet system: In this case both the dilation parameter a and the translation parameter b take only discrete values.

<u>**The continuous wavelet transform**</u> Here we give a quick overview of the theory of the continuous wavelet transform of L^2-functions on $\mathbb{R}$ from the point of view of representation theory. The definition of dilation operators D_a and translation operators L_b for any $(b, a) \in \mathbb{R} \times \mathbb{R}^*$ allows to define a group multiplication on $G := \mathbb{R} \times \mathbb{R}^*$ by

$$(b_0, a_0)(b, a) = (a_0bb_0, a_0a). \tag{1.2}$$

The non-unimodular group $G = \mathbb{R} \times \mathbb{R}^*$ is called *"affine group"* with associated to the group multiplication (1.2). The left Haar measure is then $|a|^{-2}\,dadb$ and the right Haar measure is $|a|^{-1}\,dadb$ on $\mathbb{R} \times \mathbb{R}^*$. Define the representation π of $\mathbb{R} \times \mathbb{R}^*$ on $L^2(\mathbb{R})$ by

$$\pi(b, a)\psi(x) = L_bD_a\psi(x) = |a|^{-1/2}\,\psi(\frac{x-b}{a}) \quad (b, a) \in \mathbb{R} \times \mathbb{R}^*. \tag{1.3}$$

π is unitary and irreducible. For a signal $f \in L^2(\mathbb{R})$ and fixed selected $\psi \in L^2(\mathbb{R})$ we define

$$V_\psi f(b, a) := |a|^{-1/2}\int f(x)\overline{\psi(\frac{x-b}{a})}dx \quad \forall a > 0, b \in \mathbb{R}. \tag{1.4}$$

With the help of $L_bD_a\psi$ in (1.3), the definition (1.4) can be read as

$$V_\psi f(b,a) = \langle f, \pi(b,a)\psi\rangle \tag{1.5}$$

which are called *wavelet coefficients* of f. $V_\psi f$ defined over $\mathbb{R}\times\mathbb{R}^*$ is a bounded function, as the Cauchy-Schwartz inequality implies

$$|V_\psi f(b,a)| \leq \parallel f \parallel\parallel \psi \parallel \quad \forall\ (b,a) \in \mathbb{R}\times\mathbb{R}^*.$$

Thus V_ψ maps space $L^2(\mathbb{R})$ into the set of bounded functions over $\mathbb{R}\times\mathbb{R}^*$. ψ is called *admissible* when the operator

$$V_\psi :\ L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}\times\mathbb{R}^*, |a|^{-2}\, dadb)$$

defined via (1.5) is an isometry up to a constant, i.e., the equality

$$\parallel f \parallel^2 = const. \int_{-\infty}^{\infty}\int_0^{\infty} |\langle f, \pi(b,a)\psi\rangle|\, |a|^{-2}\, dadb \tag{1.6}$$

holds for any $f \in L^2(\mathbb{R})$, where the constant depends only on ψ. Thus V_ψ is called the *continuous wavelet transform*; $V_\psi(f)$ is called continuous wavelet transform of function f associated to the wavelet ψ.

By the isometry given in (1.6), a function f can be recovered by its wavelet coefficients (1.5) by means of the *resolution of identity*

$$f = const. \int_{-\infty}^{\infty}\int_0^{\infty} \langle f, \pi(b,a)\psi\rangle \pi(b,a)\psi\, |a|^{-2}\, dadb, \tag{1.7}$$

where the integral is understood in the weak sense.

Using the Fourier transform, there is a characterization for function ψ to be an admissible vector, which can be read as below:

A vector $\psi \in L^2(\mathbb{R})$ is admissible if and only if it satisfies the condition

$$C_\psi = \int_{-\infty}^{\infty} \left|\hat{\psi}(\xi)\right|^2 |\xi|^{-1}\, d\xi < \infty. \tag{1.8}$$

Discrete wavelet transform: Orthonormal wavelet bases As mentioned before,- in the discrete wavelet transform, the dilations and translations parameters both take only discrete values. For dilation parameter one chooses the integer powers of one fixed dilation parameter $a_0 > 1$, i.e., a_0^m and hence b can be discretized by $nb_0a_0^m$ for some fixed b_0 and for all $n \in \mathbb{Z}$; usually $a_0 = 2$. The corresponding discrete wavelet system to the parameters a_0, b_0 is $\{\psi_{n,m}\}_{(n,m)\in\mathbb{Z}\times\mathbb{Z}}$ where

$$\psi_{n,m}(x) = |a_0|^{-m/2}\,\psi(a_0^{-m}x - nb_0) \quad \forall(n,m) \in \mathbb{Z}\times\mathbb{Z},\ \forall x \in \mathbb{R}.$$

For a fixed function f, the inner products $\{\langle f, \psi_{n,m}\rangle\}_{(n,m)}$, called the discrete wavelet coefficients of f , are given by

$$\langle f, \psi_{n,m}\rangle = |a_0|^{-m/2} \int f(x)\overline{\psi(a_0^{-m}x - nb_0)}dx.$$

Note that in general it is not easy to construct a discrete wavelet system $\{\psi_{n,m}\}_{(n,m)}$ that constitutes an orthonormal basis for $L^2(\mathbb{R})$ for any suitable of a_0, b_0.

One of the constructive methods for orthonormal wavelet bases is *"multiresolution analysis"*, abbreviated by MRA: A multiresolution analysis consists of a sequence of closed and nested subspaces $\{V_j\}$, approximation spaces, in $L^2(\mathbb{R})$, whose union is dense in $L^2(\mathbb{R})$ and intersection is trivial. Moreover there must exist a function ϕ in the central space V_0, so that its translations under $\mathbb{Z}$ constitutes an orthonormal basis for V_0. The function ϕ is called *"scaling function"* of the multiresolution analysis MRA. An MRA provides an orthogonal decomposition of $L^2(\mathbb{R})$. Using the scaling function one can construct a function ψ in W_0, the orthogonal component of V_0 in V_1, such that the set of its translations under $\mathbb{Z}$ constitutes an ONB for W_0. Hence using the orthogonal decomposition of $L^2(\mathbb{R})$ the wavelet system $\{\psi_{n,m}\}_{(n,m)}$ forms an orthonormal wavelet basis of $L^2(\mathbb{R})$. Therefore any function f in $L^2(\mathbb{R})$ can be recovered from its discrete wavelet coefficients by the discrete inversion formula:

$$f = \sum_{n,m} \langle f, \psi_{n,m}\rangle \psi_{n,m}. \tag{1.9}$$

Remarks

1) *The construction of wavelet ONB's is considerably more difficult than that of functions fulfilling the admissibility condition (1.8). This motivates the interest in constructions such as MRA's.*

2) *The wavelets $\psi_{b,a} := L_b D_a \psi$, as the component of the position b and scale a, constructed from the mother wavelet ψ, are applied to the analysis of signals f in $L^2(\mathbb{R})$. The "well localized" wavelets in both time and frequency are considered. This kind of wavelets provide a good localization of informations about the signals. More precisely, the L^2-inner product $\langle f, \psi_{b,a} \rangle$ contains local information about the regularity of f at scale a and centered at position b. For example, the fast decay of the absolute value of the wavelet coefficients $\langle f, \psi_{b,a} \rangle$ as $a \to 0$ provides the smoothness of function f at the point b.*

The purpose of the thesis

Let $\mathbb{H} = \mathbb{R}^3$ denote the Heisenberg Lie group with non-commutative group operation defined by

$$(p_1, q_1, t_1) * (p_2, q_2, t_2) = \left(p_1 + p_2, q_1 + q_2, t_1 + t_2 + \frac{(p_1 q_2 - q_1 p_2)}{2} \right).$$

The Haar measure on $\mathbb{H}$ is the usual Lebesgue measure on $\mathbb{R}^3$. Note that sometimes we use the identification $\mathbb{H} = \mathbb{C} \times \mathbb{R}$ in our work.

The definition of the continuous wavelet transform respectively an admissible vector is associated to one-parameter dilation group of $\mathbb{H}$, i.e., $H = (0, \infty)$, where any $a > 0$ defines an automorphism of $\mathbb{H}$, by

$$a.(p, q, t) = (ap, aq, a^2 t) \qquad \forall (p, q, t) \in \mathbb{H}. \tag{1.10}$$

Adapting the notation of dilation and translation operators on $L^2(\mathbb{R})$, for some $a > 0$, D_a is a unitary operator on $L^2(\mathbb{H})$ given by

$$D_a f(p, q, t) = a^2 f(a.(p, q, t)) = a^2 f(ap, aq, a^2 t) \quad \forall f \in L^2(\mathbb{H}),$$

and for any $\upsilon \in \mathbb{H}$, L_ω is defined by

$$L_\omega f(\upsilon) = f(\omega^{-1}\upsilon) \quad \forall \upsilon \in \mathbb{H}.$$

Using the dilation and translation operators, the *quasiregular representation* π of the semidirect product $G := \mathbb{H} \rtimes (0, \infty)$ acts on $L^2(\mathbb{H})$ by

$$(\pi(\omega, a)f)(\upsilon) := L_\omega D_a f(\upsilon) = a^{-2} f(a^{-1}.(\omega^{-1}\upsilon))$$

for any $f \in L^2(\mathbb{H})$ and $(\omega, a) \in G$ and for all $\upsilon \in \mathbb{H}$. G is non-nunimodular, with left and right Haar measures are given by $a^{-5} da d\omega$, $a^{-1} da d\omega$ respectively.

The principal purpose of this work is the construction of discrete and continuous wavelet systems on the Heisenberg group arising from a "well localized" wavelet. The chief tool for approaching our aim will be the quasiregular representation of $G = \mathbb{H} \rtimes (0, \infty)$ on $L^2(\mathbb{H})$, and methods of Fourier analysis on $\mathbb{H}$.

Structure of the Thesis

This thesis consists of 5 chapters. The contexts and new results contained in the thesis are organized as follows:

- *Chapter 2* contains an overview of the basics concerning the Heisenberg group $\mathbb{H}$ and analysis on $\mathbb{H}$,
- *Chapter 3* provides the construction of normalized tight wavelet frame on $\mathbb{H}$,
- *Chapter 4* presents the complete characterization of radial Schwartz functions as well as radial admissible functions,
- *Chapter 5* illustrates the *Calderón* admissibility condition for vectors in $L^2(\mathbb{H})$ and provides the *"Mexican-Hat"*- wavelet on $\mathbb{H}$ as an example of a radial admissible Schwartz function.

Overview of the results

Chapter **2** provides some background for the analysis to be presented in subsequent chapters. The key information for later use is the Fourier and wavelet transform on the

Heisenberg group. We explain the basic concepts and results around abstract Fourier analysis and representation theory on the Heisenberg group. This includes the most important *Stone and von Neumann* and *Plancherel* theorems. This material is, for the most part, available from books [16],[15], [20] and [48]. This chapter also serves to establish our main notations.

We conclude this chapter with fundamental terms and results which are needed in connection with continuous and discrete wavelet analysis.

Next we introduce the concept of frames of a Hilbert space. Moreover using the translation and dilation operators we define a discrete wavelet system $\{L_{2^{-j}\gamma}D_{2^{-j}}\psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ for $L^2(\mathbb{H})$ for semidirect product $G = \mathbb{H} \rtimes (0,\infty)$, which is associated with suitable $\psi \in L^2(\mathbb{H})$ and some lattice Γ in $\mathbb{H}$. The key step is the existence of a suitable lattice Γ and ψ such that the related wavelet system forms a normalized tight frame. More about frames is available in the books of Christensen [6] and Gröchening [24]; for more examples of semidirect products and their frames see for instance [1].

Chapter **3** concerns the construction of discrete wavelet system in $L^2(\mathbb{H})$. We show the existence of a normalized tight wavelet frame, and exhibit certain similarities to multiresolution analyses in $L^2(\mathbb{R})$. The main result of this chapter can be found in Section 3.3. First, we describe the notions from multiresolution analysis in $L^2(\mathbb{R})$ that are needed to understand the remainder of this chapter. We use the following convention for Fourier transform of functions in $L^1(\mathbb{R})$:

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

We start by introducing the Shannon theorem for $L^2(\mathbb{R})$ which states that a slowly varying signal can be interpolated from a knowledge of its value at a discrete set of points. More precisely for a *bandlimited* function $f \in L^2(\mathbb{R})$, i.e its Fourier transform $\hat{f}$ has support in some compact interval, then for some $b > 0$:

$$f(x) = \sum_{n\in b\mathbb{Z}} f(\frac{n\pi}{b})\frac{sin(bx - n\pi)}{bx - n\pi}. \tag{1.11}$$

The statement (1.11) means that f can be recovered from the samples $\{f(\frac{n\pi}{b})\}_{n\in\mathbb{Z}}$.

Shannon MRA on $\mathbb{H}$: By analyzing the Shannon basis, we can explain the concept of multiresolution analysis for $L^2(\mathbb{R})$ which is hidden in the Shannon basis, i.e., we show the existence of a sequence of closed linear and left shift-invariant nested subspaces V_j of $L^2(\mathbb{R})$ $(j \in \mathbb{Z})$ such that their union is dense in $L^2(\mathbb{R})$ and their intersection is trivial. Moreover $f \in V_j \Leftrightarrow f(2.) \in V_{j+1}$. And one can see that the function $\phi = sinc \in V_0$, so called the scaling function, has the property that its left translations under $\mathbb{Z}$ forms an orthonormal basis of V_0. Consequently using the scaling function and the orthogonal decomposition of $L^2(\mathbb{R})$ under W_js, the orthogonal complements of V_j in V_{j+1}, one can show that the translations and dilations of function $\psi := 2\phi(2.) - \phi = \sqrt{2}D_{1/2}\phi - \phi \in W_0$ with integer powers of $a = 2$, (see (1.10)), forms an orthogonal basis for $L^2(\mathbb{R})$.
Note that the Fourier transform of function ϕ has support in compact interval $[-\pi, \pi]$ and hence ψ is bandlimited and $\hat{\psi}$ has support in the set $[-2\pi, -\pi] \cup [\pi, 2\pi]$.

The reason that we choose the Shannon basis for illustration is the simplicity of its Fourier transform. The observations made by analyzing the Shannon basis for $L^2(\mathbb{R})$ lead us to formulate the definition of a Shannon multiresolution analysis for the space $L^2(\mathbb{H})$.
Let us start with the definition of MRA in general: Adapting the definition of MRA for $L^2(\mathbb{R})$ to one for $L^2(\mathbb{H})$, a *frame multiresolution analysis* for $L^2(\mathbb{H})$ associated to a lattice Γ in $\mathbb{H}$ and dilation given by $a > 0$ can be expressed in the following way:

Definition of frame multiresolution analysis (frame-MRA): *We say that a sequence of closed subspace $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{H})$ forms a frame-MRA of $L^2(\mathbb{H})$ if the following conditions are satisfied:*

1. $V_j \subseteq V_{j+1} \quad \forall j \in \mathbb{Z}$,

2. $\overline{\bigcup V_j} = L^2(\mathbb{H})$,

3. $\bigcap V_j = \{0\}$,

4. *$f \in V_j \Leftrightarrow f(a.) \in V_{j+1}$,*

5. *V_0 is left shift-invariant under Γ, and consequently V_j is shift-invariant under $(a^{-j}.\Gamma)$ (note that $L_{(a^{-j}.\gamma)}(\phi(a^j.)) = (L_\gamma \phi)(a^j.)$ for any $\gamma \in \Gamma$),*

6. *there exists a function $\phi \in V_0$, the so-called* **scaling function**, *or generator of the $frame - MRA$, such that the set $L_\Gamma(\phi) = \{\phi(\gamma^{-1}.) : \ \gamma \in \Gamma\}$ constitutes a normalized tight frame for V_0.*

Here we address the main topic, i.e., to answer the basic question, how can a function on $\mathbb{H}$ produce a Shannon multiresolution analysis of $L^2(\mathbb{H})$. We shall consider two basic issues: the union density and trivial intersection properties. Normally, the trivial intersection property is less important because it is the consequence of the other conditions. We shall present a function S for which the nested sequence of subspaces generated by S has dense union. Remark that we do not suppose that the function S is our scaling function.
For simplicity, from now on we take $a = 2$ in the above definition.
The idea of this chapter is to apply an approach to find a suitable analogue of the *sinc* function in $L^2(\mathbb{R})$, $S \in L^2(\mathbb{H})$, as a starting point, whose Plancherel transform is supported in a bounded interval, i.e. for any $\lambda \neq 0$, $\hat{S}(\lambda)$ is a finite rank projection of $L^2(\mathbb{R})$ with respect to some orthonormal basis $\{e_n^\lambda\}_{n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. We then construct a Shannon-MRA on $\mathbb{H}$ with applying the function S, for which the properties $1-6$ in the definition of frame-MRA hold. Note that the construction of a Shannon-MRA and also the existence of the scaling function ϕ in the property 6 strongly depends on the structure of the function S and ϕ is necessarily not equal to S.
Further on we show the existence of function ψ, so-called wavelet, in W_0, the orthogonal component of V_0 in V_1, which is bandlimited and the set of its left translations under some other suitable lattice forms a normalized tight frame for W_0. Consequently using the orthogonal decomposition $L^2(\mathbb{H}) = \bigoplus_{j\in\mathbb{Z}} W_j$, where W_j is the orthogonal component of V_j in V_{j+1}, we show that the set of $\{L_{\delta_{2^{-j}}\gamma} D_{2^{-j}}\psi\}_{(j,\gamma)\in\mathbb{Z}\times\Gamma}$ constitutes a normalized tight frame of $L^2(\mathbb{H})$, Theorem 3.18.

Remark *Observe that the wavelets obtained by the above method may have poor decay properties.*

Chapter **4** concentrates on the study and characterization of *radial admissible Schwartz* functions on the Heisenberg group. To enter into the details of this chapter we need some preparation:

A rather new development, which we consider in this chapter, is the construction of continuous wavelet transformation based on the quasiregular representation of the group $\mathbb{H} \rtimes (0, \infty)$. In general the admissible respectively wavelet functions are referred to that kind of functions which satisfies the condition of (1.12) below. Recall that G is a non-unimodular group and its left Haar measure is given by $d\mu_G(\omega, a) = a^{-5} da d\omega$. Then the admissible vector in $L^2(\mathbb{H})$ and continuous wavelet transform on $L^2(\mathbb{H})$ is defined as follows:

$(*)$ **Definition:** *For any $\psi \in L^2(\mathbb{H})$ the coefficient operator V_ψ defined by*

$$V_\psi : L^2(\mathbb{H}) \to L^2(G) \quad \textit{by} \quad V_\psi(f)(\omega, a) = \langle f, \pi(\omega, a)\psi \rangle \tag{1.12}$$

is called **continuous wavelet transform** *if $V_\psi : L^2(\mathbb{H}) \to L^2(G)$ is an isometric operator up to a scalar, i.e*

$$\| f \|^2 = const. \int_{\mathbb{H}} \int_0^\infty | V_\psi(f)(\omega, a) |^2 a^{-5} da d\omega \quad \forall f \in L^2(\mathbb{H}), \tag{1.13}$$

where the constant depends only on ψ. The function ψ for which (1.13) holds for any $f \in L^2(\mathbb{H})$ is called **admissible**.

The importance of the isometry given by formula (1.13) is that a function $f \in L^2(\mathbb{H})$ can be reconstructed from its wavelet coefficients $V_\psi(f)(\omega, a) = \langle f, \pi(\omega, a)\psi \rangle$ by means of the *"resolution identity" ("Calderón's formula")* , i.e, formula (1.13) can be read as

$$f = const. \int_{\mathbb{H}} \int_0^\infty \langle f, \pi(\omega, a)\psi \rangle \pi(\omega, a)\psi \; a^{-5} da d\omega \quad \forall f \in L^2(\mathbb{H}),$$

with the convergence of the integral in the weak sense.

The family of wavelets $\{\pi(\omega, a)\psi\}_{(\omega,a)\in\mathbb{H}\times(0,\infty)}$ are constructed from the admissible vector

ψ, so-called mother wavelet, by dilation a and translation ω respectively.

The motivation for studying of continuous wavelets is again that dilated and translated copies of wavelets can be used for the analysis of signals and should provide localized information about signals when the wavelet is suitably chosen; e.g has many vanishing moments and is smooth enough. More precisely, the L^2-inner product $\langle f, \pi(\omega, a)\psi\rangle$ contains local information about the regularity of f at scale a and centered at ω. For example, following the case of wavelet analysis on $\mathbb{R}$ one expects that the fast decay of the absolute value of the wavelet coefficients $\langle f, \pi(\omega, a)\psi\rangle$ as $a \to 0$ provides the smoothness of function f at a neighborhood of point ω.

These observations provided the motivation for the study and construction of fast decaying wavelets in this thesis. However, we will not study the use of wavelets for the analysis of local smoothness properties, and rather focus on construction issues.

To understand the remainder of this chapter we give here some basic definitions and notations:

Definition of radial functions on $\mathbb{H}$: *Using coordinates* (z, t) *on Heisenberg group* $\mathbb{H}$*, where* $z \in \mathbb{C}$ *and* $t \in \mathbb{R}$ *, we say a function* f *on* $\mathbb{H}$ *is radial if* $f = f \circ R_\theta$ *in the* L^2*-sense for every* $\theta \in [0, 2\pi)$*, where* R_θ *is a rotation operator on* $\mathbb{H}$ *and is given by* $R_\theta(z, t) = (\tilde{R}_\theta z, t)$*, and* $\tilde{R}_\theta$ *is the rotation operator on* $\mathbb{R}^2$ *by angle* θ*.*

Note that a continuous function f is radial if and only if $f(z, t)$ depends only on $\mid z \mid$ and t and we may also write $f(z, t) = f_0(\mid z \mid, t)$ with $f_0 : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$. Here the equality is understood pointwise.

On the nilpotent Lie group $\mathbb{H}$ we consider the space $\mathcal{S}(\mathbb{H})$ of Schwartz functions, see [7], [14]. Then the class of radial Schwartz function on the Heisenberg group has an alternative description as below:

Lemma: *The set* $\mathcal{S}_r(\mathbb{H})$ *of radial Schwartz functions on* $\mathbb{H}$ *has an alternative characterization given by*

$$\mathcal{S}_r(\mathbb{H}) = \Big\{ f \in C_r^\infty(\mathbb{H}) : \ (|z|^{2k} \mid t \mid^s)(\partial_z \partial_{\bar{z}})^d \partial_t^l f \in C_b(\mathbb{H}) \ \textit{for every} \ \ d, l, k, s \in \mathbb{N}_0 \Big\},$$

where C_r^∞ *stands for the set of smooth and radial functions and* C_b *denotes for the set of bounded and continuous functions on* $\mathbb{H}$.

(For the proof see Appendix A).

We restrict our study to the class of radial functions on $\mathbb{H}$ to obtain a complete characterization of the set $\widehat{\mathcal{S}_r}(\mathbb{H})$ of Fourier transforms of radial Schwartz functions on $\mathbb{H}$. We show that a function f is contained in $\mathcal{S}_r(\mathbb{H})$ if and only if its radial Fourier transform satisfies suitable decay conditions by applying certain derivative and difference operators. The reason for restricting our study of smooth wavelets to the class of radial functions is the following statement:

Suppose $f \in L^2(\mathbb{H})$. *Then* f *is radial if and only if for almost every* $\lambda \neq 0$ *its Plancherel transform* $\hat{f}(\lambda)$ *is given by*

$$\hat{f}(\lambda) = \sum_n R_f(n, \lambda)\ \phi_n^\lambda \otimes \phi_n^\lambda$$

for a suitable function R_f *defined on* $\mathbb{N}_0 \times \mathbb{R}^*$ *and the orthonormal basis* $\{\phi_n^\lambda\}_{n\in\mathbb{N}_0}$ *for* $L^2(\mathbb{R})$ *consisting of scaled Hermitian functions.*

The proof of the statement will be given in Theorem 4.11. The theorem presents a simple representation of group Fourier transform for radial functions, which allows us to characterize all smooth function in terms of their Fourier transform.

The motivation for the study of $\mathcal{S}_r(\mathbb{H})$ is the fact that $\mathcal{S}_r(\mathbb{R}^3)$ is preserved by the Fourier transform. A related result for the case $\mathbb{H}$ can be found in Geller's paper [21].

The characterization of $\widehat{\mathcal{S}_r}(\mathbb{H})$ enables one to construct smooth radial functions f on $\mathbb{H}$ which decay rapidly at infinity whose radial Fourier transforms R_f are prescribed in advance subject to some conditions. We will study it in Theorem 4.36 in detail, which provides both necessary and sufficient conditions for a function R on $\mathbb{R}^* \times \mathbb{N}_0$ to belong to space $\widehat{\mathcal{S}_r}(\mathbb{H})$:

i) R is rapidly decreasing on $\mathbb{R}^* \times \mathbb{N}_0$ and for any n the function $R(n, .)$ is continuous on $\mathbb{R}^*$,

ii) $R(n, .) \in C^\infty(\mathbb{R}^*) \quad \forall n \in \mathbb{N}_0$ and $\forall m \in \mathbb{N}_0$ the functions $\lambda^m \partial_\lambda^m R$ satisfy certain decay conditions. In particular, $\lambda^m \partial_\lambda^m R(n,\lambda)$ is a rapidly decreasing sequence in n for each fixed $\lambda \in \mathbb{R}^*$,

iii) Certain derivatives of R also satisfy the two conditions above. They are defined on $\mathbb{R}^* \times \mathbb{N}_0$ as specific combination of $\frac{d}{d\lambda}$ and difference operators which play the role of differentiation in the discrete parameter $n \in \mathbb{N}_0$.

The precise formulation of these conditions can be found in Section 4.5.4. The derivatives of functions in $\widehat{\mathcal{S}}_r(\mathbb{H})$ referred to above are operators corresponding to multiplication of functions in $\mathcal{S}_r(\mathbb{H})$ by certain polynomials. The difference operators in the discrete parameter $n \in \mathbb{N}_0$ are linear operators. As summary of their properties is given in Section 4.4.

One consequence of the estimates involved in our characterization of $\widehat{\mathcal{S}}_r(\mathbb{H})$ is that $f \in \mathcal{S}_r(\mathbb{H})$ can been recovered from its radial Fourier transform $R = \hat{f}$ via the inversion formula

$$f(z,t) = \sum_n \int_{\lambda \in \mathbb{R}^*} R(n,\lambda) \overline{\Phi_{n,n}^\lambda(z)} e^{-i\lambda t} d\mu(\lambda),$$

where $\Phi_{n,n}^\lambda$ are dilated special Hermit functions and $d\mu(\lambda) = (2\pi)^{-2} |\lambda| \, d\lambda$.

As mentioned before, in this chapter we also consider the problem of characterizing admissible functions in the space $\mathcal{S}_r(\mathbb{H})$ via their radial Fourier transform. The reason for restricting our study of admissible vectors to the class of radial function is again Theorem 4.11. As it turns out, this result allows to derive a simplified admissibility condition for radial functions.

For approaching this aim we first provide a complete characterization of admissible functions in space $L_r^2(\mathbb{H})$. We show that a function f in $L_r^2(\mathbb{H})$ is admissible if and only if the related radial Fourier transform R_f is square integrable in the continuous variable λ with respect to a suitable positive measure on the space $\mathbb{R}^*$, and the value of the integral is independent of discrete values $n \in \mathbb{N}_0$, i.e.,

$f \in L^2_r(\mathbb{H})$ is admissible if and only if for its radial Fourier transform $\{R(n,\lambda)\}_{n,\lambda}$ is

$$\int_{\lambda=0}^{\infty} \mid R_f(n,\lambda) \mid^2 \lambda^{-1} d\lambda = c \qquad \forall\, n \in \mathbb{N}$$

for some positive constant c .

The complete proof of this result is given in Theorem 4.37, which provides both necessary and sufficient conditions for a function R to be the radial Fourier coefficient of an admissible function.

We conclude this chapter with a result which is connected to our characterization of radial functions. The result shows that the characterization of admissible radial functions simplifies for a special class of radial functions constructed on the Fourier side:

Radial function f on $\mathbb{H}$ is admissible if its corresponded Fourier transform R_f can be constructed by $R_f(n,\lambda) = \tilde{R}((2n+1)\,|\lambda|)$, where $\tilde{R} \in L^2(\mathbb{R}^+, \lambda d\lambda)$ and

$$\int_{0}^{\infty} \mid \tilde{R}(\lambda) \mid^2 \lambda^{-1} d\lambda < \infty.$$

The complete proof of the conclusion is given in Theorem 4.39. Hence we have obtained simplified criteria for radial functions to be admissible, as well as for membership in $\mathcal{S}_r(\mathbb{H})$. We expect that these results can be employed to show that there exist functions satisfying both criteria, i.e., radial admissible Schwartz functions. However, in this thesis, we will obtain an example by a different construction, presented in Chapter 5.

In Chapter **5** we characterize the space of admissible Schwartz function on the Heisenberg group **without** applying the Fourier transform but using the *Calderón* reproducing formula instead. We show that a function ϕ in $\mathcal{S}(\mathbb{H})$ with mean value zero is admissible in the sense of Definition $(*)$ if and only if it is *Calderón* admissible.

We say a function $\phi \in \mathcal{S}(\mathbb{H})$ with $\int \phi = 0$ is **Calderón** admissible if for any $0 < \varepsilon < A$ and $g \in \mathcal{S}(\mathbb{H})$

$$g * \int_{\varepsilon}^{A} \tilde{\phi}_a * \phi_a \, a^{-1} da \to cg \quad \text{as } \varepsilon \to 0 \,;\, A \to \infty$$

holds in the sense of tempered distributions, where c is a nonzero constant, and for $a > 0$ is $\phi_a(\omega) = a^{-4}\phi(a^{-1}\omega)$ and $\widetilde{\psi}(\omega) = \overline{\psi(\omega^{-1})}$.

One of the main results of this chapter is Theorem 5.4. The theorem shows that the admissibility in the usual sense (i.e.,(1.12)) and in the sense of Calderón is equivalent as long as the function is Schwartz:

Function ϕ in $\mathcal{S}(\mathbb{H})$ is Calderón admissible if and only if for any $g \in \mathcal{S}(\mathbb{H})$ is

$$\int_0^\infty \int_{\mathbb{H}} |\langle g, L_\omega D_a \phi\rangle|^2 d\omega a^{-5} da = c \parallel g \parallel_2^2,$$

where the operators L and D are translation and dilation operators respectively.

In this chapter we also consider the problem of existence of such functions as a **main consequence**, which involves with our results. We present an example of a *Calderón* admissible function in the class $\mathcal{S}(\mathbb{H})$. Applying the definition of *Calderón* admissibility, we prove the existence of **"Mexican-Hat"** wavelet for the Heisenberg group obtained from the **Heat** kernel, defined in a complete analogous way to the *"Mexican-Hat"* wavelet on $\mathbb{R}$. (Theorem 5.11). This provides an example of an admissible radial Schwartz function.

Chapter 2

Notations and Preliminaries

In this chapter, the basic concepts and results centered around Fourier analysis and wavelet analysis on the Heisenberg group are presented. After introducing some notations, we present the basic notations, regarding group representations, Hilbert-Schmidt operators and trace-class operators, tensor products of Hilbert spaces, and direct integrals of Hilbert spaces. In the last section, we provide the operator-valued Fourier analysis for the Heisenberg group $\mathbb{H}$ including the most important Plancherel Theorem. This material may, for the most part, be found in [16], and will be applied in the sequel without further explanation.

Let G be a locally compact group. A positive Borel measure μ on G is called a left Haar measure if: (i) μ is a nonzero Radon measure on G, (ii) $\mu(xE) = \mu(E)$ for any $x \in G$, and any Borel subset $E \subseteq G$. σ is called right Haar measure if (ii) is replaced by : (ii)' $\sigma(Ex) = \sigma(E)$. One of the fundamental results in harmonic analysis is that every locally compact group G has a left Haar measure which is unique up to multiplication by a constant. G is unimodular if the left Haar measure is right Haar measure. See [16], §2.2 for further details.

Let G be a locally compact group with a fixed left Haar measure μ. We shall generally write dx for $d\mu(x)$. Let $C_c(G)$ denote the function space consisting of continuous compactly supported complex-valued functions on G. For $f \in C_c(G)$, let

$$\| f \|_p = \Big(\int_G | f(x) |^p \, dx \Big)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. Let $L^p(G)$ denote the completion of the normed linear space $(C_c(G), \| . \|_p)$. We are most interested in $L^1(G)$ and $L^2(G)$.

If $f, g \in L^1(G)$, the convolution of f and g is the function defined by

$$f * g(x) = \int_G f(y) g(y^{-1}x) dy.$$

If $f \in L^1(G)$, the involution of f, $\tilde{f}$, is defined by the relation

$$f^*(x) = \overline{f(x^{-1})}.$$

If f is a function on G and $y \in G$, we define the *left* and *right translations* of f through y by

$$L_y f(x) = f(y^{-1}x), \qquad R_y f(x) = f(xy).$$

The reason for using y^{-1} in L_y and y in R_y is to make the maps $y \to L_y$ and $y \to R_y$ group homomorphisms:

$$L_{yz} = L_y L_z, \qquad R_{yz} = R_y R_z.$$

2.1 Group Representation

Let G be a locally compact group. A continuous unitary representation of G is a pair $(\pi, \mathcal{H}_\pi)$, where $\mathcal{H}_\pi$ is a Hilbert space (the representation space of π) and π is a homomorphism from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators that is continuous with the respect to the strong operator topology. More precisely, $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ satisfies $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, and $x \to \pi(x)\xi$ is continuous from G to $\mathcal{H}_\pi$, for any $\xi \in \mathcal{H}_\pi$.

Suppose $\mathcal{K}$ is a closed subspace of $\mathcal{H}_\pi$. $\mathcal{K}$ is called an invariant subspace for π if $\pi(x)\mathcal{K} \subseteq \mathcal{K}$

for all $x \in G$. If $\mathcal{K}$ is invariant and nontrivial, the restriction of π to $\mathcal{K}$, $\pi^{\mathcal{K}}(x) := \pi(x)\Big|_{\mathcal{K}}$, defines a representation of G on $\mathcal{K}$, called a subrepresentation of π. If π admits an invariant nontrivial subspace of $\mathcal{H}_\pi$, then π is called *reducible*, otherwise π is called *irreducible*. If $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ are two representations of G and $T \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_\sigma)$, where $\mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_\sigma)$ denotes all bounded linear operators from $\mathcal{H}_\pi$ to $\mathcal{H}_\sigma$, satisfies $T\pi(x) = \sigma(x)T$ for all $x \in G$, then T is said to be an intertwining operator for π and σ. If there exists a unitary map $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ which intertwines π and σ, then we say that π is equivalent to σ and write $\pi \sim \sigma$. The dual space $\widehat{G}$ of G is the set of equivalence classes of irreducible unitary representation of G. See [16] §7.2 for a discussion of the space $\widehat{G}$.
The space of all intertwining operators for π and σ is denoted by $Hom(\pi, \sigma)$. Irreducibility of π is related to the structure of $Hom(\pi, \pi)$ by a fundamental result:

Lemma 2.1. ***Schur's Lemma:*** *A unitary representation π on $\mathcal{H}_\pi$ is irreducible if and only if $Hom(\pi, \pi)$ contains only scalar multiples of the identity.*

For the proof of this result, see [16] §3.1.

2.2 Hilbert-Schmidt and Trace-Class Operators

Let us start by recalling the definition of Hilbert-Schmidt norm of an operator in a finite dimensional Hilbert space $\mathcal{H}$. Let $T \in B(\mathcal{H})$ be any endomorphism in $\mathcal{H}$. Take any orthonormal basis $\{e_k\}_{k=1}^d$ of $\mathcal{H}$, where $d = dim(\mathcal{H})$, and assume that T is replaced by the matrix $(t_{k,l})$ in the basis $\{e_k\}$; obviously $t_{k,l} = \langle Te_k, e_l \rangle$ and

$$\| T \|_{H.S} = \Big(\sum_{k,l} | t_{k,l} |^2 \Big)^{\frac{1}{2}} \tag{2.1}$$

defines a norm on $B(\mathcal{H})$, the set of endomorphisms in $\mathcal{H}$. It is called the *Hilbert-Schmidt* norm of T. If S is another endomorphism, represented by the matrix (s_{kl}) with respect to the same basis, a computation shows that

$$tr(S^*T) = \sum_{k,l} t_{kl}\overline{s_{kl}}.$$

This shows that the Hilbert-Schmidt norm (2.1) is derived from the following inner product

$$\langle T, S\rangle = \sum_{k,l}\sum_{k,l} t_{kl}\overline{s_{kl}} = Tr(S^*T)$$

on $B(\mathcal{H})$. One can show that $\| T \|_{H.S}$ and $\langle T, S\rangle$ are independent of the choice of orthonormal basis $\{e_k\}$ of $\mathcal{H}$ ([16], Appendix 2).

Now we need some analogous results in arbitrary Hilbert space. Let $\mathcal{H}$ be a separable Hilbert space and $T \in B(\mathcal{H})$ be a continuous linear operator on $\mathcal{H}$. Let us take an orthonormal basis $\{e_k\}$ of $\mathcal{H}$. Then $\sum_k \| Te_k \|^2$ is independent of the choice of orthonormal basis $\{e_k\}$ of $\mathcal{H}$.

An operator $T \in B(\mathcal{H})$ is called *Hilbert-Schmidt* operator if for one, hence for any orthonormal basis $\{e_k\}$, $\sum_k \| Te_k \|^2 < \infty$. By the preceding argument, this is well-defined. We use $HS(\mathcal{H})$ to denote the set of all Hilbert-Schmidt operators on $\mathcal{H}$. For $T \in HS(\mathcal{H})$, define the Hilbert-Schmidt norm of T as

$$\| T \|_{H.S}^2 = (\sum_k \| Te_k \|^2)^{\frac{1}{2}}.$$

We have the following properties: If T is a Hilbert-Schmidt operator, so is T^*. If T and S are Hilbert-Schmidt operators, so is $aT + bS$, for any constants a, b. So we see that all the Hilbert-Schmidt operators on $\mathcal{H}$ form a normed linear space. Moreover, for any $T \in HS(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, TS and ST are both in $HS(\mathcal{H})$. Thus $HS(\mathcal{H})$ is also a two-sided ideal in $\mathcal{B}(\mathcal{H})$.

We call a product of two operators in $HS(\mathcal{H})$ a *trace-class* operator. By the preceding argument, if T is trace-class operator and $\{e_k\}_k$ is any orthonormal basis for $\mathcal{H}$, then

$$\| T \|_1 = tr(T) := \sum_k \langle Te_k, e_k\rangle$$

is well-defined and it is independent of $\{e_k\}_k$. We have the following properties for trace-class operators: every trace-class operator is a Hilbert-Schmidt operator and T is a Hilbert-Schmidt operator if and only if T^*T is a trace-class operator. T is trace-class if and only if T^* is a trace-class operator.
We use the abbreviation **ONB** for orthonormal bases and the word **projection** for self-adjoint projection operator on a Hilbert space.
For more about Hilbert-Schmidt operators and trace-class operators, see [16], Appendix 2, and [47], §2 and §3.

2.3 Tensor Products of Hilbert Spaces

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. We define the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ to be the set $\mathcal{H}_1 \otimes \mathcal{H}_2$ of all linear operators $T : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\sum_k \| Te_k \|^2 < \infty$ for some, hence any, orthonormal basis $\{e_k\}$ for $\mathcal{H}_2$. If we set

$$\| T \|_{H.S} = (\sum_k \| Te_k \|^2)^{\frac{1}{2}},$$

then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space with the norm $\| \, . \, \|_{H.S}$ and associated inner product

$$\langle T, S \rangle = \sum_k \langle Te_k, Se_k \rangle,$$

where $\{e_k\}$ is any orthonormal basis of $\mathcal{H}_2$.
If $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$, the map $\omega \to \langle \omega, \eta \rangle \xi \ \ (\omega \in \mathcal{H}_2)$ belongs to $\mathcal{H}_1 \otimes \mathcal{H}_2$; we denote it by $\xi \otimes \eta$:

$$(\xi \otimes \eta)(\omega) = \langle \omega, \eta \rangle \xi.$$

Note that $(\xi \otimes \eta)^* = \eta \otimes \xi$ and for any operators T and W on $\mathcal{H}_1$ and repectively on $\mathcal{H}_2$ we have $T \circ (\xi \otimes \eta) = T\xi \otimes \eta$, $(\xi \otimes \eta) \circ W = \xi \otimes W^*\eta$.
For $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ we shall denote $\mathcal{H} \otimes \mathcal{H} = HS(\mathcal{H})$. For more information and detail, see [16] §7.3 or [36] §2.6.

2.4 Direct Integral of Hilbert Spaces

In the following, we outline some notations and results concerning direct integrals. For further information on direct integrals, we refer the reader to [16] §7.4.

A family $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ of nonzero separable Hilbert spaces indexed by A will be called a field of Hilbert spaces over A, where a Borel σ-algebra is supposed on A. A map f on A such that $f(\alpha) \in \mathcal{H}_\alpha$ for each $\alpha \in A$ will be called a *vector field* on A. We denote the inner product and norm on $\mathcal{H}_\alpha$ by $\langle\ ,\ \rangle_\alpha$ and $\|\ .\ \|_\alpha$. A measurable field of Hilbert space over A is a field of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ together with a countable family $\{e_j\}_1^\infty$ of vector fields with the following properties:

(i): the functions $\alpha \mapsto \langle e_j(\alpha), e_k(\alpha)\rangle_\alpha$ are measurable for all j, k.

(ii): the linear span of $\{e_j(\alpha)\}_1^\infty$ is dense in $\mathcal{H}_\alpha$, for each α.

Given a measurable field of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$, $\{e_j\}$ on A, a vector field f on A will be called **measurable** if the function $\alpha \to \langle f(\alpha), e_j(\alpha)\rangle_\alpha$ is measurable function on A, for each j. Finally, we are ready to define direct integrals. Suppose $\{\mathcal{H}_\alpha\}_{\alpha\in A}$, $\{e_j\}_1^\infty$ is a measurable field of Hilbert spaces over A, and suppose μ is a measure on A. The direct integral of the spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ with respect to μ is denoted by

$$\int_A^\oplus \mathcal{H}_\alpha d\mu(\alpha).$$

This is the space of measurable vector fields f on A such that

$$\| f \|^2 = \int_A \| f(\alpha) \|_\alpha^2 \, d\mu(\alpha) < \infty,$$

where two vector fields agreeing almost everywhere are identified. Then it easily follows that $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_A \langle f(\alpha), g(\alpha)\rangle_\alpha d\mu(\alpha).$$

In case of a constant field, that is, $\mathcal{H}_\alpha = \mathcal{H}$ for all $\alpha \in A$, $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha) = L^2(A, \mu, \mathcal{H})$, all the measurable functions $f : A \to \mathcal{H}$ defined on a measurable space (A, μ) with values in

$\mathcal{H}$ such that

$$\parallel f \parallel^2 = \int_A \parallel f(\alpha) \parallel^2 d\mu(\alpha) < \infty.$$

Here $\mathcal{H}$ is considered as a Borel space with the Borel-σ-algebra of the norm topology.

2.5 The Heisenberg Group $\mathbb{H}$

The Heisenberg group $\mathbb{H}$ is a Lie group with underlying manifold $\mathbb{R}^3$. We denote points in $\mathbb{H}$ by (p, q, t) with $p, q, t \in \mathbb{R}$, and define the group operation by

$$(p_1, q_1, t_1) * (p_2, q_2, t_2) = \big(p_1 + p_2, q_1 + q_2, t_1 + t_2 + \frac{1}{2}(p_1 q_2 - q_1 p_2)\big). \tag{2.2}$$

It is straightforward to verify that this is a group operation, with the origin $0 = (0, 0, 0)$ as the identity element. Note that the inverse of (p, q, t) is given by $(-p, -q, -t)$.
We can identify both $\mathbb{H}$ and its Lie algebra $\mathfrak{h}$ with $\mathbb{R}^3$, with group operation given by (2.2) and Lie bracket given by

$$[(p_1, q_1, t_1), (p_2, q_2, t_2)] = (0, 0, p_1 q_2 - q_1 p_2). \tag{2.3}$$

The Haar measure on the Heisenberg group $\mathbb{H} = \mathbb{R}^3$ is the usual Lebesgue measure. The Lie algebra $\mathfrak{h}$ of the Heisenberg group $\mathbb{H}$ has a basis $\{X, Y, T\}$ with $[X, Y] = T$ and all other brackets are zero, such that the exponential function $exp : \mathfrak{h} \to \mathbb{H}$ becomes identity, i.e.,

$$exp\ (pX + qY + tT) = (p, q, t).$$

We define the action of $\mathfrak{h}$ on space $C^\infty(\mathbb{H})$ via left invariant differential operators by the following formula:

Suppose $f \in C^\infty(\mathbb{H})$, then

$$\begin{aligned}(Xf)(p,q,t) =& \frac{d}{ds}\big(f(p,q,t).exp(sX)\big)\Big|_{s=0}\\ =& \frac{d}{ds}\big(f(p,q,t).(s,0,0)\big)\Big|_{s=0}\\ =& \frac{d}{ds}\big(f(p+s,q,t-\frac{1}{2}sq)\big)\Big|_{s=0}\\ =& \frac{d}{dp}f(p,q,t)-\frac{1}{2}q\frac{d}{dt}f(p,q,t).\end{aligned}$$

Likewise

$$\begin{aligned}(Yf)(p,q,t) =& \frac{d}{ds}\big(f((p,q,t).exp(sY)\big)\Big|_{s=0}\\ =& \frac{d}{ds}\big(f(p,q,t).(0,s,0)\big)\Big|_{s=0}\\ =& \frac{d}{ds}\big(f(p,q+s,t+\frac{1}{2}ps)\big)\Big|_{s=0}\\ =& \frac{d}{dq}f(p,q,t)+\frac{1}{2}p\frac{d}{dt}f(p,q,t),\end{aligned}$$

and

$$(Tf)(p,q,t) = \frac{d}{dt}f(p,q,t).$$

Consequently the elements $X+iY$ and $X-iY$ act as follow:

$$\begin{aligned}((X+iY)f)(p,q,t) &= (Xf)(p,q,t)+i(Yf)(p,q,t)\\ &= \frac{d}{dp}f(p,q,t)-\frac{q}{2}\frac{d}{dt}f(p,q,t)+i\frac{d}{dq}f(p,q,t)+i\frac{p}{2}\frac{d}{dt}f(p,q,t) \qquad (2.4)\\ &= \left(\frac{d}{dp}+i\frac{d}{dq}\right)f(p,q,t)+\frac{i}{2}(p-iq)\frac{d}{dt}f(p,q,t) \qquad (2.5)\end{aligned}$$

and

$$((X-iY)f)(p,q,t) = \left(\frac{d}{dp}-i\frac{d}{dq}\right)f(p,q,t)-\frac{i}{2}(p+iq)\frac{d}{dt}f(p,q,t).$$

One basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}$ is written as $\{Z,\bar{Z},T\}$ where

$$Z = X+iY = \frac{\partial}{\partial z}+i\frac{\bar{z}}{2}\frac{\partial}{\partial t}, \quad \bar{Z} = X-iY = \frac{\partial}{\partial \bar{z}}-i\frac{z}{2}\frac{\partial}{\partial t}, \qquad (2.6)$$

and

$$T = \frac{\partial}{\partial t}. \tag{2.7}$$

With these conventions one has $\left[Z, \bar{Z}\right] = -i\frac{\partial}{\partial t}$.

2.6 Fourier Analysis on the Heisenberg Group

This section contains a brief review of Fourier analysis on the Heisenberg group $\mathbb{H}$, including the most important Plancherel Theorem. For details, we refer the reader to [16] §7.5 and §7.6.

2.6.1 The Representations of the Heisenberg Group

The Heisenberg group is the best known example from the realm of nilpotent Lie groups. The representation theory of $\mathbb{H}$ is simple and well understood. Using the fundamental theorem, due to *Stone and von Neumann*, we can give a complete classification of all the irreducible unitary representation of $\mathbb{H}$.

For the Heisenberg group there are two families of irreducible unitary representations, at least up to unitary equivalence. One family, giving all infinite-dimensional irreducible unitary representations, is parametrized by nonzero real numbers λ; the other family, giving all one-dimensional representations, is parametrized by $(b, \beta) \in \mathbb{R} \times \mathbb{R}$. We will see below that the one-dimensional representations have no contribution to the Plancherel formula and Fourier inversion transform, i.e. they form a set of representations that has zero Plancherel measure. Hence we will focus on the **Schrödinger** representation, defined next:

The infinite-dimensional irreducible unitary representations of Heisenberg group are called **Schrödinger** representations, all of them are realised on $L^2(\mathbb{R})$ as follows:

For each $\lambda \in \mathbb{R}^*(= \mathbb{R} - \{0\})$, consider for any $(p, q, t) \in \mathbb{H}$ the operator $\rho_\lambda(p, q, t)$ acting on $L^2(\mathbb{R})$ by

$$\rho_\lambda(p,q,t)\phi(x) = e^{i\lambda t}e^{i\lambda(px+\frac{1}{2}(pq))}\phi(x+q) \tag{2.8}$$

where $\phi \in L^2(\mathbb{R})$. It is easy to see that $\rho_\lambda(p,q,t)$ is a unitary operator satisfying

$$\rho_\lambda\big((p_1,q_1,t_1)(p_2,q_2,t_2)\big) = \rho_\lambda(p_1,q_1,t_1)\rho_\lambda(p_2,q_2,t_2).$$

Thus each ρ_λ is a strongly continuous unitary representation of $\mathbb{H}$, i.e. for any $f \in L^2(\mathbb{R})$, $\rho_\lambda(x_n)f \to \rho_\lambda(x)f$ as $x_n \to x$. Note that each ρ_λ is irreducible [16].

A theorem of Stone and von Neumann says that up to unitary equivalence these are all the irreducible unitary representations of $\mathbb{H}$ that are nontrivial at the center.

Theorem 2.2. ***(Stone and von Neumann)*** *The representations ρ_λ, $\lambda \neq 0$ are irreducible. If π is any irreducible unitary representation of $\mathbb{H}$ on a Hilbert space $\mathcal{H}$ such that $\pi(0,t) = e^{i\lambda t}I$ for some $\lambda \neq 0$, then π is unitary equivalent to ρ_λ.*

For the proof of the theorem see Folland [16].

2.6.2 Fourier Transform on the Heisenberg Group

In this section we define the group Fourier transform for functions on $\mathbb{H}$ and introduce the inversion and Plancherel theorems for the Fourier transform. Recall that the Haar measure on the Heisenberg group $\mathbb{H} = \mathbb{R}^3$ is the usual Lebesgue measure. It is also easy to show that it is both left and right invariant under the group multiplication defined by (2.2), i.e. $\mathbb{H}$ is unimodular.

Next we introduce the convolution of two functions f and g on $\mathbb{H}$. For f and g in $L^1(\mathbb{H})$, the **convolution** of f and g is the function defined by

$$f * g(\omega) = \int_{\mathbb{H}} f(\nu)g(\nu^{-1}\omega)d\nu. \tag{2.9}$$

By Fubini's theorem, the integral is absolutely convergent for almost every ω and

$$\| f * g \|_1 \leq \| f \|_1 \| g \|_1 .$$

Convolution can be extended from L^1 to L^2 space. More precisely, if $f \in L^1$ and $g \in L^2$, then the integral in (2.9) converges absolutely for almost every ω and one has $f * g \in L^2$, and

$$\| f * g \|_2 \leq \| f \|_1 \| g \|_2$$

Moreover for any pair $f, g \in L^2(\mathbb{H})$ is $f * \tilde{g} \in C_b(\mathbb{H})$, where $\tilde{g}(\omega) = \overline{g(\omega^{-1})}$. For more details about convolution of functions see for example [16] Proposition (2.39).

Definition 2.3. *$f \in L^2(\mathbb{H})$ is called* ***selfadjoint convolution idempotent*** *if $f = \tilde{f} = f * f$.*

The selfadjoint convolution idempotents and their support properties are studied in detail by Führ [20] in §2.5.

Next we begin with Fourier transform of integrable functions on $\mathbb{H}$. If $f \in L^1(\mathbb{H})$, we define the Fourier transform of f to be the measurable field of operators over $\widehat{\mathbb{H}}$ given by the weak operator integrals, as follow:

$$\widehat{f}(\lambda) = \int_{\mathbb{H}} f(\omega)\rho_\lambda(\omega)d\omega. \tag{2.10}$$

For short, we write $\widehat{f}(\lambda)$ instead of $\widehat{f}(\rho_\lambda)$. Note that the Fourier transform $\widehat{f}(\lambda)$ is an operator-valued function, which for any $\phi, \psi \in L^2(\mathbb{R})$ fulfils

$$\langle \widehat{f}(\lambda)\phi, \psi \rangle = \int_{\mathbb{H}} f(p, q, t)\langle \rho_\lambda(p, q, t)\phi, \psi \rangle dpdqdt,$$

by definition of the weak operator integral. The operator $\widehat{f}(\lambda)$ is bounded on $L^2(\mathbb{R})$ with the operator norm satisfying

$$\| \widehat{f}(\lambda) \| \leq \| f \|_1 .$$

If $f \in L^1 \cap L^2(\mathbb{H})$, $\widehat{f}(\lambda)$ is actually a Hilbert-Schmidt operator and a Fourier transform can be extended for all $f \in L^2(\mathbb{H})$.

Let $\mathcal{M} := (L^1 \cap L^2)(\mathbb{H})$ and $\mathcal{N} :=$ linear span of $\{f * g \big| f, g \in \mathcal{M}\}$. By definition of convolution in harmonic analysis, for any $f, g \in L^2(\mathbb{H})$ and $\omega \in \mathbb{H}$ we can write $(f * g)(\omega) = \langle f, L_\omega \tilde{g}\rangle$. Therefore $f * g \in C_0(\mathbb{H})$ [16]. Moreover $\mathcal{N}$ is a vector space of functions which can be shown to be dense in both $L^1(\mathbb{H})$ and $L^2(\mathbb{H})$. With the notations set as above, we have the following abstract Plancherel theorem. The proof may be found in [52] for the Heisenberg group and for more general case of groups see for example [16].

Theorem 2.4. ***Plancherel Theorem*** *The Fourier transform $f \to \widehat{f}$ maps $\mathcal{M}$ into $\int_{\lambda\in\mathbb{R}^*}^{\oplus} L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) d\mu(\lambda)$, where $d\mu(\lambda) = (2\pi)^{-2} |\lambda| \, d\lambda$ is Plancherel measure given on $\widehat{\mathbb{H}}$. This map extends to a unitary map from $L^2(\mathbb{H})$ onto $\int_{\lambda\in\mathbb{R}^*}^{\oplus} L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) d\mu(\lambda)$ i.e., it sets up an isometric isomorphism between $L^2(\mathbb{H})$ and the Hilbert space*

$$L^2\Big(\mathbb{R}^*, d\mu(\lambda), L^2(\mathbb{R}) \otimes L^2(\mathbb{R})\Big),$$

i.e., the space of functions on $\mathbb{R}^*$ taking values in $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ which are square integrable with respect to $d\mu(\lambda)$.
For $f, g \in \mathcal{M}$ one has the Parseval formula

$$\int_{\mathbb{H}} f(\omega)\overline{g(\omega)} dx = \int_{\lambda\in\mathbb{R}^*} tr\Big(\widehat{g}(\lambda)^* \widehat{f}(\lambda)\Big) d\mu(\lambda).$$

And for $f \in \mathcal{N}$ one has the Fourier inversion formula

$$f(\omega) = \int_{\lambda\in\mathbb{R}^*} tr\Big(\rho_\lambda(\omega)^* \widehat{f}(\lambda)\Big) d\mu(\lambda).$$

Simple computations show that the basic properties of the Fourier transform remain valid for $f, g \in (L^1 \cap L^2)(\mathbb{H})$:

$$\begin{aligned}
\widehat{(af + bg)}(\lambda) &= a\widehat{f}(\lambda) + b\widehat{g}(\lambda) \\
\widehat{(f * g)}(\lambda) &= \widehat{f}(\lambda)\widehat{g}(\lambda) \\
\widehat{(L_\omega f)}(\lambda) &= \rho_\lambda(\omega)\widehat{f}(\lambda) \qquad (2.11) \\
\widehat{(\tilde{f})}(\lambda) &= \widehat{f}(\lambda)^*, \qquad (2.12)
\end{aligned}$$

where $\widehat{f}(\lambda)^*$ is the adjoint operator of $\widehat{f}(\lambda)$ and $\omega \in \mathbb{H}$.

We conclude this section with computation of Fourier transform of $f(a.)$. Recall that the dilation operator given by $a > 0$ is defined on $\mathbb{H}$ as follow:

$$a: \ (p,q,t) \to a.(p,q,t) = (ap, aq, a^2t) \quad \forall\, (p,q,t) \in \mathbb{H}.$$

Suppose $\lambda \neq 0$. Then from definition of Schrödinger representation we obtain the following equality:

$$\rho_\lambda(a^{-1}.(0,0,t)) = e^{i\lambda a^{-2}t} I = \rho_{a^{-2}\lambda}(0,0,t) \qquad \forall t \in \mathbb{R}.$$

Hence from Theorem 2.2, the representations $\rho_\lambda(a^{-1}.)$ and $\rho_{a^{-2}\lambda}$ are unitary equivalent. It means there exists a unitary operator $U_{a,\lambda} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, so that

$$\rho_\lambda(a^{-1}.(p,q,t)) = U_{a,\lambda}\rho_{a^{-2}\lambda}(p,q,t)U^*_{a,\lambda} \quad \forall (p,q,t) \in \mathbb{H}. \tag{2.13}$$

With computation it is easy to see that for any $a > 0$ and $\lambda \neq 0$ is $U_{a,\lambda} = D_a^*$ where $D_a f(.) = a^{-1/2} f(a^{-1}.)$. Since $D_a^* = D_{a^{-1}}$ then in (2.13) is

$$\rho_\lambda(a^{-1}.(p,q,t)) = D_{a^{-1}}\rho_{a^{-2}\lambda}(p,q,t)D_a \quad \forall (p,q,t) \in \mathbb{H}. \tag{2.14}$$

After this preparation, we are ready to compute the Fourier transform of function $f \circ \delta_a$ in the following lemma where $f \in L^2(\mathbb{H})$:

Lemma 2.5. *For any $f \in L^2(\mathbb{H})$ is*

$$\widehat{f(a.)}(\lambda) = a^{-4} D_{a^{-1}} \widehat{f(a^{-2}\lambda)} D_a.$$

Proof: From definition of Fourier transform (2.10), for $\lambda \neq 0$ we have

$$\begin{aligned}
\widehat{f(a.)}(\lambda) &= \int_\lambda f(a.(p,q,t))\rho_\lambda(p,q,t)\,dpdqdt \\
&= \int_\lambda f(ap, aq, a^2t)\rho_\lambda(p,q,t)\,dpdqdt \\
&= a^{-4}\int_\lambda f(p,q,t)\rho_\lambda(a^{-1}p, a^{-1}q, a^{-2}t)\,dpdqdt \\
&= a^{-4}\int_\lambda f(p,q,t)\left(\rho_\lambda(a^{-1}.(p,q,t))\right)dpdqdt,
\end{aligned}$$

now inserting (2.14), we derive the following relation:

$$\begin{aligned}\widehat{f(a.)}(\lambda) &= a^{-4} D_{a^{-1}} \Big(\int_{\lambda} f(p,q,t) \rho_{a^{-2}\lambda}(p,q,t) dpdqdt \Big) D_a \\ &= a^{-4} D_{a^{-1}} \widehat{f(a^{-2}\lambda)} D_a.\end{aligned} \tag{2.15}$$

□

2.7 Wavelet Analysis on the Heisenberg Group

In this section, the basic concepts concerning wavelet analysis on the Heisenberg group from the discrete and continuous point of view are presented.

2.7.1 Continuous Wavelet Analysis: A Representation Point of View

To understand the concept of continuous wavelet transformation on the Heisenberg group, first we present the wavelet transform on $\mathbb{R}$. The wavelet transform on $\mathbb{R}$ can be defined from the representation-theoretic view as follow:

Suppose $a > 0$ and $b \in \mathbb{R}$. Then L_b is translation operators on $L^2(\mathbb{R})$, which acts by

$$L_b : \ f \rightarrow f(. - b) \quad \forall f \in L^2(\mathbb{R}).$$

We already introduced the dilation operators D_a on $L^2(\mathbb{R})$.

The definition of dilation and translation operators suggests defining a group multiplication on $\mathbb{R} \times \mathbb{R}^*$ by

$$(b_1, a_1).(b_2, a_2) = (a_1 b_2 + b_1, a_1 a_2). \tag{2.16}$$

One obtains the so-called "affine group". The left Haar measure is then $|a|^{-2}\, dbda$ on $\mathbb{R} \times \mathbb{R}^*$. Now define the representation π of $\mathbb{R} \times \mathbb{R}^*$ on $L^2(\mathbb{R})$ by letting, for any $(b, a) \in \mathbb{R} \times \mathbb{R}^*$

$$\pi(b,a) f(x) = T_b D_a f(x) = |a|^{\frac{-1}{2}} f(a^{-1}(x - b)) \quad \forall f \in L^2(\mathbb{R}) \ x \in \mathbb{R}.$$

π is homomorphism with respect to the group multiplication in (2.16) and is unitary and irreducible representation of $\mathbb{R} \times \mathbb{R}^*$. With above preparation we give the definition of an admissible vector in $L^2(\mathbb{R})$, with respect to the representation π, and hence continuous wavelet transform. We shall say a function $\phi \in L^2(\mathbb{R})$ is an admissible vector when the operator

$$V_\phi : \ L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R} \times \mathbb{R}^*, |a|^2 \, dadb), \quad V_\phi(f)(b,a) = \langle f, \pi(b,a)\phi \rangle$$

is an isometry up to a constant, i.e.,

$$\| \ f \ \|^2 = const. \int_{\mathbb{R}} \int_0^\infty \mid V_\phi(f)(b,a) \mid^2 |a|^{-2} \, dadb \quad \forall \ f \in L^2(\mathbb{R}),$$

where the constant is non-negative and only depends on ϕ . Then ϕ is called **wavelet** and $V_\phi(f)$ is called continuous wavelet transform of function f.

There already exist many introductions to wavelet theory, written from various points of view and for audiences on all levels. The books by Y.Meyer [41] and I.Daubechies [9], are still unsurpassed.

Our definition of continuous wavelet transform for the Heisenberg group will be from a representation-theoretic point of view, adapted from the case $\mathbb{R}$. For the construction of wavelet transform one needs a one-parameter group of dilations for $\mathbb{H}$. Here we consider $H := (0, \infty)$ as an one-parameter dilation group of $\mathbb{H}$ which is defined as follows:
Suppose $a > 0$. Then $a.$ denotes an automorphism of $\mathbb{H}$ given by

$$a.(p,q,t) = (ap, aq, a^2t) \qquad \forall (p,q,t) \in \mathbb{H}. \tag{2.17}$$

The set $(0, \infty)$ forms a group of automorphisms of $\mathbb{H}$, called dilation group for $\mathbb{H}$ (for more details about such dilation groups see for example [14]). From now on, $a > 0$ refers to the automorphism $a \rightarrow a.\omega$ for all $\omega \in \mathbb{H}$. For the remainder of the work, $H = (0, \infty)$ denotes a group of automorphisms of $\mathbb{H}$ with operation in (2.17).
Since $H = (0, \infty)$ operates continuously by topological automorphisms on the locally compact group $\mathbb{H}$, we can define the semidirect product $G := \mathbb{H} \rtimes (0, \infty)$, which is a

locally compact topological group with product topology. Elements of G can be written as $(\omega, a) \in \mathbb{H} \times (0, \infty)$ and the group operation is defined by

$$(\omega, a)(\acute{\omega}, \acute{a}) = (\omega(a.\acute{\omega}), a\acute{a}) \quad \forall \omega, \acute{\omega} \in \mathbb{H} \ \ and \ \forall a, \acute{a} > 0.$$

G is a non-unimodular group, with left Haar measure given by $d\mu_G(\omega, a) = a^{-5} d\omega da$ and modular function $\Delta_G(\omega, a) = a^{-4}$ for any $(\omega, a) \in G$. Analogous to above, for $a > 0$ the dilation operator D_a on $L^2(\mathbb{H})$ is defined by $D_a f(.) = a^{-2} f(a^{-1}.)$ and for $\omega \in \mathbb{H}$, L_ω denotes the left translation where $L_\omega f(.) = f(\omega^{-1}.)$ for any f defined on $\mathbb{H}$.

Definition 2.6. ***(quasiregular representation)*** *For any $(\omega, a) \in G$ and $f \in L^2(\mathbb{H})$ define*

$$(\pi(\omega, a)f)(\upsilon) := L_\omega D_a f(\upsilon) = a^{-2} f(a^{-1}(\omega^{-1}.\upsilon)) \quad \forall \ \upsilon \in \mathbb{H}. \tag{2.18}$$

It is easy to prove that the map $\pi : G \to \mathcal{U}(L^2(\mathbb{H}))$ is a strongly continuous unitary representation of G. This representation is called "quasiregular representation".
Recall that $\mathcal{U}(L^2(\mathbb{H}))$ is the set of the unitary operators defined on $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$.

Next we give the definition of admissible vectors in $L^2(\mathbb{H})$, associated to the quasiregular representation on $L^2(\mathbb{H})$ by (2.18).

Definition 2.7. ***(admissible vector)*** *Let $(\pi, \mathcal{H}_\pi = L^2(\mathbb{H}))$ denote the strongly continuous unitary representation of the locally compact group $G := \mathbb{H} \rtimes (0, \infty)$. G is considered with left Haar measure $d\mu(\omega, a) = a^{-5} d\omega da$. For any $\phi \in L^2(\mathbb{H})$ the coefficient operator V_ϕ is defined as follows:*

$$V_\phi : L^2(\mathbb{H}) \to L^2(G) \quad by \quad V_\phi(f)(\omega, a) = \langle f, \pi(\omega, a)\phi \rangle.$$

ϕ is called **admissible** *if $V_\phi : L^2(\mathbb{H}) \to L^2(G)$ is an isometric operator up to a constant, i.e,*

$$\| f \|^2 = const. \int_{\mathbb{H}} \int_0^\infty | V_\psi(f)(\omega, a) |^2 \, a^{-5} da d\omega \quad \forall \ f \in L^2(\mathbb{H}), \tag{2.19}$$

where the constant only depends on ψ. Then V_ϕ is called ***continuous wavelet transform***.

One of the important consequence of the isometry given by formula (2.19) is that a function can be reconstructed from its wavelet transform by means of the *"resolution identity"*(*"inversion formula"*) , i.e, formula (2.19) can be read as

$$f = const. \int_{\mathbb{H}} \int_0^\infty \langle f, \pi(\omega, a)\psi\rangle \pi(\omega, a)\psi \; a^{-5} da d\omega \quad \forall\, f \in L^2(\mathbb{H}),$$

with the convergence of the integral in the weak sense.
An important aspect of wavelet theory is its microscope effect, i.e, by choosing a suitable wavelet ψ, as the lense, one can obtain local information about the argument function $f \in L^2(\mathbb{H})$. This information is obtained from the Fourier coefficients $\langle f, \pi(\omega, a)\psi\rangle$ when for instance the coefficients have a fast decay for $a \to 0$. This property can happen for example if the wavelet is in the Schwartz space $\mathcal{S}(\mathbb{H})$ with several vanishing moments.
The existence of admissible vectors for quasiregular representation of $G := N \rtimes H$ on $L^2(N)$ is already proved in Führ's book [20]; §5.4, where N is a homogeneous Lie group and H is a one-parameter group of dilations for N (for the definition of homogeneous groups see for example [14], $\mathbb{H}$ is one example of such a group). However, the existence of fast-decaying wavelets was left open.
Our work establishes existence of admissible radial Schwartz vectors for case $N = \mathbb{H}$ and $H = (0, \infty)$. In fact we give an answer to this question in Chapter 4 by characterizing the class of admissible radial Schwartz functions, and give an example of such a wavelet in Chapter 5.

The existence of admissible vectors in closed subspaces of $L^2(\mathbb{H}^n)$ was studied for example in [34], where $\mathbb{H}^n$ is n-dimensional Heisenberg group with the underlying manifold $\mathbb{C}^n \times \mathbb{R}$. The authors consider the unitary reducible representation U of a non-unimodular group P on $L^2(\mathbb{H}^n)$. A closer look reveals that, for $n = 1$, Liu and Peng are concerned precisely with our setting: As formula (1.10) in [34] shows, their group is isomorphic to the group G from §2.7.1, and the representation U defined in Formula (1.11) of [34] is precisely the associated quasiregular representation, which we defined in 2.6. The only slight difference consists in the parametrization of the dilations.

The authors then decompose $L^2(\mathbb{H}^n)$ into a direct (infinite) sum of irreducible invariant closed subspaces, $\mathcal{M}_m$, under the representation U on $L^2(\mathbb{H}^n)$. They proceed to show that the restriction of U to these subspaces is square-integrable, i.e, each subspace $\mathcal{M}_m$ contains at last one nonzero wavelet vector with respect to U. Furthermore the authors give a characterization of the admissibility condition in each irreducible invariant closed subspace $\mathcal{M}_m$ in the terms of Fourier transform. But they did not show the existence of an admissible vector for all of $L^2(\mathbb{H}^n)$, unlike our results.

Observe that the fact that the representation U in [34] is unitary equivalent to the direct sum of irreducible representations, which are all square integrable, entails by Corollary 4.27 in [20], that there exists an admissible vector in $L^2(\mathbb{H})$. Therefore we already know of the existence of an admissible vector; this was pointed out following Corollary 4.27 in [20]. However, this source does not contain any concrete description of admissible vectors, of the kind we obtain in this thesis. In particular, the existence of well-localized wavelets, which is one of the main goals of this thesis, has not been previously investigated.

2.7.2 Discrete Wavelet Analysis

This section introduces *frames* and some related notations, which will be used in the context of frame-MRA, see Sections 3.2 and 3.3 of this thesis. The concept of frames is a generalization of orthonormal bases, defined as follow:

Definition 2.8. *A countable subset $\{e_n\}_{n\in I}$ of a Hilbert space $\mathcal{H}$ is said to be a* ***frame*** *of $\mathcal{H}$ if there exist two numbers $0 < a \leq b$ so that, for any $f \in \mathcal{H}$,*

$$a \parallel f \parallel^2 \leq \sum_{n\in I} \mid \langle f, e_n\rangle \mid^2 \leq b \parallel f \parallel^2 .$$

The positive number ***a*** *and* ***b*** *are called* ***frame bounds****. Note that the frame bounds are not unique. The optimal lower frame bound is the supremum over all lower frame bounds, and the optimal upper frame bound is the infimum over all upper frame bounds. The optimal frame bounds are actually frame bounds. The frame is called* ***tight frame*** *when $a = b$ and* ***normalized tight frame*** *when $a = b = 1$. Frames were introduced by [11].*

For our purpose, in Chapter 3 we consider the wavelet frames which are produced from one function using a countable family of dilation and left translation operators. The generator function is called *"discrete wavelet"*.
Below we will give a concrete example of wavelet frames with respect to a very special lattice as the translation set. Suppose Γ is a lattice in $\mathbb{H}$ and $a > 0$ refers to the automorphism $a : \omega \to a.\omega$ of $\mathbb{H}$. And, suppose $\mathcal{H}$ be a subspace of $L^2(\mathbb{H})$ and $\psi \in \mathcal{H}$. Then the discrete system $\{L_{a^{-j}\gamma} D_{a^{-j}} \psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ in $\mathcal{H}$ is called *discrete wavelet system* generated by ψ, where $D_{a^{-j}}$ stands for the unitary dilation operator obtained by $a^j : \omega \to a^j.\omega$ and L_γ is the left translation operator with regard to γ. The discrete wavelet system $\{L_{a^{-j}\gamma} D_{a^{-j}} \psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ is called *(tight, normalized tight) wavelet frame* if it forms a (tight, normalized tight) frame.

Chapter 3

Wavelet Frames on the Heisenberg Group

3.1 Introduction

Multiresolution analysis (MRA) is an important mathematical tool because it provides a natural framework for understanding and constructing discrete wavelet systems.
In the present chapter, we shall consider the concrete example of building an MRA on the Heisenberg group $\mathbb{H}$.

Our main contributions are:
(i) For the Heisenberg group $\mathbb{H}$, we formulate the definition of a *frame multiresolution analysis (frame-MRA)* for $L^2(\mathbb{H})$, by adapting the notation of MRA of $L^2(\mathbb{R})$. There are three things in MRA that mainly concern us: the density of the union, the triviality of the intersection of the nested sequence of closed subspaces and the existence of refinable functions, i.e., the functions, which have an expansion in their scaling. The triviality of the intersection is derived from the other conditions of MRA. To get the density of the union, we have to generalize the concept of the *"support"* of the Fourier transform. The new concepts, such as "*bandlimited*" in $L^2(\mathbb{H})$, arise in this generalization. As to

refinability, it depends very much on the individual function ϕ, so-called scaling function. An example of a scaling function is presented below.

(ii) We provide a concrete example of frame-MRA on Heisenberg group (so-called Shannon-MRA), for which we prove the existence of wavelet functions. This wavelet function is related to a certain lattice of $\mathbb{H}$.

In Section 3.2 we introduce the (Whittaker-) Shannon sampling theorem for $L^2(\mathbb{R})$, which addresses the question: how can one reconstruct a function $f : \mathbb{R} \to \mathbb{C}$ from a countable set of function values $\{f(k)\}_{k\in\mathbb{Z}}$? Then we show that this can be done by requiring f to belong to a certain function space.

Then, we provide the definition of multiresolution analysis for $L^2(\mathbb{R})$ and show how the (Whittaker -) Shannon's sampling theorem relates to the (Shannon) multiresolution analysis for $L^2(\mathbb{R})$. We exhibit this fact by introducing the scaling function (sinc function) and arbitrary interpolation property of a special closed subspace of $L^2(\mathbb{R})$. This leads us to the definition of the Shannon multiresolution analysis for the space $L^2(\mathbb{H})$. In Section 3.3 we introduce the general MRA (**frame-MRA**) on $\mathbb{H}$, i.e., the concept of orthonormal basis will be replaced by frames. Then we present a concrete example of frame-MRA on $\mathbb{H}$, Shannon MRA, and hence we prove the existence of a scaling and wavelet function for the Heisenberg group.

Finally, we consider the existence of Shannon normalized tight frame on $\mathbb{H}$, i.e., existence of a *bandlimited* function on $\mathbb{H}$ such that its translations under an appropriate lattice in $\mathbb{H}$ and its dilations with respect to the integer powers of a suitable automorphism of $\mathbb{H}$ yields a normalized tight frame for $L^2(\mathbb{H})$.

3.2 Multiresolution analysis in $L^2(\mathbb{R})$

Before introducing the multiresolution analysis for $L^2(\mathbb{R})$, we present a known theorem, which provides our motivation for developing a similar kind of MRA on the Heisenberg group.

3.2.1 The (Whittaker-) Shannon Sampling Theorem for $L^2(\mathbb{R})$: A motivating Example

As mentioned before, we use the following convention for Fourier transform (in one dimension):

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

A signal f on $\mathbb{R}$ is called *band-limited* if its Fourier transform $\widehat{f}$ vanishes outside of some bounded interval, say $[-b, b]$. The smallest such b is then called the *bandwidth* of f. Since the frequency content of f is limited, f can be expected to vary slowly, its precise degree of slowness being governed by b: The smaller b is, the slower the variation. In turn, we expect that a slowly varying signal can be *interpolated* from a knowledge of its values at a discrete set of points, i.e., by *sampling.* The slower the variations, the less frequently the signal needs to be sampled. This is the intuition behind *Shannon's sampling theorem* (see [49]), which states that the interpolation can, in fact, be made *exact.* It uses the *sinc- function*, which is defined by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

with $\widehat{sinc} = \chi_{[-\pi,\pi]}$.

Theorem 3.1. *(Sampling Theorem) Assume that $f \in L^2(\mathbb{R})$ such that $\hat{f}$ has support in $[-b, b]$ for some $b > 0$, i.e, $\hat{f}(\xi) = 0$ for any $\mid \xi \mid > b$. Then f can be recovered pointwise from the samples $\{f(\frac{n\pi}{b})\}_{n\in\mathbb{Z}}$ via*

$$f(x) = \sum_{n\in b\mathbb{Z}} f(\frac{n\pi}{b})\frac{\sin(bx - n\pi)}{bx - n\pi}. \tag{3.1}$$

Proof: For simplicity we take $b = \pi$ and suppose $supp\hat{f} \subset [-\pi, \pi]$. The general case follows with a dilation argument. Since the set $\{\frac{1}{\sqrt{2\pi}}e^{-int}\}_{n\in\mathbb{Z}}$ is a complete orthonormal set in $L^2([-\pi,\pi))$, then $\hat{f}$ can be expanded in a Fourier series in the interval $[-\pi,\pi]$:

$$\hat{f}(\omega)=\sum_{n\in\mathbb{Z}} c_n e^{-in\omega}, \tag{3.2}$$

where

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)e^{in\omega}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{in\omega}d\omega=f(n).$$

That is, the Fourier coefficients in (3.2) are samples of f. (For more details see for example [9],[43].) Thus using inversion Fourier transform

$$f(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\omega)e^{ix\omega}d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)e^{ix\omega}d\omega \tag{3.3}$$

and with the exponential type assumption and substitute equation (3.2) into (3.3) we get:

$$f(x)=\sum_{n\in\mathbb{Z}} f(n)\frac{\sin\pi(x-n)}{\pi(x-n)}.$$

Now for any $b>0$ one can get:

$$f(x)=\sum_{n\in\mathbb{Z}} f(n\pi/b)\frac{\sin b(x-n\pi/b)}{b(x-n\pi/b)}. \tag{3.4}$$

Consider that the equality holds in the sense of $L^2(\mathbb{R})$. But since the Fourier series in (3.2) converges in $L^2([-b,b])$ and $L^2([-b,b])\subset L^1([-b,b])$, hence the series in (3.4) converges even uniformly, by the Riemann-Lebesgue theorem. $\square$

Observation: The Shannon sampling theorem shows that f is determined by the discrete set of values $\{f(\frac{n\pi}{b})\}_{n\in\mathbb{Z}}$. The set of $\{\frac{n\pi}{b}\}_{n\in\mathbb{Z}}$ is called *sampling lattice*. The statement is false without the exponential type assumption. This means that a band-limited signal can be recovered from its sample values with sampling density inversely related to the exponential type. This fact underlies digital processing of audio signals, which are assumed band-limited because our ears hear only a finite bandwidth.

In the following section, we will define a ladder of closed left shift-invariant subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$, and use V_j to approximate general functions in $L^2(\mathbb{R})$. The results yield a so-called MRA. Next the definition of a shift-invariant subspace:

Definition 3.2. *Suppose M is a subset of $\mathbb{R}$ and $\mathcal{H}$ is a subspace of $L^2(\mathbb{R})$. Then we say $\mathcal{H}$ is left shift-invariant under M if for any $m \in M$ is $L_m(\mathcal{H}) \subseteq \mathcal{H}$, i.e., $L_m f(.) = f(. - m) \in \mathcal{H}$ for any $f \in \mathcal{H}$.*

3.2.2 Definition of Multiresolution Analysis of $L^2(\mathbb{R})$

We start by first investigating the definition of multiresolution analysis of $L^2(\mathbb{R})$ and finding some key points in the definition by properly interpreting it within a more general context.

Definition 3.3. *(**Multiresolution Analysis**) A multiresolution analysis (MRA) of $L^2(\mathbb{R})$ consists of a sequence of closed linear subspaces $V_j,\ j \in \mathbb{Z}$, for $L^2(\mathbb{R})$ with the following properties:*

1. $V_j \subset V_{j+1},\ j \in \mathbb{Z}$,
2. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
4. $f \in V_j \Leftrightarrow f(2.) \in V_{j+1}$,
5. *V_0 is shift-invariant under $\mathbb{Z}$.*
6. *There is a function $\phi \in V_0$, called the scaling function or generator of the MRA, such that the collection $\{L_k\phi;\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_0.*

Because of property 4, the MRA is often referred to as a dyadic MRA.

Remarks 3.4. *(a) Observe that property 4 in Definition 3.3 implies that*

$$f \in V_j \Leftrightarrow f(2^{-j}.) \in V_0. \tag{3.5}$$

It follows that an MRA is essentially completely determined by the closed subspace V_0. But from property 6, V_0 is the closure of the linear span of the $\mathbb{Z}$-translations *of the scaling function ϕ. Thus the starting point of the construction of MRA is the existence of the scaling function ϕ. Therefore, it is especially important to give some conditions under which an initial function ϕ generates an MRA.*

(b) *Because of equation (3.5) this implies that if $f \in V_j$, then $f(2^{-j}. - n) \in V_0$ for all $n \in \mathbb{Z}$. Finally property 6 in Definition 3.3 and equation (3.5) implies that the system $\{L_{2^{-j}n}D_{2^{-j}}\phi\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_j for all $j \in \mathbb{Z}$, where $\forall n \in \mathbb{Z}\,\forall x \in \mathbb{R}$ is $L_{2^{-j}n}D_{2^{-j}}\phi(x) = 2^{j/2}\phi(2^j x - n)$*

(c) *The basic property of multiresolution analysis is that whenever a collection of closed subspaces satisfies properties 1-6 in Definition 3.3, then there exists an orthonormal wavelet basis $\{L_{2^{-j}n}D_{2^{-j}}\psi;\ j, n \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$, such that for all $f \in L^2(\mathbb{R})$*

$$P_{j+1}f = P_j f + \sum_{n\in\mathbb{Z}} \langle f, L_{2^{-j}n}D_{2^{-j}}\psi\rangle L_{2^{-j}n}D_{2^{-j}}\psi,$$

where P_j is the orthogonal projection of $L^2(\mathbb{R})$ onto V_j.

Notation: One should think of the V_j's as approximation subspaces that contain details up to a resolution 2^{-j}. The orthogonal projection $P_j f$ of f into V_j is an approximation of f that keeps details of size up to 2^{-j} and smaller details. The difference $P_{j+1}f - P_j f$ then contains the details that are added by refining the resolution from 2^{-j} to 2^{-j-1}. The subspace $W_j = (P_{j+1} - P_j)(L^2(\mathbb{R}))$ is the so-called detail space. $\lim\limits_{j\to\infty} P_j f = f$ by property 2 in Definition 3.3, and hence L^2 is the direct sum $L^2(\mathbb{R}) = \bigoplus\limits_{j\in\mathbb{Z}} W_j$.

For instance, the *"sinc basis"* is associated with the multiresolution analysis of band-limited functions, which we will consider in the following subsection.

3.2.3 Multiresolution Analysis hidden in the Shannon (Sinc) Bases for $L^2(\mathbb{R})$

In this section we try to illustrate the concept of MRA by analyzing the Shannon basis. For $L^2(\mathbb{R})$, we start at scale of 1 by considering ϕ. For $j \in \mathbb{Z}$, let

$$V_j := \{f \in L^2(\mathbb{R});\ supp(\hat{f}) \subset [-2^j\pi, 2^j\pi]\}.$$

It is easy to see that each V_j is a closed subspace of $L^2(\mathbb{R})$. By the definition of V_j's, the property 1 in Definition 3.3 is trivial. For property 6, observe that

$$\phi(x) = \frac{1}{\sqrt{2\pi}}(\chi_{[-\pi,\pi]})\check{}\,(x) = \frac{1}{\sqrt{2\pi}}\frac{\sin(\pi x)}{\pi x}.$$

Taking $g(\omega) = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}(\omega)$ and $g_n(\omega) = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}(\omega)e^{-in\omega}$, since $\{g_n\}_{n\in\mathbb{Z}}$ constitutes an ONB for $L^2([-\pi,\pi])$, then $\{L_n\phi\}_{n\in\mathbb{Z}}$ is an ONB for V_0. For property 2, suppose $f \in L^2(\mathbb{R})$ and let f_j be given by $\hat{f}_j(\omega) = \chi_{[-2^j\pi,2^j\pi]}(\omega)\hat{f}(\omega)$. Applying the dominated convergence theorem it is easy to see that $\| \hat{f}_j - \hat{f} \|_2 \to 0$ as $j \to \infty$ and hence applying Plancherel's formula we are done.

If $f \in V_0$, we can expand $\hat{f}$ in a Fourier series on $[-\pi,\pi)$ (see the proof of Theorem 3.1):

$$\hat{f}(\omega) = \sum_n c_n e^{-in\omega},$$

where $c_n = f(n)$. Note that for every $f \in L^2(\mathbb{R})$ one has

$$\widehat{P_j f} = \chi_{[-2^j\pi,2^j\pi]}(\omega)\hat{f}(\omega),$$

where P_j is the orthogonal projection onto V_j. In other word, for any $f \in L^2(\mathbb{R})$

$$P_j f = f * (2^j sinc(2^j.))\ , \qquad \text{where } \ sinc(x) = \frac{\sin(\pi x)}{\pi x} \in V_0.$$

One may prove that $V_0 = L^2(\mathbb{R}) * sinc$ and using the convolution theorem, for any $f \in V_0$ is $f * sinc = f$. As well for any $j \in \mathbb{Z}$, is $V_j = L^2(\mathbb{R}) * (2^j sinc(2^j.))$.

The properties 3 and 4 in Definition 3.3 follow directly from the definition of V_j's.

The space V_0 is the so-called **Paley-Wiener space** of $\mathbb{R}$. Note that V_0 is an example of a *sampling subspace* in $L^2(\mathbb{R})$, i.e., for any $f \in V_0$ we have:

(i) $\sum_n | f(n) |^2 = \| f \|^2$

(ii) $f(x) = \sum_n f(n)\ sinc(x-n)$ for $b=\pi$, with convergence both in L^2 sense and uniformly (see Theorem 3.1).
(The existence of sampling subspaces in general has been studied by Führ in [20] §2.6.)

(iii) The space V_0 has arbitrary interpolation, i.e, for any sequence $\{a_n\}_{n\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$ there exists a function f in V_0, so that $f(n) = a_n$ for all $n \in \mathbb{Z}$.

The construction of *Shannon wavelets* from Shannon MRA begins by considering the orthogonal complements of V_j in V_{j+1}, i.e. W_j. One can show that the property 4 in Definition 3.3 holds for closed shift-invariant subspaces $\{W_j\}$. Using property 2 in Definition 3.3 gives the orthogonal decomposition

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Observe that the translations of function $\psi = 2\phi(2.)-\phi$, $\{L_n\psi\}_{n\in\mathbb{Z}}$, provides an orthonormal basis for W_0. Therefore by orthogonal decomposition of $L^2(\mathbb{R})$ the set $\{L_{2^{-j}n}D_{2^j}\psi\}_{n,j}$ forms an orthonormal basis for $L^2(\mathbb{R})$, which is known as *Shannon (sinc) basis,* and the function ψ is called the *Shannon wavelet* in $L^2(\mathbb{R})$. We will prove the above expression in Theorem 3.5 below. But first note that the projection operators of $L^2(\mathbb{R})$ onto W_j are given by:

$$Q_j :\ f \to f * \psi_j,$$

where $\psi_0 := \psi$ and $\psi_j = 2^{j+1}\phi(2^{j+1}.) - 2^j\phi(2^j.)$. This fact can be seen immediately from definition of the projections P_j. Using it we can state the following theorem:

Theorem 3.5. *The wavelet system* $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}\times\mathbb{Z}}$ *constitute an ONB for* $L^2(\mathbb{R})$*, where* $\psi_{j,n}(x) = L_{2^{-j}n}D_{2^{-j}}\psi(x)$.

Proof: To prove our assertion, first we shall show that the translations of the function $\psi(x) := 2\phi(2.)-\phi \in W_0$, $\{L_n\psi\}_{n\in\mathbb{Z}}$, is an ONB of W_0. Suppose $f \in W_0$. Since $f = f*\psi*\psi$, then by the convolution theorem $\hat{f} = \widehat{(f*\psi)}\ \hat{\psi}$ and since $supp\hat{\psi} \subset \{\omega :\ \pi \leq |\omega| \leq 2\pi\}$ then

$$supp\widehat{(f*\psi)} \subset \{\omega :\ \pi \leq |\omega| \leq 2\pi\} \subset \{\omega :\ -2\pi \leq \omega \leq 2\pi\},$$

hence $\widehat{(f*\psi)} \in L^2([-2\pi, 2\pi])$. Using the relation (3.2) for $b = 2\pi$ the equality

$$\widehat{(f*\psi)}(\omega) = \frac{1}{2}\sum_n f*\psi(n)e^{-in\omega} = \frac{1}{2}\sum_n \langle f, \psi_n\rangle e^{-in\omega} \tag{3.6}$$

holds (note that here $\widetilde{\psi} = \psi$). Therefore using the relation (3.6) one has

$$\begin{aligned}\hat{f}(\omega) = \widehat{(f*\psi)}(\omega)\ \hat{\psi}(\omega) &= \frac{1}{2}\sum_n \langle f, \psi_n\rangle e^{-in\omega}\hat{\psi}(\omega)\\ &= \frac{1}{2}\sum_n \langle f, \psi_n\rangle \hat{\psi_n}(\omega),\end{aligned}$$

and hence the inverse Fourier transform implies

$$f(x) = \frac{1}{2}\sum_n \langle f, \psi_n\rangle \psi_n(x). \tag{3.7}$$

The relation in (3.7) shows that the members of W_0 have an expansion in $\{\psi_n\}$. Observe that for any n is $\hat{\psi}_n = \frac{1}{\sqrt{2\pi}}\chi_{[-2\pi,-\pi)\cup(\pi,2\pi]}e^{-in.}$. Therefore the orthogonality of $\{\psi_n\}_{n\in\mathbb{Z}}$ follows by using the Parseval theorem and orthogonality of $\{\frac{1}{\sqrt{2\pi}}\chi_{[-2\pi,-\pi)\cup(\pi,2\pi]}e^{-in.}\}_{n\in\mathbb{Z}}$ with $L^2([-2\pi,-\pi)\cup(\pi,2\pi])$-norm. Since $\parallel \psi \parallel = 1$ and $\{\psi_n\}_{n\in\mathbb{Z}}$ is complete then $\{\psi_n\}_{n\in\mathbb{Z}}$ constitutes an ONB for W_0.

Similarly for any $j \in \mathbb{Z}$, $\{\psi_{n,j}\}_{n\in\mathbb{Z}}$ is an ONB of W_j and any $f \in W_j$ can be represented as

$$f(x) = \sum_{n\in\mathbb{Z}} \langle f, \psi_{n,j}\rangle \psi_{n,j}(x).$$

Consequently using the orthogonal decomposition of $L^2(\mathbb{R})$ under subspaces W_j, for any $f \in L^2(\mathbb{R})$, the equality

$$f(x) = \sum_j Q_j(f) = const. \sum_j\sum_n \langle Q_j(f), \psi_{n,j}\rangle\psi_{n,j}(x) = \sum_j\sum_n \langle f, \psi_{n,j}\rangle\psi_{n,j}(x)$$

holds, as desired. $\square$

Note: This kind of orthonormal basis of $L^2(\mathbb{R})$ is known as *"Shannon (sinc) bases"* and hence the function ψ is called *Shannon (sinc) wavelet* in $L^2(\mathbb{R})$.

3.3 Construction of Shannon Multiresolution Analysis for the Heisenberg group

3.3.1 Introduction

Analogous to $\mathbb{R}$, wavelets in $L^2(\mathbb{H})$ are functions ψ with the properties that their appropriate translates and dilates defined with respect to the Lie structure of the Heisenberg group can be used to approximate any L^2-function . But here the special concept of multiresolution analysis needs to be appropriately adapted.

3.3.2 Definition of Frame Multiresolution Analysis for the Heisenberg group (frame-MRA)

By analogy to the above, we can adapt the definition of MRA for $L^2(\mathbb{R})$ to one for $L^2(\mathbb{H})$, replacing the concept of orthonormal basis by frames.

Since the triviality of the intersection is a direct consequence of the other conditions of the definition of an MRA, we prove this property immediately after we give the definition of an MRA.

We begin by properly interpreting the concept of MRA of $L^2(\mathbb{R})$. It is obvious that $\mathbb{Z}$ is a lattice subgroup of $\mathbb{R}$. The shift-invariance of V_0 in Definition 3.3 can be interpreted as an invariance property with respect to the action of the discrete lattice subgroup $\mathbb{Z}$ of $\mathbb{R}$. The scaling operator D can be viewed as the action of some group automorphism of $\mathbb{R}$, with the property $D\mathbb{Z} \subset \mathbb{Z}$.

With this in mind, it is not difficult to conjecture the correct generalization of MRA to Heisenberg group :

- First, a discrete subgroup Γ of $\mathbb{H}$ will play the same role in $\mathbb{H}$ as $\mathbb{Z}$ in $\mathbb{R}$. Γ is discrete means that the topology on Γ induced from $\mathbb{H}$ is the discrete topology.

- We are now dealing with a non-abelian group. Hence, there are two kinds of translations: left translation $L_{\mathbb{H}}$ and right translation $R_{\mathbb{H}}$. We choose left translation in accordance with the continuous wavelet transform.

Definition 3.6. *Suppose Ω is a subset of $\mathbb{H}$ and $\mathcal{H}$ is a subspace of $L^2(\mathbb{H})$. We say $\mathcal{H}$ is left shift-invariant under Ω, if for any $\omega \in \Omega$ we have $L_\omega \mathcal{H} \subseteq \mathcal{H}$.*

After this preparation, we can give a definition of **frame-MRA** for $L^2(\mathbb{H})$ related to an automorphism of $\mathbb{H}$ caused by $a > 0$(see (1.10)) and a lattice Γ in $\mathbb{H}$.

Definition 3.7. ***(frame-MRA)*** *We say that a sequence of closed subspace $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{H})$ forms a frame-MRA of $L^2(\mathbb{H})$, associated to an automorphism $a \in Aut(\mathbb{H})$ and a lattice Γ in $\mathbb{H}$, if the following conditions are satisfied:*

1. $V_j \subseteq V_{j+1} \quad \forall j \in \mathbb{Z}$,

2. $\overline{\bigcup V_j} = L^2(\mathbb{H})$,

3. $\bigcap V_j = \{0\}$,

4. $f \in V_j \Leftrightarrow f(a.) \in V_{j+1}$,

5. *V_0 is left shift-invariant under Γ, and consequently V_j is left shift-invariant under $a^{-j}.\Gamma$, and*

6. *there exist a function $\phi \in V_0$, called the scaling function, or generator of the $frame-$ MRA, such that the set $L_\Gamma(\phi)$ constitutes a normalized tight frame for V_0.*

Remarks 3.8. (a) *Here, for the scaling function we do not impose regularity and decay condition on ϕ. In our case to make the argument simple and general, we require only that $\phi \in L^2(\mathbb{H})$.*

(b) *In analogy with $L^2(\mathbb{R})$, we say V_0 is refinable if $D_{a^{-1}}(V_0) \subseteq V_0$. Thus the condition*

1 *in Definition 3.7 is equivalent to saying that* V_0 *is refinable. Thus, the basic question concerning* $frame - MRA$ *is whether the scaling function exists. We will see later that such scaling functions do exist. We will enter into details in Section 3.3.3 for a very special case .*
(c) *To have a sequence of nested closed subspaces, we must find a refinable function like* ϕ *in* V_0*. It is already known by Boor, DeVore and Ron in [5] for the real case that the refinability of* ϕ *is not enough to generate an* MRA*. Hence we need other requirements. We will consider this in detail later.*

3.3.3 Constructing of Shannon MRA for the Heisenberg Group

The approach via the solution of scaling equation, with methods of Lawton [32], leads to difficult analytical problems. Therefore we follow a new approach, which is based on the point of view of *Shannon multiresolution analysis* . This will allow us to derive the existence of a Shannon wavelet in $L^2(\mathbb{H})$.
The following example of $L^2(\mathbb{R})$, the canonical construction of wavelet bases starts with a multiresolution analysis $\{V_j\}_j$. In $L^2(\mathbb{R})$ one proves the existence of a wavelet $\psi \in W_0$, such that $\{L_k\psi,\ k \in \mathbb{Z}\}$ is an orthonormal basis for W_0.
Consequently the set $\{L_{2^{-j}k}D_{2^j}\psi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j. By the orthogonal decomposition $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$, the wavelet system $\{L_{2^{-j}k}D_{2^j}\psi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.
In this section, we shall construct on $\mathbb{H}$ an analog of Shannon-MRA from Section 3.2.3. In contrast of the case $\mathbb{R}$, the construction of the scaling function is not our starting point for obtaining of a frame-MRA, but first we intend to construct a special function which implies the existence of the scaling function in some closed subspace of $L^2(\mathbb{H})$. Furthermore for the constructing, we shall consider the automorphism $a = 2$ of $\mathbb{H}$ which is given by:

$$a.(x, y, t) = (2x, 2y, 2^2t) \qquad \forall (x, y, t) \in \mathbb{H}.$$

As remarked before, we shall construct a function in some closed and shift-invariant subspace of $L^2(\mathbb{H})$, such that its translations and dilations yields a normalized tight frame of $L^2(\mathbb{H})$. We will do it by analogy to Section 3.2.3. For this reason, first we choose the dilation operator $D_a = D_2$ and try to associate a space V_0 which has similar properties as the Paley-Wiener space on $\mathbb{R}$, with the aim of constructing a Shannon multiresolution analysis similar to that of $\mathbb{R}$. We start with the definition of a bandlimited function on $\mathbb{H}$:

Definition 3.9. *Suppose $\mathcal{I}$ is some bounded subset of $\mathbb{R}^*$ and S is a function in $L^2(\mathbb{H})$. We say S is $\mathcal{I}$-bandlimited if $\widehat{S}(\lambda) = 0$ for all $\lambda \notin \mathcal{I}$.*

We will need the following definition in the next theorem:

Definition 3.10. *A function S in $L^2(\mathbb{H})$ is called selfadjoint convolution idempotent if $S = \tilde{S} = S * S$.*

Convolution idempotents in L^1 have been studied for instance in [26]. For properties of selfadjoint convolution idempotents in L^2 we refer the reader to [20], §2.5.

Theorem 3.11. *Let $d \in \mathbb{N}$. There exists a selfadjoint convolution idempotent function S in $L^2(\mathbb{H})$ which is $\mathcal{I}$-bandlimited for $\mathcal{I} = \left[-\frac{\pi}{2d}, \frac{\pi}{2d}\right] - \{0\}$. Define $S_j = 2^{4j} S(2^j.)$ for $j \in \mathbb{Z}$. Then S_j is $\mathcal{I}_j$-bandlimited for $\mathcal{I}_j = \left[-\frac{4^j\pi}{2d}, \frac{4^j\pi}{2d}\right] - \{0\}$ in $\mathbb{R}^*$ and the following consequences hold:*

a) $S * S_j = S \ \ \forall j > 0$ and $S_j * S = S_j \ \ \forall j < 0$,

b) $f * S_j \to 0$ in L^2-norm as $j \to -\infty \ \forall f \in L^2(\mathbb{H})$,

c) $f * S_j \to f$ in $L^2 -$ norm as $j \to \infty \ \forall f \in L^2(\mathbb{H})$ and

d) $S_j = \widetilde{S_j} = S_j * S_j$.

Proof: Let $\mathcal{I}_0 := \mathcal{I}_0$. We intend to show that there exists a function S which is $\mathcal{I}_0$-bandlimited and satisfies the assertion of the theorem. We start from the Plancherel side, i.e, construction of Hilbert-Schmidt operators $\hat{S}(\lambda)$ with associated to $\lambda \in \mathbb{R}^*$. For this purpose we choose an orthonormal basis $\{e_i\}_{i\in\mathbb{N}_0}$ in $L^2(\mathbb{R})$. For any $\lambda \neq 0$ define $e_i^\lambda = D_{|\lambda|^{-1/2}} e_i$. Observe that for any λ, $\{e_i^\lambda\}_i$ is an ONB of $L^2(\mathbb{R})$ since the dilation operators $D_{|\lambda|^{-1/2}}$ are unitary. Therefore $\{\{e_i^\lambda\}_i\}_\lambda$ is a measurable family of orthonormal bases in $L^2(\mathbb{R})$. (For instance one can take the orthonormal basis of $\{\phi_n\}_{n\in\mathbb{N}_0}$ in $L^2(\mathbb{R})$, where ϕ_n are Hermite functions, and for any $\lambda \neq 0$, ϕ_n^λ are given by $\phi_n^\lambda(x) = D_{|\lambda|^{-1/2}}\phi_n = |\lambda|^{\frac{1}{4}}\, \phi_n(\sqrt{|\,\lambda\,|}x)$ for all $x \in \mathbb{R}$. For more details about the Hermite functions see for example [52] or Chapter 4 of the work.)

Let $\lambda \neq 0$ such that $\lambda \in \mathcal{I}_0$. Define $\widehat{S}(\lambda)$ as follow:

$$\widehat{S}(\lambda) = \begin{cases} \sum_{i=0}^{4^k} \left(e_i^{\frac{\lambda}{2\pi}} \otimes e_i^{\frac{\lambda}{2\pi}}\right) & \text{if } \frac{2\pi}{4^{(k+2)}d} < |\lambda| \leq \frac{2\pi}{4^{k+1}d} \text{ for some } k \in \mathbb{N}_0, \\ 0 & |\lambda| > \frac{\pi}{2d}. \end{cases}$$

Therefore for any λ, where $\frac{2\pi}{4^{(k+2)}d} < |\lambda| \leq \frac{2\pi}{4^{k+1}d}$, the operator $\widehat{S}(\lambda)$ is a projection operator on the first $4^k + 1$ elements of the orthonormal basis $\{e_i^{\frac{\lambda}{2\pi}}\}_{i\in\mathbb{N}_0}$, where $e_i^{\frac{\lambda}{2\pi}} = D_{|\frac{\lambda}{2\pi}|^{-1/2}} e_i$. The definition of $\widehat{S}$ entails the following consequences:

(i) $\forall\lambda \ \| \widehat{S}(\lambda) \|_{H.S}^2 = 4^k + 1$ where $k \geq 0$ and satisfies $\frac{2\pi}{4^{(k+2)}d} < |\lambda| \leq \frac{2\pi}{4^{k+1}d}$

(ii) $\int_{|\lambda|\leq\frac{\pi}{2d}} \| \widehat{S}(\lambda) \|_{H.S}^2 \, d\mu(\lambda) = \sum_{k=0} \int_{\frac{2\pi}{4^{(k+2)}d}<|\lambda|\leq\frac{2\pi}{4^{k+1}d}} (4^j + 1) d\mu(\lambda) < \infty$,
where $d\mu(\lambda) = (2\pi)^{-2}\, |\lambda|\, d\lambda$, *and*

(iii) $\widehat{S}(\lambda) = \widehat{S}(\lambda)^* = \widehat{S}(\lambda) \circ \widehat{S}(\lambda)$, $\forall\lambda \neq 0$.

Observe that (ii) implies this point that the vector field $\{\widehat{S}(\lambda)\}_\lambda$ on $\mathbb{R}^*$ is contained in $\int_{\mathbb{R}^*}^{\oplus} L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) d\mu(\lambda)$ (for the definition of the direct integral see Section 2.4) and hence since the Plancherel theorem is surjective then $\widehat{S}$ has a preimage S in $L^2(\mathbb{R})$ with the Plancherel transform $\widehat{S}$, given as above. The property (iii) implies that S is selfadjoint convolution idempotent by the convolution theorem.

Suppose $j \in \mathbb{Z}$ and $S_j := 2^{4j} S(2^j.)$. Using the equivalence of representations ρ_λ and $\rho_{2^{-2j}\lambda}$ and the relation (2.15) we obtain

$$\widehat{S_j}(\lambda) = D_{2^{-j}} \widehat{S}(4^{-j}\lambda) D_{2^j}. \tag{3.8}$$

Therefore from the definition of S_j, $\widehat{S}_j(\lambda) = 0$ for any $|\lambda| > \frac{4^j\pi}{2d}$ (note that $D^*_{2^j} = D_{2^{-j}}$). Hence the S_j is $\mathcal{I}_j$-bandlimited, where $\mathcal{I}_j = [-\frac{4^j\pi}{2d}, 0)\bigcup(0, \frac{4^j\pi}{2d}]$. According to the consequence of (iii) , the relation (3.8) shows that S_j is selfadjoint and convolution idempotent, hence $d)$ is proved.

To prove $a)$, suppose $j > 0$ and $\lambda \in \mathcal{I}_j$. Then $4^{-j}\lambda \in \mathcal{I}_0$, hence there exists a non-negative integer k_j such that

$$\frac{2\pi}{4^{(k_j+2)}d} < \left|4^{-j}\lambda\right| \leq \frac{2\pi}{4^{(k_j+1)}d},$$

or equivalently

$$\frac{2\pi}{4^{(k_j-j+2)}d} < |\lambda| \leq \frac{2\pi}{4^{(k_j-j+1)}d}.$$

For the case $k_j < j$, observe that $\widehat{S}(\lambda) = 0$. For the case $k_j \geq j$, from the definition of $\widehat{S}$ we have the followings:

$$\widehat{S}(\lambda) = \sum_{i=0}^{4^{k_j-j}} e_i^{\frac{\lambda}{2\pi}} \otimes e_i^{\frac{\lambda}{2\pi}} \quad and$$

$$\widehat{S}(4^{-j}\lambda) = \sum_{i=0}^{4^{k_j}} e_i^{4^{-j}(\frac{\lambda}{2\pi})} \otimes e_i^{4^{-j}(\frac{\lambda}{2\pi})}. \tag{3.9}$$

Recall that, from the definition of the family of orthonormal bases $\{e_i^\lambda\}_i$, $e_i^{4^{-j}(\frac{\lambda}{2\pi})}$ can be read as below:

$$e_i^{4^{-j}(\frac{\lambda}{2\pi})} = D_{\left|4^{-j}\frac{\lambda}{2\pi}\right|^{-1/2}} e_i = D_{2^j}\left(D_{\left|\frac{\lambda}{2\pi}\right|^{-1/2}} e_i\right) = D_{2^j} e_i^{\frac{\lambda}{2\pi}}, \tag{3.10}$$

replacing (3.10) into (3.9) we get

$$\widehat{S}(4^{-j}\lambda) = \sum_{i=0}^{4^{k_j}} D_{2^j} e_i^{\frac{\lambda}{2\pi}} \otimes D_{2^j} e_i^{\frac{\lambda}{2\pi}},$$

and hence

$$\widehat{S}_j(\lambda) = D_{2^{-j}}\widehat{S}(4^{-j}\lambda)D_{2^j} = \sum_{i=0}^{4^{k_j}} e_i^{\frac{\lambda}{2\pi}} \otimes e_i^{\frac{\lambda}{2\pi}}. \tag{3.11}$$

Observe that for any $\lambda \in \mathcal{I}_j$, the operator $\widehat{S}(\lambda)$ is a projection on the first $4^{k_j-j}+1$ elements of orthonormal basis $\{e_i^{\frac{\lambda}{2\pi}}\}$ for some suitable $k_j \geq j$, whereas $\widehat{S}_j(\lambda)$ is a projection on the first $4^{k_j}+1$ elements of the same orthonormal basis. Hence we get

$$\widehat{S}(\lambda)\circ\widehat{S}_j(\lambda)=\widehat{S}_j(\lambda)\circ\widehat{S}(\lambda)=\sum_{i=0}^{4^{k_j-j}} e_i^{\frac{\lambda}{2\pi}}\otimes e_i^{\frac{\lambda}{2\pi}}=\widehat{S}(\lambda), \tag{3.12}$$

which is a projection on the first $4^{k_j-j}+1$ elements of the orthonormal basis $\{e_i^{\frac{\lambda}{2\pi}}\}$. For fixed $j>0$ since the relation (3.12) holds for any $\lambda\in\mathcal{I}_j$, then by applying the convolution and the Plancherel theorem respectively we obtain $S*S_j=S$.

Likewise for $j<0$, suppose $\lambda\in\mathcal{I}_j$. Then for some $k_j\in\mathbb{N}_0$ is $\frac{2\pi}{4^{(k_j+2)}d}<|4^{-j}\lambda|\leq\frac{2\pi}{4^{(k_j+1)}d}$. Analogous to the previous case, the operator $\widehat{S}(\lambda)$ is a projection on the first $4^{k_j-j}+1$ elements of orthonormal basis $\{e_i^{\frac{\lambda}{2\pi}}\}$ and $\widehat{S}_j(\lambda)$ is a projection on the first $4^{k_j}+1$ elements of the same orthonormal basis. Thus

$$\widehat{S}(\lambda)\circ\widehat{S}_j(\lambda)=\widehat{S}_j(\lambda)\circ\widehat{S}(\lambda)=\sum_{i=0}^{4^{k_j}} e_i^{\frac{\lambda}{2\pi}}\otimes e_i^{\frac{\lambda}{2\pi}}=\widehat{S}_j(\lambda). \tag{3.13}$$

Once again applying convolution and Plancherel theorems in the relation (3.13) yields $S*S_j=S_j$ and hence $a)$ is proved.

To prove $b)$, Suppose $j\in\mathbb{Z}$ and $f\in L^2(\mathbb{H})$. Then from the structure and properties of function S, $f*S_j\in L^2(\mathbb{H})$. Before starting to give the proof of this part, observe that because of the consequence in (iii), for any $\lambda\neq 0$ the operator $\widehat{S}(\lambda)$ is bounded and has operator norm less than 1. Hence for any $j\in\mathbb{Z}$ and $\lambda\neq 0$ is $\|\ \widehat{S}_j(\lambda)\ \|_\infty=\|\ D_{2^{-j}}\widehat{S}(4^{-j}\lambda)D_{2^j}\ \|_\infty\leq 1$. Using the inequality and applying the Plancherel and convolution theorems respectively we get the followings:

$$\begin{aligned}
\|\ f*S_j\ \|_2^2=\|\ \widehat{(f*S_j)}\ \|_{H.S}^2 &= \int_{\mathbb{R}^*}\|\ \widehat{(f*S_j)}(\lambda)\ \|_2^2\, d\mu(\lambda) &(3.14)\\
&= \int_{0<|4^{-j}\lambda|\leq\frac{\pi}{2d}}\|\ \widehat{f}(\lambda)\circ\widehat{S}_j(\lambda)\ \|_{H.S}^2\, d\mu(\lambda)\\
&\leq \int_{0<|4^{-j}\lambda|\leq\frac{\pi}{2d}}\|\ \widehat{f}(\lambda)\ \|_{H.S}^2\|\ \widehat{S}_j(\lambda)\ \|_\infty^2\, d\mu(\lambda)\\
&\leq \int_{0<|4^{-j}\lambda|\leq\frac{\pi}{2d}}\|\ \widehat{f}(\lambda)\ \|_{H.S}^2\, d\mu(\lambda)\\
&= \int_{\mathbb{R}^*}\|\ \widehat{f}(\lambda)\ \|_{H.S}^2\,\chi_{[-\frac{4^j\pi}{2d},0)\cup(0,\frac{4^j\pi}{2d}]}(\lambda)d\mu(\lambda), &(3.15)
\end{aligned}$$

where χ denotes the characteristic function and $d\mu(\lambda) = (2\pi)^{-2} |\lambda| \, d\lambda$. If we take the limit of the right hand side in (3.15), since $\int_\lambda \parallel \widehat{f}(\lambda) \parallel^2_{H.S} d\mu(\lambda) < \infty$, then by the dominated convergence theorem we have the permission to pass over the limit into the integral and hence

$$\begin{aligned}&\lim_{j \to -\infty} \int_{\mathbb{R}^*} \parallel \widehat{f}(\lambda) \parallel^2_{H.S} \chi_{[-\frac{4^j\pi}{2d},0)\cup(0,\frac{4^j\pi}{2d}]}(\lambda) d\mu(\lambda) \\ &= \int_{\mathbb{R}^*} \parallel \widehat{f}(\lambda) \parallel^2_{H.S} \lim_{j \to -\infty} \chi_{[-\frac{4^j\pi}{2d},0)\cup(0,\frac{4^j\pi}{2d}]}(\lambda) d\mu(\lambda) = 0,\end{aligned}$$

The latter implies that the limit of the left hand side in the relation (3.14) is also zero as $j \to -\infty$, i.e, $\lim\limits_{j \to -\infty} \parallel f * S_j \parallel_2 = 0$, which proves b).

To prove c) suppose f is in $L^2(\mathbb{H})$. Recall that for any fixed λ, $\{e_i^{\frac{\lambda}{2\pi}}\}_{i=0}^{\infty}$ constitutes an ONB for $L^2(\mathbb{R})$. Therefore the identity operator I on $L^2(\mathbb{R})$ can be read as

$$I = \sum_{i=0}^{\infty} e_i^{\frac{\lambda}{2\pi}} \otimes e_i^{\frac{\lambda}{2\pi}}$$

and hence the operator $\hat{f}(\lambda)$ can be represented as

$$\widehat{f}(\lambda) = \sum_{i=0}^{\infty} \left(\widehat{f}(\lambda) e_i^{\frac{\lambda}{2\pi}} \right) \otimes e_i^{\frac{\lambda}{2\pi}}. \tag{3.16}$$

Therefore for any $j \in \mathbb{Z}$, according to the representation of $\widehat{f}(\lambda)$ in (3.16) and the representation of operator $D_{2^{-j}} \widehat{S}(4^{-j}\lambda) D_{2^j}$ in (3.11), for some $k_j \geq j$ we obtain the followings:

$$\begin{aligned}&\|\widehat{f}(\lambda) \circ D_{2^{-j}} \widehat{S}(4^{-j}\lambda) D_{2^j} - \widehat{f}(\lambda) \parallel^2_{H.S} \\ &= \parallel \sum_{i=4^{k_j}+1}^{\infty} \left(\widehat{f}(\lambda) e_i^{\frac{\lambda}{2\pi}} \right) \otimes e_i^{\frac{\lambda}{2\pi}} \parallel^2_{H.S} \\ &= \sum_{i=4^{k_j}+1}^{\infty} \left\| \widehat{f}(\lambda) e_i^{\frac{\lambda}{2\pi}} \right\|^2_2.\end{aligned} \tag{3.17}$$

Letting $j \to \infty$ (hence $k_j \to \infty$), the right hand side of (3.17) goes to zero. From the other side using the Plancherel theorem we have

$$\begin{aligned}\parallel f * S_j - f \parallel^2_2 &= \int_{\mathbb{R}^*} \parallel \widehat{f}(\lambda) \circ D_{2^{-j}} \widehat{S}(4^{-j}\lambda) D_{2^j} - \widehat{f}(\lambda) \parallel^2_{H.S} d\mu(\lambda) \qquad (3.18) \\ &= \int_{\mathbb{R}^*} \sum_{i=4^{k_j}+1}^{\infty} \left\| \widehat{f}(\lambda) e_i^{\frac{\lambda}{2\pi}} \right\|^2_2 d\mu(\lambda).\end{aligned}$$

In the same way as in the argument for b), using the dominated convergence theorem for the relation (3.18) one gets:

$$\lim_{j\to\infty} \| f * S_j - f \|_2^2 = \int \lim_{j\to\infty} \| \widehat{f}(\lambda) \circ D_{2^{-j}} \widehat{S}(4^{-j}\lambda) D_{2^j} - \widehat{f}(\lambda) \|_{H.S}^2 \, d\mu(\lambda)$$
$$= \int \lim_{j\to\infty} \sum_{i=4^{(k_j+j)}+1}^{\infty} \left\| \widehat{f}(\lambda) e_i^{\frac{\lambda}{2\pi}} \right\|_2^2 d\mu(\lambda) = 0,$$

as desired, which completes the proof of the theorem. □

The next step to approach to the construction of an MRA via the function S is that we start with the definition of a closed left invariant subspace of $L^2(\mathbb{H})$, V_0. Define $V_0 = L^2(\mathbb{H}) * S$. It is clear that V_0 is closed, and with the following additional properties:

1. V_0 is contained in the set of all bounded and continuous functions in $L^2(\mathbb{H})$. Hence V_0 is a proper subspace of $L^2(\mathbb{H})$. The boundedness of elements is easy to see from the definition of convolution operator and Cauchy-Schwartz inequality:

$$| g * S(x) | \leq \| f \|_2 \| S \|_2 \quad \forall x \in \mathbb{H} \quad g \in L^2(\mathbb{R}),$$

2. Since S is convolution idempotent then for any $f \in V_0$ is $f * S = f$,

3. Suppose Γ is any lattice in $\mathbb{H}$. Then $L_\gamma(g * S) = L_\gamma g * S$ which shows V_0 is left shift-invariant under Γ.

4. For any $g \in L^2(\mathbb{H})$ and $j \in \mathbb{Z}$ is $D_{2^j}(g * S) = D_{2^j} g * D_{2^j} S$.

Note: Observe that not every space $L^2(\mathbb{H}) * S$ with $S = \tilde{S} = S * S$ owns a normalized tight frame of the form $\{L_\gamma \phi\}_\gamma$ for some $\phi \in L^2(\mathbb{H}) * S$. As will be seen later, this depends on the *multiplicity function* associated to S, see Definition 3.13 and Theorem 3.14. For more details see [20] Corollary 6.8 and Theorem 6.4.

Recall that $L_{2^{-j}\gamma} D_{2^{-j}} S(\omega) = 2^{j/2} S(\gamma^{-1}(2^j.\omega)) \quad \forall j \in \mathbb{Z}\ , \gamma \in \Gamma,\ x \in \mathbb{H}$. Next define $V_1 = L^2(\mathbb{H}) * (2^4 S(2.))$. As well, V_1 is left invariant under $2^{-1}\Gamma$ and closed subspace of $L^2(\mathbb{H})$. The functions in V_1 are continuous bounded function and from (3.8) are $\mathcal{I}_1$-bandlimited. With regard to the consequence $a)$ of Theorem 3.11, for any $f \in V_0$ we have

$$f = f * S = f * (S * 2^4 S(2.)) = (f * S) * (2^4 S(2.)).$$

The latter shows that the conclusion $V_0 \subseteq V_1$ holds.
By continuing in this manner, we define $V_2 = L^2(\mathbb{H}) * (2^8 S(2^2.))$ to be the closed subspace of functions which are $\mathcal{I}_2$-bandlimited. Obviously with the similar argument as above one can easily prove that $V_1 \subseteq V_2$.
Similarly, one can define subspaces $V_3 \subseteq V_4 \subseteq \cdots$. On the other hand one may define negatively indexed subspaces. For example, we define $V_{-1} = L^2(\mathbb{H}) * 2^{-4} S(2^{-1}.)$. This space contains the functions which are $\mathcal{I}_{-1}$-bandlimited and obviously $V_{-1} \subseteq V_0$. Again, one may continue in this way to construct the sequence of closed and left $(2^{-j}\Gamma)$-shift-invariant subspaces of $L^2(\mathbb{H})$:

$$\{0\} \subseteq \cdots V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq L^2(\mathbb{H}), \tag{3.19}$$

which are scaled versions of the central space V_0. Our next aim is to show that, in the sense of Definition 3.7, the sequence of closed subspaces $\{V_j\}$ forms a frame-MRA of $L^2(\mathbb{H})$. For this reason we must show that the all properties $1 - 6$ in Definition 3.7 hold for the sequence $\{V_j\}$. But (3.19) shows that V_j's satisfy the first property (the nested property). For the other properties we state the next remark:

Remarks 3.12. *(1) To show the density of $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{H})$, i.e., $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{H})$, suppose P_j denotes the projection operator of $L^2(\mathbb{H})$ onto V_j. Then P_j is given by*

$$P_j: \ f \to f * 2^{4j} S(2^j.). \tag{3.20}$$

Therefore the density of $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{H})$ is equivalent to say that for any $f \in L^2(\mathbb{H})$ and j with $P_j f = 0 \ \forall j \in \mathbb{Z}$ is $f = 0$. It follows directly from Theorem 3.11, c). More precisely

$$0 = P_j f = f * S_j \to f \ as \ j \to \infty$$

which implies $f = 0$.

(2) *For triviality of intersection, observe that for any $f \in V_j$ is $f * 2^{4j}S(2^j.) = f$. Therefore for any $f \in \bigcap V_j$ is $f * 2^{4j}S(2^j.) = f$ for all j. Therefore by b) in Theorem 3.11, it implies that $f = 0$ as desired.*

(3) *Property 4 in Definition 3.7 is clear from the construction of V_j's. This property enables us to pass up and down among the spaces V_j by scaling*

$$f \in V_j \Longleftrightarrow f(2^{k-j}.) \in V_k.$$

(4) *Generally when V_0 is left shift-invariant under some lattice Γ, then the spaces V_j are shift-invariant under $(2^{-j}\Gamma)$. We will return to this fact later and will show how one can choose an appropriate lattice Γ such that it allows the construction of a wavelet frame on $\mathbb{H}$.*

(5) *Observe that, by contrast to the Shannon multiresolution analysis on $\mathbb{R}$, condition 6 in Definition 3.7 requires the existence of some frame generator ϕ, not necessarily $\phi = S$. This is due to the fact that we did not suppose any other conditions for the selection of orthonormal basis $\{e_i^\lambda\}_i$ for the constructions of Hilbert-Schmidt operators $\widehat{S}(\lambda)$ respectively S. This is one of the disagreement between our defined sinc-MRA of $L^2(\mathbb{H})$ and the sinc-MRA of $L^2(\mathbb{R})$. In the case $\mathbb{R}$ the sinc function by which the subspaces V_j's are defined, generates an ONB for V_0 and hence for all V_j, under some other suitable discrete subgroups of $\mathbb{R}$. In our case on the Heisenberg group we will show the existence of a function ϕ in V_0 such that its left translations under a suitable Γ forms a normalized tight frame for V_0 and hence for all V_j under $2^{-j}\Gamma$.*

As we briefly mentioned in Remark 3.12, we shall show the existence of a function ϕ in V_0 with which the property 6 in Definition 3.7 holds for V_0. We will observe below that this fact strongly depends on the structure of S and definition of V_0.

To achieve this goal, we recall one definition and one theorem of Führ's book [20] below. They can be found in Section 6.2 of this book.

Definition 3.13. *Suppose $\mathcal{H}$ be a left-invariant subspace of $L^2(\mathbb{H})$ and P be the projection operator of $L^2(\mathbb{H})$ onto $\mathcal{H}$. There exists a unique associated projection field $(\widehat{P}_\lambda)_\lambda$ satisfying*

$$\widehat{P(f)}(\lambda) = \hat{f}(\lambda) \circ \widehat{P}_\lambda \quad \forall\, f \in L^2(\mathbb{H}).$$

The associated multiplicity function $m_{\mathcal{H}}$ is then defined by

$$m_{\mathcal{H}} : \ \mathbb{R}^* \to \mathbb{N} \cup \{\infty\}; \quad m_{\mathcal{H}}(\lambda) = rank(\widehat{P}_\lambda).$$

$\mathcal{H}$ is called bandlimited if the support of its associated multiplicity function $m_{\mathcal{H}}$, $\Sigma(\mathcal{H})$, is bounded in $\mathbb{R}^*$.

The next theorem provides a characterization of closed left shift-invariant subspaces of $L^2(\mathbb{H})$ which admit a tight frame. But before we state this theorem we need to introduce two numbers associated to lattice Γ. The number $d(\Gamma)$ refer to a positive integer number d for which $\alpha(\Gamma_d) = \Gamma$ for some $\alpha \in Aut(\mathbb{H})$, where Γ_d is a lattice in $\mathbb{H}$ and is defined by

$$\Gamma_d := \big\{(m, dk, l + \frac{1}{2}dmk) : \ m, k, l \in \mathbb{Z}\big\}. \tag{3.21}$$

It is easy to check that Γ_d forms a group under the group operation (2.2) (this kind of lattices are presented in [20], §6.1). Observe that due to Theorem 6.2 in [20], such strictly positive number d exists and is uniquely determined. As well, we define $r(\Gamma)$ be the unique positive real satisfying

$$\Gamma \cap Z(\mathbb{H}) = \{(0, 0, r(\Gamma)k); \ k \in \mathbb{Z}\},$$

where $Z(\mathbb{H})$ denotes the center of $\mathbb{H}$, $Z(\mathbb{H}) = \{0\} \times \{0\} \times \mathbb{R} \subset \mathbb{H}$. With above notations we state Theorem 6.4 from [20].

Theorem 3.14. *Suppose $\mathcal{H}$ is a left-invariant subspace of $L^2(\mathbb{H})$ and $m_{\mathcal{H}}$ is its associated multiplicity function. Then there exists a tight frame (hence normalized tight frame) of* *the form $\{L_\gamma\phi\}_{\gamma\in\Gamma}$ with an appropriate $\phi \in \mathcal{H}$ if and only if the inequality*

$$m_{\mathcal{H}}(2\pi\lambda)\,|2\pi\lambda| + m_{\mathcal{H}}\left(2\pi\lambda - \frac{1}{r(\Gamma)}\right)\left|2\pi\lambda - \frac{1}{r(\Gamma)}\right| \leq \frac{1}{d(\Gamma)r(\Gamma)} \tag{3.22}$$

holds for $m_{\mathcal{H}}$ almost everywhere. From the inequality (3.22) it can be read that the support of $m_{\mathcal{H}}$ is bounded and in particular it is contained in the interval $\left[-\frac{1}{d(\Gamma)r(\Gamma)}, \frac{1}{d(\Gamma)r(\Gamma)}\right]$ up to a set of measure zero. Therefore $\mathcal{H}$ is bandlimited.

Remark 3.15. *Note that Theorem 6.4 in [20] refers to different realizations of the Schrödinger representations, hence we have the additional factor 2π in the relation (3.22).*

This theorem as a main tool enables us to show the existence of a function ϕ in V_0 which provides a tight frame for V_0. Therefore as a consequence we have:

Theorem 3.16. *There exists a normalized tight frame of the form $\{L_\gamma\phi\}_{\gamma\in\Gamma}$ for an appropriate $\phi \in V_0$ and a suitable lattice Γ in $\mathbb{H}$.*

Proof: For our purpose we pick a lattice with $r(\Gamma) = \frac{1}{2\pi}$ and $d(\Gamma) = d$. (Observe that it is possible due to Theorem 6.2 in [20] to select a lattice with the desired associated numbers r and d.) From the definition of V_0, $\{\widehat{S}(\lambda)\}_{\lambda\in\mathbb{R}^*}$ is the associated projection field of V_0 with the multiplicity function m_{V_0} which is given by

$$m_{V_0}(2\pi\lambda) = rank(\widehat{S}(2\pi\lambda)) = \begin{cases} 4^k + 1 & if \;\; \frac{1}{4^{(k+2)}d} \leq |\lambda| \leq \frac{1}{4^{k+1}d} \text{ for some } k \in \mathbb{N}_0 \\ 0 & \quad elsewhere. \end{cases}$$

One can easily prove that the inequality in (3.22) holds for m_{V_0}. By the construction of S in Theorem 3.11, $\hat{S}(\lambda) = 0$ for any $|\lambda| > \frac{\pi/2}{d}$ which provides:

$$\Sigma(m_{V_0}) \subset \left[-\frac{\pi/2}{d}, \frac{\pi/2}{d}\right] \subset \left[-\frac{2\pi}{d}, \frac{2\pi}{d}\right] = \left[-\frac{1}{d(\Gamma)r(\Gamma)}, \frac{1}{d(\Gamma)r(\Gamma)}\right]$$

Therefore the all conditions of Theorem 3.14 hold for V_0. Hence there exists a function ϕ, so-called scaling function, such that for our selected lattice Γ, $L_\Gamma\phi$ forms a normalized tight frame for V_0. From there the property 6 of Definition 3.7 is satisfied. $\square$

Corollary 3.17. *For any $j \in \mathbb{Z}$, $\{L_{2^{-j}\gamma} D_{2^{-j}}\phi\}_\gamma$ constitutes a normalized tight frame of V_j.*

As already mentioned, the MRA for the L^2 space is considered with the aim to find an associated discrete wavelet system in L^2. More precisely, we want to construct a discrete wavelet system for $L^2(\mathbb{H})$ which is a normalized tight frame. We will study our statement in details in the next Section by considering a "scaling function" ϕ in V_0.

3.3.4 Existence of Shannon n.t Wavelet Frame for the Heisenberg group

It is natural to try to obtain one normalized tight frame (n.t frame) for $L^2(\mathbb{H})$ by combining all the n.t frame $\{L_{2^{-j}\gamma} D_{2^{-j}}\phi\}_{\gamma\in\Gamma}$ of V_js. But although $V_j \subseteq V_{j+1}$, the n.t frame for V_j is not contained in the n.t frame $\{L_{2^{-(j+1)}\gamma} D_{2^{-(j+1)}}\phi\}_{\gamma\in\Gamma}$ of V_{j+1}. Therefore it shows that the union of all n.t frames for V_is does not constitute a n.t frame for $L^2(\mathbb{H})$.
To find an n.t frame for $L^2(\mathbb{H})$, we use the following (general) method. For every $j \in \mathbb{Z}$, use W_j to denote the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1} = V_j \oplus W_j$, where the symbol $\oplus$ stands for orthogonal closed subspace. Suppose Q_j denotes the orthogonal projection of $L^2(\mathbb{H})$ onto W_j. Then $P_{j+1} = P_j + Q_j$ and apparently is:

$$V_j = \bigoplus_{k\leq j-1} W_k.$$

The most important thing remaining unchanged is that, the spaces W_j, $j \in \mathbb{Z}$, still keep the scaling property from V_j:

$$f \in W_j \Longleftrightarrow f(2^{k-j}.) \in W_k. \tag{3.23}$$

Consequently we get

$$L^2(\mathbb{H}) = \bigoplus_{j\in\mathbb{Z}} W_j. \tag{3.24}$$

The orthogonal decomposition of $L^2(\mathbb{H})$ under W_j's follows that each $f \in L^2(\mathbb{H})$ has a representation $f = \sum_j Q_j f$, where $Q_j f \perp Q_k f$ for any pair of j, k, $j \neq k$.

Our goal is reduced to finding an n.t frame for W_0. If we can find such a n.t frame for W_0, then by the scaling property (3.23) and orthogonal decomposition of $L^2(\mathbb{H})$ under W_js in (3.24), we can easily get a n.t frame for space $L^2(\mathbb{H})$. We study it in the next Lemma in details:

Lemma 3.18. *Suppose $\psi \in W_0$ and Γ is a lattice in $\mathbb{H}$ such that $\{L_\gamma \psi\}_{\gamma\in\Gamma}$ constitutes a n.t frame of W_0. Then $\{L_{2^{-j}\gamma} D_{2^{-j}}\psi\}_{\gamma,j}$ is a n.t frame of $L^2(\mathbb{H})$.*

Proof: Observe that this lemma is a consequence of orthogonal decomposition of $L^2(\mathbb{H})$ under W_j's. Suppose $f \in L^2(\mathbb{H})$. From (3.24), f can be presented as

$$f = \sum_j Q_j(f).$$

Therefore to prove that the system $\{L_{2^{-j}\gamma} D_{2^{-j}}\psi\}_{\gamma,j}$ forms a n.t frame of $L^2(\mathbb{H})$, it is sufficient to show that for any j the system $\{L_{2^{-j}\gamma} D_{2^{-j}}\psi\}_{\gamma}$ is a n.t frame of W_j. From the scaling property of spaces W_j's (3.23) we have $Q_j(f)(2^{-j}.) \in W_0$. Take $Q_j(f) = f_j$. Therefore from the assertion of Lemma

$$\|f_j(2^{-j}.)\|^2 = \sum_\gamma \left|\langle f_j(2^{-j}.), L_\gamma\psi\rangle\right|^2.$$

Replacing $2^{2j} D_{2^j} f_j(.) = f_j(2^{-j}.)$ in above we get

$$\|f_j\|^2 = \|D_{2^j} f_j\|^2 = \sum_\gamma \left|\langle D_{2^j} f_j, L_\gamma\psi\rangle\right|^2 = \sum_\gamma \left|\langle f_j, L_{2^{-j}\gamma} D_{2^{-j}}\psi\rangle\right|^2. \tag{3.25}$$

Summing over j in (3.25) yields:

$$\|f\|^2 = \sum_j \|f_j\|^2 = \sum_{j,\gamma} \left|\langle f_j, L_{2^{-j}\gamma} D_{2^{-j}}\psi\rangle\right|^2 = \sum_{j,\gamma} \left|\langle f, L_{2^{-j}\gamma} D_{2^{-j}}\psi\rangle\right|^2,$$

as desired. □

By Lemma 3.18 it remains to show that the space W_0 contains a function ψ generating a normalized tight frame of W_0.

Remark 3.19. *By the definition of orthogonal projections P_1, P_0 in (3.20) for any $f \in L^2(\mathbb{H})$ we have*

$$Q_0(f) = P_1(f) - P_0(f) = f * (2^4 S(2.)) - f * S = f * \left[(2^4 S(2.)) - S\right],$$

and hence is

$$W_0 = L^2(\mathbb{H}) * \left[(2^4 S(2.)) - S\right]. \tag{3.26}$$

Likewise for any j one can see that

$$W_j = L^2(\mathbb{H}) * \left[(2^{4j} S(2^j.)) - (2^{4(j-1)} S(2^{j-1}.))\right] \quad and,$$
$$Q_j(f) = f * \left[(2^{4j} S(2^j.)) - (2^{4(j-1)} S(2^{j-1})\right] \quad \forall\, f \in L^2(\mathbb{H}),$$

where Q_j, as earlier mentioned, is the projection operator of $L^2(\mathbb{H})$ onto W_j.

The representation of the space W_0 in (3.26) suggests to get a n.t frame for W_0 by applying Theorem 3.14. We study it in the next theorem. Recall that $\sum(W_0)$ denotes the support of space W_0.

Theorem 3.20. *W_0 is bandlimited and contains a function ψ such that its left translations under a suitable lattice Γ forms a n.t frame of W_0.*

Proof: Due to the support of S, we have

$$\sum \left[(2^4 S(2.)) - S\right] \subset \left[-\frac{\pi}{d}, \frac{\pi}{d}\right], \tag{3.27}$$

where Σ stands for the support of the function $(2^4 S(2.)) - S$ in the Plancherel side. Hence W_0 is bandlimited. To prove that the space W_0 contains a n.t frame, observe that by Lemma (3.17) the set $\{L_{2^{-1}\gamma} D_{2^{-1}}\phi\}_{\gamma\in\Gamma}$ is a n.t frame of V_1 for a suitable Γ. From the other side the projection of V_1 onto W_0, Q_0, is left invariant and hence for any $\gamma \in \Gamma$ is $Q_0(L_{2^{-1}\gamma} D_{2^{-1}}\phi) = L_{2^{-1}\gamma}\big(Q_0(D_{2^{-1}}\phi)\big)$. Since the image of a n.t frame under a left shift-invariant projection is again a n.t frame of the image space, then the set $\{Q_0\big(L_{2^{-1}\gamma} D_{2^{-1}}\phi\big)\}_\gamma = \{L_{2^{-1}\gamma}\big(Q_0(D_{2^{-1}}\phi)\big\}_\gamma$ constitutes a n.t frame for W_0, as desired. □

Corollary 3.21. *There exists a bandlimited function $\psi \in L^2(\mathbb{H})$ and a lattice Γ in $\mathbb{H}$ such that the discrete wavelet system $\{L_{2^{-j}\gamma} D_{2^{-j}\psi}\}_{j,\gamma}$ forms a n.t frame of $L^2(\mathbb{H})$.*

Proof: Using Lemma 3.18 and Theorem 3.20 the assertion follows. $\square$

Remarks 3.22. *1) Observe that in this work, in contrast to the case of $\mathbb{R}$, it is not desired that the wavelet function ψ contained in W_0 is constructed by so-called scaling function ϕ in V_0.*

2) The role of the multiresolution concept in this chapter is somewhat different to the case of wavelets in $L^2(\mathbb{R})$. On $\mathbb{R}$, scaling equations and associated multiresolution analysis are a useful tool for the explicit construction of well-localized wavelets, resulting in a convenient discretization and fast decomposition algorithems. By contrast, the multiresolution analysis on $L^2(\mathbb{H})$ is rather an interesting byproduct of the construction of Shannon-wavelets, serving as a motivations of some of its features. It is doubtful that one can construct well-localized wavelets with this approach. At least the Shannon wavelet on $L^2(\mathbb{R})$, which is not even integrable, suggests this. Bad localization is also a handicap for decomposition algorithms that one can (in principle) derive from a multiresolution on $L^2(\mathbb{H})$.

Chapter 4

Admissibility of Radial Schwartz Functions on the Heisenberg Group

4.1 Introduction

The first aim of this chapter is to characterize the space of radial Schwartz functions on the Heisenberg group by applying the Fourier transform. We show that a radial function on the Heisenberg group is Schwartz if and only if its radial Fourier transform is rapidly decreasing. The main result is Theorem 4.36.

The reason for considering radial functions is the following: Given such a function f, its operator valued Fourier transform $\widehat{f}(\lambda)$ turns out to be diagonal in the dilated Hermite basis $\{\phi_n^\lambda\}_{n\in\mathbb{N}_0}$, for $\lambda \in \mathbb{R}^*$. This fact allows a convenient treatment of radial functions on the Fourier side; it is achieved by first showing that for radial function f, the special Hermite expansion of f^λ reduces to a Laguerre expansion. For that reason, in section 4.2 we show some more properties of Hermite and Laguerre function and in Section 4.4 we calculate some useful relations concerning these special functions, which we need for the characterization of radial Schwartz functions. We then provide sufficient and necessary conditions for the radial Fourier transform of a radial function to be a Schwartz function. In this section we also consider the characterization of admissible Schwartz functions via their radial Fourier transforms, which is done in the main result, Theorem 4.37. Let us start with the definition of a radial function:

Definition 4.1. ***(radial function)*** *Using coordinates (z,t) on Heisenberg group $\mathbb{H}$, where $z \in \mathbb{C}$ and $t \in \mathbb{R}$, we say a function $f \in L^p(\mathbb{H})$ is radial if $f = f \circ R_\theta$ in the L^p-sense for every $\theta \in [0, 2\pi)$, where R_θ is the rotation operator on $\mathbb{H}$ with respect to θ and is given by $R_\theta(z,t) = (\tilde{R}_\theta z, t)$, $\tilde{R}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.*

Remark 4.2. *A continuous function $f \in L^p(\mathbb{H})$ is radial if and only if $f(z,t)$ depends only on $|\, z \,|$ and t. In this case we may write $f(z,t) = f_0(|\, z \,|, t)$ with $f_0 : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$, where the latter equality is understood pointwise.*

Schwartz functions have played an important role in harmonic analysis with nilpotent groups. Let $\mathbb{H}$ be the Heisenberg group with Lie algebra $\mathfrak{h}$, see §2.5. The exponential map $\exp : \mathfrak{h} \to \mathbb{H}$ is a polynomial diffeomorphism and one defines the (Fréchet) space $\mathcal{S}(\mathbb{H})$ of Schwartz functions on $\mathcal{S}(\mathbb{H})$ via identification with the usual space $\mathcal{S}(\mathfrak{h})$ of Schwartz functions on the vector space $\mathfrak{h}$:

$$\mathcal{S}(\mathbb{H}) := \{f : \ \mathbb{H} \to \mathbb{C};\ f \circ \exp \in \mathcal{S}(\mathfrak{h})\}.$$

Observe that with identifying $\mathbb{H}$ with its underlying manifold $\mathbb{R}^3$ one gets $\mathcal{S}(\mathbb{H}) = \mathcal{S}(\mathbb{R}^3)$ which we will use this in the remainder of the our work. For more details see for instance [7], [14].

Notation: $L^2_r(\mathbb{H})$ denotes the space of radial functions in $L^2(\mathbb{H})$, and $\mathcal{S}_r(\mathbb{H})$ the space of radial Schwartz functions on $\mathbb{H}$.

4.2 Preliminaries and Notations

In this section we recapitulate Hermite, Laguerre and special Hermite functions and show some of their properties which are needed in studying the expansions of group Fourier transform of radial functions on the Heisenberg group in terms of special Hermite functions. Also, in the study of special Hermite functions we shall recall briefly some results from the representation theory of the Heisenberg group. We also define Weyl transforms which we need later.

4.2.1 Hermite and Laguerre Functions

The importance of Hermite functions in the theory of Fourier integrals was realized by N.Wiener. They play an important role not only in Euclidean Fourier analysis but also in harmonic analysis on the Heisenberg group. As we need several properties of these distinguished offspring of the Gaussian in this chapter, we think it is appropriate to introduce them here.

Hermite polynomials $H_n(x)$, are defined on the real line by:

$$H_n(x) = (-1)^n \frac{d^n}{dx^n}(e^{-x^2})e^{x^2}, \quad n = 0, 1, 2, \cdots$$

We then define the normalized Hermite functions ϕ_n by setting

$$\phi_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(x) e^{-\frac{1}{2}x^2}, \quad n = 0, 1, 2, \cdots$$

Many properties of the Hermite functions follow directly from the above definition. We record here some properties which are needed in the sequel.

The following relations hold for Hermite polynomials:

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x), \quad H_n(x) = 2xH_{n-1}(x) - \frac{d}{dx}H_{n-1}(x).$$

For the Hermite functions ϕ_n these relations take the form

$$(-\frac{d}{dx} + x)\phi_n(x) = \phi_{n+1}(x), \quad \text{and} \quad (\frac{d}{dx} + x)\phi_n(x) = 2n\phi_{n-1}(x)$$

The operators $A = -\frac{d}{dx} + x$ and $A^* = \frac{d}{dx} + x$ are called the **creation** and **annihilation** operators.

The family $\{\phi_n\}_{n\in\mathbb{N}_0}$ is an orthonormal system in $L^2(\mathbb{R})$. But we can say more.

Theorem 4.3. *The system $\{\phi_n\}_{n\in\mathbb{N}_0}$ is an orthonormal basis for $L^2(\mathbb{R})$. Consequently, every $f \in L^2(\mathbb{R})$ has an expansion*

$$f(x) = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n(x),$$

where the series converges to f in the L^2-norm.

For $\lambda \in \mathbb{R}^*$, ϕ_n^λ is given by

$$\phi_n^\lambda(x) = \mid \lambda \mid^{\frac{1}{4}} \phi_n(\sqrt{\mid \lambda \mid} x) \qquad \forall x \in \mathbb{R}.$$

For the Fourier analysis of radial functions, dilated version of the Hermite functions play an important role:

Corollary 4.4. *For any $\lambda \neq 0$, the system $\{\phi_n^\lambda\}_{n\in\mathbb{N}_0}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$.*

For more information about Hermite functions see for example [7], [15], [52].

Next, we consider *Laguerre* polynomials. Laguerre polynomials are defined by:

$$L_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) e^x.$$

Here $x > 0$ and $n = 0, 1, 2, \cdots$. Each L_n is a polynomial of degree n. It is explicitly given by

$$L_n(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-x)^j}{j!} \qquad x > 0, \;\; n = 0, 1, 2, \ldots$$

We then define the Laguerre functions Ψ_n by setting

$$\Psi_n(x) = L_n(\frac{1}{2}x^2) e^{-\frac{1}{4}x^2}, \quad n = 0, 1, 2, \cdots, \; x > 0$$

Then we have the following important Theorem:

Theorem 4.5. *The system of functions $\{\Psi_n\}_{n\in\mathbb{N}_0}$ forms an orthonormal basis for $L^2(\mathbb{R}^+, tdt)$.*

Proof: see for example [15] or [52]. □

Note: For each $\lambda \in \mathbb{R}^*$ we can dilate the functions $\{\Psi_n\}_{n\in\mathbb{N}_0}$ and obtain

$$\Psi_{n,\lambda}(t) = \mid \lambda \mid^{1/2} \Psi_n(\mid \lambda \mid^{1/2} t)$$

which constitutes another orthonormal basis for $L^2(\mathbb{R}^+, tdt)$.

4.2.2 Weyl Transform on $L^1(\mathbb{C})$

For $\lambda \neq 0$ let ρ_λ be the Schrödinger representation, as introduced earlier in (2.8). Then for $f \in L^1(\mathbb{H})$, $\widehat{f}(\lambda)$ is a bounded operator on $L^2(\mathbb{R})$ with the operator norm satisfying $\parallel \widehat{f}(\lambda) \parallel \leq \parallel f \parallel_1$. For $f \in (L^1 \cap L^2)(\mathbb{H})$, $\widehat{f}(\lambda)$ is actually a Hilbert-Schmidt operator and a Fourier transform can be extended for all $f \in L^2(\mathbb{H})$ by Plancherel theorem.
Let us define $\rho_\lambda(z) := \rho_\lambda(x, y, 0)$, where $z := x + iy$. Then $\rho_\lambda(x, y, t) = e^{i\lambda t}\rho_\lambda(z)$, and set

$$f^\lambda(z) = \int_{-\infty}^{\infty} f(z,t)e^{i\lambda t}dt \tag{4.1}$$

to be the inverse Fourier transform of f in the t-variable. Observe that f^λ is in $(L^1 \cap L^2)(\mathbb{H})$ if $f \in (L^1 \cap L^2)(\mathbb{H})$. Then from (4.1) and the definition of $\hat{f}(\lambda)$ it follows that

$$\hat{f}(\lambda) = \int_{\mathbb{C}} f^\lambda(z)\rho_\lambda(z)dz.s$$

Therefore, it is natural to consider operators of the form

$$W_\lambda(g) = \int_{\mathbb{C}} g(z)\rho_\lambda(z)dz$$

for functions $g \in (L^1 \cap L^2)(\mathbb{C})$. For $g \in (L^1 \cap L^2)(\mathbb{C})$, $W_\lambda(g)$ is an integral operator and is called **Weyl transform** of g. The operator $W_\lambda(g)$ is a Hilbert-Schmidt operator whose norm is given by

$$\parallel W_\lambda(g) \parallel_{H.S}^2 = (2\pi)\,|\lambda|^{-1} \int_{\mathbb{C}} |g(z)|^2\, dz$$

(see [52]). Observe that since the operator W_λ is a bounded linear operator on normed space $(L^1 \cap L^2)(\mathbb{C})$ with L^2-norm to the complete space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ with Hilbert-Schmidt norm, then W_λ is uniquely extended to a bounded linear (with the same bound) from $\overline{(L^1 \cap L^2)(\mathbb{C})} = L^2(\mathbb{C})$ to $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, which is still denoted by W_λ.
By definition we see that

$$W_\lambda(f^\lambda) = \hat{f}(\lambda), \tag{4.2}$$

therefore, as far as the t-variable is concerned the group Fourier transform is nothing but the Euclidean Fourier transform. In many problems on the Heisenberg group, an important technique is to take the partial Fourier transform in the t-variable to reduce matters to the case of $\mathbb{C}$. So, it would be reasonable to introduce a special convolution of two integrable functions on $\mathbb{C}$.

Definition 4.6. ***(λ-twisted convolution)*** *For any given two functions F and G in $L^1(\mathbb{C})$ and $\lambda \neq 0$, $F *_\lambda G$ is defined by*

$$F *_\lambda G(z) = \int_{\mathbb{C}} F(z-w)G(w)e^{i\frac{\lambda}{2}Im(z.\bar{w})}dw.$$

*We call this the λ-**twisted** convolution of F and G.*

It follows that for any F and G in $L^1(\mathbb{C})$ is

$$W_\lambda(F *_\lambda G) = W_\lambda(F)W_\lambda(G).$$

Observe that the definition of λ-twisted convolution can be extended between two functions in $L^2(\mathbb{C})$ (see [24]). For some interesting properties of the twisted convolution see the monograph [15].

4.2.3 Hermite functions on $\mathbb{C}$

Here we introduce certain auxiliary functions and study several of their important properties. These functions are defined as matrix coefficients of the Schrödinger representation

ρ_λ where $\lambda \neq 0$.

For each pair of indices j and k we define

$$\Phi^\lambda_{j,k}(z) = \langle \rho_\lambda(z)\phi^\lambda_k, \phi^\lambda_j \rangle, \tag{4.3}$$

where $\rho_\lambda(z)$ stands for $\rho_\lambda(z,0)$. The functions $\{(2\pi)^{-\frac{1}{2}} |\lambda|^{\frac{1}{2}} \Phi^\lambda_{j,k}\}$ are called the Hermite functions on $\mathbb{C}$. The importance of the Hermite functions on $\mathbb{C}$ is recorded in the following result:

Theorem 4.7. *([52] Theorem 2.3.1) The Hermite functions on $\mathbb{C}$, $\{(2\pi)^{-\frac{1}{2}} |\lambda|^{\frac{1}{2}} \Phi^\lambda_{j,k}\}_{j,k\in\mathbb{N}_0}$ form an orthonormal basis for $L^2(\mathbb{C})$.*

Next proposition shows that the Hermite functions $\Phi^\lambda_{n,n}$ are expressible in terms of Laguerre functions.

Proposition 4.8. *([52] Proposition 2.3.2) For any $\lambda \neq 0$ and $n \in \mathbb{N}_0$ is*

$$\Phi^\lambda_{n,n}(z) = L_n(\frac{1}{2} |\lambda| |z|^2) e^{-\frac{1}{4}|\lambda| |z|^2}.$$

Let us now define the Laguerre functions $l_n(z)$ on $\mathbb{C}$ by

$$l_n(z) = L_n(\frac{1}{2} |z|^2) e^{-\frac{1}{4}|z|^2} = \Phi_{n,n}(z) \tag{4.4}$$

We also define $l_{n,\lambda}(z) = l_n(|\lambda|^{1/2} z) = \Phi^\lambda_{n,n}(z)$ for $\lambda \in \mathbb{R}^*$, therefore we have the following result:

Proposition 4.9. *([52] Proposition 2.3.3) $W_\lambda(l_{n,\lambda}) = (2\pi) |\lambda|^{-1} \phi^\lambda_n \otimes \phi^\lambda_n$.*

4.3 Fourier Transform of Radial Functions

In this section we present the group Fourier transform for radial functions on $\mathbb{H}$. The following proposition contains the main reason why we consider radial functions on $\mathbb{H}$: The operator valued Fourier transform of radial functions consists of operators that are *"diagonal"* in the scaled Hermite basis. The precise proof of Proposition 4.10 can be found in [52]:

Proposition 4.10. *Let $f \in (L^1 \cap L^2)(\mathbb{H})$ be a radial function, then its Fourier transformation is given by:*

$$\hat{f}(\lambda) = \sum_n R_f(n,\lambda)\ \phi_n^\lambda \otimes \phi_n^\lambda \tag{4.5}$$

where

$$R_f(n,\lambda) = \int_{\mathbb{H}} f(z,t)\langle \rho_\lambda(z,t)\phi_n^\lambda, \phi_n^\lambda\rangle dtdz \tag{4.6}$$

$$= \int_{\mathbb{C}} f^\lambda(z)\langle \rho_\lambda(z)\phi_n^\lambda, \phi_n^\lambda\rangle dz \tag{4.7}$$

$$= \int_0^\infty f^\lambda(t) l_{n,\lambda}(t) tdt.$$

The function R_f is called *radial Fourier transform* of f.

Proof: Suppose $f \in (L^1 \cap L^2)(\mathbb{H})$ be radial. Then the function function f^λ on $\mathbb{C}$ defined by

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t)e^{i\lambda t}dt$$

is a radial function in $L^2(\mathbb{C})$. Then using polar coordinates we see that f^λ is in $L^2(\mathbb{R}^+, tdt)$ and hence we have the expansion

$$f^\lambda(r) = \sum_n \left(\int_0^\infty f^\lambda(t)\Psi_{n,\lambda}(t)tdt\right)\ \Psi_{n,\lambda}(r).$$

Written in terms of Laguerre function $l_{n,\lambda}$ on $\mathbb{C}$, this takes the form

$$f^\lambda(z) = \sum_n \mid \lambda \mid \left(\int_0^\infty f^\lambda(t) l_{n,\lambda}(t)tdt\right)\ l_{n,\lambda}(z) \tag{4.8}$$

where $l_{n,\lambda}(t)$ stands for $l_{n,\lambda}(z)$ with $\mid z \mid= t$. The above series converges in $L^2(\mathbb{C})$ and taking Weyl transform of both side and using relation (4.2) and Proposition 4.9 we have:

$$\widehat{f}(\lambda) = \sum_n \left(\int_0^\infty f^\lambda(t) l_{n,\lambda}(t)tdt\right)\ \phi_n^\lambda \otimes \phi_n^\lambda.$$

Now, taking

$$R_f(n,\lambda) = \int_0^\infty f^\lambda(t) l_{n,\lambda}(t)tdt \tag{4.9}$$

the proof follows. □

From this theorem we infer that the elements in subspace $L^1_r(\mathbb{H})$, consisting of all radial functions in $L^1(\mathbb{H})$, are commutative with respect to the convolution operator. Indeed for every pair of radial functions $f, g \in L^1(\mathbb{H})$ the convolution theorem and the fact that their Fourier transforms are diagonal in a common ONB yields that

$$\widehat{(f * g)}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda) = \widehat{g}(\lambda)\widehat{f}(\lambda) = \widehat{(g * f)}(\lambda).$$

Since the Fourier transform is injective, we see that $f * g = g * f$ and $f * g =$ is radial again. Moreover $L^p_r(\mathbb{H}) \subset L^p(\mathbb{H})$ is closed $(p = 1, 2)$, since the rotation operators on L^p are continuous, hence $L^1_r(\mathbb{H})$ is a Banach algebra.

Proposition 4.10 can be expanded to $L^2(\mathbb{H})$ as below:

Theorem 4.11. *$f \in L^2(\mathbb{H})$ is radial if and only if for almost every $\lambda \neq 0$ its Plancherel transformation $\hat{f}(\lambda)$ is diagonal in ONB $\{\phi^\lambda_n\}_{n\in\mathbb{N}_0}$.*

Proof: Suppose $\mathcal{H}_0$ is the set of L^2-functions f with the property that its Plancherel transformation $\hat{f}(\lambda)$ is diagonal in the ONB $\{\phi^\lambda_n\}_{n\in\mathbb{N}_0}$ for almost every $\lambda \neq 0$, i.e, can be represented as (4.5) for some coefficient R_f. We intend to show that $L^2_r(\mathbb{H}) = \mathcal{H}_0$.

To show $L^2_r(\mathbb{H}) \subset \mathcal{H}_0$, observe that $(L^1_r \cap L^2_r)(\mathbb{H})$ is densely contained in $L^2_r(\mathbb{H})$ in L^2-norm and from Proposition 4.10 is contained in $\mathcal{H}_0$. Since $\mathcal{H}_0$ is a closed subspace of $L^2(\mathbb{H})$, then it implies that $L^2_r(\mathbb{H}) \subset \mathcal{H}_0$ as desired.

Conversely, suppose $\mathcal{H}_1$ denotes the dense subspace of $\mathcal{H}_0$ of L^2-functions f for which

$$\int_\lambda \sum_n \mid R_f(n, \lambda) \mid d\mu(\lambda) < \infty.$$

By inverse Fourier transform, for any $f \in \mathcal{H}_1$ the equality

$$f(z, t) = \sum_n \int_{\lambda\in\mathbb{R}^*} R_f(\lambda, n)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda),$$

holds pointwise. It shows that f is radial, since the functions $\Phi^\lambda_{n,n}$ are radial. Therefore $\mathcal{H}_1 \subset L^2_r(\mathbb{H})$ and hence $\mathcal{H}_0 \subset L^2_r(\mathbb{H})$ as desired, since $\overline{\mathcal{H}_1} = \mathcal{H}_0$, and $L^2_r(\mathbb{H})$ is closed. $\square$

In general, analysis on the Heisenberg group and expansions in terms of Hermite and Laguerre functions are interrelated. On the one hand it became clear from the work of Geller

[22] that harmonic analysis on the Heisenberg group heavily depends on many properties of Hermite and Laguerre functions. On the other hand, analysis on the Heisenberg group also plays an important role in the study of Hermite and Laguerre expansions. For example, the first summability theorem for multiple Hermite expansions was deduced from the corresponding result on the Heisenberg group by Hulanicki and Jenkins [31].

4.4 Calculus on the Hermite and Laguerre Functions

Suppose f is a radial function contained in $L^2(\mathbb{H})$ and has group Fourier transform as in Theorem 4.10. Using the **inversion Fourier transformation** and Proposition 4.9 we have the inversion formula

$$f(z,t) = \sum_n \int_{\lambda \in \mathbb{R}^*} R_f(\lambda, n) \overline{\Phi^\lambda_{n,n}(z)} e^{-i\lambda t} d\mu(\lambda). \tag{4.10}$$

Suppose the integral in (4.10) is absolutely convergent everywhere. In the following we use (4.10) to compute the action of certain differential operator on the Fourier side.
Our characterization of Schwartz function , which will be presented in the next section , involves the application of difference operators Δ^+ and Δ^-:

Definition 4.12. *Given a function h on $\mathbb{N}_0$. Then $\Delta^+ h$ and $\Delta^- h$ are the functions on $\mathbb{N}_0$ defined by*

$$\Delta^+ h(n) = (n+1)(h(n+1) - h(n)) \ , \ \Delta^- h(n) = n(h(n) - h(n-1)) \forall n \in \mathbb{N}$$

and $\Delta^- h(0) = 0$.

Notation: The operators Δ^+ and Δ^- operate on $f : \mathbb{N} \times \mathbb{C} \to \mathbb{C}$ only in the integer variable, i.e, $\Delta^+ f = (\Delta^+ \otimes 1) f$ and likewise $\Delta^- f = (\Delta^- \otimes 1) f$.

4.4.1 Differentiation of Special Hermite Function on $\mathbb{C} \times \mathbb{R}^*$

Using the symbols in Definition 4.12 we have the next lemma, which describes differentiation of special Hermite functions. Let first give the next remark which will be used later:

Remark 4.13. *Suppose $f : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and $f(.t) : \mathbb{R} \to \mathbb{C}$ is integrable for each t. Let $F(t) = \int f(x,t)dt$. Suppose $\partial_t f$ exists and there is a $g \in L^1$ such that $|(\partial_t f)(x,t)| \leq g(x)$ for any x, t. Then F is differentiable and $\partial_t F(t) = \int (\partial_t f)(x,t)dx$.*

The proof can be found for example in [15].

Lemma 4.14. *Suppose $\lambda \neq 0$ and $\{\Phi_{j,k}\}_{j,k \in \mathbb{N}_0}$ are special functions on $\mathbb{C}$ in (4.3). Then for any $j = k = n \in \mathbb{N}_0$ is*

$$\partial_z \partial_{\bar{z}} \Phi^\lambda_{n,n} = -\frac{|\lambda|}{2} \left\{ \Delta^+ - \Delta^- + 2(2n+1) \right\} \Phi^\lambda_{n,n},$$

where Δ^+ and Δ^- operate on the n-variable.

Proof: With definition in (4.3), in the coordinates x, y we have

$$\begin{aligned}
\partial_z \partial_{\bar{z}} \Phi^\lambda_{n,n}(z) &= (\partial x + i\partial y)(\partial x - i\partial y)\Phi^\lambda_{n,n}(x,y) \\
&= (\partial x + i\partial y)(\partial x - i\partial y)\langle \rho_\lambda(x,y,0)\phi^\lambda_n, \phi^\lambda_n \langle \\
&= (\partial x + i\partial y)(\partial x - i\partial y) \int_{\mathbb{R}} e^{i\lambda x \zeta} \phi^\lambda_n(\zeta + \frac{1}{2}y)\phi^\lambda_n(\zeta - \frac{1}{2}y)d\zeta.
\end{aligned}$$

Hence, to prove the assertion we shall compute the following steps for any $\lambda \neq 0$ and $j, k \in \mathbb{N}_0$:

1. $\partial x \Phi^\lambda_{j,k}(x,y)$ and $i\partial y \Phi^\lambda_{j,k}(x,y)$,
2. $(\partial x + i\partial y)\Phi^\lambda_{j,k}(x,y)$ and $(\partial x - i\partial y)\Phi^\lambda_{j,k}(x,y)$,
3. $(\partial x + i\partial y)(\partial x - i\partial y)\Phi^\lambda_{j,k}(x,y)$,
4. $(\partial x + i\partial y)(\partial x - i\partial y)\Phi^\lambda_{j,k}(x,y)$, and we shall show
5. $(\partial x + i\partial y)(\partial x - i\partial y)\Phi^\lambda_{n,n}(x,y) = -\frac{|\lambda|}{2}\{\Delta^+ - \Delta^- + 2(2n+1)\}\Phi^\lambda_{n,n}$.

Recall that

$$\Phi^\lambda_{j,k}(z) = \int_{\mathbb{C}} e^{i\lambda x \zeta} \phi^\lambda_n(\zeta + \frac{1}{2}y)\phi^\lambda_n(\zeta - \frac{1}{2}y)d\zeta. \tag{4.11}$$

Now suppose $\lambda = 1$. Differentiating (4.11) with respect to x and writing $2\zeta = (\zeta + \frac{1}{2}y) + (\zeta - \frac{1}{2}y)$ we get

$$\begin{aligned}
\partial_x \Phi_{j,k}(x,y) =& \partial_x \langle \rho(x,y,0)\phi_k, \phi_j \rangle && (4.12)\\
=& \partial_x \int_{\mathbb{R}} e^{ix\zeta} \phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y) d\zeta && (4.13)\\
=& \int_{\mathbb{R}} (\partial_x e^{ix\zeta}) \phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y) d\zeta\\
=& \frac{i}{2} \int_{\mathbb{R}} e^{ix\zeta} \Big\{ (\zeta + \frac{1}{2}y) + (\zeta - \frac{1}{2}y) \Big\} \phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y) d\zeta\\
=& \frac{i}{2} \int_{\mathbb{R}} e^{ix\zeta} \Big\{ (\zeta + \frac{1}{2}y)\phi_k(\zeta + \frac{1}{2}y) \Big\} \phi_j(\zeta - \frac{1}{2}y) d\zeta\\
&+ \frac{i}{2} \int_{\mathbb{R}} e^{ix\zeta} \phi_k(\zeta + \frac{1}{2}y) \Big\{ (\zeta - \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y) \Big\} d\zeta.
\end{aligned}$$

Observe that the since

$$\left| \partial_x (e^{ix\zeta}) \phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y) \right| \leq g(\zeta, y) = \left| \zeta \phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y) \right|$$

for all $x, y, \zeta \in \mathbb{R}$ and g is integrable, then from Remark 4.13 we are of course passing in (4.13) from the

∂_x to the integral. With a similar calculation and the same argument in Remark 4.13 we also have:

$$\begin{aligned}
i\partial_y \Phi_{j,k}(x,y) =& i\partial_y \langle \rho(x,y,0)\phi_k, \phi_j \rangle && (4.14)\\
=& \frac{i}{2} \int_{\mathbb{R}} e^{ix\zeta} \Big\{ \partial_y \phi_k(\zeta + \frac{1}{2}y) \Big\} \phi_j(\zeta - \frac{1}{2}y) d\zeta\\
&- \frac{i}{2} \int_{\mathbb{R}} e^{ix\zeta} \phi_k(\zeta + \frac{1}{2}y) \Big\{ \partial_y \phi_j(\zeta - \frac{1}{2}y) \Big\} d\zeta.
\end{aligned}$$

Combining (4.12) and (4.14) we get

$$\begin{aligned}
(\partial_x + i\partial_y)\Phi_{j,k}(x,y) =& \frac{i}{2} \int_{\mathbb{R}} e^{ix\zeta} \Big\{ (\zeta + \frac{1}{2}y)\phi_k(\zeta + \frac{1}{2}y) + \partial_y \phi_k(\zeta + \frac{1}{2}y) \Big\} \phi_j(\zeta - \frac{1}{2}y) d\zeta\\
&+ \frac{i}{2} \int_{\mathbb{R}} e^{ix\zeta} \phi_k(\zeta + \frac{1}{2}y) \Big\{ (\zeta - \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y) - \partial_y \phi_j(\zeta - \frac{1}{2}y) \Big\} d\zeta
\end{aligned} \tag{4.15}$$

and

$$
\begin{aligned}
(\partial_x - i\partial_y)\Phi_{j,k}(x,y) = &\frac{i}{2}\int_{\mathbb{R}} e^{ix\zeta}\Big\{(\zeta+\frac{1}{2}y)\phi_k(\zeta+\frac{1}{2}y) - \partial_y\phi_k(\zeta+\frac{1}{2}y)\Big\}\phi_j(\zeta-\frac{1}{2}y)d\zeta \\
&+ \frac{i}{2}\int_{\mathbb{R}} e^{ix\zeta}\phi_k(\zeta+\frac{1}{2}y)\Big\{(\zeta-\frac{1}{2}y)\phi_j(\zeta-\frac{1}{2}y) + \partial_y\phi_j(\zeta-\frac{1}{2}y)\Big\}d\zeta.
\end{aligned}
\tag{4.16}
$$

Using the **creation** and **annihilation** relations for the Hermite functions ([51])

$$
\begin{aligned}
(-\partial\zeta+\zeta)\phi_n &= (2n+2)^{\frac{1}{2}}\phi_{n+1} \\
(\partial\zeta+\zeta)\phi_n &= (2n)^{\frac{1}{2}}\phi_{n-1},
\end{aligned}
$$

we obtain from the equalities (4.60) and (4.16):

$$
\begin{aligned}
(\partial_x + i\partial_y)\Phi_{j,k}(x,y) &= \frac{i}{2}\Big\{(2k)^{\frac{1}{2}}\Phi_{j,k-1}(x,y) + (2j+2)^{\frac{1}{2}}\Phi_{j+1,k}(x,y)\Big\} \\
(\partial_x - i\partial_y)\Phi_{j,k}(x,y) &= \frac{i}{2}\Big\{(2k+2)^{\frac{1}{2}}\Phi_{j,k+1}(x,y) + (2j)^{\frac{1}{2}}\Phi_{j-1,k}(x,y)\Big\}.
\end{aligned}
\tag{4.17}
$$

Applying the last two equalities we derive

$$
\begin{aligned}
\partial_z\partial_{\bar{z}}\Phi_{j,k} &= (\partial_x - i\partial_y)(\partial_x + i\partial_y)\Phi_{j,k} \\
&= -\frac{1}{4}\Big\{(2j+2)\Phi_{j,k} + ((2k+2)(2j+2))^{\frac{1}{2}}\Phi_{j+1,k+1} \\
&+ ((2k)(2j))^{\frac{1}{2}}\Phi_{j-1,k-1} + (2k)\Phi_{j,k}\Big\}.
\end{aligned}
\tag{4.18}
$$

Now for $\lambda \neq 0$ in $\mathbb{R}^*$ and j,k in $\mathbb{Z}$ we have:

$$
\begin{aligned}
\Phi^{\lambda}_{j,k}(x,y) =&\langle\rho_\lambda(x,y,0)\phi^{\lambda}_k,\phi^{\lambda}_j\rangle \\
=&\int_{\mathbb{R}} e^{i\lambda x\zeta}\phi^{\lambda}_k(\zeta+\frac{1}{2}y)\phi^{\lambda}_j(\zeta-\frac{1}{2}y)d\zeta = \Phi_{j,k}(sgn\lambda \mid \lambda\mid^{\frac{1}{2}} x, \mid\lambda\mid^{\frac{1}{2}} y).
\end{aligned}
$$

Repeating the same calculation for $\Phi^{\lambda}_{j,k}(x,y)$ and substitution $\widetilde{X} = sgn\lambda \mid \lambda \mid^{\frac{1}{2}} x$ and $\widetilde{Y} =\mid \lambda \mid^{\frac{1}{2}} y$ we have:

$$
(\partial_x - i\partial_y)(\partial_x + i\partial_y)\Phi^{\lambda}_{j,k}(x,y) = \mid\lambda\mid (\partial_{\widetilde{X}} - i\partial_{\widetilde{Y}})(\partial_{\widetilde{X}} + i\partial_{\widetilde{Y}})\Phi_{j,k}(\widetilde{X},\widetilde{Y}).
\tag{4.19}
$$

Now, applying the relations in (4.18) for (4.19) implies:

$$\begin{aligned}
&| \lambda | (\partial_{\widetilde{X}} - i\partial_{\widetilde{Y}})(\partial_{\widetilde{X}} + i\partial_{\widetilde{Y}})\Phi_{j,k}(\widetilde{X}, \widetilde{Y}) \\
&\qquad = -\frac{| \lambda |}{4}\Big\{(2j+2)\Phi_{j,k} + ((2k+2)(2j+2))^{\frac{1}{2}}\Phi_{j+1,k+1} \\
&\qquad +((2k)(2j))^{\frac{1}{2}}\Phi_{j-1,k-1} + (2k)\Phi_{j,k}\Big\}(\widetilde{X}, \widetilde{Y}) \\
&\qquad = -\frac{| \lambda |}{4}\Big\{((2j+2)\Phi^{\lambda}_{j,k} + ((2k+2)(2j+2))^{\frac{1}{2}}\Phi^{\lambda}_{j+1,k+1} \\
&\qquad +((2k)(2j))^{\frac{1}{2}}\Phi^{\lambda}_{j-1,k-1} + (2k)\Phi^{\lambda}_{j,k}\Big\}(x, y).
\end{aligned}$$

Taking $j = k = n$, then

$$\begin{aligned}
(\partial_x - i\partial_y)(\partial_x + i\partial_y)\Phi^{\lambda}_{n,n} &= -\frac{| \lambda |}{4}\Big\{(2n+2)\Phi^{\lambda}_{n,n} + (2n+2)\Phi^{\lambda}_{n+1,n+1} + (2n)\Phi^{\lambda}_{n-1,n-1}\Big\} \\
&= -\frac{| \lambda |}{2}\Big\{(n+1)(\Phi^{\lambda}_{n+1,n+1} - \Phi^{\lambda}_{n,n}) - (n)(\Phi^{\lambda}_{n,n} - \Phi^{\lambda}_{n-1,n-1}) \\
&+2(2n+1)\Phi^{\lambda}_{n,n}\Big\}.
\end{aligned}$$

Finally, using the definitions of Δ^+ and Δ^-+ in above we get

$$\partial_z\partial_{\bar{z}}\Phi^{\lambda}_{n,n} = -\frac{|\lambda|}{2}\big\{\Delta^+ - \Delta^- + 2(2n+1)\big\}\Phi^{\lambda}_{n,n},$$

as desired. $\square$

We conclude this section with the next Lemma:

Lemma 4.15. *For $\lambda \neq 0$ is*

$$\partial_\lambda \Phi^{\lambda}_{n,n}(z) = (\frac{1}{\lambda}\Delta^- - \frac{| z |^2}{4} sgn\lambda)\Phi^{\lambda}_{n,n}(z)$$

Proof: For $\lambda \neq 0$ is

$$\begin{aligned}
\Phi^{\lambda}_{n,n}(z) = \Phi_{n,n}(| \lambda |^{\frac{1}{2}} z) &= L_n(\frac{1}{2} \ | \lambda || z |^2)e^{-\frac{1}{4} |\lambda||z|^2} \\
&= \sum_k^n \binom{n}{k} \frac{(-\frac{1}{2} \ | \lambda || z |^2)^k}{k!} \ e^{-\frac{|\lambda||z|^2}{4}}
\end{aligned}$$

where $L_n(x) := \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!}$ is a Laguerre polynomial.

Let $\lambda > 0$, then

$$\partial_\lambda \Phi^\lambda_{n,n}(z) = \partial_\lambda L_n(\frac{1}{2}\ \lambda \mid z \mid^2)\ e^{-\frac{1}{4}\ \lambda|z|^2} - \frac{1}{4}\ \mid z \mid^2 L_n(\frac{1}{2}\ \lambda \mid z \mid^2) e^{-\frac{1}{4}\ \lambda|z|^2} \tag{4.20}$$

and hence the following computation shows that

$$\begin{aligned}
\partial_\lambda L_n(\frac{1}{2}\ \lambda \mid z \mid^2) &= \partial_\lambda \sum_{k=0}^{n} \binom{n}{k} \frac{(-\frac{1}{2}\ \lambda \mid z \mid^2)^k}{k!} \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{(-\frac{1}{2}\ \mid z \mid^2)^k}{k!}\ k\lambda^{k-1} \\
&= \frac{1}{\lambda} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\frac{1}{2}\ \mid z \mid^2)^k}{k!}\ k\lambda^{k} \qquad\qquad k = n-(n-k) \\
&= \frac{n}{\lambda} \sum_{k=0}^{n} \binom{n}{k} (-\frac{1}{2}\ \mid z \mid^2)^k\ \lambda^k - \frac{1}{\lambda} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\frac{1}{2}\ \mid z \mid^2)^k}{k!}\ (n-k)\lambda^k \\
&= \frac{n}{\lambda} L_n(\frac{1}{2}\ \lambda \mid z \mid^2) - \frac{n}{\lambda} L_{n-1}(\frac{1}{2}\ \lambda \mid z \mid^2) \\
&= \frac{1}{\lambda} \Delta^- L_n(\frac{1}{2}\ \lambda \mid z \mid^2).
\end{aligned} \tag{4.21}$$

Substituting (4.21) into (4.20) yields

$$\partial_\lambda \Phi^\lambda_{n,n}(z) = \frac{1}{\lambda}\Delta^- \Phi^\lambda_{n,n}(z) - \frac{1}{4} \mid z \mid^2 \Phi^\lambda_{n,n}(z) = (\frac{1}{\lambda}\Delta^- - \frac{1}{4} \mid z \mid^2)\Phi^\lambda_{n,n}(z). \tag{4.22}$$

For $\lambda < 0$ with a similar computation we have

$$\partial_\lambda \Phi^\lambda_{n,n}(z) = \frac{1}{\lambda}\Delta^- \Phi^\lambda_{n,n}(z) + \frac{1}{4} \mid z \mid^2 \Phi^\lambda_{n,n}(z). \tag{4.23}$$

Combining (4.22) and (4.23) we get

$$\boxed{\partial_\lambda \Phi^\lambda_{n,n}(z) = \Big(\frac{1}{\lambda}\Delta^- - \frac{1}{4} sgn\lambda\ \mid z \mid^2\Big)\Phi^\lambda_{n,n}(z)}$$

as desired. □

4.4.2 Multiplication of Special Hermite Functions by $\mid z \mid^2$

The first lemma of the subsection shows the interrelation of multiplication of special Hermite function by factor $\mid z \mid^2$ and the difference operators Δ^+ and Δ^-:

Lemma 4.16. *For $\lambda \neq 0$ and $n \in \mathbb{N}_0$ we have*

$$| z |^2 \, \Phi^{\lambda}_{n,n}(z) = -\frac{2}{| \lambda |}(\Delta^+ - \Delta^-)\Phi^{\lambda}_{n,n}(z). \tag{4.24}$$

Proof: First, we have to compute some elementary and preliminary relations. We start with multiplication of function $\Phi^{\lambda}_{n,n}$ with respect to $x = Re(z)$:
Writing $xe^{ix\zeta} = -i\frac{\partial}{\partial\zeta}e^{ix\zeta}$ and using integration by parts we get

$$\begin{aligned} x\Phi_{j,k}(x,y) &= x\langle \rho(x,y,0)\phi_k, \phi_j\rangle \\ &= \int_{\mathbb{R}} xe^{ix\zeta}\phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y)d\zeta \\ &= -i\int_{\mathbb{R}} \left(\frac{\partial}{\partial\zeta}e^{ix\zeta}\right) \phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y)d\zeta \\ &= i\int_{\mathbb{R}} e^{ix\zeta}\frac{\partial}{\partial\zeta}\Big\{\phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y)\Big\}d\zeta \\ &= i\int_{\mathbb{R}} e^{ix\zeta}\Big\{\frac{\partial}{\partial\zeta}\phi_k(\zeta + \frac{1}{2}y)\}\phi_j(\zeta - \frac{1}{2}y)d\zeta \\ &+ i\int_{\mathbb{R}} e^{ix\zeta}\phi_k(\zeta + \frac{1}{2}y)\frac{\partial}{\partial\zeta}\phi_j(\zeta - \frac{1}{2}y)\Big\}d\zeta. \end{aligned} \tag{4.25}$$

And also we have:

$$\begin{aligned} iy\Phi_{j,k}(x,y) &= iy\langle \rho(x,y,0)\phi_k, \phi_j\rangle \\ &= i\int_{\mathbb{R}} e^{ix\zeta}(y)\phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y)d\zeta \\ &= i\int_{\mathbb{R}} e^{ix\zeta}\left((\zeta + \frac{1}{2}y) - (\zeta - \frac{1}{2}y)\right)\phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y)d\zeta \\ &= i\int_{\mathbb{R}} e^{ix\zeta}(\zeta + \frac{1}{2}y)\phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y)d\zeta \\ &- i\int_{\mathbb{R}} e^{ix\zeta}\phi_k(\zeta + \frac{1}{2}y)(\zeta - \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y)d\zeta. \end{aligned} \tag{4.26}$$

Now taking sum and difference of (4.25) and (4.26) yields

$$\begin{aligned} (x+iy)\Phi_{j,k}(x,y) =& i\int_{\mathbb{R}} e^{ix\zeta}\Big\{\partial\zeta + (\zeta + \frac{1}{2}y)\Big\}\phi_k(\zeta + \frac{1}{2}y)\phi_j(\zeta - \frac{1}{2}y)d\zeta \\ &+ i\int_{\mathbb{R}} e^{ix\zeta}\phi_k(\zeta + \frac{1}{2}y)\Big\{\partial\zeta - (\zeta - \frac{1}{2}y)\Big\}\phi_j(\zeta - \frac{1}{2}y)d\zeta, \end{aligned} \tag{4.27}$$

$$\begin{aligned}(x-iy)\Phi_{j,k}(x,y) =& i\int_{\mathbb{R}} e^{ix\zeta}\Big\{\partial\zeta-(\zeta+\frac{1}{2}y)\Big\}\phi_k(\zeta+\frac{1}{2}y)\phi_j(\zeta-\frac{1}{2}y)d\zeta \\ &+i\int_{\mathbb{R}} e^{ix\zeta}\phi_k(\zeta+\frac{1}{2}y)\Big\{\partial\zeta+(\zeta-\frac{1}{2}y)\Big\}\phi_j(\zeta-\frac{1}{2}y)d\zeta.\end{aligned} \tag{4.28}$$

Applying the operators **creation** and **annihilation** in equations (4.27) and (4.28)

$$(-\partial\zeta+\zeta)\phi_n = (2n+2)^{\frac{1}{2}}\phi_{n+1}$$
$$(\partial\zeta+\zeta)\phi_n = (2n)^{\frac{1}{2}}\phi_{n-1}$$

we get

$$\begin{aligned}(x+iy)\Phi_{j,k}(x,y) =& i(2k)^{\frac{1}{2}}\int_{\zeta} e^{ix\zeta}\phi_{k-1}(\zeta+\frac{1}{2}y)\phi_j(\zeta-\frac{1}{2}y)d\zeta \\ &-i(2j+2)^{\frac{1}{2}}\int_{\zeta}\phi_k(\zeta+\frac{1}{2}y)\phi_{j+1}(\zeta-\frac{1}{2}y)d\zeta \\ =& i\Big\{(2k)^{\frac{1}{2}}\Phi_{j,k-1}(x,y)-(2j+2)^{\frac{1}{2}}\Phi_{j+1,k}(x,y)\Big\}\end{aligned} \tag{4.29}$$

and as well

$$(x-iy)\Phi_{j,k}(x,y) = -i\Big\{(2k+2)^{\frac{1}{2}}\Phi_{j,k+1}(x,y)-(2j)^{\frac{1}{2}}\Phi_{j-1,k}(x,y)\Big\}.$$

Combining the last equations with (4.29) one has

$$\begin{aligned}(x-iy)(x+iy)\Phi_{j,k}(x,y) =& (2k)^{\frac{1}{2}}\Big\{(2k)^{\frac{1}{2}}\Phi_{j,k}(x,y)-(2j)^{\frac{1}{2}}\Phi_{j-1,k-1}(x,y)\Big\} \\ &-(2j+2)^{\frac{1}{2}}\Big\{(2k+2)^{\frac{1}{2}}\Phi_{j+1,k+1}(x,y)-(2j+2)^{\frac{1}{2}}\Phi_{j,k}(x,y)\Big\}.\end{aligned} \tag{4.30}$$

For arbitrary $\lambda \neq 0$,

$$\begin{aligned}\mid z\mid^2\Phi^{\lambda}_{j,k}(z) =& \frac{1}{\mid\lambda\mid}(2k)^{\frac{1}{2}}\Big\{(2k)^{\frac{1}{2}}\Phi^{\lambda}_{j,k}(x,y)-(2j)^{\frac{1}{2}}\Phi^{\lambda}_{j-1,k-1}(x,y)\Big\} \\ &-\frac{1}{\mid\lambda\mid}(2j+2)^{\frac{1}{2}}\Big\{(2k+2)^{\frac{1}{2}}\Phi^{\lambda}_{j+1,k+1}(x,y)-(2j+2)^{\frac{1}{2}}\Phi^{\lambda}_{j,k}(x,y)\Big\}.\end{aligned} \tag{4.31}$$

To prove the assertion of Lemma, recall that $\Phi^{\lambda}_{j,k}(x,y) = \Phi_{j,k}(sgn\lambda\mid\lambda\mid^{\frac{1}{2}}x, |\lambda|^{\frac{1}{2}}y)$ for $\lambda \neq 0$. Replacing $\widetilde{Z} = sgn\lambda\mid\lambda\mid^{\frac{1}{2}}x + i\,|\lambda|^{\frac{1}{2}}y$ we have

$$\mid z\mid^2\Phi^{\lambda}_{j,k}(z) =\mid z\mid^2\Phi_{j,k}(\widetilde{Z}) = \frac{\mid z\mid^2}{\mid\widetilde{Z}\mid^2}\mid\widetilde{Z}\mid^2\Phi_{j,k}(Z) = \frac{1}{\mid\lambda\mid}\mid\widetilde{Z}\mid^2\Phi_{j,k}(\widetilde{Z}),$$

where $z = x + iy$. By using the equation (4.30) for $\mid \widetilde{Z} \mid^2 \Phi_{j,k}(\widetilde{Z})$ we have,

$$\begin{aligned} \mid z \mid^2 \Phi^{\lambda}_{j,k}(z) = & \frac{1}{\mid \lambda \mid}(2k)^{\frac{1}{2}}\Big\{(2k)^{\frac{1}{2}}\Phi_{j,k}(\widetilde{X},\widetilde{Y}) - (2j)^{\frac{1}{2}}\Phi_{j-1,k-1}(\widetilde{X},\widetilde{Y})\Big\} \\ & -\frac{1}{\mid \lambda \mid}(2j+2)^{\frac{1}{2}}\Big\{(2k+2)^{\frac{1}{2}}\Phi_{j+1,k+1}(\widetilde{X},\widetilde{Y}) - (2j+2)^{\frac{1}{2}}\Phi_{j,k}(\widetilde{X},\widetilde{Y})\Big\} \end{aligned} \tag{4.32}$$

or equivalently

$$\begin{aligned} \mid z \mid^2 \Phi^{\lambda}_{j,k}(z) = & \frac{1}{\mid \lambda \mid}(2k)^{\frac{1}{2}}\Big\{(2k)^{\frac{1}{2}}\Phi^{\lambda}_{j,k}(x,y) - (2j)^{\frac{1}{2}}\Phi^{\lambda}_{j-1,k-1}(x,y)\Big\} \\ & -\frac{1}{\mid \lambda \mid}(2j+2)^{\frac{1}{2}}\Big\{(2k+2)^{\frac{1}{2}}\Phi^{\lambda}_{j+1,k+1}(x,y) - (2j+2)^{\frac{1}{2}}\Phi^{\lambda}_{j,k}(x,y)\Big\} \end{aligned}$$

Put $j = k = n$, we derive

$$\boxed{\mid z \mid^2 \Phi^{\lambda}_{n,n}(z) = -\frac{2}{\mid \lambda \mid}(\Delta^+ - \Delta^-)\Phi^{\lambda}_{n,n}(z)} \tag{4.33}$$

as desired. $\square$

4.5 Radial Schwartz Functions on $\mathbb{H}$

In this section we want to characterize the class of radial Schwartz functions on the Heisenberg group in terms of their group Fourier transforms, in other words via their radial Fourier transforms.

As mentioned before, the Heisenberg group is the Lie group with underlying manifold $\mathbb{R}^3$. Identifying $z := x + iy$, then one basis for the Lie algebra $\mathbb{R}^3$ of left invariant vector fields on $\mathbb{H}$ is written as $Z, \bar{Z}, T$ where

$$Z = \partial/\partial_z + i\bar{z}\partial/\partial_t; \quad \bar{Z} = \partial/\partial_{\bar{z}} - iz\partial/\partial_t \quad \textit{and} \quad T = \partial/\partial_t.$$

With these conventions one has $\big[Z, \bar{Z}\big] = Z\bar{Z} - \bar{Z}Z = -2iT$. For more details see for example Geller's book [23].

With above notations we can state the following lemma for the class of radial Schwartz functions on the Heisenberg group:

Lemma 4.17. *The set $\mathcal{S}_r(\mathbb{H})$ of radial Schwartz functions on $\mathbb{H}$ has an alternative characterization given by*

$$\mathcal{S}_r(\mathbb{H}) = \Big\{ f \in C_r^\infty(\mathbb{H}) : \ (|z|^{2k} \mid t \mid^s)(\partial_z \partial_{\bar{z}})^d \partial_t^l f \in C_b(\mathbb{H}) \text{ for every } \ d, l, k, s \in \mathbb{N}_0 \Big\},$$

where C_r^∞ stands for the set of smooth and radial functions and C_b stands for the set of bounded and continuous functions on $\mathbb{H}$.
For $f \in \mathcal{S}_r(\mathbb{H})$ and $N \in \mathbb{N}$ define:

$$\| \ f \ \|_N = sup_{k+s \leq N, \ d+l \leq N, \ \omega=(z,t) \in \mathbb{H}} (|z|^{2k} \mid t \mid^s) \mid (\partial_z \partial_{\bar{z}})^d \partial_t^l f(\omega) \mid .$$

The $\mathcal{S}_r(\mathbb{H})$ is a Fréchet space whose topology is defined by the family of norms $\| \ . \ \|_N$.

We will present a sketch for the proof of the lemma in Appendix A.

The lemma provides the fact that the norms of any differentiation of a radial function and its multiplication with the polynomials of arbitrary degree can be controlled by N-norms, as above.
In view of the last lemma it becomes clear that first one must study the boundedness and continuity of radial functions. We will study this in the next section by use of inverse Fourier transform.

4.5.1 Bounded and Continuous radial Functions

All notations in the next definition have been introduced earlier in Section 2.2.

Definition 4.18. *Let $\mathcal{B}_1^{\oplus}$ be the space of measurable fields $(F(\lambda))_{\lambda \in \widehat{\mathbb{H}}}$ of trace-class operators, for which the following integral is finite:*

$$\int_\lambda \| \ F(\lambda) \ \|_1 \ d\mu(\lambda).$$

(Recall that $d\mu(\lambda) = (2\pi)^{-2} \, |\lambda| \, d\lambda$.) The next proposition points at the inverse of group Fourier transform for the Heisenberg group.

Proposition 4.19. *Suppose* $F = (F(\lambda)_{\lambda\in\widehat{\mathbb{H}}}) \in \mathcal{B}_1^{\oplus}$. *Then*

$$f(x) = \int_\lambda trace(F(\lambda) \circ \rho_\lambda(x)^*) d\mu(\lambda)$$

defines a function $f \in L^2(\mathbb{H}) * L^2(\mathbb{H})^*$. *Hence* f *is continuous.*

The proof of last proposition can be found for example in [33], which the author has studied the case of unimodular type I groups. Also, the proposition has been recently studied for general locally compact groups G in [20].
In the next theorem we show conditions for radial Fourier transforms of radial functions ensuring the boundedness and continuity.

Theorem 4.20. *Suppose* f *is a radial function in* $L^2(\mathbb{H})$ *and*

$$\hat{f}(\lambda) = \sum_n R_f(n,\lambda)\ \phi_n^\lambda \otimes \phi_n^\lambda$$

where R_f *is radial Fourier transform of* f. *Let* R_f *be a bounded function of variables* n *and* λ, *which for some* $d \geq 3$ *and some constant* C_d *fulfils*

$$\mid R_f(n,\lambda) \mid \leq \frac{C_d}{|\lambda|^d\,(2n+1)^d} \qquad \forall (n,\lambda). \tag{4.34}$$

Then $f \in C_b(\mathbb{H}) \cap L_r^2(\mathbb{H})$.

Proof: According to the proposition 4.19, it is sufficient to show that $(\hat{f}(\lambda))_{\lambda\in\widehat{\mathbb{H}}} \in \mathcal{B}_1^{\oplus}$, i.e, we show

$$\int_\lambda \parallel \hat{f}(\lambda) \parallel_1 d\mu(\lambda) < \infty \quad \text{respectively} \quad \int_\lambda \sum_n \mid R_f(n,\lambda) \mid d\mu(\lambda) < \infty.$$

But we can say more. We claim that

$$\int_\lambda \sum_n \mid R_f(n,\lambda) \mid^q d\mu(\lambda) < \infty$$

for $q = 1, 2$. The conjecture can be shown over two separate parts

$$\mathcal{A}_1 := \{(n,\lambda);\ |\lambda| \leq \frac{1}{(2n+1)}\} \quad \text{and} \quad \mathcal{A}_2 := \{(n,\lambda);\ |\lambda| > \frac{1}{(2n+1)}\}.$$

Then by boundedness of R_f, for a suitable $K \geq 0$ is

$$\begin{aligned}\sum_n \int_{\mathcal{A}_1} | R_f(n,\lambda) |^q \ d\mu(\lambda) &= \sum_n \int_{|\lambda| \leq \frac{1}{(2n+1)}} | R_f(n,\lambda) |^q \ d\mu(\lambda) \\ &\leq 2K^q \sum_n \int_{0<\lambda \leq \frac{1}{(2n+1)}} \lambda d\lambda = K^q \sum_n \frac{1}{(2n+1)^2} < \infty.\end{aligned}$$

Now using the assumption we get

$$\begin{aligned}\sum_n \int_{\mathcal{A}_2} | R(n,\lambda) |^q \ d\mu(\lambda) &= \sum_n \int_{|\lambda| > \frac{1}{(2n+1)}} | R_f(n,\lambda) |^q \ d\mu(\lambda) \\ &\leq 2C_d^q \sum_n \int_{\lambda > \frac{1}{2n+1}} \frac{\lambda}{\lambda^{qd}(2n+1)^{qd}} d\lambda \\ &\leq 2C_d^q \sum_n \frac{1}{(2n+1)^{qd}} \int_{\lambda > \frac{1}{(2n+1)}} \frac{1}{\lambda^2} d\lambda \\ &= \sum_n \frac{2C_d^q}{(2n+1)^{(qd+1)}} < \infty.\end{aligned}$$

The hypothesis that $qd \geq 3$ was used above to evaluate the integral of $\frac{1}{\lambda^{qd-1}}$ over $\frac{1}{2n+1} < \lambda$. Now again by Proposition 4.19 the function

$$f(z,t) = \sum_n \int_{\lambda \in \mathbb{R}^*} R_f(n,\lambda) \overline{\langle \rho_\lambda(z) \phi_n^\lambda, \phi_n^\lambda \rangle} e^{-i\lambda t} d\mu(\lambda) \quad \forall (z,t) \in \mathbb{H}$$

is a bounded and continuous radial function, i.e, $f \in C_b(\mathbb{H}) \cap L_r^2(\mathbb{H})$. □

Definition 4.21. *We define $\mathfrak{D}_0$ as the set of bounded functions R on $\mathbb{N}_0 \times \mathbb{R}^*$, such that for any nonnegative integer d exists a constant C_d so that*

$$| R(n,\lambda) | \leq \frac{C_d}{|\lambda|^d (2n+1)^d} \qquad \forall (n,\lambda).$$

Observe that according to the Theorem 4.20, functions with radial Fourier transform in $\mathfrak{D}_0$ are bounded and continuous. More precisely we have the following modification:

Corollary 4.22. *Let R be a function in $\mathfrak{D}_0$. Then there exists some function $f \in C_b(\mathbb{H}) \cap L_r^2(\mathbb{H})$, with $R_f = R$.*

Lemma 4.23. *Let Δ^+ and Δ^- be the difference operators introduced in Definition 4.12 and suppose that $R \in \mathcal{D}_0$. Then*

$$\begin{cases} (1)\ \sum_n^\infty R(n,\lambda)\Delta^+\Phi^\lambda_{n,n}(z) = & -\sum_n^\infty (\Delta^- + 1)R(n,\lambda)\Phi^\lambda_{n,n}(z) \\ (2)\ \sum_n^\infty R(n,\lambda)\Delta^-\Phi^\lambda_{n,n}(z) = & -\sum_n^\infty (\Delta^+ + 1)R(n,\lambda)\Phi^\lambda_{n,n}, \end{cases}$$

and hence combining (1) *and* (2) *we get*

$$\sum_n^\infty R(n,\lambda)(\Delta^+ - \Delta^-)\Phi^\lambda_{n,n}(z) = \sum_n^\infty (\Delta^+ - \Delta^-)R(n,\lambda)\Phi^\lambda_{n,n}(z)$$

Proof: (Observe that here all series are absolutely convergent, since for any z and $\lambda \neq 0$ the functions $n \mapsto \Phi^\lambda_{n,n}(z)$ are bounded on $\mathbb{N}_0$, and $\sum\limits_n n\,|R(n,\lambda)| < \infty$.) To the proof of (1):

$$\begin{aligned} \sum_{n=0}^\infty R(n,\lambda)\ \Delta^+\Phi^\lambda_{n,n}(z) &= \sum_{n=0}^\infty R(n,\lambda)(n+1)(\Phi^\lambda_{n+1,n+1}(z) - \Phi^\lambda_{n,n}(z)) \\ &= \sum_{n=0}^\infty (n+1)R(n,\lambda)\Phi^\lambda_{n+1,n+1}(z) - \sum_{n=0}^\infty (n+1)R(n,\lambda)\Phi^\lambda_{n,n}(z) \\ &= \sum_{n=1}^\infty nR(n-1,\lambda)\Phi^\lambda_{n,n}(z) - \sum_{n=0}^\infty (n+1)R(n,\lambda)\Phi^\lambda_{n,n}(z) \\ &= \sum_{n=1}^\infty n\big(R(n-1,\lambda) - R(n,\lambda)\big)\Phi^\lambda_{n,n}(z) - \sum_{n=0}^\infty R(n,\lambda)\Phi^\lambda_{n,n}(z) \\ &= \sum_{n=0}^\infty -\Delta^- R(n,\lambda)\Phi^\lambda_{n,n}(z) - \sum_{n=0}^\infty R(n,\lambda)\Phi^\lambda_{n,n}(z) \\ &= \sum_{n=0}^\infty -(\Delta^- + 1)R(n,\lambda)\ \Phi^\lambda_{n,n}(z), \end{aligned}$$

and to the proof of (2) :

$$\begin{aligned}
\sum_{n=0}^{\infty} R(n,\lambda)\ \Delta^{-}\Phi^{\lambda}_{n,n}(z) &= \sum_{n=0}^{\infty} R(n,\lambda)(n)(\Phi^{\lambda}_{n,n}(z) - \Phi^{\lambda}_{n-1,n-1}(z)) \\
&= \sum_{n=0}^{\infty} nR(n,\lambda)\Phi^{\lambda}_{n,n}(z) - \sum_{n=0}^{\infty} nR(n,\lambda)\Phi^{\lambda}_{n-1,n-1}(z) \\
&= \sum_{n=0}^{\infty} nR(n,\lambda)\Phi^{\lambda}_{n,n}(z) - \sum_{n=0}^{\infty}(n+1)R(n+1,\lambda)\Phi^{\lambda}_{n,n}(z) \\
&= \sum_{n=0}^{\infty}(n+1)\big(R(n,\lambda) - R(n+1,\lambda)\big)\Phi^{\lambda}_{n,n}(z) - \sum_{n=0}^{\infty} R(n,\lambda)\Phi^{\lambda}_{n,n}(z) \\
&= \sum_{n=0}^{\infty} -\Delta^{+}R(n,\lambda)\Phi^{\lambda}_{n,n}(z) - \sum_{n=0}^{\infty} R(n,\lambda)\Phi^{\lambda}_{n,n}(z) \\
&= \sum_{n=0}^{\infty} -(\Delta^{+}+1)R(n,\lambda)\ \Phi^{\lambda}_{n,n}(z),
\end{aligned}$$

which completes the proof of Lemma 4.23. □

4.5.2 Differentiation of radial Functions

Next, we want to study some properties of radial Fourier transforms of radial functions, for which the corresponding radial functions are smooth. We will see that the smoothness of a radial function directly depends on its radial Fourier transform. First of all we have the following simple lemma:

Lemma 4.24. *Suppose $R \in \mathfrak{D}_0$ and $s \in \mathbb{N}$. Then every multiplication of R with factor λ^s is contained in $\mathfrak{D}_0$.*

Proof: Since $R \in \mathfrak{D}_0$ then for positive number $s+d$ there exists some constant C_d such that

$$|\ R(n,\lambda)\ | \leq \frac{C_d}{|\ \lambda\ |^{d+s}\ (2n+1)^{d+s}} \qquad \forall\ (n,\lambda). \tag{4.35}$$

Now by multiplication of relation (4.35) by $\mid \lambda \mid^s$ from both sides we obtain:

$$\begin{aligned}\mid \lambda \mid^s \mid R(n,\lambda) \mid &\leq \frac{\mid \lambda \mid^s C_d}{\mid \lambda \mid^{d+s} (2n+1)^{d+s}} \\ &= \frac{C_d}{\mid \lambda \mid^{d} (2n+1)^{d+s}} \\ &\leq \frac{C_d}{\mid \lambda \mid^{d} (2n+1)^{d}},\end{aligned}$$

which shows $\lambda^s R$ is contained in $\mathfrak{D}_0$. □

Lemma 4.24 and Theorem 4.20 combine to yield the following result.

Theorem 4.25. *Suppose f is in $L^2_r(\mathbb{H})$ and suppose $R_f \in \mathfrak{D}_0$. Then for any nonnegative natural number s, $\partial_t^s f$ exists and is contained in $C_b(\mathbb{H}) \cap L^2_r(\mathbb{H})$. In other words $R_{\partial_t^s f} \in \mathfrak{D}_0$.*

Proof: Using Lemma 4.24 , $(i\lambda)^s R_f$ is contained in $\mathfrak{D}_0$ and using Theorem 4.20, the function $F_s(z,t)$ defined by

$$F_s(z,t) = \sum_n \int_{\mathbb{R}^*} (-i\lambda)^s R(n,\lambda) \overline{\Phi^\lambda_{n,n}(z)} e^{-i\lambda t} d\mu(\lambda)$$

is a bounded and continuous function on $\mathbb{H}$. Furthermore we have:

$$\begin{aligned}F_s(z,t) = \sum_n \int_{\mathbb{R}^*} (-i\lambda)^s R(n,\lambda) \overline{\Phi^\lambda_{n,n}(z)} e^{-i\lambda t} d\mu(\lambda) &= \sum_n \int_{\mathbb{R}^*} R(n,\lambda) \overline{\Phi^\lambda_{n,n}(z)} \partial_t^s e^{-i\lambda t} d\mu(\lambda) \\ &= \partial_t^s \sum_n \int_{\mathbb{R}^*} R(n,\lambda) \overline{\Phi^\lambda_{n,n}(z)} e^{-i\lambda t} d\mu(\lambda) \\ &= \partial_t^s f(z,t)\end{aligned}$$

by absolute convergence. Observe that above the exchange of differentiation operator with integral and sum is permissible from the same argument in Remark 4.13. Thereby the last computation shows the existence of $\partial_t^s f$, which is a bounded and continuous function with $\partial_t^s f = F_s$ and $R_{\partial_t^s f} = (-i\lambda)^s R_f \in \mathfrak{D}_0$. □

Lemma 4.26. *Suppose $R \in \mathfrak{D}_0$. Then for any nonnegative integer number d we have*

$$\mid \lambda \mid^d \left\{\Delta^+ - \Delta^- + 2(2n+1)\right\}^d R(n,\lambda) \in \mathfrak{D}_0.$$

Proof: We show the assertion of the lemma for $d = 1$. The case $d > 1$ follows by induction. For $d = 1$, the term $\mid \lambda \mid \big\{\Delta^+ - \Delta^- + 2(2n+1)\big\}R(n,\lambda)$ can be written in the following way:

$$\begin{aligned}
&\mid \lambda \mid \big\{\Delta^+ - \Delta^- + 2(2n+1)\big\}R(n,\lambda) \\
&= \mid \lambda \mid \{(n+1)R(n+1,\lambda) - (2n+1)R(n,\lambda) + nR(n-1,\lambda)\}.
\end{aligned}$$

For the proof, let $m \geq 0$. Since $R \in \mathfrak{D}_0$, then there exists some constant C_m so that

$$\mid R(n+1,\lambda) \mid \leq \frac{C_m}{\mid \lambda \mid^{m+1} (2(n+1)+1)^{m+1}}, \tag{4.36}$$

$$\mid R(n,\lambda) \mid \leq \frac{C_m}{\mid \lambda \mid^{m+1} (2n+1)^{m+1}}, \tag{4.37}$$

$$\mid R(n-1,\lambda) \mid \leq \frac{C_m}{\mid \lambda \mid^{m+1} (2(n-1)+1)^{m+1}}. \tag{4.38}$$

Using the evaluations in (4.36) – (4.38) we get

$$\begin{aligned}
\mid \lambda \mid \mid (n+1)R(n+1,&\lambda) - (2n+1)R(n,\lambda) + nR(n-1,\lambda) \mid \\
&\leq \mid \lambda \mid (n+1) \mid R(n+1,\lambda) \mid + \mid \lambda \mid (2n+1) \mid R(n,\lambda) \mid \\
&\quad + \mid \lambda \mid n \mid R(n-1,\lambda) \mid \\
&\leq \mid \lambda \mid (n+1) \frac{C_m}{\mid \lambda \mid^{m+1} (2(n+1)+1)^{m+1}} \\
&\quad + \mid \lambda \mid (2n+1)) \frac{C_m}{\mid \lambda \mid^{m+1} (2n+1)^{m+1}} \\
&\quad + \mid \lambda \mid n \frac{C_m}{\mid \lambda \mid^{m+1} (2(n-1)+1)^{m+1}} \\
&\leq \frac{C_m}{\mid \lambda \mid^{m} (2n+1)^{m}} + \frac{C_m}{\mid \lambda \mid^{m} (2n+1)^{m}} \\
&\quad + \frac{nC_m}{\mid \lambda \mid^{m+1} (2n-1)^{m+1}}.
\end{aligned} \tag{4.39}$$

Now, using the inequalities

$$\frac{n}{2n+1} \leq K_m \Big(\frac{2n-1}{2n+1}\Big)^{m+1} \quad \text{respectively} \quad \frac{n}{(2n-1)^{m+1}} \leq \frac{K_m}{(2n+1)^m} \quad \forall n \in \mathbb{N}$$

we obtain

$$(4.39) \leq \frac{C_m}{|\lambda|^m (2n+1)^m} + \frac{C_m}{|\lambda|^m (2n+1)^m} + \frac{C_m K_m}{|\lambda|^m (2n+1)^m} = \frac{(2+K_m)C_m}{|\lambda|^m (2n+1)^m}$$

and hence taking $\widetilde{C}_m = (2+K_m)C_m$ we get

$$|\lambda|\,|\big\{\Delta^+ - \Delta^- + 2(2n+1)\big\}R(n,\lambda)| \leq \frac{\widetilde{C}_m}{|\lambda|^m (2n+1)^m} \quad \forall (n,\lambda)$$

as desired. □

The next theorem deals with smoothness of radial functions on $\mathbb{C}$.

Theorem 4.27. *Suppose f is in $L^2_r(\mathbb{H})$ and $R_f \in \mathfrak{D}_0$. Then for any nonnegative number d, $(\partial_z \partial_{\bar{z}})^d f$ exists and $R_{(\partial_z\partial_{\bar{z}})^d f} \in \mathfrak{D}_0$.*

Proof: The theorem is proved with induction. Using the last lemma the following Hilbert Schmidt operator

$$\sum_n -\frac{|\lambda|}{2}\big\{\Delta^+ - \Delta^- + 2(2n+1)\big\}R_f(n,\lambda)\,\phi_n^\lambda \otimes \phi_n^\lambda$$

is well-defined and defines via pointwise Fourier inversion the following bounded and continuous radial function in $L^2(\mathbb{H})$

$$\begin{aligned}
&\sum_n \int -\frac{|\lambda|}{2}\big\{\Delta^+ - \Delta^- + 2(2n+1)\big\}R_f(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda) \\
&\qquad = \sum_n \int -\frac{|\lambda|}{2}R(n,\lambda)\big\{\Delta^+ - \Delta^- + 2(2n+1)\big\}\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda) \\
&\qquad = \sum_n \int R_f(n,\lambda)(\partial_z\partial_{\bar{z}})\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda).
\end{aligned}$$

Recall that the last two equalities can be written by Lemmas 4.23 and 4.14 respectively. By absolute convergence of the series we have

$$\begin{aligned}
&\sum_n \int R_f(n,\lambda)\,(\partial_z\partial_{\bar{z}})\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda) \\
&= (\partial_z\partial_{\bar{z}})\sum_n \int R_f(n,\lambda)\,\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda) \\
&= (\partial_z\partial_{\bar{z}})f(z,t)
\end{aligned}$$

which yields the existence of a bounded and continuous radial function $(\partial_z\partial_{\bar{z}})f$ in $L^2(\mathbb{H})$ with

$$R_{(\partial_z\partial_{\bar{z}})f} = -\frac{|\lambda|}{2}\big\{\Delta^+ - \Delta^- + 2(2n+1)\big\}R_f \in \mathfrak{D}_0.$$

Now let $d > 1$. Since

$$|\lambda|^d \left\{\Delta^+ - \Delta^- + 2(2n+1)\right\}^d R_f \in \mathfrak{D}_0.$$

Then similar to the first term of induction is

$$|\lambda|^{d+1} \left\{\Delta^+ - \Delta^- + 2(2n+1)\right\}^{d+1} R_f \in \mathfrak{D}_0$$

which proves the existence of bounded and continuous radial function $(\partial_z\partial_{\bar{z}})^d f$ in $L^2(\mathbb{H})$.

□

Here, we extract the last corollary of this section through Theorems 4.25 and 4.27.

Corollary 4.28. *Suppose f be a radial function in $L^2(\mathbb{H})$ with $R_f \in \mathfrak{D}_0$. Then for every $d, s \in \mathbb{N}_0$, $(\partial_z\partial_{\bar{z}})^d\partial_t^s f$ exists and $R_{(\partial_z\partial_{\bar{z}})^d\partial_t^s f} \in \mathfrak{D}_0$.*

4.5.3 Multiplication by Polynomials

This section is concerned with criteria on the Fourier transform of a radial function f which guarantees that the product of f with arbitrary powers of $|z|$ and t gives a bounded function. The next Theorem gives a partial answer.

Theorem 4.29. *Suppose f is in $L^2_r(\mathbb{H})$, such that its radial Fourier transform R_f satisfies the condition*

$$\frac{1}{|\lambda|^p}(\Delta^+ - \Delta^-)^p R_f(n,\lambda) \in \mathfrak{D}_0 \quad \forall\, p \in \mathbb{N}_0. \tag{4.40}$$

Then $R_{|z|^{2p}f} \in \mathfrak{D}_0$ for any nonnegative p and the property in (4.40) holds for the function $R_{|z|^{2p}f}$ as well.

Proof: Suppose $p \in \mathbb{N}_0$. To prove the assertion we show that the function $| z |^{2p} f$ is in $L^2_r(\mathbb{H})$ and has the radial Fourier transform

$$R_{|z|^{2p}f}(n,\lambda) = \frac{1}{| \lambda |^p}(\Delta^+ - \Delta^-)^p R_f(n,\lambda) \in \mathfrak{D}_0.$$

Since $\frac{1}{|\lambda|^p}(\Delta^+ - \Delta^-)^p R(n,\lambda) \in \mathfrak{D}_0$ then due to the inversion Fourier transform and by Theorem 4.20 the radial function

$$\widetilde{f}(z,t) = \sum_n \int_\lambda -\frac{2^p}{| \lambda |^p}(\Delta^+ - \Delta^-)^p R_f(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{i\lambda t}d\mu(\lambda)$$

is bounded and continuous. Applying Lemmas 4.23 and 4.16 we have :

$$\begin{aligned}\widetilde{f}(z,t) &= \sum_n \int_\lambda -\frac{2^p}{| \lambda |^p}(\Delta^+ - \Delta^-)^p R_f(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{i\lambda t}d\mu(\lambda)\\ &= \sum_n \int_\lambda R_f(n,\lambda) \mid z \mid^{2p} \overline{\Phi^\lambda_{n,n}(z)}e^{i\lambda t}d\mu(\lambda)\\ &=| z |^{2p} \sum_n R_f(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{i\lambda t}d\mu(\lambda)\\ &=| z |^{2p} f(z,t)\end{aligned}$$

Hence $| z |^{2p} f$ is bounded and continuous with $R_{|z|^{2p}f} = -\frac{2^p}{|\lambda|^p}(\Delta^+ - \Delta^-)^p R_f \in \mathfrak{D}_0$. $\square$

Notation: The set $C^1(\mathbb{R}^*)$ denotes the set of functions R which are differentiable on $\mathbb{R}^*$ and $\lim_{\lambda\to 0^+} R(\lambda)$ and $\lim_{\lambda\to 0^-} R(\lambda)$ and as well $\lim_{\lambda\to 0^+} \partial_\lambda R(\lambda)$ and $\lim_{\lambda\to 0^-} \partial_\lambda R(\lambda)$ exist and

$$\lim_{\lambda\to 0^+} R(\lambda) = \lim_{\lambda\to 0^-} R(\lambda), \quad \lim_{\lambda\to 0^+} \partial_\lambda R(\lambda) = \lim_{\lambda\to 0^-} \partial_\lambda R(\lambda).$$

Using the last notation, the following is an analog of Theorem 4.29 for the t variable.

Theorem 4.30. *Suppose f is a bounded radial function in $L^2(\mathbb{H})$ such that for any $n \in \mathbb{N}_0$, $R_f(n,.)$ is contained in $C^1(\mathbb{R}^*)$. And, suppose*

$$\frac{1}{\lambda}(\Delta^+ - \Delta^-)R_f(n,\lambda) \quad \textit{and} \quad (\partial_\lambda - \frac{1}{\lambda}\Delta^+)R_f(n,\lambda) \in \mathfrak{D}_0.$$

Then the function $(it)f$ is radial and

$$R_{(it)f}(n,\lambda) = sgn\lambda\big[-\frac{1}{2\lambda}(\Delta^+ - \Delta^-) + (\partial_\lambda - \frac{1}{\lambda}\Delta^+)\big]R_f(n,\lambda) \in \mathfrak{D}_0. \tag{4.41}$$

Note 4.31. *The relation in (4.41) implies that* $tf \in C_b(\mathbb{H}) \cap L^2_r(\mathbb{H})$.

(By the proof of Theorem 4.20 we may apply pointwise Fourier inversion.)

Proof: From inverse Fourier transform we have the following:

$$tf(z,t) = \sum_n \int_{\mathbb{R}^*} R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}\; te^{-i\lambda t} d\mu(\lambda) = i\sum_n \int_{\mathbb{R}^*} R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}\left(\partial_\lambda e^{-i\lambda t}\right) d\mu(\lambda)$$

then by partial integration and recalling that $d\mu(\lambda) = (2\pi)^{-2}\,|\lambda|\,d\lambda$ one gets

$$tf(z,t) = i(2\pi)^{-2}\sum_n \Big\{ \int_{-\infty}^0 \partial_\lambda(\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)})e^{-i\lambda t}d\lambda - \lim_{\lambda\to 0^-}(\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t})\Big\} \tag{4.42}$$

$$- i(2\pi)^{-2}\sum_n \Big\{ \int_0^{+\infty} \partial_\lambda(\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)})e^{-i\lambda t}d\lambda - \lim_{\lambda\to 0^+}(\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t})\Big\} \tag{4.43}$$

Observe that $\lim_{\lambda\to 0^-}(\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}) = \lim_{\lambda\to 0^+}(\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}) = 0$, since R is bounded and $\lim\ \Phi^\lambda_{n,n}(z)e^{-i\lambda t}$ exists as $\lambda \to 0$. Hence

$$\begin{aligned}
&(2\pi)^2(it)f(z,t) \\
&\qquad = \sum_n \left\{\int_0^{+\infty} \partial_\lambda(\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)})e^{-i\lambda t}d\lambda - \int_{-\infty}^0 \partial_\lambda(\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)})e^{-i\lambda t}d\lambda\right\} \\
&\qquad = I - II.
\end{aligned} \tag{4.44}$$

Here we compute the term I, and the term II can be done similarly,

$$\begin{aligned}
I &= \sum_n \int_0^{+\infty} \Big\{R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)} + \lambda\big(\partial_\lambda R(n,\lambda)\big)\overline{\Phi^\lambda_{n,n}(z)} + \lambda R(n,\lambda)\partial_\lambda\overline{\Phi^\lambda_{n,n}(z)}\Big\}e^{-i\lambda t}d\lambda \\
&= \sum_n \int_0^{+\infty} \Big\{R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)} + \lambda\partial_\lambda R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}\Big\}\, e^{-i\lambda t}d\lambda \\
&+ \sum_n \int_0^{+\infty} \lambda R(n,\lambda)\partial_\lambda\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\lambda
\end{aligned} \tag{4.45}$$

Using Lemmas 4.15, 4.16 and 4.23 respectively in the latter term we have:

$$\begin{aligned}
&\sum_n \int_0^{+\infty} \lambda R(n,\lambda)\partial_\lambda \overline{\Phi_{n,n}^\lambda(z)} e^{-i\lambda t} d\lambda \\
&= \sum_n \int_0^{\infty} \lambda R(n,\lambda)\big(\frac{1}{\lambda}\Delta^- - \frac{1}{4} \mid z \mid^2\big) \overline{\Phi_{n,n}^\lambda(z)} e^{-i\lambda t} d\lambda \\
&= \sum_n \int_0^{\infty} \left(-(\Delta^+ + 1) - \frac{1}{2}(\Delta^+ - \Delta^-)\right) R(n,\lambda) \overline{\Phi_{n,n}^\lambda(z)} e^{-i\lambda t} d\lambda. \qquad (4.46)
\end{aligned}$$

Replacing (4.46) in (4.45) we obtain:

$$I = \sum_n \int_0^{\infty} \left\{(\lambda\partial_\lambda - \Delta^+) - \frac{1}{2}(\Delta^+ - \Delta^-)\right\} R(n,\lambda) \overline{\Phi_{n,n}^\lambda(z)} e^{-i\lambda t} d\lambda. \qquad (4.47)$$

The similar computation for II we show that

$$-II = \sum_n \int_{-\infty}^{0} \left\{(\lambda\partial_\lambda - \Delta^+) + \frac{1}{2}(\Delta^+ - \Delta^-)\right\} R(n,\lambda) \overline{\Phi_{n,n}^\lambda(z)} e^{-i\lambda t} d\lambda. \qquad (4.48)$$

Now, substituting I and II into (4.44)

$$(it)f(z,t) = \sum_n \int sgn\lambda \Big\{(\partial_\lambda - \frac{1}{\lambda}\Delta^+) - \frac{1}{2\lambda}(\Delta^+ - \Delta^-)\Big\} R(n,\lambda) \overline{\Phi_{n,n}^\lambda(z)} e^{-i\lambda t} d\mu(\lambda)$$

as desired. □

The next Theorem lists some conditions on R_f to generalize the result in Theorem 4.30 for t^s, where $s > 1$.

Theorem 4.32. *Suppose f is a bounded function in $L_r^2(\mathbb{H})$, such that its radial Fourier transform R_f has the following properties:*

$$(i) \quad R_f \in \mathfrak{D}_0 \qquad (4.49)$$

$$(ii) \quad R_f(n,.) \in C^\infty(\mathbb{R}^*) \quad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \lambda^m \partial_\lambda^m R_f \in \mathfrak{D}_0 \ \ \forall\, m \in \mathbb{N}_0$$

$$(iii) \quad \lambda^m \partial_\lambda^m (\partial_\lambda - \frac{1}{\lambda}\Delta^+)^k \frac{1}{|\lambda|^p}(\Delta^+ - \Delta^-)^p R_f(n,\lambda) \in \mathfrak{D}_0 \quad \forall\, m,k,p \in \mathbb{N}_0$$

$$(iv) \quad \lambda^m \partial_\lambda^m \frac{1}{|\lambda|^p}(\Delta^+ - \Delta^-)^p (\partial_\lambda - \frac{1}{\lambda}\Delta^+)^k R_f(n,\lambda) \in \mathfrak{D}_0 \quad \forall\, m,k,p \in \mathbb{N}_0.$$

Then for any $s \in \mathbb{N}_0$ is $R_{(it)^s f} \in \mathfrak{D}_0$ with

$$R_{(it)^s f}(n,\lambda) = (sgn\lambda)^s \left[-\frac{1}{2\lambda}(\Delta^+ - \Delta^-) + (\partial_\lambda - \frac{1}{\lambda}\Delta^+)\right]^s R_f(n,\lambda) \in \mathfrak{D}_0,$$

and the properties $(ii) - (iv)$ hold for $R_{(it)^s f}$ also.

Proof: The statement is proved with induction on s. For the first statement $s = 1$ it is known from the assumption of the Theorem and an application of Theorem 4.30 , i.e., for $s = 1$ is

$$R_{(it)f}(n,\lambda) = (sgn\lambda)\left[-\frac{1}{2\lambda}(\Delta^+ - \Delta^-) + (\partial_\lambda - \frac{1}{\lambda}\Delta^+)\right] R_f(n,\lambda) \in \mathfrak{D}_0$$

and fulfils all conditions $(i) - (iv)$.

Assume that the statement is true for some $s > 1$. Then using the first statement of induction for the function $f_s = (it)^s f$ we are done, i.e., the function $(it)^{s+1}f$ is radial in $L^2(\mathbb{H})$ with corresponding radial Fourier transform $R_{(it)^{s+1}f}$ with

$$R_{(it)^{s+1}f}(n,\lambda) = (sgn\lambda)^s \left[-\frac{1}{2\lambda}(\Delta^+ - \Delta^-) + (\partial_\lambda - \frac{1}{\lambda}\Delta^+)\right]^s R_f(n,\lambda) \in \mathfrak{D}_0$$

which satisfies the properties in $(ii) - (iv)$. □

4.5.4 Sufficient and Necessary Conditions for a Radial Function to be Schwartz

The main result of this section is presented in Theorem 4.36. The importance of the theorem is that it provides a simple way of picking out an element of $\mathcal{S}_r(\mathbb{H})$, which is reduced to a given **rapidly decreasing** element R.

Definition 4.33. *The space $\widetilde{\mathcal{D}}$ consists of all the functions $R : \mathbb{N}_0 \times \mathbb{R}^* \longrightarrow \mathbb{C}$, for which for any $n \in \mathbb{N}_0$ $\lim\limits_{\lambda\to 0^+} R(n,\lambda)$ and $\lim\limits_{\lambda\to 0^-} R(n,\lambda)$ both exist and are equal and R fulfils the following properties:*

(i) $R \in \mathfrak{D}_0$

(ii) $R(n,.) \in C^\infty(\mathbb{R}^*)$ $\forall n \in \mathbb{N}_0$ *and* $\lambda^m \partial_\lambda^m R \in \mathfrak{D}_0$ $\forall\, m \in \mathbb{N}_0$

(iii) $\lambda^m \partial_\lambda^m\, (\partial_\lambda - \frac{1}{\lambda}\Delta^+)^k \frac{1}{|\lambda|^p}(\Delta^+ - \Delta^-)^p R \in \mathfrak{D}_0$ $\forall\, m,k,p \in \mathbb{N}_0$

(iv) $\lambda^m \partial_\lambda^m\, \frac{1}{|\lambda|^p}(\Delta^+ - \Delta^-)^p (\partial_\lambda - \frac{1}{\lambda}\Delta^+)^k R \in \mathfrak{D}_0$ $\forall\, m,k,p \in \mathbb{N}_0$.

*We shall call the elements of $\widetilde{\mathcal{D}}$ **rapidly decreasing functions** on $\mathbb{N}_0 \times \mathbb{R}^*$.*

The next corollary can immediately be extracted by Theorem 4.32 and Corollary 4.28 :

Corollary 4.34. *Suppose $f \in L^2_r(\mathbb{H})$ so that $R_f \in \widetilde{\mathcal{D}}$. Then for any $s, k, d, l \in \mathbb{N}_0$*

$$R_{(\partial_z \partial_{\bar{z}})^d \partial_t^l (|z|^{2k}|t|^s)f} \in \mathfrak{D}_0.$$

We then have the next theorem, which contains one of the central results of this section and expresses a sufficient condition for elements in $\mathcal{S}_r(\mathbb{H})$:

Theorem 4.35. *Suppose $f \in L^2_r(\mathbb{H})$, with $R_f \in \widetilde{\mathcal{D}}$. Then $f \in \mathcal{S}_r(\mathbb{H})$.*

Proof: In view of Lemma 4.17 we need to show

$$\mid z \mid^{2k} \mid t \mid^s (\partial_z \partial_{\bar{z}})^d \partial_t^l f \in C_b(\mathbb{H}) \quad \forall s, k \in \mathbb{N}_0 \text{ and } d, l \in \mathbb{N}_0.$$

The expression is proved with induction on $k + s$. For $k + s = 0$ the statement is known from Corollary 4.34. Now let it be true for $k+s = m$; we intend to show it for $k+s = m+1$. For $k + s = m + 1$ we may write

$$(\partial_z \partial_{\bar{z}})^d \partial_t^l \mid z \mid^{2k} \mid t \mid^s f = \mid z \mid^{2k} \mid t \mid^s (\partial_z \partial_{\bar{z}})^d \partial_t^l f + \sum_{j=0}^{m} P_j(z, \bar{z}, t) D_j(\partial_z \partial_{\bar{z}}, \partial_t) f \qquad (4.50)$$

where the P_j and the D_j are polynomials in $z, \bar{z}, t$ variables and respectively in $\partial_z, \partial_{\bar{z}}, \partial_t$ of maximal degree j (see the work of Geller [22]). Observe that the left hand side in above equation is in $C_b(\mathbb{H})$ form Corollary 4.34 and the sum over j on the right hand side is in $C_b(\mathbb{H})$ also because of assumption of induction. Hence we get our assertion. $\square$

Theorem 4.36. *(Main theorem) Suppose f is a function in $L^2_r(\mathbb{H})$ with corresponding radial Fourier transform R_f on $\mathbb{N}_0 \times \mathbb{R}^*$. Then*

$$f \in \mathcal{S}_r(\mathbb{H}) \Longleftrightarrow R_f \in \widetilde{\mathcal{D}}.$$

Proof: The part " $\Leftarrow$ " was already proved in Theorem 4.35. For the converse direction let f be a radial Schwartz function. Then we shall show:

a) $R_f \in \mathfrak{D}_0$ and $\lim_{\lambda \to 0^+} R(n, \lambda)$; $\lim_{\lambda \to 0^+} R(n, -\lambda)$ exists and are equal,

b) $R_f(n,.) \in C_0^\infty(\mathbb{R}^*)$ for each fixed $n \in \mathbb{N}_0$ and $\lambda^m \partial_\lambda^m R_f \in \mathfrak{D}_0$ for each $m \in \mathbb{N}$

c) $\lambda^m \partial_\lambda^m \left(\partial_\lambda - \frac{1}{\lambda}\Delta^+\right)^k \frac{1}{|\lambda|^p}(\Delta^+ - \Delta^-)^p R_f \in \mathfrak{D}_0$ for all $m, k, p \geq 0$

d) $\lambda^m \partial_\lambda^m \frac{1}{|\lambda|^p}(\Delta^+ - \Delta^-)^p \left(\partial_\lambda - \frac{1}{\lambda}\Delta^+\right)^k R_f \in \mathfrak{D}_0$ for all $m, k, p \geq 0$

Proof of a): To show that $R_f \in \mathfrak{D}_0$, we consider the **sub-Laplacian operator** on the Heisenberg group $\mathfrak{L} := -\Delta - \frac{1}{4} \mid z \mid^2 \partial_t^2 + (x\partial_y - y\partial_x)$ which has eigenfunctions $e_{n,\lambda}(z,t) = \Phi_{n,n}^\lambda(z)e^{i\lambda t}$ with eigenvalues $|\lambda|\,(2n+1)$ ([51] §1.4). Using (4.9) for any $N \in \mathbb{N}_0$ we have:

$$\begin{aligned}
|\lambda|^N (1+2n)^N |R_f(n,\lambda)| &= \left| \int\int f(z,t)\,|\lambda|^N (1+2n)^N e_{n,\lambda}(z,t)dzdt \right| \\
&= \left| \int\int f(z,t)\mathfrak{L}^N(e_{n,\lambda})(z,t)dzdt \right| \\
&= \left| \int\int \mathfrak{L}^N f(z,t)e_{n,\lambda}(z,t)dzdt \right| \\
&\leq \| \mathfrak{L}^N f \|_1 .
\end{aligned}$$

(Observe that in computation of above integrals we use the partial integration for the vector fields. For more details see for example [53].) Taking $C_N := \| \mathfrak{L}^N f \|_1$ we get

$$| R_f(n,\lambda) | \leq \frac{C_N}{|\lambda|^N (1+2n)^N} \qquad \forall (n,\lambda) \in \mathbb{N}_0 \times \mathbb{R}^*.$$

For existence and equality of limits at zero, the inverse Fourier transform allows to write

$$R_f(n,\lambda) = \int_{\mathbb{R}} \int_{\mathbb{C}} f(z,t)e_{n,\lambda}(z,t)dzdt. \tag{4.51}$$

Observe that $f(z,t)e_{n,\lambda}(z,t)$ is uniformly bounded by f. And since $\lim_{\lambda\to 0} e_{\lambda,n}(z,t) = \lim_{\lambda\to 0} \Phi_{n,n}^\lambda(z)e^{i\lambda t} = 1$ and f is integrable then by dominated convergence theorem one immediately can derive that

$$\lim_{\lambda\longrightarrow 0^+} R_f(n,\lambda) = \lim_{\lambda\longrightarrow 0^+} R_f(n,-\lambda) = \int_t \int_z f(z,t)dzdt.$$

Proof of b): To show the smoothness of function R_f we return to the equality (4.51)

$$R_f(n,\lambda) = \int_{\mathbb{R}} \int_{\mathbb{C}} f(z,t)e_{n,\lambda}(z,t)dzdt = \int\int f(z,t)\Phi_{n,n}^\lambda(z)e^{i\lambda t}dzdt.$$

Since functions $\{e_{n,\lambda}(z,t)\}_{n,\lambda}$ are smooth in $\lambda \in \mathbb{R}^*$ for fixed (z,t) then the functions $\{(fe_{n,\lambda})(z,t)\}$ are. Observe that for any $m \in \mathbb{N}$, the function $f\partial_\lambda^m e_{n,\lambda}(z,t)$ is continuous in variable λ and $f\partial_\lambda^m e_{n,\lambda}$ is absolute integrable on $\mathbb{R} \times \mathbb{C}$. Hence $\partial_\lambda^m R$ exists and

$$\partial_\lambda^m R(n,\lambda) = \int\int f(z,t)\partial_\lambda^m e_{n,\lambda}(z,t)dzdt.$$

For the second part, to show $\lambda^m \partial_\lambda^m R_f \in \mathfrak{D}_0$ for any $m \in \mathbb{N}$, we take the derivation of functions $\Phi_{n,n}^\lambda$ in variable $\lambda \in \mathbb{R}^*$. But, with the simple computation in Lemma 4.15, we saw that

$$\partial_\lambda \Phi_{n,n}^\lambda(z) = \Big(\frac{1}{\lambda}\Delta^- - \frac{|\, z\,|^2}{4} sgn\lambda\Big)\Phi_{n,n}^\lambda(z).$$

Hence, for $\lambda > 0$ and $e_{n,\lambda}(z,t) = \Phi_n^\lambda(z)e^{i\lambda t}$ we have

$$\begin{aligned}\partial_\lambda e_{n,\lambda}(z,t) &= (it)e_{n,\lambda}(z,t) + \frac{1}{\lambda}\Delta^- e_{n,\lambda}(z,t) - \frac{|\, z\,|^2}{4}e_{n,\lambda}(z,t)\\ &= (it)e_{n,\lambda}(z,t) + \frac{n}{\lambda}\Big(e_{n,\lambda}(z,t) - e_{n-1,\lambda}(z,t)\Big) - \frac{|\, z\,|^2}{4}e_{n,\lambda}(z,t).\end{aligned}$$

Then for $m = 1$:

$$\begin{aligned}\partial_\lambda R(n,\lambda) &= \int\int f(z,t)\ \partial_\lambda e_{n,\lambda}(z,t)dzdt\\ &= R_{(itf)}(n,\lambda) + \frac{n}{\lambda}\Big(R_f(n,\lambda) - R_f(n-1,\lambda)\Big) - \frac{1}{4}R_{(|z|^2 f)}(n,\lambda)\\ &= R_{(it-\frac{|z|^2}{4})f}(n,\lambda) + \frac{\Delta^-}{\lambda}R_f(n,\lambda).\end{aligned}$$

Using the hypotheses of the Theorem, for a given $N \geq 0$ there exist constants $\{C_{i,N}\}_{i=1}^4$ so that for any (n,λ)

$$\begin{aligned}|\, \partial_\lambda R_f(n,\lambda)\,| \leq & \frac{C_{1,N}}{|\lambda|^{N+1}(1+2n)^{N+1}} + \frac{C_{2,N}}{|\lambda|^{N+1}(1+2n)^N}\\ &+ \frac{C_{3,N}}{|\lambda|^{N+1}(1+2n)^N} + \frac{C_{4,N}}{|\lambda|^{N+1}(1+2n)^{N+1}}.\end{aligned}$$

Taking $C_N = \sum_{i=1}^4 C_{i,N}$ we obtain

$$|\, \partial_\lambda R_f(n,\lambda)\,| \leq \frac{C_N}{|\lambda|^{N+1}(1+2n)^N} \qquad \forall\ (n,\lambda) \quad \lambda > 0.$$

With induction on m is $\lambda^m \partial^m R_f \in \mathfrak{D}_0$.

A similar argument for $\lambda < 0$ implies the assertion of this part.

Proof of c): Since f is a Schwartz function, the inverse Fourier transform yields the following pointwise equality:

$$f(z,t) = \sum_n \int_\lambda R_f(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda).$$

Observe that for any $p \geq 0$, by applying Lemma 4.23 we get

$$\begin{aligned}&\sum_n \int_\lambda \frac{(\Delta^+ - \Delta^-)^p}{|\lambda|^p} R_f(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda)\\ &\qquad = \sum_n \int_\lambda R_f(n,\lambda)\frac{(\Delta^+ - \Delta^-)^p}{|\lambda|^p}\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda)\end{aligned}$$

and applying Lemma 4.16 yields:

$$\begin{aligned}&= \sum_n \int_\lambda R_f(n,\lambda)\frac{|\ z\ |^{2p}}{(-2)^p}\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda)\\ &= \frac{|\ z\ |^{2p}}{(-2)^p}f(z,t).\end{aligned}$$

Since $|\ z\ |^{2p} f$ is Schwartz, then from part $a)$, $R_{|z|^{2p}f} = (-2)^p \frac{(\Delta^+ - \Delta^-)^p}{|\lambda|^p} R \in \mathfrak{D}_0$.

On the other hand, for $k \geq 0$ from Theorem 4.29 and (4.41) we have:

$$\begin{aligned}&\sum_n \int_\lambda (sgn\lambda)^k(\partial_\lambda - \frac{1}{\lambda}\Delta^+)^k R(n,\lambda)\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t}d\mu(\lambda)\\ &\qquad = \sum_n \int_\lambda R(n,\lambda)(-\frac{|\ z\ |^2}{4} + it)^k(\overline{\Phi^\lambda_{n,n}(z)}e^{-i\lambda t})d\mu(\lambda)\\ &\qquad = (-\frac{|\ z\ |^2}{4} + it)^k f(z,t),\end{aligned}$$

which shows $R_{(-\frac{|z|^2}{4}+it)^k f} = (sgn\lambda)^k(\partial_\lambda - \frac{1}{\lambda}\Delta^+)^k R \in \mathfrak{D}_0$ (since $(-\frac{|z|^2}{4} + it)^k f$ is Schwartz).

Next, replacing f by $|\ z\ |^{2p}\ f$, for any $k \geq 0$ we derive:

$$(sgn\lambda)^k\big(\partial_\lambda - \frac{\Delta^+}{\lambda}\big)^k \frac{(\Delta^+ - \Delta^-)^p}{|\lambda|^p} R_f \in \mathfrak{D}_0,$$

which is the radial Fourier transform of the Schwartz function $(-\frac{|z|^2}{4} + it)^k\,|z|^{2p}\,f$. Now, applying part **b**) for that function one gets:

$$\lambda^m \partial_\lambda^m\,(sgn\lambda)^k \big(\partial_\lambda - \frac{\Delta^+}{\lambda}\big)^k \frac{(\Delta^+ - \Delta^-)^p}{|\lambda|^p} R_f \in \mathfrak{D}_0 \quad \forall m \in \mathbb{N}_0$$

as desired.

Proof of d): It can be proved in an analogous way to the part **c**). □

4.6 Admissible Radial Functions on the Heisenberg Group

As earlier introduced , $a \in (0,\infty)$ denotes an automorphism of the Heisenberg group $\mathbb{H}$, which operates on $\mathbb{H}$ as follows:

$$a : (x,y,t) \to a.(x,y,t) = (ax, ay, a^2 t) \tag{4.52}$$

and inverse given by

$$a^{-1}.(x,y,t) = (a^{-1}x, a^{-1}y, a^{-2}t).$$

For our convenience we identify the interval $(0,\infty)$ as a group of the automorphisms of $\mathbb{H}$ with Haar measure $d\mu(a) = a^{-1}da$. We consider the group $G := \mathbb{H} \rtimes (0,\infty)$ with left Haar measure $dha^{-5}da$ where dh denotes Lebesgue measure on $\mathbb{H}$. π denotes the quasi regular representation of group G, which operates by dilation and left-translation operators on $L^2(\mathbb{H})$ in the following way:

$$\begin{aligned} \pi :\ G := \mathbb{H} \rtimes (0,\infty) &\longrightarrow \mathcal{U}(L^2(\mathbb{H})) \\ (h,a) &\longmapsto \pi(h,a) = L_h D_{a^{-1}} \end{aligned}$$

where L_h is the left-translation operator, defined as follow:

$$\begin{aligned} L_h :\ L^2(\mathbb{H}) &\longrightarrow L^2(\mathbb{H}) \\ \theta &\longmapsto L_h(\theta)(.) = \theta(h^{-1}\ .) \end{aligned}$$

In this section, the existence of a function $f \in L^2(\mathbb{H})$ is considered, which is admissible with respect to the quasi regular representation of $G = \mathbb{H} \rtimes (0, \infty)$ acting on $L^2(\mathbb{H})$.

4.6.1 Admissibility of the radial Functions

Our aim in this part is to derive a criterion for a radial function on the Heisenberg group, to satisfy the admissibility condition. Let us recall the definition of an admissible vector on $\mathbb{H}$ w.r.t π, i.e., functions $f \in L^2(\mathbb{H})$ such that the operator

$$V_f : L^2(\mathbb{H}) \longrightarrow L^2(G), \tag{4.53}$$
$$V_f g(h, a) = \langle g, \pi(h, a) f \rangle = \big(g * (D_a f)^*\big)(h) = \big(g * (D_a f^*)\big)(h) \quad (h, a) \in G$$

be first well defined and second isometric, i.e., for any $g \in L^2(\mathbb{H})$ is $\| V_f g \|_{L^2(G)} = \| g \|_{L^2(\mathbb{H})}$. We restrict our investigation of such functions on the Heisenberg group to the class of radial functions. We collect the main result of this section in next theorem which presents a sufficient and necessary condition for admissibility of a radial function by dint of its Fourier transform.

Theorem 4.37. *(Main theorem) Suppose $f \in L^2(\mathbb{H})$ and radial with the radial Fourier transform R_f. Then f is admissible if and only if for some positive constant $c \neq 0$ the following holds:*

$$\int_{t=0}^{\infty} | R_f(n, t) |^2 \, t^{-1} dt = c \qquad \forall n \in \mathbb{N}.$$

For the proof of Theorem 4.37, we need the next lemma:

Lemma 4.38. *For $a > 0$ is $R_{D_{a^{-1}} f}(n, \lambda) = a^2 R_f(n, a^2 \lambda)$.*

Proof: Since f is a radial function, then obviously $D_a f$ is. Thus by the radial Fourier transform of $D_a f$ (see (4.6)) we have:

$$\begin{aligned}
R_{D_a f}(n,\lambda) &= \int D_a f(z,t) l_{n,\lambda}(z) e^{i\lambda t} dtdz \\
&= \int a^{-2} f(a^{-1}z, a^{-2}t) l_{n,\lambda}(z) e^{i\lambda t} dtdz \\
&= \int a^2 f(z,t) l_{n,\lambda}(a.z) e^{ia^2\lambda t} dtdz \\
&= a^2 \int f(z,t) l_n(|\lambda|^{\frac{1}{2}}\, a.z) e^{ia^2\lambda t} dtdz \qquad \text{from definition of } \ l_{n,\lambda} \ in \ (4.4) \\
&= a^2 \int f(z,t) l_n(\mid a^2\lambda \mid^{\frac{1}{2}} .z) e^{ia^2\lambda t} dtdz \\
&= a^2 \int f(z,t) l_{n,a^2\lambda}(z) e^{i(a^2\lambda)t} dtdz \\
&= a^2 R_f(n, a^2\lambda).
\end{aligned}$$

□

Using Lemma 4.38 the proof of Theorem 4.37 is as follow:

Proof: (of Theorem 4.37) We shall show that for some constant $c > 0$

$$\| V_f(g) \|^2_{L^2(G)} = c \| g \|^2_{L^2(\mathbb{H})} \quad \forall g \in L^2(\mathbb{H}). \tag{4.54}$$

Using the hypotheses of the theorem and the fact that $(D_a f)^* = D_a f^*$, then for any $a > 0$

and $g \in (L^1 \cap L^2)(\mathbb{H})$ we have:

$$
\begin{aligned}
\| V_f(g) \|^2_{L^2(G)} &= \int_a \int_{\mathbb{H}} | (g * D_{a^{-1}} f^*)(h) |^2 \, dh a^{-5} da \\
&= \int_a \| g * D_{a^{-1}} f^* \|^2_{L^2(\mathbb{H})} \, a^{-5} da \\
&= \int_a \| \hat{g} \circ (\widehat{D_{a^{-1}} f^*}) \|^2_{L^2(\widehat{\mathbb{H}})} \, a^{-5} da \\
&= \int_a \int_{\mathbb{R}^*} \| \hat{g}(\lambda) \circ (\widehat{D_{a^{-1}} f^*})(\lambda) \|^2_{H.S} \, d\mu(\lambda) a^{-5} da \qquad (4.55) \\
&= \int_a \int_{\mathbb{R}^*} \| \sum_n R_{D_{a^{-1}} f^*}(n, \lambda) \hat{g}(\lambda) \circ \phi_n^\lambda \otimes \phi_n^\lambda \|^2_{H.S} \, d\mu(\lambda) a^{-5} \qquad (4.56) \\
&= \int_a \int_{\mathbb{R}^*} \sum_n | R_{D_{a^{-1}} f^*}(n, \lambda) |^2 \| \hat{g}(\lambda) \circ \phi_n^\lambda \otimes \phi_n^\lambda \|^2_{H.S} \, d\mu(\lambda) a^{-5} da \qquad (4.57) \\
&= \int_a \int_{\mathbb{R}^*} \sum_n | R_{D_{a^{-1}} f^*}(n, \lambda) |^2 \| \hat{g}(\lambda) \phi_n^\lambda \|^2_2 \, d\mu(\lambda) a^{-5} da \\
&= \int_a \int_{\mathbb{R}^*} \sum_n | R_{D_{a^{-1}} f}(n, \lambda) |^2 \| \hat{g}(\lambda) \phi_n^\lambda \|^2_2 \, d\mu(\lambda) a^{-5} da \\
&= \int_{\mathbb{R}^*} \sum_n \int_a | R_{D_{a^{-1}} f}(n, \lambda) |^2 \, a^{-5} da \, \| \hat{g}(\lambda) \phi_n^\lambda \|^2_2 \, d\mu(\lambda) \\
&= \int_{\mathbb{R}^*} \sum_n \Big(\int_a | R_f(n, a) |^2 \, a^{-1} da \Big) \| \hat{g}(\lambda) \phi_n^\lambda \|^2_2 \, d\mu(\lambda) \qquad (4.58) \\
&= c \int_{\mathbb{R}^*} \sum_n \| \hat{g}(\lambda) \phi_n^\lambda \|^2_2 \, d\mu(\lambda) \\
&= c \int_{\mathbb{R}^*} \| \hat{g}(\lambda) \|^2_{H.S} \, d\mu(\lambda) \\
&= c \int_{\mathbb{H}} | g(b) |^2 \, db.
\end{aligned}
$$

The equality in (4.56) is obtained by applying Theorem 4.10. In the equality in (4.57) we use the orthogonality of operators $\{\phi_n^\lambda \otimes \phi_n^\lambda\}_n$ with respect to the Hilbert-Schmidt norm and for the equality in (4.58) we applied Lemma 4.38. The last equality is written by Plancherel theorem . Since $(L^1 \cap L^2)(\mathbb{H})$ is dense in $L^2(\mathbb{H})$ then the assertion is true for any $g \in L^2(\mathbb{H})$, since the operator V_f is closed [20]. □

One can prove Theorem 4.37 without applying Lemma 4.38. As mentioned in Section

2.6.2, for $a > 0$ and $\lambda \neq 0$ there exists a unitary operator $U_{a,\lambda}$ so that

$$\widehat{(D_a f)}^*(\lambda) = U_{a,\lambda} \widehat{f}(a^2\lambda)^* U_{a,\lambda}^*, \tag{4.59}$$

and from Theorem 4.10 we have

$$\widehat{f}(a^2\lambda)^* = \sum_n \overline{R_f(n, a^2\lambda)}\ \phi_n^{a^2\lambda} \otimes \phi_n^{a^2\lambda}. \tag{4.60}$$

Now applying the equation in (4.59), and substituting (4.60) into equation in (4.55), and using the fact that the image of an ONB under a unitary operator is again an ONB, we are done.

Theorem 4.39. *Let $\tilde{R}$ be a bounded function on $\mathbb{R}^+$ such that $\tilde{R} \in L^2(\mathbb{R}^+, ada)$. Define $R(n, \lambda) = \tilde{R}((2n+1) \mid \lambda \mid)$. Then the corresponding function to R in $L^2_r(\mathbb{H})$ is admissible if and only if $\tilde{R} \in L^2(\mathbb{R}^+, a^{-1}da)$.*

Proof: Suppose R as above and $\{F(\lambda)\}_{\lambda \in \mathbb{R}^*}$ is an associated operator field which is defined as follows:

$$F(\lambda) = \sum_{n \in \mathbb{N}_0} R(n, \lambda) \phi_n^\lambda \otimes \phi_n^\lambda. \tag{4.61}$$

Observe that from the assumptions of the theorem we have:

$$\begin{aligned} \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}^*} \mid R(n, \lambda) \mid^2 d\mu(\lambda) &= \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}^*} \mid \tilde{R}((2n+1)\,|\lambda|) \mid^2 d\mu(\lambda) \\ &= \left(\sum_{n \in \mathbb{N}_0} \frac{2}{(2\pi)^2 (2n+1)^2} \right) \int_0^\infty \mid \tilde{R}(\lambda) \mid^2 \lambda d\lambda < \infty. \end{aligned} \tag{4.62}$$

Therefore the operator field $\{F(\lambda)\}$ presents a radial function $f \in L^2(\mathbb{H})$ with Fourier coefficients $\{R(n, \lambda)\}_{(n,\lambda)}$ such that $\hat{f}(\lambda) = F(\lambda)$. From the definition of R is

$$\begin{aligned} \int_0^\infty \mid R(n, \lambda) \mid^2 \lambda^{-1} d\lambda &= \int_0^\infty \mid \tilde{R}((2n+1)\lambda) \mid^2 \lambda^{-1} d\lambda \\ &= \int_0^\infty \mid \tilde{R}(\lambda) \mid^2 \lambda^{-1} d\lambda \\ &= \parallel \tilde{R} \parallel^2_{L^2(\mathbb{R}^+, \lambda^{-1} d\lambda)} \end{aligned}$$

which implies by Theorem 4.37 that f is admissible. $\square$

It remains to be seen, whether there exist examples, for which both criteria, Theorem 4.37 and Theorem 4.36 can be checked directly. The *Mexican hat wavelet* obtained in the next Chapter will be an example of a radial function fulfilling both.

Chapter 5

Mexican Hat Wavelet on the Heisenberg Group

5.1 Introduction and Definitions

In this section the admissible vectors are studied from the point of view of *Calderón*'s formula. We shall present the notation of **Calderón admissible** vectors. Further we show in Theorem 5.4 that for the class of Schwartz functions the Calderón admissibility condition is equivalent to the usual admissibility property which has been introduced in §2.7 of this work. Furthermore we provide an example of an admissible Schwartz function on $\mathbb{H}$, which is an analog of the so called **Mexican hat** wavelet. The precise proof can be found in Theorem 5.11.

As mentioned before , the existence of an admissible vector for $L^2(N)$ is proved by Führ in [20], where N is a homogeneous group, for the quasiregular representation of $G := N \rtimes H$ on $L^2(N)$. Here H is a one-parameter group of dilations of N.
The existence of such vectors for the case $N := \mathbb{R}^k$ and $H < GL(k, \mathbb{R})$ has recently been studied by different authors. For example for the case $k = 1$ and $H := \mathbb{Z}$ see [39]. The case $N := \mathbb{H}$ and $H := \mathbb{R}$ as a one-parameter group of dilation is considered by [34].

Definition 5.1. *(***Weak integral of distributions***) let* $(\eta_a)_{a\in\mathbb{R}}$ *denote a family of distributions. If for all* $\phi \in \mathcal{S}(\mathbb{H})$ *the map* $\mathbb{R} \ni a \mapsto \langle \phi, \eta_a \rangle$ *is measurable and absolutely integrable, and moreover* $\phi \mapsto \int_{\mathbb{R}} \langle \phi, \eta_a \rangle da$ *defines a tempred distribution* ψ*, we call* ψ *the* weak integral *of the family* $(\eta_a)_{a\in\mathbb{R}}$*, and denote it with* $\psi = \int_{\mathbb{R}} \eta_a da$*.*

Before presenting the Calderón admissibility condition, let recall the following Remark:

Remark 5.2. *Note that from Theorem 1.65 in [14], for any* $\eta \in \mathcal{S}(\mathbb{H})$ *with* $\int \eta = 0$*, the vector valued integral* $\int_0^\infty \eta_a \ \ a^{-1} da$ *is convergent in weak sense, as defined in 5.3 below.*

Now we have the following definition:

Definition 5.3. *Let* $\phi \in \mathcal{S}(\mathbb{H})$ *and* $\int \phi = 0$*. Then* ϕ *is called* **Calderón** *admissible if for any* $0 < \varepsilon < A$ *and* $g \in \mathcal{S}(\mathbb{H})$

$$g * \int_\varepsilon^A \tilde{\phi}_a * \phi_a \, a^{-1} da \to cg \quad as \; \varepsilon \to 0 \, ; A \to \infty \tag{5.1}$$

holds in the sense of tempered distributions (weak sense), i.e., taking the inner product of left-hand side of (5.1) with any $f \in \mathcal{S}(\mathbb{H})$

$$\langle g * \int_\varepsilon^A \tilde{\phi}_a * \phi_a \, a^{-1} da, f \rangle = \langle \int_\varepsilon^A \tilde{\phi}_a * \phi_a \, a^{-1} da, \; \tilde{g} * f \rangle, \tag{5.2}$$

and commuting the inner product with the integral over a *in the right-hand side of (5.2), then it must converges to* $c\langle g, f \rangle$ *as* $\varepsilon \to 0$ *and* $A \to \infty$*, where* c *is a nonzero constant. Observe that for* $a > 0$ *we define* $\phi_a(\omega) = a^{-4}\phi(a^{-1}\omega)$*.*

In Lemma 5.4 below, we show that this definition of admissible is consistent with our usage of the word *admissible* as the definition in 2.7:

Theorem 5.4. *Let* $\phi \in \mathcal{S}(\mathbb{H})$*, then* ϕ *is admissible if and only if* ϕ *is Calderón admissible.*

Proof: Suppose $\phi \in \mathcal{S}(\mathbb{H})$ and $g \in \mathcal{S}(\mathbb{H})$. Then according to Definition 2.7 we may formally write:

$$\begin{aligned}
\|V_\phi g\|_2^2 &= \int_0^\infty \int_{\mathbb{H}} |\langle g, \lambda(b) D_a\phi\rangle|^2 db a^{-5} da && (5.3)\\
&= \int_0^\infty \int_{\mathbb{H}} |g * D_a\widetilde{\phi}(b)|^2 db a^{-5} da\\
&= \int_0^\infty \|g * D_a\widetilde{\phi}\|^2_{L^2(\mathbb{H})} a^{-5} da && (5.4)\\
&= \lim_{\varepsilon\to 0,\ A\to\infty} \int_\varepsilon^A \|g * D_a\widetilde{\phi}\|^2_{L^2(\mathbb{H})} a^{-5} da && (5.5)\\
&= \lim_{\varepsilon\to 0,\ A\to\infty} \int_\varepsilon^A \langle g * D_a\widetilde{\phi}\ ,\ g * D_a\widetilde{\phi}\rangle a^{-5} da\\
&= \lim_{\varepsilon\to 0,\ A\to\infty} \int_\varepsilon^A \langle g\ ,\ g * D_a\widetilde{\phi} * D_a\phi\rangle a^{-5} da\\
&= \langle g\ ,\ \lim_{\varepsilon\to 0,\ A\to\infty} g * \int_\varepsilon^A D_a\widetilde{\phi} * D_a\phi a^{-5} da\rangle\\
&= \langle g\ ,\ \lim_{\varepsilon\to 0,\ A\to\infty} g * \int_\varepsilon^A \widetilde{\phi}_a * \phi_a a^{-1} da\rangle. && (5.6)
\end{aligned}$$

Here the equalities hold in the extended sense that one side is finite iff the other is. Suppose ϕ is a *Calderón* admissible vector. Then from the definition, for some constant c, $\lim_{\varepsilon\to 0,\ A\to\infty} g * \int_\varepsilon^A \widetilde{\phi}_a * \phi_a a^{-1} da = cg$ in weak sense. Hence

$$\langle g\ ,\ \lim_{\varepsilon\to 0,\ A\to\infty} g * \int_\varepsilon^A \widetilde{\phi}_a * \phi_a a^{-1} da\rangle = c \parallel g \parallel^2$$

Going the relation (5.6) backward, it shows that the limit in (5.5) exists and is finite. Therefore $\parallel V_\phi g \parallel_2^2 = c \parallel g \parallel_2^2$. Since $\mathcal{S}(\mathbb{H})$ is a dense subspace of $L^2(\mathbb{H})$ and V_ϕ is a closed operator on $L^2(\mathbb{H})$, then V_ϕ is isometric on $L^2(\mathbb{H})$ up to the constant c, which means ϕ is admissible in the sense of Definition 2.7.

Conversely, suppose ϕ is an admissible vector. We show that ϕ is *Calderón* admissible, or equivalently the relation

$$\langle f\ ,\ \lim_{\varepsilon\to 0,\ A\to\infty} g * \int_\varepsilon^A \widetilde{\phi}_a * \phi_a a^{-1} da\rangle = c\langle f, g\rangle \tag{5.7}$$

holds for any pair $f, g \in \mathcal{S}(\mathbb{H})$ and for some non-zero constant c.

Since ϕ is admissible then for any $g \in \mathcal{S}(\mathbb{H})$ is $\parallel V_\phi g \parallel_2^2 = c \parallel g \parallel_2^2$, which shows that the integral (5.4) is finite. Now, defining $F(\varepsilon, A) := \int_\varepsilon^A \|g * D_a\widetilde{\phi}\|^2_{L^2(\mathbb{H})} a^{-5} da$, observe that, for ε fixed, $F(\varepsilon, .)$ is a positive and increasing function in variable A and is bounded from above. Hence $\lim F(\varepsilon, A)$ exists as $A \to \infty$ and by "general principle of convergence Theorem" it can be read as

$$\lim_{A\to\infty} F(\varepsilon, A) = \int_\varepsilon^\infty \|g * D_a\widetilde{\phi}\|^2_{L^2(\mathbb{H})} a^{-5} da. \tag{5.8}$$

Taking $G(\varepsilon) = \lim_{A\to\infty} F(\varepsilon, A)$, G is positive and decreasing. Hence then $\lim G(\varepsilon)$ exists as ε goes to 0. Therefore again by convergence theorem we can write:

$$\lim_{\varepsilon\to 0} G(\varepsilon) = \int_0^\infty \|g * D_a\widetilde{\phi}\|^2_{L^2(\mathbb{H})} a^{-5} da.$$

The last argument shows that the equality (5.5) holds and hence we get the following equality:

$$\langle g\ ,\ \lim_{\varepsilon\to 0,\ A\to\infty} g * \int_\varepsilon^A \widetilde{\phi}_a * \phi_a a^{-1} da\rangle = \| V_\phi g \|_2^2 = c \| g \|_2^2 \tag{5.9}$$

To show the relation (5.7) for any pair $f, g \in \mathcal{S}(\mathbb{H})$, we observe that (5.9) is (5.7) for the case $f = g$. The general case now follows by polarization; i.e.:
Taking $K_{\varepsilon,A} = \int_\varepsilon^A \widetilde{\phi}_a * \phi_a a^{-1} da$, the equation (5.9) shows that

$$\lim_{\varepsilon\to 0, A\to\infty} \langle g, g * K_{\varepsilon,A}\rangle = c \| g \|^2 \qquad \forall g \in \mathcal{S}(\mathbb{H}). \tag{5.10}$$

Now suppose $f \in \mathcal{S}(\mathbb{H})$. Then

$$\begin{aligned}\langle f+g, (f+g) * K_{\varepsilon,A}\rangle =& \langle f, f * K_{\varepsilon,A}\rangle + \langle f, g * K_{\varepsilon,A}\rangle \\ &+ \langle g, f * K_{\varepsilon,A}\rangle + \langle g, g * K_{\varepsilon,A}\rangle.\end{aligned}$$

Using (5.10), the left hand side of above equality goes to $\| f + g \|^2$ as $\varepsilon \to 0, A \to \infty$ and it implies

$$\langle f, g * K_{\varepsilon,A}\rangle + \langle g, f * K_{\varepsilon,A}\rangle \to c\,(\langle f, g\rangle + \langle g, f\rangle) \quad as\ \varepsilon \to 0, A \to \infty.$$

and hence

$$\lim_{\varepsilon\to 0, A\to\infty} Re\langle f * K_{\varepsilon,A}, g\rangle = c\ Re\langle f, g\rangle. \tag{5.11}$$

Replacing $f + ig$ by $f + g$ we get

$$\langle f, g * K_{\varepsilon,A}\rangle - \langle g, f * K_{\varepsilon,A}\rangle \to c\left(\langle f, g\rangle - \langle g, f\rangle\right) \quad as\ \varepsilon \to 0, A \to \infty.$$

which implies

$$\lim_{\varepsilon\to 0, A\to\infty} Im\langle f * K_{\varepsilon,A}, g\rangle = c\, Im\langle f, g\rangle. \tag{5.12}$$

(5.11) and (5.12) combined yield (5.7), and hence we are done. □

Remark 5.5. *Actually, if $\mathcal{H}$ is a complex Hilbert space, then for any pair of bounded operators T, S on $\mathcal{H}$ such that $\langle Tf, f\rangle = \langle Sf, f\rangle$ for all $f \in \mathcal{H}$ implies that $T = S$. To prove, replace f by $f + g$ we get*

$$\langle T(f + g), f + g\rangle = \langle Tf, f\rangle + \langle Tf, g\rangle + \langle Tg, f\rangle + \langle Tg, g\rangle,$$

to get

$$\langle Tf, g\rangle + \langle Tg, f\rangle = \langle Sf, g\rangle + \langle Sg, f\rangle.$$

Change f to (if) to find

$$\langle Tf, g\rangle - \langle Tg, f\rangle = \langle Sf, g\rangle - \langle Sg, f\rangle.$$

Add the two equations imply our assertion.

Notation 5.6. *Note that $\frac{d}{da}$ is understood as a vector derivation as follow:*
Suppose X is a topological vector space and F is a vector valued function on $\mathbb{R}^+$ such that $F(a) \in X$ for any $a \in \mathbb{R}^+$. Then we say $\frac{d}{da}F(a)$ exists at point $a = a_0$ when $\lim\limits_{h\to 0^+} \frac{F(a_0+h)-F(a_0)}{h}$ exists in the topology of X. We then shall write

$$\frac{d}{da}F(a)_{a=a_0} = \lim_{h\to 0^+} \frac{F(a_0 + h) - F(a_0)}{h}.$$

In the next Proposition we will show a sufficient condition for Schwartz functions to be admissible, which is one of the chief tools for the proof of the main theorem of this chapter.

Proposition 5.7. *Suppose $\phi, \psi \in \mathcal{S}(\mathbb{H})$, so that $\int \psi \neq 0$ and for some constants $k, c > 0$ and non-zero real number q is $\widetilde{\phi}_{a^q} * \phi_{a^q} = -ac\frac{d}{da}\psi_{ka^q}$. Then ϕ is admissible.*

Proof: Suppose $g \in \mathcal{S}(\mathbb{H})$ and $0 < \varepsilon < A < \infty$. Using the change of coordinate a to a^q and that $\widetilde{\phi}_{a^q} * \phi_{a^q} = -ac\frac{d}{da}\psi_{ka^q}$, we can write

$$\begin{aligned} g * \int_{\varepsilon}^{A} \widetilde{\phi}_a * \phi_a a^{-1} da &= q\ g * \int_{\varepsilon^{1/q}}^{A^{1/q}} \widetilde{\phi}_{a^q} * \phi_{a^q} a^{-1} da \\ &= q\ g * \int_{\varepsilon^{1/q}}^{A^{1/q}} (-a\frac{d}{da})\psi_{ka^q} a^{-1} da \\ &= q\ g * \int_{\varepsilon^{1/q}}^{A^{1/q}} (-\frac{d}{da}\psi_{ka^q}) da \\ &= -q(g * (\psi_{Ak} - \psi_{\varepsilon k})) \\ &= qg * (\psi_{\varepsilon k} - \psi_{Ak}). \end{aligned} \tag{5.13}$$

Since $\int \psi \neq 0$, then from Proposition 1.20 [14] we have:

$$\lim_{\varepsilon \to 0} g * \psi_{\varepsilon k} = g \int \psi \quad , in\ L^2 - norm. \tag{5.14}$$

On the other hand we can write:

$$\|g * \psi_{Ak}\|_2 \leq \|g\|_1 \|\psi_{kA}\|_2 = (kA)^{-\frac{1}{2}} \|g\|_1 \|\psi\|_2 \tag{5.15}$$

which shows $g * \psi_{Ak} \longrightarrow 0 \quad in\ L^2$-norm as $A \to \infty$. Now applying (5.14) and (5.15) in (5.13) one gets

$$g * \int_{\varepsilon}^{A} \phi_a * \tilde{\phi}_a a^{-1} da \to g \int \psi \quad as \quad \varepsilon \to 0,\ A \to \infty \quad , in\ L^2 - norm. \tag{5.16}$$

□

To reach the other main result of this chapter, next we need to recall some basic definitions:

Let $L = -(X^2 + Y^2)$ be the sub-Laplacian operator, where X and Y are the left-invariant vector fields on the Heisenberg group (see Section 2.5). The heat kernel operator associated to L is the differential operator $\frac{d}{dt} + L$ on $\mathbb{H} \times \mathbb{R}$, where $\frac{d}{dt}$ is the coordinate vector field on $\mathbb{R}$ (one can consider this coordinate as the time coordinate). For the heat operator we recall here Proposition 1.68 of [14].

Proposition 5.8. *There exists a unique C^∞ function h on $\mathbb{H} \times (0, \infty)$, for which the following properties hold:*

1. $(\frac{d}{dt} + L)h = 0$ *on* $\mathbb{H} \times (0, \infty)$,
2. $h(\omega, t) \geq 0$, $h(\omega, t) = h(\omega^{-1}, t)$ $\forall(\omega, t) \in \mathbb{H} \times (0, \infty)$, *and* $\int h(\omega, t)d\omega = 1$ *for* $t > 0$,
3. $h(., s) * h(., t) = h(., s + t) \quad \forall\, s, t > 0$,
4. $r^4 h(r\omega, r^2 t) = h(\omega, t) \quad \forall\ \omega \in \mathbb{H},\ t, r > 0$ *($r\omega$ denotes the operation of the automorphism $r.$ to ω).*

The solution h is called **heat kernel**.

Observe that here the interval $(0, \infty)$ has nothing to do with the group of dilations which is introduced in Section 2.7. It should be considered as time interval.
The idea of this section is to apply Proposition 5.7 to $\phi(x) = Lh(x, 1)$ to show that the function ϕ is an admissible vector. For that purpose here we need to compute the dilation of functions $h(., 1)$ and $Lh(., 1)$:

Lemma 5.9. *For any $a > 0$ and $\omega \in \mathbb{H}$ is*

$$h(\omega, 1)_a = a^2 h(\omega, a^2) \quad and \quad Lh(\omega, 1)_a = a^2 Lh(\omega, a^2).$$

Proof: Suppose $a > 0$ and $\omega \in \mathbb{H}$, then applying the property in 4 in Proposition 5.8 one gets:

$$h(\omega, 1)_a = a^{-4} h(a^{-1}\omega, 1) = a^2 h(\omega, a^2). \tag{5.17}$$

Similarly by applying the properties 1 and 4 in Proposition 5.8 for $Lh(.,1)$ we have:

$$\begin{aligned} Lh(\omega,1)_a &= a^{-4}Lh(a^{-1}\omega,1) \\ &= -a^{-4}\frac{d}{dt}h(a^{-1}\omega,t)\big|_{t=1} \\ &= -\frac{d}{dt}h(\omega,a^2t)\big|_{t=1} \\ &= a^2Lh(\omega,a^2). \end{aligned} \tag{5.18}$$

□

5.2 Theorem and Mexican Hat Wavelet on $\mathbb{H}$

The purpose of this section is to check *Calderón* admissibility for the Schwartz function $\phi = Lh(.,1)$, which we state it now in the next theorem as the main result of this chapter.

Remark 5.10. *On the real line $\mathbb{R}$, the heat kernel is given by $h(x,t) = \frac{1}{\sqrt{4\pi}}e^{-\frac{x^2}{4t}}$. In particular, $h(x,1) = \frac{1}{\sqrt{4\pi}}e^{-\frac{x^2}{4}}$. The second derivative of the Gaussian is an often employed wavelet, the Mexican-Hat wavelet. This motivates the name of the wavelet presented in the next theorem.*

Theorem 5.11. *The " Mexican hat wavelet" on $\mathbb{H}$, $\phi(\omega) = Lh(\omega,1)$, is admissible.*

Proof: Observe that by definition of L , the function $Lh(.,1)$ on $\mathbb{H}$ has mean value zero. It is also easy to see that $\widetilde{\phi} = \phi$: for any $\omega \in \mathbb{H}$ and $t > 0$ define $h_t(\omega) = h(\omega,t)$. Then by applying the property 2 in Proposition 5.8 we have:

$$\begin{aligned} \widetilde{(Lh)}(\omega,t) &= -\widetilde{\frac{d}{dt}h(\omega,t)} = -\frac{d}{dt}\widetilde{h}_t(\omega) \\ &= -\frac{d}{dt}\overline{h_t}(\omega^{-1}) = -\frac{d}{dt}h_t(\omega) = Lh(\omega,t). \end{aligned} \tag{5.19}$$

To prove the theorem it is sufficient to show that for the function $\psi = h(\omega,1) + Lh(\omega,1)$ the relation

$$\widetilde{\phi}_{\sqrt{a}} * \phi_{\sqrt{a}} = \phi_{\sqrt{a}} * \phi_{\sqrt{a}} = -ca\frac{d}{da}\psi_{\sqrt{2a}} \tag{5.20}$$

holds. Hence by applying Proposition 5.7 we will get our assertion.

Using the relations (5.17) and (5.18) we get

$$\begin{aligned} \phi_{\sqrt{a}} * \phi_{\sqrt{a}} &= (Lh(.,1))_{\sqrt{a}} * (Lh(.,1))_{\sqrt{a}} \\ &= aLh(.,a) * aLh(.,a) \\ &= a^2 L^2 h(.,2a), \end{aligned} \tag{5.21}$$

as well as

$$\begin{aligned} \psi_{\sqrt{2a}} &= (h(.,1))_{\sqrt{2a}} + (Lh(.,1))_{\sqrt{2a}} \\ &= h(.,2a) + 2aLh(.,2a). \end{aligned} \tag{5.22}$$

Observe that the vector derivation of $\psi_{\sqrt{2a}}$ with respect to the parameter a is computed as follows :

$$\begin{aligned} \frac{d}{da}\psi_{\sqrt{2a}} &= \frac{d}{da}h(.,2a) + 2Lh(.,2a) + 2a\frac{d}{da}Lh(.,2a) \\ &= 2\frac{d}{d2a}h(.,2a) + 2Lh(.,2a) + 4a\frac{d}{d2a}Lh(.,2a) \\ &= -4aL^2h(.,2a). \end{aligned} \tag{5.23}$$

Comparing the equations (5.21) and (5.23), we see that the relation (5.20) holds for ϕ and ψ and for $c = 4$, as desired. $\square$

Appendix A

Proof of Lemma 4.17

Lemma 4.17 may be considered as folklore, even though its proof is nontrivial and quite technical. In this section we include an expanded version of an argument presented to us by D.Geller (private communication).

We need the following lemma as a strong tool for the proof of Lemma 4.17:

Lemma A.1. *Suppose $\{Z_1 := Z, Z_2 := \bar{Z}, T\}$ is the basis of the left invariant vector fields algebra on $\mathbb{H}$ presented in (2.6), and $\mathfrak{L}_0 = \frac{1}{2}(Z_1Z_2+Z_2Z_1)$ is the* Heisenberg sub-Laplacian *operator which is equal to the one introduced in the proof of Theorem 4.36 a). Suppose B is the unit ball in the Heisenberg group and $\zeta \in C_c^\infty(B)$. Then there is a constant $C > 0$ such that for all smooth f on $\mathbb{H}$ the follwoing estimates hold:*

(a)

$$\|\zeta(Z_1 f)\|_2^2 + \|\zeta(Z_2 f)\|_2^2 \leq C \sum_{i=0,1} \|\zeta_i \mathfrak{L}^i f\|_2^2, \tag{A.1}$$

where $\zeta_i \in C_c^\infty(B)$ and $\zeta_i = 1$ in a neighborhood of the support of ζ.

(b) For any $N, k \in \mathbb{N}_0$

$$\sum_{j_1,\dots,j_N=1,2} \|\zeta(Z_{j_1} \dots Z_{j_N} T^k f)\|_2^2 \leq C \sum_{0\leq l+m\leq N} \|\xi_{l+m,N} \mathfrak{L}^l T^{m+k} f\|_2^2. \tag{A.2}$$

(c) For any $N, k \in \mathbb{N}_0$

$$\|Z_{j_1} \dots Z_{j_N} T^k(\xi f)\|_2^2 \leq C \sum_{0 \leq l+m \leq 2N,\ d=0,\cdots,K} \|\xi_{l+m} \mathfrak{L}^l T^{m+d} f\|_2^2. \tag{A.3}$$

Proof *Proof of (a)* Let $f \in C^\infty(\mathbb{H})$, then

$$\|\xi(Zf)\|_2^2 + \|\xi(\overline{Z}f)\|_2^2 = 2\langle \xi f, \xi \mathfrak{L}_0 f \rangle + E, \tag{A.4}$$

where

$$E = \langle \xi f, (\overline{Z}\xi)(Zf) + (Z\xi)(\overline{Z}f) \rangle - \langle (Z\xi)f, \xi(Zf) \rangle - \langle (\overline{Z}\xi)f, \xi(\overline{Z}f) \rangle \tag{A.5}$$

(This can be shown by replacing $\xi(Z_j f) = Z_j(\xi f) - (Z_j \xi) f$ in $\|\xi(Z_j f)\|_2^2 = \langle \xi(Z_j f),$- $\xi(Z_j f) \rangle$, where $Z_j = Z$ and $\overline{Z}$.) Using the Schwartz inequality, for any arbitrary small constant (s.cst) there is a sufficient large constant (l.cst) such that

$$\begin{aligned}
\left|\langle \xi f, (\overline{Z}\xi)(Zf) \rangle\right| &= \left|\langle (\overline{\overline{Z}\xi}) f, \bar{\xi}(Zf) \rangle\right| \\
&\leq \|(\overline{Z}\xi) f\|_2 \|\xi(Zf)\|_2 \\
&\leq (l.cst)\|(\overline{Z}\xi) f\|_2^2 + (s.cst)\|\xi(Zf)\|_2^2,
\end{aligned}$$

and similarly

$$\begin{aligned}
\left|\langle \xi f, (Z\xi)(\overline{Z}f) \rangle\right| &\leq (l.cst)\|(Z\xi) f)\|_2^2 + (s.cst)\|\xi(\overline{Z}f)\|_2^2 \qquad \text{(A.6)} \\
|\langle (Z\xi) f, \xi(Zf) \rangle| &\leq (l.cst)\|(Z\xi) f)\|_2^2 + (s.cst)\|\xi(Zf)\|_2^2 \\
\left|\langle (\overline{Z}\xi) f, \xi(\overline{Z}f) \rangle\right| &\leq (l.cst)\|(\overline{Z}\xi) f\|_2^2 + (s.cst)\|\xi(\overline{Z}f)\|_2^2,
\end{aligned}$$

Therefore using the above estimates, in $(A.4)$ we get

$$\begin{aligned}
\left\|\xi(Zf)\|_2^2 + \|\xi(\overline{Z}f)\right\|_2^2 &\leq C\left(\|\xi f\|_2^2 + \|\xi(\mathfrak{L}_0 f)\|_2^2\right) \qquad \text{(A.7)} \\
&\quad + (l.cst)\left(2\|(Z\xi) f\|_2^2 + \|(\overline{Z}\xi f)\|_2^2\right) \\
&\quad + (s.cst)\left(2\|\xi(\overline{Z}f)\|_2^2\right) + \|\xi(Zf)\|_2^2. \qquad \text{(A.8)}
\end{aligned}$$

Note that for a sufficiently small constant (s.cst), the term in (A.8) can be absorbed back into the left side of (A.7), and hence for a suitable constant $C > 0$ the relation (A.1) holds.

Proof of (b) We prove it by induction. For $N=1$ and $k \in \mathbb{N}_0$ the assertion is true from (a). Suppose that the relation (A.2) is true for $N-1$, then for N one has:

$$\begin{aligned}
\sum_{j_1,\dots,j_N=1,2} \|\xi(Z_{j_1}\dots Z_{j_N}T^k f)\|_2^2 &= \sum_{j_1,\dots,j_N=1,2} \|\xi Z_{j_1}\left(Z_{j_2}\dots Z_{j_N}T^k f\right)\|_2^2 \\
&\leq C \sum_{\substack{j_2,\dots,j_N=1,2 \\ 0\leq i+s\leq 1}} \|\xi_{i+s}\mathfrak{L}_0^i T^s\left(Z_{j_2}\dots Z_{j_N}T^k f\right)\|_2^2 \\
&= C \sum_{\substack{j_2,\dots,j_N=1,2 \\ 0\leq i+s\leq 1}} \|\xi_{i+s}\mathfrak{L}_0^i \left(Z_{j_2}\dots Z_{j_N}T^{k+s} f\right)\|_2^2 \\
&= C\Big[\sum_{j_1,\dots,j_N=1,2\ s=0,1} \|\xi_s Z_{j_2}\dots Z_{j_N}T^{k+s} f\|_2^2 \qquad \text{(A.9)} \\
&+ \sum_{j_1,\dots,j_N=1,2} \|\xi' \mathfrak{L}_0\left(Z_{j_2}\dots Z_{j_N}T^k f\right)\|_2^2\Big].
\end{aligned}$$

Using the assumption of the induction and the relation $\mathfrak{L}_0 Z_k = Z_k \mathfrak{L}_0 + 2iZ_K T$, for the constant $\lambda = (2i)^N$ one gets:

$$\begin{aligned}
&(const.)(A.9) \\
&= \sum_{\substack{j_1,\dots,j_N=1,2 \\ s=0,1}} \|\xi_s\left(Z_{j_2}\dots Z_{j_N}T^{k+s} f\right)\|_2^2 + \sum_{j_1,\dots,j_N=1,2} \|\xi' Z_{j_2}\dots Z_{j_N}\left(\mathfrak{L}_0+\lambda T\right)T^k f\|_2^2, \\
&\leq C \sum_{\substack{0\leq l+m\leq N-1 \\ s=0,1}} \left\|\xi_{l+m}\mathfrak{L}_0^l T^{m+k+s} f\right\|_2^2 + C' \sum_{0\leq l+m\leq N-1} \left\|\xi'_{l+m}\mathfrak{L}_0^{l+1} T^{m+k} f\right\|_2^2 \\
&+ C'' \sum_{0\leq l+m\leq N-1} \left\|\xi''_{l+m}\mathfrak{L}_0^l T^{m+k+1} f\right\|_2^2 \\
&\leq C_1 \sum_{0\leq l+m\leq N} \left\|\xi_{(l+m,N)}\mathfrak{L}_0^l T^{m+k} f\right\|_2^2,
\end{aligned}$$

which completes the proof of (b).

Proof of (c) Obviously for $N=0$

$$T^k(\xi f) = \sum_{d=0}^{k} c_d\left(T^{k-d}\xi\right)\left(T^d f\right).$$

Taking $\xi_d := T^{k-d}\xi$, then for some constant C

$$\left\|T^k(\xi f)\right\|_2^2 \leq C\sum_{d=0}^{k}\left\|\xi_d T^d f\right\|_2^2.$$

The above estimate shows that to prove the realtion (A.3) it is sufficient to show the relation for $k=0$. The proof will follow by applying the induction as follows:
Suppose $N=1$ Then for $j_1 = 0,1$

$$Z_{j_1}(\xi f) = (Z_{j_1}\xi)f + \xi(Z_{j_1}f),$$

hence from (b),

$$\begin{aligned}
\|Z_{j_1}(\xi f)\|_2^2 &\leq 2\left\|(Z_{j_1}\xi)\, f\right\|_2^2 + 2\left\|\xi\,(Z_{j_1}f)\right\|_2^2 \\
&\leq C\Big[\,\|\xi' f\|_2^2 + \sum_{0\leq l+m\leq 1}\left\|\xi_{l+m}\mathfrak{L}_0^l T^m f\right\|_2^2\Big] \\
&\leq C'\sum_{0\leq l+m\leq 1}\left\|\xi'_{l+m}\mathfrak{L}_0^l T^m f\right\|_2^2.
\end{aligned}$$

Suppose that the statement is true for $N>1$, i.e,

$$\|Z_{j_1}...Z_{j_N}(\xi f)\|_2^2 \leq C\sum_{0\leq l+m\leq 2N}\left\|\xi_{l+m}\mathfrak{L}_0^l T^m f\right\|_2^2,$$

then for $N+1$ we have

$$\begin{aligned}
\|Z_{j_1}...Z_{j_N}Z_{j_N+1}(\xi f)\|_2^2 &= \|Z_{j_1}...Z_{j_N}\left((Z_{j_N+1}\xi)f + \xi Z_{j_N+1}f\right)\|_2^2 \\
&\leq 2\left\|Z_{j_1}...Z_{j_N}\left((Z_{j_N+1}\xi)f\right)\right\|_2^2 + 2\left\|Z_{j_1}...Z_{j_N}\left(\xi Z_{j_N+1}f\right)\right\|_2^2 \\
&= 2\left\|Z_{j_1}...Z_{j_N}\left(\xi' f\right)\right\|_2^2 + 2\left\|Z_{j_1}...Z_{j_N}\left(\xi Z_{j_N+1}f\right)\right\|_2^2 \\
&\leq C_1\sum_{0\leq l+m\leq 2N}\left\|\xi'_{l+m}\mathfrak{L}_0^l T^m f\right\|_2^2 \qquad \text{(A.10)} \\
&+ C_2\sum_{0\leq l+m\leq 2N}\left\|\xi_{l+m}\mathfrak{L}_0^l T^m Z_{j_{N+1}}f\right\|_2^2. \qquad \text{(A.11)}
\end{aligned}$$

Since that $\mathfrak{L}_0^l T^m Z_{j_{N+1}} = \mathfrak{L}_0^l Z_{j_{N+1}} T^m$ and $\mathfrak{L}_0 Z_{j_{N+1}} = Z_{j_{N+1}}\mathfrak{L}_0 + 2iZ_{j_{N+1}}T$, then by induction

$$\mathfrak{L}_0^l Z_{j_{N+1}} = Z_{j_{N+1}}\sum_{r=0}^{l} c_r \mathfrak{L}_0^{l-k}T^r; \qquad \forall l \in \mathbb{N}_0$$

Sabstituting the latter relation into the (A.10), from (b) one gets the following:

$$
\begin{aligned}
&(A.10)+(A.11) \qquad\qquad (\text{A.12})\\
&\leq C_1 \sum_{0\leq l+m\leq 2N} \left\|\xi'_{l+m}\mathfrak{L}_0^l T^m f\right\|_2^2 + C'_2 \sum_{0\leq l+m\leq 2N} \sum_{r=0}^{l} \left\|\xi_{l+m,r} Z_{j_{N+1}} \mathfrak{L}_0^{l-r} T^{r+m} f\right\|_2^2\\
&\leq \sum_{0\leq l+m\leq 2N} \left\|\xi'_{l+m}\mathfrak{L}_0^l T^m f\right\|_2^2 + C \sum_{0\leq l+m\leq 2N} \sum_{r=0}^{l} \sum_{0\leq s+l\leq 1} \left\|\xi'_{(l+m,r,s)} \mathfrak{L}_0^s T^t \left(\mathfrak{L}_0^{l-r} T^{r+m} f\right)\right\|_2^2\\
&=\leq C \sum_{0\leq l+m\leq 2N} \left\|\xi'_{l+m}\mathfrak{L}_0^l T^m f\right\|_2^2 + C \sum_{\substack{0\leq l+m\leq 2N\\ r=0,\dots,l}} \sum_{0\leq s+t\leq 1} \left\|\xi'_{l+m,r,s} \mathfrak{L}_0^{l+s-r} \left(T^{t+r+m} f\right)\right\|_2^2\\
&\leq C \sum_{0\leq l+m\leq 2(N+1)} \left\|\xi_{l+m}\mathfrak{L}_0^l T^m f\right\|_2^2,
\end{aligned}
$$

as desired.

Lemma A.2. *For any $N, k \in \mathbb{N}_0$,*

$$\sup_{x\in B} \left|Z_{j_1}...Z_{j_N} T^k(\xi f(x))\right| \leq C \sum_{\substack{0\leq l+m\leq 2N\\ d=0,\dots,k}} \left\|\xi'_{l+m}\mathfrak{L}_0^l T^{m+d} f\right\|_2^2. \tag{A.13}$$

Proof: The proof follows from the Sobolev Lemma for Lie groups (see A.1.5 in [Corwin..]) and Lemma A.1 (c).

Now we are ready to state the proof of Lemma 4.17:

Proof: Suppose $f \in C_r^\infty(\mathbb{H})$ such that for any $d, s \geq 0$, $\left|(\partial_z\partial_{\bar{z}})^d \partial_t^s f(z,t)\right|$ decays rapidly as $|(z,t)| \to \infty$. We want to show that for any $N, k \geq 0$, $\left|Z_{j_1}...Z_{j_N} T^k f(z,t)\right|$ decays rapidly too as $|(z,t)| \to \infty$, which provides that f is Schwartz. Observe that for any compact subset K of $\mathbb{H}$, contained in the unit ball B, and for $f_K := f|_K$ is

$$\left\|\mathfrak{L}_0^l T^{m+d} f_K\right\|_2^2 \leq (const.) \sup_{w\in K} \left|\mathfrak{L}_0^l T^{m+d} f(w)\right|^2. \tag{A.14}$$

Combining the estimates (A.14) and (A.13) we have

$$\sup_{x\in B} \left|Z_{j_1}...Z_{j_N} T^k(\xi f(x))\right| \leq C \sum_{\substack{0\leq l+m\leq 2N\\ d=0,\dots,k}} \sup_{(z,t)\in B} \left|\xi'_{l+m}\mathfrak{L}_0^l T^{m+d} f(z,t)\right|_2^2. \tag{A.15}$$

Suppose that $\xi = 1$ in a neighborhood of 0, since the operators $\mathfrak{L}_0$ and T are left invariant, then using the left translation operator in $u = (z, t)$ one can write

$$\mathfrak{L}_0^l T^k f(u) = L_u(\mathfrak{L}_0^l T^k f)(0) = \mathfrak{L}_0^l T^k (L_u f)(0) = \mathfrak{L}_0^l T^k \xi (L_u f)(0). \tag{A.16}$$

Observe that for any $l, k \in \mathbb{N}_0$, $\left|\mathfrak{L}_0^l T^k f(z,t)\right|$ decays rapidly as $\left|(\partial_z \partial_{\overline{z}})^d \, \partial_t^s f(z,t)\right|$ decays rapidly, since $\mathfrak{L}_0^l T^k f$ can be written as a sum of $\mid z \mid^{2r} (\partial_z \partial_{\overline{z}})^d \, T^s f$ where $d \leq l$. According to the fact and applying the translation operator in $u \in \mathbb{H}$ in the both side of (A.15) and using the equality in (A.16), one can show that since the right side of (A.15) decays rapidly in $|u|$ then the left side does too, which completes the proof of the lemma.

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