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Discrete Tomography, the Instability of Point X-Rays and Separability Problems for Aperiodic Quasicrystals

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Abstract

In analogy to the fan-beam geometry used in computerized tomography we investigate point X-rays in discrete tomography and their instability behaviour. We construct arbitrarily large irreducible switching components. After applying any affine resp. projective transformation the tomograpically equivalent lattice sets overlap in at most the minimal number of lattice points to define the transformation. Thus, compared with parallel X-rays even worse instability results are discovered.

Considering the reconstruction problem in the discrete tomography of quasicystals we have to determine all subsets of a finite point set which can be separated by an up to translation fixed so-called window, i. e. a ball or polytope. We show that both cases can be dealt with in polynomial. Also, we are concerned with the characterization and the calculation of smallest separating balls and triangles.

Contents

Chapter 1

Introduction

1.1 Computational complexity

The **running time** of an algorithm to solve a problem instance usually depends on the input size which is defined by the length of the string of symbols encoding the problem instance using the **binary encoding** scheme. We speak of an **instance of a problem** if all parameters of the problem are specified.
The running time is defined by the number of steps in which the **Turing machine** reaches the end state. For more details about Turing machines and binary encoding we refer to [107], [68].
An algorithm is called a **polynomial-time algorithm** if there exists a polynomial so that for any instance of size at most n the running time is bounded from above by the polynomial at n.
The class of recognition problems that can be solved by a polynomial-time algorithm is denoted by $\mathcal{P}$.
The class of recognition problems so that for any "yes"-instance there exists a certificate, the length of which is bounded by a polynomial in the size of the "yes"-instance and which can be checked in polynomial time for validity, is denoted by $\mathcal{NP}$. Thereby, the class $\mathcal{P}$ belongs to the class $\mathcal{NP}$.
Let A_1, A_2 be two recognition problems. We say that A_1 reduces in polynomial time to A_2 if and only if there exists a polynomial-time algorithm $\mathcal{A}_1$ for A_1 that uses several times as a subroutine at unit cost a (hypothetical) algorithm $\mathcal{A}_2$ for A_2. We call $\mathcal{A}_1$ a **polynomial-time reduction** of A_1 to A_2.
We say that a recognition problem A_1 **polynomially transforms** to another recognition problem A_2 if, given any string x, we can construct a string y within polynomial time so that x is a "yes"-instance of A_1 if and only if y is a "yes"-instance of A_2.
A recognition problem $A \in \mathcal{NP}$ is said to be **$\mathcal{NP}$-complete** if all other problems in $\mathcal{NP}$ polynomially transform to A.
If all recognition problems within the class $\mathcal{NP}$ polynomially reduce to some

recognition problem A, i. e. A is as hard as any problem in $\mathcal{NP}$, but we are not able to argue that $A \in \mathcal{NP}$, the recognition problem A is said to be $\mathcal{NP}$**-hard**. Moreover, the term $\mathcal{NP}$-hard is sometimes used to describe optimization problems, the recognition problems of which are $\mathcal{NP}$-complete.

To characterize the asymptotic behaviour of the running time of an algorithm let us introduce the following notation using the **Landau symbol**:

Let $f(n)$, $g(n)$ be two positive-valued functions defined on the set $\mathbb{N}$ of natural numbers. We write

$$f(n) \in O(g(n)) \tag{1.1}$$

if there exists a constant $c > 0$ so that, for large enough n, $f(n) \leq c \cdot g(n)$.

We do not want to go in more details as the basics mentioned here suffice in the following. For more details and complexity classes, however, we refer to [107] again.

1.2 Inverse problems

An **inverse problem** is posed in a way that is inverted from that in which **direct problems** are posed determining the effect $b \in Y$ of a given cause $x \in X$ according to a definite mathematical model A, i. e. $Ax = b$.

A common feature of inverse problems is their **instability**, i. e. small changes in the data may give rise to large changes in the solution. Around the turn of the last century, Hadamard clearly formulated the concept of a **well-posed problem**. He took existence, uniqueness and stability of solutions to be the characteristics of a well-posed problem.

If at least one of the demands on the solution is not satisfied, the problem is called **ill-posed**.

Existence and uniqueness are usually guaranteed by taking the least square solution of $||Ax - b||$ of minimal norm, which exists and is uniquely determined if $b \in \text{im}(A) \oplus \text{im}(A)^\perp$, where $\text{im}(A)$ denotes the range of the operator A. The inverse mapping $A^+ : \text{im}(A) \oplus \text{im}(A)^\perp \subset Y \to X$ is called the **Moore-Penrose generalized inverse** of the operator A, which is continuous if and only if the range $\text{im}(A)$ is closed. To restore continuity, a regularization of A^+ is used, which is a family $(T_\gamma)_{\gamma>0}$ of continuous operators $T_\gamma : Y \to X$ satisfying

$$\lim_{\gamma \to 0} T_\gamma b = A^+ b \tag{1.2}$$

on the domain of A^+. Let $b^\epsilon \in Y$ be an approximation of b according to the data error $||b - b^\epsilon|| \leq \epsilon$ and let $\gamma(\epsilon)$ denote the regularization parameter so that

$$\gamma(\epsilon) \to 0, \tag{1.3}$$

$$||T_{\gamma(\epsilon)}||\epsilon \to 0 \tag{1.4}$$

for $\epsilon \to 0$. It yields that

$$\begin{aligned} ||T_{\gamma(\epsilon)} b^\epsilon - A^+ b|| &\leq ||T_{\gamma(\epsilon)}(b^\epsilon - b)|| + ||T_{\gamma(\epsilon)} b - A^+ b|| \\ &\leq ||T_{\gamma(\epsilon)}||\epsilon + ||T_{\gamma(\epsilon)} b - A^+ b|| \\ &\to 0, \end{aligned} \tag{1.5}$$

and thus the inverse problem to $Ax = b$ can approximately be solved. Determining a good regularization parameter is a major issue in the theory of ill-posed problems.

Some well-known examples for regularization methods are given by the **truncated singular value decomposition** neglecting the small singular values, which cause large error within the inversion process, the **Tikhonov-Phillips regularization**, which moves the small singular values away from 0, and iterative and projective regularizations, see for example [114], [103], [102].

In the following we introduce the field of computerized tomography as an example for an inverse problem and its discrete correspondents geometric and discrete tomography.

1.2.1 Computerized tomography

In the early 1970s **computerized tomography** was introduced in diagnostic radiology and since then, many other applications of computerized tomography have become known.

In computerized tomography as well as in geometric tomography, which we will introduce in more details later, the available data to reconstruct a function are line integrals (or more general k-dimensional hyperplane integrals) through the considered object called X-rays (or more general k-dimensional X-ray transforms).

The most prominent example of computerized tomography is transmission computer tomography in diagnostic radiology. In that field a cross-section of the human body is scanned by an X-ray beam, whose intensity loss is recorded by a detector and processed by a computer to produce a two-dimensional image. If I_0 denotes the initial intensity of the beam L, which we think of as a straight line, I_1 its intensity after having passed the body and $f(x)$ the attenuation coefficient of the object at the point x, then

$$\frac{I_1}{I_0} = e^{-\int_L f(x)dx} \tag{1.6}$$

describes the relationship between those parameters.

For the parallel scanning geometry a single source and a single detector are required, which move in parallel and rotate during the scanning process. Besides the parallel scanning geometry leading to the **Radon transform** and the **ray transform**, which integrate a function f on $\mathbb{R}^n$ over hyperplanes resp. over straight lines and coincide in the case $n = 2$, also the **fan-beam** scanning geometry is in use in computerized tomography, leading to the **cone beam** transform. In that case the source runs on a circle around the body, firing a whole fan of X-rays, which are simultaneously recorded by a linear detector array for each source position.

In 1917, the Austrian mathematician J. Radon published a paper that demonstrated that an objective function f can be recovered from all its projections, if f is infinitely differentiable and rapidly decreasing, see [112]. Later it was shown that an infinite set of projections is already enough to determine an objective function f, if the function is located within $L_0^2(\mathbb{R}^2)$, see [121], which improves the result of Radon, but is still impractical.

1.2.2 Geometric tomography

If the function to be reconstructed assumes only discrete values, the methods and algorithms which are in use are quite different than those which are used in computerized tomography and belong to fields such as combinatorics, convex analysis, linear and integer programming, and measure theory. A typical example for that case is the reconstruction of the characteristic function of a set that is 1 in the interior and 0 outside the set.

In 1949 Lorentz used the connections to the analysis of functions and gave necessary and sufficient conditions on uniqueness and consistency of a function-pair, see [95].

In the case of continuous tomography we speak of **geometric tomography**, which specially focusses on the analysis of the relevant geometric questions and studies geometric objects rather than density functions. Roughly speaking, it deals with the retrieval of information about a geometric object (e. g. a convex polytope or body, a star-shaped body, a compact or measurable set) from data about its sections, or projections, or both, see [53]. The subject has connections with convex geometry, stereology, geometric probing in robotics, computerized tomography, and other areas.

The discussion of geometric connections was started by Hammer, who in 1961 raised the problem:

When is a planar convex body uniquely determined by its X-rays?

In that context the (parallel) X-ray X_uK of a convex body K in the direction $u \in S^1$ on the unit circle S^1 is the function defined on $u^\perp$ which gives the length of each chord of K parallel to u.

Gardner and McMullen proved that it is possible to find four X-ray directions that will uniquely determine all planar convex bodies, see [53], Theorem 1.2.11.

Besides parallel X-rays, also point X-rays are investigated in geometric tomography, which correspond to the fan-beam X-rays in computerized tomography. A point X-ray X_pK resp. a directed point X-ray D_pK of a planar convex body K at a point p gives the lengths of all the intersections of the body with the lines through the point p resp. with rays issuing from the point p.

First results in that context are given by Falconer, see [33] and [48]. The main result of Volčič states that any set of four points in general position in the plane has the property that the point X-rays at these points will distinguish between any two convex bodies, see [53], Theorem 5.3.8, [33] and [129].

For directed X-rays, three noncollinear points will suffice for that purpose, see [53], Theorem 5.3.6.

1.2.3 Discrete tomography

In a further step away from continuous tomography (see **geometric tomography**), the domain of the functions to be reconstructed is often assumed to be discrete, too.

Research on **discrete tomography** was stimulated in the 1990s as a result of advances in electron microscopy that made it possible to count the number of atoms in each projected column of a crystal lattice, in several directions. The novel method was called QUantitative ANalysis of The Information provided

by Transmission Electron Microscopy, abbreviated QUANTITEM, for details see [84],[119] and [64]. Unfortunately, the signal-to-noise ratio turned out to be prohibitively large for this technique. Recently, new techniques have been developed based on high-resolution electron microscopy (HREM) image simulation and exit wave reconstruction that provide a much better signal-to-noise ratio, see [78], [79]. The improvement of field emission high-resolution transmission electron microscopes (HRTEM) including the introduction of aberration correctors extended their resolution to sub-Ångstrom values.
Five American major electron microscopy centres are teaming up for a project called Transmission Electron Aberration-corrected Microscope (TEAM), which is funded by the U.S. Department of Energy's Office of Basic Energy Sciences. One of the goals is to achieve a resolution of 0.5 Ångstrom - about one-million times smaller than the diameter of a human hair - by the end of the decade. One of the challenges is to develop a complex system of lenses to correct the aberrated images, which are created by the optical system of present microscopes, see [26], [135].
Motivated by the crystalline structure, the domain of the function to be reconstructed is mostly given by the lattice set $\mathbb{Z}^n$ or can often be reduced to it by some affine transformation. In the following we restrict to the case that a planar lattice set $F \subset \mathbb{Z}^2$ has to be reconstructed by its line sum values along a set S of lattice directions. A **lattice direction** $u \in S$ is defined by $u = (r, s) \in \mathbb{Z}^2 \backslash \{0\}$ satisfying $\gcd(r, s) = 1$.
The line sum value along some line $l := \{(i,j) \in \mathbb{Z}^2 | rj - si = t\}$ in direction u (or X-ray data in analogy to the continuous case) is given by $|F \cap l|$ resp. by $\sum_{(i,j)\in l\cap\mathbb{Z}^2} f(i,j)$ for the characteristic function f of the lattice set F.
Nonuniqueness is represented by so-called **switching components** with respect to the set S of lattice directions, which are given by those integer functions g on the lattice set $\mathbb{Z}^2$ having finite support so that all line sum values in any lattice direction $u \in S$ are equal to 0. Moreover, if the range of one of those functions g is given by $\{-1, 0, +1\}$ and if $\{(i,j)|g(i,j) = 1\} \subset F$ and $\{(i,j)|g(i,j) = -1\} \cap F = \emptyset$, the lattice set F is not uniquely determined by its line sum values. Changing all elements within a switching component is called a **switching operation**.
By **elementary switching component** we refer to the switching component

$$g(i,j) = \text{coeff}_{i,j}\Big(\prod_{u=(r,s)\in S} (x^r y^s - 1)\Big), \tag{1.7}$$

which corresponds to the smallest projection of the $|S|$-dimensional unit cube into the lattice set $\mathbb{Z}^2$, and its translations, compare [69].
Fishburn and Shepp introduce the concept of additivity, which describes the uniqueness of the reconstruction on the domain $[0, 1]$ instead of $\{0, 1\}$ for each lattice position, see [50]. Additivity is necessary and sufficient for uniqueness

in the case $m = 2$ lattice directions because of the total unimodularity of the system matrix, and is sufficient but not necessary if $m \geq 3$.

Many problems of discrete tomography are first discussed as combinatorial problems during the late 1950s and 1960s. In 1957 Ryser published a necessary and sufficient consistency condition for a pair of integral vectors being the row and column sum vectors of a binary matrix, see [115]. By giving a constructive proof of his theorem, Ryser provided the first reconstruction algorithm. In the same year Gale proved the same consistency condition as Ryser, but applying it to flows in networks, see [52].

A discrete analogue of Gardner and McMullen's theorem in geometric tomography was obtained by Gardner and Gritzmann, who showed that convex lattice sets in $\mathbb{Z}^2$ are determined by certain prescribed sets of four lattice directions, see [54].

The complexity of **consistency**, **reconstruction** and **uniqueness** is investigated in [55]. For a finite set S of lattice directions and at least $m = |S| = 3$ pairwise different lattice directions, the problems

> CONSISTENCY(S)
> Instance: For each $u \in S$ a function $f_u : \mathcal{D}(u) \to \mathbb{N}_0$, where $\mathcal{D}(u)$ is a finite set of lattice lines parallel to u.
> Question: Does there exist an $F \subset \mathbb{Z}^2$ so that $X_u F = f_u$ for all $u \in S$?

and

> UNIQUENESS(S)
> Instance: Lattice set $F \subset \mathbb{Z}^2$ of finite cardinality.
> Question: Does there exist an $F' \subset \mathbb{Z}^2$ having the same X-ray values as F according to the direction set S?

are $\mathcal{NP}$-complete and the problem

> RECONSTRUCTION(S)
> Instance: For each $u \in S$ a function $f_u : \mathcal{D}(u) \to \mathbb{N}_0$, where $\mathcal{D}(u)$ is a finite set of lattice lines parallel to u.
> Task: Construct a solution F (if one exists), which satisfies $X_u F = f_u$ for all $u \in S$.

is $\mathcal{NP}$-hard, see [55] and [73], section 4.4, whereas in the case $m = 2$ all those problems are solved in polynomial time. Moreover, the complexity for the polyatomic case is investigated in [30] and [56].

Recent results are concerned with the stability of lattice sets according to small changes within the X-ray data, see [7], [3], [6], [4], [5]. Large instability was discovered for $m \geq 3$ lattice directions. In particular, it was shown that for any $\alpha \in \mathbb{N}$ there exist two finite lattice sets F_1, F_2 satisfying

- F_k for $k = 1, 2$ is uniquely determined by the X-rays $X_{u_j} F_k$ for $j = 1, \ldots, m$,
- $F_1 \cap F_2 = \emptyset$,
- $|F_1| = |F_2| \geq \alpha$,
- $\sum_{u \in S} ||X_u F_1 - X_u F_2||_1 = 2(m-1)$,
- $|F_1 \cap t(F_2)| \leq (3 \cdot 2^{m-2}+1)^2((32m-44)^2+2)$ for each affine transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$.

Besides electron microscopy and material sciences, in particular the reconstruction of crystalline structures from images produced by quantitative high resolution transmission electron microscopy, discrete tomography has various interesting applications and connections to medical imaging, image processing, graph theory, scheduling (see [32], [109]), quality control in semiconductor industry (see [7]), data coding, statistical data security (see [77], [49]), game theory, and so on.

1.3 Separability problems

Many problems in supervised machine learning can be formulated as classifying objects into a finite set of categories, based on a given training set. Numerous powerful statistical learning techniques such as decision trees, support vector machines, logistic regression, etc. have been developed.

The geometric version of the classifying problem is formulated as the following **separability problem**:

Given a subset $S \subset P$ within the finite point set $P \subset \mathbb{R}^n$, find some ball, polytope, etc. so that the point set S is located inside, but the point set $P \setminus S$ is located outside.

We give a short overview of some results which we will refer to later:

Separation by polyhedrals or balls is discussed in literature, see [98, 106]. In [106] both detecting spherical separability and finding a smallest separating ball are answered in polynomial time in the number of points for fixed space dimension. More precisely, both can be solved in $O(3^{n^2} \cdot |P|)$ arithmetic operations for the finite point set $P \subset \mathbb{R}^n$ and the dimension n of the space. Polytopal separability is treated in [98] by deciding whether two finite point sets can be separated by a fixed number of hyperplanes.

Klee and Laskowski describe in [86] an algorithmic approach to find a minimal enclosing triangle for a planar set of points $P \subset \mathbb{R}^2$, where minimal refers to the Lebesgue measure. Enclosing a convex polytope can be done in linear time with respect to the number of vertices. Therefore, the running time of their algorithm is restricted by the algorithmic calculation of the convex hull of the point set, which can be done by Graham scan in $O(|P| \cdot \log|P|)$ arithmetic operations, see [59].

Hurtado is concerned with the problem of separating two finite point sets by wedges and strips in [75]. In the wedge case both detecting separability and determining all vertices of separating wedges is done in $O(|P| \cdot \log|P|)$ arithmetic operations.

1.4 Overview and main results

The main results of this thesis are contained in the Chapters 2 up to 7. All chapters are mainly self-contained.
Chapter 2 is concerned with the reconstruction of quasicrystals, in particular the separability problems which appear in that context. Chapter 3 and Chapter 4 examine instabilities and the possibility to locate convex lattice sets for point X-rays, in each case also surveying the consequences to the parallel case. Chapter 5 and Chapter 6 treat the characterization of small errors and the construction of large instabilities in a bounded lattice set for parallel X-rays. Chapter 7 takes underlying periodicity assumptions of a bounded lattice set into account. More details are given in the following.

Chapter 2

Considering the reconstruction problem in the discrete tomography of quasicrystalline structures, there also appears some separability question (for motivation and more details see later):
Let $W \subset \mathbb{R}^n$ be some compact subset in the Euclidean space $\mathbb{R}^n$, which is known up to translation. We have to determine all separable subsets S within the point set P, i. e. the set

$$\{S \subset P | \exists t \in \mathbb{R}^n : (t + W) \cap P = S \text{ and } (\mathbb{R}^n \backslash (t + W)) \cap S = \emptyset\}. \tag{1.8}$$

Deciding separability as considered in literature is now replaced by calculating all separable subsets. Furthermore, for any polytope W the position of the hyperplanes to each other is now fixed in contrary to the considerations in [98]. In Chapter 2 we present algorithms for the polytopal and the spherical case, which run in polynomial time $O(m^{n+1} \cdot |P|^{n+1})$ resp. $O(|P|^{n+2})$ for fixed space dimension n and the number m of facets in the polytopal case. By worst case analysis we see that there are at most $O(|P|^n)$ separable subsets in both cases. Furthermore, some annotations to the elliptic case, to the intersection and union of balls and the union of polytopes are also given.

In Chapter 2 we are also concerned with minimal separating balls and triangles. As minimality causes that both interior and exterior points are possibly located on the boundary of the separating ball or triangle, we characterize the approximability of minimal separating balls and triangles. By using ideas from both triangular enclosing in [86], [105], [85] and wedge separability in [75], we attend the generalization of minimal triangular enclosing as given in [86] to minimal triangular separability, which can be solved in $O(|P|^2 \cdot \log |P|)$ arithmetic operations.

Chapter 3

In Chapter 3 we consider point X-rays in analogy to the fan-beam geometry in computerized tomography and investigate the instability in that case. Based on projective methods as already used in [45], it turns out that point X-rays lead to even worse assertions than in the case of parallel X-rays. Besides affine transformations we also take projective transformations into account in order to exclude that the worse results are only based on the different geometries which underlie the projective mapping and the affine transformations. For both cases we can show that the number of overlappings of the constructed lattice sets is constant with respect to the number of point X-ray sources and depend only on the dimension n of the lattice set, if we consider the general case $\mathbb{Z}^n$ instead of $\mathbb{Z}^2$. In particular, we get for $m \geq 3$ point X-ray sources $p_1, \dots, p_m$ that for each $\alpha \in \mathbb{N}$ there exist two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^n$ satisfying

- F_k for $k = 1, 2$ is uniquely determined by the point X-rays $X_{p_j} F_k$ for $j = 1, \dots, m$,
- $F_1 \cap F_2 = \emptyset$,
- $|F_1| = |F_2| \geq \alpha$,
- $\sum_{i=1}^{m} ||X_{p_i} F_1 - X_{p_i} F_2|| = 2(m-1)$,
- $|F_1 \cap t(F_2)| \leq n + 1$ for each affine transformation $t : \mathbb{Z}^n \to \mathbb{Z}^n$,
- $|F_1 \cap \rho_{\mathbb{R}^n}(F_2)| \leq n + 2$ for each projective transformation $\rho : \mathbb{P}^n \to \mathbb{P}^n$.

We also take a look at some uniqueness aspects according to convexity assumptions and the consequences of our instability results to the case of parallel X-rays. We can push the upper bound of the number of overlappings with regard to affine transformations to the value $2^{m-1}-1$ for translations $t(x) := x+b$ and half-around rotations $t(x) := -x + b$ and to the value 4 in the other cases. Other non-projective methods reduce the upper bound to some multiple of the smallest possible cardinality of a switching component.

Chapter 4

In Chapter 4 the situation is a little bit changed. Now we assume that the lattice set is well-known, but not its position. Thus, we are concerned with the question how many sources are needed to uniquely determine its location with regard to affine lattice transformations by its directed and undirected point X-ray values. The analogue question is also asked for the case of parallel X-rays.

Chapter 5

We know that error values in the right hand side data can cause large changes within the original data, see [7]. Hence, some knowledge of the possible error vectors and the location of possible nearby reconstructions to a reference template can help to exclude large changes or to reject some template within the quality control in semiconductor industry on the basis of its X-ray data all the same.

Chapter 6

Instabilities as considered in [7] can reach large scale, but seem to be sparsely located. Therefore, in Chapter 6 we consider the size of instabilities for a bounded lattice set.

Chapter 7

We analyse periodicity assumptions for a bounded lattice set in Chapter 7. Motivated by the necessary width of conducting paths on wafers in semiconductor industry, jumps are admitted at certain horizontal and vertical positions of fixed interrelated distance. Necessary uniqueness conditions in the cases with and without absorption are quoted and interpreted within the process of scaling and rescaling. Moreover, some remarks about stability and instability are given, in particular possible right hand side error data are investigated.

Chapter 2

Separation of point sets

2.1 Motivation

In the following we investigate the discrete tomography of systems of **aperiodic order**, more precisely, of so-called **mathematical quasicrystals** (or **model sets**). The main motivation for our interest comes from the physical existence of quasicrystals. Model sets which arise by definition from so-called **cut and project schemes**, compare [100], are commonly accepted to be good mathematical models for quasicrystalline structures, see [122].

A well-known class of planar model sets is the class of **cyclotomic model sets** $\varLambda \subset \mathbb{Z}[\xi_k]$, where ξ_k is a primitive kth root of unity (e. g., $\xi_k = e^{\frac{2\pi i}{k}}$) and $k \notin \{1,2,3,4,6\}$, including the planar model sets associated with the Ammann-Beenker tiling ($k=8$), the Tübingen triangle tiling ($k=5$) and the shield tiling ($k=12$), for illustration see [74].

Let $k \in \mathbb{N}\backslash\{1,2,3,4,6\}$, let $\varphi(k) = \big|\{l \in \mathbb{N} \,|\, 1 \leq l \leq k, \gcd(l,k) = 1\}\big|$be the Euler's totient function and let the set $\{\sigma_i \,|\, i \in \{1,\dots,\frac{\varphi(k)}{2}\},\ \sigma_1(z) = z$ the identity$\}$ arise from the Galois group of $\mathbb{Q}(\xi_k)/\mathbb{Q}$ according to the corresponding cyclotomic field $\mathbb{Q}(\xi_k)$ by choosing exactly one automorphism for each pair of complex conjugate automorphisms. The class of **cyclotomic** model sets arises from cut and project schemes of the following form, where we follow the algebraic setting of Pleasants, see [110]:

$$
\begin{array}{ccccc}
\mathbb{R}^2 & \stackrel{\pi}{\longleftarrow} & \mathbb{R}^2 \times (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1} & \stackrel{\pi_{\text{int}}}{\longrightarrow} & (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1} \\
\cup\ {\scriptstyle\text{dense}} & & \cup\ {\scriptstyle\text{lattice}} & & \cup\ {\scriptstyle\text{dense}} \\
\underbrace{\mathbb{Z}[\xi_k]}_{=:L} & \stackrel{1\text{–}1}{\longleftrightarrow} & \underbrace{\{(z = \sigma_1(z), \underbrace{(\sigma_2(z),\dots,\sigma_{\frac{\varphi(k)}{2}}(z))}_{=:z^\star}) \,|\, z \in \mathbb{Z}[\xi_k]\}}_{=:\tilde{L}} & \stackrel{1\text{–}1}{\longleftrightarrow} & \underbrace{(\mathbb{Z}[\xi_k])^{\frac{\varphi(k)}{2}-1}}_{=L^\star}
\end{array}
$$

Given any subset $W \subset (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1}$ with $\emptyset \neq \operatorname{int}(W) \subset W \subset \operatorname{clos}(\operatorname{int}(W))$ compact (in particular, W is relatively compact) and any $t \in \mathbb{R}^2$, we obtain a planar **model set** $\Lambda_k(t,W) := t + \Lambda_k(W)$ relative to the above cut and project scheme by the setting

$$\Lambda_k(W) := \{z \in \mathbb{Z}[\xi_k] \,|\, z^\star \in W\},$$

compare [100, 110] for more details and more general settings, and [11] for general background. Further, the space $\mathbb{R}^2$ resp. the space $(\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1}$ is called the **physical** resp. the **internal** space, W is referred to as the **window** of $\Lambda_k(t,W)$ and $\star\colon\, L \longrightarrow (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1}$ is the so-called **star map**.
The class of **cyclotomic** model sets is defined as the union of all sets

$$\begin{aligned}\mathcal{M}(\mathbb{Z}[\xi_k]) := \{\Lambda_k(t,W) \,|\, t \in \mathbb{R}^2, W \subset (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1} \text{ with}\\ \emptyset \neq \operatorname{int}(W) \subset W \subset \operatorname{clos}(\operatorname{int}(W)) \text{ compact}\}\end{aligned} \tag{2.1}$$

for $k \in \mathbb{N}\backslash\{1,2,3,4,6\}$.
For example, the planar model set associated with the Ammann-Beenker tiling, see [8],[13] and [51], can be described in algebraic terms as

$$\begin{aligned}\Lambda_{\mathrm{AB}} := \{z \in \mathbb{Z}[\xi_8] \,|\, z^\star \in O \text{ the regular octagon of unit edge length}\\ \text{centred at the origin}\},\end{aligned} \tag{2.2}$$

where the star map * is the Galois automorphism within the Galois group of $\mathbb{Q}(\xi_8)/\mathbb{Q}$ defined by $\xi_8 \longmapsto \xi_8^3$.
Using standard results of elementary algebra, it is shown that the decomposition problem for cyclotomic model sets can be solved in polynomial time, see [74] and [12]:

Theorem 2.1.1
Let $k \geq 3$, let $o_1, \ldots, o_m \in \mathbb{Z}[\xi_k]$ be $m \geq 2$ (pairwise non-parallel) ***module directions*** *and let $\mathcal{L}_{o_i}$ be the set of lines in direction o_i, $i \in \{1,\ldots,m\}$. Furthermore, let $p_{o_i} : \mathcal{L}_{o_i} \to \mathbb{N}_0$, $i \in \{1,\ldots,m\}$ be functions with finite support $supp(p_{o_i}) \subset \mathcal{L}_{o_i}^{\mathbb{Z}[\xi_k]}$. The problem of decomposing the grid*

$$\mathcal{G}_{\{p_{o_i}|i\in\{1,\ldots,m\}\}} := \bigcap_{i=1}^{m} \Big(\bigcup_{l \in supp(p_{o_i})} l \Big) \tag{2.3}$$

into its equivalence classes modulo $\mathbb{Z}[\xi_k]$ according to the equivalence relation

$$g \sim g' :\Longleftrightarrow g - g' \in \mathbb{Z}[\xi_k] \tag{2.4}$$

can be solved in polynomial time. Whenever the module directions are fixed, there are only finitely many possible equivalence classes.
The phenomenon of multiple equivalence classes modulo $\mathbb{Z}[\xi_k]$ in the grid can also occur in the crystallographic cases $k = 3$ and $k = 4$.

Now we turn to the reconstruction problem for X-ray data in two module directions:
Let $o_1, o_2 \in \mathbb{Z}[\xi_k]$ be two (non-parallel) module directions, let $p_{o_1} : \mathcal{L}_{o_1} \to \mathbb{N}_0$, $p_{o_2} : \mathcal{L}_{o_2} \to \mathbb{N}_0$ be two functions with finite support and let $W \subset (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1}$ be a window satisfying $\emptyset \neq \operatorname{int}(W) \subset W \subset \operatorname{clos}(\operatorname{int}(W))$. The task is to reconstruct a finite set F if existent which is contained in a cyclotomic model set

$$\Lambda_k(t, \tau + W) \in \mathcal{M}(\mathbb{Z}[\xi_k]) \tag{2.5}$$

with $t \in \mathbb{R}^2$ and $\tau \in (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1}$ and satisfies $X_{o_1}(F) = p_{o_1}$ and $X_{o_2}(F) = p_{o_2}$.
By Theorem 2.1.1 we can assume that the equivalence classes of the grid $\mathcal{G}_{\{p_{o_1}, p_{o_2}\}}$ modulo $\mathbb{Z}[\xi_k]$ are given, say $\mathcal{G}_{\{p_{o_1}, p_{o_2}\}} = \dot\bigcup_{i=1}^{c} G_i$ where $G_i - t_i \subset \mathbb{Z}[\xi_k]$ for suitable $t_i \in \mathbb{R}^2$. Due to the definition of (cyclotomic) model sets, not every subset of the equivalence class G_i that conforms to the X-ray data is admissible, more precisely, a possible reconstruction $F \subset G_i$ must satisfy:

$$\exists\ \tau \in (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1}\ :\ (F - t_i)^\star \subset \tau + W \tag{2.6}$$

Therefore, we have to know all subsets $(F - t_i)^*$ within each set $(G_i - t_i)^*$ which can be separated by the window W.

Definition 2.1.2 (separation with respect to the point set P)
*Let $\emptyset \neq P \subset \mathbb{R}^n$ be a nonempty finite point set within the Euclidean space $\mathbb{R}^n$. We say that a subset $P' \subset P$ can be **separated with respect to the point set** P by the window $W \subset \mathbb{R}^n$, if*

$$p \in t + W \text{ for all } p \in P', \tag{2.7}$$

$$p \notin t + W \text{ for all } p \in P \backslash P' \tag{2.8}$$

for some translation vector $t \in \mathbb{R}^n$.
We write

$$\operatorname{Sep}_W(P) := \{P' \subset P \,|\, P' \text{ can be separated with respect to } P \text{ by } W\}. \tag{2.9}$$

Thus, we have to determine the set

$$\operatorname{Sep}_W((G_i - t_i)^\star) := \{(G_i - t_i)^\star \cap (\tau + W) \,|\, \tau \in (\mathbb{R}^2)^{\frac{\varphi(k)}{2}-1}\}, \tag{2.10}$$

which contains all those subsets of $(G_i - t_i)^\star$ that are separable from its complement by a translate of the window W.
Our target is to find an algorithm for the determination of the set $\operatorname{Sep}_W(P)$ that requires a polynomial number of arithmetic operations in terms of the size $|P|$ for fixed space dimension n. This problem is tractable for special classes of windows:
In the case of spherical and polytopal windows we will present algorithms which take $O(|P|^{n+2})$ resp. $O(m^{n+1} \cdot |P|^{n+1})$ arithmetic operations, where m is the number of facets in the polytopal case.

2.2 Separation by hyperplanes

The basic idea that is used to determine all sets which can be separated by hyperplanes is also used later for spherical and polytopal separation. Therefore, we want to consider the separability problem for hyperplanes first, although that case will not be directly applied to quasicrystalline reconstruction problems.

Definition 2.2.1 (hyperplane)
For $a \neq 0, b \in \mathbb{R}^n$ we define

$$H_{a,b} := \{x \in \mathbb{R}^n | a^T x = a^T b\}, \tag{2.11}$$
$$H^{\leq}_{a,b} := \{x \in \mathbb{R}^n | a^T x \leq a^T b\}. \tag{2.12}$$

The first one is the (unique) ***hyperplane*** *with normal vector a which contains the point $b \in \mathbb{R}^n$, the second one is the associated closed halfspace in the opposite direction of the normal vector a. The sets $H^{\geq}_{a,b}$, $H^{<}_{a,b}$ and $H^{>}_{a,b}$ are analogously defined.*

Definition 2.2.2 (separation by a hyperplane)
Let $\emptyset \neq P \subset \mathbb{R}^n$ be a nonempty finite point set within the Euclidean space $\mathbb{R}^n$. A subset $S \subset P$ can be ***separated by a hyperplane with respect to the point set*** *P, if there exist $a \neq 0, b \in \mathbb{R}^n$ so that*

$$S \subset H^{\leq}_{a,b}, \tag{2.13}$$
$$P \backslash S \subset H^{>}_{a,b}. \tag{2.14}$$

We write

$$\mathcal{H}(P) := \{S \subset P | S \text{ can be separated by a hyperplane with respect to } P\}. \tag{2.15}$$

Given two finite point sets $S, P\backslash S \subset \mathbb{R}^n$, the problem of finding a hyperplane H which separates the point sets S and $P\backslash S$ (called linear separability problem) can be formulated as linear programming problem for $n+1$ variables and $|P|$ constraints. Thus, for fixed space dimension n the linear separability problem can be solved in $O(|P|)$ arithmetic operations, compare [98], [97].
For fixed space dimension n and fixed number m of hyperplanes also the m-polyhedral separability problem of deciding whether the point set S can be separated with respect to the point set P by the intersection of m halfspaces is solved in polynomial time within the cardinality $|P|$ of the point set P, see [98]. But in the case of separation by polytopes in Section 2.3 the hyperplanes supporting the facets of the polytopal window cannot be chosen independently from each other. Therefore, the idea presented in [98] cannot be used for our purpose.

Moreover, our question of separation by a hyperplane is more than recognizing whether a subset $S \subset P$ can be separated with respect to a finite point set $P \subset \mathbb{R}^n$. We want to find all subsets of the point set P which can be separated with respect to the point set P.
In 1968 Vapnik and Chervonenkis discovered the connection between the concept of ***VC dimension*** *(Vapnik-Chervonenkis dimension) and the* ***growth function*** *within the context of learning theory, see [126], [127] and [125]:*
The VC dimension of a set of functions is the maximal number h of vectors $z_1, \dots, z_h$ that can be separated into two classes in all 2^h possible ways by using functions of the set. In the case of linear functions in the n-dimensional Euclidean space $\mathbb{R}^n$ the VC dimension is equal to the value $h = n + 1$. The growth function is denoted by

$$G(l) := \ln \sup_{z_1,\dots,z_l} N(z_1, \dots, z_l) \tag{2.16}$$

for the number $N(z_1, \dots, z_l)$ of different separations of the point set $\{z_1, \dots, z_l\}$ according to the given set of functions. If the VC dimension h of the fixed set of functions is finite, the growth function is bounded by the inequality

$$G(l) \leq h \cdot \ln(\frac{l}{h} + 1) \tag{2.17}$$

with $G(h) = h \cdot \ln 2$ for $l = h$. Thus, in general the number of subsets $S \subset P$ within the finite point set $P \subset \mathbb{R}^n$ exponentially grows within the space dimension n.
Without loss of generality let us assume that the affine space spanned by the point set P has dimension n, as otherwise we can reduce all considerations to the lower dimensional problem of finding all subsets of the point set P which can be separated within the affine subspace $\text{aff}(P)$. Therefore, we know that $P \cap (\mathbb{R}^n \backslash H_{a,b}) \neq \emptyset$ for any hyperplane $H_{a,b}$.
First of all, we want to give a rough idea of the algorithm which we intend to develop. Let us assume that $S \in \mathcal{H}(P)$ and that the hyperplane H separates the point set S with respect to the point set P. If the hyperplane H itself does not contain n affinely independent points of the point set P, we may rotate the hyperplane H (leaving the affine subspace $\text{aff}(H \cap P)$ fixed), until the rotated hyperplane H' hits a new point within the point set P. By iterating that process we end up with a hyperplane which is spanned by n affinely independent points within the point set P and, additionally, the point set S is contained in one of the closed halfspaces associated with the hyperplane.
The additional points in the first rotation step (and analogously in each further rotation step) can be calculated by only using the hyperplane H' and the fixed affine subspace $\text{aff}(P \cap H)$, as we will see later.
The above described procedure is helpful because there is only a polynomial number of hyperplanes which are spanned by n affinely independent points

within the point set P. Thus, let us start with all those hyperplanes, and for each of them let us try to reverse the process of iterative rotations to recover all separable subsets $S \in \mathcal{H}(P)$.

Lemma 2.2.3 *Let $\emptyset \neq S \in \mathcal{H}(P)$ and let $H_{a,b}$ be one of the hyperplanes which separates the point set S with respect to the point set P and which satisfies $b \in H_{a,b} \cap S$. (Such a hyperplane exists as we can find one by translating a separating hyperplane until it hits at least one point of the point set S.)*
If $\dim \operatorname{aff}(H_{a,b} \cap S) < n - 1$ there exists a vector $a' \in \mathbb{R}^n$ so that

1. $H^{<}_{a',b} \cap P \subset S$,
2. $H^{>}_{a',b} \cap P \subset P \backslash S$,
3. $H_{a,b} \cap S \subset H_{a',b} \cap P$,
4. $\dim \operatorname{aff}(H_{a',b} \cap P) = \dim \operatorname{aff}(H_{a,b} \cap S) + 1$.

In particular, the point set S (resp. the point set $P \backslash S$) still lies within the closed halfspace $H^{\leq}_{a',b}$ (resp. $H^{\geq}_{a',b}$), the point set $H_{a,b} \cap S$ remains on the separating hyperplane, but some of the additional points added on the hyperplane in order to increase the affine dimension by 1 possibly belong to the point set $P \backslash S$.

Proof

Since $P \cap (\mathbb{R}^n \backslash H_{a,b}) \neq \emptyset$ there is at least one vector $\bar{a} \in \mathbb{R}^n$ so that

$$\operatorname{aff}(H_{a,b} \cap S) \subset \operatorname{aff}(H_{\bar{a},b} \cap P) \tag{2.18}$$

and

$$\dim \operatorname{aff}(H_{\bar{a},b} \cap P) > \dim \operatorname{aff}(H_{a,b} \cap S). \tag{2.19}$$

Therefore, the vector

$$(1 - \lambda_{min}) \cdot a + \lambda_{min} \cdot \bar{a} \neq 0, \tag{2.20}$$

where λ_{min} is given by

$$\lambda_{min} := \min\{\lambda > 0 | \dim \operatorname{aff}(H_{(1-\lambda)\cdot a+\lambda\cdot\bar{a},b} \cap P) > \dim \operatorname{aff}(H_{a,b} \cap S)\}, \tag{2.21}$$

satisfies 1. - 3.
Now let us choose the vector a' so that 1. - 3. and

$$\dim \operatorname{aff}(H_{a',b} \cap P) \geq \dim \operatorname{aff}(H_{a,b} \cap P) + 1 \tag{2.22}$$

are satisfied and that k with respect to the setting

$$l := \dim \operatorname{aff}(H_{a,b} \cap P), \tag{2.23}$$

$$l + k := \dim \operatorname{aff}(H_{a',b} \cap P) \tag{2.24}$$

attains the minimal value that is possible for any vector a' satisfying 1. - 3. and (2.22). We claim that also 4. is fulfilled, which means that $k = 1$.
For that purpose let us assume that $k > 1$. Let the point set $\{p_1, \ldots, p_{l+1}\} \subset H_{a,b} \cap S$ be affinely independent and let us extend that point set to an affinely independent point set $\{p_1, \ldots, p_{l+1}, p_{l+2}, \ldots, p_{l+k+1}\} \subset H_{a',b} \cap P$. Without loss of generality let us assume that $p_{l+k+1} \in P \backslash S$. Then we can find some vector $v \in \mathbb{R}^n \backslash \{0\}$ so that

$$v^T(p_j - p_1) = 0 \text{ for } j = 2, \ldots, l+k, \tag{2.25}$$

$$v^T(p_{l+k+1} - p_1) > 0. \tag{2.26}$$

Thus, it yields for $\epsilon > 0$ sufficiently small and the vector $\tilde{a} := a' + \epsilon \cdot v$ that

$$\dim \operatorname{aff}(H_{a,b} \cap P) < \dim \operatorname{aff}(H_{\tilde{a},b} \cap P) < \dim \operatorname{aff}(H_{a',b} \cap P) \tag{2.27}$$

in contradiction to the choice of the vector a', as the conditions 1.-3. are still fulfilled for the vector $\tilde{a}$ because of

$$\begin{aligned} \tilde{a}^T p_{l+k+1} &= (a' + \epsilon \cdot v)^T (p_1 + (p_{l+k+1} - p_1)) = \\ &= \tilde{a}^T p_1 + \epsilon \cdot v^T (p_{l+k+1} - p_1) > \tilde{a}^T p_1. \end{aligned} \tag{2.28}$$

□

Lemma 2.2.4 *The step from the vector a to the vector a' in Lemma 2.2.3 can be reversed in the sense that the point set $S \in \mathcal{H}(P)$ is the union*

$$(H^{<}_{a',b} \cap P) \cup (H^{\leq}_{a,b} \cap H_{a',b} \cap P) \tag{2.29}$$

of the point set $H^{<}_{a',b} \cap P$ and the intersection of the point set P with the embedded closed halfspace within the affine subspace $\operatorname{aff}(H_{a',b} \cap P)$ which is given by the embedded hyperplane $\operatorname{aff}(H_{a,b} \cap P)$.

Proof
The rotation in Lemma 2.2.3 satisfies 1.-4. Thus, we get that

$$S = H^{\leq}_{a,b} \cap P = (H^{\leq}_{a',b} \cap P) \backslash (H^{>}_{a,b} \cap H_{a',b} \cap P) = \tag{2.30}$$

$$= (H^{<}_{a',b} \cap P) \cup (H^{\leq}_{a,b} \cap H_{a',b} \cap P). \tag{2.31}$$

□

Of course, we neither know the relevant affine subspace $\operatorname{aff}(H_{a,b} \cap P)$ by starting with the affine space $\operatorname{aff}(H_{a',b} \cap P)$ nor the relevant embedded halfspace within the affine space $\operatorname{aff}(H_{a',b} \cap P)$ which is given by the embedded hyperplane $\operatorname{aff}(H_{a,b} \cap P)$, in order to determine the point set S. But by treating all point sets of $\dim \operatorname{aff}(H_{a',b} \cap P)$ affinely independent points within the point set $H_{a',b} \cap P$ and both embedded halfspaces with respect to the chosen point set, we can calculate a subset of the set $\mathcal{H}(P)$ which contains the point set S. That is

used in the following algorithm:
The iterative application of Lemma 2.2.3 shows that we can start with the hyperplanes which are given by affinely independent n*-sets* $\bar{P}$ *within the point set* P*, in order to calculate all separable sets* $\emptyset \neq S \in \mathcal{H}(P)$*. Each possibly applied step according to Lemma 2.2.3 has to be reversed. After reversing one step we can either stop the procedure or reduce the further calculation to the fixed space with respect to the rotation in Lemma 2.2.3, as further additional points which have to be eliminated in the next reversing step only lie within that subspace.*
For notational purpose within the algorithm let us define the set $\mathrm{Sequ}(\bar{P})$ *of descending sequences with respect to the point set* $\bar{P}$*.*

Notation 2.2.5
For some point set $\bar{P}$ *of cardinality* $|\bar{P}| \leq n$ *let*

$$\mathrm{Sequ}(\bar{P}) := \{(\mathcal{P}_{|\bar{P}|}, \ldots, \mathcal{P}_1) | \mathcal{P}_{|\bar{P}|} = \bar{P}, \mathcal{P}_j \supset \mathcal{P}_{j-1}, |\mathcal{P}_j| = j\} \tag{2.32}$$

denote the set of all descending sequences $(\mathcal{P}_{|\bar{P}|}, \ldots, \mathcal{P}_1)$ *of point sets* $\mathcal{P}_j$ *starting with the point set* $\bar{P}$ *so that* $|\mathcal{P}_j| = j$ *for* $j = 1, \ldots, |\bar{P}|$ *and each set includes the consecutive one.*

SeparationByHyperplanes$_n(P)$
Input: finite point set $\emptyset \neq P \subset \mathbb{R}^n$ of full affine dimension
Output: $\mathcal{H}(P)$
(1) $M \leftarrow \{\emptyset\}$
(2) **foreach** $\bar{P} \subseteq P$, $\bar{P}$ affinely independent, $|\bar{P}| = n$
(3) **foreach** $\mathcal{P} = (\mathcal{P}_n, \ldots, \mathcal{P}_1) \in \mathrm{Sequ}(\bar{P})$
(4) **for** $j = n$ **to** 1
(5) $H^j_{\mathcal{P}} \leftarrow \mathrm{aff}(\mathcal{P}_j)$ which is the (embedded) hyperplane within $\mathrm{aff}(\mathcal{P}_{j+1})$ resp. $\mathbb{R}^n$ for $j = n$
(6) $H^{j,<}_{\mathcal{P}}, H^{j,>}_{\mathcal{P}} \leftarrow$ the associated (embedded) open halfspaces within $\mathrm{aff}(\mathcal{P}_{j+1})$ resp. $\mathbb{R}^n$ for $j = n$
(7) $M \leftarrow M \cup \bigcup_{i=1}^{n} \{(H^i_{\mathcal{P}} \cup \bigcup_{j=i}^{n} H_j) \cap P | H_j \in \{H^{j,<}_{\mathcal{P}}, H^{j,>}_{\mathcal{P}}\}\}$,
(8) **return** M

Theorem 2.2.6
The algorithm determines the set $\mathcal{H}(P)$ *of all subsets within a nonempty finite point set* $P \subset \mathbb{R}^n$ *which can be separated with respect to the point set* P *by a hyperplane in* $O(|P|^{n+1})$ *arithmetic operations.*

Proof
Notice, that according to Definition 2.1.2 also the empty set belongs to $\mathcal{H}(P)$*, i. e.* $\emptyset \in \mathcal{H}(P)$*.*
Using Lemma 2.2.3 and Lemma 2.2.4 all point sets $(H^i_{\mathcal{P}} \cup \bigcup_{j=i}^{n} H_j) \cap P$ *in (7) can*

be separated with respect to the point set P by a hyperplane, as any point set $(H_{\mathcal{P}}^{i} \cup \bigcup_{j=i}^{n} H_j) \cap P$ results from a sequence of rotations, see also the motivation of the algorithm before. Notice, that all point sets $\emptyset \neq S \in \mathcal{H}(P)$ are actually reached, as we consider all affinely independent subsets $\bar{P} \subset P$ of cardinality n and all sequences $(\mathcal{P}_n, \ldots, \mathcal{P}_1) \in \mathrm{Sequ}(\bar{P})$. Therefore, every calculated point set within the algorithm belongs to the set $\mathcal{H}(P)$ and every point set $S \in \mathcal{H}(P)$ is determined by the algorithm.

For the complexity assertion let us notice that the number of affinely independent subsets $\bar{P} \subset P$ of cardinality n is bounded from above by

$$\binom{|P|}{n} \in O(|P|^n). \tag{2.33}$$

For every point set $\bar{P}$ we have to consider $n!$ descending sequences $(\mathcal{P}_n, \ldots, \mathcal{P}_1) \in \mathrm{Sequ}(\bar{P})$, which does not depend on the cardinality $|P|$ of the point set P. For each descending sequence the membership of every point within the point set P to the embedded hyperplanes $H_{\mathcal{P}}^{j}$ and halfspaces H_j has to be tested, compare (7) within the algorithm. Thus, the total complexity is given by

$$O(|P|^{n+1}) \tag{2.34}$$

for fixed space dimension n and is therefore polynomial within the cardinality $|P|$ of the point set P. □

Remark 2.2.7 The algorithmic approach for polytopal separation in Section 2.3 needs all subsets P' within a finite point set P which can be separated by a hyperplane passing through some point $q \in \mathbb{R}^n$ not necessarily contained in the point set P. The algorithm presented before can be adapted to that case by

1. $M \leftarrow \emptyset$ in (1), as the empty set does not have to belong to the set of separable subsets and is calculated in (7), if it is separable,
2. $q \in \bar{P} \subset P \cup \{q\}$ in (2), because the separating hyperplanes have to pass through the point q,
3. $\mathcal{P}_1 = \{q\}$ in (3) because of the same reason as before.

Of course, complexity and correctness are given in analogy to Theorem 2.2.6.

Definition 2.2.8
For notational purpose let us write

$$\mathcal{H}_q(P) := \{S \subset P | S \text{ can be separated with respect to } P \text{ by a hyperplane passing through } q \in \mathbb{R}^n\}. \tag{2.35}$$

2.3 Separation by polytopes

Now let us consider the case of separation by a full-dimensional polytopal window $W = \{x \in \mathbb{R}^n | Ax \leq b\}$ *for* $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$*. In [98] the case of separation by a fixed number of hyperplanes, but non-specified arrangement is treated. To be more precise, for fixed space dimension* n *the question whether two given point sets can be separated by the intersection of a fixed number* m *of arbitrary halfspaces is answered in polynomial time by determining all separable subsets within the union of the two point sets.*
In our case, however, the arrangement of the hyperplanes which support the facets of the polytopal window is fixed. We will present an algorithmic approach of complexity $O(m^{n+1}|P|^{n+1})$ *for the number* m *of facets to determine all subsets within the point set* P *which can be separated with respect to the point set* P *by a given full-dimensional polytopal window.*
In many situations below the knowledge about the set of translation vectors $t \in \mathbb{R}^n$ *for some fixed point* $p \in P$ *so that* $p \in t + W$ *will help us to determine all subsets* $P' \subset P$ *which can be separated with respect to the point set* P *by the window* W*.*

Definition 2.3.1 (suspension set)
The ***suspension set*** A_Q *of a point set* $Q \subset \mathbb{R}^n$ *with respect to some window* $W \subset \mathbb{R}^n$ *is defined by*

$$A_Q := \{t \in \mathbb{R}^n | p \in t + W \text{ for all } p \in Q\}. \tag{2.36}$$

For a single point $q \in \mathbb{R}^n$ *we simply write* A_q *instead of* $A_{\{q\}}$*.*
The suspension set A_Q *of a discrete point set* Q *describes the set of all translation vectors* $t \in \mathbb{R}^n$ *so that all points* $p \in Q$ *are contained in the translated window* $t + W$*. Therefore, the set* $\mathrm{Sep}_W(P)$ *is alternatively given by*

$$\mathrm{Sep}_W(P) = \{S \subset P | A_S \setminus \bigcup_{p \in P \setminus S} A_p \neq \emptyset\}. \tag{2.37}$$

Lemma 2.3.2 *The suspension set* A_p *of a point* $p \in \mathbb{R}^n$ *with respect to the window* W *is given by*

$$A_p = p - W. \tag{2.38}$$

Especially, in the polytopal case $W = \{x \in \mathbb{R}^n | Ax \leq b\}$ *we get that*

$$A_p = p + \{-x \in \mathbb{R}^n | Ax \leq b\} = \{x \in \mathbb{R}^n | Ax \geq Ap - b\}. \tag{2.39}$$

Proof
The equivalence

$$p \in t + W \Leftrightarrow p - t \in W \Leftrightarrow t - p \in -W \Leftrightarrow t \in p - W \tag{2.40}$$

leads to the general assertion.
In the case of polytopal windows we calculate that

$$A_p = p + \{-x \in \mathbb{R}^n | Ax \leq b\} = \{y := p - x \in \mathbb{R}^n | A(p - y) \leq b\} = \\ = \{y \in \mathbb{R}^n | Ay \geq Ap - b\}. \tag{2.41}$$

□

Notation 2.3.3
For later purpose let us denote by a_k the kth row of the matrix A and let us define b_k^p by

$$b_k^p := (Ap - b)_k = a_k^T p - b_k \tag{2.42}$$

for $k = 1, \ldots, m$ and $p \in P$.
Let $S \in \mathrm{Sep}_W(P)$ be a separable subset of the point set $P \subset \mathbb{R}^n$ so that the translated window $t + W$ separates the point set S with respect to the point set P. As we will see later, the translation vector t can be identified by another translation vector $t + \triangle t$ so that the translation vector $t + \triangle t$ is given by the intersection of n hyperplanes in general position within the set of all supporting hyperplanes of the polytopal suspension sets A_p for $p \in P$ and satisfies

- $a_k^T t \left\{ \begin{array}{c} > \\ < \\ = \end{array} \right\} b_k^p \Longrightarrow a_k^T (t + \triangle t) \left\{ \begin{array}{c} \geq \\ \leq \\ = \end{array} \right\} b_k^p$ *for all $k = 1, \ldots, m$ and $p \in P$,*
- $\dim \mathrm{span}\{a_k | a_k^T (t + \triangle t) = b_k^p$ *for some* $p \in P\} = n$.

Therefore, the idea of the presented algorithm is to consider all intersection points of n hyperplanes in general position within the set of the at most $m \cdot |P|$ supporting hyperplanes of the polytopal suspension sets. In order to consider all classes of translation vectors $-\triangle t$ back from the translation vector $t + \triangle t$ to the original translation vector t to determine the associated separable point sets, the set $\mathcal{H}_0(\{a_1, \ldots, a_m\})$ is calculated and the relationship of its members to the set of closed halfspaces given by the hyperplanes $\{a_k^T (t + \triangle t) = b_k^p\}$ for $k = 1, \ldots, m$ and $p \in P$ which are not left in direction $-\triangle t$ is used (for details see also later).

SeparationByPolytopes$_n(P, W)$
Input: finite point set $\emptyset \neq P \subset \mathbb{R}^n$, window $W = \{x \in \mathbb{R}^n |$ $a_k^T x \leq b_k$ for $k = 1, \ldots, m\}$, $\dim(W) = n$
Output: $\mathrm{Sep}_W(P)$

$M \leftarrow \{\emptyset\}$
calculate $\mathcal{H}_0(\{a_1, \ldots, a_m\})$
foreach $k \in \{1, \ldots, m\}$ and $p \in P$
  $b_k^p \leftarrow a_k^T p - b_k$
foreach $\mathcal{U} := \{(a_{k_1}, p_{k_1}), \ldots (a_{k_n}, p_{k_n})\}$ with $p_{k_j} \in P$, $\{a_{k_1}, \ldots, a_{k_n}\} \subset \{a_1, \ldots, a_m\}$ linearly independent
  $S_\mathcal{U} \leftarrow \bigcap_{j=1}^n \{x \in \mathbb{R}^n | a_{k_j}^T x = b_{k_j}^{p_{k_j}}\}$
  foreach $N \in \mathcal{H}_0(\{a_1, \ldots, a_m\}) \cup \{\{a_1, \ldots, a_m\}\}$
    $M \leftarrow M \cup \{p \in P |\ a_k^T S_\mathcal{U} > b_k^p$ or $(a_k^T S_\mathcal{U} = b_k^p$ and $a_k \in N)$ for $k = 1, \ldots, m\}$
return M

Theorem 2.3.4
The algorithm determines the set $\mathrm{Sep}_W(P)$ of all subsets within a nonempty finite point set $P \subset \mathbb{R}^n$ which can be separated with respect to the point set P by a fixed polytopal window W of dimension n with m facets in $O(m^{2n+1} \cdot |P|^{n+1})$ arithmetic operations.

Proof
Let $S \in \mathrm{Sep}_W(P)$ and let $t \in \mathbb{R}^n$ be a translation vector so that $(t+W) \cap P = S$. In the case that $\dim \mathrm{span}\{a_k | a_k^T t = b_k^p$ for some $p \in P\} < n$ there is a further translation vector $\triangle t$ so that

- $a_k^T t \left\{ \begin{array}{c} > \\ < \\ = \end{array} \right\} b_k^p \Longrightarrow a_k^T (t + \triangle t) \left\{ \begin{array}{c} \geq \\ \leq \\ = \end{array} \right\} b_k^p$ *for all $k = 1, \ldots, m$ and $p \in P$,*
- $\dim \mathrm{span}\{a_k | a_k^T (t + \triangle t) = b_k^p$ *for some $p \in P\}$* $> \dim \mathrm{span}\{a_k | a_k^T t = b_k^p$ *for some $p \in P\}$,*

as we will show in the following:
Let $l := \dim \mathrm{span}\{a_k | a_k^T t = b_k^p$ for some $p \in P\} < n$ and let $\{a_{k_1}, \ldots, a_{k_l}\} \subset \{a_1, \ldots, a_m\}$ be a basis of the linear space $\mathrm{span}\{a_k | a_k^T t = b_k^p$ for some $p \in P\}$. There exists a vector $0 \neq v \in \{a_{k_1}, \ldots, a_{k_l}\}^\perp$ so that

$$\dim \mathrm{span}\{a_k | a_k^T (t + v) = b_k^p \text{ for some } p \in P\} > l, \tag{2.43}$$

as the polytopal window W is assumed to be full-dimensional by $\dim(W) = n$. By the setting $\triangle t := \lambda_{min} \cdot v$ for

$$\lambda_{min} := \min\{\lambda > 0 |\ \text{criterion (2.43) is satisfied for } \lambda \cdot v \text{ instead of } v\ \} \tag{2.44}$$

we have found the translation vector $\triangle t$ as mentioned before.
Using inductive arguments the translation vector $t + \triangle t$ is given by the intersection of n supporting hyperplanes in general position.
In the case that $a_k^T(t + \triangle t) = a_k^T t + a_k^T \triangle t = b_k^p$ for $\triangle t \neq 0$ we calculate that

$$a_k^T t \geq b_k^p \Longleftrightarrow a_k^T \triangle t \leq 0. \tag{2.45}$$

Therefore, we have to determine the set $\mathcal{H}_0(\{a_1, \ldots, a_m\})$ of all subsets of the point set $\{a_1, \ldots, a_m\}$ which can be separated by a hyperplane passing through the point $0 \in \mathbb{R}^n$ in order to decide which halfspaces $\{x \in \mathbb{R}^n | a_k^T x \geq b_k^p\}$ so that $a_k^T(t + \triangle t) = b_k^p$ are not left by the translation $-\triangle t$ of the translation vector $t + \triangle t$ back to the translation vector t.
For the complexity assertion let us notice that at most

$$\begin{pmatrix} m \cdot |P| \\ n \end{pmatrix} \in O((m \cdot |P|)^n) \tag{2.46}$$

n-sets of hyperplanes in general position have to be considered in (5). Calculating the intersection point is done in constant time for fixed space dimension n. The cardinality of $\mathcal{H}_0(\{a_1, \ldots, a_m\})$ is given by $O(m^n)$, its calculation needs $O(m^{n+1})$ arithmetic operations according to the results of separation by hyperplanes before. The calculation of the values b_k^p for $k \in \{1, \ldots, m\}$ and $p \in P$ needs $O(m|P|)$ arithmetic operations in total. For every point $p \in P$ and every $k \in \{1, \ldots, m\}$ we have to decide to which halfspace given by the hyperplane $\{a_k^T x = b_k^p\}$ the intersection point S_U as given in the algorithm belongs. The total complexity is therefore given by

$$O(m^{n+1} + m|P| + (m|P|)^n \cdot m^n \cdot m|P|) = O(m^{2n+1} \cdot |P|^{n+1}) \tag{2.47}$$

for fixed space dimension n and is thus polynomial within the number m of facets and the cardinality $|P|$ of the point set P. □

Remark 2.3.5 The condition $\dim(W) = n$ above is not really restrictive. In the case that $\dim(W) < n$ let us assume that $0 \in \text{aff}(W)$ by applying some translation on the polytopal window W if necessary. Now we only have to divide the point set P into its equivalence classes modulo the affine space $\text{aff}(W)$ denoted by $\text{Equ}_W(P)$ in the following, i. e. the points $p_1, p_2 \in P$ lie in the same equivalence class $P' \in \text{Equ}_W(P)$ if and only if $p_1 - p_2 \in \text{aff}(W)$. For each equivalence class $P' \in \text{Equ}_W(P)$ we have to apply the algorithm above on the affine space $\text{aff}(P')$ instead of the Euclidean space $\mathbb{R}^n$. Thus, the total amount of arithmetic operations is bounded by

$$\begin{aligned} &\sum_{P' \in \text{Equ}_W(P)} O(m^{2 \cdot \dim(W)+1} \cdot |P'|^{\dim(W)+1}) \\ &\leq O(m^{2 \cdot \dim(W)+1} \cdot (\sum_{P' \in \text{Equ}_W(P)} |P'|)^{\dim(W)+1}) \\ &\leq O(m^{2 \cdot \dim(W)+1} \cdot |P|^{\dim(W)+1}). \end{aligned} \tag{2.48}$$

The complexity in Theorem 2.3.4 and Remark 2.3.5 can be reduced to $O(m^{n+1} \cdot |P|^{n+1})$ resp. to $O(m^{\dim(W)+1} \cdot |P|^{\dim(W)+1})$ arithmetic operations. For that purpose let us modify the algorithm of separation by hyperplanes:

SeparationByHyperplanes(P, P')
Input: finite point sets $\emptyset \neq P' \subset P \subset \mathbb{R}^n$, $\dim \operatorname{aff}(P') = \dim \operatorname{aff}(P)$
Output: $\mathcal{H}(P, P')$
(1) $M \leftarrow \{\emptyset\}$
(2) **foreach** $\bar{P} \subseteq P'$, $\bar{P}$ affinely independent, $|\bar{P}| = \dim \operatorname{aff}(P)$
(3) **foreach** $\mathcal{P} = (\mathcal{P}_{\dim \operatorname{aff}(P)}, \ldots, \mathcal{P}_1) \in \operatorname{Sequ}(\bar{P})$
(4) **for** $j = \dim \operatorname{aff}(P)$ **to** 1
(5) $H_{\mathcal{P}}^{j} \leftarrow \operatorname{aff}(P_j)$ the (embedded) hyperplane within $\operatorname{aff}(P_{j+1})$ resp. $\operatorname{aff}(P)$
(6) $H_{\mathcal{P}}^{j,<}$, $H_{\mathcal{P}}^{j,>}$ $\leftarrow$ the associated (embedded) open halfspaces within $\operatorname{aff}(P_{j+1})$ resp. $\operatorname{aff}(P)$
(7) $M \leftarrow M \cup \bigcup_{i=1}^{\dim \operatorname{aff}(P)} \{(H_{\mathcal{P}}^{i} \cup \bigcup_{j=i}^{\dim \operatorname{aff}(P)} H_j) \cap P | H_j \in \{H_{\mathcal{P}}^{j,<}, H_{\mathcal{P}}^{j,>}\}\}$,
(8) **return** M

Notice, that the set $\mathcal{H}(P)$ is given by

$$\mathcal{H}(P) = \bigcup_{\substack{P' \subset P \text{ affinely independent,} \\ \dim \operatorname{aff}(P') = \dim \operatorname{aff}(P)}} \mathcal{H}(P, P'), \qquad (2.49)$$

where the set $\mathcal{H}(P, P')$ denotes the output of the algorithm above.
Moreover, by the reformulation the algorithmic approach for separation by hyperplanes is generalized to the case that we do not consider point sets of full dimension for later purpose in Section 2.4.

Remark 2.3.6 The algorithm above can also be adapted to the case of separation by hyperplanes passing through some point $q \in \mathbb{R}^n$ by

1. $M \leftarrow \emptyset$ in (1), as the empty set does not have to belong to the set of separable subsets and is calculated in (7), if it is separable,

2. $q \in \bar{P} \subset P' \cup \{q\}$ in (2), because the separating hyperplanes have to pass through the point q, and $|\bar{P}| = \dim \operatorname{aff}(P \cup \{q\})$, because the point q does possibly not lie within the affine space $\operatorname{aff}(P)$,

3. $\mathcal{P}_1 = \{q\}$ in (3), because the separating hyperplanes have to pass through the point q.

It yields also in that case that

$$\mathcal{H}_q(P) = \bigcup_{\substack{P' \subset P \text{ affinely independent}, \\ \dim \operatorname{aff}(P') = \dim \operatorname{aff}(P)}} \mathcal{H}_q(P, P'), \tag{2.50}$$

where the set $\mathcal{H}_q(P, P')$ denotes the output of the modified algorithm.
Now we are prepared to formulate the new complexity assertion.

Corollary 2.3.7 *The set* $\mathrm{Sep}_W(P)$ *of all subsets within a nonempty finite point set* $P \subset \mathbb{R}^n$ *which can be separated with respect to the point set* P *by a fixed polytopal window* W *of dimension* n *with* m *facets is determined in* $O(m^{n+1} \cdot |P|^{n+1})$ *arithmetic operations.*
If $\dim(W) < n$ *we need at most* $O(m^{\dim(W)+1} \cdot |P|^{\dim(W)+1})$ *arithmetic operations.*

Proof
Let us reformulate the algorithm presented for the separation by polytopes before by

1. *canceling* (2) *and inserting*

   $(5')$ *calculate* $\mathcal{H}_0(\{a_1, \ldots, a_m\}, \{a_{k_1}, \ldots, a_{k_n}\})$,

   which takes $O(n! \cdot m)$ *arithmetic operations instead of* $O(m^{n+1})$ *and calculates at most* $O(n!)$ *different subsets of the set* $\{a_1, \ldots, a_m\}$,

2. $N \in \mathcal{H}_0(\{a_1, \ldots, a_m\}, \{a_{k_1}, \ldots, a_{k_n}\}) \cup \{\{a_1, \ldots, a_m\}\}$ *in (7) instead of* $N \in \mathcal{H}_0(\{a_1, \ldots, a_m\}) \cup \{\{a_1, \ldots, a_m\}\}$*, which reduces the number* $O(m^n)$ *of loops to the amount* $O(n!)$.

Thus, the complexity of the modified algorithm is given by

$$O(m|P| + (m|P|)^n \cdot [n! \cdot m + n! \cdot m|P|]) = O((m|P|)^n \cdot m|P|) \tag{2.51}$$

for fixed space dimension n*. Because of Remark 2.3.6 the modifications preserve the correctness of the algorithm.*
For lower-dimensional windows we again divide the point set P *into its equivalence classes, see Remark 2.3.5, and apply the modified algorithm to each of those equivalence classes.* □

Remark 2.3.8 Separation by open polytopal windows $W = \{Ax < b\}$ is similarly treated by making some small modifications. We only have to replace (8) within the algorithm by

$$M \leftarrow M \cup \{p \in P | a_k^T S_{\mathcal{U}} > b_k^p \text{ or } (a_k^T S_{\mathcal{U}} = b_k^p \text{ and } a_k \notin N) \\ \text{for } k = 1, \ldots, m\}, \tag{2.52}$$

as we calculate that $a_k^T t > b_k^p$ if and only if $a_k^T \triangle t < 0$ in the case that $a_k^T(t+\triangle t) = b_k^p$. Thus, all open halfspaces $\{x \in \mathbb{R}^n | a_k^T x > b_k^p,\, a_k^T(t+\triangle t) = b_k^p\}$, which are reached by the translation $-\triangle t$ of the translation vector $t + \triangle t$ back to the translation vector t, are determined using the complementary sets of the point sets $N \in \mathcal{H}_0(\{a_1, \ldots, a_m\})$.

Remark 2.3.9 The algorithm for the polytopal case can be adapted to treat windows W which are finite unions of polytopes, for example star-shaped windows like the 6-star given in Figure 2.1.
We again use the suspension sets of every point $p \in P$ with respect to the window W and their supporting hyperplanes, the dotted lines in Figure 2.1 for the 6-star example. The intersection points as well as all classes of possible translation vectors have to be determined again. To calculate the set of all separable subsets within the point set P we now have to decide for each point $p \in P$ whether the translation vector t lies in one of the parts of its suspension set (i. e. in the hexagon or in one of the six triangles in the 6-star case) or in none of them.

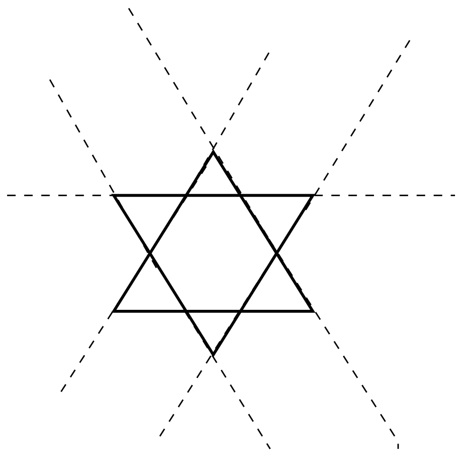

Figure 2.1: 6-star window as union of polytopes

2.4 Separation by balls of fixed radius

Let $B_R(M)$ denote the ball with centre M and radius R. In the following considerations we will investigate the separation problem for the spherical window $W = B_R(0)$ of fixed radius R.

We know from [106] that both detecting spherical separability of two finite point sets $S, P\backslash S \subset \mathbb{R}^n$ and finding the smallest separating sphere can be done in $O(3^{n^2}|P|)$ arithmetic operations by solving the linear program

$$\max_{a_1,\ldots,a_n,c,d} d \text{ s. t.} \tag{2.53}$$

$$\sum_{i=1}^n a_i p_i + \sum_{i=1}^n p_i^2 = \sum_{i=1}^n (p_i + \frac{1}{2}a_i)^2 - \frac{1}{4}\sum_{i=1}^n a_i^2 \leq c \text{ for } p = (p_1, \ldots, p_n) \in S, \tag{2.54}$$

$$\sum_{i=1}^n a_i p_i + \sum_{i=1}^n p_i^2 \geq c + d \text{ for } p = (p_1, \ldots, p_n) \in P\backslash S, \tag{2.55}$$

within the Euclidean space $\mathbb{R}^{n+1}$ after the embedding

$$\mathbb{R}^n \to \mathbb{R}^{n+1}, \tag{2.56}$$

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \sum_{i=1}^n x_i^2) \tag{2.57}$$

resp. the convex quadratic program with linear constraints

$$\min_{a_1,\ldots,a_n,c} \frac{1}{4}\sum_{i=1}^n a_i^2 + c \text{ s. t.} \tag{2.58}$$

$$\sum_{i=1}^n a_i p_i + \sum_{i=1}^n p_i^2 = \sum_{i=1}^n (p_i + \frac{1}{2}a_i)^2 - \frac{1}{4}\sum_{i=1}^n a_i^2 \leq c \text{ for } p = (p_1, \ldots, p_n) \in S, \tag{2.59}$$

$$\sum_{i=1}^n a_i p_i + \sum_{i=1}^n p_i^2 \geq c \text{ for } p = (p_1, \ldots, p_n) \in P\backslash S, \tag{2.60}$$

using the techniques presented in [97].

The largest separating sphere is found in $O(|P|^{\lfloor \frac{n}{2} \rfloor + 1})$ arithmetic operations by combinatorial methods. As the adequate optimization problem to (2.58)-(2.60) to find the largest separating sphere results in a concave quadratic program, Megiddo's algorithm in [97] cannot be applied. The ideas used to find the largest separating sphere can suitably be modified in order to also determine smallest circles in $O(|P|^{\lfloor \frac{n}{2} \rfloor + 1})$ arithmetic operations. Therefore, we can decide in $O(|P|^{\lfloor \frac{n}{2} \rfloor + 1})$ arithmetic operations whether the point sets S, $P\backslash S \subset \mathbb{R}^n$ can be separated by a ball of fixed radius R, which is also best possible in the space dimension n for the methods used in [106].

However, the separability question we will answer in the following is again concerned with the determination of all subsets within a finite point set P which can be separated with respect to the point set P by a ball of fixed radius R.

We want to start with a rough description of how the later presented algorithm works. Let us assume that $S \in \text{Sep}_{B_R}(P)$ and that $t \in \mathbb{R}^n$ is some translation vector so that the translated window $t + B_R(0)$ separates the point set S with respect to the point set P. If the ball $t + B_R(0)$ is not uniquely determined by its boundary points $\text{bd}(t + B_R(0)) \cap P$ within the point set P, we can try to rotate the ball $t + B_R(0)$ while leaving the boundary points $\text{bd}(t + B_R(0)) \cap P$ on the boundary of the rotated ball, until a new point hits the boundary. By iteration we end up with a ball that contains the point set S, and the dimension of the affine space spanned by the boundary points within the point set P is maximal.

We will see later that any rotational process as described before can be represented by a small translation vector. To reverse the rotational process, the non-preserved boundary points are calculated by the point sets which are separated with respect to all boundary points within the point set P by hyperplanes passing through the centre of the ball.

The following definitions will help us within the later algorithm to decide whether a given affinely independent point set $P' \subset P$ can be further extended to a larger affinely independent subset P'' of the point set P so that $P'' \subset \text{bd}(B_R(M))$ for some suitably chosen centre M.

Definition 2.4.1 (minimal radius $R_{min}^{P'}$)
*Let $P' \subset \mathbb{R}^n$ be a finite point set within the Euclidean space $\mathbb{R}^n$. The **minimal radius** $R_{min}^{P'}$ with respect to the point set P' is defined by*

$$R_{min}^{P'} := \begin{cases} \infty, & \text{if the point set } P' \text{ is affinely dependent} \\ \min\{R \mid \exists M : P' \subset \text{bd}(B_R(M))\}, & \text{otherwise} \end{cases} . \tag{2.61}$$

Remark 2.4.2 If $P' = \{p\}$, the minimal radius with respect to the point set P' is given by $R_{min}^{\{p\}} = 0$.

Definition 2.4.3 (R-maximal affinely independent set)
*Let $P' \subset P$ be a subset of the point set P. The point set P' is called an **R-maximal affinely independent subset** of cardinality $|P'|$, if*

$$R_{min}^{P'} \leq R, \tag{2.62}$$

$$R_{min}^{P' \cup \{p\}} > R \text{ for every } p \in P \setminus P'. \tag{2.63}$$

In order to calculate the inner points $\text{int}(B_R(M)) \cap P$ and the boundary points $\text{bd}(B_R(M)) \cap P$ of the ball $B_R(M)$ which lie within the point set P, we will use the following assertions within the later presented algorithm.

Lemma 2.4.4 *In the following let P' be an R-maximal affinely independent subset of the point set P.*
If $R_{min}^{P'} = R$ and $2 \leq |P'| \leq n+1$, the centre of the ball is uniquely determined by the intersection of hyperplanes within the affine subspace $\operatorname{aff}(P')$. The centre of the ball is calculated in constant time for fixed space dimension n.
If $R_{min}^{P'} < R$ and $|P'| < n$, it yields that

1. $\operatorname{bd}(B_R(M)) \cap P \subset \operatorname{aff}(P')$ *for every M satisfying $P' \subset \operatorname{bd}(B_R(M))$,*
2. $\operatorname{int}(B_R(M)) \cap P = \operatorname{int}(B_{R_{min}^{P'}}(\bar{M})) \cap P$ *for every M satisfying $P' \subset \operatorname{bd}(B_R(M))$ and $\bar{M}$ uniquely specified by $P' \subset \operatorname{bd}(B_{R_{min}^{P'}}(\bar{M}))$.*

If $R_{min}^{P'} < R$ and $|P'| = n$, we have to calculate two balls centred in M_1, M_2. Let us define

$$\bar{p} := \arg \min_{p \in P \setminus P'} |R_{min}^{P' \cup \{p\}} - R|, \tag{2.64}$$

$$\bar{R} := R_{min}^{P' \cup \{\bar{p}\}}, \tag{2.65}$$

$$\bar{M}_1 \text{ the centre uniquely determined by } P' \cup \{\bar{p}\} \subset \operatorname{bd}(B_{\bar{R}}(\bar{M}_1)) \tag{2.66}$$

and let $\bar{M}_2$ be the reflection of the point $\bar{M}_1$ on the affine subspace $\operatorname{aff}(P')$. Let us assume that the points M_k, $\bar{M}_k$ lie on the same side of the affine subspace $\operatorname{aff}(P')$ for $k = 1, 2$.
We get that

1. $\operatorname{bd}(B_R(M_1)) \cap P = \operatorname{bd}(B_R(M_2)) \cap P = \operatorname{bd}(B_{R_{min}^{P'}}(\bar{M}))$ *for $\bar{M}$ the orthogonal projection of the point $\bar{M}_1$ resp. of the point $\bar{M}_2$ on the affine subspace $\operatorname{aff}(P')$,*
2. $$\operatorname{int}(B_R(M_k)) = \begin{cases} \operatorname{int}(B_{\bar{R}}(\bar{M}_k)) \cup (\operatorname{bd}(B_{\bar{R}}(\bar{M}_k)) \cap H^{<}_{\bar{M}-\bar{M}_k, p \in P'}) & \text{if } \bar{R} < R \\ \operatorname{int}(B_{\bar{R}}(\bar{M}_k)) \cup (\operatorname{bd}(B_{\bar{R}}(\bar{M}_k)) \cap H^{>}_{\bar{M}-\bar{M}_k, p \in P'}) & \text{if } \bar{R} > R \end{cases}$$
   $$= \operatorname{int}(B_{\bar{R}}(\bar{M}_k)) \cup (\operatorname{bd}(B_{\bar{R}}(\bar{M}_k)) \cap H^{<}_{sgn(R-\bar{R})(\bar{M}-\bar{M}_k), p \in P'}).$$

Proof
The assertion for $R_{min}^{P'} = R$ and $2 \leq |P'| \leq n+1$ is clear.
Let us look at the case that $R_{min}^{P'} < R$ and that $|P'| < n$.
Assuming that assertion 1. is not fulfilled, there is at least one point $p \in P \setminus \operatorname{aff}(P')$ which satisfies $R_{min}^{P' \cup \{p\}} \leq R$ in contradiction to the R-maximal affine independency of the point set P'.
For assertion 2. let us assume that $\operatorname{int}(B_R(M_1)) \cap P \neq \operatorname{int}(B_R(M_2)) \cap P$ for M_1, M_2 satisfying $P' \subset \operatorname{bd}(B_R(M_1))$ and $P' \subset \operatorname{bd}(B_R(M_2))$. For the parameterization

$$\begin{aligned} &\gamma : [0,1] \to \{x \in \mathbb{R}^n \,|\, \|p - x\| = R \text{ for all } p \in P'\} =: \operatorname{bd}(K), \\ &\gamma(0) = M_1, \\ &\gamma(1) = M_2 \end{aligned} \tag{2.67}$$

and for

$$\tau := \min\{\lambda \in [0,1] | \operatorname{int}(B_R(\gamma(\lambda))) \cap P \neq \operatorname{int}(B_R(M_1)) \cap P\} \tag{2.68}$$

we get that there is an R-maximal affinely independent subset P'' *within the point set* $\operatorname{bd}(B_R(\gamma(\tau))) \cap P$ *which satisfies* $P' \subsetneq P'' \subset P$*, in contradiction to the assumption that the point set* P' *is R-maximal.*
Therefore, if we can show that

$$\bigcap_{M \in \operatorname{bd}(K)} \operatorname{int}(B_R(M)) \cap P \subset \operatorname{int}(B_{R_{min}^{P'}}(\bar{M})) \cap P \subset \bigcup_{M \in \operatorname{bd}(K)} \operatorname{int}(B_R(M)) \cap P, \tag{2.69}$$

we result in assertion 2. by using that $\operatorname{int}(B_R(M_1)) \cap P = \operatorname{int}(B_R(M_2)) \cap P$ *for every two points* M_1, M_2 *satisfying* $P' \subset \operatorname{bd}(B_R(M_1))$ *and* $P' \subset \operatorname{bd}(B_R(M_2))$.
For the first inclusion let $p \in \bigcap_{M \in \operatorname{bd}(K)} \operatorname{int}(B_R(M)) \cap P$ *and let us choose some* $M \in \operatorname{bd}(K)$ *satisfying* $(M - \bar{M})^T(p - \bar{M}) \leq 0$*. Using the cosinus theorem, we calculate that*

$$||p - \bar{M}||^2 < R^2 - ||M - \bar{M}||^2 = (R_{min}^{P'})^2 \tag{2.70}$$

and therefore $p \in \operatorname{int}(B_{R_{min}^{P'}}(\bar{M}))$.
For the second inclusion let $p \in \operatorname{int}(B_{R_{min}^{P'}}(\bar{M})) \cap P$ *and let us choose some* $M \in \operatorname{bd}(K)$ *satisfying* $(M - \bar{M})^T(p - \bar{M}) \geq 0$*. It yields that*

$$||p - M||^2 < (R_{min}^{P'})^2 + ||M - \bar{M}||^2 = R^2, \tag{2.71}$$

using the cosinus theorem again, and therefore $p \in \bigcup_{M \in \operatorname{bd}(K)} \operatorname{int}(B_R(M)) \cap P$.
Now let us look at the remaining case that $R_{min}^{P'} < R$ *and that* $|P'| = n$*. Because of the R-maximal affine independency of the point set* P'*, assertion 1. is clear. Thus, assertion 2. is left to show. We will restrict the following considerations to the case* $k = 1$*, i. e. we will only look at the situation for the points* M_1 *and* $\bar{M}_1$.
We know that

$$||M_1 - p|| = R \text{ for } p \in \operatorname{aff}(P') \cap \operatorname{bd}(B_{\bar{R}}(\bar{M}_1)) \tag{2.72}$$

by the definition of the point $\bar{M}_1$*. Let the point* $p^<$ *be defined by*

$$p^< := \operatorname{bd}(B_{\bar{R}}(\bar{M}_1)) \cap (M_1 + \mathbb{R}(\bar{M} - M_1)) \cap H^{\leq}_{\bar{M}-\bar{M}_1, p \in P'}. \tag{2.73}$$

By using the cosinus theorem, we deduce that

$$||M_1 - p^<|| \left\{ \begin{array}{c} < \\ > \end{array} \right\} ||\bar{M}_1 - p^<|| = \bar{R} \left\{ \begin{array}{c} < \\ > \end{array} \right\} R \tag{2.74}$$

in the case that $\bar{R} < R$ *resp. in the case that* $\bar{R} > R$ *and that the distance* $||M_1 - p||$ *defined on the boundary* $\mathrm{bd}(\bar{B}_R(\bar{M}_1))$ *of the ball* $B_{\bar{R}}(\bar{M}_1)$ *is strictly monoton within the angle between* $p - M_1$ *and* $\bar{M}_1 - M_1$*. Thus, we result in assertion 2.* □

In the following algorithm we calculate the R*-maximal affinely independent subsets* $P' \subset P$ *within the point set* P *as well as their associated balls. Notice, that in the case that* $\dim \mathrm{aff}(P') = n-1$*, in particular* $|P'| = n$*, we have to consider two balls in general. Furthermore, in the case that* $\dim \mathrm{aff}(P') = n$*, in particular* $|P'| = n+1$*, we also have to calculate the ball itself and its reflection on the hyperplane* $\mathrm{aff}(\bar{P})$ *for any* n*-subset* $\bar{P} \subset P'$*, as every* n*-point set* $\bar{P}$ *is not* R*-maximal, but the point set* P' *cannot be reached as boundary points by any rotation of the reflected ball.*

In order to determine all separable subsets of the point set P *which result from small translations of the centre of the ball afterwards, we have to calculate all sets of boundary points which can be separated by a hyperplane passing through the centre of the ball, for details see later. Because of (2.49) resp. of (2.50) that calculation can be started with the* R*-maximal affinely independent point set* P' *instead of the complete set of boundary points within the point set* P*.*

SeparationByBalls$_n(P)$
Input: finite point set $\emptyset \neq P \subseteq \mathbb{R}^n$, radius R
Output: $\mathrm{Sep}_{B_R}(P)$

$T \leftarrow \{\emptyset\}$
foreach R-maximal affinely independent subset $P' \subset P$
  if $R_{min}^{P'} = R$
    $M \leftarrow$ the uniquely determined centre of the ball
    foreach $N \in \mathcal{H}_M(\mathrm{bd}(B_R(M)) \cap P, P') \cup \{\emptyset\}$
      $T \leftarrow T \cup \{(B_R(M) \cap P) \backslash N\}$
    if $|P'| = n+1$
      foreach $\bar{P} \subset P'$ with $|\bar{P}| = n$
        $\bar{M} \leftarrow$ the reflection of M on $\mathrm{aff}(\bar{P})$
        if $\dim \mathrm{aff}(\mathrm{bd}(B_R(\bar{M})) \cap P) < n$
          foreach $N \in \mathcal{H}(\mathrm{bd}(B_R(\bar{M})) \cap P, \bar{P}) \cup \{\emptyset\}$
            $T \leftarrow T \cup \{(B_R(\bar{M}) \cap P) \backslash N\}$
  if $R_{min}^{P'} < R$ and $|P'| < n$
    $\bar{M} \leftarrow$ the uniquely determined projection of any centre on $\mathrm{aff}(P')$
    foreach $N \in \mathcal{H}(\mathrm{bd}(B_{R_{min}^{P'}}(\bar{M})) \cap P, P') \cup \{\emptyset\}$
      $T \leftarrow T \cup \{(B_{R_{\mathrm{min}}^{P'}}(\bar{M}) \cap P) \backslash N\}$
  if $R_{min}^{P'} < R$ and $|P'| = n$
    $\bar{p} \leftarrow \arg\min_{p \in P \backslash P'} |R_{min}^{P' \cup \{p\}} - R|$
    $\bar{R} \leftarrow R_{min}^{P' \cup \{\bar{p}\}}$
    $\bar{M}_1 \leftarrow$ the centre uniquely determined by $P' \cup \{\bar{p}\} \subset \mathrm{bd}(B_{\bar{R}}(\bar{M}_1))$
    $\bar{M}_2 \leftarrow$ the reflection of $\bar{M}_1$ on $\mathrm{aff}(P')$
    $\bar{M} \leftarrow$ the uniquely determined projection of $\bar{M}_1$ resp. of $\bar{M}_2$ on $\mathrm{aff}(P')$
    foreach $k \in \{1, 2\}$
      $\mathrm{int}(B_R(M_k)) \leftarrow \mathrm{int}(B_{\bar{R}}(\bar{M}_k)) \cup (\mathrm{bd}(B_{\bar{R}}(\bar{M}_k)) \cap H^{<}_{sgn(R-\bar{R})(\bar{M}-\bar{M}_k), p \in P'})$
    foreach $N \in \mathcal{H}(\mathrm{bd}(B_{R_{min}^{P'}}(\bar{M})) \cap P, P') \cup \{\emptyset\}$
      $T \leftarrow T \cup \{((\mathrm{int}(B_R(M_1)) \cup \mathrm{bd}(B_{R_{min}^{P'}}(\bar{M}))) \cap P) \backslash N\}$
      $T \leftarrow T \cup \{((\mathrm{int}(B_R(M_2)) \cup \mathrm{bd}(B_{R_{min}^{P'}}(\bar{M}))) \cap P) \backslash N\}$
return T

In the spherical case the suspension set A_S of some subset $S \subset P$ of the finite point set P is given by

$$A_S = \bigcap_{p \in S} B_R(p) \cap \bigcap_{p \in P \backslash S} \mathbb{R}^n \backslash B_R(p). \tag{2.75}$$

The following lemma characterizes the situation that the closure of the suspension set A_S has less connected components than its interior.

Lemma 2.4.5 *Let $P \subset \mathbb{R}^n$ be a finite point set within the Euclidean space $\mathbb{R}^n$ and let $S \subset P$ be a subset of the point set P. Let the open set G be defined by*

$$G := \bigcap_{p \in S} \mathrm{int}(B_R(p)) \cap \bigcap_{p \in P \backslash S} \mathbb{R}^n \backslash B_R(p) \tag{2.76}$$

and let $G_1, \ldots, G_k$ be its connected components. If

$$x \in \mathrm{bd}(G_i) \cap \mathrm{bd}(G_j) \text{ for } 1 \leq i \neq j \leq k, \tag{2.77}$$

i. e. the closure $\mathrm{clos}(G) = G \cup \mathrm{bd}(G)$ of the open set G has less than k connected components, then there exists a maximal affinely independent subset $P' \subset \bar{P}$ within the point set

$$\bar{P} := \{p \in P | x \in \mathrm{bd}(B_R(p))\} \tag{2.78}$$

satisfying $R_{min}^{P'} = R$.

Proof

Locally considered within a small area $B_\epsilon(x)$ around the point x, we result for $\epsilon \to 0$ in the notation of tangential spaces and can therefore describe the sets $\mathrm{int}(B_R(p))$, $(\mathbb{R}^n \backslash B_R(p)) \cup \{x\}$ for $p \in \bar{P}$ and the set G by

$$\mathrm{int}(B_R(p)) \approx H^{>}_{p-x,x} \text{ for } p \in \bar{P} \cap S, \tag{2.79}$$

$$(\mathbb{R}^n \backslash B_R(p)) \cup \{x\} \approx H^{\leq}_{p-x,x} \text{ for } p \in \bar{P} \cap (P \backslash S), \tag{2.80}$$

$$G \approx (\bigcap_{p \in \bar{P} \cap S} H^{>}_{p-x,x} \cap \bigcap_{p \in \bar{P} \cap (P \backslash S)} H^{\leq}_{p-x,x}) \backslash \{x\}. \tag{2.81}$$

The $\approx$-notation shall denote the focus on the local consideration within the area $B_\epsilon(x)$ and the process $\epsilon \to 0$.
If the convex polyhedron $I := \bigcap_{p \in \bar{P} \cap S} H^{>}_{p-x,x} \cap \bigcap_{p \in \bar{P} \cap (P \backslash S)} H^{\leq}_{p-x,x}$ has two connected components after the deletion of the point x, the condition $\dim I = 1$ has to be fulfilled, in particular it yields that

$$I = \bigcap_{p \in \bar{P} \cap (P \backslash S)} H^{\leq}_{p-x,x} \tag{2.82}$$

is a line within the Euclidean space $\mathbb{R}^n$. Therefore, we necessarily have that $I \subset H_{p-x,x}$ for every $p \in \bar{P}$, in particular all normal vectors $p-x$ are orthogonal

to the line I.
Let us define the radius $\bar{R}$ by

$$\bar{R} := \min\{R | \exists M \in \mathbb{R}^n : \bar{P} \subset \text{bd}(B_R(M))\} \tag{2.83}$$

and let $\bar{M} \in \text{aff}(\bar{P})$ denote the centre of the ball $B_{\bar{R}}(\bar{M})$ which satisfies $\bar{P} \subset \text{bd}(B_{\bar{R}}(\bar{M}))$.
Let us assume that $\bar{R} < R$ and that there exists some point $p \in \bar{P} \cap H^{\leq}_{\bar{M}-x,x}$.
By the cosinus theorem we calculate that

$$\bar{R}^2 = ||p - \bar{M}||^2 \geq ||p - x||^2 + ||x - \bar{M}||^2 = R^2 + ||x - \bar{M}||^2, \tag{2.84}$$

in contradiction to $\bar{R} < R$. Therefore, we get that $\bar{P} \subset H^{\geq}_{\bar{M}-x,x}$ in the case that $\bar{R} < R$.
For $\bar{P} \subset H^{\geq}_{\bar{M}-x,x}$ we conclude that the point $x + (x - \bar{M}) \notin I$ lies within the closed halfspace $H^{\leq}_{p-x,x}$ for every point $p \in \bar{P}$ in contradiction to (2.82).
Thus, it yields that $\bar{R} = R$ and that $R^{P'}_{min} = R$ for some maximal affinely independent subset $P' \subset \bar{P}$ within the point set $\bar{P}$. □

Remark 2.4.6 According to Lemma 2.4.5 any point x which satisfies (2.77) is determined by an R-maximal affinely independent subset $P' \subset P$ within the point set $\bar{P}$ as defined in (2.78).

Theorem 2.4.7
The algorithm presented before determines the set $\text{Sep}_{B_R}(P)$ of all subsets within a nonempty finite point set $P \subset \mathbb{R}^n$ which can be separated with respect to the point set P by a ball of radius R in $O(|P|^{n+2})$ arithmetic operations.
Proof
Let $S \in \text{Sep}_{B_R}(P)$ and let $B := B_R(M)$ be a ball which separates the point set S with respect to the point set P. We will show that the point set S is calculated by the algorithm.
If $\text{bd}(B) \cap P \neq \emptyset$ and $\min\{\tilde{R} | \exists \tilde{M} : \text{bd}(B) \cap P \subset \text{bd}(B_{\tilde{R}}(\tilde{M}))\} = R$, then there exists an R-maximal affinely independent point set $P' \subset (\text{bd}(B) \cap P)$ within the set of boundary points $\text{bd}(B) \cap P$, and the point set S is calculated in (3)-(6) for $N = \emptyset$.
If $\text{bd}(B) \cap P \neq \emptyset$ and $\min\{\tilde{R} | \exists \tilde{M} : \text{bd}(B) \cap P \subset \text{bd}(B_{\tilde{R}}(\tilde{M}))\} < R$, then it yields that $\dim \text{aff}(\text{bd}(B) \cap P) < n$. Therefore, there is at least one further centre point M' which satisfies $(\text{bd}(B) \cap P) \subset \text{bd}(B_R(M'))$. Let the point $\bar{M} := M + \epsilon(M' - M)$ be defined for a sufficiently small $\epsilon > 0$ so that

$$P \cap int(B) \subset int(B_R(\bar{M})), \tag{2.85}$$
$$P \cap (\mathbb{R}^n \backslash B) \subset \mathbb{R}^n \backslash B_R(\bar{M}), \tag{2.86}$$

i. e. interior and exterior points remain interior and exterior points. It yields that $p \in int(B_R(\bar{M}))$ *for every point* $p \in P \cap \mathrm{bd}(B)$*, as*

$$M, M' \in \bigcap_{p \in \mathrm{bd}(B) \cap P} \mathrm{bd}(A_p) = \bigcap_{p \in \mathrm{bd}(B) \cap P} \mathrm{bd}(B_R(p)), \tag{2.87}$$

but the line segment given by the boundary points $M, M' \in \mathrm{bd}(A_p) = \mathrm{bd}(B_R(p))$ *except the points* M, M' *themselves lies within the interior of the suspension set* $A_p = B_R(p)$ *for every* $p \in \mathrm{bd}(B) \cap P$*. Therefore, the case that* $\mathrm{bd}(B) \cap P \neq \emptyset$ *and* $\min\{\tilde{R} | \exists \tilde{M} : \mathrm{bd}(B) \cap P \subset \mathrm{bd}(B_{\tilde{R}}(\tilde{M}))\} < R$ *can be reduced to the remaining case that* $\mathrm{bd}(B) \cap P = \emptyset$*.*
The idea behind the following procedure is to successively move the centre M *of the ball* B *to a new centre* M' *which satisfies*

1. $||p - M|| \left\{ \begin{array}{c} < \\ > \\ = \end{array} \right\} R \Longrightarrow ||p - M'|| \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} R$ *for every* $p \in P$*, i. e.* $S \subset B_R(M') \cap P$ *and* $(\mathbb{R}^n \backslash B_R(M')) \cap P \subset P \backslash S$*,*
2. $\dim \mathrm{aff}(\mathrm{bd}(B_R(M')) \cap P) > \dim \mathrm{aff}(\mathrm{bd}(B_R(M)) \cap P)$

after each motion.
By considering the point $M' := M + sgn(\mathrm{dist}(p_\mu, M) - R)\mu \frac{p_\mu - M}{||p_\mu - M||}$ *for*

$$\mu := \min_{p \in P} \mathrm{dist}(p, \mathrm{bd}(B)), \tag{2.88}$$

$$p_\mu := \arg\min_{p \in P} \mathrm{dist}(p, \mathrm{bd}(B)) \tag{2.89}$$

instead of the point M*, we can guarantee that at least one point* $p \in P$ *lies on the sphere of the ball* $B_R(M')$*. Condition 1. is satisfied because of the choice of the value* μ *and the point* p_μ *in (2.88)-(2.89).*
Thus, let us assume that $\min\{\tilde{R} | \exists \tilde{M} : \mathrm{bd}(B_R(M)) \cap P \subset \mathrm{bd}(B_{\tilde{R}}(\tilde{M}))\} < R$ *and that* $0 \leq k := \dim \mathrm{aff}(\mathrm{bd}(B_R(M)) \cap P) < n - 1$*.*
If any affinely independent point set $P' \subset \mathrm{bd}(B_R(M)) \cap P$ *is not R-maximal, then there exists another centre point*

$$\bar{M} \in \{x \in \mathbb{R}^n | ||p - x|| = R \text{ for all } p \in \mathrm{bd}(B_R(M)) \cap P\} =: \mathrm{bd}(K) \tag{2.90}$$

within the set $\mathrm{bd}(K)$ *of possible centre points (the sphere of an at least 2-dimensional ball) which satisfies*

$$\dim \mathrm{aff}(\mathrm{bd}(B_R(\bar{M})) \cap P) > \dim \mathrm{aff}(\mathrm{bd}(B_R(M)) \cap P). \tag{2.91}$$

For the parameterization

$$\gamma : [0, 1] \to \mathrm{bd}(K), \tag{2.92}$$

$$\gamma(0) = M, \tag{2.93}$$

$$\gamma(1) = \bar{M} \tag{2.94}$$

and for

$$\tau := \min\{\lambda \in [0,1] | \dim \operatorname{aff}(\operatorname{bd}(B_R(\gamma(\lambda))) \cap P) > \dim \operatorname{aff}(\operatorname{bd}(B_R(M)) \cap P)\} \quad (2.95)$$

the new centre M' may be chosen by $M' := \gamma(\tau)$.
After iteratively applying the procedure as often as possible, let M' be the new centre. We end up with some R-maximal affinely independent subset $P' \subset \operatorname{bd}(B_R(M'))$ or in the case that $\min\{\tilde{R} | \exists \tilde{M} : \operatorname{bd}(B_R(M')) \cap P \subset \operatorname{bd}(B_{\tilde{R}}(\tilde{M}))\} < R$ and $\dim \operatorname{aff}(\operatorname{bd}(B_R(M')) \cap P) = n-1$. In particular, in the second case the two possible centre points are uniquely specified by the boundary points $\operatorname{bd}(B_R(M')) \cap P$ and the radius R, and for none of the two centre points a further motion step can be applied. The reflection of the ball centred in the point M' on the affine subspace $\operatorname{aff}(\operatorname{bd}(B_R(M')) \cap P)$ is characterized by an R-maximal affinely independent subset of boundary points, if the ball itself is not, and we result in (8)-(10).
According to Lemma 2.4.5 and Remark 2.4.6 the centre M of any separating ball $B_R(M)$ can directly be moved to the new centre $M' \neq M$ by some vector $\epsilon \cdot \triangle t$ for $\epsilon > 0$ sufficiently small. In the case that $(M'-p)^T(M'-p) = R^2$ we calculate that

$$[(M' - \epsilon \cdot \triangle t) - p]^T [(M' - \epsilon \cdot \triangle t) - p] \leq R^2 \quad (2.96)$$

$$\Leftrightarrow 2\epsilon \triangle t^T (M' - p) \geq \epsilon^2 \triangle t^T \triangle t \quad (2.97)$$

$$\Leftrightarrow \triangle t^T (M' - p) > 0. \quad (2.98)$$

Therefore, we have to determine all subsets within the set $\operatorname{bd}(B_R(M')) \cap P$ of boundary points which can be separated by a hyperplane passing through the point M', in order to decide which boundary points of the ball $B_R(M')$ do not lie within the ball $B_R(M' - \triangle\bar{t})$ for some small translation vector $\triangle\bar{t}$. Therefore, any point set $S \in \operatorname{Sep}_W(P)$ is calculated by the algorithm.
Otherwise, every point set which is calculated within the algorithm can also be separated with respect to the point set P. That is clear for the subsets of the point set P which are separated by any ball specified by an R-maximal affinely independent subset $P' \subset P$ of the point set P. Furthermore, all small motions of the centre of each ball are taken into account by the calculation of the set $\mathcal{H}_M(.)$ resp. of the set $\mathcal{H}(.)$ as just mentioned.
For the complexity assertion let us notice that the number of all subsets at most of cardinality $n+1$ within the point set P and therefore the number of all R-maximal affinely independent subsets $P' \subset P$ within the point set P is bounded by

$$\binom{|P|}{n+1} \in O(|P|^{n+1}). \quad (2.99)$$

The calculation of the sets $\mathcal{H}_M(.,.)$ and $\mathcal{H}(.,.)$ without testing the points $p \in P$ only depends on the space dimension n, as we start with $|P'| \leq n+1$, but the space dimension n is considered to be fixed. For every point $p \in P$ we have to determine to which sets the point p belongs. Therefore, the total complexity is given by

$$O(|P|^{n+2}) \tag{2.100}$$

for fixed space dimension n and thus polynomial within the cardinality $|P|$ of the point set P. □

Remark 2.4.8 Because of the fixed radius R it would actually be enough to treat all affinely independent subsets $P' \subset P$ of cardinality $|P'| \leq n$ within the point set P which satisfy $R_{min}^{P'} \leq R$. But in the case that $|P'| = n$ and that $R_{min}^{P'} < R$ the exact determination of the centres of the two balls (for inner, outer and boundary points) needs the calculation of square roots, which can efficiently be done only approximately by Heron's method.

Remark 2.4.9 By only small modifications we can also treat the case of open balls. In (6), (12), (16), (26) and (27) within the algorithm for closed balls we only have to take the complementary of the set N within the set of boundary points instead of the set itself.

Remark 2.4.10 The algorithmic idea can easily be adapted to the case of ellipsoids instead of balls by a linear transformation applied to the point set P.

2.5 Separation by balls of arbitrary radius

In the following we will treat the problem of separation by balls of arbitrary radius. Detecting spherical separability for two finite point sets $S, P \backslash S \subset \mathbb{R}^n$ is possible in $O(3^{n^2}|P|)$ resp. in $O(|P|)$ arithmetic operations for fixed space dimension n, see [106]. But we are again interested in the question how to determine all separable subsets S within a finite point set $P \subset \mathbb{R}^n$.

In the following we will introduce two methods to solve the separation problem using different embeddings in a higher dimensional Euclidean space and the separation algorithm for balls of fixed radius resp. the separation algorithm for hyperplanes.

2.5.1 Separation by balls of fixed radius after embedding

The idea behind the first algorithmic treatment is to embed the point set $P \subset \mathbb{R}^n$ in the Euclidean space $\mathbb{R}^{n+1}$ by

$$\begin{aligned} &\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}, \\ &x \mapsto \begin{pmatrix} x \\ 1 \end{pmatrix} \end{aligned} \tag{2.101}$$

and use the algorithm for balls of fixed radius for some suitable radius R. All subsets $S \subset P$ which can be separated with respect to the point set P by any ball of radius $r \leq R$ are calculated by that method. Thus, by choosing the radius R sufficiently large, the calculation of all subsets of the point set P which can be separated by some ball of arbitrary radius takes $O(|P|^{n+3})$ arithmetic operations.

The following definition will help us to determine the radius R suitably large within the lemma afterwards.

Definition 2.5.1 (minimal radius $\bar{R}^{P'}_{min}$)
Let $P' \subset \mathbb{R}^n$ be a set of points within the Euclidean space $\mathbb{R}^n$ of cardinality $|P'| \leq n+1$. The ***minimal radius*** *$\bar{R}^{P'}_{min}$ with respect to the point set P' is defined by*

$$\bar{R}^{P'}_{min} := \begin{cases} 0, \text{ if the point set } P' \text{ is affinely dependent} \\ \min\{R | \exists M : P' \subset \mathrm{bd}(B_R(M))\}, \text{ otherwise} \end{cases} . \tag{2.102}$$

Lemma 2.5.2 *Let the Euclidean space $\mathbb{R}^n$ be embedded in the Euclidean space $\mathbb{R}^{n+1}$ by*

$$\begin{aligned} &\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}, \\ &x \mapsto \begin{pmatrix} x \\ 1 \end{pmatrix} \end{aligned} \tag{2.103}$$

and let the radius R be chosen large enough by

$$R > \max_{P' \subset P, |P'| \leq n+1} \bar{R}^{P'}_{min}. \tag{2.104}$$

Then the problem of separation by balls of arbitrary radius with respect to the finite point set $P \subset \mathbb{R}^n$ is solved by the separation algorithm for balls of fixed radius R applied to the embedded point set $\{ \begin{pmatrix} p \\ 1 \end{pmatrix} | p \in P\}$.

Proof

Because of the used embedding we have to show that any subset $S \subset P$ within the point set P which can be separated with respect to the point set P by a ball of arbitrary radius is also separated by a ball of radius at most of value R.
Let the subset $S \subset P$ be separated with respect to the point set P by the ball $B := B_r(M)$. After some translation if necessary we can assume without loss of generality that the set of points within the point set P which lie on the sphere of the ball B is not empty. Let $A := \mathrm{aff}(P \cap \mathrm{bd}(B))$ be the affine subspace spanned by the point set $P \cap \mathrm{bd}(B)$. If $P \cap (\mathbb{R}^n \backslash A) = \emptyset$, then there is an affinely independent subset $P' \subset \mathrm{bd}(B) \cap P$ within the set of boundary points $\mathrm{bd}(B) \cap P$ so that $\bar{R}^{P'}_{min} < R$ by the choice of the value R in (2.104).

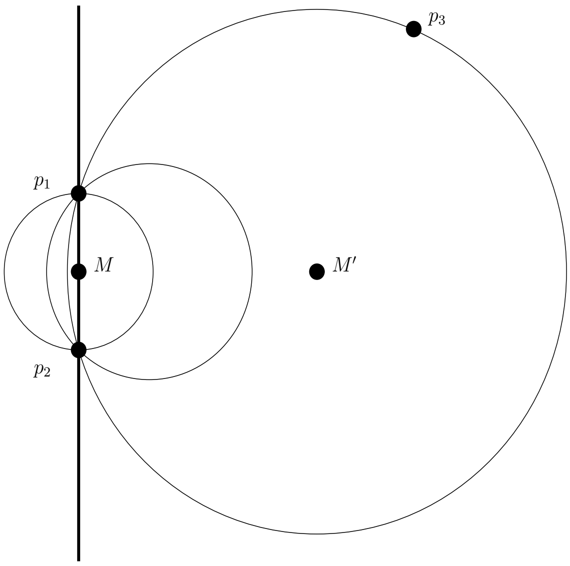

Figure 2.2: Variation of the radius in order to extend the set of boundary points within the point set P

Thus, let us consider the case that $P \cap (\mathbb{R}^n \backslash A) \neq \emptyset$. By variation of the radius r (see Figure 2.2) so that the boundary points within the point set P still remain boundary points, we can find a ball $B_{r'}(M')$ so that $\mathrm{bd}(B) \cap P \subsetneq \mathrm{bd}(B_{r'}(M')) \cap P$. In particular, the set of boundary points within the point set P is extended, but for every ball $B_{\bar{r}_\lambda}(\bar{M}_\lambda)$ for $\bar{M}_\lambda := M + \lambda(M' - M), \lambda \in [0, 1[$

and $\bar{r}_\lambda := ||p - \bar{M}_\lambda||$ *for some boundary point* $p \in \mathrm{bd}(B) \cap P$ *satisfying*

$$\mathrm{bd}(B) \cap P \subset \mathrm{bd}(B_{\bar{r}_\lambda}(\bar{M}_\lambda)) \tag{2.105}$$

we have that

$$\mathrm{bd}(B_{\bar{r}_\lambda}(\bar{M}_\lambda)) \cap P = \mathrm{bd}(B) \cap P. \tag{2.106}$$

The choice of the radius r' *and the point* M' *implies that*

$$int(B_{r'}(M')) \cap P \subset S, \tag{2.107}$$

$$(\mathbb{R}^n \backslash B_{r'}(M')) \cap S = \emptyset. \tag{2.108}$$

Thus, we return to the separation of the set S *by a tiny reverse variation of the radius* r'*.*
Using inductive arguments the radius R *as chosen in (2.104) is therefore large enough to calculate the point set* S *by some ball of radius at most of value* R*.*
□

Remark 2.5.3 The determination of the radius R within Lemma 2.5.2 takes $O(|P|^{n+1})$ arithmetic operations as for all (affinely independent) subsets $P' \subset P$ within the point set P of cardinality lower or equal to the value $n+1$ the minimal radius $\bar{R}^{P'}_{min}$ has to be calculated.

Theorem 2.5.4
Let $\emptyset \neq P \subset \mathbb{R}^n$ *be a nonempty finite point set within the Euclidean space* $\mathbb{R}^n$*. All subsets which can be separated with respect to the point set* P *by a ball of arbitrary radius are determined in* $O(|P|^{n+2})$ *arithmetic operations.*
Proof
Because of the embedding (2.103) and because of Lemma 2.5.2 all subsets which can be separated with respect to the point set P *by any ball are calculated by applying the algorithm for balls of fixed radius on the embedded point set.*
Notice, that the embedded point set does not contain any R*-maximal affinely independent subset of cardinality* $n+2$*, as*

$$\dim \mathrm{aff}(\{\begin{pmatrix} p \\ 1 \end{pmatrix} | p \in P\}) = \dim \mathrm{aff}(P) \leq n. \tag{2.109}$$

Therefore, taking also Remark 2.5.3 into account, the complexity of the complete algorithm is actually given by $O(|P|^{n+2})$ *(instead of* $O(|P|^{n+3})$*).* □

2.5.2 Separation by hyperplanes after embedding

The determination of all subsets within the point set P which can be separated with respect to the point set P by any ball is solved more elegant and also in $O(|P|^{n+2})$ arithmetic operations by using another kind of embedding.
Let the embedding be given by

$$\begin{aligned} &\mathbb{R}^n \to \mathbb{R}^{n+1}, \\ &(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \sum_{i=1}^{n} x_i^2), \end{aligned} \tag{2.110}$$

which is already used in [106]. The plane

$$-2\sum_{i=1}^{n} a_i x_i + x_{n+1} = R^2 - \sum_{i=1}^{n} a_i^2 \text{ for } x_{n+1} := \sum_{i=1}^{n} x_i^2 \tag{2.111}$$

within the transformed Euclidean space $\mathbb{R}^{n+1}$ can be rewritten by

$$\sum_{i=1}^{n} (x_i - a_i)^2 = R^2. \tag{2.112}$$

Thus, the separation problem for balls of arbitrary radius within the Euclidean space $\mathbb{R}^n$ is solved by the separation algorithm for hyperplanes within the Euclidean space $\mathbb{R}^{n+1}$ in $O(|P|^{n+2})$ arithmetic operations for fixed space dimension n.

2.6 Separation by the intersection of two balls

The general case of separation by m balls each of arbitrary radius can easily be reduced to the case of separation by hyperplanes using the embedding as denoted in (2.110). Of course, all $O(|P|^{n+1})$ possibilities for each of the m hyperplanes in the Euclidean space $\mathbb{R}^{n+1}$ have to be combined so that we result in $(O(|P|^{n+1}))^m = O(|P|^{m(n+1)})$ separation possibilities. Each point $p \in P$ has to be tested for each of those separation possiblities in order to decide to which part the point belongs. Altogether, we have to invest $O(|P|^{m(n+1)+1})$ arithmetic operations. Thus, for fixed number m of balls and fixed space dimension n the determination of all subsets within the point set P which can be separated with respect to the point set P are determined in polynomial time within the cardinality $|P|$ of the point set P.

In the following the determination of all subsets within the point set P which can be separated with respect to the point set P by the intersection $W = B_R(0) \cap B_R(M)$ of two balls of equal radius R will be discussed. The separation algorithm for one ball can be adapted by considering both the points within the point set P themselves and the points shifted by the vector $-M$.

Lemma 2.6.1 *Let $P \subset \mathbb{R}^n$ be a finite point set within the Euclidean space $\mathbb{R}^n$, let $S_1 \dot\cup S_2 \dot\cup S_3 \dot\cup S_4 = P$ be a partition of the point set P and let the point $M \in \mathbb{R}^n$ be fixed. It yields that*

$$B_R(t) \cap B_R(t+M) \cap P = S_1, \tag{2.113}$$
$$(B_R(t) \backslash B_R(t+M)) \cap P = S_2, \tag{2.114}$$
$$(B_R(t+M) \backslash B_R(t)) \cap P = S_3, \tag{2.115}$$
$$(\mathbb{R}^n \backslash (B_R(t) \cup B_R(t+M))) \cap P = S_4 \tag{2.116}$$

for some translation vector $t \in \mathbb{R}^n$ if and only if

$$S_1 \cup (-M + S_1) \cup S_2 \cup (-M + S_3) \subset B_R(t), \tag{2.117}$$
$$((-M + S_2) \cup S_3 \cup S_4 \cup (-M + S_4)) \cap B_R(t) = \emptyset. \tag{2.118}$$

Proof

First of all let us assume that the translation vector $t \in \mathbb{R}^n$ satisfies (2.113)-(2.116). As

$$S_1 \cup S_3 \subset B_R(t+M) \Longleftrightarrow (-M + S_1) \cup (-M + S_3) \subset B_R(t), \tag{2.119}$$
$$(S_2 \cup S_4) \cap B_R(t+M) = \emptyset \Longleftrightarrow ((-M + S_2) \cup (-M + S_4)) \cap B_R(t) = \emptyset, \tag{2.120}$$

the conditions (2.117) and (2.118) are also satisfied for the translation vector t.

Now let us assume that (2.117)-(2.118) are fulfilled for some translation vector $t \in \mathbb{R}^n$. *We again use (2.119) and (2.120) to deduce that*

$$S_1 \subset B_R(t) \cap B_R(t+M), \tag{2.121}$$

$$S_2 \subset B_R(t), S_2 \cap B_R(t+M) = \emptyset, \tag{2.122}$$

$$S_3 \subset B_R(t+M), S_3 \cap B_R(t) = \emptyset, \tag{2.123}$$

$$S_4 \cap (B_R(t) \cup B_R(t+M)) = \emptyset. \tag{2.124}$$

Therefore, the conditions (2.113)-(2.116) are also satisfied for the translation vector t. □

By using Lemma 2.6.1 we can determine all subsets $S \subset P$ *within a finite point set* P *which can be separated with respect to the point set* P *by the window* $W = B_R(0) \cap B_R(M)$ *for fixed radius* R *and fixed* M *in* $O(|P|^{n+2})$ *arithmetic operations.*

Theorem 2.6.2
Let $\emptyset \neq P \subset \mathbb{R}^n$ *be a nonempty finite point set within the Euclidean space* $\mathbb{R}^n$ *and let the window* $W = B_R(0) \cap B_R(M)$ *be given for fixed radius* R *and fixed* $M \in \mathbb{R}^n$. *All subsets* $S \subset P$ *within the point set* P *which can be separated with respect to the point set* P *by the window* W *are calculated in* $O(|P|^{n+2})$ *arithmetic operations.*

Proof
Let us define the point set $\bar{P} := P \cup (-M + P)$ *by the union of the point set* P *and its translate by the vector* $-M$. *All subsets* $\bar{S}$ *within the point set* $\bar{P}$ *which can be separated with respect to the point set* $\bar{P}$ *by the window* $\bar{W} := B_R(0)$ *are calculated in* $O((2|P|)^{n+2}) = O(|P|^{n+2})$ *arithmetic operations according to Theorem 2.4.7. By using Lemma 2.6.1 the set of all subsets* $S \subset P$ *which can be separated within the original problem is given by the set*

$$\{S_{\bar{S}} | \bar{S} \subset \bar{P} \text{ can be separated with respect to } P \text{ by } \bar{W}\}, \tag{2.125}$$

where the point set $S_{\bar{S}} \subset P$ *is defined by*

$$S_{\bar{S}} := \{p \in P | p \in \bar{S} \text{ and } -M + p \in \bar{S}\}. \tag{2.126}$$

□

Remark 2.6.3 Of course, the result of Theorem 2.6.2 can be generalized to m balls of fixed radius and fixed $M_0 := 0, M_1, M_2, \ldots, M_{m-1} \in \mathbb{R}^n$ if we solve the separability problem for the point set $\bar{P} := \bigcup_{l=0}^{m-1} -M_l + P$ and the window $\bar{W} := B_R(0)$ and replace the definition (2.126) of the point set $S_{\bar{S}}$ by

$$S_{\bar{S}} := \{p \in P | -M_l + p \in \bar{S} \text{ for } l = 0, \ldots, m-1\}. \tag{2.127}$$

Thus, all subsets $S \subset P$ within the point set $P \subset \mathbb{R}^n$ which can be separated with respect to the point set P by the window $W := \bigcap_{i=1}^{m} B_R(M_{i-1})$ are determined in $O(m^{n+2}|P|^{n+2})$ arithmetic operations.
The union of m balls instead of their intersection is treated, if we replace the definition (2.126) of the point set $S_{\bar{S}}$ by

$$S_{\bar{S}} := \{p \in P| - M_l + p \in \bar{S} \text{ for some } l \in \{0, \dots, m-1\}\}. \qquad (2.128)$$

Corollary 2.6.4 *Let $\emptyset \neq P \subset \mathbb{R}^n$ be a nonempty finite point set within the Euclidean space $\mathbb{R}^n$ and let the window $W(R) = B_R(0) \cap B_R(M)$ be given in dependence on some non-specified radius R, but for fixed $M \in \mathbb{R}^n$. All subsets $S \subset P$ within the point set P which can be separated with respect to the point set P by the window $W(R)$ for some radius $R > 0$ are calculated in $O(|P|^{n+2})$ arithmetic operations.*

Proof

As before in Theorem 2.6.2 the problem can be reduced to the case of separation by one ball with now arbitrary radius. □

Remark 2.6.5 Generalizations to more than two balls and to the union instead of the intersection of the balls are also possible in the case of arbitrary radius.

2.7 Worst-case analysis

2.7.1 Worst-case analysis for the separation by balls and hyperplanes

In the following we will examine the worst-case situation within the determination of all separable subsets with respect to balls and hyperplanes.

Lemma 2.7.1 *There exists a finite point configuration $P \subset \mathbb{R}^n$ within the Euclidean space $\mathbb{R}^n$ that contains $O(|P|^{n+1})$ subsets $S \subset P$ within the point set P which can be separated with respect to the point set P by balls of arbitrary radius resp. that contains $O(|P|^n)$ subsets $S \subset P$ within the point set P which can be separated with respect to the point set P by balls of fixed radius resp. by hyperplanes.*

Proof

Let the point set P be defined by

$$P := \{p_{i,0} := -i \cdot e_1 | i = 1, \ldots, k\} \cup \bigcup_{l=1}^{n} \{p_{i,l} := i \cdot e_l | i = 1, \ldots, k\}. \quad (2.129)$$

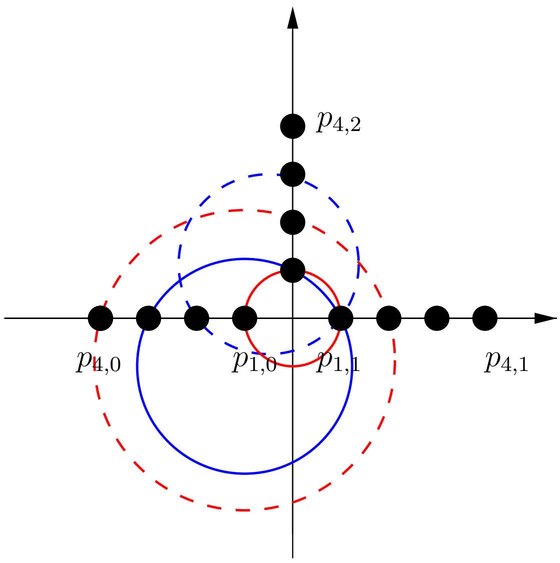

Figure 2.3: Worst-case construction for balls ($k = 4$, $n = 2$)

There are $k^{n+1} = (\frac{|P|}{n+1})^{n+1} \in O(|P|^{n+1})$ different subsets $\bar{P} := \{\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_n\}$ within the point set P with $\bar{p}_0 \in \{-i \cdot e_1 | i = 1, \ldots, k\}$ and $\bar{p}_l \in \{i \cdot e_l | i = 1, \ldots, k\}$ for $1 \leq l \leq n$, which uniquely determine balls $B_{\bar{P}}$ with spherical points $\bar{P}$, as each point set $\bar{P}$ is affinely independent. Notice, that the ball $B_{\bar{P}}$ separates the point set

$$\{p_{1,0}, p_{2,0}, \ldots, \bar{p}_0\} \cup \{p_{1,1}, p_{2,1}, \ldots, \bar{p}_1\} \cup \{p_{1,2}, p_{2,2}, \ldots, \bar{p}_2\} \cup \\ \cdots \cup \{p_{1,n}, p_{2,n}, \ldots, \bar{p}_n\} \quad (2.130)$$

with respect to the point set P, as the point $0 \in \mathbb{R}^n$ and therefore all points within the set in (2.130) lie within the ball $B_{\bar{P}}$ because of convexity arguments, and the other points within the point set P lie outside the ball $B_{\bar{P}}$, for illustration see Figure 2.3. Therefore, there are at least $O(|P|^{n+1})$ subsets within the point set P which can be separated with respect to the point set P by a ball of arbitrary radius.
The assertions for the cases of separation by hyperplanes and by balls of fixed radius are implied by the embeddings (2.101) and (2.110) and the assertion for the case of separation by balls of arbitrary radius. □

Remark 2.7.2 The complexity of any algorithm which determines all separable subsets within a finite point set P is bounded from below by the number of separable subsets. Therefore, according to the results in Lemma 2.7.1 the presented separation algorithms for balls and hyperplanes are best possible up to multiplication by some factor $O(|P|)$ resp. by some factor $O(|P|^2)$. Notice, that we have to expend $O(|P|)$ arithmetic operations to decide which points within the point set P belong to the separable set. Treating balls of fixed radius the further expenditure by the factor $O(|P|)$ is caused by the fact that we also have to consider $(n+1)$-subsets within the point set P in order to avoid calculating square roots to determine the centres of the balls.

2.7.2 Worst-case analysis for the separation by polytopes

Now we will examine the worst-case situation within the determination of all subsets which can be separated by polytopal windows.

Lemma 2.7.3 *There exists a finite point configuration $P \subset \mathbb{R}^n$ within the Euclidean space $\mathbb{R}^n$ that contains $O(|P|^n)$ subsets $S \subset P$ within the point set P which can be separated with respect to the point set P by a fixed polytope.*

Proof

Without loss of generality (as we will explain later) let us look at the case that the polytopal window W is given by the cube $W := [0,1]^n$. Let the point set $P := \{p_1, \ldots, p_{nk}\}$ be defined by

$$p_i := (\frac{i}{2k}, \frac{3}{4}, \ldots, \frac{3}{4}) \text{ for } i = 1, \ldots, k, \tag{2.131}$$

$$p_{k+i} := (\frac{3}{4}, \frac{i}{2k}, \frac{3}{4}, \ldots, \frac{3}{4}) \text{ for } i = 1, \ldots, k, \tag{2.132}$$

$$\vdots$$

$$p_{(n-1)k+i} := (\frac{3}{4}, \ldots, \frac{3}{4}, \frac{i}{2k}) \text{ for } i = 1, \ldots, k. \tag{2.133}$$

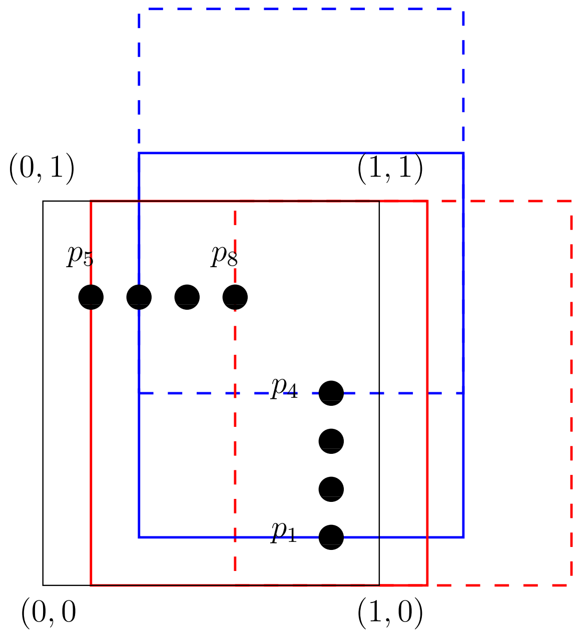

Figure 2.4: Worst-case construction for cubes ($k = 4$, $n = 2$)

For every translation vector $t = (t_1, \ldots, t_n)$ so that $t_i \in \{\frac{j}{2k} | j = 1, \ldots, k+1\}$ the point set

$$\bigcup_{i=1}^{n} \{p_{(i-1)n+j} | 2k \cdot t_i \leq j \leq k\} \tag{2.134}$$

is separated by the window $t+W$ with respect to the point set P, see Figure 2.4 for illustration. Therefore, there are at least $(k+1)^n = (\frac{|P|}{n}+1)^n \in O(|P|^n)$ separable subsets within the point set P.

Now let us dicuss the case of general polytopal windows:

Let $a_1, \ldots, a_n$ be the linearly independent normal vectors of n hyperplanes defining one of the vertices of some polytopal window. Without loss of generality let us assume that the vertex is given by the point $0 \in \mathbb{R}^n$. By a linear transformation which maps the standard orthonormal basis vectors $e_1 = (1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1)$ to the suitably ordered and linearly independent vectors $-\lambda a_1, \ldots, -\lambda a_n$ for some sufficiently small factor $\lambda > 0$ the vertex 0 of the cube $[0,1]^n$ and its n determining facets (more precisely, their supporting hyperplanes) are mapped to the polytopal vertex 0 and n of its incident facets (more precisely, their supporting hyperplanes). Therefore, the general polytopal case is reduced to the case $W = [0,1]^n$ and its consideration before. □

Remark 2.7.4 Unions of polytopes lead to the same worst-case result by considering some appropriate vertex of some of the united polytopes.

2.7.3 Worst-case analysis for the separation by the intersection of two balls

Finally, we show that also in the case of separation by the intersection of balls the number of separable subsets can reach the complexity of the algorithm up to some factor $O(|P|^2)$.

Lemma 2.7.5 *There exists a finite point configuration $P \subset \mathbb{R}^n$ within the Euclidean space $\mathbb{R}^n$ that contains $O(|P|^n)$ subsets $S \subset P$ within the point set P which can be separated with respect to the point set P by the intersection of two balls $B_R(0) \cap B_R(\triangle t)$ for fixed radius R and fixed relational position $\triangle t$ of their centres.*

Proof

Without loss of generality let us assume that the relational position $\triangle t$ of the two ball centres is given by $\triangle t := \lambda e_1$ for some $0 < \lambda < 2 \cdot R$. Let us define the points

$$p_i := -i \cdot \frac{1}{2k} e_2 \text{ for } i = 1, \ldots, k, \tag{2.135}$$

$$p_{(j-1)k+i} := i \cdot \frac{1}{2k} e_j \text{ for } i = 1, \ldots, k \text{ and } j = 2, \ldots, n \tag{2.136}$$

and let us fix the parameter $0 < \mu < \frac{1}{S} \cdot \sqrt{R^2 - (\frac{\lambda}{2})^2}$, where the value S is defined by

$$S := \max\{R_{min}^{\{\bar{p}_1,\ldots,\bar{p}_n\}} | \bar{p}_i \in \{p_{(i-1)k+1}, \ldots, p_{ik}\}\}. \tag{2.137}$$

Let the point set P be defined by $P := \{q_i := \mu p_i | i = 1, \ldots, nk\}$. For every subset $\{\bar{q}_1, \ldots, \bar{q}_n\} \subset P$ within the point set P so that $\bar{q}_i \in \{q_{(i-1)k+1}, \ldots, q_{ik}\}$ for $i = 1, \ldots, n$ we calculate by using the cosinus theorem that the first component of the centre M of each of the two balls determined by the spherical points $\bar{q}_i, \ldots, \bar{q}_n$ and the radius R is bounded from below by the absolut value

$$\begin{aligned} \text{dist}(M, \text{aff}(\{\bar{q}_1, \ldots, \bar{q}_n\})) &= \sqrt{R^2 - (R_{min}^{\{\bar{q}_1,\ldots,\bar{q}_n\}})^2} = \\ &= \sqrt{R^2 - (\mu \cdot R_{min}^{\{\bar{p}_1,\ldots,\bar{p}_n\}})^2} > \sqrt{R^2 - (R^2 - (\frac{\lambda}{2})^2)} = \frac{\lambda}{2}. \end{aligned} \tag{2.138}$$

If the first component of the centre M is positive, we calculate that

$$\text{dist}(M - \triangle t, q) < \sqrt{((\frac{\lambda}{2})^2 + (R_{min}^{\{\bar{q}_1,\ldots,\bar{q}_n\}})^2)} < \sqrt{(\frac{\lambda}{2})^2 + (R^2 - (\frac{\lambda}{2})^2)} = R \tag{2.139}$$

for every point $q \in \{\bar{q}_i, \ldots, \bar{q}_n\}$, which implies that

$$\bigcup_{i=1}^{n} \{q_{(i-1)k+1}, \ldots, \bar{q}_i\} = (B_R(M) \cap B_R(M - \triangle t)) \cap P. \tag{2.140}$$

Analogous to Lemma 2.7.1 the number of subsets within the point set P which can be separated with respect to the point set P is therefore given at least by the value $k^n = (\frac{|P|}{n})^n \in O(|P|^n)$. □

2.8 Minimal separating balls and (planar) polytopes

We have discussed the algorithmic determination of all subsets $S \subset P$ within a finite point set $P \subset \mathbb{R}^n$ which can be separated with respect to the point set P before. In the polytopal case the separating polytope was assumed to be known up to translation.

Now let some subset $S \subset P$ within the finite point set P be specified. In the following we attend the determination of a ***minimal ball*** *resp. of a* ***minimal polytope*** *with fixed number m of facets which separates the subset $S \subset P$ with respect to the point set P, if spherical resp. polytopal separation is possible at all. Minimal refers to the volume given by the Lebesgue n-measure.*

The smallest separating ball is determined in $O(|P|)$ arithmetic operations for fixed space dimension n by solving a convex quadratic minimization problem in the Euclidean space $\mathbb{R}^{n+1}$ with linear constraints by the techniques presented in [97], see [106] and compare the beginning of Section 2.4. In general, points within both the point set S and its complementary point set $P \backslash S$ are located on the boundary of the calculated ball. We characterize in which cases a minimal separating ball resp. a locally minimal separating triangle can at least be approximated in the case that its boundary also contains points within the point set $P \backslash S$.

The smallest triangle which contains a finite point set $S \subset \mathbb{R}^2$ is attained in $O(|S| \cdot \log |S|)$ arithmetic operations, as the convex hull of the point set S is determined in $O(|S| \cdot \log |S|)$ arithmetic operations and all local minimal triangles which enclose the convex polygon $\operatorname{conv}(S)$ are calculated in linear time within the number of vertices of the polygon $\operatorname{conv}(S)$, see [86] and [85].

We extend the triangular enclosing problem to minimal triangular separability by generalizing some assertions in [86], [105] and formulate an algorithm based on wedge separability to determine a minimal separating triangle in $O(|P|^2 \cdot \log |P|)$ arithmetic operations.

2.8.1 Minimal separation and approximation

In the present subsection we will take a look at the cases in which a minimal separating window can be approximated. Notice, that minimality will actually have to be replaced by the term infimum, if boundary points of the window belong to both point sets S and $P\backslash S$. But we will continue speaking of minimality in the following.
First of all let us consider in which cases the minimal separating ball can at least be approximated if boundary points also belong to the complementary point set $P\backslash S$ of the point set S.

Lemma 2.8.1 *Let the subset $S \subset P$ within the finite point set $P \subset \mathbb{R}^n$ be specified. The ball $B_{r_{min}}(M_{min})$ of minimal radius which separates the point set S with respect to the point set P, i. e.*

$$S \subset B_{r_{min}}(M_{min}), \tag{2.141}$$
$$(P\backslash S) \cap \operatorname{int}(B_{r_{min}}(M_{min})) = \emptyset, \tag{2.142}$$

can be approximated by a sequence $(B_{r_i}(M_i))_{i\in\mathbb{N}}$ of balls which separate the point set S with respect to the point set P, i. e.

$$S \subset B_{r_i}(M_i), \tag{2.143}$$
$$(P\backslash S) \cap B_{r_i}(M_i) = \emptyset, \tag{2.144}$$

if and only if

$$\bigcap_{(p,q)\in\mathcal{T}} H^{<}_{q-p,\frac{1}{2}(p+q)} \neq \emptyset \tag{2.145}$$

for the set $\mathcal{T} := \{(p,q) \in (\operatorname{bd}(B_{r_{min}}(M_{min})))^2 | p \in S, q \in P\backslash S\}$.
Proof
Let us assume that the minimal separating ball $B_{r_{min}}(M_{min})$ can be approximated by a sequence $(B_{r_i}(M_i))_{i\in\mathbb{N}}$ of balls so that

$$M_i \to M_{min}, \tag{2.146}$$
$$r_i \to r_{min} \tag{2.147}$$

for $i \to \infty$. In particular, let $M_i \in H^{<}_{(q-p),\frac{1}{2}(p+q)}$ for every $p \in S$ and $q \in P\backslash S$, as $||p - M_i|| < ||q - M_i||$ has to be fulfilled. Therefore, we conclude that

$$M_i \in \bigcap_{p\in S,q\in P\backslash S} H^{<}_{(q-p),\frac{1}{2}(p+q)} \subset \bigcap_{(p,q)\in\mathcal{T}} H^{<}_{q-p,\frac{1}{2}(p+q)} \neq \emptyset. \tag{2.148}$$

For the reverse direction let us assume that condition (2.145) is satisfied.
For every sufficiently small value $\epsilon > 0$ we can find some point $M_\epsilon \in \bigcap_{(p,q)\in\mathcal{T}} H^{<}_{q-p,\frac{1}{2}(p+q)}$ so that $||M_\epsilon - M_{min}|| = \epsilon$, which satisfies that the

ball $B_{r_\epsilon}(M_\epsilon)$ *separates the point set* S *with respect to the point set* P *for the radius*

$$r_\epsilon := \max_{p \in S \cap \mathrm{bd}(B_{r_{\min}}(M_{\min}))} ||p - M_\epsilon||, \tag{2.149}$$

as $||q - M_\epsilon|| > r_\epsilon$ *for* $q \in \mathrm{bd}(B_{r_{min}}(M_{min})) \cap P \backslash S$, *if the value* $\epsilon > 0$ *is chosen sufficiently small, and then also both interior and exterior points are preserved.* □

Some consequences for the planar case are formulated within the following corollary.

Corollary 2.8.2 *Let the subset* $S \subset P$ *within the finite planar point set* $P \subset \mathbb{R}^2$ *be specified.*

1. *If* $|S| \geq 2$, *the minimal separating ball* $B_{r_{min}}(M_{min})$ *has at least two boundary points within the point set* S, *i. e.*

$$|\,\mathrm{bd}(B_{r_{min}}(M_{min})) \cap S| \geq 2. \tag{2.150}$$

2. *The minimal separating ball* $B_{r_{min}}(M_{min})$ *cannot be approximated by a sequence of actually separating balls if and only if there are two boundary points* $p_1, p_2 \in \mathrm{bd}(B_{r_{min}}(M_{min})) \cap S$ *within the point set* S *so that both open halfspaces* $H^<_{(p_2-p_1)^\perp, p_1}$ *and* $H^>_{(p_2-p_1)^\perp, p_1}$ *contain boundary points within the point set* $P \backslash S$, *i. e.*

$$H^<_{(p_2-p_1)^\perp, p_1} \cap \mathrm{bd}(B_{r_{min}}(M_{min})) \cap P \backslash S \neq \emptyset, \tag{2.151}$$
$$H^>_{(p_2-p_1)^\perp, p_1} \cap \mathrm{bd}(B_{r_{min}}(M_{min})) \cap P \backslash S \neq \emptyset. \tag{2.152}$$

Proof

The minimality of the separating ball $B_{r_{min}}(M_{min})$ *implies that*

$$\mathrm{bd}(B_{r_{min}}(M_{min})) \cap S \neq \emptyset. \tag{2.153}$$

Thus, for the first assertion let us assume that $\mathrm{bd}(B_{r_{min}}(M_{min})) \cap S = \{p_0\}$ *and let the value* $\bar{r}$ *be defined by*

$$\bar{r} := \max_{p \in S \backslash \{p_0\}} ||p - M_{min}||. \tag{2.154}$$

Then for every sufficiently small

$$0 < \epsilon < \frac{r_{min} - \bar{r}}{2} \tag{2.155}$$

we get that $B_{r_{min}-\epsilon}(M_{min}+\epsilon\cdot\frac{p_0-M_{min}}{||p_0-M_{min}||})\cap P=S$ *as*

$$||p-(M_{min}+\epsilon\cdot\frac{p_0-M_{min}}{||p_0-M_{min}||}||\leq||p-M_{min}||+\epsilon \tag{2.156}$$

$$\leq\bar{r}+\epsilon<r_{min}-\epsilon \tag{2.157}$$

for every point $p\in S$ *and* $B_{r_{\min}-\epsilon}(M_{min}+\epsilon\cdot\frac{p_0-M_{min}}{||p_0-M_{min}||})\subset B_{r_{\min}}(M_{\min})$*, which contradicts the minimality of the radius* r_{min}*.*

For the equivalence in assertion 2. let us assume that the points $p_1,p_2\in\mathrm{bd}(B_{r_{min}}(M_{min}))\cap S$ *satisfy (2.151)-(2.152) and that*

$$q_1\in H^{<}_{(p_2-p_1)^{\perp},p_1}\cap\mathrm{bd}(B_{r_{min}}(M_{min}))\cap P\backslash S, \tag{2.158}$$

$$q_2\in H^{>}_{(p_2-p_1)^{\perp},p_1}\cap\mathrm{bd}(B_{r_{min}}(M_{min}))\cap P\backslash S. \tag{2.159}$$

Thus, for every point $\bar{p}_1\in\mathrm{bd}(B_{r_{min}}(M_{min}))\cap S\cap H^{<}_{(q_1-q_2)^{\perp},q_1}$ *we can find some point* $\bar{p}_2\in\mathrm{bd}(B_{r_{min}}(M_{min}))\cap S\cap H^{>}_{(q_1-q_2)^{\perp},q_1}$ *(by taking one of the points* p_1*,* p_2*) so that the points* q_1,q_2 *lie on different sides of the line passing through the points* $\bar{p}_1$*,* $\bar{p}_2$*. (Similar arguments work for every point* $\bar{p}_1\in\mathrm{bd}(B_{r_{min}}(M_{min}))\cap S\cap H^{>}_{(q_1-q_2)^{\perp},q_1}$ *and some point* $\bar{p}_2\in\mathrm{bd}(B_{r_{min}}(M_{min}))\cap S\cap H^{<}_{(q_1-q_2)^{\perp},q_1}$*.)*

For every point $\bar{q}_1\in H^{<}_{(\bar{p}_2-\bar{p}_1)^{\perp},\bar{p}_1}\cap P\backslash S\cap\mathrm{bd}(B_{r_{min}}(M_{min}))$ *and every point* $\bar{q}_2\in H^{>}_{(\bar{p}_2-\bar{p}_1)^{\perp},\bar{p}_1}\cap P\backslash S\cap\mathrm{bd}(B_{r_{min}}(M_{min}))$ *the set* $H^{<}_{\bar{q}_1-\bar{p}_1,\frac{1}{2}(\bar{p}_1+\bar{q}_1)}\cap H^{<}_{\bar{q}_2-\bar{p}_1,\frac{1}{2}(\bar{p}_1+\bar{q}_2)}$ *describes the interior of a cone with apex* M_{min}*, which contains the point* $\bar{p}_1$*, but not the points* $\bar{p}_2,\bar{q}_1$ *and* $\bar{q}_2$*. Therefore, we conclude that*

$$\bigcap_{(p,q)\in\mathcal{T}}H^{<}_{q-p,\frac{1}{2}(p+q)} \tag{2.160}$$

$$\subset\bigcap_{(\bar{q}_1,\bar{q}_2)\in D,k=1,2}H^{<}_{\bar{q}_1-\bar{p}_k,\frac{1}{2}(\bar{p}_k+\bar{q}_1)}\cap H^{<}_{\bar{q}_2-\bar{p}_k,\frac{1}{2}(\bar{p}_k+\bar{q}_2)}=\emptyset$$

for the set $\mathcal{T}$ *as defined in Lemma 2.8.1 and the set*

$$D:=\{(\bar{q}_1,\bar{q}_2)\in((P\backslash S)\cap\mathrm{bd}(B_{r_{min}}(M_{min})))^2|\bar{q}_1\in H^{<}_{(\bar{p}_2-\bar{p}_1)^{\perp},\bar{p}_1},$$

$$\bar{q}_2\in H^{>}_{(\bar{p}_2-\bar{p}_1)^{\perp},\bar{p}_1}\}. \tag{2.161}$$

Thus, condition (2.145) in Lemma 2.8.1 is not satisfied.

In the case that no boundary points within the point set $\mathrm{bd}(B_{r_{min}}(M_{min}))\cap S$ *satisfy the conditions (2.151)-(2.152) let us choose the points* $\bar{p}_1$*,* $\bar{p}_2\in\mathrm{bd}(B_{r_{min}}(M_{min}))\cap S$ *so that*

$$H^{\leq}_{(\bar{p}_2-\bar{p}_1)^{\perp},\bar{p}_1}\cap\mathrm{bd}(B_{r_{min}}(M_{min}))\cap P\subset S, \tag{2.162}$$

$$H^{>}_{(\bar{p}_2-\bar{p}_1)^{\perp},\bar{p}_1}\cap\mathrm{bd}(B_{r_{min}}(M_{min}))\cap P\subset P\backslash S. \tag{2.163}$$

The minimal separating ball $B_{r_{min}}(M_{min})$ *can be approximated by a sequence of balls* $B_{r_\epsilon}(M_\epsilon)$ *for*

$$M_\epsilon:=M_{min}-\epsilon\cdot\frac{(\bar{p}_2-\bar{p}_1)^{\perp}}{||(\bar{p}_2-\bar{p}_1)^{\perp}||}, \tag{2.164}$$

$$r_\epsilon:=||\bar{p}_1-M_\epsilon||<r_{\min}+\epsilon \tag{2.165}$$

and the value $\epsilon > 0$ sufficiently small so that interior and exterior points are preserved, as the ball $B_{r_\epsilon}(M_\epsilon)$ actually separates the point set S with respect to the point set P. □

Next let us look at minimal separating triangles. The following equivalence characterizes the case that a locally minimal separating triangle can be approximated by a sequence of triangles which actually separate the point set S with respect to the point set P.

Lemma 2.8.3 *Let the subset $S \subset P$ within the finite planar point set $P \subset \mathbb{R}^2$ be specified. A locally minimal triangle T with vertices A, B, C can be approximated by a sequence of triangles which actually separate the point set S with respect to the point set P if and only if the boundary $\mathrm{bd}(T)$ of the triangle T can be divided into line segments s_1, s_2, s_3 so that*

1. $\bigcup_{i=1}^{3} s_i = \mathrm{bd}(T)$,
2. $\bigcup_{i \neq j} s_i \cap s_j = \{A, B, C\} \cap S$, *in particular, vertices of the triangle T are taken twice if and only if they belong to the point set S,*
3. *for $i = 1, 2, 3$ the point set $S \cap s_i$ can be separated with respect to the point set $P \cap s_i$ by a line.*

Proof

Let the line segments s_1, s_2, s_3 satisfy conditions 1.-3. and let the point q_i be the intersection point of the line segment s_i and one of the lines which separates the point set $S \cap s_i$ with respect to the point set $P \cap s_i$ for $i = 1, 2, 3$. If $s_i \cap (P \backslash S) \neq \emptyset$ for $i \in \{1, 2, 3\}$, we apply a small rotation on the line containing the line segment s_i which remains the point q_i fixed so that interior and exterior points are not concerned. Because of the conditions 1.-3. also the boundary points $\mathrm{bd}(T) \cap P$ of the triangle T within the point set P are separated now. By choosing each rotation angle sufficiently small the area of the new triangle is arbitrarily close to the area of the original triangle T.

For the reverse direction let us also restrict to the set of boundary points $\mathrm{bd}(T) \cap P$, as interior and exterior points are not concerned by any sufficiently nearby approximation T' of the triangle T. Let e denote some edge of the original triangle T and let e' be its approximation within the triangle T'. As the approximating triangle T' separates the point set S with respect to the point set P, the approximating edge e' of the edge e has to separate the point set $S \cap e$ with respect to the point set $P \cap e$. Notice, that every vertex of the triangle T which lies within the point set S can be added to both adjacent edges so that we result in the conditions 1.-3. □

Example 2.8.4 To illustrate Lemma 2.8.3 let us look at the situation as given in Figure 2.5.

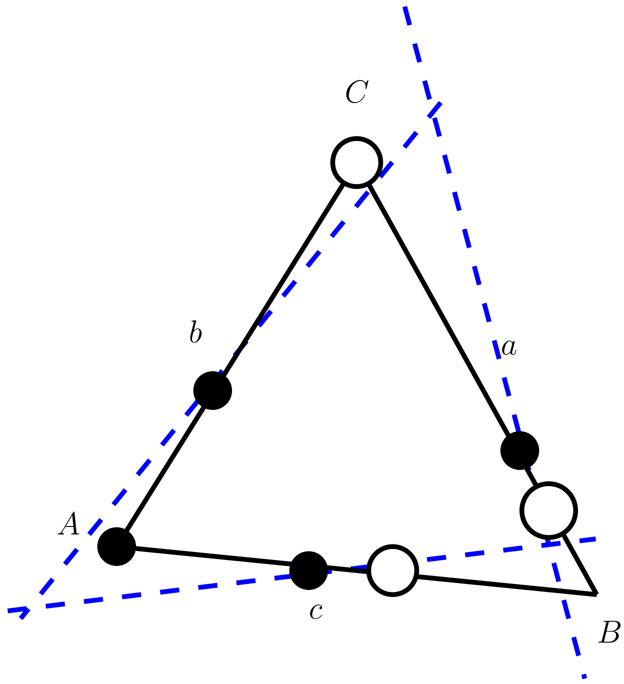

Figure 2.5: Approximating locally minimal triangles

The filled circles belong to the point set S, the non-filled circles to its complementary point set $P \backslash S$. The line segments s_1, s_2, s_3 with respect to Lemma 2.8.3 are given by

$$s_1 := a \backslash \{C\} \cup \{B\}, \tag{2.166}$$

$$s_2 := b \cup \{A, C\}, \tag{2.167}$$

$$s_3 := c \backslash \{B\} \cup \{A\}. \tag{2.168}$$

2.8.2 Algorithmic determination of minimal separating triangles

In order to determine a minimal resp. an infimium triangle which separates the point set $S \subset P$ with respect to the finite planar point set $P \subset \mathbb{R}^2$, we will calculate all locally minimal triangles. Finding a minimal enclosing triangle of a finite point set S, which is also based on the idea of calculating all locally minimal enclosing triangles, is considered in [86] and [105]. We will generalize some results of [86] and [105] in the following in order to get an algorithmic approach for our problem of minimal triangular separability.
For further considerations let us assume that the convex hull $\text{conv}(S)$ of the point set S does not contain any point within its complementary point set $P \backslash S$, as otherwise separation by any convex polygon is not possible at all.

Lemma 2.8.5 *Let the subset $S \subset P$ within the finite planar point set $P \subset \mathbb{R}^2$ be specified and let T denote a (locally) minimal separating triangle, which satisfies*

$$\text{int}(T) \cap P \subset S, \tag{2.169}$$
$$(\mathbb{R}^2 \backslash T) \cap S = \emptyset. \tag{2.170}$$

1. *Every edge of the triangle T contains at least one point within the point set S.*

2. *If the centre point c of an edge e does not lie within the convex hull $conv(S)$ of the point set S, then there exist a point $p \in S \cap e$ and a point $q \in (P \backslash S) \cap e$ so that the point p lies between the two points c and q.*

3. *Every locally minimal separating triangle possesses at least one edge which is incident to at least two points within the point set P.*
   Moreover, there exists another triangle T' of same area having at least two edges each of which contains at least two points within the point set P.

Proof
Assuming that condition 1. is not satisfied, the area of the triangle T is reduced by moving the edge in the opposite direction of its normal vector, which contradicts the local minimality of the triangle T.
Let e denote some edge of the triangle T. If neither the centre point c of the edge e lies within the polygon $\text{conv}(S)$ *nor condition 2. is satisfied, the edge e can be rotated by fixing some point which separates the point set $S \cap e$ and the point set $\{c\} \cup ((P \backslash S) \cap e)$ so that by choosing the rotation angle sufficiently small both condition (2.169) and condition (2.170) remain fulfilled, but the area of the new triangle is smaller than the area of the original triangle T:*
The added area and the subtracted area are described by two triangles which have the same angle (the rotation angle) within the fixed point. The adjacent

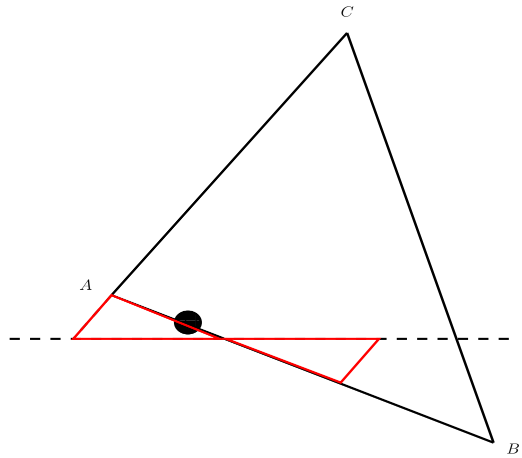

Figure 2.6: Edge rotation so that the resulting triangle has a smaller area

edges of the added triangle are shorter than the adjacent edges of the subtracted triangle, see Figure 2.6.

The triangle T is also not locally minimal according to [86] if only the centre points of the edges belong to the point set P. Thus, to show assertion 3. let us assume that $|e_3 \cap P| \geq 2$ and that $|e_1 \cap P| = |e_2 \cap P| = 1$. Because of the conditions 1. and 2. the centre points c_1, c_2 of the edges e_1, e_2 belong to the point set S. Analogous to [105] let us move the vertex C which is incident to the edges e_1 and e_2, while the distance $\mathrm{dist}(C, e_3)$ of the vertex C to the edge e_3 and the contact points c_1, c_2 of the triangle with the convex polygon $\mathrm{conv}(S)$ remain fixed. We stop the motion by the time that one of the edges e_1, e_2 is incident to at least two points within the point set P. As the base length of value $2 \cdot dist(c_1, c_2)$ and the height $dist(C, e_3)$ of the triangle are not changed during the motion of the vertex C, the area remains fixed. □

The following result will be used to compute all locally minimal triangles within the later presented algorithmic approach.

Lemma 2.8.6 *Let the subset $S \subset P$ within the finite planar point set $P \subset \mathbb{R}^2$ be specified. The number of lines which contain two points within the point set S resp. one point within each of the two point sets S and $P \backslash S$ and which possibly support one of the locally minimal triangles lies within $O(|P|)$. Those lines are determined in $O(|P| \cdot \log |P|)$ resp. in $O(|P|^2)$ arithmetic operations.*

Proof

Each line which is incident to two points within the point set S and which contains an edge of a locally minimal separating triangle supports the convex hull $\mathrm{conv}(S)$ of the point set S. The number of edges of the convex hull $\mathrm{conv}(S)$ is bounded by the value $|S| \leq |P|$. The convex hull $\mathrm{conv}(S)$ is calculated in $O(|S| \cdot \log |S|) \leq O(|P| \cdot \log |P|)$ arithmetic operations by Graham scan, see [59].

For any point $q \in P \backslash S$ there are exactly two lines l_1, l_2 incident to the point q and to at least one point within the point set S and possibly supporting an edge of a locally minimal separating triangle. The lines l_1, l_2 are given by those lines which pass through the point q and which support the convex polygon $\operatorname{conv}(S)$*. Their total number is bounded from above by the value $2 \cdot |P \backslash S| \leq 2 \cdot |P|$.*
For the complexity assertion let $p \in \operatorname{conv}(S)$, let the edges e_1, e_2 of the convex polygon $\operatorname{conv}(S)$ be incident to the point p and let $l(e_1), l(e_2)$ denote their supporting lines. The line which passes through the point $p \in \operatorname{conv}(S)$ and some point $q \in P \backslash S$ is identical to the line l_1 resp. to the line l_2 if and only if the point q and the convex polygon $\operatorname{conv}(S)$ once lie on the same, once on opposite sides of the lines $l(e_1)$, $l(e_2)$, see Figure 2.7 for illustration.

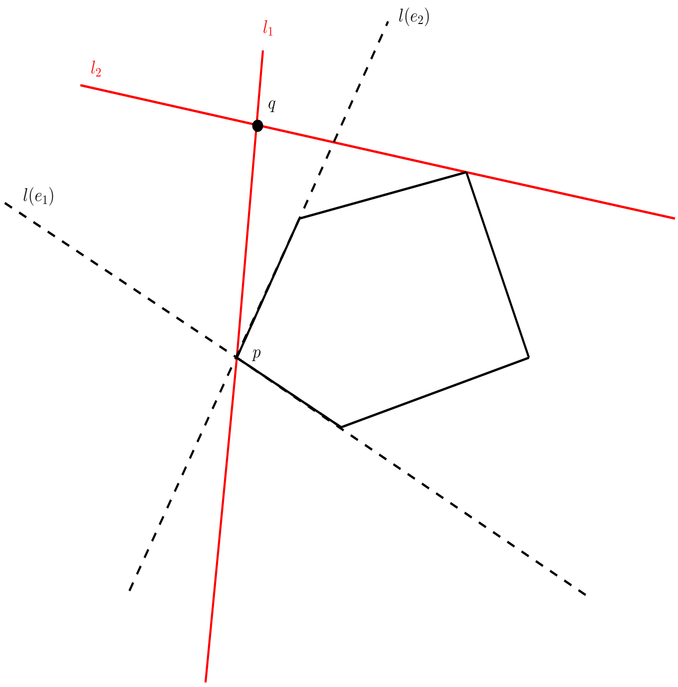

Figure 2.7: The convex hull $\operatorname{conv}(S)$ of the point set S and the lines which possibly contain an edge of a locally minimal triangle

Thus, we have to determine the location of each point $q \in P \backslash S$ with respect to the consecutively ordered lines supporting the edges of the convex polygon $\operatorname{conv}(S)$*. That takes at most $O(|P|^2)$ arithmetic operations in total.* □
If we fix one of the lines which possibly contain an edge of a locally minimal triangle, see Lemma 2.8.6, the remaining separability problem is reduced to the question of wedge separability. Both the decision of whether two finite point sets are wedge separable and the computation of the regions of vertices which belong to separating wedges are discussed in [75]. Similar techniques as applied for wedge separability are already used in [46] in order to obtain a convex polygon which separates two finite point sets and is minimal within the number of edges.

Definition 2.8.7 (wedge separability)
*A **wedge** within the Euclidean plane $\mathbb{R}^2$ is bounded by two rays which start from some common vertex.*
*A subset $S \subset P$ within the finite planar point set $P \subset \mathbb{R}^2$ is called **wedge separable with respect to the point set** P, if there exists a wedge which contains all the points of the point set S, but none of its complementary point set $P \backslash S$.*

Theorem 2.8.8
Let the subset $S \subset P$ within the finite planar point set $P \subset \mathbb{R}^2$ be specified. Wedge separability for the point set S with respect to the point set P can be decided in $O(|P| \cdot \log |P|)$ arithmetic operations. Moreover, the regions of vertices which belong to the separating wedges are also calculated in $O(|P| \cdot \log |P|)$ arithmetic operations.
The number of segments and rays of those regions is bounded from above by the value $4 \cdot |P|$. Each of those segments and rays is contained within one of the lines as described in Lemma 2.8.6.
The number of intersection points of each segment or ray with some line which contains one of the edges of a locally minimal separating triangle is given by $O(|P|)$.

Proof
For details we refer to [75].
The last assertion is trivial as every line which contains an edge of the convex polygon $\text{conv}(S)$ possibly intersects the segment or the ray. □

For ordering purpose let us introduce the following notation.

Notation 2.8.9
Let l_1, l_2 be two lines within the Euclidean plane $\mathbb{R}^2$ with common point p. The angle between the two lines l_1, l_2 denoted by $\angle(l_1, l_2)$ is given by the rotation angle in clockwise direction around the fixed point p.

Lemma 2.8.10 *Let the subset $S \subset P$ within the finite planar point set $P \subset \mathbb{R}^2$ be specified and let the vertices of the convex polygon $\text{conv}(S)$ be numbered in clockwise direction. The lines in Lemma 2.8.6 are sorted in $O(|P| \cdot \log |P|)$ arithmetic operations according to*

$$l_1 < l_2 :\Leftrightarrow \begin{cases} p_1 < p_2 \text{ for } p_1 \in l_1 \cap S, p_2 \in l_2 \cap S \\ \text{or} \\ p_1 = p_2 \text{ and } \angle(\bar{l}, l_1) < \angle(\bar{l}, l_2) \end{cases}, \tag{2.171}$$

if $\bar{l}$ denotes the line which passes through the point p_1 and through the point $p_1 - 1$ (modulo the number of vertices of the convex polygon $\text{conv}(S)$).

Proof
Every line l in Lemma 2.8.6 is characterized by two points within the point set P.

At least one of them lies within the point set S. In the case that $l_1 \cap S = l_2 \cap S$ we have to decide whether $\angle(\bar{l}, l_1) < \angle(\bar{l}, l_2)$ is fulfilled. That can be done in constant time by examining the position of the second point incident to the line l_2 with respect to the lines $\bar{l}$ and l_1. □

Now we are ready to calculate minimal separating triangles on the basis of the wedge separability problem.

Theorem 2.8.11

Let the subset $S \subset P$ within the finite planar point set $P \subset \mathbb{R}^2$ be specified. A minimal resp. infimum triangle which can be approximated by triangles actually separating the point set S with respect to the point set P is determined in $O(|P|^2 \cdot \log |P|)$ arithmetic operations.

Proof

Deciding for each point $p \in P$ and each line l in Lemma 2.8.6 whether the point p is incident to the line l or to which of the associated halfspaces the point p belongs takes $O(|P|^2)$ arithmetic operations in total.

The point set $P \cap l$ which is incident to some line l in Lemma 2.8.6 is sorted in $O(|P \cap l| \cdot \log |P \cap l|)$ arithmetic operations, which takes at most $O(|P|^2 \cdot \log |P|)$ arithmetic operations in total.

According to Lemma 2.8.5 at least one edge of a locally minimal separating triangle is contained in one of the lines given in Lemma 2.8.6. Thus, let us fix some of those lines and refer to it by l in the following. Let

$$\mathcal{M}_l := (p_1, p_2, \ldots, p_{k_1}, \ldots, p_{k_2}, \ldots, p_{L_l}) \tag{2.172}$$

denote the sortation of the point set $P \cap l$ and let $\bar{l}_1$ and $\bar{l}_2$ be those subsequences of the sequence $\mathcal{M}_l$ which are characterized by

$$\bar{l}_1 := \{p_1, \ldots, p_{k_1}\} \subset P \backslash S, \tag{2.173}$$

$$\bar{l}_2 := \{p_{k_2}, \ldots, p_{L_l}\} \subset P \backslash S, \tag{2.174}$$

$$(l \backslash (\bar{l}_1 \cup \bar{l}_2)) \cap P \backslash S = \emptyset. \tag{2.175}$$

The line l can be rotated so that either the point sets $\bar{l}_1$ and $\mathcal{M}_l \backslash \bar{l}_1$ or the point sets $\bar{l}_2$ and $\mathcal{M}_l \backslash \bar{l}_2$ lie on different sides of the line l afterwards. Thus, the triangular separability problem is reduced to one of the following wedge separability problems, compare Lemma 2.8.3, as we are only interested in locally minimal triangles which can be approximated by actually separating ones:

1. *Separate the point set S with respect to the point set $(P \cap (l^+ \cup l)) \backslash \bar{l}_1$*
2. *Separate the point set S with respect to the point set $(P \cap (l^+ \cup l)) \backslash \bar{l}_2$*

In both cases l^+ denotes the open halfspace with respect to the line l so that the closed halfspace $l^+ \cup l$ contains the convex polygon $\mathrm{conv}(S)$.

Let s be one of the at most $4 \cdot |P|$ segments or rays (the blue-coloured ray in Figure 2.8) which border the regions of vertices belonging to separating wedges, compare Theorem 2.8.8. Let l_s denote the line which contains the segment or the ray s.

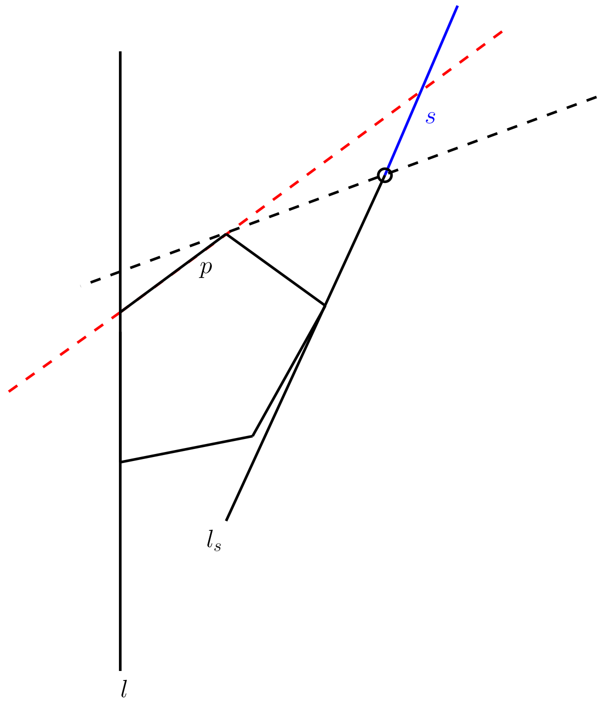

Figure 2.8: Partial segments and contact points

According to assertion 3. in Lemma 2.8.5 two lines each possibly containing an edge of one of the locally minimal separating triangles are fixed by the lines l and l_s. Thus, it remains to treat all possible locations of the third edge. Regarding the structure of the regions of wedge vertices, compare [75], criterion 2. in Lemma 2.8.5 can only be applied to those lines which pass through one of the end points of the segment or the ray s and its associated contact point with the convex polygon $\mathrm{conv}(S)$.
Thus, it remains to decide whether the contact point of the third edge with the convex polygon $\mathrm{conv}(S)$ is centred within the third edge. For that purpose let p be one of the possible contact points within the third edge and let l_p denote the line which is parallel to the line l and which satisfies $d(l, l_p) = 2 \cdot d(p, l) = 2 \cdot d(p, l_p)$. We have to decide whether the line l_p passes through the segment or ray s. Thus, using binary search techniques, a locally minimal triangle can be found in $O(\log |P|)$, if one exists at all for fixed lines l and l_s. The vertex of the wedge belongs to the boundary of one of the regions of wedge vertices. Thus, it can be approximated by a sequence of actually separating wedge vertices. Therefore, each calculated triangle can be approximated by triangles which actually separate the point set S with respect to the point set P.

The area of each locally minimal triangle is calculated in constant time within the cardinality $|P|$ of the point set P and helps us to determine one of the minimal separating triangles amongst all locally minimal ones. □

MimimalSeparatingTriangle(S, P)
Input: finite point set $\emptyset \neq P \subset \mathbb{R}^2$, subset $S \subset P$
Output: minimal/infimum separating triangle T

$T \leftarrow 0$
$area(T) \leftarrow \infty$
calculate conv(S) by Graham scan
$\mathcal{L} \leftarrow \{\text{lines } l \subset \mathbb{R}^2 | e \subset l \text{ for some edge } e \text{ of conv}(S)\}$
$\mathcal{L} \leftarrow \mathcal{L} \cup \{\text{lines } l \subset \mathbb{R}^2 | q, p_q \in l \text{ for } q \in P \backslash S \text{ and } p_q \in S$ one of the contact points with conv$(S)\}$
$\mathcal{L}^{sort} \leftarrow$ sortation of $\mathcal{L}$ according to Lemma 2.8.10
foreach $l \in \mathcal{L}$
  $\mathcal{M}_l \leftarrow (p_1, p_2, \ldots, p_{k_1}, \ldots, p_{k_2}, \ldots, p_{L_l})$ the sortation of the point set $P \cap l$
  $\bar{l}_1 \leftarrow \{p_1, \ldots, p_{k_1}\}$,
  $\bar{l}_2 \leftarrow \{p_{k_2}, \ldots, p_{L_l}\}$ according to (2.173)-(2.175)
  $l^+ \leftarrow$ open halfspace associated with the line l so that conv$(S) \subset (l \cup l^+)$
  for $i = 1$ **to** 2
    calculate the regions of vertices of wedges which separate S with respect to $P \cap (l^+ \cup l \backslash \bar{l}_i)$
    $\mathcal{S}_l^i \leftarrow \{$segments and rays of the regions$\}$
    foreach $s \in \mathcal{S}_l^i$
      $l_s \leftarrow$ the line which contains s
      $\bar{l}_s \leftarrow$ the line found by binary search in $\mathcal{L}^{sort}$ according to criterion 1. and 2. in Lemma 2.8.5
      $T_s \leftarrow$ the triangle given by the lines l, l_s and $\bar{l}_s$
      if $area(T_s) < area(T)$
        $T \leftarrow T_s$
        $area(T) \leftarrow area(T_s)$
return T

Remark 2.8.12 Separability problems within the Euclidean space $\mathbb{R}^3$ are considered in [76]. Amongst others, separability by a constant number of planes is discussed by the authors. However, up to now it is not clear whether it might be possible to extend the presented algorithm in order to answer the question of minimal prismatic or minimal tetrahedral separability. That would extend minimal triangular separability to dimension 3.

Chapter 3

Point X-rays and instability

Different scanning geometries are used in the field of computerized tomography. Besides the parallel scanning geometry also the cone beam scanning geometry is in application. With the attempt to transfer some more results of computerized tomography to the field of discrete tomography, let us examine the situation using point X-rays instead of parallel X-rays.

Definition 3.0.1 (undirected point X-ray)
Let $p \in \mathbb{Z}^2$ be a lattice point and let $F \subset \mathbb{Z}^2$ be a finite lattice set. The ***(undirected) point X-ray*** *of the lattice set F at the lattice point p is defined by*

$$X_pF(u) := |\{q \in F | q - p = \lambda \cdot u \text{ for } \lambda \in \mathbb{R}\}| \tag{3.1}$$

for any lattice direction $u = (r, s) \in \mathbb{Z}^2 \backslash \{0\}$ so that $\gcd(r, s) = 1$.

Remark 3.0.2 For general purpose in Section 3.5, the **(undirected) point X-ray** of some lattice set $F \subset \mathbb{Z}^n$ at some lattice point $p \in \mathbb{Z}^n$ is defined by

$$X_pF(u) := |\{q \in F | q - p = \lambda \cdot u \text{ for } \lambda \in \mathbb{R}\}| \tag{3.2}$$

for any lattice direction $u = (u_1, \ldots, u_n) \neq 0$ so that $\gcd(u_1, \ldots, u_n) = 1$.

It is shown in [45] that for any set of point X-ray sources there are tomographically equivalent and distinct lattice sets. Adapting some ideas used in [45] leads us also in the case of point X-rays to strong instability assertions analogous to [7]:

We construct arbitrarily large irreducible switching components to show strong instability results. By having Ryser's Theorem (see [115] and [73], Chapter 3) in mind, which states for the parallel case and two projection directions that tomographically equivalent lattice sets can be transformed into each other by elementary switching operations, we consider $m > n$ different X-ray sources to guarantee the irreducibility of the switching components, if n denotes the dimension of the lattice set $\mathbb{Z}^n$. Moreover, we guarantee that some pair of arbitrarily large and tomographically equivalent lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ overlap in at most 3 lattice points after applying any affine transformation resp. in at most $n+1$ lattice points for the generalized case that the lattice sets $F_1, F_2 \subset \mathbb{Z}^n$ are located within the lattice set $\mathbb{Z}^n$.

In order to exclude the case that the results are only based on perspective aspects, we also examine the projective besides the affine difference of the constructed lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ and guarantee that the lattice sets overlap in at most 4 lattice points after any projective transformation resp. in at most $n+2$ lattice points, if the lattice sets $F_1, F_2 \subset \mathbb{Z}^n$ are located within the lattice set $\mathbb{Z}^n$.

Furthermore, we will take a look at the consequences for the case that the point X-ray sources are located within the convex hull of the two lattice sets as well as the uniqueness of convex lattice sets according to point X-rays.

Finally, we turn back to the case of parallel X-rays. By applying the techniques which are used for the case of point X-rays and which are based on projective transformations as well as the knowledge of the solvability of equation systems, we show that some pair of arbitrarily large and tomographically equivalent lattice sets $F_1, F_2 \in \mathbb{Z}^2$ overlap in more than 4 lattice points after some affine transformation only in the case of translation $t(x) := x + b$ or in the case of half-around rotation $t(x) := -x + b$. In that case the upper bound of the number of overlaps is decreased compared with the results in [7], [5], [3], but still exponential within the number m of point X-ray sources.

Afterwards, we take a look at non-projective construction methods and reduce further attempts of strengthening the affine dissimilarity assertion to the search of switching components of minimal cardinality.

3.1 Basic construction

First of all we will concentrate on the basic construction, which is based on ideas in [45]. Some modifications will later be applied in order to show affine and perspective dissimilarity.

Theorem 3.1.1
Let $m \geq 3$ and suppose that $p_j = (p_{1j}, p_{2j}) \in \mathbb{Z}^2$, $j = 1, \ldots, m$ are distinct lattice points within the lattice set $\mathbb{Z}^2$. For any $\alpha \in \mathbb{N}$ there exist two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ satisfying

- *F_k for $k = 1, 2$ is uniquely determined by the point X-rays $X_{p_j} F_k$ for $j = 1, \ldots, m$,*
- *$F_1 \cap F_2 = \emptyset$,*
- *$|F_1| = |F_2| \geq \alpha$,*
- *$\sum_{i=1}^{m} |X_{p_i} F_1 - X_{p_i} F_2| = 2(m-1)$.*

Proof
Let the projective transformation φ be defined by

$$\varphi : \mathbb{P}^m \to \mathbb{P}^2, \tag{3.3}$$

$$\varphi(x_1, \ldots, x_{m+1}) := \tag{3.4}$$

$$(\sum_{i=1}^{m} \kappa^{i-1} b p_{1i} x_i + c_1 x_{m+1}, \sum_{i=1}^{m} \kappa^{i-1} b p_{2i} x_i + c_2 x_{m+1}, \sum_{i=1}^{m} \kappa^{i-1} b x_i + a x_{m+1}),$$

using homogeneous coordinates in both projective spaces $\mathbb{P}^m$ and $\mathbb{P}^2$. Its restriction $\varphi_{\mathbb{R}^m}$ as mapping between the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^2$ is given by

$$\varphi_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^2, \tag{3.5}$$

$$\varphi_{\mathbb{R}^m}(x) = \left(\frac{\sum_{i=1}^{m} \kappa^{i-1} b p_{1i} x_i + c_1}{\sum_{i=1}^{m} \kappa^{i-1} b x_i + a}, \frac{\sum_{i=1}^{m} \kappa^{i-1} b p_{2i} x_i + c_2}{\sum_{i=1}^{m} \kappa^{i-1} b x_i + a}\right). \tag{3.6}$$

The parameters

$$\kappa = \kappa(p_{11}, p_{12}, \ldots, p_{1m}, p_{21}, p_{22}, \ldots, p_{2m}) \in \mathbb{N} \backslash \{1\}, \tag{3.7}$$

$$a = a(\kappa), b = b(\kappa) \in \mathbb{N}, \tag{3.8}$$

$$c_1 = c_1(\kappa, a, b), c_2 = c_2(\kappa, a, b) \in \mathbb{Z} \tag{3.9}$$

will be suitably specified later. The jth vector e_j in the standard orthonormal basis of the Euclidean space $\mathbb{R}^{m+1}$ is mapped to $\varphi(e_j) = (\kappa^{j-1} b p_{1j}, \kappa^{j-1} b p_{2j}, \kappa^{j-1} b)$ for $j = 1, \ldots, m$, which shows that the projective transformation φ maps the jth coordinate direction within the Euclidean space $\mathbb{R}^m$ to the lattice *point p_j for $j = 1, \dots, m$. In the case that $\alpha < (l+1) \cdot 2^{m-2}$ let us recursively define the staircase-like switching component $G_1^{(m)} \cup G_2^{(m)} \subset \mathbb{Z}^m$ by*

$$G_1^{(2)} := \{(j, j, 0, \dots, 0) \in \mathbb{Z}^m | j = 0, \dots, l\}, \tag{3.10}$$

$$G_2^{(2)} := \{(j+1, j, 0, \dots, 0) \in \mathbb{Z}^m | j = 0, \dots, l-1\} \cup \{(0, l, 0 \dots, 0)\}, \tag{3.11}$$

$$G_1^{(j)} := G_1^{(j-1)} \cup (e_j + G_2^{(j-1)}) \text{ for } j = 3, \dots, m, \tag{3.12}$$

$$G_2^{(j)} := G_2^{(j-1)} \cup (e_j + G_1^{(j-1)}) \text{ for } j = 3, \dots, m. \tag{3.13}$$

The lattice sets F_1, F_2 are given by

$$F_1 := \varphi_{\mathbb{R}^m}(G_1^{(m)} \backslash \{0\}), \tag{3.14}$$

$$F_2 := \varphi_{\mathbb{R}^m}(G_2^{(m)} \backslash \{(0, l, 0 \dots, 0)\}) \tag{3.15}$$

for suitably large chosen parameter $\kappa > l$ and suitably chosen parameters a, b, c_1, c_2 with respect to the dependencies as given in (3.7)-(3.9).

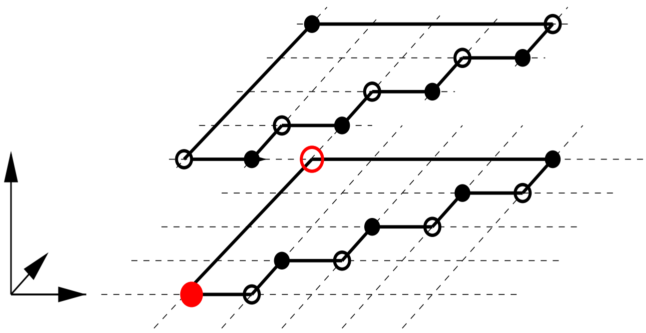

Figure 3.1: The lattice sets $G_1^{(3)}$ (filled circles) and $G_2^{(3)}$ (non-filled circles) for $l = 4$, the red-coloured lattice points indicate the eliminated ones for the lattice sets F_1, F_2

The following lemmata will show that the parameters κ, a, b, c_1, c_2 can be chosen so that

1. *$\varphi_{\mathbb{R}^m}(G_1^{(m)}) \cup \varphi_{\mathbb{R}^m}(G_2^{(m)}) \subset \mathbb{Z}^2$, i. e. integrality is guaranteed for the lattice sets F_1, F_2 (see Lemma 3.2.3),*

2. *$F_1 \cap F_2 \subset \varphi_{\mathbb{R}^m}(G_1^{(m)}) \cap \varphi_{\mathbb{R}^m}(G_2^{(m)}) = \emptyset$, i. e. the lattice sets F_1, F_2 are distinct (see Lemma 3.2.4),*

3. *$X_{p_j} F_k(u) \in \{0, 1\}$ for $j = 1, \dots, m$, $k = 1, 2$ and any lattice direction $u \in \mathbb{Z}^2 \backslash \{0\}$ (see Lemma 3.2.5),*

4. *$(\bigcap_{j=1}^m \bigcup_{(p_j,u) \in M_k} p_j + \mathbb{R} \cdot u) \cap \mathbb{Z}^2 = \varphi_{\mathbb{R}^m}(G_1^{(m)}) \cup \varphi_{\mathbb{R}^m}(G_2^{(m)})$ for $k = 1, 2$ and the set $M_k := \{(p, u) \in \{p_1, \dots, p_m\} \times \mathbb{Z}^2 \backslash \{0\} | X_p(\varphi_{\mathbb{R}^m}(G_k^{(m)}))(u) \neq 0\}$, i. e. the grid is given by $\varphi_{\mathbb{R}^m}(G_1^{(m)}) \cup \varphi_{\mathbb{R}^m}(G_2^{(m)}) = \varphi_{\mathbb{R}^m}(G_1^{(m)} \cup G_2^{(m)})$ (see Lemma 3.2.6).*

For that purpose we use polynomial theory in each lemma. To avoid equality for two different polynomials within the parameters, we only have to forbid some parameter combinations, which we will call combinations of ***forbidden parameters****. These elimination strategies can be suitably combined (see some later remarks within the proofs of the lemmata).*
Notice, that for any lattice points $x, y \in G_1^{(m)} \cup G_2^{(m)}$ *there is an alternating path* $(x_0 = x, x_1, \ldots, x_q = y)$ *according to the construction of the lattice sets* $G_1^{(m)}, G_2^{(m)}$ *so that* $x_i \in G_1^{(m)} \cup G_2^{(m)}$, $x_{i+1} - x_i \in \operatorname{span}(e_{j_i})$ *for some vector* e_{j_i} *in the standard orthonormal basis of the Euclidean space* $\mathbb{R}^m$ *and the consecutive lattice points* x_i, x_{i+1} *do not both belong to the lattice set* $G_1^{(m)}$ *resp. to the lattice set* $G_2^{(m)}$*. Therefore, the uniqueness of the lattice sets* F_1 *and* F_2 *is implied by 3. and 4. The assertion of Theorem 3.1.1 follows immediately.* □

Remark 3.1.2 The argument using the existence of an alternating path is necessary. For example in the case that $m = 3$ and that

$$G_1^{(3)} = \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\} \cup((5,5,0) + \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}), \quad (3.16)$$

$$G_2^{(3)} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\} \cup((5,5,0) + \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}) \quad (3.17)$$

as illustrated in Figure 3.2 the grid within the lattice set $\mathbb{Z}^3$ is given by the lattice set $G_1^{(3)} \cup G_2^{(3)} = \{0,1\}^3 \cup ((5,5,0) + \{0,1\}^3)$. The projection data in the directions e_1, e_2 and e_3 are given by the values 0 and 1. A suitable choice of the parameters for the projective transformation φ remains those properties. Nevertheless, the lattice sets $F_1 := \varphi_{\mathbb{R}^m}(G_1^{(3)} \backslash \{(0,0,0)\})$ and $F_2 := \varphi_{\mathbb{R}^m}(G_2^{(3)} \backslash \{(0,1,0)\})$ are not uniquely determined by their point X-rays.

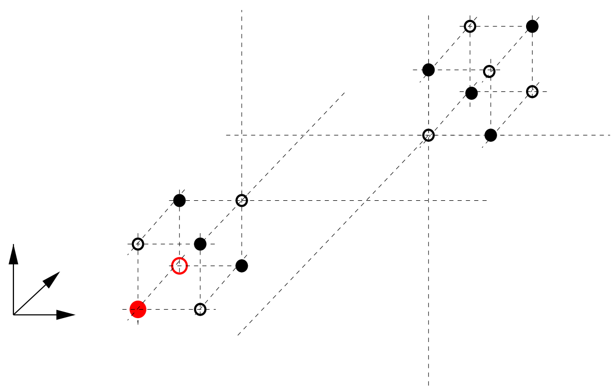

Figure 3.2: The lattice sets $G_1^{(3)}$ (filled circles) and $G_2^{(3)}$ (non-filled circles) within the counter-example, the red-coloured lattice points indicate the eliminated ones for the lattice sets F_1, F_2

3.2 Technical lemmata

The basic ideas to prove Theorem 3.1.1 are given before. Now we will work on the technical details.
The following lemma constitutes some later assumptions on the location of the point X-ray sources $p_1, \ldots, p_m$ by applying some translation on the lattice set $\mathbb{Z}^2$.

Lemma 3.2.1 *Any translation*

$$t : \mathbb{R}^2 \to \mathbb{R}^2, \quad (3.18)$$
$$t(x) := x + b \quad (3.19)$$

for $b \in \mathbb{Z}^2$ applied to the point X-ray sources $p_1, \ldots, p_m$ and to any finite lattice set $F \subset \mathbb{Z}^2$ results in

$$X_{t(p_j)} t(F) = X_{p_j} F \quad (3.20)$$

for $j = 1, \ldots, m$. Therefore, we can assume that

1. *$p_{ij} \neq 0$ for $i = 1, 2$ and $j = 1, \ldots, m$, in particular, that $p_j \neq 0$ for $j = 1, \ldots, m$,*
2. $\det \begin{pmatrix} p_{1i} & p_{1j} \\ p_{2i} & p_{2j} \end{pmatrix} = 0 \Leftrightarrow i = j$

by translation.

Proof
For any $x \in F$ and for $j = 1, \ldots, m$ we calculate that

$$t(x) - t(p_j) = \lambda \cdot u \text{ for } \lambda \in \mathbb{Z} \quad (3.21)$$
$$\Leftrightarrow (x + b) - (p_j + b) = x - p_j = \lambda \cdot u. \quad (3.22)$$

Therefore, we can assume 1. by translation.
For 2. we conclude that

$$\det \begin{pmatrix} p_{1i} & p_{1j} \\ p_{2i} + b & p_{2j} + b \end{pmatrix} = b \cdot \det \begin{pmatrix} p_{1i} & p_{1j} \\ 1 & 1 \end{pmatrix} + \det \begin{pmatrix} p_{1i} & p_{1j} \\ p_{2i} & p_{2j} \end{pmatrix} = 0 \quad (3.23)$$

is possible for all $b \in \mathbb{Z}$ if and only if $p_{1i} = p_{1j}$ in a first step and then also $p_{2i} = p_{2j}$. □
The next lemma helps us to make some more later used assumptions on the location of the point X-ray sources.

Lemma 3.2.2 *Any linear transformation*

$$t : \mathbb{R}^2 \to \mathbb{R}^2, \tag{3.24}$$
$$t(x) := Ax \tag{3.25}$$

for $A \in \mathbb{Z}^{2\times 2}$ *and* $\det(A) = \pm 1$ *applied to the point* X*-ray sources* $p_1, \ldots, p_m$ *and any lattice set* $F \subset \mathbb{Z}^2$ *results in*

$$X_{t(p_i)} t(F)(u) = X_{p_i} F(t^{-1}(u)) \tag{3.26}$$

for $i = 1, \ldots, m$ *and every lattice direction* $u \in \mathbb{Z}^2 \backslash \{0\}$. *Moreover, we get that* $t(\mathbb{Z}^2) = \mathbb{Z}^2$. *Therefore, the transformation* t *can be inverted on the lattice set* $\mathbb{Z}^2$.
Thus, we can assume that

$$p_i - p_j \notin \mathbb{Z} \cdot e_1 \cup \mathbb{Z} \cdot e_2, \tag{3.27}$$

for $i \neq j$.
Proof
For any $x \in F$ *we calculate that*

$$t(x) - t(p_i) = \lambda \cdot u \text{ for } \lambda \in \mathbb{Z} \tag{3.28}$$
$$\Leftrightarrow Ax - Ap_i = A(x - p_i) = \lambda \cdot u \Leftrightarrow x - p_i = \lambda \cdot A^{-1} \cdot u \tag{3.29}$$

for $i = 1, \ldots, m$ *and* $A^{-1} \cdot u \in \mathbb{Z}^2$, *as* $\det(A) = \pm 1$. *Therefore, we can assume without loss of generality that (3.27) is fulfilled by applying a linear transformation on the* X*-ray sources if necessary:*
Let $(1, s) \in \mathbb{Z}^2 \backslash \{(1, 0), (0, 1)\}$ *be some lattice direction so that* $p_i - p_j \notin \mathbb{R} \cdot (1, s)$ *for all* $i \neq j$. *There are infinitely many lattice directions* $(r', s') := (r', s \cdot r' + 1)$ *so that* $\gcd(r', s') = 1$, $p_i - p_j \notin \mathbb{R} \cdot (r', s')$ *and* $\det \begin{pmatrix} 1 & r' \\ s & s' \end{pmatrix} = 1$. □

The following lemma shows that suitably chosen parameter values lead to the fact that the projected lattice sets F_1, F_2 *actually lie within the lattice set* $\mathbb{Z}^2$. *There is even some degree of freedom left within the choice of the parameters* c_1, c_2.

Lemma 3.2.3 *Let the lattice sets* F_1, F_2 *be defined by* $F_1 := \varphi_{\mathbb{R}^m}(G_1^{(m)} \backslash \{0\})$, $F_2 := \varphi_{\mathbb{R}^m}(G_2^{(m)} \backslash \{(0, l, 0, \ldots, 0)\})$ *as in the proof of Theorem 3.1.1.*
For fixed parameter κ *we can find integer values* a, b *so that* $F_1, F_2 \subset \mathbb{Z}^2$. *The set of possible parameter combinations* (c_1, c_2) *with respect to fixed parameters* κ, a, b *has infinite cardinality.*
Proof
The ideas are similar to those in [45]. Notice, that

$$\{\sum_{i=1}^{m} \kappa^{i-1} x_i | x \in \{0, 1, \ldots, \kappa - 1\}^m\} = \{0, 1, \ldots, \kappa^m - 1\}. \tag{3.30}$$

By a result of Green and Tao (see [60]) we can choose $a,\ b \in \mathbb{N}$ *so that each member of the arithmetic progression*

$$\{a, a+b, \ldots, a+(\kappa^m-1)b\} \tag{3.31}$$

is a prime. Therefore, using the Chinese Remainder Theorem, we can find infinitely many pairs of parameters (c_1, c_2) *satisfying*

$$c_k \equiv -\sum_{i=1}^{m} \kappa^{i-1} b p_{ki} x_i \mod (\sum_{i=1}^{m} \kappa^{i-1} b x_i + a) \tag{3.32}$$

for any $x \in \{0, \ldots, \kappa-1\}^m$ *and* $k = 1, 2$, *in order to guarantee that the lattice sets* F_1 *and* F_2 *are subsets of the lattice set* $\mathbb{Z}^2$ *for* $\kappa > l$. □

The next lemma guarantees that the lattice sets F_1, F_2 *are distinct, if we only choose at least one of the parameters* $c_1,\ c_2$ *large enough in absolute value.*

Lemma 3.2.4 *Let the lattice sets* $F_1,\ F_2 \subset \mathbb{Z}^2$ *be defined as in the proof of Theorem 3.1.1 by* $F_1 := \varphi_{\mathbb{R}^m}(G_1^{(m)} \backslash \{0\})$, $F_2 := \varphi_{\mathbb{R}^m}(G_2^{(m)} \backslash \{(0, l, 0, \ldots, 0)\})$. *Let the parameters* κ, a, b *be fixed and let the parameters* c_1, c_2 *be given up to modularity with respect to Lemma 3.2.3. The parameters* c_1,c_2 *can be chosen so that* $F_1 \cap F_2 = \emptyset$.

Proof

With similar arguments as in [45] we conclude for $x^1, x^2 \in \{0, \ldots, \kappa-1\}^m$, $x_1 \neq x_2$ *that*

$$\varphi_{\mathbb{R}^m}(x^1) = \varphi_{\mathbb{R}^m}(x^2) \tag{3.33}$$

$$\Leftrightarrow (\sum_{i=1}^{m} \kappa^{i-1} b x_i^1 + a)(\sum_{i=1}^{m} \kappa^{i-1} b p_{ki} x_i^2 + c_k) = \tag{3.34}$$

$$(\sum_{i=1}^{m} \kappa^{i-1} b x_i^2 + a)(\sum_{i=1}^{m} \kappa^{i-1} b p_{ki} x_i^1 + c_k) \text{ for } k = 1, 2 \tag{3.35}$$

$$\Rightarrow |c_k| \leq |(\sum_{i=1}^{m} \kappa^{i-1} b x_i^1 + a)(\sum_{i=1}^{m} \kappa^{i-1} b p_{ki} x_i^2) -$$

$$(\sum_{i=1}^{m} \kappa^{i-1} b x_i^2 + a)(\sum_{i=1}^{m} \kappa^{i-1} b p_{ki} x_i^1)| \tag{3.36}$$

$$\leq 2(a + (\kappa^m - 1)b)(\kappa^m - 1) b \max |p_{ij}| \text{ for } k = 1, 2. \tag{3.37}$$

Therefore, we result in contradiction by choosing one of the parameters c_1, c_2 *large enough in absolute value.* □

The following lemma assures that the point X*-ray data of the lattice sets* F_1, F_2 *do not extend the value 1 for suitably chosen parameters* κ *and* c_1, c_2.

Lemma 3.2.5 *Let the lattice sets* F_1, $F_2 \subset \mathbb{Z}^2$ *be defined as in the proof of Theorem 3.1.1 by* $F_1 := \varphi_{\mathbb{R}^m}(G_1^{(m)} \backslash \{0\})$, $F_2 := \varphi_{\mathbb{R}^m}(G_2^{(m)} \backslash \{(0, l, 0, \ldots, 0)\})$. *For suitably large chosen parameter* $\kappa > l$ *within the projective transformation (3.3)-(3.4) we can find a pair of parameters* (c_1, c_2) *so that*

$$X_{p_j} F_k(u) \in \{0, 1\} \tag{3.38}$$

for $j = 1, \ldots, m$, $k = 1, 2$ *and any lattice direction* $u \in \mathbb{Z}^2 \backslash \{0\}$.

Proof

Notice, that because of Lemma 3.2.3 there are infinitely many possible pairs of parameters (c_1, c_2) *for fixed values* κ, a, b.

The lattice points $p_j, \varphi_{\mathbb{R}^m}(x^1), \varphi_{\mathbb{R}^m}(x^2)$ *are collinear for the lattice points* $x^1, x^2 \in G_1^{(m)} \cup G_2^{(m)}$ *if and only if* $\varphi_{\mathbb{R}^m}(x^1) - p_j$, $\varphi_{\mathbb{R}^m}(x^2) - p_j$ *are linearly dependent, which is again equivalent to*

$$\det \begin{pmatrix} \sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1j}) x_i^1 + \bar{c}_1 & \sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1j}) x_i^2 + \bar{c}_1 \\ \sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2j}) x_i^1 + \bar{c}_2 & \sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2j}) x_i^2 + \bar{c}_2 \end{pmatrix} = \tag{3.39}$$

$$= \bar{c}_1 (\sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2j}) x_i^2 - \sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2j}) x_i^1) + \tag{3.40}$$

$$+ \bar{c}_2 (\sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1j}) x_i^1 - \sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1j}) x_i^2)$$

$$+ \textit{ terms of lower degree } = 0 \tag{3.41}$$

for $\bar{c}_1 := c_1 - a \cdot p_{1j}$, $\bar{c}_2 := c_2 - a \cdot p_{2j}$. *If*

$$\sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2j}) x_i^2 - \sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2j}) x_i^1 \neq 0 \tag{3.42}$$

and if the parameter c_2 *is fixed, there is exactly one solution for the parameter* $\bar{c}_1$ *resp. for the parameter* c_1 *within the infinite set of possibilities. The situation is similar in the case that*

$$\sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1j}) x_i^1 - \sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1j}) x_i^2 \neq 0 \tag{3.43}$$

and that the roles of the parameters c_1, c_2 *are changed.*

There are still infinitely many possibilities for the tuple (c_1, c_2), *even if we eliminate all those pairs of the parameters* c_1, c_2 *according to all possible combinations of the point X-ray source* p_j *and the lattice points* $\varphi_{\mathbb{R}^m}(x^1)$, $\varphi_{\mathbb{R}^m}(x^2)$ *for some lattice points* $x^1, x^2 \in G_1^{(m)} \cup G_2^{(m)}$, *as the elimination corresponds to the elimination of a finite number of lattice lines within the two-dimensional lattice set* $\{(c_1, c_2) | c_1, c_2$ *permissible according to Lemma 3.2.3*$\}$, *if (3.42) or (3.43) is*

satisfied.
Thus, we have to exclude the case that both

$$\sum_{i=1}^{m} \kappa^{i-1} b(p_{2i} - p_{2j}) x_i^2 - \sum_{i=1}^{m} \kappa^{i-1} b(p_{2i} - p_{2j}) x_i^1 = 0 \tag{3.44}$$

and

$$\sum_{i=1}^{m} \kappa^{i-1} b(p_{1i} - p_{1j}) x_i^1 - \sum_{i=1}^{m} \kappa^{i-1} b(p_{1i} - p_{1j}) x_i^2 = 0. \tag{3.45}$$

The parameter κ can be assumed to be chosen sufficiently large, i.e.

$$\kappa > 4 \cdot m \cdot l \cdot \max(|p_{ij}|). \tag{3.46}$$

Therefore, we state that

$$(p_i - p_j) x_i^1 = (p_i - p_j) x_i^2 \textit{ for } i = 1, \dots, m \tag{3.47}$$

has to be necessarily satisfied, as it iteratively yields for $k = m-1, m-2, \dots, 1$ that

$$\begin{aligned}
&\max_{s=1,2} \Big| \sum_{i=1}^{k} \kappa^{i-1} (p_{si} - p_{sj}) x_i^1 - \sum_{i=1}^{k} \kappa^{i-1} (p_{si} - p_{sj}) x_i^2 \Big| \\
&\leq 4 \cdot k \cdot l \cdot \kappa^{k-1} \cdot \max |p_{ij}| && (3.48)\\
&\leq 4 \cdot m \cdot l \cdot \kappa^{k-1} \cdot \max |p_{ij}| && (3.49)\\
&< \kappa^k \leq \max_{s=1,2} |\kappa^k (p_{s(k+1)} - p_{sj}) x_{k+1}^1 - \kappa^k (p_{s(k+1)} - p_{sj}) x_{k+1}^2|
\end{aligned}$$

for $x_{k+1}^1 \neq x_{k+1}^2$ and $p_{k+1} \neq p_j$.
As $p_i \neq p_j$ for $i \neq j$, the case $x_i^1 \neq x_i^2$ within the equation system (3.47) is only possible for $i = j$. Therefore, because of the definition of the lattice sets $G_1^{(m)}$ and $G_2^{(m)}$ we conclude that either $x^1 = x^2$ or the lattice points x^1, x^2 do not both belong to the lattice set $G_1^{(m)}$ resp. to the lattice set $G_2^{(m)}$, as different elements within the lattice set $G_1^{(m)}$ resp. the lattice set $G_2^{(m)}$ differ in at least two components. □

Now we will see that the lattice set $\varphi_{\mathbb{R}^m}(G_1) \cup \varphi_{\mathbb{R}^m}(G_2)$ is closed with respect to the calculation of its grid.

Lemma 3.2.6 *Let the lattice sets $G_1^{(m)}$, $G_2^{(m)} \subset \mathbb{Z}^m$ be defined as in the proof of Theorem 3.1.1.*
For suitably large chosen parameter κ within the projective transformation (3.3)-(3.4) we can find a pair of parameters (c_1, c_2) in dependence on suitable parameters a, b so that the grid is given by

$$(\bigcap_{j=1}^{m} \bigcup_{(p_j,u) \in M_k} p_j + \mathbb{R} \cdot u) \cap \mathbb{Z}^2 = \varphi_{\mathbb{R}^m}(G_1^{(m)}) \cup \varphi_{\mathbb{R}^m}(G_2^{(m)}) \tag{3.50}$$

for $k = 1, 2$ *and the set*

$$M_k := \{(p, u) \in \{p_1, \ldots, p_m\} \times \mathbb{Z}^2 \backslash \{0\} | X_p(\varphi_{\mathbb{R}^m}(G_k^{(m)}))(u) \neq 0\}. \quad (3.51)$$

Proof

Let us assume that there is some lattice point $s = (s_1, s_2)$ *within the grid*

$$(\bigcap_{j=1}^{m} \bigcup_{(p_j,u) \in M_1} p_j + \mathbb{R} \cdot u) \cap \mathbb{Z}^2 = (\bigcap_{j=1}^{m} \bigcup_{(p_j,u) \in M_2} p_j + \mathbb{R} \cdot u) \cap \mathbb{Z}^2 \quad (3.52)$$

which does not belong to the lattice set $\varphi_{\mathbb{R}^m}(G_1^{(m)} \cup G_2^{(m)})$*. Then the lattice point* s *is the intersection point of* m *lines passing through the lattice points* $p_k, \varphi_{\mathbb{R}^m}(x^k)$ *for* $k = 1, \ldots, m$ *and some lattice point* $x^k \in G_1^{(m)}$*. The lattice point* s *lies on the lines passing through the lattice points* $p_k, \varphi_{\mathbb{R}^m}(x^k)$ *for* $k = 1, 2, 3$ *(and similar for any index set* $\{i_1, i_2, i_3\} \subset \{1, \ldots, m\}$ *of cardinality* 3*) if and only if*

$$\det(s - p_k, \varphi_{\mathbb{R}^m}(x^k) - p_k) = 0 \text{ for } k = 1, 2, 3 \quad (3.53)$$

$$\Leftrightarrow (s_1 - p_{1k})A_{2k} - (s_2 - p_{2k})A_{1k} = 0 \text{ for } k = 1, 2, 3 \quad (3.54)$$

$$\Leftrightarrow s_1 A_{2k} - s_2 A_{1k} = B_k \text{ for } k = 1, 2, 3 \quad (3.55)$$

for A_{lk}, B_k *defined by*

$$A_{lk} := [\sum_{i=1}^{m} \kappa^{i-1} b(p_{li} - p_{lk}) x_i^k + (c_l - a p_{lk})], \quad (3.56)$$

$$B_k := p_{1k} A_{2k} - p_{2k} A_{1k}. \quad (3.57)$$

Let us look at the case that the lines passing through the lattice points $p_1, \varphi_{\mathbb{R}^m}(x^1)$ *resp. through the lattice points* p_2*,* $\varphi_{\mathbb{R}^m}(x^2)$ *are identical (parallelism does not matter, as in that case the intersection point* s *does not exist at all), which occurs if and only if* $x^1 = x^2$ *by using the result of Lemma 3.2.5 for the parameter* κ *chosen sufficiently large. The vectors* $\varphi_{\mathbb{R}^m}(x^1) - p_1$*,* $\varphi_{\mathbb{R}^m}(x^1) - p_2$ *are linearly dependent if and only if*

$$\det \begin{pmatrix} q^1 - p_{11}q & q^1 - p_{12}q \\ q^2 - p_{21}q & q^2 - p_{22}q \end{pmatrix} =$$

$$= c_1 q(p_{21} - p_{22}) + c_2 q(p_{12} - p_{11}) + \cdots = 0 \quad (3.58)$$

for

$$q^k := \sum_{i=1}^{m} \kappa^{i-1} b p_{ki} x_i^1 + c_k, \quad (3.59)$$

$$q := \sum_{i=1}^{m} \kappa^{i-1} b x_i^1 + a \neq 0. \quad (3.60)$$

As $p_1 \neq p_2$ we can assume by again eliminating forbidden parameter tuples (c_1, c_2) (compare the proof of Lemma 3.2.5) that the case of identical lines does not occur. Thus, the rank of the linear equation system (3.55) is at least 2. (Notice for general purpose, that the elimination of the forbidden parameter combinations (c_1, c_2) in Lemma 3.2.5 and the eliminations here as well as later can suitably be combined.) Therefore, the linear equation system (3.55) can be solved if and only if

$$\det\begin{pmatrix} A_{21} & A_{11} & B_1 \\ A_{22} & A_{12} & B_2 \\ A_{23} & A_{13} & B_3 \end{pmatrix} = \det\begin{pmatrix} A_{21} & A_{11} & B_1 \\ A_{22}-A_{21} & A_{12}-A_{11} & B_2-B_1 \\ A_{23}-A_{21} & A_{13}-A_{11} & B_3-B_1 \end{pmatrix} = \tag{3.61}$$

$$\begin{aligned} &= c_2^2[(p_{13}-p_{11})(A_{12}-A_{11})-(p_{12}-p_{11})(A_{13}-A_{11})]+ \\ &\quad +c_1^2[(p_{23}-p_{21})(A_{22}-A_{21})-(p_{22}-p_{21})(A_{23}-A_{21})]+ \\ &\quad + \text{ terms of lower degree } = \end{aligned} \tag{3.62}$$

$$\begin{aligned} &= c_2^2[(p_{13}-p_{11})(\sum_{i=1}^m \kappa^{i-1}b(p_{1i}-p_{12})x_i^2 - \sum_{i=1}^m \kappa^{i-1}b(p_{1i}-p_{11})x_i^1)- \\ &\quad -(p_{12}-p_{11})(\sum_{i=1}^m \kappa^{i-1}b(p_{1i}-p_{13})x_i^3 - \sum_{i=1}^m \kappa^{i-1}b(p_{1i}-p_{11})x_i^1)]+ \\ &+c_1^2[(p_{23}-p_{21})(\sum_{i=1}^m \kappa^{i-1}b(p_{2i}-p_{22})x_i^2 - \sum_{i=1}^m \kappa^{i-1}b(p_{2i}-p_{21})x_i^1) \\ &\quad -(p_{22}-p_{21})(\sum_{i=1}^m \kappa^{i-1}b(p_{2i}-p_{23})x_i^3 - \sum_{i=1}^m \kappa^{i-1}b(p_{2i}-p_{21})x_i^1)]+ \\ &\quad + \text{ terms of lower degree } = 0. \end{aligned} \tag{3.63}$$

We have to check in which cases it is possible that both the coefficient of c_1^2 and the coefficient of c_2^2 are equal to 0, as otherwise we simply reduce the set of admissible parameter combinations (c_1, c_2) again. Similar as in the proof of Lemma 3.2.5 we reduce to equality for each coefficient of κ^i by assuming that the parameter κ is chosen sufficiently large by

$$\kappa > 16 \cdot l \cdot m \cdot (\max |p_{ij}|)^2. \tag{3.64}$$

Notice, that if $x_i^1 = x_i^2 = x_i^3$ for any $i = 1, 2, \ldots, m$, the coefficient of κ^{i-1} equals 0. Thus, considering the case $i > 3$ we reduce our examinations without loss of generality to the case that $x_i^1 = 1$ and $x_i^2 = x_i^3 = 0$, as $x_i^1, x_i^2, x_i^3 \in \{0, 1\}$. The equation system reduces to

$$(p_{k2} - p_{k3})(p_{ki} - p_{k1}) = 0 \text{ for } k = 1, 2, \tag{3.65}$$

but the case that $p_{ki} - p_{kj} = 0$ for any $k \in \{1, 2\}$ and $i \neq j$ is excluded by Lemma 3.2.2.

In the case that $i \in \{1,2,3\}$ *the value of* x_i^i *can be arbitrarily chosen, as the coefficient of* x_i^i *is equal to zero. Thus, it remains to consider the case that*

$$x_i^j \begin{cases} \neq 0 \text{ for exactly one } j \neq i \\ = 0 \text{ otherwise (in particular, } x_i^i = 0) \end{cases}, \tag{3.66}$$

using the arguments of equality for $x_i^1 = x_i^2 = x_i^3$ *as before. We analogously result in contradiction. Thus,* $x_i^j \neq 0$ *is only possible for* $i = j$.
Therefore, we deduce that

$$x^k = x^0 + \lambda_k e_k \text{ for } k = 1,2,3 \tag{3.67}$$

for some lattice point $x^0 \in G_2^{(m)}$ *because of the definition of the lattice sets* $G_1^{(m)}$ *and* $G_2^{(m)}$. *By Lemma 3.2.5 the lines given by the lattice points* $p_k, \varphi_{\mathbb{R}^m}(x^k)$ *have at most one common intersection point, as (3.46) is satisfied by (3.64) and as the parameter tuple* (c_1, c_2) *can be chosen besides the eliminated pairs of parameters in Lemma 3.2.5. Thus, we get that* $s = \varphi_{\mathbb{R}^m}(x^0) \in \varphi_{\mathbb{R}^m}(G_1^{(m)} \cup G_2^{(m)})$ *in contradiction to our assumption at the beginning.* □

The following theorem will show that already the basic construction as given in Theorem 3.1.1 leads to some affine dissimilarity assertion.

Theorem 3.2.7
Let the lattice sets F_1, $F_2 \subset \mathbb{Z}^2$ *be defined as in the proof of Theorem 3.1.1 by* $F_1 := \varphi_{\mathbb{R}^m}(G_1^{(m)} \backslash \{0\})$, $F_2 := \varphi_{\mathbb{R}^m}(G_2^{(m)} \backslash \{(0,l,0,\dots,0)\})$.
Let the parameters κ, a, b *be fixed and let the parameters* c_1, c_2 *be given up to modularity with respect to Lemma 3.2.3. The parameters* c_1, c_2 *can be chosen so that*

$$|F_1 \cap t(F_2)| \leq |\varphi_{\mathbb{R}^m}(G_1^{(m)}) \cap t(\varphi_{\mathbb{R}^m}(G_2^{(m)}))| \leq 2 \tag{3.68}$$

for any affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$t(x) := \begin{pmatrix} A & -B \\ B & A \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, A^2 + B^2 \neq 0 \tag{3.69}$$

resp. by

$$t(x) := \begin{pmatrix} A & B \\ B & -A \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, A^2 + B^2 \neq 0. \tag{3.70}$$

Proof
We will only consider the case that the affine transformation t *is given by (3.69), as the arguments for the affine transformation in (3.70) are similar. Let us assume that the lattice point* $\varphi_{\mathbb{R}^m}(x^k)$ *is mapped to the lattice point* $\varphi_{\mathbb{R}^m}(x^{k+1})$

for $k = 1, 3, 5$ and $x^1, x^3, x^5 \in G_1^{(m)}$, $x^2, x^4, x^6 \in G_2^{(m)}$ pairwise distinct lattice points. Necessarily, the equation system

$$\begin{cases} t(\varphi_{\mathbb{R}^m}(x^1)) - \varphi_{\mathbb{R}^m}(x^2) = t(\varphi_{\mathbb{R}^m}(x^3)) - \varphi_{\mathbb{R}^m}(x^4) \\ t(\varphi_{\mathbb{R}^m}(x^1)) - \varphi_{\mathbb{R}^m}(x^2) = t(\varphi_{\mathbb{R}^m}(x^5)) - \varphi_{\mathbb{R}^m}(x^6) \end{cases} \iff \tag{3.71}$$

$$\begin{cases} A \cdot [\frac{q_1^1}{q_1} - \frac{q_3^1}{q_3} + c_1(\frac{1}{q_1} - \frac{1}{q_3})] - B \cdot [\frac{q_1^2}{q_1} - \frac{q_3^2}{q_3} + c_2(\frac{1}{q_1} - \frac{1}{q_3})] = \\ \qquad = c_1(\frac{1}{q_2} - \frac{1}{q_4}) + \frac{q_2^1}{q_2} - \frac{q_4^1}{q_4} \\ A \cdot [\frac{q_1^2}{q_1} - \frac{q_3^2}{q_3} + c_2(\frac{1}{q_1} - \frac{1}{q_3})] + B \cdot [\frac{q_1^1}{q_1} - \frac{q_3^1}{q_3} + c_1(\frac{1}{q_1} - \frac{1}{q_3})] = \\ \qquad = c_2(\frac{1}{q_2} - \frac{1}{q_4}) + \frac{q_2^2}{q_2} - \frac{q_4^2}{q_4} \\ A \cdot [\frac{q_1^1}{q_1} - \frac{q_5^1}{q_5} + c_1(\frac{1}{q_1} - \frac{1}{q_5})] - B \cdot [\frac{q_1^2}{q_1} - \frac{q_5^2}{q_5} + c_2(\frac{1}{q_1} - \frac{1}{q_5})] = \\ \qquad = c_1(\frac{1}{q_2} - \frac{1}{q_6}) + \frac{q_2^1}{q_2} - \frac{q_6^1}{q_6} \\ A \cdot [\frac{q_1^2}{q_1} - \frac{q_5^2}{q_5} + c_2(\frac{1}{q_1} - \frac{1}{q_5})] + B \cdot [\frac{q_1^1}{q_1} - \frac{q_5^1}{q_5} + c_1(\frac{1}{q_1} - \frac{1}{q_5})] = \\ \qquad = c_2(\frac{1}{q_2} - \frac{1}{q_6}) + \frac{q_2^2}{q_2} - \frac{q_6^2}{q_6} \end{cases} \tag{3.72}$$

for q_k, q_k^l defined by

$$q_k := \sum_{i=1}^{m} \kappa^{i-1} b x_i^k + a \text{ prime}, \tag{3.73}$$

$$q_k^l := \sum_{i=1}^{m} \kappa^{i-1} b p_{li} x_i^k \tag{3.74}$$

has to be solved for the variables A, B.
Let us consider the first two equations for the variables A, B. As the determinant of the coefficient matrix is given by

$$(c_1^2 + c_2^2)(\frac{1}{q_1} - \frac{1}{q_3})^2 + \text{ terms of lower degree} \tag{3.75}$$

as polynomial within the parameters c_1, c_2 and as q_1, q_3 are different primes, we can assume in general (by a suitable choice for the parameter tuple (c_1, c_2)) that the rank of the complete equation system (3.72) is at least 2. Let us look at the extended coefficient matrix with respect to the first three equations. Its determinant is calculated by

$$\begin{aligned} c_1^3(\frac{1}{q_1} - \frac{1}{q_3})[(\frac{1}{q_1} - \frac{1}{q_3})(\frac{1}{q_2} - \frac{1}{q_6}) - (\frac{1}{q_1} - \frac{1}{q_5})(\frac{1}{q_2} - \frac{1}{q_4})] + \\ + \text{ lower terms in } c_1. \end{aligned} \tag{3.76}$$

The coefficient of c_1^3 is equal to zero if and only if

$$q_4 q_5 (q_3 - q_1)(q_6 - q_2) = q_3 q_6 (q_5 - q_1)(q_4 - q_2). \tag{3.77}$$

Without loss of generality we can assume that $q_6 > q_i$ for $i = 1, \dots, 5$, as otherwise we use the same suitable permutation for both the triple $(1, 3, 5)$ and the triple $(2, 4, 6)$ of indices within the equation system (3.71). Therefore, condition (3.77) cannot be satisfied for the primes $q_1, \dots, q_6$, as the prime q_6 does not divide any of the left hand side factors. □

3.3 Affine, projective and perspective dissimilarity

Now we will treat affine and projective aspects, in order to show both affine and perspective dissimilarity for instable lattice sets.

Definition 3.3.1 (perspective transformation)
A projective transformation $\rho : \mathbb{P}^2 \to \mathbb{P}^2$, $[x] \mapsto [Ax]$ is called ***perspective****, if there are two planes $\tilde{\pi}$ and π within the Euclidean space $\mathbb{R}^3$ so that the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$ maps the plane $\tilde{\pi}$ rigidly on the plane π, which means that any two points separated by a distance d in the plane π are mapped to points separated by the same distance d in the plane $\tilde{\pi}$.*

Remark 3.3.2 Notice, that not every projective transformation has to be perspective, but every projective transformation can be expressed as the composite of at most three perspective transformations, see [24].
The following considerations of projective dissimilarity aspects are motivated by the case of perspectivity. In the practical background of discrete tomography in semiconductor industry perspectivity aspects can be interpreted by tilting the template, which is shot by a point source in the production process. Furthermore, by considering projective transformations we also want to exclude that the large affine dissimilarity is only based on the different geometries which underlie the projective mapping within the construction of the lattice sets and the affine transformations.
For further purpose let us redefine the lattice sets

$$G_1^{(2)} := \{(j, j^2, 0, \dots, 0) | j = 0, 1, \dots, l\}, \tag{3.78}$$

$$G_2^{(2)} := \{(j+1, j^2, 0, \dots, 0) | j = 0, 1, \dots, l-1\} \cup \{(0, l^2, 0, \dots, 0)\} \tag{3.79}$$

and the lattice sets $G_1^{(j)}, G_2^{(j)}$ for $j = 3, \dots, m$ iteratively as before.

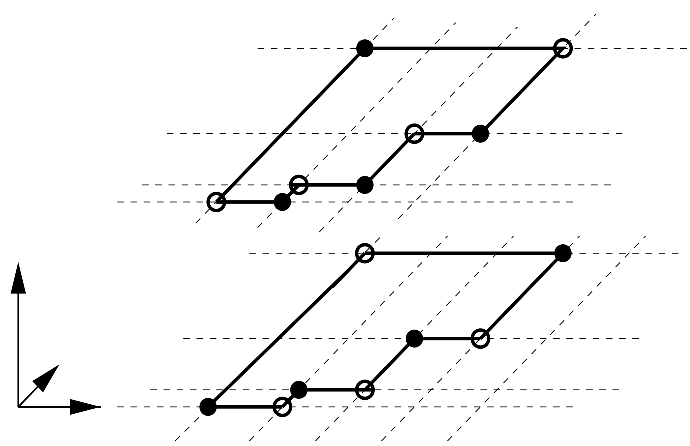

Figure 3.3: The lattice sets $G_1^{(3)}$ (filled circles) and $G_2^{(3)}$ (non-filled circles) for $l = 3$

The following two lemmata discuss collinearity within both the Euclidean space $\mathbb{R}^2$ and the projective space $\mathbb{P}^2$.

Lemma 3.3.3 *Let the lattice sets $G_1^{(m)}, G_2^{(m)} \subset \mathbb{Z}^m$ be defined with respect to the redefinition (3.78)-(3.79) and let φ be the projective transformation as given by (3.3)-(3.4).*
Let $t : \mathbb{R}^2 \to \mathbb{R}^2$ be any affine transformation which maps the lattice point $\varphi_{\mathbb{R}^m}(x^k)$ to the lattice point $\varphi_{\mathbb{R}^m}(x^{k+1})$ for $k = 1, 3, 5$ and for pairwise distinct lattice points x^1, x^3, $x^5 \in G_1^{(m)}$ and x^2, x^4, $x^6 \in G_2^{(m)}$. Then the lattice points $\varphi_{\mathbb{R}^m}(x^1)$, $\varphi_{\mathbb{R}^m}(x^3)$ and $\varphi_{\mathbb{R}^m}(x^5)$ are in general noncollinear by elimination of forbidden parameter tuples (c_1, c_2).

Proof
In the case that the lattice points $\varphi_{\mathbb{R}^m}(x^1)$, $\varphi_{\mathbb{R}^m}(x^3)$ and $\varphi_{\mathbb{R}^m}(x^5)$ are collinear for $x^1, x^3, x^5 \in G_1^{(m)}$, i. e.

$$\varphi_{\mathbb{R}^m}(x^1) + \lambda(\varphi_{\mathbb{R}^m}(x^3) - \varphi_{\mathbb{R}^m}(x^1)) = \varphi_{\mathbb{R}^m}(x^5) \text{ for some } \lambda \in \mathbb{R}, \tag{3.80}$$

we calculate $\lambda = \lambda(c_1)$ by the first line of the equation system (3.80) in dependence on the parameter c_1 and $\lambda = \lambda(c_2)$ by the second line in dependence on the parameter c_2. But the value λ cannot really depend on the parameters c_1 and c_2 because of elimination arguments as before. Thus, as the affine transformation t maps the lattice point $\varphi_{\mathbb{R}^m}(x^k)$ to the lattice point $\varphi_{\mathbb{R}^m}(x^{k+1})$ for $k = 1, 3, 5$, we result in the necessary condition that

$$\frac{1}{q_{1+l}} + \lambda\left(\frac{1}{q_{3+l}} - \frac{1}{q_{1+l}}\right) = \frac{1}{q_{5+l}} \text{ for } l = 0, 1$$

$$\text{and } q_k := \sum_{i=1}^{m} \kappa^{i-1} b x_i^k + a \text{ prime} \tag{3.81}$$

$$\Leftrightarrow \lambda = \frac{\frac{1}{q_5} - \frac{1}{q_1}}{\frac{1}{q_3} - \frac{1}{q_1}} = \frac{\frac{1}{q_6} - \frac{1}{q_2}}{\frac{1}{q_4} - \frac{1}{q_2}} \Leftrightarrow (3.77). \tag{3.82}$$

Therefore, similar arguments as in the proof of Lemma 3.2.7 lead to contradiction. Thus, the lattice points $\varphi_{\mathbb{R}^m}(x^1)$, $\varphi_{\mathbb{R}^m}(x^3)$ and $\varphi_{\mathbb{R}^m}(x^5)$ are in general noncollinear. □

Lemma 3.3.4 *Let the lattice sets $G_1^{(m)}$, $G_2^{(m)} \subset \mathbb{Z}^m$ be defined with respect to the redefinition (3.78)-(3.79) and let φ be the projective transformation as given by (3.3)-(3.4).*
Let $x^1, x^2, x^3 \in G_k^{(m)}$ be pairwise distinct lattice points for some $k \in \{1, 2\}$. The projective points $\varphi(x^1)$, $\varphi(x^2)$, $\varphi(x^3) \in \mathbb{P}^2$ are in general noncollinear by elimination of forbidden parameter tuples (c_1, c_2).

Proof
Let us assume that the projective points $\varphi(x^1), \varphi(x^2), \varphi(x^3) \in \mathbb{P}^2$ are collinear,

which is the case if and only if the vectors $\varphi(x^1), \varphi(x^2), \varphi(x^3) \in \mathbb{R}^3$ *are linearly dependent, i.e.*

$$\det\begin{pmatrix} q_1^1+c_1 & q_2^1+c_1 & q_3^1+c_1 \\ q_1^2+c_2 & q_2^2+c_2 & q_3^2+c_2 \\ q_1+a & q_2+a & q_3+a \end{pmatrix} = \det\begin{pmatrix} q_1^1+c_1 & q_2^1-q_1^1 & q_3^1-q_1^1 \\ q_1^2+c_2 & q_2^2-q_1^2 & q_3^2-q_1^2 \\ q_1+a & q_2-q_1 & q_3-q_1 \end{pmatrix}$$

$$= c_1 \cdot \det\begin{pmatrix} q_2^2-q_1^2 & q_3^2-q_1^2 \\ q_2-q_1 & q_3-q_1 \end{pmatrix} - c_2 \cdot \det\begin{pmatrix} q_2^1-q_1^1 & q_3^1-q_1^1 \\ q_2-q_1 & q_3-q_1 \end{pmatrix} +$$

$$+a \cdot \det\begin{pmatrix} q_2^1-q_1^1 & q_3^1-q_1^1 \\ q_2^2-q_1^2 & q_3^2-q_1^2 \end{pmatrix} + \textit{ terms of lower degree} = 0 \quad (3.83)$$

for

$$q_k^l := \sum_{i=1}^m \kappa^{i-1} b p_{li} x_i^k, \quad (3.84)$$

$$q_k := \sum_{i=1}^m \kappa^{i-1} b x_i^k. \quad (3.85)$$

Because of the same arguments as before we have to examine the case that the coefficients of both monomials c_1 *and* c_2 *are equal to* 0. *Let the values* i_{max}, j_{max} *be defined by*

$$i_{max} := \max\{i | (x_i^2 - x_i^1) \neq 0\}, \quad (3.86)$$

$$j_{max} := \max\{j | (x_j^3 - x_j^1) \neq 0\}. \quad (3.87)$$

By permuting the index set $\{1,2,3\}$ *within (3.83) if necessary, we can assume that* $i_{max} \neq j_{max}$ *if* $\max(i_{max}, j_{max}) \geq 3$, *as* $x_i^k \in \{0,1\}$ *for* $i \geq 3$ *and* $k = 1,2,3$. *Thus, we calculate that*

$$\det\begin{pmatrix} q_2^l-q_1^l & q_3^l-q_1^l \\ q_2-q_1 & q_3-q_1 \end{pmatrix} = \quad (3.88)$$

$$= \kappa^{i_{max}+j_{max}-2} b^2 (x_{i_{max}}^2 - x_{i_{max}}^1)(x_{j_{max}}^3 - x_{j_{max}}^1)(p_{li_{max}} - p_{lj_{max}}) + \textit{terms of lower degree in } \kappa. \quad (3.89)$$

Therefore, we result in $p_{li_{max}} = p_{lj_{max}}$ *for* $l = 1,2$, *which contradicts the fact that* $p_i \neq p_j$ *for* $i \neq j$.
In the case that $\max(i_{max}, j_{max}) \leq 2$ *we calculate that*

$$\det\begin{pmatrix} q_2^l-q_1^l & q_3^l-q_1^l \\ q_2-q_1 & q_3-q_1 \end{pmatrix} = \quad (3.90)$$

$$= \kappa b^2((x_1^2 - x_1^1)(x_2^3 - x_2^1) - (x_2^2 - x_2^1)(x_1^3 - x_1^1))(p_{l1} - p_{l2}) = 0 \textit{ for } l = 1,2$$

$$\Leftrightarrow (x_1^2 - x_1^1)(x_2^3 - x_2^1) - (x_2^2 - x_2^1)(x_1^3 - x_1^1) = 0 \quad (3.91)$$

$$\Leftrightarrow x_1^3 + x_1^1 = \frac{x_2^3 - x_2^1}{x_1^3 - x_1^1} = \frac{x_2^2 - x_2^1}{x_1^2 - x_1^1} = x_1^2 + x_1^1 \quad (3.92)$$

$$\Leftrightarrow x_1^2 = x_1^3 \textit{ and } x_2^2 = x_2^3 \quad (3.93)$$

in contradiction to the fact that $x^2 \neq x^3$. Therefore, the projective points $\varphi(x^1)$, $\varphi(x^2)$, $\varphi(x^3)$ are in general noncollinear. □

Remark 3.3.5 If the parameters a, b are chosen so that the length of the arithmetic progression with respect to the parameters a, b extends the needed length by r, then the parameter a can be replaced by one of the values $a + b$, $a+2\cdot b, \dots, a+r\cdot b$. Thus, within the lemma before the case that the coefficient of the parameter a is equal to 0 can alternatively be considered:
For $\max(i_{max}, j_{max}) \geq 3$ and $i_{max} \neq j_{max}$ we also calculate that

$$\det \begin{pmatrix} q_2^1 - q_1^1 & q_3^1 - q_1^1 \\ q_2^2 - q_1^2 & q_3^2 - q_1^2 \end{pmatrix} = \tag{3.94}$$

$$= \kappa^{i_{max}+j_{max}-2} b^2 (x^2_{i_{max}} - x^1_{i_{max}})(x^3_{j_{max}} - x^1_{j_{max}}) \det(p_{i_{max}}, p_{j_{max}}) + \\ + \text{ terms of lower degree in } \kappa \neq 0, \tag{3.95}$$

as $\det(p_{i_{max}}, p_{j_{max}}) \neq 0$ according to Lemma 3.2.1.
Analogously, the case $i_{max} = j_{max} = 2$ is treated by

$$\det \begin{pmatrix} q_2^1 - q_1^1 & q_3^1 - q_1^1 \\ q_2^2 - q_1^2 & q_3^2 - q_1^2 \end{pmatrix} = \tag{3.96}$$

$$= \kappa^1 b^2 [(x_1^2 - x_1^1)(x_2^3 - x_2^1) - (x_2^2 - x_2^1)(x_1^3 - x_1^1)] \det(p_1, p_2) = 0 \tag{3.97}$$

$$\Leftrightarrow (x_1^2 - x_1^1)(x_2^3 - x_2^1) - (x_2^2 - x_2^1)(x_1^3 - x_1^1) = 0 \tag{3.98}$$

$$\Leftrightarrow x^2 = x^3. \tag{3.99}$$

Now we can formulate a stronger version of Theorem 3.1.1 by stating large affine dissimilarity for all affine transformations.

Theorem 3.3.6
The assertion of Theorem 3.1.1 can be extended so that in addition

$$|F_1 \cap t(F_2)| \leq |\varphi_{\mathbb{R}^m}(G_1^{(m)}) \cap t(\varphi_{\mathbb{R}^m}(G_2^{(m)}))| \leq 3 \tag{3.100}$$

for any affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$.
Proof
Let φ be the projective transformation as given in (3.3)-(3.4) and let the lattice sets $G_1^{(m)}$, $G_2^{(m)}$ be defined with respect to the redefinition (3.78)-(3.79) and the lattice sets F_1, F_2 by

$$F_1 := \varphi_{\mathbb{R}^m}(G_1^{(m)} \backslash \{0\}), \tag{3.101}$$

$$F_2 := \varphi_{\mathbb{R}^m}(G_2^{(m)} \backslash \{0, l^2, 0, \dots, 0\}). \tag{3.102}$$

As in fact the staircase-like construction of the lattice set $G_1^{(m)} \cup G_2^{(m)}$ is used in Section 3.2, we only have to concentrate on the extention with respect to the *affine dissimilarity assertion. But notice that in some cases, for example for the lower bound of the parameter κ, we have to replace the term l by the term l^2. Let us assume that*

$$t : \varphi_{\mathbb{R}^m}(x^k) \mapsto \varphi_{\mathbb{R}^m}(x^{k+1}) \text{ for } k = 1,3,5,7 \tag{3.103}$$

and $x^1, x^3, x^5, x^7 \in G_1^{(m)}$, $x^2, x^4, x^6, x^8 \in G_2^{(m)}$ pairwise distinct lattice points. According to Lemma 3.3.3 none triple of the lattice points $\varphi_{\mathbb{R}^m}(x^k)$ for $k = 1,3,5,7$ resp. for $k = 2,4,6,8$ is collinear. Using the Fundamental Theorem of Affine Geometry (see for example [24]) and the construction strategy for unique affine transformations, the linear equation system

$$\lambda_1(\varphi_{\mathbb{R}^m}(x^3) - \varphi_{\mathbb{R}^m}(x^1)) + \lambda_2(\varphi_{\mathbb{R}^m}(x^5) - \varphi_{\mathbb{R}^m}(x^1)) = \varphi_{\mathbb{R}^m}(x^7) - \varphi_{\mathbb{R}^m}(x^1), \tag{3.104}$$

$$\lambda_1(\varphi_{\mathbb{R}^m}(x^4) - \varphi_{\mathbb{R}^m}(x^2)) + \lambda_2(\varphi_{\mathbb{R}^m}(x^6) - \varphi_{\mathbb{R}^m}(x^2)) = \varphi_{\mathbb{R}^m}(x^8) - \varphi_{\mathbb{R}^m}(x^2) \tag{3.105}$$

is uniquely solved according to our mapping assumption (3.103).
Let us look at the extended coefficient matrix of the equation system (3.104)-(3.105) with respect to the first component equations resp. with respect to the second component equations. Let us assume that the extended coefficient matrix has rank 1 for sufficiently many values of the parameter c_1 resp. of the parameter c_2, which implies that $\frac{\frac{1}{q_3}-\frac{1}{q_1}}{\frac{1}{q_5}-\frac{1}{q_1}} = \frac{\frac{1}{q_4}-\frac{1}{q_2}}{\frac{1}{q_6}-\frac{1}{q_2}}$ if and only if (3.77), but that cannot be the case.
Therefore, the parameters λ_1, λ_2 are uniquely determined, but calculated in dependence on the parameter c_1 for the first component equations resp. in dependence on the parameter c_2 for the second component equations in general. Let us assume that the values λ_1, λ_2 are independent on the parameters c_1, c_2, as otherwise we can eliminate bad parameter combinations again. Both equations within the equation system (3.104) together imply that

$$\lambda_1 = \frac{\det\begin{pmatrix} \varphi_{\mathbb{R}^m}(x^7) - \varphi_{\mathbb{R}^m}(x^1) & \varphi_{\mathbb{R}^m}(x^5) - \varphi_{\mathbb{R}^m}(x^1) \end{pmatrix}}{\det\begin{pmatrix} \varphi_{\mathbb{R}^m}(x^3) - \varphi_{\mathbb{R}^m}(x^1) & \varphi_{\mathbb{R}^m}(x^5) - \varphi_{\mathbb{R}^m}(x^1) \end{pmatrix}} = \tag{3.106}$$

$$= \frac{\det\begin{pmatrix} \frac{q_7^1-q_1^1}{q_7} + \bar{c}_1(\frac{1}{q_7}-\frac{1}{q_1}) & \frac{q_5^1-q_1^1}{q_5} + \bar{c}_1(\frac{1}{q_5}-\frac{1}{q_1}) \\ \frac{q_7^2-q_1^2}{q_7} + \bar{c}_2(\frac{1}{q_7}-\frac{1}{q_1}) & \frac{q_5^2-q_1^2}{q_5} + \bar{c}_2(\frac{1}{q_5}-\frac{1}{q_1}) \end{pmatrix}}{\det\begin{pmatrix} \frac{q_3^1-q_1^1}{q_3} + \bar{c}_1(\frac{1}{q_3}-\frac{1}{q_1}) & \frac{q_5^1-q_1^1}{q_5} + \bar{c}_1(\frac{1}{q_5}-\frac{1}{q_1}) \\ \frac{q_3^2-q_1^2}{q_3} + \bar{c}_2(\frac{1}{q_3}-\frac{1}{q_1}) & \frac{q_5^2-q_1^2}{q_5} + \bar{c}_2(\frac{1}{q_5}-\frac{1}{q_1}) \end{pmatrix}} \tag{3.107}$$

$$= \frac{\frac{1}{q_1q_5q_7}[\bar{c}_1C_{75}^2 + \bar{c}_2C_{75}^1 + D_{75}]}{\frac{1}{q_1q_3q_5}[\bar{c}_1C_{35}^2 + \bar{c}_2C_{35}^1 + D_{35}]} = \frac{q_3}{q_7}\text{const}_1 \tag{3.108}$$

for

$$q_k := \sum_{i=1}^{m} \kappa^{i-1} b x_i^k + a \text{ prime}, \tag{3.109}$$

$$q_k^l := \sum_{i=1}^{m} \kappa^{i-1} b p_{li} x_i^k, \tag{3.110}$$

$$\bar{c}_l := c_l + q_1^l, \tag{3.111}$$

$$C_{mn}^l := (q_1 - q_m)(q_n^l - q_1^l) - (q_1 - q_n)(q_m^l - q_1^l), \tag{3.112}$$

$$D_{mn} := q_1[(q_m^1 - q_1^1)(q_n^2 - q_1^2) - (q_m^2 - q_1^2)(q_n^1 - q_1^1)]. \tag{3.113}$$

Notice, that $\text{const}_1 \neq 0$ *as* $C_{75}^2 \neq 0$ *or* $C_{75}^1 \neq 0$, *using the same arguments as in Lemma 3.3.4. Furthermore, the value* C_{mn}^l *does not depend on the choice of the parameter* a, *and therefore also the value* const_1 *does not depend on the parameter* a.

Analogously, both equations within the equation system (3.105) together imply that

$$\lambda_1 = \frac{\frac{1}{q_2 q_6 q_8}[\bar{c}_1 C_{86}^2 + \bar{c}_2 C_{86}^1 + D_{86}]}{\frac{1}{q_2 q_4 q_6}[\bar{c}_1 C_{46}^2 + \bar{c}_2 C_{46}^1 + D_{46}]} = \frac{q_4}{q_8} \text{const}_2 \tag{3.114}$$

for $\text{const}_2 \neq 0$ *independent on* a.

We calculate by taking the statement at the beginning of Remark 3.3.5 into account that

$$\frac{q_3}{q_7} \text{const}_1 = \frac{q_4}{q_8} \text{const}_2 \tag{3.115}$$

$$\Leftrightarrow q_3 q_8 \text{const}_1 = q_4 q_7 \text{const}_2 \tag{3.116}$$

$$\Leftrightarrow \text{const}_1 = \text{const}_2 \text{ and } q_3 q_8 = q_4 q_7, \tag{3.117}$$

as $q_k \in \mathbb{Z}[a]$ *has leading coefficient* 1 *for* $k = 1, \ldots, 8$. *As all values* q_k *are prime, equation* $q_3 q_8 = q_4 q_7$ *cannot be fulfilled for different values* q_k, *which contradicts our assumption that the parameter values* λ_1, λ_2 *are independent on the parameters* c_1, c_2. □

In order to formulate projective dissimilarity assertions as well, let us now extend Lemma 3.3.4 by some translational arguments for later purpose.

Lemma 3.3.7 *Let the lattice set* $G_1^{(m)} \cup G_2^{(m)}$ *be defined with respect to the redefinition in (3.79) and let* $x^1, x^2, \ldots, x^5 \in G_1^{(m)} \cup G_2^{(m)}$ *be distinct lattice points. Let the values* q_k^l, q_k *be defined by (3.84), (3.85). For the parameter setting*

$$c_1 := -q_5^1, \tag{3.118}$$

$$c_2 := -q_5^2, \tag{3.119}$$

$$a := -q_4 \tag{3.120}$$

we calculate that

$$\det\begin{pmatrix} q_1^1+c_1 & q_2^1+c_1 & q_3^1+c_1 \\ q_1^2+c_2 & q_2^2+c_2 & q_3^2+c_2 \\ q_1+a & q_2+a & q_3+a \end{pmatrix} =$$

$$= \det\begin{pmatrix} q_1^1-q_5^1 & q_2^1-q_1^1 & q_3^1-q_1^1 \\ q_1^2-q_5^2 & q_2^2-q_1^2 & q_3^2-q_1^2 \\ q_1-q_4 & q_2-q_1 & q_3-q_1 \end{pmatrix} \neq 0, \tag{3.121}$$

$$\det\begin{pmatrix} q_4^1+c_1 & q_2^1+c_1 & q_3^1+c_1 \\ q_4^2+c_2 & q_2^2+c_2 & q_3^2+c_2 \\ q_4+a & q_2+a & q_3+a \end{pmatrix} =$$

$$= \det\begin{pmatrix} q_4^1-q_5^1 & q_2^1-q_4^1 & q_3^1-q_4^1 \\ q_4^2-q_5^2 & q_2^2-q_4^2 & q_3^2-q_4^2 \\ 0 & q_2-q_4 & q_3-q_4 \end{pmatrix} \neq 0, \tag{3.122}$$

$$\det\begin{pmatrix} q_5^1+c_1 & q_2^1+c_1 & q_3^1+c_1 \\ q_5^2+c_2 & q_2^2+c_2 & q_3^2+c_2 \\ q_5+a & q_2+a & q_3+a \end{pmatrix} =$$

$$= \det\begin{pmatrix} 0 & q_2^1-q_5^1 & q_3^1-q_5^1 \\ 0 & q_2^2-q_5^2 & q_3^2-q_5^2 \\ q_5-q_4 & q_2-q_5 & q_3-q_5 \end{pmatrix} \neq 0 \tag{3.123}$$

in general by applying some translation if necessary.
Considering also the index sets $\{1,2,4\}$, $\{1,3,4\}$ instead of $\{2,3,4\}$ in (3.122) resp. the index sets $\{1,2,5\}$, $\{1,3,5\}$ instead of $\{2,3,5\}$ in (3.123), each triple of the projective points $\varphi(x^1), \varphi(x^2), \varphi(x^3), \varphi(x^4) \in \mathbb{P}^2$ resp. of the projective points $\varphi(x^1), \varphi(x^2), \varphi(x^3), \varphi(x^5) \in \mathbb{P}^2$ is noncollinear for the parameter setting (3.118)-(3.120).

Proof

We calculate that

$$\det\begin{pmatrix} q_1^1-q_5^1 & q_2^1-q_1^1 & q_3^1-q_1^1 \\ q_1^2-q_5^2+k(q_1-q_5) & q_2^2-q_1^2+k(q_2-q_1) & q_3^2-q_1^2+k(q_3-q_1) \\ q_1-q_4 & q_2-q_1 & q_3-q_1 \end{pmatrix}$$

$$= \det\begin{pmatrix} q_1^1-q_5^1 & q_2^1-q_1^1 & q_3^1-q_1^1 \\ q_1^2-q_5^2+k(q_4-q_5) & q_2^2-q_1^2 & q_3^2-q_1^2 \\ q_1-q_4 & q_2-q_1 & q_3-q_1 \end{pmatrix} = \tag{3.124}$$

$$= k\cdot(q_5-q_4)\cdot\det\begin{pmatrix} q_2^1-q_1^1 & q_3^1-q_1^1 \\ q_2-q_1 & q_3-q_1 \end{pmatrix} + \textit{ terms of lower degree in } k$$

for the translation

$$(p_{1j}, p_{2j}) \in \mathbb{R}^2 \mapsto (p_{1j}, p_{2j}+k) \in \mathbb{R}^2. \tag{3.125}$$

Taking also the translation

$$(p_{1j}, p_{2j}) \mapsto (p_{1j} + k, p_{2j}) \tag{3.126}$$

into account, we result in assertion (3.121), as the case that

$$\det \begin{pmatrix} q_2^1 - q_1^1 & q_3^1 - q_1^1 \\ q_2 - q_1 & q_3 - q_1 \end{pmatrix} = \det \begin{pmatrix} q_2^2 - q_1^2 & q_3^2 - q_1^2 \\ q_2 - q_1 & q_3 - q_1 \end{pmatrix} = 0 \tag{3.127}$$

has already been excluded within the proof of Lemma 3.3.4 before.
To show assertion (3.122) similar arguments are applied.
Assertion (3.123) is given for pairwise distinct lattice points x^2, x^3, x^4, x^5 *as*

$$\det \begin{pmatrix} q_2^1 - q_5^1 & q_3^1 - q_5^1 \\ q_2^2 - q_5^2 & q_3^2 - q_5^2 \end{pmatrix} \neq 0 \tag{3.128}$$

by Remark 3.3.5. □

Now we are ready to strengthen Theorem 3.3.6 by showing large projective and therefore large perspective dissimilarity for the lattice sets F_1, F_2.

Theorem 3.3.8
The assertion of Theorem 3.3.6 can be extended so that in addition

$$|F_1 \cap \rho_{\mathbb{R}^2}(F_2)| \leq 4 \tag{3.129}$$

for any projective and therefore also perspective transformation $\rho : \mathbb{P}^2 \to \mathbb{P}^2$ *and* $\rho_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ *its restriction to the Euclidean space* $\mathbb{R}^2$.

Proof
Let φ *be the projective transformation as given in (3.3)-(3.4) and let the lattice sets* $G_1^{(m)}$, $G_2^{(m)}$ *be defined with respect to the redefinition (3.78)-(3.79) and the lattice sets* F_1, F_2 *by*

$$F_1 := \varphi_{\mathbb{R}^m}(G_1^{(m)} \backslash \{0\}), \tag{3.130}$$

$$F_2 := \varphi_{\mathbb{R}^m}(G_2^{(m)} \backslash \{0, l^2, 0, \ldots, 0\}). \tag{3.131}$$

Again, we only have to concentrate on the extention with respect to the projective dissimilarity assertion.
In the following we will use notation (3.84) and notation (3.85).
Let us assume that

$$[\varphi_{\mathbb{R}^m}(x^k), 1] \in \mathbb{P}^2 \mapsto [\varphi_{\mathbb{R}^m}(x^{k+1}), 1] \in \mathbb{P}^2 \text{ for } k = 1, 3, \ldots, 9 \tag{3.132}$$

by some projective transformation

$$\rho : \mathbb{P}^2 \to \mathbb{P}^2, \tag{3.133}$$

$$\rho : \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} A \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \end{bmatrix}, \ A = (a_{ij})_{i,j=1,2,3} \text{ nonsingular}, \tag{3.134}$$

which is the case if and only if the homogeneous linear equation system

$$A\begin{pmatrix} q_k^1+c_1 \\ q_k^2+c_2 \\ q_k+a \end{pmatrix} = \lambda_{k+1}\begin{pmatrix} q_{k+1}^1+c_1 \\ q_{k+1}^2+c_2 \\ q_{k+1}+a \end{pmatrix} \quad \text{for } k=1,3,\ldots,9 \tag{3.135}$$

of dimension 15×14 *within the variables* a_{ij}, $i,j = 1,2,3$ *and* λ_{k+1}, $k=1,3,5,7,9$ *is nontrivially solved by* $\lambda_{k+1}\neq 0$ *for* $k=1,3,5,7,9$ *and the matrix* $A=(a_{ij})_{i,j=1,2,3}$ *nonsingular. For the further argumentation let the linear equation system (3.135) be denoted by*

$$B\left(a_{11},a_{12},a_{13},a_{21},\ldots,a_{33},-\lambda_2,\ldots,-\lambda_{10}\right)^T=0, \tag{3.136}$$

for the (15×14)*-dimensional matrix*

$$B=(b_{ij})_{i=1,\ldots,15,j=1,\ldots,14} := \tag{3.137}$$

$$= \begin{pmatrix}
b_{1,1} & b_{1,2} & b_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & b_{1,10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_{2,4} & b_{2,5} & b_{2,6} & 0 & 0 & 0 & b_{2,10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_{3,7} & b_{3,8} & b_{3,9} & b_{3,10} & 0 & 0 & 0 & 0 \\
b_{4,1} & b_{4,2} & b_{4,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{4,11} & 0 & 0 & 0 \\
0 & 0 & 0 & b_{5,4} & b_{5,5} & b_{5,6} & 0 & 0 & 0 & 0 & b_{5,11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_{6,7} & b_{6,8} & b_{6,9} & 0 & b_{6,11} & 0 & 0 & 0 \\
b_{7,1} & b_{7,2} & b_{7,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{7,12} & 0 & 0 \\
0 & 0 & 0 & b_{8,4} & b_{8,5} & b_{8,6} & 0 & 0 & 0 & 0 & 0 & b_{8,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_{9,7} & b_{9,8} & b_{9,9} & 0 & 0 & b_{9,12} & 0 & 0 \\
b_{10,1} & b_{10,2} & b_{10,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{10,13} & 0 \\
0 & 0 & 0 & b_{11,4} & b_{11,5} & b_{11,6} & 0 & 0 & 0 & 0 & 0 & 0 & b_{11,13} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_{12,7} & b_{12,8} & b_{12,9} & 0 & 0 & 0 & b_{12,13} & 0 \\
b_{13,1} & b_{13,2} & b_{13,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{13,14} \\
0 & 0 & 0 & b_{14,4} & b_{14,5} & b_{14,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{14,14} \\
0 & 0 & 0 & 0 & 0 & 0 & b_{15,7} & b_{15,8} & b_{15,9} & 0 & 0 & 0 & 0 & b_{15,14}
\end{pmatrix}$$

and for

$$\begin{aligned}
(b_{1,1},b_{1,2},b_{1,3}) &= (b_{2,4},b_{2,5},b_{2,6}) = (b_{3,7},b_{3,8},b_{3,9}) := \\
&= (q_1^1+c_1,q_1^2+c_2,q_1+a), \\
(b_{4,1},b_{4,2},b_{4,3}) &= (b_{5,4},b_{5,5},b_{5,6}) = (b_{6,7},b_{6,8},b_{6,9}) := \\
&= (q_3^1+c_1,q_3^2+c_2,q_3+a), \\
(b_{7,1},b_{7,2},b_{7,3}) &= (b_{8,4},b_{8,5},b_{8,6}) = (b_{9,7},b_{9,8},b_{9,9}) := \\
&= (q_5^1+c_1,q_5^2+c_2,q_5+a), \\
(b_{10,1},b_{10,2},b_{10,3}) &= (b_{11,4},b_{11,5},b_{11,6}) = (b_{12,7},b_{12,8},b_{12,9}) := \\
&= (q_7^1+c_1,q_7^2+c_2,q_7+a), \\
(b_{13,1},b_{13,2},b_{13,3}) &= (b_{14,4},b_{14,5},b_{14,6}) = (b_{15,7},b_{15,8},b_{15,9}) := \\
&= (q_9^1+c_1,q_9^2+c_2,q_9+a),
\end{aligned}$$

$$(b_{1,10}, b_{2,10}, b_{3,10}) := (q_2^1 + c_1, q_2^2 + c_2, q_2 + a),$$
$$(b_{4,11}, b_{5,11}, b_{6,11}) := (q_4^1 + c_1, q_4^2 + c_2, q_4 + a),$$
$$(b_{7,12}, b_{8,12}, b_{9,12}) := (q_6^1 + c_1, q_6^2 + c_2, q_6 + a),$$
$$(b_{10,13}, b_{11,13}, b_{12,13}) := (q_8^1 + c_1, q_8^2 + c_2, q_8 + a),$$
$$(b_{13,14}, b_{14,14}, b_{15,14}) := (q_{10}^1 + c_1, q_{10}^2 + c_2, q_{10} + a).$$

Using both the Fundamental Theorem of Projective Geometry (see for example [24]) and Lemma 3.3.7, we know that the 12×12 *submatrices* $(b_{ij})_{i,j=1,\ldots,12}$ *and* $(b_{ij})_{i=1,\ldots,9,13,14,15,j=1,\ldots,12}$ *are regular for the parameter setting* $c_1 := -q_8^1$, $c_2 := -q_8^2$ *and* $a := -q_{10}$*:*
As no triple of the projective points $\varphi(x^1), \varphi(x^3), \varphi(x^5), \varphi(x^7) \in \mathbb{P}^2$ *and no triple of the projective points* $\varphi(x^2), \varphi(x^4), \varphi(x^6), \varphi(x^8) \in \mathbb{P}^2$ *are collinear, the projective transformation*

$$\varphi(x^k) \mapsto \varphi(x^{k+1}) \text{ for } k = 1, 3, 5, 7 \tag{3.138}$$

is uniquely determined. In particular, the values $a_{i,j}$ *for* $i, j = 1, 2, 3$ *and the values* λ_{k+1} *for* $k = 1, 3, 5, 7$ *are uniquely specified up to some common multiple. Thus, if the variable* λ_8 *is fixed to the value* 1*, the other variables* $a_{i,j}$ *for* $i, j = 1, 2, 3$ *and* λ_{k+1} *for* $k = 1, 3, 5$ *are uniquely determined by the linear equation system*

$$(b_{i,j})_{i,j=1,\ldots,12}(a_{1,1}, \ldots, a_{3,3}, -\lambda_2, \ldots, -\lambda_6)^T = \\ = (0, \ldots, 0, q_8 - q_{10})^T \tag{3.139}$$

if and only if the matrix $(b_{i,j})_{i,j=1,\ldots,12}$ *is regular. Similar arguments work to show the regularity of the matrix* $(b_{ij})_{i=1,\ldots,9,13,14,15,j=1,\ldots,12}$*.*
Let the vectors $B_i \in \mathbb{R}^{12}$, $\bar{B}_i \in \mathbb{R}^{14}$ *for* $i = 1, \ldots, 15$ *be defined by*

$$B_i := (b_{i,1}, \ldots, b_{i,12}), \tag{3.140}$$
$$\bar{B}_i := (b_{i,1}, \ldots, b_{i,14}). \tag{3.141}$$

Notice, that it yields that

$$B_{12} \in \operatorname{span}\{B_1, \ldots, B_9, B_{13}, (q_{10}^1 - q_8^1)B_{14} - (q_{10}^2 - q_8^2)B_{13}, B_{15}\}\setminus \\ \operatorname{span}\{B_1, \ldots, B_9\} = \\ = \operatorname{span}\{B_1, \ldots, B_9, (q_{10}^1 - q_8^1)B_{14} - (q_{10}^2 - q_8^2)B_{13}, B_{14}, B_{15}\}\setminus \\ \operatorname{span}\{B_1, \ldots, B_9\} \tag{3.142}$$

because of $q_{10}^1 - q_8^1 \neq 0$ *and* $q_{10}^2 - q_8^2 \neq 0$*, as we have that* $x^8 \neq x^{10}$ *and* $p_{ij} \neq 0$ *according to Lemma 3.2.1. Thus, by the Exchange Theorem of linear algebra the vector* B_{12} *within the basis* $\{B_1, \ldots, B_{12}\}$ *of the 12-dimensional vector space* $\operatorname{span}\{B_1, \ldots, B_{12}\}$ *spanned by the vectors* $B_1, \ldots, B_{12}$ *can be replaced by the* *vector B_{15} or by the vector $(q_{10}^1 - q_8^1)B_{14} - (q_{10}^2 - q_8^2)B_{13}$. Therefore, we conclude that*

$$\operatorname{span}\{\bar{B}_1, \ldots, \bar{B}_{11}, \bar{B}_{15}\} \subset \{x \in \mathbb{R}^{14} | x_{13} = x_{14} = 0\}, \tag{3.143}$$

$$\dim \operatorname{span}\{\bar{B}_1, \ldots, \bar{B}_{11}, \bar{B}_{15}\} = 12 \tag{3.144}$$

resp. that

$$\operatorname{span}\{\bar{B}_1, \ldots, \bar{B}_{11}, (q_{10}^1 - q_8^1)\bar{B}_{14} - (q_{10}^2 - q_8^2)\bar{B}_{13}\} \subset \{x \in \mathbb{R}^{14} | x_{13} = x_{14} = 0\}, \tag{3.145}$$

$$\dim \operatorname{span}\{\bar{B}_1, \ldots, \bar{B}_{11}, (q_{10}^1 - q_8^1)\bar{B}_{14} - (q_{10}^2 - q_8^2)\bar{B}_{13}\} = 12. \tag{3.146}$$

Because of $q_8 - q_{10} \neq 0$, $q_8^1 - q_{10}^1 \neq 0$ and $q_8^2 - q_{10}^2 \neq 0$ we get that $(\bar{B}_{12})_{13} \neq 0$, $(\bar{B}_{13})_{14} \neq 0$ and $(\bar{B}_{14})_{14} \neq 0$. Therefore, the matrix B has full rank for the parameter setting $c_1 := -q_8^1$, $c_2 := -q_8^2$ and $a := -q_{10}$. Thus, we conclude that the determinant of at least one quadratic (14×14)-submatrix of the matrix B has to be polynomial in at least one of the three parameters c_1, c_2 and a. Thus, in general the matrix B has full rank, which implies that the equation system (3.136) is only solved by $a_{ij} = 0$ for $i, j = 1, \ldots, 3$ and $\lambda_{k+1} = 0$ for $k = 1, 3, 5, 7, 9$. That fact leads to contradiction to assumption (3.132) at the beginning. □

3.4 Point X-rays located within the convex hull

Several uniqueness and nonuniqueness results with respect to convex lattice sets and point X-rays are given in [45]. Especially for collinear point X-ray sources some results of [54] for the parallel X-ray geometry are transferred. But now we concentrate on the class of lattice sets satisfying that the point X-ray sources are located within their convex hulls.
In geometric tomography a convex body is determined by two point X-ray sources within the interior of a convex body. We will see that in discrete tomography convex lattice sets are uniquely determined by three noncollinear point X-ray sources by the further information that at least two of them lie within the lattice set.
That is not the case, if we weaken the assumption. For any finite number m of point X-ray sources $p_1, \dots, p_m$ we can construct two tomographically equivalent lattice sets F_1, F_2 so that $\{p_1, \dots, p_m\} \subset \mathrm{conv}(F_1 \backslash F_2) \cap \mathrm{conv}(F_2 \backslash F_1)$.
The switching construction will be used to also show strong instability under the assumption that the point X-ray sources $p_1, \dots, p_m$ are located within the convex hull of each considered lattice set.

3.4.1 Convex lattice sets uniquely determined by $m = 3$ point X-ray sources

In geometric tomography two point X-ray sources suffice to determine convex bodies, if they are located within the interior of the convex bodies, see [53]. In contrast to that we will see that in discrete tomography convex lattice sets are uniquely determined by three noncollinear point X-ray sources, if we further know that at least two of them lie within the lattice sets, but two point X-rays are not enough in general.

Lemma 3.4.1 *Let $F_1, F_2 \subset \mathbb{Z}^2$ be two finite lattice sets and let $p \in \mathbb{Z}^2$ be a lattice point. The lattice point p either belongs to both lattice sets F_1, F_2 or to none of them, if the lattice sets F_1, F_2 have the same point X-ray with respect to the lattice point p.*

Proof
The finiteness of the lattice set F_1 implies that $X_pF_1(u) \geq 1$ for all lattice directions $u \in \mathbb{Z}^2 \backslash \{0\}$ is only possible if and only if the lattice point p belongs to the lattice set F_1:
If $p \in F_1$ it is easy to see that $X_pF_1(u) \geq 1$ is implied for all lattice directions u. In the case that $p \notin F_1$ there is at least one line l passing through the lattice point p which, however, does not pass through the lattice set F_1.
The same is true for the lattice set F_2. □

Lemma 3.4.2 *Any convex finite lattice set $F \subset \mathbb{Z}^2$ is uniquely determined by the point X-rays with respect to three distinct and noncollinear lattice points*

$p_1, p_2, p_3 \in F$ within the lattice set F.
By considering the point X-rays with respect to two distinct lattice points $p_1, p_2 \in F$ within the lattice set F, uniqueness is given besides the lattice set $\{x \in F \cap \mathbb{Z}^2 | x - p_1 = \lambda(p_2 - p_1)$ for $\lambda \in \mathbb{R}\}$.

Proof

Let us assume that the lattice sets $F', F \subset \mathbb{Z}^2$ are tomographically equivalent with respect to the lattice points p_1, p_2. Without loss of generality let us assume that the line $l := \{x \in \mathbb{R}^2 | x - p_1 = \lambda(p_2 - p_1)$ for $\lambda \in \mathbb{R}^2\}$ is given by $l = \{x \in \mathbb{R}^2 | x_2 = 0\}$. Let us further assume that $(F \triangle F') \backslash l \neq \emptyset$. For $i, j = 1, 2$ let us rotate the line l so that the lattice point p_i remains fixed, until the rotated halfline which has not contained the lattice points p_1, p_2 at the beginning passes through some lattice point within the lattice set $F \triangle F' \cap \{(-1)^j x_2 < 0\}$, if $F \triangle F' \cap \{(-1)^j x_2 < 0\} \neq \emptyset$. We will denote that lattice point by q_{ij} in the following. Notice, that the lattice points q_{11}, q_{21} or the lattice points q_{12}, q_{22} exist because of the assumption that $(F \triangle F') \backslash l \neq \emptyset$. Without loss of generality let us assume in the following that all four lattice points exist. Let the open cones G_1, G_2 be defined by

$$G_i := \{\lambda_1(q_{i1} - p_i) + \lambda_2(q_{i2} - p_i) | \lambda_1, \lambda_2 > 0\} \tag{3.147}$$

for $i = 1, 2$ and notice that $(G_1 \cup G_2) \cap (F \triangle F') \subset l$.

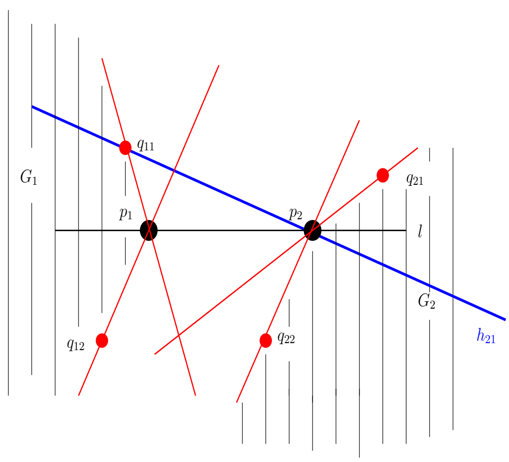

Figure 3.4: Uniqueness except for one line by two point X-rays

Now let us look at the lines $h_{i,j}$ for $i, j = 1, 2$ which are incident to the lattice points $p_i, q_{i',j}$ for $i' \in \{1, 2\} \backslash \{i\}$. Because of the convexity of the lattice sets F and F' and as $p_i \in F \cap F'$ and $q_{i',j} \in F \triangle F'$, all lattice points incident to the line $h_{i,j}$ which lie between the lattice points p_i and $q_{i',j}$ have to belong to the lattice set F (resp. to the lattice set F') if $q_{i',j} \in F$ (resp. if $q_{i',j} \in F'$).
Let $h \in \{h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}\}$ denote that line which encloses the smallest angle with the line l and let $i(h)$, $i'(h)$ and $j(h)$ denote the indices with respect to

the line h. Then it yields that $h \backslash \{p_{i(h)} + \lambda \cdot (q_{i'(h),j(h)} - p_{i(h)}) | 0 \leq \lambda \leq 1\} \subset G_1 \cup G_2$. That is the case as each line passing through the lattice points p_i and $q_{i',j}$ has a smaller angle with the line l than the line passing through the lattice points $p_{i'}$ and $q_{i',j}$, and thus $F \triangle F' \cap (h \backslash \{p_{i(h)} + \lambda \cdot (q_{i'(h),j(h)} - p_{i(h)}) | 0 \leq \lambda \leq 1\}) = \emptyset$.
Therefore, we result in contradiction to the tomographic equivalence of the lattice sets F, F', as the lattice point $q_{i'(h),j(h)}$ belongs to only one of the two lattice sets F, F'. Thus, the lattice set F is uniquely determined besides those lattice points which lie on the line l.
If the lattice points p_1, p_2, p_3 are noncollinear, every lattice point within the lattice set $\mathbb{Z}^2$ is not incident to at least one of the three lines $\{x \in \mathbb{R}^2 | x - p_i = \lambda(p_j - p_i) \text{ for } \lambda \in \mathbb{R}\}$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. Thus, any convex finite lattice set $F \subset \mathbb{Z}^2$ is uniquely determined by three distinct and noncollinear lattice points $p_1, p_2, p_3 \in F$ within the lattice set F. □

Remark 3.4.3 In geometric tomography any convex body is uniquely determined by the point X-rays with respect to two distinct points p_1, p_2, if they are located within the interior of the convex body, see [53], Theorem 5.3.3. The difference between geometric and discrete tomography is caused by the fact that the line passing through the points p_1 and p_2 has measure 0 in geometric tomography, but not in discrete tomography.

Corollary 3.4.4 *Any convex finite lattice set $F \subset \mathbb{Z}^2$ is uniquely determined by the point X-rays with respect to three noncollinear lattice points $p_1, p_2, p_3 \in \mathbb{Z}^2$, if at least two of them belong to the lattice set F.*

Proof
In the case that $p_1, p_2 \in F$ uniqueness is given up to the lattice points which are incident to the line $l = \{x \in \mathbb{R}^2 | x - p_1 = \lambda(p_2 - p_1) \text{ for } \lambda \in \mathbb{R}^2\}$ passing through the lattice points p_1 and p_2. As the lattice points p_1, p_2, p_3 are assumed to be noncollinear, the point X-ray with respect to the lattice point p_3 determines the remaining lattice points, which are located on the line l. □

3.4.2 Nonuniqueness according to point X-ray sources located within the convex hull

As we will show now, the uniqueness result for convex lattice sets before cannot be transferred to the case that the point X-ray sources are located within the convex hull of the finite lattice set. The following nonuniqueness result with respect to point X-rays is formulated in order to extend the instability assertion to the case that the point X-ray sources $p_1, \ldots, p_m$ lie within the convex hull of the finite lattice sets in the next subsection.

Theorem 3.4.5
Let $m \geq 1$ and suppose that $p_j = (p_{1j}, p_{2j}) \in \mathbb{Z}^2$, $j = 1, \ldots, m$ are distinct lattice points within the lattice set $\mathbb{Z}^2$. For any $\alpha \in \mathbb{N}$ there exist two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ which are tomographically equivalent and satisfy

$$F_1 \cap F_2 = \emptyset, \tag{3.148}$$

$$|F_1| = |F_2| \geq \alpha, \tag{3.149}$$

$$\{p_1, \ldots, p_m\} \subset \operatorname{conv}(F_1) \cap \operatorname{conv}(F_2). \tag{3.150}$$

Proof
For $m = 2$ resp. for $m = 1$ arbitrarily large and tomographically equivalent lattice sets are located on the line which passes through the lattice points p_1 and p_2 resp. on one of the lines which pass through the lattice point p_1. Thus, it remains to treat the case that $m \geq 3$.
For that purpose let the value $l \in \mathbb{N}$ be chosen sufficiently large with respect to the value α and let the lattice sets $G_1^{(m)}$ and $G_2^{(m)}$ be defined with respect to (3.10)-(3.13) resp. with respect to the redefinition in (3.78)-(3.79). To indicate the choice of the parameters a, b, c_1 and c_2 let the restriction of the projective transformation as given in (3.5)-(3.6) be denoted by $\varphi_{\mathbb{R}^m}^{a,b,c_1,c_2}$ in the following. Let the lattice sets F_1, F_2 be defined by

$$F_1 := \bigcup_{j=1}^{4} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_1^{(m)}), \tag{3.151}$$

$$F_2 := \bigcup_{j=1}^{4} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_2^{(m)}). \tag{3.152}$$

The parameters b and a_j, c_1^j, c_2^j for $j = 1, 2, 3, 4$ are chosen according to Lemma 3.2.3 and Lemma 3.2.4 to assure the following facts:

1. *By choosing a sufficiently large arithmetic progression characterized by the parameters a, b, all values $\sum_{i=1}^{m} \kappa^{i-1} b x_i^k + a_j$ for $x^k \in G_1^{(m)} \cup G_2^{(m)}$ and $j = 1, 2, 3, 4$ are different primes by the setting $a_1 := a$ and $a_j := a + \lambda_j \cdot b$, $j = 2, 3, 4$ for some integer value $\lambda_j > 0$ so that $a_j > \sum_{i=1}^{m} \kappa^{i-1} b x_i^k + a_{j-1}$*

for every lattice point $x^k \in G_1^{(m)} \cup G_2^{(m)}$. *Thus, integrality is guaranteed for the lattice sets* F_1, F_2.

2. *For any lattice point* $x^k \in G_1^{(m)} \cup G_2^{(m)}$ *we can assure that*
   - $(\varphi_{\mathbb{R}^m}^{a_1,b,c_1^1,c_2^1}(x^k))_i > \max|p_{i,j}|$ *for* $i = 1,2$ *by choosing some parameters* $c_1^1, c_2^1 > 0$,
   - $(\varphi_{\mathbb{R}^m}^{a_2,b,c_1^2,c_2^2}(x^k))_1 > \max|p_{i,j}|$ *and* $(\varphi_{\mathbb{R}^m}^{a_2,b,c_1^2,c_2^2}(x^k))_2 < -\max|p_{i,j}|$ *by choosing some parameters* $c_1^2 > 0, c_2^2 < 0$,
   - $(\varphi_{\mathbb{R}^m}^{a_3,b,c_1^3,c_2^3}(x^k))_1 < -\max|p_{i,j}|$ *and* $(\varphi_{\mathbb{R}^m}^{a_3,b,c_1^3,c_2^3}(x^k))_2 > \max|p_{i,j}|$ *by choosing some parameters* $c_1^3 < 0, c_2^3 > 0$,
   - $(\varphi_{\mathbb{R}^m}^{a_4,b,c_1^4,c_2^4}(x^k))_i < -\max|p_{i,j}|$ *for* $i = 1,2$ *by choosing some parameters* $c_1^4, c_2^4 < 0$.

   Therefore, we result in $\{p_1, \dots, p_m\} \subset \operatorname{conv}(F_1 \backslash F_2) \cap \operatorname{conv}(F_2 \backslash F_1)$.

3. *It yields that* $F_1 \cap F_2 = \bigcup_{j=1}^4 \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_1^m) \cap \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G^m) = \emptyset$ *by choosing* $|c_1^j|, |c_2^j|$ *large enough for* $j = 1,2,3,4$.

□

Remark 3.4.6 In the case that $m \geq 3$ we can also define the lattice sets F_1 and F_2 by

$$F_1 := \bigcup_{j=1}^{3} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_1^{(m)}), \tag{3.153}$$

$$F_2 := \bigcup_{j=1}^{3} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_2^{(m)}) \tag{3.154}$$

by choosing the parameters b and a_j, c_1^j, c_2^j for $j = 1,2,3$ as follows:
Let the parameters $c_1^1, c_2^1, c_1^2, c_2^2, c_1^3, c_2^3$ be chosen as in 2. and 3. within the proof of Theorem 3.4.5. If we further guarantee that both

$$(\varphi_{\mathbb{R}^m}^{a_2,b,c_1^2,c_2^2}(x^k))_2 < -5 \cdot \max|p_{ij}| \tag{3.155}$$

$$\Leftrightarrow -c_2^2 > 5 \cdot \max|p_{ij}|(\sum_{i=1}^{m} \kappa^{i-1} b x_i + a_2) + \sum_{i=1}^{m} \kappa^{i-1} b p_{2i} x_i \tag{3.156}$$

$$\Leftarrow -c_2^2 > \max|p_{ij}| \cdot (6 \cdot \max_{x^k \in G_1^{(m)} \cup G_2^{(m)}} \sum_{i=1}^{m} \kappa^{i-1} b x_i^k + 5 \cdot a_2) \tag{3.157}$$

and

$$-(\varphi_{\mathbb{R}^m}^{a_2,b,c_1^2,c_2^2}(x^k))_2 - (\varphi_{\mathbb{R}^m}^{a_2,b,c_1^2,c_2^2}(x^k))_1 > 2 \cdot \max|p_{ij}| \tag{3.158}$$

$$\Leftrightarrow -c_2^2 > 2 \cdot \max|p_{ij}|(\sum_{i=1}^{m} \kappa^{i-1} b x_i + a_2) + \sum_{i=1}^{m} \kappa^{i-1} b p_{1i} x_i + c_1^2 + $$

$$+ \sum_{i=1}^{m} \kappa^{i-1} b p_{2i} x_i \tag{3.159}$$

$$\Leftarrow -c_2^2 > 2 \cdot \max|p_{ij}| \cdot (2 \cdot \max_{x^k \in G_1^{(m)} \cup G_2^{(m)}} (\sum_{i=1}^{m} \kappa^{i-1} b x_i^k) + a_2) + c_1^2$$

$$\text{for } c_1^2 > 0 \tag{3.160}$$

and analogously that

$$-c_1^3 > \max|p_{ij}| \cdot (6 \cdot \max_{x^k \in G_1^{(m)} \cup G_2^{(m)}} \sum_{i=1}^{m} \kappa^{i-1} b x_i^k + 5 \cdot a_3), \tag{3.161}$$

$$-c_1^3 > 2 \cdot \max|p_{ij}| \cdot (2 \cdot \max_{x^k \in G_1^{(m)} \cup G_2^{(m)}} (\sum_{i=1}^{m} \kappa^{i-1} b x_i^k) + a_3) + c_2^3 \text{ for } c_2^3 > 0 \tag{3.162}$$

are satisfied, the hyperplane

$$\{x \in \mathbb{R}^2 | \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} -\max|p_{ij}| \\ -\max|p_{ij}| \end{pmatrix}\} \tag{3.163}$$

separates the lattice sets

$$\bigcup_{j=2,3} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_1^{(m)} \cup G_2^{(m)}), \tag{3.164}$$

$$\{p_1, \dots, p_m\} \cup \varphi_{\mathbb{R}^m}^{a_1,b,c_1^1,c_2^1}(G_1^{(m)} \cup G_2^{(m)}), \tag{3.165}$$

the hyperplane $\{x_1 = \max|p_{ij}|\}$ the lattice sets

$$\bigcup_{j=1,2} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_1^{(m)} \cup G_2^{(m)}), \tag{3.166}$$

$$\{p_1, \dots, p_m\} \cup \varphi_{\mathbb{R}^m}^{a_3,b,c_1^3,c_2^3}(G_1^{(m)} \cup G_2^{(m)}), \tag{3.167}$$

and the hyperplane $\{x_2 = \max|p_{ij}|\}$ the lattice sets

$$\bigcup_{j=1,3} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_1^{(m)} \cup G_2^{(m)}), \tag{3.168}$$

$$\{p_1, \dots, p_m\} \cup \varphi_{\mathbb{R}^m}^{a_2,b,c_1^2,c_2^2}(G_1^{(m)} \cup G_2^{(m)}), \tag{3.169}$$

see Figure 3.5 for illustration.

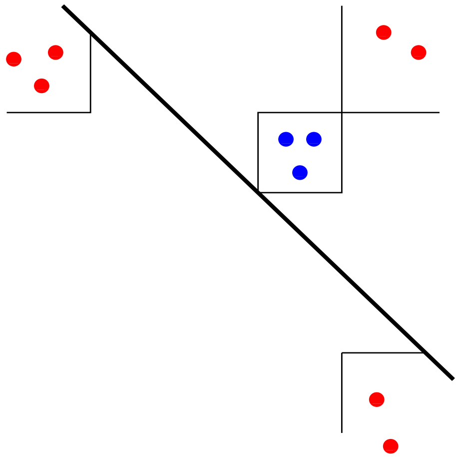

Figure 3.5: Separation by the hyperplane (the point X-ray sources p_j are blue-coloured and the points of the lattice set $F_1 \cup F_2$ are red-coloured)

Thus, we can still assure that $\{p_1, \ldots, p_m\} \subset \mathrm{conv}(F_1 \backslash F_2) \cap \mathrm{conv}(F_2 \backslash F_1)$ while respecting the general demands on the parameters c_i^j for $i = 1, 2$ and $j = 1, 2, 3$. That observation will help us in Section 3.4.3 to construct lattice sets of instability with respect to the error value $3 \cdot 2(m-1)$.

An interesting, but open question is the classification of uniquely determined subclasses of lattice sets with respect to the demand that the point X-ray sources are located within the convex hull of the lattice sets.

3.4.3 Instability for point X-ray sources located within the convex hull

Now let us strengthen the instability results for point X-rays by the further demand that the point X-ray sources lie within the convex hull of the considered lattice sets. Also for that case large instability is shown for the error value $3 \cdot 2(m-1)$ instead of the error value $2(m-1)$, if $m \geq 3$ denotes the number of point X-rays. For that purpose we use Remark 3.4.6 and combine three switching components as constructed in Section 3.3. Affine and projective dissimilarity assertions are enlarged to the complete construction.

Theorem 3.4.7
Let $m \geq 3$ and suppose that $p_j = (p_{1j}, p_{2j}) \in \mathbb{Z}^2$, $j = 1, \dots, m$ are distinct lattice points within the lattice set $\mathbb{Z}^2$. For any $\alpha \in \mathbb{N}$ there exist two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ satisfying

- *F_k for $k = 1, 2$ is uniquely determined by the point X-rays $X_{p_j} F_k$ for $j = 1, \dots, m$,*
- *$F_1 \cap F_2 = \emptyset$,*
- *$|F_1| = |F_2| \geq \alpha$,*
- *$\{p_1, \dots, p_m\} \subset \operatorname{conv}(F_1) \cap \operatorname{conv}(F_2)$,*
- *$\sum_{i=1}^{m} |X_{p_i} F_1 - X_{p_i} F_2| = 3 \cdot 2(m-1)$.*

Moreover, for any affine transformation

$$t : \mathbb{R}^2 \to \mathbb{R}^2, \tag{3.170}$$
$$x \mapsto Ax + b \tag{3.171}$$

so that the matrix $A \in \mathbb{R}^{2\times 2}$ is nonsingular, $b \in \mathbb{R}^2$ and any projective transformation

$$\rho : \mathbb{P}^2 \to \mathbb{P}^2, \tag{3.172}$$
$$x = (x_1, x_2, x_3) \mapsto Dx \tag{3.173}$$

so that the matrix $D \in \mathbb{R}^{3\times 3}$ is nonsingular and its restriction to the Euclidean space $\mathbb{R}^2$ is given by

$$\rho_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2, \tag{3.174}$$
$$x = (x_1, x_2) \mapsto \left(\frac{\rho(x,1)_1}{\rho(x,1)_3}, \frac{\rho(x,1)_2}{\rho(x,1)_3}\right) \tag{3.175}$$

we get that

$$|F_1 \cap t(F_2)| \leq 3, \tag{3.176}$$
$$|F_1 \cap \rho_{\mathbb{R}^2}(F_2)| \leq 4. \tag{3.177}$$

Proof

Let the lattice sets F_1, F_2 be defined by

$$F_1 := \bigcup_{j=1}^{3} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_1^{(m)}), \tag{3.178}$$

$$F_2 := \bigcup_{j=1}^{3} \varphi_{\mathbb{R}^m}^{a_j,b,c_1^j,c_2^j}(G_2^{(m)}) \tag{3.179}$$

subject to the redefinition of the lattice sets $G_1^{(m)}$ and $G_2^{(m)}$ in (3.78)-(3.79). For the integrality of the lattice sets F_1, F_2 and for both the condition that $F_1 \cap F_2 = \emptyset$ and the condition that $\{p_1, \ldots, p_m\} \subset \operatorname{conv}(F_1) \cap \operatorname{conv}(F_2)$ we refer to Theorem 3.4.5 and Remark 3.4.6. Thus, it remains to show that Lemma 3.2.5, Lemma 3.2.6, Theorem 3.3.6 and Theorem 3.3.8 can be enlarged to the combined lattice sets F_1, F_2 by successively choosing the parameters c_1^1, c_2^1, $c_1^2 = c_1^2(c_1^1, c_2^1)$, $c_2^2 = c_2^2(c_1^1, c_2^1)$ and $c_1^3 = c_1^3(c_1^1, c_2^1, c_1^2, c_2^2)$, $c_2^3 = c_2^3(c_1^1, c_2^1, c_1^2, c_2^2)$ in dependence on the parameters fixed before:

1. *To extend Lemma 3.2.5 we have to avoid the additional case of collinearity for the lattice points $p_k, \varphi_{\mathbb{R}^m}^{a_{j_1},b,c_1^{j_1},c_2^{j_1}}(x^1), \varphi_{\mathbb{R}^m}^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2)$ and $j_2 > j_1$, which is equivalent to*

$$\det\left(\varphi_{\mathbb{R}^m}^{a_{j_1},b,c_1^{j_1},c_2^{j_1}}(x^1) - p_k, \varphi_{\mathbb{R}^m}^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2) - p_k\right) = \tag{3.180}$$

$$= \det\begin{pmatrix} \sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1k})x_i^1 + \bar{c}_1^{j_1} & \sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1k})x_i^2 + \bar{c}_1^{j_2} \\ \sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2k})x_i^1 + \bar{c}_2^{j_1} & \sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2k})x_i^2 + \bar{c}_2^{j_2} \end{pmatrix} \tag{3.181}$$

$$= \bar{c}_2^{j_2}(\sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1k})x_i^1 + \bar{c}_1^{j_1}) - \bar{c}_1^{j_2}(\sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2k})x_i^1 + \bar{c}_2^{j_1})$$
$$+ \textit{ constant term } = 0 \tag{3.182}$$

   for $\bar{c}_1^j := c_1^j - a_j \cdot p_{1k}$, $\bar{c}_2^j := c_1^j - a_j \cdot p_{2k}$. In particular, the determinant is a linear polynomial within both parameters $c_1^{j_2}$ and $c_2^{j_2}$, if the parameters $c_1^{j_1}$, $c_2^{j_1}$ are fixed so that both conditions $\sum_{i=1}^m \kappa^{i-1} b(p_{1i} - p_{1k})x_i^1 + \bar{c}_1^{j_1} \neq 0$ and $\sum_{i=1}^m \kappa^{i-1} b(p_{2i} - p_{2k})x_i^1 + \bar{c}_2^{j_1} \neq 0$ are satisfied for every lattice point $x^1 \in G_1^{(m)} \cup G_2^{(m)}$.

2. *To extend Lemma 3.2.6 we have to consider the determinant of the extended coefficient matrix of the linear equation system*

$$\det\left(s - p_k, \varphi_{\mathbb{R}^m}^{a_{j_k},b,c_1^{j_k},c_2^{j_k}}(x^k) - p_k\right) = 0 \textit{ for } k = 1, 2, 3 \tag{3.183}$$

$$\Leftrightarrow s_1 A_{2k} - s_2 A_{1k} = B_k \textit{ for } k = 1, 2, 3 \tag{3.184}$$

within the variables $s_1, -s_2$, where A_{lk}, B_k are defined by

$$A_{lk} := \sum_{i=1}^{m} \kappa^{i-1} b(p_{li} - p_{lk}) x_i^k + (c_l^{j_k} - a_{j_k} p_{lk}), \tag{3.185}$$

$$B_k := p_{1k} A_{2k} - p_{2k} A_{1k}. \tag{3.186}$$

The equation system again corresponds to the intersection of three lines passing through the lattice points $p_k, \varphi_{\mathbb{R}^m}^{a_{j_k}, b, c_1^{j_k}, c_2^{j_k}}(x^k)$ for $k = 1, 2, 3$ and $j_k \in \{1, 2, 3\}$. It remains to examine the equation system (3.184) for the additional cases $j_1 > j_2 = j_3$, $j_1 = j_2 > j_3$ and $j_1 > j_2 > j_3$ by successively using the elimination arguments of forbidden parameter tuples:

(a) *In the case that $j_1 > j_2 = j_3$ the determinant of the extended coefficient matrix in (3.184) is calculated by*

$$\begin{aligned} & c_2^{j_1}[\det\begin{pmatrix} A_{12} & B_2 \\ A_{13} & B_3 \end{pmatrix} + p_{11} \cdot \det\begin{pmatrix} A_{22} & A_{12} \\ A_{23} & A_{13} \end{pmatrix}] - \\ & \quad - c_1^{j_1}[\det\begin{pmatrix} A_{22} & B_2 \\ A_{23} & B_3 \end{pmatrix} + p_{21} \cdot \det\begin{pmatrix} A_{22} & A_{12} \\ A_{23} & A_{13} \end{pmatrix}] + \\ & \quad + \text{ terms of lower degree in } c_1^{j_1} \text{ and } c_2^{j_1} = \qquad (3.187) \\ & = c_2^{j_1}[(c_1^{j_2})^2 \cdot (p_{22} - p_{23}) + \text{ terms of lower degree in } c_1^{j_2}] - \\ & \quad - c_1^{j_1}[(c_2^{j_2})^2 \cdot (p_{13} - p_{12}) + \text{ terms of lower degree in } c_2^{j_2}] + \ldots \end{aligned}$$

Thus, as $p_2 \neq p_3$ the determinant is polynomial of degree 1 within the parameters $c_1^{j_1}$, $c_2^{j_1}$ in general.

(b) *In the case that $j_1 = j_2 > j_3$ we calculate that*

$$\begin{aligned} & (c_1^{j_1})^2 (p_{21} - p_{22}) A_{23} - (c_2^{j_1})^2 (p_{12} - p_{11}) A_{13} + \\ & \quad + \text{ terms of lower degree in } c_1^{j_1} \text{ and } c_2^{j_1}. \end{aligned} \tag{3.188}$$

Thus, as $p_1 \neq p_2$ and by choosing the parameters $c_1^{j_2}, c_2^{j_2}$ so that $A_{13} \neq 0$ and $A_{23} \neq 0$, the determinant is polynomial of degree 2 within the parameters $c_1^{j_1}$, $c_2^{j_1}$.

(c) *In the last case that $j_1 > j_2 > j_3$ we calculate that*

$$c_2^{j_1}[\det\begin{pmatrix} A_{12} & B_2 \\ A_{13} & B_3 \end{pmatrix} + p_{11} \cdot \det\begin{pmatrix} A_{22} & A_{12} \\ A_{23} & A_{13} \end{pmatrix}] - \tag{3.189}$$

$$\begin{aligned} & - c_1^{j_1}[\det\begin{pmatrix} A_{22} & B_2 \\ A_{23} & B_3 \end{pmatrix} + p_{21} \cdot \det\begin{pmatrix} A_{22} & A_{12} \\ A_{23} & A_{13} \end{pmatrix}] + \\ & + \text{ terms of lower degree in } c_1^{j_1} \text{ and } c_2^{j_1} = \\ & = c_2^{j_1}[c_2^{j_2}(p_{11} - p_{12}) A_{13} + c_1^{j_2}(\ldots) + \ldots] - \\ & \quad c_1^{j_1}[c_1^{j_2}(p_{22} - p_{21}) A_{23} + c_2^{j_2}(\ldots) + \ldots] + \ldots \end{aligned} \tag{3.190}$$

Thus, as $p_1 \neq p_2$ *and by choosing the parameters* $c_1^{j_3}, c_2^{j_3}$ *so that* $A_{13} \neq 0$ *and* $A_{23} \neq 0$*, the determinant is polynomial of degree* 1 *within the parameters* $c_1^{j_1}, c_2^{j_1}$.

3. *Next we want to extend Lemma 3.3.3. The lattice points* $\varphi_{\mathbb{R}^m}^{a_{j_1},b,c_1^{j_1},c_2^{j_1}}(x^1)$, $\varphi_{\mathbb{R}^m}^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2)$ *and* $\varphi_{\mathbb{R}^m}^{a_{j_3},b,c_1^{j_3},c_2^{j_3}}(x^3)$ *are collinear for the lattice points* $x^1, x^2, x^3 \in G_1^{(m)} \cup G_2^{(m)}$ *if and only if*

$$\det\Big(\varphi_{\mathbb{R}^m}^{a_{j_3},b,c_1^{j_3},c_2^{j_3}}(x^3) - \varphi_{\mathbb{R}^m}^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2), \varphi_{\mathbb{R}^m}^{a_{j_1},b,c_1^{j_1},c_2^{j_1}}(x^1) - \varphi_{\mathbb{R}^m}^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2)\Big) = 0. \tag{3.191}$$

It remains to consider the additional cases $j_1 > j_2, j_3$ *and* $j_1 = j_2 > j_3$ *within equation (3.191).*
In the case that $j_1 > j_2, j_3$ *the determinant in (3.191) is calculated by*

$$\begin{aligned}&\frac{c_2^{j_1}}{q_1}(\varphi_{\mathbb{R}^m}^{a_{j_3},b,c_1^{j_3},c_2^{j_3}}(x^3) - \varphi_{\mathbb{R}^m}^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2))_1 - \\ &\quad -\frac{c_1^{j_1}}{q_1}(\varphi_{\mathbb{R}^m}^{a_{j_3},b,c_1^{j_3},c_2^{j_3}}(x^3) - \varphi_{\mathbb{R}^m}^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2))_2 + \\ &\quad + \text{terms of lower degree}\end{aligned} \tag{3.192}$$

for

$$q_k := \sum_{i=1}^{m} \kappa^{i-1} b x_i^k + a_{j_k}. \tag{3.193}$$

Thus, as $\varphi_{\mathbb{R}^m}^{a_{j_3},b,c_1^{j_3},c_2^{j_3}}(x^1) \neq \varphi_{\mathbb{R}^m}^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2)$ *the determinant is linear as polynomial within the parameters* $c_1^{j_1}, c_2^{j_1}$.
In the case that $j_1 = j_2 > j_3$ *the determinant is calculated by*

$$\begin{aligned}&c_2^{j_1}(\frac{1}{q_1} - \frac{1}{q_2})[c_1^{j_3}\frac{1}{q_3} - c_1^{j_1}\frac{1}{q_2} + \ldots] - \\ &\quad -c_1^{j_1}(\frac{1}{q_1} - \frac{1}{q_2})[c_2^{j_3}\frac{1}{q_3} - c_2^{j_1}\frac{1}{q_2} + \ldots] + \ldots\end{aligned} \tag{3.194}$$

Thus, the determinant is quadratic within the parameters $c_1^{j_1}, c_2^{j_1}$ *in general.*

4. *In a further step let us extend Lemma 3.3.4. The lattice points* $\varphi^{a_{j_1},b,c_1^{j_1},c_2^{j_1}}(x^1)$, $\varphi^{a_{j_2},b,c_1^{j_2},c_2^{j_2}}(x^2)$ *and* $\varphi^{a_{j_3},b,c_1^{j_3},c_2^{j_3}}(x^3)$ *are collinear if and only if*

$$\det\begin{pmatrix} q_1^1 + c_1^{j_1} & q_2^1 + c_1^{j_2} & q_3^1 + c_1^{j_3} \\ q_1^2 + c_2^{j_1} & q_2^2 + c_2^{j_2} & q_3^2 + c_2^{j_3} \\ q_1 & q_2 & q_3 \end{pmatrix} = 0 \tag{3.195}$$

for q_k *defined as in (3.193) and for* q_k^l *defined by*

$$q_k^l := \sum_{i=1}^{m} \kappa^{i-1} b p_{li} x_i^k \tag{3.196}$$

for $k = 1, 2, 3$.
In the case that $j_1 > j_2 = j_3$ *we calculate that*

$$c_1^{j_1}[c_2^{j_2}(q_3 - q_2) + \ldots] - c_2^{j_1}[c_1^{j_2}(q_3 - q_2) + \ldots] + \ldots, \tag{3.197}$$

in the case that $j_1 > j_2 > j_3$ *we calculate that*

$$c_1^{j_1}[c_2^{j_2}q_3 - c_2^{j_3}q_2 + \ldots] - c_2^{j_1}[c_1^{j_2}q_3 - \\ - c_1^{j_3}q_2 + \ldots] + \ldots, \tag{3.198}$$

and in the case that $j_1 = j_2 > j_3$ *we calculate that*

$$c_2^{j_1}[c_1^{j_3}(q_2 - q_1) + \ldots] - c_1^{j_1}[c_2^{j_3}(q_2 - q_1) + \ldots] + \ldots \tag{3.199}$$

Thus, in each case the determinant is polynomial of degree 1 *within the parameters* $c_1^{j_1}, c_2^{j_1}$, *as the values* q_k *are different primes.*

5. *To show the affine dissimilarity assertion let the parameters* λ_1, λ_2 *be calculated analogously to Theorem 3.3.6. (By the setting* $c_1^1 = c_1^2 = c_1^3$, $c_2^1 = c_2^2 = c_2^3$ *we again argue that the parameters* λ_1, λ_2 *are uniquely determined by that calculation.) Because of the same arguments as there we can assume that the parameters* λ_1, λ_2 *do not really depend on the parameters* c_1^1, c_2^1, c_1^2, c_2^2 *and* c_1^3, c_2^3.

   (a) *In the case that* $j_1 = j_3 = j_5 = j_7$ *and* $j_2 = j_4 = j_6 = j_8$ *the parameter* λ_1 *is calculated by* $\lambda_1 = \mathrm{const}_1 \frac{q_3}{q_7} = \mathrm{const}_2 \frac{q_4}{q_8}$ *for* $\mathrm{const}_1, \mathrm{const}_2 \neq 0$ *using the same arguments as before. Therefore, the parameter* a *can suitably be adapted.*

   (b) *Let us take a look at the case that* $j_l \notin \{j_1, j_3, j_5, j_7\} \backslash \{j_l\}$ *for* $l \in \{1, 3, 5, 7\}$ *(resp. that* $j_l \notin \{j_2, j_4, j_6, j_8\} \backslash \{j_l\}$ *for* $l \in \{2, 4, 6, 8\}$*): Without loss of generality let us assume that* $j_7 \notin \{j_1, j_3, j_5\}$. *But then the parameter* λ_1 *which is calculated analogously to (3.106) is linear in the parameters* $c_1^{j_7}, c_2^{j_7}$, *as the lattice points* $\varphi_{\mathbb{R}^m}^{a_{j_1}, b, c_1^{j_1}, c_2^{j_1}}(x^1)$, $\varphi_{\mathbb{R}^m}^{a_{j_5}, b, c_1^{j_5}, c_2^{j_5}}(x^5)$ *are different.*

   (c) *It is left to consider the case that without loss of generality* $j_3 = j_7$, $j_1 = j_5$ *and that the indices* j_2, j_4, j_6, j_8 *do not fit to the case (b): The parameter* λ_1 *is calculated by* $\lambda_1 = \frac{q_3}{q_7}\mathrm{const}_1$ *for* $\mathrm{const}_1 \neq 0$, *as the lattice points* $\varphi_{\mathbb{R}^m}^{a_{j_1}, b, c_1^{j_1}, c_2^{j_1}}(x^1)$, $\varphi_{\mathbb{R}^m}^{a_{j_5}, b, c_1^{j_5}, c_2^{j_5}}(x^5)$ *are different. Otherwise, the parameter* λ_1 *is calculated by some multiple of a rational polynomial within the linear factors* $q_2, q_4, q_6, q_8 \in \mathbb{R}[a]$. *Again, the parameter* a *can suitably be adapted.*

6. *For the projective dissimilarity assertion we have to generalize Lemma 3.3.7 to the successive choice of the parameter tuples* (c_1^1, c_2^1), (c_1^2, c_2^2) *and* (c_1^3, c_2^3)*:*
Let the projective points $\varphi^{a_{j_k}, b, c_1^{j_k}, c_2^{j_k}}(x^k)$ *be given for* $k = 1, \ldots, 5$ *satisfying* $j_5 \geq j_4 \geq j_1, j_2, j_3$*. Let* q_k *and* q_k^l *be defined by (3.193), (3.196) and let us denote*

$$\bar{q}_k := q_k - a_{j_k}, \tag{3.200}$$

$$c_1^{j_5} := -q_5^1, \tag{3.201}$$

$$c_2^{j_5} := -q_5^2, \tag{3.202}$$

$$a_{j_4} := -\bar{q}_4. \tag{3.203}$$

Notice, that $\bar{q}_{k_1}, \bar{q}_{k_2}$ *are different numbers, if* $j_{k_1} = j_{k_2}$*.*

(a) *In the case that* $j_l = j_2 > j_3$ *for* $l \in \{1, 4, 5\}$ *we calculate analogously to Lemma 3.3.7 for the translation* $(p_{1j}, p_{2j}) \mapsto (p_{1j}, p_{2j} + k)$ *that*

$$\det \begin{pmatrix} q_l^1 + c_1^{j_l} & q_2^1 + c_1^{j_2} & q_3^1 + c_1^{j_3} \\ q_l^2 + c_2^{j_l} + k \cdot \bar{q}_l & q_2^2 + c_2^{j_2} + k \cdot \bar{q}_2 & q_3^2 + c_2^{j_3} + k \cdot \bar{q}_3 \\ \bar{q}_l + a_{j_l} & \bar{q}_2 + a_{j_2} & \bar{q}_3 + a_{j_3} \end{pmatrix} = \tag{3.204}$$

$$= \det \begin{pmatrix} q_l^1 + c_1^{j_l} & q_2^1 - q_l^1 & q_3^1 + c_1^{j_3} \\ q_l^2 + c_2^{j_l} + k \cdot \bar{q}_l & q_2^2 - q_l^2 + k(\cdot\bar{q}_2 - \bar{q}_l) & q_3^2 + c_2^{j_3} + k \cdot \bar{q}_3 \\ \bar{q}_l + a_{j_l} & \bar{q}_2 - \bar{q}_l & \bar{q}_3 + a_{j_3} \end{pmatrix}$$

$$= \det \begin{pmatrix} q_l^1 + c_1^{j_l} & q_2^1 - q_l^1 & q_3^1 + c_1^{j_3} \\ q_l^2 + c_2^{j_l} - k \cdot a_{j_l} & q_2^2 - q_l^2 & q_3^2 + c_2^{j_3} - k \cdot a_{j_3} \\ \bar{q}_l + a_{j_l} & \bar{q}_2 - \bar{q}_l & \bar{q}_3 + a_{j_3} \end{pmatrix} =$$

$$= (c_2^{j_l} - k \cdot a_{j_l}) \cdot c_1^{j_3} \cdot (\bar{q}_2 - \bar{q}_l) + \text{terms at most of degree 1 within } k,\ c_1^{j_3} \tag{3.205}$$

$$\begin{cases} -k \cdot q_l \cdot c_1^{j_3} \cdot (\bar{q}_2 - \bar{q}_l) + \ldots & \text{for } j_l = j_5 \\ -k \cdot a_{j_l} \cdot c_1^{j_3} \cdot (\bar{q}_2 - \bar{q}_l) + \ldots & \text{for } j_l < j_5 \end{cases}, \tag{3.206}$$

which is in general linear within the parameter k *except for one value of the parameter* $c_1^{j_3}$ *so that possibly the previous fixed parameter* $c_1^{j_3}$ *has to be adapted.*

(b) *In the case that* $j_l > j_2 = j_3$ *for* $l \in \{1, 4, 5\}$ *we calculate for the translation* $(p_{1j}, p_{2j}) \mapsto (p_{1j}, p_{2j} + k)$ *that*

$$\det \begin{pmatrix} q_l^1 + c_1^{j_l} & q_2^1 + c_1^{j_2} & q_3^1 + c_1^{j_3} \\ q_l^2 + c_2^{j_l} + k \cdot \bar{q}_l & q_2^2 + c_2^{j_2} + k \cdot \bar{q}_2 & q_3^2 + c_2^{j_3} + k \cdot \bar{q}_3 \\ \bar{q}_l + a_{j_l} & \bar{q}_2 + a_{j_2} & \bar{q}_3 + a_{j_3} \end{pmatrix} = \tag{3.207}$$

$$= \det \begin{pmatrix} q_l^1 + c_1^{j_l} & q_2^1 - q_3^1 & q_3^1 + c_1^{j_3} \\ q_l^2 + c_2^{j_l} + k \cdot \bar{q}_l & q_2^2 - q_3^2 + k \cdot (\bar{q}_2 - \bar{q}_3) & q_3^2 + c_2^{j_3} + k \cdot \bar{q}_3 \\ \bar{q}_l + a_{j_l} & \bar{q}_2 - \bar{q}_3 & \bar{q}_3 + a_{j_3} \end{pmatrix}$$

$$= \det \begin{pmatrix} q_l^1 + c_1^{j_l} & q_2^1 - q_3^1 & q_3^1 + c_1^{j_3} \\ q_l^2 + c_2^{j_l} - k \cdot a_{j_l} & q_2^2 - q_3^2 & q_3^2 + c_2^{j_3} - k \cdot a_{j_3} \\ \bar{q}_l + a_{j_l} & \bar{q}_2 - \bar{q}_3 & \bar{q}_3 + a_{j_3} \end{pmatrix} =$$

$$= (c_2^{j_l} - k \cdot a_{j_l}) \cdot c_1^{j_3} \cdot (\bar{q}_2 - \bar{q}_3) + \text{terms at most of degree 1 within } k,\ c_1^{j_3}, \tag{3.208}$$

which is again linear within the parameter k in general because of the same arguments as before.

(c) *In the case that $j_l > j_2 > j_3$ for $l \in \{1,4,5\}$ we calculate for the translation $(p_{1j}, p_{2j}) \mapsto (p_{1j}, p_{2j} + k)$ that*

$$\det \begin{pmatrix} q_l^1 + c_1^{j_l} & q_2^1 + c_1^{j_2} & q_3^1 + c_1^{j_3} \\ q_l^2 + c_2^{j_l} + k \cdot \bar{q}_l & q_2^2 + c_2^{j_2} + k \cdot \bar{q}_2 & q_3^2 + c_2^{j_3} + k \cdot \bar{q}_3 \\ \bar{q}_l + a_{j_l} & \bar{q}_2 + a_{j_2} & \bar{q}_3 + a_{j_3} \end{pmatrix} = \tag{3.209}$$

$$= -(c_2^{j_l} + k \cdot \bar{q}_l) \cdot c_1^{j_2} \cdot (\bar{q}_3 + a_{j_3}) + \text{terms at most of degree 1 within } k,\ c_1^{j_2} \tag{3.210}$$

$$= \begin{cases} k \cdot (\bar{q}_5 - \bar{q}_l) \cdot c_1^{j_2} \cdot (\bar{q}_3 + a_{j_3}) + \dots & \text{for } j_l = j_5 \\ -k \cdot \bar{q}_l \cdot c_1^{j_2} \cdot (\bar{q}_3 + a_{j_3}) + \dots & \text{for } j_l < j_5 \end{cases}, \tag{3.211}$$

which is again polynomial within the parameter k by possibly adapting the previously fixed parameter $c_1^{j_2}$.

Therefore, we can successively choose the parameter tuples (c_1^1, c_2^1), $(c_1^2(c_1^1, c_2^1), c_2^2(c_1^1, c_2^1))$ and $(c_1^3(c_1^1, c_2^1, c_1^2, c_2^2), c_2^3(c_1^1, c_2^1, c_1^2, c_2^2))$ in order to guarantee noncollinearity for the parameter setting (3.201)-(3.203), which is necessary to extend Theorem 3.3.8. (Possibly we have to apply some translation or have to adapt previously fixed parameters.) For the extention of Theorem 3.3.8 we use analogous arguments as before in the proof of Theorem 3.3.8 itself.

□

Remark 3.4.8 Notice, that the error value according to our construction is given by the value $3 \cdot 2(m-1)$ instead of the minimal data error of value $2(m-1)$. The smallest possible error value, however, is not known for the case that the point X-ray sources are located within the convex hull of the lattice sets.

3.5 On the generalization to the lattice set $\mathbb{Z}^n$

In the following we will focus on some generalized version of the instability results before by considering the lattice set $\mathbb{Z}^n$ for $n \geq 2$ instead of only the planar lattice set $\mathbb{Z}^2$.
Without loss of generality let us assume that the number $m \geq 3$ of point X-ray sources is greater than the dimension $n \geq 2$ of the considered lattice set $\mathbb{Z}^n$, as otherwise we project on the affine span of the m point X-ray sources instead of the lattice set $\mathbb{Z}^n$. Let the projective transformation φ^n be defined by

$$\varphi^n : \mathbb{P}^m \to \mathbb{P}^n, \tag{3.212}$$
$$\varphi^n(x_1, \ldots, x_{m+1})_j := \begin{cases} \sum_{i=1}^m \kappa^{i-1} b p_{ji} x_i + c_j x_{m+1} & \text{for } j = 1, \ldots, n \\ \sum_{i=1}^m \kappa^{i-1} b x_i + a x_{m+1} & \text{for } j = n+1 \end{cases}$$

and its restriction $\varphi^n_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^n$ between the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ analogously as before.
As similar methods are used as before, we will not describe every argument in detail for the generalized assertions in the following.

1. For the integrality and the distinctness of the lattice sets $F_1 := \varphi^n_{\mathbb{R}^m}(G_1^{(m)} \backslash \{0\})$, $F_2 := \varphi^n_{\mathbb{R}^m}(G_2^{(m)} \backslash \{(0, l^2, 0, \ldots, 0\}$ we recall the arguments in Lemma 3.2.3 and Lemma 3.2.4.

2. We also show that $X_{p_j} F_k(u) \in \{0, 1\}$ is satisfied for $j = 1, \ldots, m$, $k = 1, 2$ and any lattice direction u, as the collinearity of the lattice points p_j, $\varphi^n_{\mathbb{R}^m}(x^1)$, $\varphi^n_{\mathbb{R}^m}(x^2)$ leads to n simultaneous conditions for every component similar to those in (3.44)-(3.45). The same arguments as applied to the case $n = 2$ before work to show that the lattice points x^1, x^2 cannot both belong to the lattice set F_1 resp. to the lattice set F_2.

3. We adapt Lemma 3.2.6 to the case $n \geq 2$ by demanding that the determinant of every (2×2)-submatrix within the $(n \times 2)$-matrices $[s - p_k, \varphi^n_{\mathbb{R}^m}(x^k) - p_k]$ for $k = 1, \ldots, m$ equals 0, in order to generalize the equation system (3.53). Similar arguments as used there assure that the lines passing through the lattice points $\varphi^n_{\mathbb{R}^m}(x^k)$ and p_k are not identical. By looking at each (3×3)-submatrix within the extended coefficient matrix which belongs to the determinant equations for the variables s_1, $-s_2$ and by examining the situation that all coefficients of any monomial c_k^2 for $k = 1, \ldots, n$ equal zero in all occuring equations (3.63), we result in (3.65) for $k = 1, \ldots, n$. The further argumentation again leads to contradiction, as Lemma 3.2.2 can be generalized to the case $n \geq 2$ by choosing $a_k := e_k + \sum_{i<k} \lambda_i \cdot e_i$, $k = 3, \ldots, n$ for suitable parameters λ_i.

4. In order to treat the affine dissimilarity assertion, let us assume in the following that the lattice points $\varphi^n_{\mathbb{R}^m}(x^1), \varphi^n_{\mathbb{R}^m}(x^3), \ldots, \varphi^n_{\mathbb{R}^m}(x^{2n+1}) \in F_1$

as well as the lattice points $\varphi^n_{\mathbb{R}^m}(x^2), \varphi^n_{\mathbb{R}^m}(x^4), \ldots, \varphi^n_{\mathbb{R}^m}(x^{2n+2}) \in F_2$ are affinely independent. Let us apply the Fundamental Theorem of Affine Geometry as in Theorem 3.3.6 by assuming that

$$\varphi^n_{\mathbb{R}^m}(x^k) \mapsto \varphi^n_{\mathbb{R}^m}(x^{k+1}) \text{ for } k = 1, 3, \ldots, 2n+3 \tag{3.213}$$

by some affine transformation $t : \mathbb{R}^n \to \mathbb{R}^n$, $t(x) := Ax + b$ for $x^1, x^3, \ldots, x^{2n+3} \in G_1^{(m)}$, $x^2, x^4, \ldots, x^{2n+4} \in G_2^{(m)}$ pairwise distinct lattice points if and only if the equation system

$$\sum_{i=1}^{n} \lambda_i(\varphi^n_{\mathbb{R}^m}(x^{2i+1}) - \varphi^n_{\mathbb{R}^m}(x^1)) = \varphi^n_{\mathbb{R}^m}(x^{2n+3}) - \varphi^n_{\mathbb{R}^m}(x^1), \tag{3.214}$$

$$\sum_{i=1}^{n} \lambda_i(\varphi^n_{\mathbb{R}^m}(x^{2i+2}) - \varphi^n_{\mathbb{R}^m}(x^2)) = \varphi^n_{\mathbb{R}^m}(x^{2n+4}) - \varphi^n_{\mathbb{R}^m}(x^2) \tag{3.215}$$

within the variables $\lambda_1, \ldots, \lambda_n$ is uniquely solved.
Let the matrix $A = (a_{i,j})_{i,j=1,\ldots,n}$ be defined by

$$a_{i,j} := \frac{q^i_{2j+1} - q^i_1}{q_{2j+1}} + \bar{c}_i\left(\frac{1}{q_{2j+1}} - \frac{1}{q_1}\right), \tag{3.216}$$

for $q_k, q^l_k, \bar{c}_k$ given by (3.109)-(3.111). Notice, that

$$\det A = \frac{1}{(q_1)^n \prod_{j=1}^{n} q_{2j+1}} \cdot \cdot \det\left(q_1(q^i_{2j+1} - q^i_1) + \bar{c}_i(q_1 - q_{2j+1})\right)_{i,j=1,\ldots,n} \tag{3.217}$$

$$= \frac{C_1(A)\bar{c}_1 + \cdots + C_n(A)\bar{c}_n + C_0(A)q_1}{\prod_{j=0}^{n} q_{2j+1}}, C_i(A) \in \mathbb{R} \text{ for } i = 0, \ldots, n \tag{3.218}$$

is at most of degree 1 as polynomial in the ring $\mathbb{R}[\bar{c}_1, \ldots, \bar{c}_n]$, as the determinant of any $(k \times k)$-dimensional submatrix of the matrix

$$\left(a_{i,j} - \frac{q^i_{2j+1} - q^i_1}{q_{2j+1}}\right)_{i,j=1,\ldots,n} = \left(\bar{c}_i \cdot \left(\frac{1}{q_{2j+1}} - \frac{1}{q_1}\right)\right)_{i,j=1,\ldots,n} \tag{3.219}$$

equals 0 for $k \geq 2$. Moreover, the coefficients $C_i(A)$ for $i = 0, \ldots, n$ are independent on the parameter a, as the differences $q_1 - q_{2j+1}$ for $j = 1, \ldots, n$ are independent on the parameter a.

The same arguments work to show that any component

$$\operatorname{adj}(A)_{k,l} = (-1)^{k+l} \det(a_{i,j})_{i \neq l, j \neq k} = \tag{3.220}$$

$$= \frac{(-1)^{k+l}}{(q_1)^{n-1} \prod_{j=1,\ldots,n, j\neq k} q_{2j+1}} \cdot \cdot \det(q_1(q^i_{2j+1} - q^i_1) + \bar{c}_i(q_1 - q_{2j+1}))_{i\neq l, j\neq k} = \tag{3.221}$$

$$= \frac{\sum_{i\neq l} C^{k,l}_i(\operatorname{adj}(A))\bar{c}_i + C^{k,l}_0(\operatorname{adj}(A))q_1}{\prod_{j=0,\ldots,n, j\neq k} q_{2j+1}}, C^{k,l}_i(\operatorname{adj}(A)) \in \mathbb{R} \tag{3.222}$$

of the adjacency matrix $\mathrm{adj}(A)$ to the matrix A is at most of degree 1 as polynomial in the ring $\mathbb{R}[\bar{c}_1, \ldots, \bar{c}_n]$ as well and that the coefficients $C_i^{k,l}(\mathrm{adj}(A))$ for $i = 0, 1, \ldots, n$ and $k, l = 1, \ldots, n$ are also independent on the parameter a.
We calculate that

$$\mathrm{adj}(A) \begin{pmatrix} \frac{q_{2n+3}^1 - q_1^1}{q_{2n+3}} + \bar{c}_1\left(\frac{1}{q_{2n+3}} - \frac{1}{q_1}\right) \\ \cdots \\ \frac{q_{2n+3}^n - q_1^n}{q_{2n+3}} + \bar{c}_n\left(\frac{1}{q_{2n+3}} - \frac{1}{q_1}\right) \end{pmatrix} = \tag{3.223}$$

$$= \left(\frac{1}{q_1 q_{2n+3}} \sum_{k=1}^{n} \mathrm{adj}(A)_{l,k} (q_1(q_{2n+3}^k - q_1^k) + \bar{c}_k(q_1 - q_{2n+3})) \right)_{l=1,\ldots,n} \tag{3.224}$$

$$= \Big(\frac{1}{q_1 q_{2n+3}} \sum_{k=1}^{n} \frac{(-1)^{l+k}}{(q_1)^{n-1} \prod_{j=1,\ldots,n, j\neq l} q_{2j+1}} \cdot \\ \cdot \det\left((q_1(q_{2j+1}^i - q_1^i) + \bar{c}_i(q_1 - q_{2j+1}))_{i\neq k, j\neq l} \right) \cdot \tag{3.225} \\ \cdot (q_1(q_{2n+3}^k - q_1^k) + \bar{c}_k(q_1 - q_{2n+3})) \Big)_{l=1,\ldots,n} =$$

$$= \left(\frac{\sum_{i=1}^{n} D_i^l(A)\bar{c}_i + D_0^l(A) q_1}{\prod_{j=0,\ldots,n+1, j\neq l} q_{2j+1}} \right)_{l=1,\ldots,n}, \; D_i^l(A) \in \mathbb{R}, \tag{3.226}$$

as any (mixed) quadratic term in the parameters $\bar{c}_1, \ldots, \bar{c}_n$ appears twice, but different signed:
For some fixed index l let us consider the quadratic term with respect to the monomial $\bar{c}_{k_1}\bar{c}_{k_2}$ for $k_1 \neq k_2$ so that $|k_1 - k_2| = 1$. Notice, that the coefficient of $\bar{c}_{k_1}(q_1 - q_{2\bar{j}+1})$ in $\det(q_1(q_{2j+1}^i - q_1^i) + \bar{c}_i(q_1 - q_{2j+1}))_{i\neq k_2, j\neq l}$ and the coefficient of $\bar{c}_{k_2}(q_1 - q_{2\bar{j}+1})$ in $\det(q_1(q_{2j+1}^i - q_1^i) + \bar{c}_i(q_1 - q_{2j+1}))_{i\neq k_1, j\neq l}$ are equal for every $\bar{j} \neq l$. Thus, the factors $(-1)^{l+k_2}$, $(-1)^{l+k_1}$ are responsible for the cancelation. To reduce the general case $|k_1 - k_2| > 1$ to the case considered before, we have to apply $|k_1 - k_2| - 1$ transpositions preserving the calculation, as $(-1)^{l+k_1} \cdot (-1)^{l+k_2} \cdot (-1)^{|k_1-k_2|-1} = -1$. Thus, also in the case that $|k_1 - k_2| > 1$ the (mixed) quadratic terms are canceled.
Therefore, the parameters $\lambda_1, \ldots, \lambda_n$ in (3.214) are calculated as rational polynomials over the ring $\mathbb{R}[\bar{c}_1, \ldots, \bar{c}_n, a]$ by

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \frac{1}{\det(A)} \mathrm{adj}(A) \begin{pmatrix} \frac{q_{2n+3}^1 - q_1^1}{q_{2n+3}} + \bar{c}_1\left(\frac{1}{q_{2n+3}} - \frac{1}{q_1}\right) \\ \vdots \\ \frac{q_{2n+3}^n - q_1^n}{q_{2n+3}} + \bar{c}_n\left(\frac{1}{q_{2n+3}} - \frac{1}{q_1}\right) \end{pmatrix}. \tag{3.227}$$

By (3.226) and by (3.218) we know the factorization into the linear factors. Similar results are obtained using the equation system (3.215) instead of the equation system (3.214).

As $q_1, \ldots, q_{2n+4}$ are different primes, we conclude analogously as before in the case $n = 2$ that the equation system (3.214)-(3.215) cannot be solved for the parameters $\lambda_1, \ldots, \lambda_n$ in general. Thus, we result in contradiction to our mapping assumption (3.213).
Therefore, any kind of affine overlapping for the lattice sets F_1, F_2 is bounded from above by

$$|F_1 \cap t(F_2)| \leq n + 1 \tag{3.228}$$

for any affine transformation $t : \mathbb{R}^n \to \mathbb{R}^n$.
The assumption of affine independency at the beginning is not really restrictive, as any affine transformation obtains affine dependencies. Thus, we can restrict the calculation of the dependency parameters λ_i to a minimal set of affinely dependent lattice points and lower dimension.

5. Next we will show the projective dissimilarity assertion for the generalized situation within the lattice set $\mathbb{Z}^n$ for $n \geq 2$. For that purpose let us assume that

$$[\varphi^n_{\mathbb{R}^m}(x^k), 1] \in \mathbb{P}^n \mapsto [\varphi^n_{\mathbb{R}^m}(x^{k+1}), 1] \in \mathbb{P}^n \text{ for } k = 1, 3, \ldots, 2n + 5 \tag{3.229}$$

by some projective transformation

$$\rho : \mathbb{P}^n \to \mathbb{P}^n, \tag{3.230}$$

$$\rho : \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} \end{bmatrix}, A = (a_{i,j})_{i,j=1,\ldots,n+1} \text{ nonsingular.} \tag{3.231}$$

As in the case $n = 2$ let us look at the associated homogeneous linear equation system

$$B(a_{1,1}, a_{1,2}, \ldots, a_{1,n+1}, a_{2,1}, \ldots, a_{n+1,n+1}, -\lambda_2, \ldots, -\lambda_{2n+6})^T = 0, \tag{3.232}$$

which has to be nontrivially solved by $\lambda_{k+1} \neq 0$ for $k = 1, 3, \ldots, 2n+5$ and the matrix $A = (a_{i,j})_{i,j=1,\ldots,n+1}$ nonsingular. Let us look at the parameter setting $c_l := -q^l_{2(n+2)}$ for $l = 1, \ldots, n$ and $a := -q_{2(n+3)}$. Each subset of $n+1$ projective points within the set $\{\varphi^n(x^k) | k = 1, 2, \ldots, 2n + 6\} \subset \mathbb{P}^n$ which does not include both the projective point $\varphi^n(x^{2(n+2)})$ and the projective point $\varphi^n(x^{2(n+3)})$ can be assumed to be noncollinear:
Without loss of generality let us assume that

$$\varphi^n(x^1) \in \operatorname{span}\{\varphi^n(x^2), \ldots, \varphi^n(x^{l+1})\} \subsetneq \mathbb{R}^{n+1} \tag{3.233}$$

and

$$\dim \operatorname{span}\{\varphi^n(x^2), \ldots, \varphi^n(x^{l+1})\} = l, \tag{3.234}$$

the value $2 \leq l \leq n$ (compare the distinctness assertion in Lemma 3.2.4) attaining the minimal value. In the following argumentation it is only important that neither the projective point $\varphi^n(x^{2(n+2)})$ nor the projective point $\varphi^n(x^{2(n+3)})$ occupies the role of $\varphi^n(x^1)$ in (3.233). We calculate for the linear space $\operatorname{span}\{\varphi^n(x^1), \ldots, \varphi^n(x^{l+1})\}$ that

$$\operatorname{span}\{\varphi^n(x^1), \ldots, \varphi^n(x^{l+1})\} = \tag{3.235}$$
$$= \operatorname{span}\{\varphi^n(x^2) - \varphi^n(x^1), \ldots, \varphi^n(x^{l+1}) - \varphi^n(x^1)\} \tag{3.236}$$
$$\neq \{x_{n+1} = 0\} \tag{3.237}$$

for the linearly independent vectors $\varphi^n(x^2)-\varphi^n(x^1), \ldots, \varphi^n(x^{l+1})-\varphi^n(x^1)$, as $q_k \neq q_1$ for $k = 2, \ldots, l+1$. For further considerations let us assume that $l = n$, as otherwise we reduce to lower dimension.
As in the case $n = 2$ let us consider the translations $p_j \in \mathbb{R}^n \mapsto p_j + k \cdot e_r \in \mathbb{R}^n$ for $r = 1, \ldots, n$ and the calculation of the determinant after some row and column manipulations adequate to (3.124). As the vector space $\{x_{n+1} = 0\}$ represents any combination of translations, the determinant is not equal to zero in general by having (3.237) in mind.

Thus, the submatrices

$$(b_{i,j})_{i=1,\ldots,(n+1)(n+2),j=1,\ldots,(n+1)(n+2)} \tag{3.238}$$

and

$$(b_{i,j})_{i=1,\ldots,(n+1)(n+1),(n+1)(n+2)+1,\ldots,(n+1)(n+2)+n+1,j=1,\ldots,(n+1)(n+2)} \tag{3.239}$$

of the matrix $B = (b_{ij})_{i=1,\ldots,(n+1)(n+3),j=1,\ldots,(n+1)(n+1)+n+2}$ are regular. As before we use the Exchange Theorem of linear algebra in order to replace the $(n+1)(n+2)$-th row within the first matrix to find a basis of $\{x \in \mathbb{R}^{(n+1)(n+1)+n+2} | x_{(n+1)(n+1)+n+2} = x_{(n+1)(n+1)+n+1} = 0\}$, which can be completed to a basis of the Euclidean space $\mathbb{R}^{(n+1)(n+1)+n+2}$ by two rows or manipulated rows within the matrix B afterwards. Thus, the setting $c_l := -q^l_{2(n+2)}$ for $l = 1, \ldots, n$ and $a := -q_{2(n+3)}$ shows that the determinant of at least one of the maximal quadratic submatrices of the matrix B has to be polynomial in at least one of the parameters $c_1, \ldots, c_n, a$. Therefore, the mapping assumption (3.229) cannot be fulfilled besides some parameter combinations, which we have to eliminate.

Now we are ready to formulate the generalized version of the instability result for point X-rays.

Theorem 3.5.1

Let $m \geq 3$ and $n \geq 2$ satisfying $m > n$ and suppose that $p_j = (p_{1j}, \ldots, p_{nj}) \in \mathbb{Z}^n$, $j = 1, \ldots, m$ are distinct lattice points within the lattice set $\mathbb{Z}^n$. For any $\alpha \in \mathbb{N}$ there exist two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^n$ which satisfy

- *F_k for $k = 1, 2$ is uniquely determined by the point X-rays $X_{p_j} F_k$ for $j = 1, \ldots, m$,*
- *$F_1 \cap F_2 = \emptyset$,*
- *$|F_1| = |F_2| \geq \alpha$,*
- *$\sum_{i=1}^{m} |X_{p_i} F_1 - X_{p_i} F_2| = 2(m-1)$.*

Moreover, it yields that

$$|F_1 \cap t(F_2)| \leq n + 1 \tag{3.240}$$

for any affine transformation $t : \mathbb{R}^n \to \mathbb{R}^n$ and

$$|F_1 \cap \rho_{\mathbb{R}^n}(F_2)| \leq n + 2 \tag{3.241}$$

for any projective transformation $\rho : \mathbb{P}^n \to \mathbb{P}^n$ and $\rho_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ its restriction to the Euclidean space $\mathbb{R}^n$.

Proof

All necessary generalizations are mentioned before. □

3.6 Consequences for the parallel geometry

The usage of projective transformations to get instability results for point X-ray sources and the technical methods used before motivate us to look at parallel X-rays again. We will show that for any set of m different lattice directions there are two uniquely determined and distinct finite lattice sets F_1, $F_2 \subset \mathbb{Z}^2$ of same, but arbitrarily large cardinality which have distance $2(m-1)$ within the right hand side data and satisfy

$$|F_1 \cap t(F_2)| \leq 2^{m-1} - 1 \tag{3.242}$$

for any affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$. Therefore, the result in [7], [5], [3] also based on projection methods can be strengthened without using algebraic curves, but the knowledge of the solvability of equation systems, as the upper bound in [5], [3] is given by the value $(3 \cdot 2^{m-2} + 1)^2((32m - 44)^2 + 2) =: f(m)$ instead of the value $2^{m-1} - 1 =: g(m)$. The new upper bound $g(m)$ is still exponential within the number m of X-ray directions, but it yields that $g(m) \in O(\sqrt{f(m)})$.
Moreover, we can guarantee that $|F_1 \cap t(F_2)|$ rises above the value 4 only in the translational case $t(x) := x + b$ for $b \in \mathbb{Z}^2$ or for some half-around rotation $t(x) := -x + b$ for $b \in \mathbb{Z}^2$.
In the following the elimination of forbidden parameter values will not be discussed in detail, as the strategy is the same as in the sections before.
Later, non-projective construction methods reduce further attempts of strengthening the affine dissimilarity to the search of switching components of minimal cardinality.

3.6.1 Technical details

Let $u_j = (u_{1j}, u_{2j}) \in \mathbb{Z}^2\backslash\{0\}$, $j = 1, \ldots, m$ be m different X-ray directions satisfying $\gcd(u_{1j}, u_{2j}) = 1$ and let the projective transformation φ be defined by

$$\varphi : \mathbb{P}^m \to \mathbb{P}^2, \tag{3.243}$$

$$\varphi(x_1, \ldots, x_{m+1}) := \tag{3.244}$$

$$(\sum_{i=1}^{m} \mu_i u_{1i} x_i + x_{m+1}, \sum_{i=1}^{m} \mu_i u_{2i} x_i + x_{m+1}, x_{m+1})$$

for suitably chosen parameters $\mu_1, \ldots, \mu_m \in \mathbb{N}$, using homogeneous coordinates in both projective spaces $\mathbb{P}^m$ and $\mathbb{P}^2$. The restriction of the projective transformation φ as mapping between the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^2$ is denoted by $\varphi_{\mathbb{R}^m}$ again.
The parameters $\mu_i \in \mathbb{N}$ will be suitably specified by choosing non-forbidden parameter tuples.
The following assertions can be formulated:

1. *The jth vector e_j in the standard orthonormal basis of the Euclidean space $\mathbb{R}^{m+1}$ is mapped to $\varphi(e_j) = (\mu_j u_{1j}, \mu_j u_{2j}, 0)$ for $j = 1, \ldots, m$, which shows that the projective transformation φ maps the jth coordinate direction within the Euclidean space $\mathbb{R}^m$ to the lattice direction u_j for $j = 1, \ldots, m$.*

2. *Integrality for the projected lattice set $\varphi_{\mathbb{R}^m}(\mathbb{Z}^m)$ is trivially given because of $u_{ij}, \mu_j \in \mathbb{Z}$ for $i = 1, 2$ and $j = 1, \ldots, m$.*

3. *Let us assume that*

$$\varphi_{\mathbb{R}^m}(x^1) = \varphi_{\mathbb{R}^m}(x^2) \tag{3.245}$$

$$\Leftrightarrow \sum_{i=1}^{m} \mu_i u_{ki} x_i^1 = \sum_{i=1}^{m} \mu_i u_{ki} x_i^2 \text{ for } k = 1, 2 \tag{3.246}$$

$$\Leftrightarrow x_i^1 = x_i^2 \text{ for } i = 1, 2, \ldots, m \tag{3.247}$$

by choosing $\mu_i \ggg \mu_{i-1}$ sufficiently large with respect to the values $\max |u_{ij}|$ and $\max |x_i^k|$.

4. *Let us assume that*

$$\varphi_{\mathbb{R}^m}(x^1) - \varphi_{\mathbb{R}^m}(x^2) = \lambda u_j \tag{3.248}$$

$$\Leftrightarrow \det \begin{pmatrix} \sum_{i=1}^m \mu_i u_{1i}(x_i^1 - x_i^2) & u_{1j} \\ \sum_{i=1}^m \mu_i u_{2i}(x_i^1 - x_i^2) & u_{2j} \end{pmatrix} =$$

$$= \sum_{i=1}^{m} \mu_i (x_i^1 - x_i^2) \det (u_i, u_j) = 0 \tag{3.249}$$

$$\Leftrightarrow (x_i^1 - x_i^2) \det(u_i, u_j) = 0 \text{ for } i = 1, 2, \ldots, m \tag{3.250}$$

by choosing $\mu_i \ggg \mu_{i-1}$ sufficiently large with respect to the values $\max |\det(u_i, u_j)|$ and $\max |x_i^1 - x_i^2|$.
Therefore, it yields that $X_{u_j} F_k \in \{0, 1\}$, if we assume that the lattice points $x^1 \neq x^2$ differ in at least two components (not only in the jth component) for all lattice points $x^1, x^2 \in F_k$ for $k = 1, 2$.

5. *Let some lattice point s be the intersection point of m lines in the directions u_j for $j = 1, \ldots, m$ passing through the lattice points $\varphi_{\mathbb{R}^m}(x^j)$, i. e.*

$$\det\left(s - \varphi_{\mathbb{R}^m}(x^j), u_j\right) = \det \begin{pmatrix} s_1 - (\sum_{i=1}^m \mu_i u_{1i} x_i^j + 1) & u_{1j} \\ s_2 - (\sum_{i=1}^m \mu_i u_{2i} x_i^j + 1) & u_{2j} \end{pmatrix} = 0 \tag{3.251}$$

$$\Leftrightarrow s_1 u_{2j} - s_2 u_{1j} = u_{2j}(\sum_{i=1}^{m} \mu_i u_{1i} x_i^j + 1) - u_{1j}(\sum_{i=1}^{m} \mu_i u_{2i} x_i^j + 1) \tag{3.252}$$

for $j = 1, 2, \ldots, m$. According to the proof of Theorem 2.5 in [73] we can assume that the lattice directions u_1, u_2 and u_3 are given by

$$u_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_3 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3.253}$$

as parallelism and intersection points are preserved by any affine transformation, in particular by the affine transformation used there. To guarantee the solvability of the equation system (3.252), we necessarily have to demand that the determinant of the extended coefficient matrix with respect to the first three equations is equal to zero, i. e.

$$\det \begin{pmatrix} 0 & -1 & -(\sum_{i=1}^m \mu_i u_{2i} x_i^1 + 1) \\ 1 & 0 & \sum_{i=1}^m \mu_i u_{1i} x_i^2 + 1 \\ 1 & -1 & \sum_{i=1}^m \mu_i u_{1i} x_i^3 - \sum_{i=1}^m \mu_i u_{2i} x_i^3 \end{pmatrix} = \tag{3.254}$$

$$= (\sum_{i=1}^m \mu_i u_{2i} x_i^1 + 1) - (\sum_{i=1}^m \mu_i u_{1i} x_i^2 + 1) +$$

$$+(\sum_{i=1}^m \mu_i u_{1i} x_i^3 - \sum_{i=1}^m \mu_i u_{2i} x_i^3) = 0 \tag{3.255}$$

$$\Leftrightarrow u_{2i}(x_i^1 - x_i^3) = u_{1i}(x_i^2 - x_i^3) \text{ for } i = 1, 2, \ldots, m \tag{3.256}$$

by choosing $\mu_i \ggg \mu_{i-1}$ sufficiently large with respect to the values $\max |u_{ij}|$ and $\max |x_i^1 - x_i^2|$. For $i = 1, 2, 3$ we conclude that

$$x_1^2 = x_1^3, \tag{3.257}$$

$$x_2^1 = x_2^3, \tag{3.258}$$

$$x_3^1 = x_3^2. \tag{3.259}$$

In the case that $i > 3$ we have to demand that

$$\det \begin{pmatrix} u_{1i} & x_i^1 - x_i^3 \\ u_{2i} & x_i^2 - x_i^3 \end{pmatrix} = 0. \tag{3.260}$$

Thus, the set of possible values for each component i has to be suitably restricted in order to exclude that condition (3.260) is satisfied besides the case that $x_i^1 = x_i^2 = x_i^3$. If that has been done, then there exists a lattice point q within the original lattice set $\mathbb{Z}^m$ which is mapped to the lattice point s by the restriction $\varphi_{\mathbb{R}^m}$ of the projective transformation φ and which is given by the intersection of the lines in the directions e_j passing through the lattice points x^j for $j = 1, 2, 3$ because of (3.257)-(3.260).

3.6.2 Main result

The technical details before will help us to strengthen the instability results in [5], [3] by a suitable definition of recursively defined lattice sets $G_1^{(l)}, G_2^{(l)} \subset \mathbb{Z}^m$ *within the original lattice set* $\mathbb{Z}^m$ *for sufficiently large* $l \in \mathbb{N}$.

Theorem 3.6.1
Let $m \geq 3$ *and suppose that* $u_j = (u_{1j}, u_{2j}) \in \mathbb{Z}^2 \backslash \{0\}$, $j = 1, \ldots, m$ *are different* X*-ray directions satisfying* $\gcd(u_{1j}, u_{2j}) = 1$. *For any* $\alpha \in \mathbb{N}$ *there exist two finite lattice sets* $F_1, F_2 \subset \mathbb{Z}^2$ *so that*

- F_k *for* $k = 1, 2$ *is uniquely determined by the* X*-rays* $X_{u_j} F_k$ *for* $j = 1, \ldots, m$,
- $F_1 \cap F_2 = \emptyset$,
- $|F_1| = |F_2| \geq \alpha$,
- $\sum_{i=1}^{m} |X_{u_i} F_1 - X_{u_i} F_2| = 2(m-1)$.

Moreover, it yields that

$$|F_1 \cap t(F_2)| \leq 4 \tag{3.261}$$

for any affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto Ax + b$ *with* $\pm I \neq A \in \mathbb{R}^{2\times 2}$ *nonsingular,* $b \in \mathbb{R}^2$. *For any translation* $t : \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto x + b$ *we can guarantee that*

$$|F_1 \cap t(F_2)| \leq 2^{m-2} - 1. \tag{3.262}$$

The worst-case situation with respect to the used construction of the lattice sets F_1, F_2 *occurs for the translation* $t(x) := x + u_j$ *in the* X*-ray direction* u_j *for* $j = 1, \ldots, m$.
For any rotation $t : \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto -x + b$ *we can guarantee that*

$$|F_1 \cap t(F_2)| \leq 2^{m-1} - 1. \tag{3.263}$$

The upper bound $2^{m-1} - 1$ *is reached by the affine transformation* $t(x) := -x + 2 \cdot \varphi_{\mathbb{R}^m}(q)$ *and* m *odd, if* q *denotes the center of some cuboid within the construction of the lattice sets which are projected by* $\varphi_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^2$ *into the plane.*

Proof
The recursive construction of the lattice sets $G_1^{(j)}, G_2^{(j)} \subset \mathbb{Z}^m$ *for* $j = 1, \ldots, l$ *below is based on* m*-dimensional cuboids, which are combined to a big switching component. First the general idea is pointed out by constructing the lattice sets* $G_1^{(l)}, G_2^{(l)}$ *and proving the affine dissimilarity assertion for the projections of those sets. The lattice sets* $\tilde{G}_1^{(l)}, \tilde{G}_2^{(l)}$ *result from the lattice sets* $G_1^{(l)}, G_2^{(l)}$

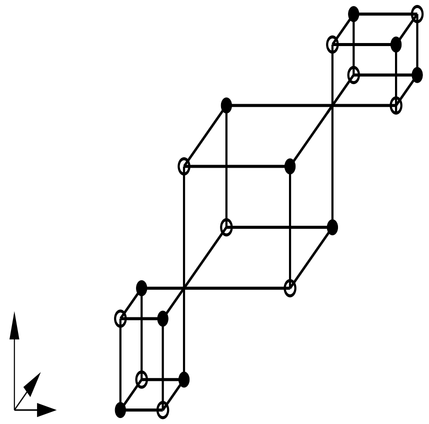

Figure 3.6: The lattice sets $G_1^{(3)}$ (filled circles) and $G_2^{(3)}$ (non-filled circles) for $l = 3$ and $m = 3$ resp. the lattice sets $G_1^{(3)} \cap \{x \in \mathbb{R}^4 | x_4 = 0\}$ and $G_2^{(3)} \cap \{x \in \mathbb{R}^4 | x_4 = 0\}$, if $m = 4$

by small modifications to also guarantee the uniqueness of their projections $F_1 := \varphi_{\mathbb{R}^m}(\tilde{G}_1^{(l)})$, $F_2 := \varphi_{\mathbb{R}^m}(\tilde{G}_2^{(l)})$.
Let us recursively define the lattice sets $G_1^{(j)}, G_2^{(j)} \subset \mathbb{Z}^m$ *for* $j = 1, \ldots, l$ *by*

$$G_1^{(1)} := \{\sum_{i \in I} a_i^1 e_i | I \subset \{1, 2, \ldots, m\}, 2||I|\}, \tag{3.264}$$

$$G_2^{(1)} := \{\sum_{i \in I} a_i^1 e_i | I \subset \{1, 2, \ldots, m\}, \gcd(2, |I|) = 1\}, \tag{3.265}$$

$$\begin{aligned} G_1^{(j)} &:= G_1^{(j-1)} \cup \{\sum_{k=1}^{j-1} \sum_{i=1}^{m} a_i^k e_i + \sum_{i \in I} a_i^j e_i | \emptyset \neq I \subset \{1, 2, \ldots, m\}, 2||I|\}, \\ G_2^{(j)} &:= G_2^{(j-1)} \backslash \{\sum_{k=1}^{j-1} \sum_{i=1}^{m} a_i^k e_i\} \\ &\quad \cup \{\sum_{k=1}^{j-1} \sum_{i=1}^{m} a_i^k e_i + \sum_{i \in I} a_i^j e_i | I \subset \{1, 2, \ldots, m\}, \gcd(2, |I|) = 1\} \end{aligned} \tag{3.266}$$

for m *odd resp. by*

$$\begin{aligned} G_1^{(j)} &:= G_1^{(j-1)} \cup \{\sum_{k=1}^{j-1} \sum_{i=1}^{m-1} a_i^k e_i + \sum_{i \in I} a_i^j e_i | \emptyset \neq I \subset \{1, 2, \ldots, m\}, 2||I|\}, \\ G_2^{(j)} &:= G_2^{(j-1)} \backslash \{\sum_{k=1}^{j-1} \sum_{i=1}^{m-1} a_i^k e_i\} \\ &\quad \cup \{\sum_{k=1}^{j-1} \sum_{i=1}^{m-1} a_i^k e_i + \sum_{i \in I} a_i^j e_i | I \subset \{1, 2, \ldots, m\}, \gcd(2, |I|) = 1\} \end{aligned} \tag{3.267}$$

for m *even and the integer value* l *chosen large enough by*

$$l > \max(\frac{\alpha}{2^{m-1} - 1}, 9). \tag{3.268}$$

The integer numbers $a_i^k \in \mathbb{N}$ are suitably specified besides forbidden values with respect to condition (3.260) and the following solvability restrictions:
The definition of the lattice sets $G_1^{(l)}$, $G_2^{(l)}$ and the facts in Section 3.6.1 with respect to the suitable choice of the parameters $\mu_1, \dots, \mu_m$ imply analogously to the case of point X-rays the assertion of Theorem 3.6.1 besides the uniqueness and the affine dissimilarity of the lattice sets $F_1 := \varphi_{\mathbb{R}^m}(G_1^{(l)}) \backslash \{p_1\}$, $F_2 := \varphi_{\mathbb{R}^m}(G_2^{(l)}) \backslash \{p_2\} \subset \mathbb{Z}^2$ and the lattice points $p_1 \in \varphi_{\mathbb{R}^m}(G_1^{(l)} \cap G_1^{(1)})$, $p_2 \in \varphi_{\mathbb{R}^m}(G_2^{(l)} \cap G_2^{(1)})$ lying on the same line in direction u_1.
To show the affine dissimilarity assertion let us assume that

$$t : \varphi_{\mathbb{R}^m}(x^k) \mapsto \varphi_{\mathbb{R}^m}(x^{k+1}) \text{ for } k = 1, 3, \dots, 9 \tag{3.269}$$

by some affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$, $t(x) := Ax + b$ for $x^1, x^3, \dots, x^9 \in G_1^{(l)}$, $x^2, x^4, \dots, x^{10} \in G_2^{(l)}$ pairwise distinct lattice points, which implies that

$$A(\varphi_{\mathbb{R}^m}(x^k) - \varphi_{\mathbb{R}^m}(x^1)) = \varphi_{\mathbb{R}^m}(x^{k+1}) - \varphi_{\mathbb{R}^m}(x^2) \text{ for } k = 3, 5, \dots, 9. \tag{3.270}$$

We can assume by elimination of some parameter combinations $(\mu_1, \dots, \mu_m, a_1^1, \dots, a_m^1, \dots, a_1^l, \dots, a_m^l)$ that the matrix $A = (a_{ij})_{i,j=1,2}$ is uniquely specified by the equation system (3.270) for $k = 3, 5$ (resp. for any subset $\{j_1, j_2\} \subset \{3, 5, \dots, 9\}$ of cardinality 2). That is the case as the matrix

$$\begin{pmatrix} \triangle(x^3)_1 & \triangle(x^3)_2 & 0 & 0 \\ 0 & 0 & \triangle(x^3)_1 & \triangle(x^3)_2 \\ \triangle(x^5)_1 & \triangle(x^5)_2 & 0 & 0 \\ 0 & 0 & \triangle(x^5)_1 & \triangle(x^5)_2 \end{pmatrix} \tag{3.271}$$

is in general regular for

$$\triangle(x^k)_q := \begin{cases} \varphi(x^k)_q - \varphi(x^1)_q & \text{for } k \text{ odd} \\ \varphi(x^k)_q - \varphi(x^2)_q & \text{for } k \text{ even} \end{cases} \tag{3.272}$$

and $q = 1, 2$. For that purpose let us assume without loss of generality that the difference $x^5 - x^1$ depends on the parameter a_i^k, but not the difference $x^3 - x^1$. Because of 4. in Section 3.6.1 we calculate that

$$\det \begin{pmatrix} \triangle(x^3)_1 & \triangle(x^3)_2 \\ \triangle(x^5)_1 & \triangle(x^5)_2 \end{pmatrix} = \tag{3.273}$$

$$= \pm\mu_i \cdot \det \begin{pmatrix} \triangle(x^3)_1 & \triangle(x^3)_2 \\ u_{1i} & u_{2i} \end{pmatrix} \cdot a_i^k + \text{ terms of lower degree} \tag{3.274}$$

is polynomial of degree 1 within the parameter a_i^k. Thus, we only have to adapt the parameter a_i^k to guarantee the regularity of the matrix (3.271).

We necessarily have to demand for the solvability of the equation system (3.270) that the extended coefficient matrix

$$\begin{pmatrix} \triangle(x^3)_1 & \triangle(x^3)_2 & 0 & 0 & \triangle(x^4)_1 \\ 0 & 0 & \triangle(x^3)_1 & \triangle(x^3)_2 & \triangle(x^4)_2 \\ \triangle(x^5)_1 & \triangle(x^5)_2 & 0 & 0 & \triangle(x^6)_1 \\ 0 & 0 & \triangle(x^5)_1 & \triangle(x^5)_2 & \triangle(x^6)_2 \\ \triangle(x^7)_1 & \triangle(x^7)_2 & 0 & 0 & \triangle(x^8)_1 \end{pmatrix} \tag{3.275}$$

of the equation system for $k = 3, 5, 7$ *(resp. for any subset* $\{j_1, j_2, j_3\} \subset \{3, 5, \ldots, 9\}$ *of cardinality* 3*) is singular. Let the coefficients* $\mathrm{coeff}_{i,k}(x^q) \in \{0, 1\}$ *of the lattice point* x^q *be defined by the representation*

$$x^q := \sum_{k=1}^{l} \sum_{i=1}^{m} \mathrm{coeff}_{i,k}(x^q) a_i^k e_i \in \mathbb{Z}[a_1^1, \ldots, a_m^1, \ldots, a_1^l, \ldots, a_m^l]. \tag{3.276}$$

Because of the regularity of the submatrix (3.271) we have to demand that $\mathrm{coeff}_{i,k}(x^8 - x^2) \neq 0$ *implies that* $\mathrm{coeff}_{i,k}(x^7 - x^1) \neq 0$*:*
First of all, let us look at the case that $\mathrm{coeff}_{i,k}(x^6 - x^2) = \mathrm{coeff}_{i,k}(x^4 - x^2) = 0$*. Let us assume that* $\mathrm{coeff}_{i,k}(x^7 - x^1) = 0$*. Because of the regularity of the submatrix (3.271) the determinant of the complete matrix (3.275) is polynomial at least of degree 1 within the parameter* a_i^k*. Therefore, the matrix (3.275) cannot be singular in general.*
It is left to treat the case that $\mathrm{coeff}_{i,k}(x^8 - x^2) \neq 0$, $\mathrm{coeff}_{i,k}(x^4 - x^2) \neq 0$ *and* $\mathrm{coeff}_{i,k}(x^6 - x^2) \neq 0$ *(by some permutation of the indices if necessary). Notice, that* $\mathrm{coeff}_{i,k}(x^8 - x^2) \neq 0$ *implies that either the lattice point* x^8 *or the lattice point* x^2 *depends on the parameter* a_i^k*. Thus, it yields that* $\mathrm{coeff}_{i,k}(x^8 - x^2) = \mathrm{coeff}_{i,k}(x^4 - x^2) = \mathrm{coeff}_{i,k}(x^6 - x^2) \in \{\pm 1\}$*. By changing the roles of the indices* 1 *and* 7 *and the roles of the indices* 2 *and* 8 *we reduce to the first case again, as* $\mathrm{coeff}_{i,k}(x^2 - x^8) \neq 0$ *and* $\mathrm{coeff}_{i,k}(x^4 - x^8) = \mathrm{coeff}_{i,k}(x^6 - x^8) = 0$*.*
By considering all differences $x^{j_1} - x^{j_2}$ *for* $j_1, j_2 \in \{2, 4, \ldots, 10\}$ *and by using similar arguments for the inverse affine transformation* t^{-1}*, we conclude that the difference vector* $x^{j_1} - x^{j_2}$ *depends on some parameter* a_i^k *if and only if the difference vector* $x^{j_1+1} - x^{j_2+1}$ *depends on the parameter* a_i^k*.*
Let us assume that without loss of generality

$$\mathrm{coeff}_{i_1,k_1}(x^8 - x^2) = \mathrm{coeff}_{i_2,k_2}(x^8 - x^2) = \tag{3.277}$$

$$= \mathrm{coeff}_{i_1,k_1}(x^7 - x^1) = -\,\mathrm{coeff}_{i_2,k_2}(x^7 - x^1) \tag{3.278}$$

for $i_1 \neq i_2$ *and let*

$$\mathrm{coeff}_{i_3,k_3}(x^{j_1+1} - x^{j_2+1}) = \mathrm{coeff}_{i_3,k_3}(x^{j_1} - x^{j_2}) \tag{3.279}$$

for $i_3 \notin \{i_1, i_2\}$ *and* $j_1, j_2 \in \{1, 3, 5, 7\}$*, i. e. the signs of at least* 1*, but at most* 2 *pairs of correspondent coefficients with respect to* 3 *distinct indices* $\{i_1, i_2, i_3\}$

differ. That can be done, as the lattice points $x^1, x^3, x^5, x^7 \in \tilde{G}_1^{(l)}$ *do not lie on the same line in direction* e_i *for any* $i = 1, \ldots, m$ *and as the lattice points* $x^1, x^3, x^5, x^7 \in \tilde{G}_1^{(l)}$ *do not lie on the same* 2*-dimensional affine subspace parallel to* $\operatorname{span}\{e_{i_1}, e_{i_2}\}$ *because of the construction of the lattice sets* $\tilde{G}_1^{(l)}, \tilde{G}_2^{(l)}$*. As*

$$A(\varphi_{\mathbb{R}^m}(x^{j_1}) - \varphi_{\mathbb{R}^m}(x^{j_2})) = \varphi_{\mathbb{R}^m}(x^{j_1+1}) - \varphi_{\mathbb{R}^m}(x^{j_2+1}) \tag{3.280}$$

for all $j_1 \neq j_2$*,* $j_1, j_2 \in \{1, 3, 5, 7\}$ *and all parameter combinations* $(\mu_1, \ldots, \mu_m, a_1^1, \ldots, a_m^1, \ldots, a_1^l, \ldots, a_m^l)$*, especially for all besides the parameters* $\mu_{i_j}, a_{i_j}^{k_j}$*,* $j \in \{1, 2, 3\}$ *set to zero, we conclude that*

$$u_{i_1} \mapsto u_{i_1}, \tag{3.281}$$
$$u_{i_2} \mapsto -u_{i_2}, \tag{3.282}$$
$$u_{i_3} \mapsto u_{i_3} \tag{3.283}$$

by the linear transformation $l(x) := A \cdot x$*, which cannot be the case for pairwise linearly independent vectors* $u_{i_1}, u_{i_2}, u_{i_3}$*.*
Thus, the affine transformation t *in (3.269) simply describes the rigid translation* $t(x) := x + b$ *for*

$$\begin{aligned} b := \varphi_{\mathbb{R}^m}(x^2) - \varphi_{\mathbb{R}^m}(x^1) = \varphi_{\mathbb{R}^m}(x^4) - \varphi_{\mathbb{R}^m}(x^3) = \varphi_{\mathbb{R}^m}(x^6) - \varphi_{\mathbb{R}^m}(x^5) = \\ = \varphi_{\mathbb{R}^m}(x^8) - \varphi_{\mathbb{R}^m}(x^7) = \varphi_{\mathbb{R}^m}(x^{10}) - \varphi_{\mathbb{R}^m}(x^9) \end{aligned} \tag{3.284}$$

or the rigid half-around rotation $t(x) := -x + b$ *for*

$$\begin{aligned} b := \varphi_{\mathbb{R}^m}(x^1) + \varphi_{\mathbb{R}^m}(x^2) = \varphi_{\mathbb{R}^m}(x^3) + \varphi_{\mathbb{R}^m}(x^4) = \varphi_{\mathbb{R}^m}(x^5) + \varphi_{\mathbb{R}^m}(x^6) = \\ = \varphi_{\mathbb{R}^m}(x^7) + \varphi_{\mathbb{R}^m}(x^8) = \varphi_{\mathbb{R}^m}(x^9) + \varphi_{\mathbb{R}^m}(x^{10}). \end{aligned} \tag{3.285}$$

In the first case we conclude that

$$\begin{aligned} \operatorname{coeff}_{i,k}(x^2 - x^1) = \operatorname{coeff}_{i,k}(x^4 - x^3) = \operatorname{coeff}_{i,k}(x^6 - x^5) = \\ = \operatorname{coeff}_{i,k}(x^8 - x^7) = \operatorname{coeff}_{i,k}(x^{10} - x^9) \text{ for all } i, k \end{aligned} \tag{3.286}$$

in general besides some parameter combinations. Therefore, the lattice points $x^1, x^3, \ldots, x^9$ *resp. the lattice points* $x^2, x^4, \ldots, x^{10}$ *are vertices of the same cuboid* $\sum_{k=1}^{q-1} \sum_{i=1}^m a_i^k e_i + [0, a_1^q] \times \cdots \times [0, a_m^q]$*. The cuboids of the two sets of lattice points* $x^1, x^3, \ldots, x^9$ *and* $x^2, x^4, \ldots, x^{10}$ *are even identical because of the rigidity of the translation* t*. Thus, we get that* $|F_1 \cap t(F_2)| \leq \frac{2^{m-1}}{2} - 1 = 2^{m-2} - 1$ *in the translational case* $t(x) := x + b$*.*
In the rotational case we also claim that the lattice points $x^1, x^2, \ldots, x^{10}$ *are vertices of the same cuboid* $\sum_{k=1}^{q-1} \sum_{i=1}^m a_i^k e_i + [0, a_1^q] \times \cdots \times [0, a_m^q]$*:*
Because of (3.285) and the definition of the lattice sets $G_1^{(l)}, G_2^{(l)} \subset \mathbb{Z}^m$ *as combinations of cuboids we conclude that either both the lattice points* x^1*,* x^3 *and the lattice points* x^2*,* x^4 *or both the lattice points* x^1*,* x^4 *and the lattice points* x^2*,* x^3 *belong to the same cuboid. In the case that those cuboids*

do not coincide, there exists some parameter a_i^k because of the distinctness of the lattice points and the construction of the lattice sets $G_1^{(l)}, G_2^{(l)}$ so that only one of the difference vectors $x^1 - x^3$, $x^2 - x^4$ depends on the parameter a_i^k in contradiction to the rigidity of the rotation t. Thus, we result in $|F_1 \cap t(F_2)| \leq \frac{2^m}{2} - 1 = 2^{m-1} - 1$ for the rotational case $t(x) := -x + b$.
For the other cases we conclude that $|F_1 \cap t(F_2)| \leq 4$.
The uniqueness of the lattice sets F_1, F_2 is left to show. For that purpose let us define the modified lattice sets $\tilde{G}_1^{(l)}, \tilde{G}_2^{(l)} \subset \mathbb{Z}^m$ by

$$\begin{aligned} \tilde{G}_1^{(l)} := (G_1^{(l)} \cup \{\sum_{i\in I}(\sum_{j=1}^{l} a_i^j)e_i | I \subset \{1,2,\ldots,m\}, \gcd(2,|I|) = 1\}) \\ \backslash\{0, \sum_{i=1}^{m}(\sum_{j=1}^{l} a_i^j)e_i, \sum_{j=1}^{l} a_1^j e_1\}, \end{aligned} \tag{3.287}$$

$$\begin{aligned} \tilde{G}_2^{(l)} := (G_2^{(l)} \cup \{\sum_{i\in I}(\sum_{j=1}^{l} a_i^j)e_i | I \subset \{1,2,\ldots,m\}, 2||I|\}) \\ \backslash\{0, \sum_{i=1}^{m}(\sum_{j=1}^{l} a_i^j)e_i, a_1^1 e_1\} \end{aligned} \tag{3.288}$$

in the case that m is odd resp. by

$$\begin{aligned} \tilde{G}_1^{(l)} := (G_1^{(l)} \cup \{\sum_{i\in I} \bar{a}_i e_i | I \subset \{1,2,\ldots,m\}, \gcd(2,|I|) = 1\}) \\ \backslash\{0, \sum_{i=1}^{m-1} \bar{a}_i e_i, \bar{a}_1 e_1\}, \end{aligned} \tag{3.289}$$

$$\begin{aligned} \tilde{G}_2^{(l)} := (G_2^{(l)} \cup \{\sum_{i\in I} \bar{a}_i e_i | I \subset \{1,2,\ldots,m\}, 2||I|\}) \\ \backslash\{0, \sum_{i=1}^{m-1} \bar{a}_i e_i, a_1^1 e_1\} \end{aligned} \tag{3.290}$$

in the case that m is even for $\bar{a}_i := \sum_{j=1}^{l} a_i^j$, $i \in \{1,2,\ldots,m-1\}$ and $\bar{a}_m \in \mathbb{Z}\backslash\{0\}$, see Figure 3.7 for illustration.
The lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ are redefined by

$$F_1 := \varphi_{\mathbb{R}^m}(\tilde{G}_1^{(l)}), \tag{3.291}$$

$$F_2 := \varphi_{\mathbb{R}^m}(\tilde{G}_2^{(l)}). \tag{3.292}$$

Because of $l \geq 10$ the lattice points $x^1, \ldots, x^{10}$ within the mapping assumption (3.269) can be assumed to be actually located within the lattice set $G_1^{(l)} \cup G_2^{(l)}$ by suitably breaking the construction of the lattice set $\tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)}$. Therefore, the dissimilarity assertion is shown in the same way as before.

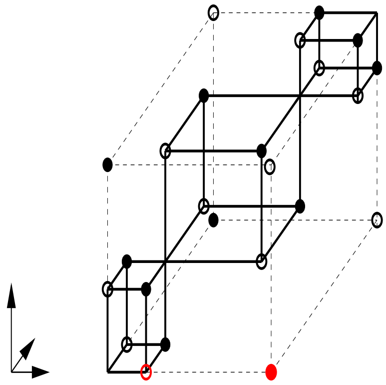

Figure 3.7: The lattice sets $\tilde{G}_1^{(3)}$ (filled circles) and $\tilde{G}_2^{(3)}$ (non-filled circles) for $l = 3$ and $m = 3$ resp. the lattice sets $\tilde{G}_1^{(3)} \cap \{x \in \mathbb{R}^4 | x_4 = 0\}$ and $\tilde{G}_2^{(3)} \cap \{x \in \mathbb{R}^4 | x_4 = 0\}$ if $m = 4$, the red-coloured lattice points would close the switching component

According to 5. in Section 3.6.1 the uniqueness of the lattice sets F_1, F_2 can be shown by determining the grid of the lattice set $\tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)}$ and showing the uniqueness of the lattice set $\tilde{G}_1^{(l)}$ resp. of the lattice set $\tilde{G}_2^{(l)}$.
For m odd let the lattice point y lie within the grid of the lattice set $\tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)}$. Then there are lattice points $x^1, \ldots, x^m \in \tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)}$ satisfying $y - x^i \in \mathbb{R} \cdot e_i$ for $i = 1, \ldots, m$. By considering the orthogonal projections onto all subspaces $\mathrm{span}\{e_{i_1}, e_{i_2}\}$ *for pairwise distinct indices $i_1, i_2 \in \{1, 2, \ldots, m\}$, we conclude in the case that m is odd that any grid point $y = \bigcap_{i=1}^m (x^i + \mathbb{R} \cdot e_i)$ for $x^i \in \tilde{G}_1^{(l)}$ resp. for $x^i \in \tilde{G}_2^{(l)}$ has to be a vertex of one of the cuboids according to the combination of the cuboids within the definition of the lattice sets $\tilde{G}_1^{(l)}$, $\tilde{G}_2^{(l)}$ and thus*

$$y \in \tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)} \cup \{\sum_{i=1}^m a_i^1 e_i, \sum_{i=1}^m (a_i^1 + a_i^2) e_i, \ldots, \sum_{i=1}^m (a_i^1 + \cdots + a_i^l) e_i\}. \tag{3.293}$$

To be more precise, let us assume that the lattice point $y = (x^1 + \mathbb{R} \cdot e_1) \cap (x^2 + \mathbb{R} \cdot e_2)$ is not a vertex of any of the cuboids. Thus, the lattice points x^1, x^2 belong to different, but - to guarantee the existence of the lattice point y - neighboured cuboids so that

$$y \in (\sum_{i=1}^m \sum_{j=1}^q a_i^j + \{(a_1^{q+1} \cdot e_1 - a_2^q \cdot e_2), (a_2^{q+1} \cdot e_2 - a_1^q \cdot e_1)\}). \tag{3.294}$$

But $(y + \mathbb{R} \cdot e_3) \cap (\tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)}) = \emptyset$ in both cases.
As the lattice points $a_1^1 e_1$ and $\sum_{j=1}^l a_1^j e_1$ are eliminated from the lattice set $\tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)}$, we can decide by the line sum values of the lattice set $\tilde{G}_1^{(l)}$ resp. of the lattice set $\tilde{G}_2^{(l)}$ whether the lattice points of the first cuboid

$\{\sum_{i\in I} a_i^1 e_i | I \subset \{1,2,\dots,m\}\} \cap (\tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)})$ belong to the reconstructed lattice set.
If the lattice points $\sum_{i\in I} a_i^1 e_i$, $I \subset \{1,2,\dots,m\}, |I| = m-1$ belong to the reconstructed lattice set, then the membership of the lattice points $\sum_{i=1}^m a_i^1 e_i + \sum_I a_i^2 e_i$, $I \subset \{1,2,\dots,m\}$, $|I| = 2$ to the reconstructed lattice set is determined as well. Using inductive arguments we see that the complete lattice set is uniquely determined by its line sum values in the directions $e_1,\dots,e_m$.
Otherwise, if the lattice points $\sum_{i\in I} a_i^1 e_i \neq a_1^1 e_1$, $I \subset \{1,2,\dots,m\}, |I| = m-2$ belong to the reconstructed lattice set, we have to consider both the case that the lattice point $\sum_{i=1}^m a_i^1 e_i$ belongs to the lattice set and the case that the lattice points $\sum_{i=1}^m a_i^1 e_i + a_k^2 e_k$ for $k = 1,\dots,m$ belong to the lattice set. In the first case also the lattice points $\sum_{i=1}^m a_i^1 e_i + \sum_{i\in I} a_i^2 e_i$, $I \subset \{1,2,\dots,m\}, |I| = m-1$ are reconstructed. Then the further reconstruction process is uniquely given by the same arguments as before. Thus, both the lattice point $a_m^1 e_m$ and the lattice point $\sum_{j=1}^l a_m^j e_m$ belong to the reconstructed lattice set. But that is not possible as the line sum values do not extend the value 1. Thus, the lattice points $\sum_{i=1}^m a_i^1 e_i + a_k^2 e_k$ for $k = 1,\dots,m$ have to belong to the reconstructed lattice set. Now we again apply inductive arguments. Thus, uniqueness is shown for the lattice sets F_1, F_2 in the case that m is odd.
If m is even, we conclude by using the same arguments as before after projection onto the hyperplane $\{x_m = 0\}$ that any grid point $y = \bigcap_{i=1}^m (x^i + \mathbb{R}\cdot e_i)$ for $x^i \in \tilde{G}_1^{(l)}$ resp. $x^i \in \tilde{G}_2^{(l)}$ is given by $y = \bar{y} + b\cdot e_m$, $b \in \{0, a_m^1, \dots, a_m^l, \bar{a}_m\}$ for $\bar{y}$ some grid point of the lattice set $(\tilde{G}_1 \cup \tilde{G}_2) \cap \{x_m = 0\}$. Taking into account how the cuboids are combined, the value of b can further be specified in dependence on the lattice point $\bar{y}$ so that the lattice point y has to lie within the lattice set

$$\tilde{G}_1^{(l)} \cup \tilde{G}_2^{(l)} \cup \{\sum_{i=1}^{m-1} a_i^1 e_i, \sum_{i=1}^{m-1} (a_i^1 + a_i^2) e_i, \dots, \sum_{i=1}^{m-1} (a_i^1 + \dots + a_i^l) e_i\}. \quad (3.295)$$

To be more precise, the fact that $y = (x^1 + \mathbb{R}\cdot e_1) \cap (x^2 + \mathbb{R}\cdot e_2)$ for $x^1, x^2 \in \tilde{G}_1^{(l)}$ resp. $x^1, x^2 \in \tilde{G}_2^{(l)}$ implies that $(x^1)_4 = (x^2)_4$. Therefore, the lattice points x^1, x^2 and then also the lattice point y are vertices of the same cuboid, if $(x^1)_4 = (x^2)_4 \neq 0$, as the values $a_m^1, \dots, a_m^l, \bar{a}_m$ are pairwise different in general. If $(x^1)_4 = (x^2)_4 = 0$, the situation is reduced to the case that m is odd.
The further uniqueness consideration is reduced to the case that m is odd by looking at the implications for the lattice points within the hyperplane $\{x_m = 0\}$. The remaining lattice points within the lattice set are then determined by the line sum values for the lines in direction e_m and the implications within each cuboid separately. □

Remark 3.6.2 In the case of point X-rays we have seen that the number of overlappings after some affine transformation can be bounded from above by

the value 3 in the planar case, see Subsection 3.3. For parallel X-rays, however, that bound cannot be reached by using the construction of the lattice sets F_1, F_2 as in the proof of Theorem 3.6.1, even if we restrict our consideration to affine transformations $t(x) := Ax + b$ for $A \neq \pm I$:
Let the number of X-ray directions be given by $m = 4$ and let the affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto Ax + b$ be defined by

$$A \cdot v_1 = v_2, \tag{3.296}$$
$$A \cdot v_3 = v_3, \tag{3.297}$$
$$b := t - A \cdot t \tag{3.298}$$

for the point

$$s := \sum_{k=1}^{q-1} \sum_{i=1}^{m} a_i^k e_i, \tag{3.299}$$

the projection

$$t := \frac{1}{2}(\varphi_{\mathbb{R}^m}(s + \sum_{i=1}^{4} a_i^q e_i) + \varphi_{\mathbb{R}^m}(s)) \tag{3.300}$$

of the center of the qth cuboid and the vectors

$$v_1 := \varphi_{\mathbb{R}^m}(s + a_1^q e_1) - t, \tag{3.301}$$
$$v_2 := \varphi_{\mathbb{R}^m}(s + a_1^q e_1 + a_3^q e_3) - t, \tag{3.302}$$
$$v_3 := \varphi_{\mathbb{R}^m}(s + a_2^q e_2) - \varphi_{\mathbb{R}^m}(s + a_1^q e_1). \tag{3.303}$$

For illustration see Figure 3.8.

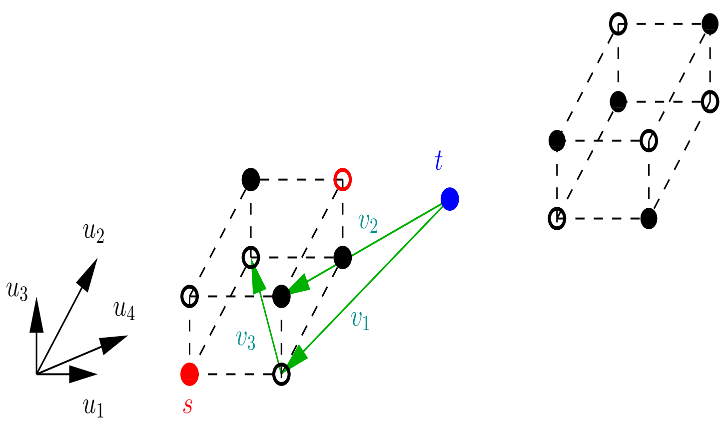

Figure 3.8: The projection of the qth cuboid. The blue-coloured point t denotes the projection of the center of the cuboid, the red-coloured points do not actually occur in the whole construction.

Thus, in total $|F_1 \cap t(F_2)| = 4$ lattice points overlap after applying the affine transformation t on the lattice set F_2.

3.6.3 Non-projective construction methods

Non-projective switching constructions reduce further attempts of strengthening the affine dissimilarity to the search of switching components of minimal cardinality.

Theorem 3.6.3
Let $m \geq 3$ and suppose that $u_j = (u_{1j}, u_{2j}) \in \mathbb{Z}^2 \setminus \{0\}$, $j = 1, \ldots, m$ are pairwise distinct lattice directions satisfying $\gcd(u_{1j}, u_{2j}) = 1$. Let G_1, $G_2 \subset \mathbb{Z}^2$ be two tomographically equivalent lattice sets of minimal cardinality which satisfy

$$0 \leq X_{u_1} G_k \leq 1 \text{ for } k = 1, 2. \tag{3.304}$$

Then there are two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ so that

- *F_k for $k = 1, 2$ is uniquely determined by the X-rays $X_{u_j} F_k$ for $j = 1, \ldots, m$,*
- *$F_1 \cap F_2 = \emptyset$,*
- *$|F_1| = |F_2| \geq \alpha$,*
- *$\sum_{i=1}^{m} |X_{u_i} F_1 - X_{u_i} F_2| = 2(m-1)$*

and

$$|F_1 \cap t(F_2)| \leq 9 \cdot |G_1| \leq \frac{9}{2} \cdot 2^m \tag{3.305}$$

for any affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto Ax + b$ with $A \in \mathbb{R}^{2\times 2}$ nonsingular and $b \in \mathbb{R}^2$.

Proof
Let $p_1 \in G_1, q_1, q_2, q_3 \in G_2$ be different lattice points satisfying $q_i \in p_1 + \mathbb{R} \cdot u_i$ for $i = 1, 2, 3$. Notice, that the vectors $q_1 - p_1, q_2 - p_1$ and $q_3 - p_1$ are pairwise linearly independent. Without loss of generality let us assume that $p_1 = 0$.
Let the value $l \in \mathbb{N}$ be chosen sufficiently large by $l > \max(\frac{\alpha}{|G_1|-1}, 9)$ and let us define the lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ by

$$F_1 := (\bigcup_{i=1}^{l} \bar{v}_i + \lambda_i \cdot G_1) \setminus \{\bar{v}_1, \ldots, \bar{v}_l\}, \tag{3.306}$$

$$F_2 := (\bigcup_{i=1}^{l} \bar{v}_i + \lambda_i \cdot G_2) \setminus \{q_3, \bar{v}_2, \ldots, \bar{v}_l\} \tag{3.307}$$

for

$$\bar{v}_1 := p_1 = 0, \tag{3.308}$$

$$\bar{v}_i := \bar{v}_{i-1} + \lambda_{i-1} \cdot q_{(i+1 \mod 3)+1} \text{ for } i = 2, \ldots, l \tag{3.309}$$

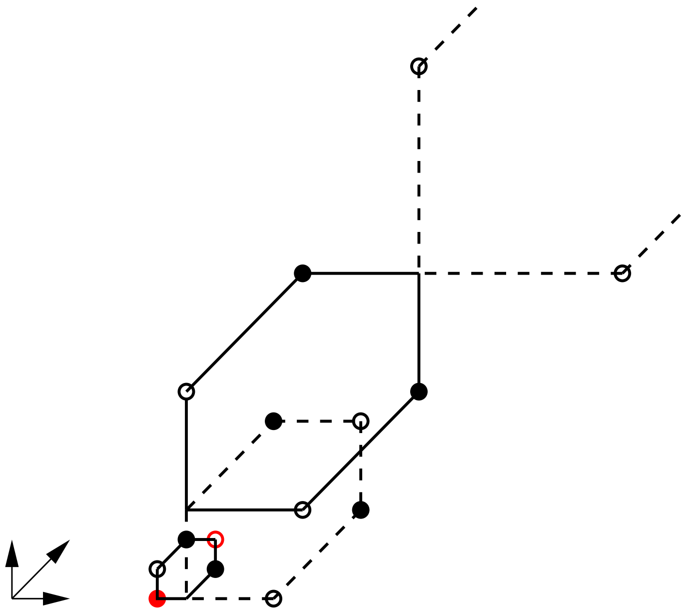

Figure 3.9: The lattice sets F_1 (filled circles) and F_2 (non-filled circles) for $m = 3$ and $u_1 = (1, 0)$, $u_2 = (0, 1)$, $u_3 = (1, 1)$, the red-coloured lattice points would close the switching component

and $\lambda_1, \ldots, \lambda_l \in \mathbb{Z}$ suitably chosen in the following.
We can guarantee that in general the grid $\mathcal{G}(F_1)$ of the lattice set F_1 satisfies

$$\mathcal{G}(F_1) \subset \bigcup_{i=1}^{l} \mathcal{G}_i \tag{3.310}$$

for the ith subgrid

$$\mathcal{G}_i := \mathcal{G}(\bar{v}_i + \lambda_i \cdot G_1) \tag{3.311}$$

of the lattice set $\bar{v}_i + \lambda_i \cdot G_1$ for $i = 1, \ldots, l$ by suitably choosing the parameter λ_i in dependence on the parameters $\lambda_1, \ldots, \lambda_{i-1}$ for $i = 2, \ldots, l$:
Without loss of generality let us restrict to the case $m = 3$. We will apply inductive arguments on the integer $k = 1, \ldots, l$. The case $k = 1$ is clear. Thus, let us conclude from $k - 1$ to k for fixed parameters $\lambda_1, \ldots, \lambda_{k-1}$. Let y be the intersection point of some line in direction u_3 passing through some point within the subgrid $\mathcal{G}_k$ and two further lines in direction u_1 and in direction u_2 each passing through some point within the union $\bigcup_{i=1}^{k-1} \mathcal{G}_i$ of subgrids. Because of the finiteness of the lattice sets there is only a finite number of values for the parameter λ_k which we have to exclude in order to assure that $y \in \bigcup_{i=1}^{k-1} \mathcal{G}_i$, as the line in direction u_3 has to pass through the lattice point $\bar{v}_k$ to guarantee the existence of the intersection point y of the three lines in general. The same is true permuting the lattice directions or changing the roles of the two lattice sets $\mathcal{G}_k$ and $\bigcup_{i=1}^{k-1} \mathcal{G}_i$.

Therefore, any lattice point $p = \bigcap_{i=1}^{m} w_i + \mathbb{R} \cdot u_i$ *for* $w_i \in F_1$ *within the grid* $\mathcal{G}(F_1)$ *of the lattice set* F_1 *which is not equal to* $\bar{v}_i$ *for* $i = 2, \ldots, l$ *can uniquely be mapped to one of the subgrids* $\mathcal{G}_i$*. For the lattice points* $\bar{v}_i$*,* $i = 2, \ldots, l$ *we cannot differentiate between the subgrids* $\mathcal{G}_{i-1}$ *and* $\mathcal{G}_i$*.*
In the following we will show the uniqueness of the lattice set F_1*. For that purpose let us assume that the lattice set* F_1 *is not uniquely determined by its* X*-ray data. Let the value* L *be defined by*

$$L := \min\{i \in \{1, \ldots, l\} | F_1 \triangle \bar{F}_1 \cap \mathcal{G}_i \neq \emptyset\} \tag{3.312}$$

for some lattice set $\bar{F}_1 \sim_{tom} F_1$ *which is tomographically equivalent to the lattice set* F_1*. We conclude that*

$$\begin{aligned} &|(F_1 \backslash \bar{F}_1) \cap \mathcal{G}_L \cap (\bar{v}_{L+1} + \mathbb{R} \cdot u_j)| - |(\bar{F}_1 \backslash F_1) \cap \mathcal{G}_L \cap (\bar{v}_{L+1} + \mathbb{R} \cdot u_j)| = \\ &= |(F_1 \backslash \bar{F}_1) \cap \mathcal{G}_L| - |(\bar{F}_1 \backslash F_1) \cap \mathcal{G}_L|\} \end{aligned} \tag{3.313}$$

for $j = 1, \ldots, m$*, as* $|(F_1 \backslash \bar{F}_1) \cap \mathcal{G}_L \cap l| - |(\bar{F}_1 \backslash F_1) \cap \mathcal{G}_L \cap l| = 0$ *for any line* $l \neq \bar{v}_{L+1} + \mathbb{R} \cdot u_j$ *in direction* u_j*.*
Furthermore, notice that

$$\begin{aligned} &|(F_1 \backslash \bar{F}_1) \cap \mathcal{G}_L \cap (\bar{v}_{L+1} + \mathbb{R} \cdot u_1)| - |(\bar{F}_1 \backslash F_1) \cap \mathcal{G}_L \cap (\bar{v}_{L+1} + \mathbb{R} \cdot u_1)| \\ &\qquad =: D \in \{0, +1\} \end{aligned} \tag{3.314}$$

is implied by assumption (3.304) and the construction of the lattice sets F_1, F_2*, as* $p_1 \in G_1$ *and* $q_1, q_2, q_3 \in G_2$*.*
In the case that $D = 0$ *we result in contradiction to the minimality of the lattice sets* G_1, G_2 *by the tomographic equivalence of the lattice sets*

$$\tilde{G}_1 := (F_1 \backslash \bar{F}_1) \cap \mathcal{G}_L \subset \bar{v}_L + \lambda_L \cdot (G_1 \backslash \{p_1\}), \tag{3.315}$$
$$\tilde{G}_2 := (\bar{F}_1 \backslash F_1) \cap \mathcal{G}_L \tag{3.316}$$

of cardinality $|\tilde{G}_1| = |\tilde{G}_2| \leq |G_1| - 1$*, as* $X_{u_1} \tilde{G}_j \leq X_{u_1} F_1 \leq 1$ *for* $j = 1, 2$*.*
In the case that $D = +1$ *let us add the lattice point* $\bar{v}_{L+1}$ *to both lattice sets* F_1 *and* $\bar{F}_1$ *and continue further considerations on the subgrid* $\mathcal{G}_L$ *and the union* $\bigcup_{j=L+1}^{l} \mathcal{G}_j$ *of subgrids separately. According to the definition of the value* L *we again result in contradiction to the minimality of the lattice sets* G_1, G_2 *by the tomographic equivalence of the lattice sets*

$$\tilde{G}_1 := (F_1 \backslash (\bar{F}_1 \cup \{\bar{v}_{L+1}\})) \cap \mathcal{G}_L \subset \bar{v}_L + \lambda_L \cdot (G_1 \backslash \{p_1\}), \tag{3.317}$$
$$\tilde{G}_2 := (((\bar{F}_1 \cup \{\bar{v}_{L+1}\}) \backslash F_1) \cap \mathcal{G}_L) \tag{3.318}$$

of cardinality $|\tilde{G}_1| = |\tilde{G}_2| \leq |G_1| - 1$*, as* $X_{u_1} \tilde{G}_j \leq X_{u_1} F_1 \leq 1$ *for* $j = 1, 2$*.*

If the lattice set F_2 is not uniquely determined by its X-ray data, we have to modify the construction of the lattice sets F_1, F_2 by

$$\tilde{F}_1 := (F_1 \cup \bigcup_{i=l+1}^{\tilde{l}} \bar{v}_i + \lambda_i \cdot G_1) \backslash \{\bar{v}_{l+1}, \ldots, \bar{v}_{\tilde{l}}, \tilde{q}_3\} \tag{3.319}$$

$$\tilde{F}_2 := (F_2 \cup \bigcup_{i=l+1}^{\tilde{l}} \bar{v}_i + \lambda_i \cdot G_2) \backslash \{\bar{v}_1, \bar{v}_{l+1}, \ldots, \bar{v}_{\tilde{l}}\} \tag{3.320}$$

for $\tilde{l} = 2 \mod 3$ so that

$$\bar{v}_{\tilde{l}} + \lambda_{\tilde{l}} \cdot q_{[(\tilde{l}+1)+1 \mod 3]+1} = \bar{v}_{\tilde{l}} + \lambda_{\tilde{l}} \cdot q_2 = \bar{v}_1, \tag{3.321}$$

i. e. the switching construction is closed, and the lattice point $\tilde{q}_3$ is defined by

$$\tilde{q}_3 := (\bar{v}_{\tilde{l}} + \lambda_{\tilde{l}} \cdot G_2) \cap (\bar{v}_1 + \mathbb{R} \cdot u_3). \tag{3.322}$$

Now, the same arguments as before (and with respect to the ordering of the indices inversely applied to the lattice set $\tilde{F}_2$) help us to show that both the lattice set $\tilde{F}_1$ and the lattice set $\tilde{F}_2$ are uniquely determined by having (3.304) and q_3, $\tilde{q}_3 \in \bar{v}_1 + \mathbb{R} \cdot u_3$ in mind.
Finally, to show the upper bound (3.305) of the affine dissimilarity let us assume that

$$t : x^k \mapsto x^{k+1} \text{ for } k = 1, 3, 5, 7 \tag{3.323}$$

by some affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$, $t(x) := Ax + b$ and pairwise distinct lattice points $x^1, x^3, x^5, x^7 \in F_1$ and $x^2, x^4, x^6, x^8 \in F_2$. Let us further assume that $x^1 \in \mathcal{G}_{i_1}, x^3 \in \mathcal{G}_{i_3}, x^5 \in \mathcal{G}_{i_5}, x^7 \in \mathcal{G}_{i_7}$ lie in different subgrids and that $|i_{j_1} - i_{j_2}| \geq 3$ for all indices $j_1, j_2 \in \{1, 3, 5, 7\}$, $j_1 \neq j_2$ by the assumption that $|F_1 \cap t(F_2)| > 9 \cdot |G_1|$. Notice, that in general none triple of the lattice points $x^1, x^3, x^5, x^7 \in F_1$ and therefore also none triple of the lattice points $x^2, x^4, x^6, x^8 \in F_2$ is collinear:
Without loss of generality let us consider the lattice points x^1, x^3, x^5 and let us assume that $i_1 = 1$, $i_3 = 4$ and $i_5 = 7$. For fixed parameters $\lambda_1, \ldots, \lambda_4$ we can choose the parameters λ_5, λ_6 so that the lattice points x^1, x^3 and $\bar{v}_7$ do not lie on the same line by using the fact that the vectors $q_i = q_i - p_1$ for $i = 1, 2, 3$ are pairwise linearly independent. Therefore, there is only a finite number of values for the parameter λ_7 so that the lattice points x^1, x^3 and x^5 do not lie in general position.
Now let us assume that $i_7 > i_j$ for $j \in \{1, \ldots, 6, 8\}$ and that $(x^7 - x^1)_1$ depends on the parameter λ_{i_7}. (Otherwise, $(x^7 - x^1)_2$ has to depend on the parame-

ter λ_{i_7}.) Then the determinant of the matrix

$$\begin{pmatrix} (x^4-x^2)_1 & (x^4-x^2)_2 & 0 & 0 & (x^3-x^1)_1 \\ 0 & 0 & (x^4-x^2)_1 & (x^4-x^2)_2 & (x^3-x^1)_2 \\ (x^6-x^2)_1 & (x^6-x^2)_2 & 0 & 0 & (x^5-x^1)_1 \\ 0 & 0 & (x^6-x^2)_1 & (x^6-x^2)_2 & (x^5-x^1)_2 \\ (x^8-x^2)_1 & (x^8-x^2)_2 & 0 & 0 & (x^7-x^1)_1 \end{pmatrix} \tag{3.324}$$

is polynomial of degree 1 within the parameter λ_{i_7}, as the lattice points x^2, x^4, x^6 are not collinear. Thus, the matrix (3.324) is in general regular in contradiction to the mapping assumption (3.323). Similar arguments work to show that $i_7 < i_j$ for $j \in \{1, \dots, 6, 8\}$ is also not possible.
Let r be any index value so that $\{x^1, \dots, x^8\} \cap (\mathcal{G}_{r+1} \cup \mathcal{G}_{r+2}) = \emptyset$ using the fact that $|i_{j_1} - i_{j_2}| \geq 3$ for any indices $j_1, j_2 \in \{1,3,5,7\}$, $j_1 \neq j_2$. Let us break the construction by eliminating the subgrids $\mathcal{G}_{r+1}$, $\mathcal{G}_{r+2}$ and let us close it again by further subgrids $\mathcal{G}_{l+1}, \mathcal{G}_{l+2}, \mathcal{G}_{l+3}$. The arguments before can now be applied to finish proving that $\{i_1, i_3, i_5, i_7\} = \{i_2, i_4, i_6, i_8\}$.
Without loss of generality let us assume that $i_7 > i_j$ for $j = 1,3,5$, that $i_2 \neq i_7$ (by taking the index i_4 or the index i_6 otherwise) and that $(x^7 - x^1)_1$ depends on the parameter λ_{i_7}. If $x^8 - x^2$ does not depend on the parameter λ_{i_7}, the determinant of the matrix (3.324) is polynomial within the parameter λ_{i_7}:
The determinant of the submatrix

$$\begin{pmatrix} (x^4-x^2)_1 & (x^4-x^2)_2 \\ (x^6-x^2)_1 & (x^6-x^2)_2 \end{pmatrix} \tag{3.325}$$

is not equal to zero as polynomial within the parameter λ_{i_7}, as either the difference $x^6 - x^2$ or the difference $x^4 - x^2$ has to depend on the parameter λ_{i_7} and by using the fact that $\{i_1, i_3, i_5, i_7\} = \{i_2, i_4, i_6, i_8\}$ and that the lattice points x^2, x^4, x^6 do not lie on the same line in general because of the considerations before.
Thus, we result in $i_k = i_{k+1}$ for $k = 1,3,5,7$ by using similar arguments of breaking and closing the grid as before.
Let us assume that $i_7 = i_8 > i_j$ for $j = 1, \dots, 6$ and let the affine transformation $t(x) := Ax + b$ be uniquely determined by $x^k \mapsto x^{k+1}$ for $k = 1,3,5$. Let the lattice point x^7 be independently on the parameter λ_{i_7} mapped on the lattice point x^8, which implies that

$$A\bar{v}_{i_7} + b = \bar{v}_{i_7}, \tag{3.326}$$
$$A(x^7 - \bar{v}_{i_7}) = x^8 - \bar{v}_{i_7}, \tag{3.327}$$

i. e. the lattice point $\bar{v}_{i_7}$ is fixed by the affine transformation t. Similar arguments work to show that the lattice points $\bar{v}_{i_j}$ for $j = 1,3,5$ are also fixed by *the affine transformation t. But the lattice points $\bar{v}_{i_1}$, $\bar{v}_{i_3}$ and $\bar{v}_{i_5}$ lie in general position besides some eliminated parameter combinations. Therefore, the affine transformation fixes all lattice points, which implies that $x^1 = x^2 \in F_1 \cap F_2 \neq \emptyset$ in contradiction to the choice of the lattice points $x^1, \ldots, x^8$ as pairwise distinct.* □

Remark 3.6.4 The cardinality of the lattice set G_1 in Theorem 3.6.3 can be bounded by $|G_1| \leq \frac{1}{2} \cdot 6^{\lceil \frac{m}{3} \rceil}$:
For that purpose let us define the lattice sets $\tilde{G}_1 := \{(0,0),(2,1),(1,2)\}$, $\tilde{G}_2 := \{(1,0),(0,1),(2,2)\}$. According to the proof of Theorem 2.5 in [73] there is an affine transformation $T(i,j) = (i\alpha_1 c_1 + j\gamma_1 c_2, i\alpha_2 c_1 + j\gamma_2 c_2)$ for $c_1 := \beta_1\gamma_2 - \beta_2\gamma_1$, $c_2 := \alpha_1\beta_2 - \alpha_2\beta_1$ and any set of linearly independent lattice directions $u_1 = (\alpha_1, \alpha_2)$, $u_2 = (\beta_1, \beta_2)$, $u_3 = (\gamma_1, \gamma_2)$ so that the lattice sets $\bar{G}_1 := T(\tilde{G}_1), \bar{G}_2 := T(\tilde{G}_2)$ are tomographically equivalent according to the lattice directions u_1, u_2, u_3. For every set of three further lattice directions u_4, u_5, $u_6 \in \mathbb{Z}^2 \backslash \{0\}$ we construct six translates of the construction before. The translates are arranged according to some multiple of the transformation of the lattice set $\tilde{G}_1 \cup \tilde{G}_2$ with respect to the lattice directions u_4, u_5, u_6 to avoid any kind of overlapping. In some components the roles of the lattice sets $\tilde{G}_1, \tilde{G}_2$ have to be changed.
The construction is illustrated in Figure 3.10 for $m = 6$ and the lattice directions $u_1 = (1,0), u_2 = (0,1), u_3 = (1,1),\ u_4 = (3,1), u_5 = (2,1), u_6 = (1,2)$. The transformation for the three lattice directions $u_4 = (3,1), u_5 = (2,1), u_6 = (1,2)$ is given by $T(i,j) = (9 \cdot i + 1 \cdot j, 3 \cdot i + 2 \cdot j)$.

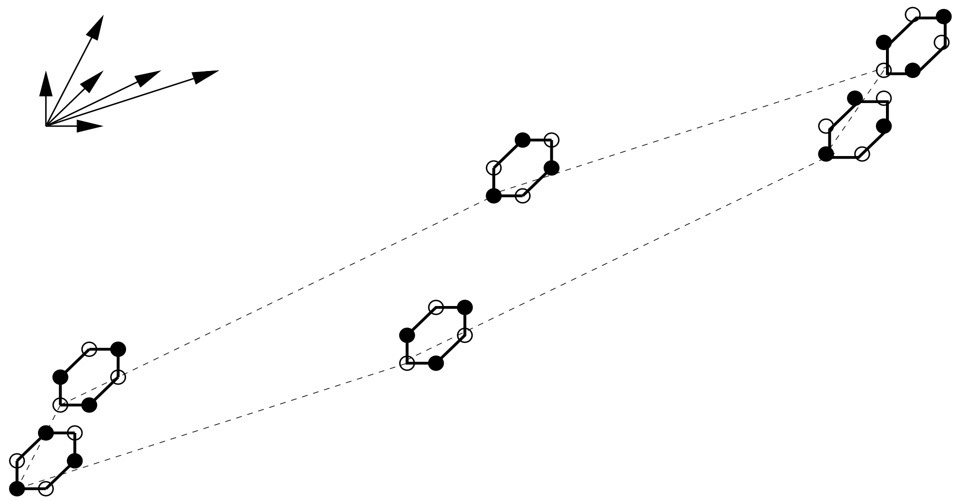

Figure 3.10: The construction for $m = 6$ and $u_1 = (1,0), u_2 = (0,1), u_3 = (1,1)$, $u_4 = (3,1), u_5 = (2,1), u_6 = (1,2)$

Remark 3.6.5 The assumption (3.304) in Theorem 3.6.3 is actually used to show the uniqueness of the two lattice sets F_1, F_2. Because of (3.304) we are able

to reduce any switching component which consists of the lattice set F_1 (resp. of the lattice set F_2) and one of its tomographically equivalent lattice sets to the subgrids $\mathcal{G}_i$ for $i = 1, \ldots, l$ by adding some lattice point $\bar{v}_i$ for $i \in \{2, \ldots, l\}$ to both lattice sets if necessary and consider the switching component on the union $\bigcup_{j=1}^{i-1} \mathcal{G}_j$ of subgrids and on the union $\bigcup_{j=i}^{l} \mathcal{G}_j$ of subgrids separately.
If we do not demand restriction (3.304), we possibly have to add some lattice point $\bar{v}_i$ more than once for $i \in \{2, \ldots, l\}$.
We do not know whether the minimality of the tomographically equivalent lattice sets G_1, G_2 already implies nonadditivity and thus that any lattice point $\bar{v}_i$ is actually added at most once. Therefore, the arguments within the proof before do not work anymore without assumption (3.304).
It is not clear whether the lattice sets G_1, G_2 of minimal cardinality in Theorem 3.6.3 resp. their cardinality $|G_1| = |G_2|$ can efficiently be determined for any set of lattice directions $u_1, \ldots, u_m$.
In the following let us look at some related problem of finding the shortest vector v within a finite dimensional lattice set $\Gamma = \mathbb{Z} \cdot v_1 + \cdots + \mathbb{Z} \cdot v_n$ which satisfies $||v||_\infty = 1$.

Lemma 3.6.6 *The recognition problem*

> *SHORTEST VECTOR, $L_\infty = 1$*
>
> *Instance: A basis $v_1, \ldots, v_n \in \mathbb{Z}^n$ of a lattice set Γ, $w \in \mathbb{N}$.*
>
> *Question: Is there a nonzero vector $x \in \Gamma$ satisfying $||x||_2^2 \le w$, $||x||_\infty = 1$?*

is $\mathcal{NP}$-complete.

Proof

For any vector $x \in \Gamma$ both conditions $||x||_2^2 \le w$ and $||x||_\infty = 1$ can be checked in linear time within the length of the vector x. Therefore, it remains to show that every problem within the class $\mathcal{NP}$ polynomially transforms to SHORTEST VECTOR, $L_\infty = 1$.
It suffices to polynomially transform the recognition problem

> *MINIMAL DISTANCE*
>
> *Instance: A binary $m \times n$-matrix H, $w \in \mathbb{N}$.*
>
> *Question: Is there a vector $x = (x_1, \ldots, x_n) \in \mathbb{Z}_2^n \backslash \{0\}$ at most of weight w satisfying $Hx^t = 0$?*

established in coding theory, which is shown to be $\mathcal{NP}$-complete in [128].
For that purpose let H be a binary $m \times n$-matrix of rank m. Let us determine a basis $v_1, \ldots v_{n-m} \in \mathbb{Z}_2^n$ of the kernel $\{x \in \mathbb{Z}_2^n | Hx^t = 0\}$ by Gaussian elimination. As all code vectors c with respect to the control matrix H are integral linear combinations of the vectors $v_1, \ldots, v_{n-m}$ modulo 2, they are given

by integral linear combinations of the vectors $v_1, \ldots, v_{n-m}, 2 \cdot e_1, \ldots, 2 \cdot e_n$ *under the condition that* $||c||_\infty \leq 1$*, see [21]. The deterministic polynomial-time LLL-algorithm, see [94], is extended to not necessarily independent vectors by Pohst, see [111], [31]. Using the modified LLL-algorithm, we finish the reduction of MINIMAL DISTANCE to SHORTEST VECTOR,* $L_\infty = 1$ *by determining an integer basis of the lattice set* $\Gamma = \{\sum_{i=1}^{m-n} \lambda_i v_i + \sum_{i=1}^{n} 2\mu_i \cdot e_i | \lambda_i, \mu_i \in \mathbb{Z}\}$.
□

Remark 3.6.7 The recognition problem SHORTEST VECTOR, $L_\infty = 1$ is related to the question whether we can find a switching component at most of size w in discrete tomography with respect to a given set of lattice directions. But notice that the set of input vectors related to discrete tomography is more specific, because the vectors arise from the elementary integer switching components as given in [69] by applying all valid translations.
Furthermore, restriction (3.304) within Theorem 3.6.3 is not considered by SHORTEST VECTOR, $L_\infty = 1$.

Chapter 4

Locating lattice sets

Within the present chapter let us assume that the planar lattice set F is well-known, but not its location within the lattice set $\mathbb{Z}^2$. Now our aim is to observe the translations, rotations and reflections of the planar lattice set F within the lattice set $\mathbb{Z}^2$ by analysing the projection data of the planar lattice set F with respect to parallel or point X-rays. We have to examine in which cases the lattice set F and its Euclidean transformation are different, but have the same projection data. Some results are based on ideas which are already used in [45].

4.1 Locating lattice sets by point X-rays

In the present section we will look at both directed and undirected point X-rays. To complete the definition of point X-rays in Chapter 3, let us define the directed point X-ray of a lattice set $F \subset \mathbb{Z}^2$ at some lattice point $p \in \mathbb{Z}^2$.

Definition 4.1.1 (directed point X-ray)
Let $p \in \mathbb{Z}^2$ be a lattice point and let $F \subset \mathbb{Z}^2$ be a finite lattice set. The ***directed point X-ray*** *of the lattice set F at the lattice point p is defined by*

$$D_pF(u) := |\{q \in F | q - p = \lambda \cdot u \text{ for } \lambda \geq 0\}| \tag{4.1}$$

for any lattice direction $u = (r, s) \in \mathbb{Z}^2 \backslash \{0\}$, $\gcd(r, s) = 1$.
The following lemma describes the directed and undirected point X-rays of the transformed lattice set $t(F) \subset \mathbb{Z}^2$ with respect to some affine lattice transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ in terms of the X-rays of the original lattice set $F \subset \mathbb{Z}^2$.

Lemma 4.1.2 *Let $t : \mathbb{R}^2 \to \mathbb{R}^2, t(x) := Ax + b$ be some affine transformation for $A \in \mathbb{Z}^{2\times2}$, $b \in \mathbb{Z}^2$ and $\det(A) = \pm1$. We calculate that $t(\mathbb{Z}^2) = \mathbb{Z}^2$ and that*

$$X_pt(F)(u) = X_{A^{-1}(p-b)}F(A^{-1}u), \tag{4.2}$$

$$D_pt(F)(u) = D_{A^{-1}(p-b)}F(A^{-1}u) \tag{4.3}$$

for any lattice point $p \in \mathbb{Z}^2$.

Proof

For the first part of the assertion we refer to Lemma 3.2.2.
The second part is implied by the equivalence

$$(Ax + b) - p = \lambda \cdot u \tag{4.4}$$

$$\Leftrightarrow A^{-1}(Ax + b - p) = x - A^{-1}(p - b) = \lambda \cdot A^{-1}u \tag{4.5}$$

for any lattice point $x \in F$. □

4.1.1 Lattice translations

First of all let us consider point X-rays and lattice translations. The following theorem shows that a single point X-ray does not determine the translated position of a lattice set in general.

Theorem 4.1.3

Let $p \in \mathbb{Z}^2$ be a lattice point and let $b \in \mathbb{Z}^2 \backslash \{0\}$. For any lattice set $F = F_1 \cup F_2 \subset \mathbb{Z}^2$ satisfying

$$F_1 - p = p + b - F_1, \tag{4.6}$$

$$F_2 \subset (p + \mathbb{R} \cdot b) \backslash \{p, p + b\}, \tag{4.7}$$

i. e. the lattice set $F \backslash (p + \mathbb{R} \cdot b)$ is symmetric with respect to the point $p + \frac{1}{2}b$, we get that

$$X_p F(u) = X_{p+b} F(u) = X_p(F - b)(u) \tag{4.8}$$

for every lattice direction $u \in \mathbb{Z}^2 \backslash \{0\}$, i. e. the translation by the vector $-b$ leads to equal X-ray data. In particular, for any lattice point $\bar{p} \in \mathbb{Z}^2$ the two translates $(\bar{p} - p) + F$, $(\bar{p} - p - b) + F \subset \mathbb{Z}^2$ of the lattice set F have equal X-ray values with respect to the lattice point $\bar{p}$.
On the other hand, if the lattice set F is convex and fulfills condition (4.8) for some $b \in \mathbb{Z}^2 \backslash \{0\}$, then the lattice set F can be written by $F = F_1 \cup F_2$ satisfying (4.6)-(4.7).

Proof

Assuming (4.6)-(4.7) we calculate that

$$X_p F_2(u) = X_{p+b} F_2(u) = \begin{cases} |F_2|, & \text{for } 0 \neq u \in \mathbb{R} \cdot b \\ 0, & \text{for } 0 \neq u \notin \mathbb{R} \cdot b \end{cases} \tag{4.9}$$

and that

$$F_1 - p = p + b - F_1 \tag{4.10}$$

$$\Leftrightarrow \forall x \in F_1 \exists x' \in F_1 : x - p = p + b - x' \tag{4.11}$$

$$\Leftrightarrow \forall x \in F_1 \exists x' \in F_1 : x - p =: u \text{ and } x' - (p + b) = -u \tag{4.12}$$

$$\Leftrightarrow X_p F_1(u) = X_{p+b} F_1(u) \text{ for all } u \in \mathbb{Z}^2 \backslash \{0\}. \tag{4.13}$$

The second equality in (4.8) follows immediately by Lemma 4.1.2.
As

$$X_{\bar{p}}(F+\bar{p}-p)(u) = X_{\bar{p}-(\bar{p}-p)}F(u) = X_pF(u), \tag{4.14}$$
$$X_{\bar{p}}(F+\bar{p}-p-b)(u) = X_{\bar{p}-(\bar{p}-p-b)}F(u) = X_{p+b}F(u), \tag{4.15}$$

we get the assertion for any lattice point $\bar{p} \in \mathbb{Z}^2$.
For the last assertion let us assume that the lattice set F *is convex and fulfills*

$$X_pF = X_{p+b}F \tag{4.16}$$

for some $b \in \mathbb{Z}^2 \backslash \{0\}$. *Let us consider both the case that* $p \notin F$ *and the case that* $p \in F$ *separately in the following.*

1. *In the case that* $p \notin F$ *there exists a line* j *passing through the lattice point* p *so that because of its convexity the lattice set* F *completely lies on one of the sides of the line* j *and both the lattice set* F *and the lattice point* $p+b$ *lie on the same side of the line* j, *if* $F\backslash(p+\mathbb{R}\cdot b) \neq \emptyset$ *and the lattice set* $F\backslash(p+\mathbb{R}\cdot b)$ *is not located on one line:*
   Let us assume that $F\backslash(p+\mathbb{R}\cdot b) \neq \emptyset$ *and that the lattice set* F *and the lattice point* $p+b$ *lie on different sides of the line* j *(including the case that* $p+b \in j$*). Let us determine the extremal lines* $l_l := p+\mathbb{R}\cdot u_l$ *and* $l_r := p+\mathbb{R}\cdot u_l$ *with respect to the lattice point* p *and the lattice set* F *so that*

$$l_r \cap F \neq \emptyset, \tag{4.17}$$
$$l_l \cap F \neq \emptyset, \tag{4.18}$$
$$F \subset p + \{\lambda_1 u_l + \lambda_2 u_r | \lambda_1, \lambda_2 \geq 0\}, \tag{4.19}$$
$$u_r^T u_l > 0, \tag{4.20}$$

   see Figure 4.1.

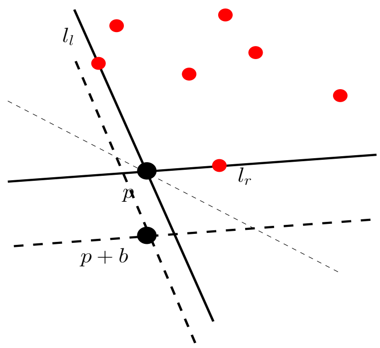

Figure 4.1: Extremal lines with respect to the lattice point p and the lattice set F (red-coloured points)

Notice, that $u_r \notin \mathbb{R} \cdot u_l$*, if the lattice set* $F \backslash (p + \mathbb{R} \cdot b)$ *is not located on one line. As* $F \backslash (p + \mathbb{R} \cdot b) \neq \emptyset$ *and as the lattice point* $p + b$ *and the lattice set* F *lie on different sides of the line* j*, there is always a lattice direction* $u \in \{u_r, u_l\}$ *so that* $X_p(u)(F) \neq 0$*, but* $X_{p+b}(u)(F) = 0$ *in contradiction to assumption (4.16).*
Because of $p + b \notin F$ *using Lemma 3.4.1, we can also determine a line* j' *passing through the lattice point* $p + b$ *so that the lattice set* F *and the lattice point* p *lie on the same side of the line* j'*.*
Because of assumption (4.16) the lattice set $\bar{F} := F \backslash (p + \mathbb{R} \cdot b)$ *and its symmetric copy* $G := -(\bar{F} - (p + \frac{1}{2}b)) + (p + \frac{1}{2}b)$ *with respect to the point* $p + \frac{1}{2}b$ *have the same X-ray values with respect to the lattice points* p *and* $p + b$*. Now let us assume that* $\bar{F} \triangle G \neq \emptyset$*. Based on the idea used to prove Theorem 4.2 in [45], let us define the line* l *by* $l := p + \mathbb{R} \cdot b$ *and let* $i \neq l$ *be that line which encloses the smallest angle with the line* l *so that*

$$i \cap (\bar{F} \triangle G) \neq \emptyset, \tag{4.21}$$

$$i \cap \{p, p + b\} \neq \emptyset. \tag{4.22}$$

By using the tomographic equivalence of the lattice sets $\bar{F}$ *and* G*, let us assume that without loss of generality the lattice points* p*,* $v_1 \in \bar{F} \backslash G$ *and* $v_2 \in G \backslash \bar{F}$ *are located in that order on the line* i*, as because of the minimality of the enclosed angle both lattice points* $v_1 \in \bar{F} \backslash G$ *and* $v_2 \in G \backslash \bar{F}$ *have to lie on the same side of the lattice point* p*. (In the contrary case the line passing through the lattice points* $p + b$ *and* v_1 *resp.* v_2 *would enclose a smaller angle.)*

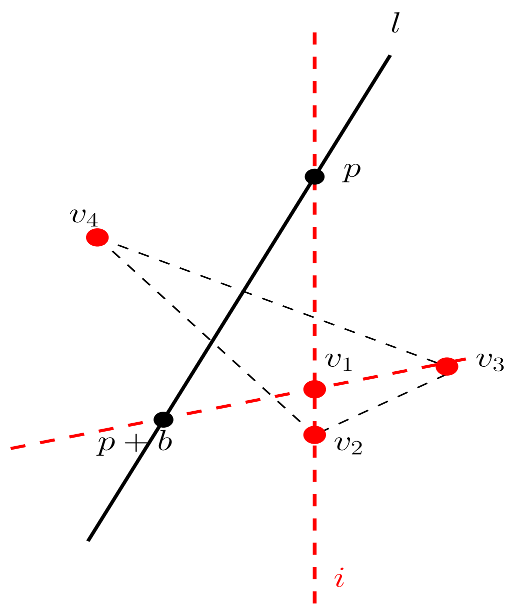

Figure 4.2: Unique determination of the lattice set $\bar{F}$

Using similar tomographic arguments the lattice points $p+b$, v_1, $v_3 \in G\backslash\bar{F}$ are located in that order on the line which passes through the lattice points $p+b$ and v_1. That is the case as $v_3 \neq (p+\frac{1}{2}b)-(v_1-(p+\frac{1}{2}b)) \in G\backslash\bar{F}$ and thus the lattice set G is not located on one single line. By assuming the ordering $v_3, p+b, v_1$, there exists no line passing through the lattice point $p+b$ so that both the lattice point p and the lattice set $G \ni v_2, v_3$ lie on the same side of that line.
Thus, the lattice point $v_1 \in \bar{F}\backslash G$ lies within the triangle which is given by the vertices $v_2, v_3, (p+\frac{1}{2}b)-(v_1-(p+\frac{1}{2}b)) \in G\backslash\bar{F}$ in contradiction to the convexity assumption on the lattice set F.

2. *In the case that $p \in F$ the lattice point $p+b$ also has to lie within the lattice set F because of Lemma 3.4.1. Let us define the lattice sets F_1, F_2 by*

$$F_2 := F \cap (p+\mathbb{R}\cdot b), \tag{4.23}$$
$$F_1 := F\backslash F_2 \tag{4.24}$$

and the lines l, g_1 and g_2 by

$$l := p+\mathbb{R}\cdot b, \tag{4.25}$$
$$g_1, g_2 \perp l \text{ so that } p \in g_1, p+b \in g_2, \tag{4.26}$$

see Figure 4.3.

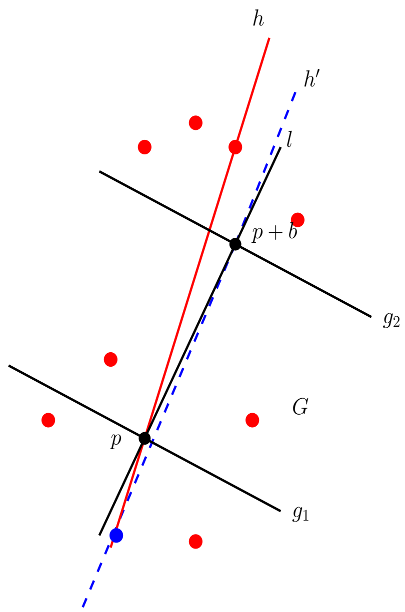

Figure 4.3: The lines l, g_1, g_2, h and h'

Let the line h be given by

$$h \cap F_1 \neq \emptyset, \tag{4.27}$$
$$h \cap \{p, p+b\} \neq \emptyset \tag{4.28}$$

enclosing the smallest angle with the line l under all those lines. All lattice points within the lattice set $F \cap h$ are uniquely determined by the X-ray value along the line h because of the convexity of the lattice set F:
The lattice set $F \cap h$ has to completely lie on the same side with respect to the lattice point p (in the illustrated case) resp. with respect to the lattice point $p+b$, as in the case that the line g_1 separates the lattice point $p+b$ and some lattice point $q \in h \cap F$ resp. in the case that the line g_2 separates the lattice point p and some lattice point $q \in h \cap F$ the line h' passing through the lattice point $p+b$ resp. through the lattice point p and the lattice point q also satisfies (4.27)-(4.28) and encloses a smaller angle with the line l.
Because of assumption (4.16) the same arguments can also be applied to the line $\bar{h}||h$ passing through the lattice point $p+b$ resp. through the lattice point p. Thus, we actually determine a set of lattice points which is symmetric with respect to the point $p + \frac{1}{2}b$. Repeating the arguments we result in (4.6) for the lattice set F_1.

□

The consequences of Theorem 4.1.3 for directed point X-rays are formulated in the following corollary.

Corollary 4.1.4 *Let $p \in \mathbb{Z}^2$ be a lattice point and let $b \in \mathbb{Z}^2 \backslash \{0\}$. If*

$$D_p F = D_{p+b} F = D_p(F - b), \tag{4.29}$$

we result in $F \subset l := p + \mathbb{R} \cdot b$.

Proof

Let us assume that $F \backslash (p + \mathbb{R} \cdot b) \neq \emptyset$. As we consider directed point X-rays, we can assume without loss of generality that the lattice set $F \backslash (p + \mathbb{R} \cdot b)$ completely lies on one side of the line l. Thus, the extremal lattice directions u_l and u_r as specified in Theorem 4.1.3 differ for the lattice points p and $p+b$ in contradiction to assumption (4.29). □

Now let us consider two undirected point X-rays and extend Theorem 4.1.3 to that case.

Corollary 4.1.5 *Let $p_1, p_2 \in \mathbb{Z}^2$ be two distinct lattice points, let $F \subset \mathbb{Z}^2$ be a convex lattice set and let $b \in \mathbb{Z}^2 \backslash \{0\}$. If*

$$X_{p_k} F = X_{p_k+b} F = X_{p_k}(F - b) \text{ for } k = 1, 2, \tag{4.30}$$

we conclude that $F \subset t + \mathbb{R} \cdot (p_2 - p_1)$ *for some* $t \in \mathbb{Z}^2$.

Proof

Let $l := p_1 + \mathbb{R} \cdot (p_2 - p_1)$ *be that line which passes through both lattice points* p_1 *and* p_2. *Let us assume that* $F \backslash (t + l) \neq \emptyset$ *for every translation vector* $t \in \mathbb{Z}^2$ *and let us choose that line* $h \notin \{p_1 + \mathbb{R} \cdot b, p_2 + \mathbb{R} \cdot b\}$ *which passes through one of the points* $p_1 + \frac{1}{2}b$, $p_2 + \frac{1}{2}b$ *and some lattice point* $x \in F$ *so that the angle between the line* h *and the line* l *gets minimal. Notice, that the lattice set* $F \backslash (p_k + \mathbb{R} \cdot b)$ *is symmetric with respect to the point* $p_k + \frac{1}{2}b$ *for* $k = 1, 2$, *compare Theorem 4.1.3. Thus, the line* h *contains pairs of points which are symmetric with respect to the point* $p_1 + \frac{1}{2}b$ *resp. with respect to the point* $p_2 + \frac{1}{2}b$. *Therefore, as the lines* h *and* l *are not parallel, we can always find another line* h' *passing through the point* $p_2 + \frac{1}{2}b$, *if the line* h *passes through the point* $p_1 + \frac{1}{2}b$ *(and vice versa), and one of those pairwise points (that one which has larger distance to the point* $p_2 + \frac{1}{2}b$ *resp. to the point* $p_1 + \frac{1}{2}b$*) so that* $h \cap h' \cap F \neq \emptyset$ *and the line* h' *encloses a smaller angle with the line* l *in contradiction to the choice of the line* h, *see Figure 4.4.* □

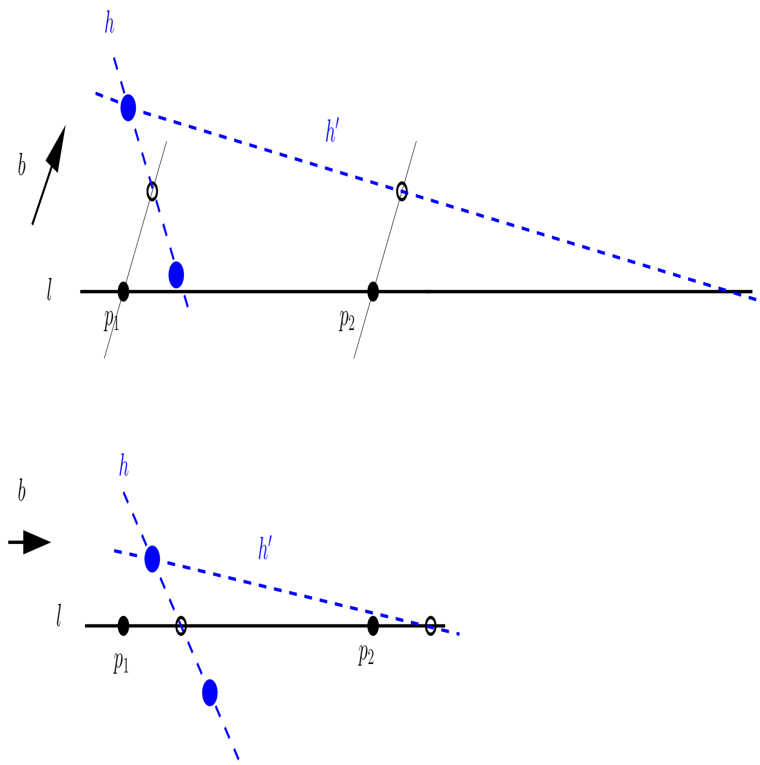

Figure 4.4: The line h enclosing the smallest angle with the line l and the construction of the line h'

Remark 4.1.6 Because of Corollary 4.1.5 and Lemma 4.1.2 three noncollinear point X-ray sources uniquely determine the translated position of any convex lattice set $F \subset \mathbb{Z}^2$.

4.1.2 Lattice rotations and reflections

Now let us consider lattice rotations and lattice reflections.
The next two lemmata are concerned with the Euclidean transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$, $t(x) := -Ix + b$ for some $b \in \mathbb{Z}^2$ and both undirected and directed X-rays.

Lemma 4.1.7 *Let $p \in \mathbb{Z}^2$ be a lattice point and let $F \subset \mathbb{Z}^2$ be some lattice set. Then there exists an Euclidean transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ so that the lattice set F and the lattice set $t(F)$ have equal (undirected) X-ray values within the lattice point p.*

Proof

Because of $x - p = -(p - x) = -[(2 \cdot p - x) - p]$ the Euclidean transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by

$$t(x) := -I(x - p) + p = 2 \cdot p - x \tag{4.31}$$

implies the assertion. □

Lemma 4.1.8 *Let $p \in \mathbb{Z}^2$ be some lattice point and let the Euclidean transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ be defined by $t(x) := -Ix + b$ for some $b \in \mathbb{Z}^2$. For any lattice set $F = F_1 \cup F_2 \subset \mathbb{Z}^2$ satisfying*

$$F_1 = b - F_1 \text{ (i. e. } F_1 \text{ is symmetric with respect to the point } \frac{1}{2}b\text{)}, \tag{4.32}$$

$$F_2 \subset]p, b - p[\tag{4.33}$$

we get that

$$D_pF = D_pt(F). \tag{4.34}$$

On the other hand, if the lattice set F is convex and fulfills condition (4.34), then the lattice set F is given by $F = F_1 \cup F_2$ satisfying (4.32)-(4.32).

Proof

According to Lemma 4.1.2 we calculate that

$$D_pt(F)(u) = D_{A^{-1}(p-b)}F(A^{-1}u) = D_{(b-p)}F(-u) \tag{4.35}$$

for $t(x) := -Ix + b$, i. e. for $A := -I$. Therefore, the lattice sets F_1, F_2 and thus also the lattice set $F = F_1 \cup F_2$ fulfill (4.34).
For the second assertion let us assume that the lattice set F is convex and fulfills condition (4.34). Using (4.35) the lattice sets $\bar{F} := F \backslash (]p, b - p[)$ and $\bar{G} := t(\bar{F})$ have the same X-ray values within the lattice points p and $b - p$. Notice, that condition (4.34) and the convexity of the lattice set F imply that $(\bar{F} \triangle \bar{G}) \cap (p + \mathbb{R} \cdot (b - 2 \cdot p) = \emptyset$. Thus, if $\bar{F} \triangle \bar{G} \neq \emptyset$ we can use the same arguments as in Theorem 4.1.3 for the lattice point $b - p$ instead of the lattice

point $p - b$ and we again result in contradition to the convexity assumption on the lattice set F, compare Figure 4.2:
As directed X-rays are considered, we can treat $\bar{F} \triangle \bar{G}$ on each side of the line $l := p + \mathbb{R} \cdot (b - 2 \cdot p)$ separately. Thus, the lattice points v_1, v_2 and v_3 also have to lie on the same side of the line l in the now considered case and some lattice point v_4 exists on the other side by using the symmetry of the lattice set $\bar{F} \triangle \bar{G}$. □
The directed point X-ray within a single lattice point cannot determine the rotation or reflection of a convex lattice set in general.

Lemma 4.1.9 *Let $p \in \mathbb{Z}^2$ be a lattice point and let $A \in \mathrm{Sym}(\{-1,+1\}^2) \backslash \{\pm I\}$ be some matrix within the symmetric group*

$$\mathrm{Sym}(\{-1,+1\}^2) := < \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} > \tag{4.36}$$

of the square $\{-1,+1\}^2 \subset \mathbb{Z}^2$. Then there exist a convex lattice set $F \in \mathbb{Z}^2$ and some $b \in \mathbb{Z}^2$ so that

$$D_p F = D_p t(F) \tag{4.37}$$

for the Euclidean transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$, $t(x) := Ax + b$, but $F \neq t(F)$. Moreover, the lattice points within the lattice set $F \triangle t(F)$ are not located on the line $p + \mathbb{R} \cdot (A^{-1}(p - b) - p)$.

Proof
Let us assume that $p = (0, 0)$.

1. *In the case of reflection let the matrix A be given without loss of generality by*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.38}$$

and let us define the lattice sets F_1, F_2 by

$$F_1 := \{(30, 30), (-45, 35), (24, 84)\}, \tag{4.39}$$
$$F_2 := \{(-30, 30), (45, 35), (-24, 84)\} \tag{4.40}$$

having the same projection values within the lattice points $p_1 = (-60, 0)$ and $p_2 = (60, 0)$ for symmetric lattice directions u_1, $u_2 := A^{-1}u_1$. Thus, we calculate for the lattice set F and the Euclidean transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by

$$F := (60, 0) + \mathrm{conv}(F_1 \cup F_2) \backslash F_1, \tag{4.41}$$
$$t(x) := Ax + b \text{ for } b := -A(120, 0)^T \tag{4.42}$$

that

$$D_{(0,0)}F(u) = D_{(120,0)}F(A^{-1}u) \tag{4.43}$$
$$\Leftrightarrow D_{(0,0)}F(u) = D_{A^{-1}(-b)}F(A^{-1}u) \tag{4.44}$$
$$\Leftrightarrow D_{(0,0)}F(u) = D_{(0,0)}t(F)(u) \tag{4.45}$$

and that $F \triangle t(F) = (60,0) + (F_1 \cup F_2) \not\subseteq \mathbb{R} \cdot (120,0)$.

2. *In the case of rotation let us consider without loss of generality the matrix*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.46}$$

Let the convex lattice set F *and the Euclidean transformation* $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ *be defined by*

$$F := \{(2,1),(2,2)\}, \tag{4.47}$$
$$t(x) := Ax + b \text{ for } b := -A(3,0)^T. \tag{4.48}$$

We calculate that

$$D_{(0,0)}F(u) = D_{-A^{-1}b}F(A^{-1}u) = D_{(0,0)}t(F)(u), \tag{4.49}$$

as

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{4.50}$$
$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{2} \cdot A^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \tag{4.51}$$

but $F \triangle t(F) = \{(2,1),(2,2)\} \triangle \{(1,1),(2,1)\} = \{(1,1),(2,2)\} \not\subseteq \mathbb{R} \cdot (3,0)$.

□

The following two lemmata consider the case of two point X*-ray sources and reflections* $t : \mathbb{Z}^2 \to \mathbb{Z}^2, t(x) := Ax + b$ *specified by* $A \in \mathrm{Sym}(\{-1,+1\}^2)$ *and* $b \perp \{x \in \mathbb{R}^2 | Ax = x\}$.

Lemma 4.1.10 *Let* $F \subset \mathbb{Z}^2$ *be a convex lattice set and let the Euclidean transformation* $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ *be defined by*

$$t(x) := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x + d \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.52}$$

for $d \in \mathbb{Z}$. *There exist two distinct point* X*-ray sources* $p_1, p_2 \in \mathbb{Z}^2$ *so that the position of the lattice set* $t(F \setminus (\mathbb{R} \cdot e_1))$ *is uniquely determined by its directed point* X*-rays with respect to the lattice points* p_1, p_2.

Proof

Let us define $p_1 := (0,0)$ and $p_2 := (1,0)$. According to Lemma 4.1.2 we have to examine whether it is possible that

$$D_{p_1}F(u) = D_{p_4}F(A^{-1}u), \tag{4.53}$$

$$D_{p_2}F(u) = D_{p_3}F(A^{-1}u) \tag{4.54}$$

for some lattice set F satisfying $t(F) \neq F$ and $p_3 := (a,0)$, $p_4 := (a+1,0)$ for some $a := d-1$.

In the case that $p_i \neq p_j$ for $i \neq j$ we have to consider four distinct lattice points p_1, p_2, p_3 and p_4. Thus, we make use of Corollary 6.6 in [45], which states that the undirected (instead of directed) X-ray values of a set of four lattice points within the lattice set $\mathbb{Z}^2$ incident to one line uniquely determine the set of convex lattice sets which do not meet that line, if there is no ordering of the four lattice points so that their cross ratio is equal to $2, 3$ or 4.

Therefore, let us calculate the cross ratio

$$[p_1,p_2,p_3,p_4] = \frac{\det\begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} 1 & a+1 \\ 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 0 & a+1 \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}} = \frac{a^2}{a^2-1} \tag{4.55}$$

of the four lattice points p_1, p_2, p_3 and p_4 for a fixed ordering and examine the case of equality to some value within the set

$$S := \{k, \frac{1}{k}, 1-k, 1-\frac{1}{k}, \frac{1}{1-k}, \frac{k}{k-1} | k \in \{2,3,4\}\} = \tag{4.56}$$

$$= \{2, \frac{1}{2}, -1; 3, \frac{1}{3}, -2, \frac{2}{3}, -\frac{1}{2}, \frac{3}{2}; 4, \frac{1}{4}, -3, \frac{3}{4}, -\frac{1}{3}, \frac{4}{3}\}, \tag{4.57}$$

see [20], Proposition 6.3.2. Because of

$$\frac{a^2}{a^2-1} = b \Leftrightarrow \frac{1}{a^2} = 1 - \frac{1}{b} \Leftrightarrow \pm a = \sqrt{\frac{b}{b-1}} \tag{4.58}$$

the value a and therefore also the value d is not integral, if $b \in S \backslash \{\frac{4}{3}\}$.

Therefore, we conclude by using Corollary 6.6 within [45] that equal directed point X-rays within the lattice points p_1, p_2 for the lattice sets F and $t(F)$ imply that $t(F \backslash (\mathbb{R} \cdot e_1)) = F \backslash (\mathbb{R} \cdot e_1)$ if $d \in \mathbb{Z} \backslash \{-1,0,1,2,3\}$.

Thus, it remains to treat the case that

$$p_1 = (0,0), p_2 = p_3 = (1,0), p_4 = (2,0) \tag{4.59}$$

for $d = 2$ (and similar for $d = 0$), the case that

$$p_1 = p_3 = (0,0), p_2 = p_4 = (1,0) \tag{4.60}$$

for $d = 1$ *and the case that*

$$p_1 = (0,0), p_2 = (1,0), p_3 = (2,0), p_4 = (3,0) \tag{4.61}$$

for $d = 3$ *(and similar for* $d = -1$*). In each case we can treat the lattice sets* $F \cap \{x_2 \geq 0\}$ *and* $F \cap \{x_2 \leq 0\}$ *separately, as we consider directed point* X*-rays.*

1. *In the case (4.59) we show in a first step that the lattice set* $F \cap (\{0,1,2\} \times \mathbb{N})$ *for any convex lattice set* $F \subset \mathbb{Z}^2$ *is uniquely determined by the directed point* X*-rays within the lattice points* $p_1, p_2 = p_3, p_4$*:*
   As the lattice set F *is convex, we know that the points* $F \cap \{x_1 = 0\}$, $F \cap \{x_1 = 1\}$ *and* $F \cap \{x_1 = 2\}$ *lie consecutively on each corresponding vertical line. Let us define the values*

   $$t_i := \max\{0\} \cup \{x_2 | (i, x_2) \in F \cap \{x_2 > 0\}\} \tag{4.62}$$

   for $i = 0, 1, 2$*. We will show that* $t_0 = t_2 \leq t_1$*:*
   If $0 < t_0 < t_2$ *there exists a lattice point* $(1,0) + \lambda(-1, t_2) \in F$ *for some* $\lambda \in \mathbb{N} \backslash \{1\}$ *so that* $(0, t_2) \in \text{conv}\{(1,0) + \lambda(-1,t_2), (2,t_2), (0,t_0)\} \cap \mathbb{Z}^2$, *which implies that* $(0, t_2) \in F$ *in contradiction to the definition of the value* t_0*.*
   The convexity of the lattice set F *implies that if* $(0, t_0 = t_2), (2, t_2) \in F$ *then also the lattice point* $(1, t_0 = t_2)$ *has to belong to the lattice set* F *and thus* $t_1 \geq t_0 = t_2$*.*
   To show for $t_0 = t_2 \neq 0$ *or* $t_1 \neq 0$ *that the values* $t_0 = t_2$*,* t_1 *are determined by the maximal values* $u_{\max}, v_{\max}$ *which satisfy that*

   $$D_{p_2}F(-1,u) = D_{p_2}F(1,u) \neq 0, \tag{4.63}$$
   $$D_{p_1}F(1,v) = D_{p_4}F(-1,v) \neq 0, \tag{4.64}$$

   let us assume that $0 < t_0 = t_2 < u_{\max}$*. Then there exist two lattice points*

   $$q_1 := (1,0) + \mu_1 \cdot (-1, u_{\max}) \in F, \tag{4.65}$$
   $$q_2 := (1,0) + \mu_2 \cdot (1, u_{\max}) \in F \tag{4.66}$$

   for some $\mu_1, \mu_2 \in \mathbb{N} \backslash \{1\}$ *and*

   $$(0, u_{\max}), (2, u_{\max}) \in \text{conv}\{q_1, q_2, (0,t_0), (2,t_2)\} \cap \mathbb{Z}^2 \tag{4.67}$$

   in contradiction to the assumption that $t_0 = t_2 < u_{\max}$*.*
   Similar convexity arguments work to show that the value $v_{\max}$ *determines the value* t_1*, as*

   $$(1, v_{\max}) \in \text{conv}\{(1,t_1), (0,0) + \mu_1 \cdot (1, v_{\max}), (2,0) + \mu_2 \cdot (-1, v_{\max})\} \tag{4.68}$$

for $\mu_1, \mu_2 \in \mathbb{N}\backslash\{1\}$ and $t_1 < v_{\max}$.

In order to show that the complete lattice set $F \cap \{x_2 > 0\}$ is uniquely determined by the X-rays with respect to the lattice points $p_1, p_2 = p_3, p_4$, let us assume that there exists a convex lattice set $\bar{F} \subset \mathbb{Z}^2$ which is tomographically equivalent to the convex lattice set F so that $(F \triangle \bar{F}) \cap \{x_2 > 0\} \neq \emptyset$. Let us assume that $(F \triangle \bar{F}) \cap \{x_2 > 0\} \cap \{x_1 > 2\} \neq \emptyset$ and let us choose the line g passing through the lattice point $p_4 = (2, 0)$ and through the lattice set $F \triangle \bar{F} \cap \{x_2 > 0\} \cap \{x_1 > 2\}$ which encloses the smallest angle with the vertical line $\{x_1 = 2\}$. Without loss of generality let us assume that the lattice points p_4, $v_1 \in F\backslash\bar{F}$, $v_2 \in \bar{F}\backslash F$ lie in that order on the line g. Let h be the line which passes through the lattice points p_1 and v_2. Because of the tomographic equivalence of the lattice sets F and $\bar{F}$ there exists a further lattice point $v_3 \in F\backslash\bar{F}$ so that the lattice points p_1, v_2, v_3 lie in that order on the line h. Notice, that because of (4.53)-(4.54) there exists some lattice point $v_4 \in F \cap \{x_2 > 0\} \cap \{x_1 < 0\}$ so that $v_2 \in \operatorname{conv}\{v_1, v_3, v_4\}$ in contradiction to the convexity of the lattice set F.

As we treat directed X-rays and as the lattice points p_1, $p_2 = p_3$, p_4 are neighboured on the line $\mathbb{R} \cdot e_1$, uniqueness is also given for the lattice set $F \cap \{x_2 = 0\}$ because of (4.53)-(4.54) and the convexity of the lattice set F.

Similar arguments also work for the case (4.60).

2. *For the case (4.61) let the values t_i for $i = 0, 1, 2, 3$ be defined as before in (4.62). If we assume that $0 < t_0 < t_3$ the lattice point $(0, t_3)$ lies within the triangle $\operatorname{conv}\{(0, t_0), (3, t_3), (1, 0) + \lambda \cdot (-1, t_3)\}$ for some $\lambda \in \mathbb{N}\backslash\{1\}$ in contradiction to the convexity of the lattice set F. If we assume that $0 < t_1 < t_2$ the lattice point $(1, t_2)$ lies within the triangle $\operatorname{conv}\{(1, t_1), (2, t_2), (2, 0) + \mu \cdot (-1, t_2)\}$ for some $\mu \in \mathbb{N}\backslash\{1\}$ in contradiction to the convexity of the lattice set F again. Further convexity arguments then imply that $t_0 = t_3 \leq t_1 = t_2$.*
   The values $t_0 = t_3$ and $t_1 = t_2$ (if they are greater than 0) are specified in the same manner as in the case (4.59) by the maximal arguments $u_{\max}$, $v_{\max}$ which satisfy

   $$D_{p_2}F(-1, u) = D_{p_3}F(1, u) \neq 0, \tag{4.69}$$

   $$D_{p_2}F(1, v) = D_{p_3}F(-1, v) \neq 0. \tag{4.70}$$

   By applying the further arguments already used for the case (4.59), we conclude that also in the case (4.61) the lattice set $F \cap \{x_2 \geq 0\}$ is uniquely determined.

Altogether, by using Lemma 4.1.2 the lattice points $p_1 = (0, 0)$ and $p_2 = (1, 0)$ uniquely determine the location of the lattice set $F\backslash\mathbb{R} \cdot e_1$ for any convex lattice set $F \subset \mathbb{Z}^2$ after any reflection (4.52). □

Lemma 4.1.11 *Let $F \subset \mathbb{Z}^2$ be a convex lattice set and let the Euclidean transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ be defined by*

$$t(x) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + d \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.71}$$

for $d \in \mathbb{Z}$. There exist three distinct point X-ray sources $p_1, p_2, q \in \mathbb{Z}^2$ so that the position of the lattice set $t(F \backslash (p_1 + \mathbb{R} \cdot (p_2 - p_1)))$ is uniquely determined by its directed point X-rays with respect to the lattice points p_1, p_2 and q.

Proof

Let us define $p_1 := (0,1)$, $p_2 := (1,0)$ and $q := (1,1)$. Using the results of Corollary 6.6 in [45] and the considerations within the proof of Lemma 4.1.10, it remains to look at the cases

$$\begin{cases} p_1 = (0,1), p_2 = p_3 = (1,0), p_4 = (2,-1), \\ q_1 := q = (1,1), q_2 = (2,0), \end{cases} \tag{4.72}$$

$$\begin{cases} p_1 = p_3 = (0,1), p_2 = p_4 = (1,0), \\ q_1 := q = q_2 = (1,1) \end{cases} \tag{4.73}$$

$$\begin{cases} p_1 = (0,1), p_2 = (1,0), p_3 = (2,-1), p_4 = (3,-2), \\ q_1 := q = (1,1), q_2 = (3,-1). \end{cases} \tag{4.74}$$

Let the values t_i be defined by

$$t_i := \max\{0\} \cup \{\lambda \in \mathbb{N} | p(i) + \lambda \cdot (1,1) \in F\} \tag{4.75}$$

for $p(i) \in \{p_1, p_2, p_3, p_4, q_1, q_2\}$ satisfying $(p(i) + \mathbb{R} \cdot (1,1)) \cap (\mathbb{R} \cdot e_1) = (i, 0)$.

1. *In the case (4.72) the same arguments as used in Lemma 4.1.10 help us to show that $t_{-1} = t_3 \le t_1$ and $t_0 = t_2$. The values $t_{-1}, \ldots, t_3$ (if they are greater than 0) are determined by the maximal arguments $u_{\max}^{p,v}$ for $p \in \{p_2 = p_3, q_1, q_2\}$ and $v \in \{(1,0), (0,1)\}$ which satisfy*

   $$D_p(v + u \cdot (1,1)) \neq 0 \tag{4.76}$$

   by using similar arguments as before. In particular, the values $u_{\max}^{p_2,(1,0)}$, $u_{\max}^{p_2,(0,1)}$ determine the value $t_0 = t_2$, the values $u_{\max}^{q_1,(1,0)}$, $u_{\max}^{q_2,(0,1)}$ determine the value t_1, and the values $u_{\max}^{q_1,(0,1)}$, $u_{\max}^{q_2,(1,0)}$ determine the value $t_{-1} = t_3$. The final considerations to show that the lattice set $t(F \backslash (p_1 + \mathbb{R} \cdot (p_2 - p_1)))$ is uniquely determined are analogous to those in Lemma 4.1.10 before.

   The case (4.73) is treated in the same manner.

2. *In the case (4.74) we analogously conclude that $t_{-1} = t_5$, $t_0 = t_4$. The values $t_{-1} = t_5$ and $t_0 = t_4$ are specified by the maximal arguments $u_{\max}, v_{\max}$ satisfying*

   $$D_{q_1}((0,1) + u \cdot (1,1)) = D_{q_2}((1,0) + u \cdot (1,1)) \neq 0, \tag{4.77}$$

   $$D_{p_2}((0,1) + v \cdot (1,1)) = D_{p_3}((1,0) + v \cdot (1,1)) \neq 0 \tag{4.78}$$

as in Lemma 4.1.10.
Analogous to Lemma 4.1.10 we show that the lattice set $F \backslash (\{p_2 = (1,0), p_3 = (2,-1), (2,0)\} + \mathbb{R} \cdot (1,1))$ *is uniquely determined by the directed point X-rays with respect to the lattice points* p_1, p_2, p_3, p_4, q_1 *and* q_2 *using the convexity of the lattice set* F*.*
Thus, let us finally assume that the lattice set $F \cap (\{p_2, p_3, (2,0)\} \times \mathbb{R} \cdot (1,1))$ *is not uniquely determined. In particular, let the lattice sets* $\bar{F}$ *and* F *be tomographically equivalent. Let* g *be that line which passes through the lattice point* p_2 *or the lattice point* p_3 *and the lattice set* $F \triangle \bar{F}$ *enclosing the smallest angle with the line* $p_2 + \mathbb{R} \cdot (p_3 - p_2)$*. Without loss of generality let us assume that the lattice points* p_2*,* $v_1 \in F \backslash \bar{F}$ *and* $v_2 \in \bar{F} \backslash F$ *lie in that order on the line* g*. Thus, the line which passes through the lattice points* p_1 *and* v_1 *also has to pass through some lattice point* $v_3 \in (\bar{F} \backslash F) \cap (p_2 + \mathbb{R} \cdot (1,1))$*. Therefore, we result in contradiction to the choice of the line* g*, as the line incident to the lattice points* p_3 *and* v_3 *encloses a smaller angle with the line* $p_2 + \mathbb{R} \cdot (p_3 - p_2)$*.*

□

Remark 4.1.12 As convex lattice sets are uniquely determined by four specific point X-ray sources located on one line, see [45], the location of any convex lattice set after any Euclidean lattice transformation is also determined by those points. But it is not clear if possibly three point X-ray sources are enough for that purpose.

4.2 Locating lattice sets by parallel X-rays

In the case of parallel X-rays the location of any translated lattice set is already determined by two distinct lattice directions. Thus, let us immediately extend our consideration to the class of Euclidean lattice transformations.

Lemma 4.2.1 *Let $F \subset \mathbb{Z}^2$ be a convex lattice set and let $A \in \mathrm{Sym}(\{-1,+1\}^2)\backslash\{-I\}$ be some matrix within the symmetric group of the square $\{-1,+1\}^2 \subset \mathbb{Z}^2$. There are two distinct lattice directions $u_1 = (r_1, s_1)$, $u_2 = (r_2, s_2) \in \mathbb{Z}^2$, $\gcd(r_k, s_k) = 1$ so that $F = t(F)$ is implied by*

$$X_{u_k}F = X_{u_k}t(F) \text{ for } k = 1, 2 \tag{4.79}$$

for the Euclidean transformation $t : \mathbb{Z}^2 \to \mathbb{Z}^2$, $t(x) := Ax + b$, $b \in \mathbb{Z}^2$.
In the case that $A := -I$ and that $u_1, u_2, u_3 \in \mathbb{Z}^2$ are three distinct lattice directions there exist a lattice set $F \subset \mathbb{Z}^2$ and some $b \in \mathbb{Z}^2$ so that

$$X_{u_k}F = X_{u_k}t(F) \text{ for } k = 1, 2, 3, \tag{4.80}$$

but $F \neq t(F)$.

Proof
Let the lattice directions u_1, u_2 be defined by $u_1 := (1, 2)$, $u_2 := (1, 3)$. Using Lemma 4.1.2 and the results of [54] for $A \neq \pm I$, we have to calculate the cross ratio

$$[u_1, u_2, u_3, u_4] = \frac{\det(u_1, u_3)\det(u_2, u_4)}{\det(u_2, u_3)\det(u_1, u_4)} \tag{4.81}$$

for $u_1, u_2, u_3 := Au_1, u_4 := Au_2$ after rearranging in order of increasing angle with the positive x-axis, see [20]. The cross ratio does not equal $\frac{4}{3}, \frac{3}{2}, 2, 3$ or 4 for any matrix $A \in \mathrm{Sym}(\{-1,+1\}^2)\backslash\{\pm I\}$, as

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow Au_1 = (-1, 2), Au_2 = (-1, 3)$$

$$\Rightarrow [u_1, u_2, Au_2, Au_1] = \frac{25}{24}, \tag{4.82}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow Au_1 = (1, -2), Au_2 = (1, -3)$$

$$\Rightarrow [u_1, u_2, Au_2, Au_1] = \frac{25}{24}, \tag{4.83}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow Au_1 = (-2, 1), Au_2 = (-3, 1)$$

$$\Rightarrow [u_1, u_2, Au_1, Au_2] = \frac{50}{49}, \tag{4.84}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow Au_1 = (2, -1), Au_2 = (3, -1)$$

$$\Rightarrow [u_1, u_2, Au_1, Au_2] = \frac{50}{49}, \tag{4.85}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow Au_1 = (2, 1), Au_2 = (3, 1)$$

$$\Rightarrow [Au_2, Au_1, u_1, u_2] = \frac{25}{24}, \tag{4.86}$$

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \Rightarrow Au_1 = (-2, -1), Au_2 = (-3, -1)$$

$$\Rightarrow [u_1, u_2, Au_2, Au_1,] = \frac{25}{24}. \tag{4.87}$$

Thus, the first part of the assertion is implied by [54].
Let the lattice set $F_1 \cup F_2$ *be given by the vertex set of the lattice* U*-polygon with respect to the lattice directions* $u_1, u_2, u_3 \in \mathbb{Z}^2$*, see [54], so that no two vertices within the lattice set* F_1 *resp. within the lattice set* F_2 *are neighboured. Let the convex lattice set* F *and the Euclidean transformation* $t : \mathbb{Z}^2 \to \mathbb{Z}^2$ *be defined by*

$$F := (\mathrm{conv}(F_1 \cup F_2) \cap \mathbb{Z}^2) \backslash F_1, \tag{4.88}$$

$$t(x) := -Ix + 2 \cdot b \tag{4.89}$$

for the center b *of the convex lattice set* $\mathrm{conv}(F_1 \cup F_2)$*. As the lattice sets* F_1*,* F_2 *are tomographically equivalent, it yields that*

$$X_{u_k} F = X_{u_k} t(F) \text{ for } k = 1, 2, 3, \tag{4.90}$$

but $t(F) \triangle F = F_1 \cup F_2$ *because of the definition of the lattice sets* F_1, F_2 *and* F*.* □

Remark 4.2.2 According to [54] any set of four distinct lattice directions with cross ratio not equal to $\frac{4}{3}, \frac{3}{2}, 2, 3$ or 4 uniquely determines the class of convex lattice sets. Thus, also the lattice motions of a convex lattice set $F \subset \mathbb{Z}^2$ are uniquely determined by those four lattice directions.

Chapter 5

Characterizing small error values

According to [7], [5] and [3] we know that small error within the X-ray values can cause large changes within the original data and that the smallest right hand side difference for two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ of same cardinality and m distinct X-ray directions is given by $2(m-1)$.
In the following we will take a closer look at the right hand side data. We are interested in possible combinations of directions on which different X-ray values can occur for two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ of same cardinality. Furthermore, we want to detect some characteristics arising within the error values.
That information possibly helps us to decide which templates can be rejected on the basis of its X-ray data within the quality control in semiconductor industry.
To guarantee stability for any quality control algorithm, let us assume that no admissible template is rejected within the quality control in semiconductor industry. Then using Bayes decision theory (see for example [43]) the performance of any control algorithm is characterized by the conditional probability of rejecting a nonadmissible template. Therefore, in order to optimize the performance of the algorithm by suitable rejection rules, we have to characterize possible right hand side differences for global assertions resp. possible right hand side differences according to an a priori known reference template for local assertions.
For further purpose let us define the **error partition** of two finite lattice sets $F_1, F_2 \subset \mathbb{Z}^2$, which describes the partitioning of the complete right hand side error $\sum_{u \in S} ||X_u F_1 - X_u F_2||_1$ on the lattice directions within the direction set S.

Definition 5.0.1 (error partition)
Let $F_1, F_2 \subset \mathbb{Z}^2$ be two finite lattice sets and let $S = \{(r_1, s_1), \ldots, (r_m, s_m)\} \subset \mathbb{Z}^2 \backslash \{0\}$ be a set of distinct lattice directions satisfying $\gcd(r_i, s_i) = 1$ for $i = 1, \ldots, m$. The ***error partition*** $\operatorname{part}(F_1, F_2)$ *of the lattice sets F_1 and F_2*

with respect to the set S of lattice directions is defined by

$$\begin{aligned}&\operatorname{part}(F_1, F_2) := \\ &(||X_{(r_{i_1},s_{i_1})}F_1 - X_{(r_{i_1},s_{i_1})}F_2||_1, \ldots, ||X_{(r_{i_m},s_{i_m})}F_1 - X_{(r_{i_m},s_{i_m})}F_2||_1)\end{aligned} \tag{5.1}$$

so that $||X_{(r_{i_j},s_{i_j})}F_1 - X_{(r_{i_j},s_{i_j})}F_2||_1 \geq ||X_{(r_{i_{j+1}},s_{i_{j+1}})}F_1 - X_{(r_{i_{j+1}},s_{i_{j+1}})}F_2||_1$ *for* $j = 1, \ldots, m-1$.

5.1 The case of $m = 3$ X-ray directions

First of all let us consider the application of $m = 3$ X-ray directions. We will differentiate between the case that exactly one of the directions is afflicted with error and the case that two directions are afflicted with error.

5.1.1 Error values arising in one direction

In the case of $m = 3$ X-ray directions the location of the error values is characterized by the following lemma, if the error values arise in exactly one of the directions.

Lemma 5.1.1 *Let F_1, $F_2 \subset \mathbb{Z}^2$ be two finite lattice sets of same cardinality so that*

$$||X_{(0,1)}F_1 - X_{(0,1)}F_2||_1 = ||X_{(1,0)}F_1 - X_{(1,0)}F_2||_1 = 0, \tag{5.2}$$

$$||X_{(1,1)}F_1 - X_{(1,1)}F_2||_1 = 4. \tag{5.3}$$

Then the nonzero values of the difference vector $X_{(1,1)}F_1 - X_{(1,1)}F_2$ are ordered either by

$$1, -1, -1, 1 \text{ (resp. } -1, 1, 1, -1) \tag{5.4}$$

or by

$$1, -2, 1 \text{ (resp. } -1, 2, -1). \tag{5.5}$$

In each case the distances between the first two and the last two error lines defined by the number of lines which have to be passed from one line to the other one are equal.

Proof

Because of (5.2) the signed lattice set $F_2 - F_1$ can be written as an integral linear combination of elementary switching components with respect to the horizontal and the vertical direction, each of which is represented by the polynomial

$$\tilde{p}_{sw}(x, y) := (x-1)(y-1) \tag{5.6}$$

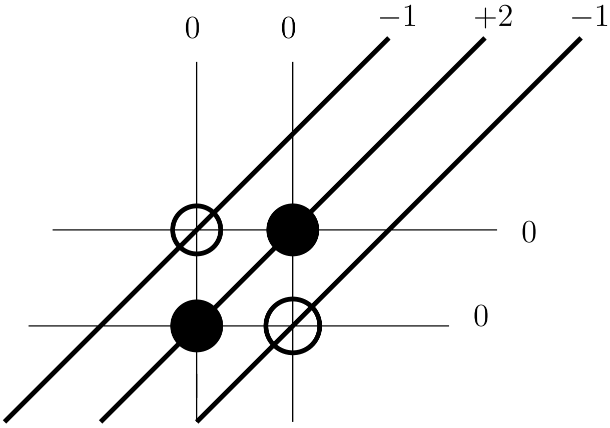

Figure 5.1: Elementary switching component according to horizontal and vertical directions and its diagonal line sums

up to translation. The diagonal projection values along direction $(1, 1)$ *are given by*

$$\begin{aligned} p_{sw}(z) &:= \tilde{p}_{sw}(x = z, y = z^{-1}) = -z + 2 - z^{-1} = \\ &= -z^{-1} \cdot (z-1)^2, \end{aligned} \tag{5.7}$$

i. e. nonzero line sum values are located on neighboured lines as indicated by the difference 1 *between the exponents within the polynomial* p_{sw} *and have values* -1, $+2$, -1, *see Figure 5.1.*
Because of (5.3) the diagonal projection values of the signed lattice set $F_2 - F_1$ *are represented by the polynomial*

$$p(z) := z^{a_1} + z^{a_2} - z^{a_3} - z^{a_4} \tag{5.8}$$

for $a_1, a_2, a_3, a_4 \in \mathbb{N}_0$ *without loss of generality and* $a_i \neq a_j$ *for* $i \in \{1, 2\}$, $j \in \{3, 4\}$. *We calculate that*

$$p_{sw} | p \tag{5.9}$$

$$\begin{aligned} \iff (z-1) | \operatorname{sgn}(a_1 - a_3) z^{\min(a_1,a_3)} (z^{|a_1-a_3|-1} + \cdots + z + 1) \\ + \operatorname{sgn}(a_2 - a_4) z^{\min(a_2,a_4)} (z^{|a_2-a_4|-1} + \cdots + z + 1) \end{aligned} \tag{5.10}$$

$$\iff \operatorname{sgn}(a_1 - a_3)|a_1 - a_3| + \operatorname{sgn}(a_2 - a_4)|a_2 - a_4| = 0 \quad \textit{by setting } z = 1 \tag{5.11}$$

$$\iff \operatorname{sgn}(a_1 - a_3) \operatorname{sgn}(a_2 - a_4) = -1 \textit{ and } |a_1 - a_3| = |a_2 - a_4| \quad \textit{as } a_1 - a_3, a_2 - a_4 \neq 0. \tag{5.12}$$

Thus, the polynomial p *is given by*

$$p(z) = \pm z^c (z^{a+b} - z^b - z^a + 1) \tag{5.13}$$

for $a := |a_1 - a_3| = |a_2 - a_4|$, $b := |a_1 - a_4| > 0$ *and* $c := \min_{i=1,\dots,4} a_i$. □

Remark 5.1.2 The assertion of Lemma 5.1.1 can be generalized to any set of $m = 3$ X-ray directions and one faulty direction:
Without loss of generality the horizontal and the vertical direction belong to the set of X-ray directions and the lines in those directions are not afflicted with error by applying a nonsingular transformation if necessary, see for example [73], Chapter 2. The diagonal projection values of an elementary switching component with respect to the horizontal and the vertical direction along the third lattice direction (r_3, s_3) with $r_3, s_3 \geq 0$ without loss of generality (the other case is similarly treated) are represented by

$$(z^{r_3} - 1)(z^{-s_3} - 1) = -z^{-s_3}(z-1)^2 \cdot (z^{r_3-1} + \cdots + 1)(z^{s_3-1} + \cdots + 1), \tag{5.14}$$

i. e. a multiple of the polynomial $(z-1)^2$. That is the case as the value $j - \frac{s_3}{r_3}i = \frac{1}{r_3}(jr_3 - is_3)$ describes the ordinate distance for the line incident to any lattice point (i, j) and thus the value $jr_3 - is_3$ gives the distance of the lattice point (i, j) to the line incident to the lattice point $(0, 0)$, as $\gcd(r_3, s_3) = 1$.

Remark 5.1.3 Analogous to Lemma 5.1.1 let us now assume that an error of value 6 occurs in direction $(1, 1)$ and is represented by the polynomial

$$p(z) = z^{a_1} + z^{a_2} + z^{a_3} - z^{a_4} - z^{a_5} - z^{a_6} \tag{5.15}$$

for $a_1, \ldots, a_6 \in \mathbb{N}_0$ and $a_i \neq a_j$ for $i \in \{1, 2, 3\}$, $j \in \{4, 5, 6\}$. We calculate that

$$p_{sw}|p \Longleftrightarrow \sum_{i=1}^{3} \operatorname{sgn}(a_i - a_{3+i})|a_i - a_{3+i}| = 0, \tag{5.16}$$

which implies that

$$\pm z^e \cdot p(z) = (z^{a+b} - 1) - z^c(z^a - 1) - z^d(z^b - 1) = \tag{5.17}$$
$$= z^{a+b} - z^{a+c} - z^{b+d} + z^c + z^d - 1 + z^b - z^b = \tag{5.18}$$
$$= (z^{a+b} - z^{a+c} - z^b + z^c) - (z^{b+d} - z^b - z^d + 1) \tag{5.19}$$

for $a, b, c, d, e \in \mathbb{Z}$, $a, b > 0$ so that none of the monomials within (5.17) is canceled. In particular, the representation of the error values can be expressed by the sum of two polynomials each representing an error of value 4, see (5.19).

5.1.2 Error values arising in two directions

The following lemma characterizes the location of the error values in the case of $m = 3$ X-ray directions and two directions afflicted with error. The results for one afflicted direction above will help us to prove the following assertion.

Lemma 5.1.4 *Let F_1, $F_2 \subset \mathbb{Z}^2$ be two finite lattice sets of same cardinality so that*

$$||X_{(0,1)}F_1 - X_{(0,1)}F_2||_1 = ||X_{(1,1)}F_1 - X_{(1,1)}F_2||_1 = 2, \tag{5.20}$$

$$||X_{(1,0)}F_1 - X_{(1,0)}F_2||_1 = 0. \tag{5.21}$$

Then the signed lattice set $F_1 - F_2$ is tomographically equivalent to two differently signed lattice points (the intersection points of the $(+1)$- resp. the (-1)-error lines), which lie on the same horizontal line.

Proof

In order to reduce the described situation to the case in Lemma 5.1.1, let us add two signed lattice points which are located on the same horizontal line, but not on any diagonal error line and cancel the vertical error values, see Figure 5.2.

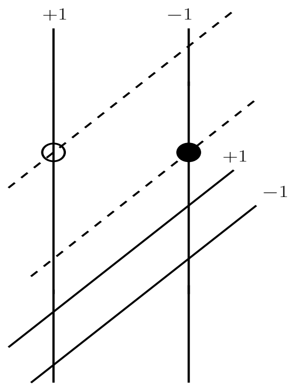

Figure 5.2: Shifting the vertical error in direction $(1, 1)$

Applying the knowledge of Lemma 5.1.1 to the modified situation, we see that the added lattice points can be shifted in vertical direction so that both signed lattice points also lie on the diagonal error lines and thus cancel any kind of error. □

Remark 5.1.5 The assertion of Lemma 5.1.4 can be generalized to any set of $m = 3$ X-ray directions by similar arguments as in Remark 5.1.2 again.

5.2 The case of $m = 4$ X-ray directions

The results above for $m = 3$ X-ray directions help us to make some assertions for the case of $m = 4$ X-ray directions. Let us assume that the direction set is given by $S = \{(0,1), (1,0), (1,1), (1,-1)\}$, i. e. we consider horizontal, vertical and diagonal projections. In the following we examine the case of error value $2(m-1) = 6$ by treating all possible error partitions separately.

5.2.1 The case $\text{part}(F_1, F_2) = (6, 0, 0, 0)$

After applying an orthogonal transformation if necessary, error occurs along direction $(1,1)$ and is represented according to Remark 5.1.3 by

$$q_1(z) = (z^{a+b} - 1) - z^c(z^a - 1) - z^d(z^b - 1) \tag{5.22}$$

or by

$$q_2(z) = z^c(z^{a+b} - 1) - (z^a - 1) - z^d(z^b - 1) \tag{5.23}$$

for $a, b, c, d \in \mathbb{N}_0$ without loss of generality and $a, b > 0$ so that none of the monomials is canceled.

The signed lattice set $F_2 - F_1$ is represented by an integral linear combination of elementary switching components with respect to the directions $(1,0), (0,1)$ and $(1,-1)$, each of which is represented by the polynomial

$$\tilde{p}_{sw}(x, y) := (x-1)(y-1)(x-y) = y^2 + x + x^2y - y - xy^2 - x^2 \tag{5.24}$$

up to translation. The projection values along direction $(1,1)$ are given by

$$p_{sw}(z) := \tilde{p}_{sw}(x = z, y = z^{-1}) = -z^{-2} \cdot (z^4 - 2z^3 + 2z - 1) = \tag{5.25}$$

$$= -z^{-2} \cdot (z^2 - 2z + 1)(z^2 - 1) = -z^{-2} \cdot (z-1)^3(z+1). \tag{5.26}$$

We calculate that

$$p_{sw} | q_1 \tag{5.27}$$

$$\iff (z-1)^2(z+1) | (z^b - z^c)(z^{a-1} + \cdots + 1) - (z^d - 1)(z^{b-1} + \cdots + 1) \tag{5.28}$$

$$\iff (z-1)(z+1) | \tag{5.29}$$

$$\begin{cases} z^c(z^{b-c-1} + \cdots + 1)(z^{a-1} + \cdots + 1) - (z^{d-1} + \cdots + 1)(z^{b-1} + \cdots + 1) \\ \qquad\qquad \text{if } b > c \\ -z^b(z^{c-b-1} + \cdots + 1)(z^{a-1} + \cdots + 1) - (z^{d-1} + \cdots + 1)(z^{b-1} + \cdots + 1) \\ \qquad\qquad \text{if } c > b \end{cases} \tag{5.30}$$

$$\Longrightarrow \begin{cases} (b-c)a = bd & \text{by setting } z = 1 \\ \begin{cases} 2|b \text{ or } 2|d \Longrightarrow 2|a \text{ or } 2|(b-c), \\ \gcd(2, bd) = 1 \Longrightarrow \gcd(2, a(b-c)) = 1 \end{cases} & \text{by setting } z = -1 \end{cases} \tag{5.31}$$

$$\iff (b-c)a = bd \tag{5.32}$$

and analogously that

$$p_{sw}|q_2 \Longrightarrow a(b+c) = b(d-c). \quad (5.33)$$

Lemma 5.2.1 *Let F_1, F_2 be two finite lattice sets of same cardinality and of error partition* $\mathrm{part}(F_1, F_2) = (6,0,0,0)$ *with respect to the direction set* $S = \{(0,1),(1,0),(1,1),(1,-1)\}$. *Let the error values along direction* $(1,1)$ *be represented by the polynomial*

$$q_1(z) = (z^{2b}-1) - z^c(z^b-1) - z^{b-c}(z^b-1) = \quad (5.34)$$
$$= z^{2b} - z^{b+c} - z^{2b-c} + z^c + z^{b-c} - 1 \quad (5.35)$$

for $b > b-c > 0$. There are two finite lattice sets $\bar{F}_1$, $\bar{F}_2$ of same cardinality so that $|\bar{F}_1 - \bar{F}_2| = 6$ and the signed lattice set $\bar{F}_1 - \bar{F}_2$ has the same X-ray data as the signed lattice set $F_1 - F_2$.
The same is true if the error values along direction $(1,1)$ are represented by the polynomial

$$q_2(z) = z^c(z^{2b}-1) - (z^b-1) - z^{b+2c}(z^b-1) \quad (5.36)$$

for $b + 2c > b > 0$.

Proof

Let the lattice sets $\tilde{F}_1, \tilde{F}_2$ be defined by

$$\tilde{F}_1 := \{(i,j) \in \mathbb{Z}^2 | \, \mathrm{coeff}_{i,j}(p_{\tilde{F}_1}) = 1\}, \quad (5.37)$$
$$\tilde{F}_2 := \{(i,j) \in \mathbb{Z}^2 | \, \mathrm{coeff}_{i,j}(p_{\tilde{F}_2}) = 1\} \quad (5.38)$$

for the polynomials

$$p_{\tilde{F}_1}(x,y) := y^b + x^c + x^b y^c, \quad (5.39)$$
$$p_{\tilde{F}_2}(x,y) := y^c + x^c y^b + x^b, \quad (5.40)$$

see Figure 5.3 for illustration.

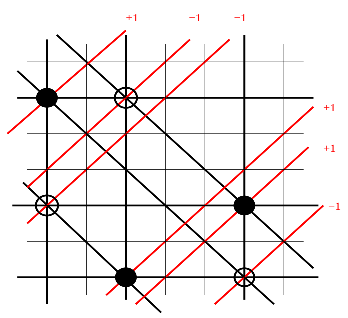

Figure 5.3: Signed lattice set $\tilde{F}_1 - \tilde{F}_2$ for $q_1(z) = (z^{10}-1) - z^2(z^5-1) - z^3(z^5-1)$ and $p_{\tilde{F}_1}(x,y) = y^5 + x^2 + x^5y^2$, $p_{\tilde{F}_2}(x,y) = y^2 + x^2y^5 + x^5$

We calculate that

$$(p_{\tilde{F}_2} - p_{\tilde{F}_1})(x = z, y = z) = (p_{\tilde{F}_2} - p_{\tilde{F}_1})(x = 1, y = z) = \\ = (p_{\tilde{F}_2} - p_{\tilde{F}_1})(x = z, y = 1) = 0, \tag{5.41}$$

$$(p_{\tilde{F}_2} - p_{\tilde{F}_1})(x = z, y = z^{-1}) = (z^{-c} + z^{c-b} + z^b) - (z^{-b} + z^c + z^{b-c}) = \\ = z^{-b} q_1(z). \tag{5.42}$$

Thus, the lattice sets $\bar{F}_1, \bar{F}_2$ with respect to the assertion of the lemma are given by the lattice sets $\tilde{F}_1, \tilde{F}_2$ up to some translation.
For the second case we define the lattice sets $\tilde{F}_1, \tilde{F}_2$ by (5.37)-(5.38) for the polynomials

$$p_{\tilde{F}_1}(x, y) := y^c + x^c y^{b+c} + x^{b+c}, \tag{5.43}$$

$$p_{\tilde{F}_2}(x, y) := y^{b+c} + x^c + x^{b+c} y^c. \tag{5.44}$$

□

The following lemma shows that the error representations (5.34), (5.36) are also necessary for the assertion within the lemma before.

Lemma 5.2.2 *Let $F_1, F_2 \subset \mathbb{Z}^2$ be two finite lattice sets of same cardinality, of error partition $\text{part}(F_1, F_2) = (6, 0, 0, 0)$ with respect to the direction set $S = \{(0,1), (1,0), (1,1), (1,-1)\}$ and with error along direction $(1,1)$. There are two finite lattice sets $\tilde{F}_1, \tilde{F}_2 \subset \mathbb{Z}^2$ of cardinality 3 so that the signed lattice sets $F_1 - F_2$ and $\tilde{F}_1 - \tilde{F}_2$ are tomographically equivalent if and only if the error along direction $(1,1)$ is represented by (5.34) resp. by (5.36) up to some translation.*

Proof

Because of Lemma 5.2.1 it is left to show that the error along direction $(1,1)$ is always represented by (5.34) resp. by (5.36) for two lattice sets $\tilde{F}_1, \tilde{F}_1$ of cardinality 3 which are tomographically equivalent with respect to the direction set $S = \{(0,1), (1,0), (-1,1)\}$.
For that purpose let us assume that $(0,0) \in \tilde{F}_1$ and $(A,0), (0,B), (C,-C) \in \tilde{F}_2$ (or vice versa) for $A, B, C \neq 0$ and $C \neq A, C \neq -B$, as otherwise the switching component with respect to the direction set $\{(1,0), (0,1), (1,-1)\}$ cannot be closed by the remaining two lattice points within the lattice set $\tilde{F}_1$, for illustration see Figure 5.4.
Let us intersect the horizontal and the vertical line passing through the lattice point $(C, -C)$ with the diagonal lines passing through the lattice points $(A,0), (0,B)$. The case

$$(0, B) + \lambda(1, -1) = (C, \dots) \Leftrightarrow \lambda = C, \tag{5.45}$$

$$(A, 0) + \mu(1, -1) = (\dots, -C) \Leftrightarrow \mu = C \tag{5.46}$$

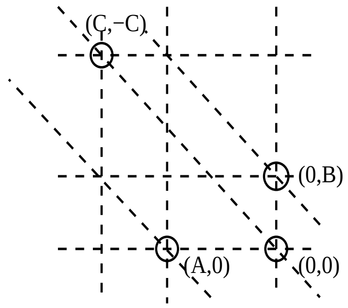

Figure 5.4: The lattice sets $\tilde{F}_1, \tilde{F}_2$ satisfying $|\tilde{F}_1| = |\tilde{F}_2|$ for $\text{part}(F_1, F_2) = (6, 0, 0, 0)$

implies that $\tilde{F}_1 = \{(0,0), (C, B-C), (A+C, -C)\}$. *But as*

$$\{(C, B-C), (A+C, -C)\} \cap ((A, 0) + \mathbb{R} \cdot (0, 1)) = \emptyset \tag{5.47}$$

for $A \neq C$ *and* $C \neq 0$, *the lattice set* $\tilde{F}_1$ *is not tomographically equivalent to the lattice set* $\tilde{F}_2$.
The case

$$(0, B) + \lambda(1, -1) = (\dots, -C) \Leftrightarrow \lambda = B + C, \tag{5.48}$$

$$(A, 0) + \mu(1, -1) = (C, \dots) \Leftrightarrow \mu = C - A \tag{5.49}$$

implies that $\tilde{F}_1 = \{(0,0), (B+C, -C), (C, A-C)\}$. *As we have to demand that*

$$(B + C, -C) \in (A, 0) + \mathbb{R} \cdot (0, 1), \tag{5.50}$$

$$(C, A - C) \in (0, B) + \mathbb{R} \cdot (1, 0), \tag{5.51}$$

we conclude that

$$A = B + C. \tag{5.52}$$

Therefore, the signed lattice set $\tilde{F}_2 - \tilde{F}_1$ *is represented by the polynomial*

$$\begin{aligned}(p_{\tilde{F}_2} - p_{\tilde{F}_1})(x, y) &= x^A + y^B + x^C y^{-C} - 1 - x^{B+C} y^{-C} - x^C y^{A-C} = \\ &= -1 + x^A + y^B + x^{A-B} y^{B-A} - x^A y^{B-A} - x^{A-B} y^B \end{aligned} \tag{5.53}$$

and the error values along direction $(1, 1)$ *by the polynomial*

$$p_{\tilde{F}_2} - p_{\tilde{F}_1}(x = z, y = z^{-1}) = \tag{5.54}$$

$$= -1 + z^A + z^{-B} + z^{2A-2B} - z^{2A-B} - z^{A-2B} = \tag{5.55}$$

$$= (z^{2(A-B)} - 1) - z^A(z^{A-B} - 1) - z^{-B}(z^{A-B} - 1). \tag{5.56}$$

□

5.2.2 The case $\text{part}(F_1, F_2) = (4, 2, 0, 0)$

The case $\text{part}(F_1, F_2) = (4, 2, 0, 0)$ *is not possible as* $\text{part}(F_1, F_2) = (2, 0, 0)$ *does not occur for* $m = 3$ *lattice directions, see [6], [3].*

5.2.3 The case $\mathrm{part}(F_1, F_2) = (2, 2, 2, 0)$

By considering triples of directions each of error partition $\mathrm{part}(F_1, F_2) = (2, 2, 0)$ *the distances* x_1*,* x_2 *and* x_3 *between the parallel error lines along each of the three directions afflicted with error are implied to be equal, see Figure 5.5. Therefore, as indicated in Figure 5.5 on the right hand side, the error values along the horizontal and the vertical direction can be shifted to direction* $(1, 1)$ *(or to direction* $(1, -1)$*) by two signed lattice points, which are the intersection points of the horizontal and the vertical* $(+1)$*-error lines resp.* (-1)*-error lines. The* X*-ray data along direction* $(1, -1)$ *(resp. along direction* $(1, 1)$*) are not changed by that procedure. Because of [6], [3] the smallest error value for* $m = 4$ X*-ray directions is equal to the value* $2(m - 1) = 6$*. Thus, the inserted signed lattice points also have to lie on the diagonal error lines.*

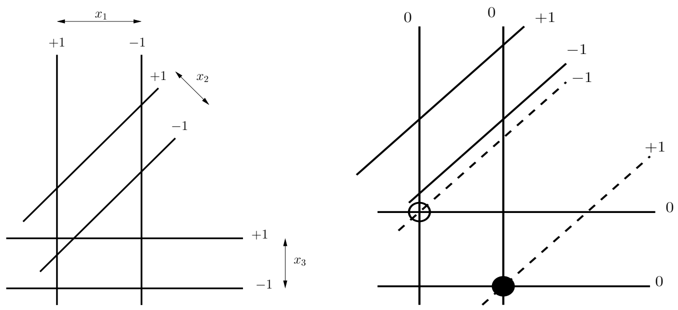

Figure 5.5: Shifting the horizontal and the vertical error in direction $(1, 1)$

5.3 Rejecting lattice sets

In the following we are interested in the question whether an unknown template $F \subset \mathbb{Z}^2$ can be algorithmically rejected with respect to a reference template $F_0 \subset \mathbb{Z}^2$ on the basis of its X-ray data b_F along the directions $(1,0),(0,1),(1,1)$ for $m = 3$ resp. along the directions $(1,0),(0,1),(1,1),(1,-1)$ for $m = 4$ X-ray directions, i. e. whether the difference vector $b_{F_0} - b_F$ of the right hand side data implies a large symmetric difference $F \triangle F_0$ for the templates F and F_0.

5.3.1 The case $m = 3$ and error partition $\mathrm{part}(F, F_0) = (4, 0, 0)$

If the error values $b_{F_0} - b_F$ are represented up to translation by the polynomial $p(z) = z^{a+b} - z^a - z^b + 1$ for $a, b > 0$ and the error lines are denoted by l_1, l_2, l_3, l_4, we have to decide whether the case

$$(x,y),(x-a,y+b) \in F_0 \cap (\bigcup_{i=1}^{4} l_i) \text{ and } (x-a,y),(x,y+b) \neq F_0 \quad (5.57)$$

or the case

$$(x,y),(x-b,y+a) \in F_0 \cap (\bigcup_{i=1}^{4} l_i) \text{ and } (x-b,y),(x,y+a) \neq F_0 \quad (5.58)$$

is possible for some $(x,y) \in \mathbb{Z}^2$.

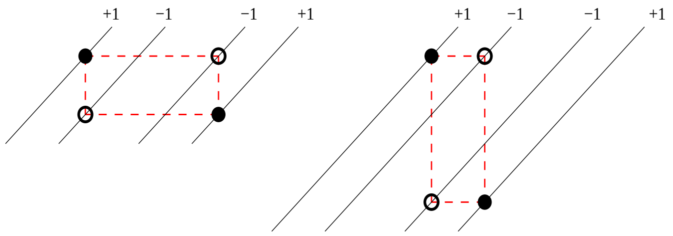

Figure 5.6: Error rejection for $m = 3$ and error partition $\mathrm{part}(F, F_0)$ =(4, 0, 0)

Otherwise, the template F can be rejected on the basis of its X-ray data b_F with respect to an error tolerance $|F \triangle F_0|$ up to the value 5.

5.3.2 The case $m = 3$ and error partition $\mathrm{part}(F, F_0) = (2, 2, 0)$

If the intersection point S_1 of the (+1)-error lines lies within the set F_0, but not the intersection point S_2 of the (−1)-error lines, the template F cannot be rejected with respect to an error tolerance of value 2.
Let us examine the situation that the lattice set $F \subset \mathbb{Z}^2$ satisfies $|F \triangle F_0| = 4$ in the following.

Lemma 5.3.1 *Let the error partition of two finite lattice sets* $F, F_0 \subset \mathbb{Z}^2$ *be given by* $\mathrm{part}(F, F_0) = (2, 2, 0)$ *and let the horizontal direction not be afflicted with error. Let* $S_1 = (x_{S_1}, y_{S_1})$ *denote the intersection point of the* $(+1)$*-error lines and* $S_2 = (x_{S_2}, y_{S_2})$ *the intersection point of the* (-1)*-error lines so that* $y_{S_1} = y_{S_2}$ *according to Lemma 5.1.4 and* $a := x_{S_2} - x_{S_1} > 0$.
There exists a lattice set $\bar{F} \subset \mathbb{Z}^2$ *which is tomographically equivalent to the lattice set* F *and satisfies* $|\bar{F} \triangle F_0| = 4$ *if and only if*

$$\begin{cases} (x_{S_1} - b, y_{S_1} - b), (x_{S_1}, y_{S_1} - a - b) \in F_0, \\ (x_{S_1} + a, y_{S_1} - b), (x_{S_1} - b, y_{S_1} - a - b) \neq F_0 \end{cases} \tag{5.59}$$

or

$$\begin{cases} (x_{S_1}, y_{S_1} + b), (x_{S_1} + a + b, y_{S_1} + a + b) \in F_0, \\ (x_{S_1} + a, y_{S_1} + a + b), (x_{S_1} + a + b, y_{S_1} + b) \neq F_0 \end{cases} \tag{5.60}$$

for $b \in \mathbb{Z} \backslash \{0\}, b > -a$.

Proof

The signed lattice sets in (5.59) resp. in (5.60) fit to the error lines. Therefore, it remains to show that all signed lattice sets of cardinality 4 *which fit to the error lines are given by (5.59) and (5.60).*
Because of the error partition, the four lattice points of the signed lattice set $\bar{F} - F_0$ *lie on two horizontal lines and one pair of lattice points additionally on the same vertical, another pair of lattice points on the same diagonal line. The possible cases are illustrated in Figure 5.7, in which the first case is described by (5.59) for* $b > 0$*, the second case by (5.60) for* $b > 0$*, and the third case both by (5.60) for* $-a < b = -b_1 < 0$ *and by (5.59) for* $-a < b = -b_2 < 0$. □

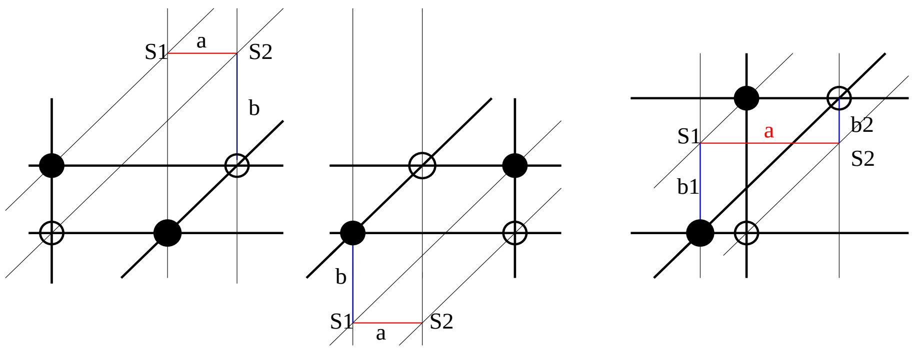

Figure 5.7: Error rejection for $m = 3$ and error partition $\mathrm{part}(F, F_0) = (2, 2, 0)$

5.3.3 The case $m = 4$ and error partition $\mathrm{part}(F, F_0) = (6, 0, 0, 0)$

In the proof of Lemma 5.2.2 all signed lattice sets of cardinality 6 *are characterized. Therefore, if either the case of cardinality* 6 *is excluded at all by the* *error representation or it does not occur that all elements within any signed lattice set in Lemma 5.2.2 which lie on the $(+1)$-error lines and none of those lattice points which lie on the (-1)-error lines belong to the lattice set F_0, the template F can be rejected on the basis of its X-ray data with respect to an error tolerance $|F \triangle F_0|$ up to the value 7.*

5.3.4 The case $m = 4$ and error partition $\text{part}(F, F_0) = (2, 2, 2, 0)$

We already know that the template F can be rejected on the basis of its X-ray data with respect to an error tolerance up to the value 3, if the intersection point S_1 of the $(+1)$-error lines does not belong to the lattice set F_0 or the intersection point S_2 of the (-1)-error lines belongs to the lattice set F_0. The next theorem extends the rejection possibility to an error tolerance up to the value 7.

Theorem 5.3.2
Let the error partition of two finite lattice sets $F, F_0 \subset \mathbb{Z}^2$ of same cardinality be given by $\text{part}(F, F_0) = (2, 2, 2, 0)$ and let direction $(1, -1)$ be not afflicted with error. Without loss of generality let us assume that the intersection point of the $(+1)$-error lines is given by $S_1 = (0, 1)$ and the intersection point of the (-1)-error lines by $S_2 = (1, 0)$.
If the case

$$S_1 \in F_0 \text{ and } S_2 \notin F_0 \tag{5.61}$$

and the case

$$\begin{cases} (0, 1+t), (1+t, 2+t), (2+t, 1) \in F_0 \\ (1+t, 0), (2+t, 1+t), (1, 2+t) \notin F_0 \end{cases} \tag{5.62}$$

for $t \in \mathbb{R} \backslash \{0\}$ are excluded, the template F can be rejected on the basis of its X-ray data with respect to an error tolerance up to the value 7.

Proof
To exclude the situation (5.61) let us assume that the lattice point S_1 does not belong to any signed lattice set with respect to the X-ray data $b_{F_0} - b_F$. Thus, let the lattice point $(0, 1+t)$ belong to the signed lattice set for some $t \neq 0$. As direction $(1, -1)$ is not afflicted with error, we conclude that the lattice point $(1, t)$, the lattice point $(1 + \frac{t}{2}, \frac{t}{2})$ or the lattice point $(1 + t, 0)$ also belongs to the signed lattice set.

1. *In the first case horizontal and after that vertical line sum conditions imply that the lattice points*

$$(2+t, 1+t), (t-1, t), (2+t, 1), (t-1, 0) \tag{5.63}$$

also belong to the signed lattice set, see Figure 5.8, which is represented by the polynomial

$$q_t(x,y) = +y^{1+t} - xy^t - x^{2+t}y^{1+t} + x^{t-1}y^t + x^{2+t}y - x^{t-1}, \quad (5.64)$$

but $q_t(x=z, y=z) = -z^{3+2t} + z^{2t-1} + z^{3+t} - z^{t-1} = 0$ *is satisfied if and only if* $t = 0$.

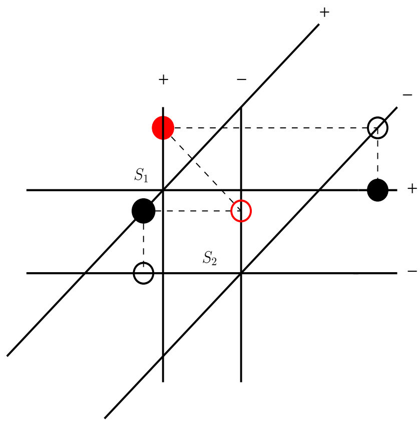

Figure 5.8: Error rejection for $m = 4$, error partition part$(F, F_0) = (2, 2, 2, 0)$ and the lattice points $(0, 1+t), (1, t)$ belong to the signed lattice set

2. *In the second case horizontal and after that vertical line sum conditions imply that the signed lattice set is represented by the polynomial*

$$q_t(x,y) = y^{1+t} - x^{1+\frac{t}{2}}y^{\frac{t}{2}} - xy^{1+t} + x^{-1+\frac{t}{2}}y^{\frac{t}{2}} + x^{1+\frac{t}{2}}y - x^{-1+\frac{t}{2}}, \quad (5.65)$$

see Figure 5.9, but we again calculate that $q_t(x=z, y=z) \neq 0$ *for* $t \neq 0$.

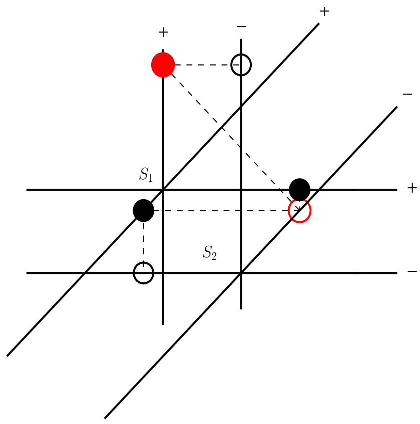

Figure 5.9: Error rejection for $m = 4$, error partition part$(F, F_0) = (2, 2, 2, 0)$ and the lattice points $(0, 1+t), (1+\frac{t}{2}, \frac{t}{2})$ belong to the signed lattice set

3. *In the third case successively applying the line sum conditions for direction* $(1,1)$*, direction* $(1,0)$*, direction* $(1,-1)$ *and direction* $(0,1)$*, the signed lattice set is represented by the polynomial*

$$q_t(x,y) = y^{1+t} - x^{1+t} - xy^{2+t} + x^{1+t}y^{2+t} - x^{2+t}y^{1+t} + x^{2+t}y, \tag{5.66}$$

see Figure 5.10. Thus, the case (5.62) has to be excluded for the possibility of rejection.

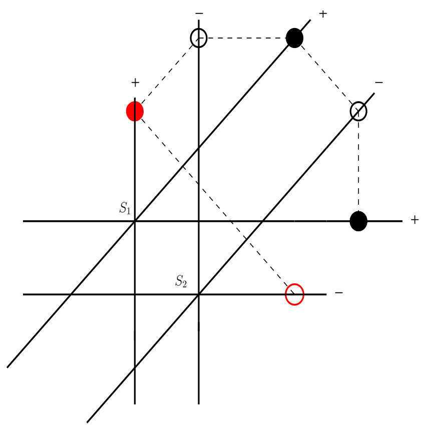

Figure 5.10: Error rejection for $m = 4$, error partition $\mathrm{part}(F, F_0) = (2, 2, 2, 0)$ and the lattice points $(0, 1+t), (1+t, 0)$ belong to the signed lattice set

□

Chapter 6

Instability within a finite lattice set

Even small differences in the right hand side data possibly lead to large discrepancy in the original data, see [7] and also compare the constructions for the parallel case in Section 3.6.
But the constructions used to prove that assertion seem to be very sparse distributed. Thus, we want to construct instabilities within a finite lattice set with respect to the four standard directions, i. e. the horizontal, the vertical and the two diagonal lattice directions.

Lemma 6.0.1 *Let $G = \{(i,j) \in \mathbb{Z}^2 | 0 \le i < 4\bar{n}, 0 \le j < 4\bar{n}\}$ be the finite lattice set called reference area in the following and let the set S of lattice directions be given by $S = \{(1,0),(0,1),(1,1),(1,-1)\}$. Then there are two lattice sets $F_1, F_2 \subset G$ within the reference area G satisfying*

- *F_1, F_2 are uniquely determined by their X-rays,*
- $\sum_{u \in S} ||X_u F_1 - X_u F_2|| = 6$,
- $F_1 \cap F_2 = \emptyset$,
- $|F_1| = |F_2| = 4\bar{n} - 1$.

Proof
Let the polynomial $g = g(x,y)$ and the lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ be defined by

$$\begin{aligned} g := & ((xy)^{\bar{n}} - 1)(x^{2\bar{n}} y^{-2\bar{n}} - 1) \cdot \qquad (6.1) \\ & \cdot [y^{2\bar{n}}(1 + xy + \cdots + (xy)^{\bar{n}-1}) - xy^{2\bar{n}}(1 + xy + \cdots + (xy)^{\bar{n}-2}) - y^{3\bar{n}-1}] - \\ & - y^{3\bar{n}-1}((xy)^{\bar{n}} - 1), \end{aligned}$$

$$F_1 := \{(i,j) \in \mathbb{Z}^2 | \operatorname{coeff}_{i,j}(g) = +1\}, \qquad (6.2)$$

$$F_2 := \{(i,j) \in \mathbb{Z}^2 | \operatorname{coeff}_{i,j}(g) = -1\} \qquad (6.3)$$

in dependence on the value $\bar{n}$*, i. e. the lattice set* $F_1 \cup F_2 \cup \{(0, 3\bar{n}-1), (\bar{n}, 4\bar{n}-1)\}$ *corresponds to a switching component with respect to the horizontal and the vertical direction and its translates in both direction* $(1, 1)$ *and direction* $(1, -1)$*.*
For illustration look at Figure 6.1:
The red lattice points belong to the lattice set F_1*, the blue ones to the lattice set* F_2*. The additional lattice points* $(0, 3\bar{n}-1)$*,* $(\bar{n}, 4\bar{n}-1)$ *to close the switching component are black-coloured.*

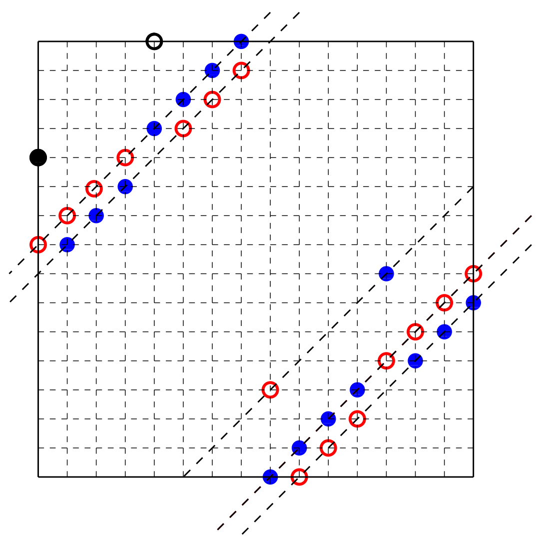

Figure 6.1: The lattice sets F_1, F_2 for $\bar{n} = 4$

The assertion of the lemma besides the uniqueness of the lattice sets F_1, F_2 *is directly implied by the definition of the lattice sets* F_1, F_2*. Thus, let us concentrate on the uniqueness of the lattice sets* F_1, F_2 *in the following:*
First of all $(0, 2\bar{n}) \in F_1$ *is determined because of the directions* $(0, 1), (1, 1)$*,* $(1, -1)$*. (Notice, that the horizontal and the vertical line sum values restrict our consideration to the reference area* G*. Taking the* X*-rays in direction* $(0, 1)$ *and in direction* $(1, 1)$ *into account, the lattice point* $(0, 2\bar{n} - 1)$ *could alternatively belong to the lattice set* F_1*, but that is not possible because of the line sums in direction* $(1, -1)$*.)*
After that the X*-ray data according to the directions* $(1, 0)$*,* $(1, 1)$ *and* $(1, -1)$ *help us to decide that the lattice point* $(2\bar{n} + 1, 0)$ *belongs to the lattice set* F_1*. Because of the directions* $(0, 1)$*,* $(1, 1)$ *and* $(1, 0)$ *we can fix* $(2\bar{n}, \bar{n} - 1) \in F_1$ *next. Now we successively apply arguments which are similar to the arguments for the lattice points* $(0, 2\bar{n})$*,* $(2\bar{n} + 1, 0)$ *before, in order to determine that the lattice points* $(1, 2\bar{n}+1), (2\bar{n}+2, 1), \ldots, (k, 2\bar{n}+k)$*,* $(2\bar{n}+1+k, k), \ldots,$ $(\bar{n} - 2, 3\bar{n} - 2), (3\bar{n} - 1, \bar{n} - 2), (\bar{n} - 1, 3\bar{n} - 1)$ *belong to the lattice set* F_1*.*
Because of the line sum values according to the directions $(0, 1)$ *and* $(1, 1)$ *the lattice point* $(3\bar{n}, \bar{n})$ *belongs to the lattice set* F_1*, because of the directions* $(1, 0)$ *and* $(1, 1)$ *we can fix* $(\bar{n} + 1, 3\bar{n}) \in F_1$*. Again, successively applying*

similar arguments to the lattice points $(3\bar{n}+1, \bar{n}+1), (\bar{n}+2, 3\bar{n}+1), \ldots, (3\bar{n}+k, \bar{n}+k), (\bar{n}+1+k, 3\bar{n}+k), \ldots, (4\bar{n}-2, 2\bar{n}-2), (2\bar{n}-1, 4\bar{n}-2), (4\bar{n}-1, 2\bar{n}-1)$, *the complete lattice set* F_1 *is uniquely determined.*
Similar arguments also work to show the uniqueness of the lattice set F_2, *as the lattice sets* F_1, F_2 *are symmetric.* □
Now we try to get an even better lower bound for the largest appearing instability with respect to the direction set $S = \{(1,0),(0,1),(1,1),(1,-1)\}$ *by the usage of the following lemma and the remark afterwards.*

Lemma 6.0.2 *It yields that*

$$\sum_{i=0}^{k-2} x^i \sum_{j=i}^{k-2} y^j (x-1)(y-1) = \sum_{i=0}^{k-1} (xy)^i - x\sum_{i=0}^{k-2}(xy)^i - y^{k-1} \tag{6.4}$$

for $k \geq 2$.

Proof

For $k=2$ *we calculate that*

$$\sum_{i=0}^{0} x^i \sum_{j=i}^{0} y^j (x-1)(y-1) = (x-1)(y-1) = xy - x - y + 1 = \tag{6.5}$$

$$= \sum_{i=0}^{1}(xy)^i - x\sum_{i=0}^{0}(xy)^i - y^1. \tag{6.6}$$

Now we want to conclude from k *to* $k+1$*:*

$$\sum_{i=0}^{(k+1)-2} x^i \sum_{j=i}^{(k+1)-2} y^j (x-1)(y-1) = \tag{6.7}$$

$$= \sum_{i=0}^{k-2} x^i \sum_{j=i}^{k-2} y^j (x-1)(y-1) + \sum_{i=0}^{k-2} x^i y^{k-1}(x-1)(y-1) + \tag{6.8}$$

$$+ x^{k-1}y^{k-1}(x-1)(y-1) = \tag{6.9}$$

$$= \sum_{i=0}^{k-1}(xy)^i - x\sum_{i=0}^{k-2}(xy)^i - y^{k-1} + y^{k-1}(x^{k-1}-1)(y-1) + \tag{6.10}$$

$$+ x^k y^k - x^k y^{k-1} - x^{k-1}y^k + x^{k-1}y^{k-1} = \tag{6.11}$$

$$= \sum_{i=0}^{(k+1)-1}(xy)^i - x\sum_{i=0}^{(k+1)-2}(xy)^i - y^{(k+1)-1} \tag{6.12}$$

□

Remark 6.0.3 Because of Lemma 6.0.2 we transform

$$g := \sum_{i=0}^{k-2} x^i \sum_{j=i}^{k-2} y^j (x-1)(y-1) \cdot ((xy)^k - 1)(x-y) = \tag{6.13}$$

$$= h \cdot [((xy)^k - 1)(x-y)] \tag{6.14}$$

for

$$h := \sum_{i=0}^{k-1}(xy)^i - x\sum_{i=0}^{k-2}(xy)^i - y^{k-1}. \tag{6.15}$$

The components $x(xy)^k \cdot h, -y(xy)^k \cdot h, -x \cdot h$ and $y \cdot h$ of the polynomial g do not overlap for $k \geq 4$, as $((xy)^k - 1)$ within (6.14) denotes a sufficiently large shift in direction $(1,1)$ and the components $-x \cdot h$ and $y \cdot h$ do not overlap if $y^{k-1} \cdot xy^{-1} \notin \{(xy)^i | i = 0, \ldots, k-1\} \cup \{x(xy)^i | i = 0, \ldots, k-2\}$, which is the case if and only if $k \notin \{2, 3\}$, for illustration see Figure 6.2.

Theorem 6.0.4
Let $G = \{(i,j) \in \mathbb{Z}^2 | 0 \leq i < 2\bar{n}+1, 0 \leq j < 2\bar{n}+1\}$ denote the reference area for $\bar{n} \geq 4$ and let $S := \{(1,0),(0,1),(1,1),(1-1)\}$ be the set of lattice directions. Then there are two lattice sets $F_1, F_2 \subset G$ within the reference area G satisfying

- *F_1, F_2 are uniquely determined by their X-rays,*
- $\sum_{u \in S} ||X_u F_1 - X_u F_2|| = 6$,
- $F_1 \cap F_2 = \emptyset$,
- $|F_1| = |F_2| = 4\bar{n} - 1$.

Proof
Let the polynomials $g = g(x,y), \bar{g} = \bar{g}(x,y)$ and the lattice sets $F_1, F_2 \subset \mathbb{Z}^2$ be defined by

$$g := \sum_{i=0}^{\bar{n}-2} x^i \sum_{j=i}^{\bar{n}-2} y^j (x-1)(y-1) \cdot ((xy)^{\bar{n}} - 1)(x-y), \tag{6.16}$$

$$\bar{g} := g - ((xy)^{\bar{n}} - 1)y^{\bar{n}}, \tag{6.17}$$

$$F_1 := \{(i,j) \in \mathbb{Z}^2 | \operatorname{coeff}_{i,j}(\bar{g}) = +1\}, \tag{6.18}$$

$$F_2 := \{(i,j) \in \mathbb{Z}^2 | \operatorname{coeff}_{i,j}(\bar{g}) = -1\} \tag{6.19}$$

in dependence on the value $\bar{n}$. For illustration of the lattice sets in the case that $\bar{n} = 4$ see Figure 6.2.
Because of the definition of the lattice sets F_1, F_2 and because of Remark 6.0.3 it remains to show the uniqueness of the lattice set F_2 as in Lemma 6.0.1. The uniqueness of the lattice set F_2 then implies the uniqueness of the symmetric lattice set F_1.
Taking step by step the line sums of the lines

$$l_1 := \{(i,j) \in \mathbb{Z}^2 | i = 0\}, \tag{6.20}$$

$$l_2 := \{(i,j) \in \mathbb{Z}^2 | i + j = 1\}, \tag{6.21}$$

$$l_3 := \{(i,j) \in \mathbb{Z}^2 | j = 0\}, \tag{6.22}$$

$$l_4 := \{(i,j) \in \mathbb{Z}^2 | i + j = 2\} \tag{6.23}$$

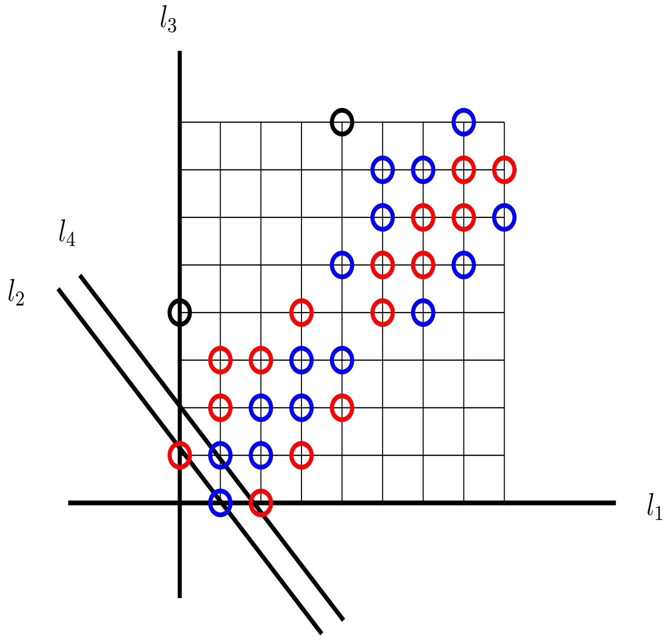

Figure 6.2: Instability for horizontal, vertical and diagonal directions, the lattice set F_1 is red-coloured, the lattice set F_2 is blue-coloured

into account, we conclude that $(1,0),(1,1) \in F_2$, *but no further point incident to the lines* $l_1,\ldots,l_4$ *belongs to the lattice set* F_2. *For inductive purpose let us define the lines*

$$l_{4p-3} := \{(i,j) \in \mathbb{Z}^2 | i = p-1\}, \tag{6.24}$$

$$l_{4p-2} := \{(i,j) \in \mathbb{Z}^2 | i+j = 2p-1\}, \tag{6.25}$$

$$l_{4p-1} := \{(i,j) \in \mathbb{Z}^2 | j = p-1\}, \tag{6.26}$$

$$l_{4p} := \{(i,j) \in \mathbb{Z}^2 | i+j = 2p\} \tag{6.27}$$

for $p = 2,\ldots,\bar{n}-1$. *We want to conclude from* $p-1$ *to* p *for* $p < \bar{n}$.
Because of the fact that the lattice points

$$(1,0),(1,1),\ldots,(p-1,p-2),(p-1,p-1) \in F_2 \tag{6.28}$$

are already determined, we can also fix $(p,p-1),(p,p) \in F_2$ *by taking a look at the line sums of the lines* $l_{4p-3},\ldots,l_{4p}$ *in the mentioned order. To determine* $(\bar{n},\bar{n}-1) \in F_2$ *we additionally use the lines* $l_{4\bar{n}-3} := \{(i,j) \in \mathbb{Z}^2 | j = \bar{n}-1\}$ *and* $l_{4\bar{n}-2} := \{(i,j) \in \mathbb{Z}^2 | i+j = 2\bar{n}-1\}$.
Now let us define the lines $l_{4\bar{n}-1},\ldots,l_{4\bar{n}+3}$ *by*

$$l_{4\bar{n}-1} := \{(i,j) \in \mathbb{Z}^2 | i+j = 2\bar{n}\}, \tag{6.29}$$

$$l_{4\bar{n}+1+l} := \{(i,j) \in \mathbb{Z}^2 | i-j = l\} \text{ for } l = -1,\ldots,2. \tag{6.30}$$

The line sum values of the lines $l_{4\bar{n}-1},\ldots,l_{4\bar{n}+3}$ *imply that we can fix* $F_2 \backslash \{(\bar{n}+1, 2\bar{n}-1)\} \subset F_2$. *After that the difference values within the*

X-rays of the lattice set $F_2 \backslash \{(\bar{n}+1, 2\bar{n}-1)\}$ and the lattice set F_2 determine the remaining lattice point $(\bar{n}+1, 2\bar{n}-1)$. □

Remark 6.0.5 If we define the **density $\mathcal{M}_S$ of instability** with respect to the reference area $G := \{(i,j) \in \mathbb{Z}^2 | 0 \le i < n_1, 0 \le j < n_2\}$ and the set S of m distinct lattice directions by

$$\mathcal{M}_S := \max_{(F_1,F_2)\in\mathcal{F}} \frac{|F_1 \triangle F_2|}{n_1 n_2} \tag{6.31}$$

for $(F_1, F_2) \in \mathcal{F}$ if and only if

1. F_1, F_2 are uniquely determined by their X-rays,
2. $|F_1| = |F_2|$,
3. $F_1 \cap F_2 = \emptyset$,
4. $\sum_{u\in S} ||X_u F_1 - X_u F_2||_1 = 2(m-1)$,

then Theorem 6.0.4 shows that

$$\mathcal{M}_{\{(1,0),(0,1),(1,1),(1,-1)\}} \ge \frac{2(4\bar{n}-1)}{(2\bar{n}+1)(2\bar{n}+1)} \approx \frac{4}{\sqrt{n_1 \cdot n_2}}. \tag{6.32}$$

Remark 6.0.6 To show the uniqueness of the lattice sets F_1, F_2 in Theorem 6.0.4, we have not actually used the binarity of the lattice sets. Thus, the assertion can be extended to additive lattice sets as defined in [50] instead of uniquely determined lattice sets.

Remark 6.0.7 The construction of the lattice sets F_1, F_2 in Theorem 6.0.4 can be generalized to any direction set $S = \{(1,0), (0,1), (r_1, s_1), (r_2, s_2)\}$ for $r_1, r_2, s_1 > 0$ and $s_2 < 0$. For that purpose let us define the polynomials $g = g(x,y)$, $h = h(x,y)$ by

$$h := \sum_{i=0}^{k-1} (x^{r_1} y^{s_1})^i - x^{r_1} \sum_{i=0}^{k-2} (x^{r_1} y^{s_1})^i - (y^{s_1})^{k-1}, \tag{6.33}$$

$$g := h \cdot ((x^{r_1} y^{s_1})^k - 1)(x^{l\cdot r_2} - y^{l\cdot s_2}) \tag{6.34}$$

for some parameters $k, l \in \mathbb{N}$. By the choice of l we have to guarantee that the components

$$h \cdot ((x^{r_1} y^{s_1})^k - 1) x^{l\cdot r_2}, \tag{6.35}$$

$$h \cdot ((x^{r_1} y^{s_1})^k - 1)(-y^{l\cdot s_2}) \tag{6.36}$$

of the polynomial g do not overlap.

Chapter 7

Weaving patterns in discrete tomography

In order to detect errors within templates in semiconductor industry, modelling the conductor paths with weaving patterns is motivated by the following assumptions:

- The conductor path must have both minimal width and height in order to be permeable.
- The conductor path is assumed to have only horizontal and vertical changes of direction.
- Moreover, the changes are assumed to arise only in prescribed distances, the characteristic parameters of the weaving patterns.

7.1 Preliminaries and definitions

We have to introduce what we are talking about if we say that we examine weaving patterns as special underlying structure of the considered lattice set. In order to define weaving patterns we use some definitions from coding theory.

Definition 7.1.1 (direct product of codes)
Let q be a prime number and let $\mathbb{F}_q$ be the finite field with q elements. Let $\gamma : (\mathbb{F}_q)^k \to (\mathbb{F}_q)^n$ be an injective linear transformation and let $d : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{N}_0$, $d(w, w') := |\{i | w_i \neq w'_i\}|$ define a metric on the vector space $(\mathbb{F}_q)^n$, i. e.

1. *$d(w, w') = 0 \Leftrightarrow w = w'$ (identity of indiscernibles),*
2. *$d(w, w') = d(w', w)$ (symmetry),*
3. *$d(w, w'') + d(w'', w') \geq d(w, w')$ (triangle inequality).*

*Then $\gamma((\mathbb{F}_q)^k)$ is called a **linear** $(n, k, d)_q$**-code** with **minimal distance** $d := \min_{w,w' \in \gamma((\mathbb{F}_q)^k), w \neq w'} d(w, w')$.*
Let $\mathcal{C}_1$ be a linear $(n_1, k_1, d_1)_q$-code and let $\mathcal{C}_2$ be a linear $(n_2, k_2, d_2)_q$-code. Then the linear $(n_1 n_2, k_1 k_2, d_1 d_2)_q$-code

$$\mathcal{C}_1 \otimes \mathcal{C}_2 := \{R \in \mathbb{F}_q^{n_2 \times n_1} |\text{ any row/column of } R \text{ is a codeword in } \mathcal{C}_1/\mathcal{C}_2\} \quad (7.1)$$

*is called the **direct product** of the codes $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Definition 7.1.2 (weaving patterns)
*For $q = 2$ within the previous definition any element $R \in \mathbb{F}_2^{n_2 \times n_1}$ can be (canonically) identified with a lattice set. If we speak of the **set of weaving patterns with row structure** $\mathcal{C}_1$ **and column structure** $\mathcal{C}_2$, then we mean the set $\mathcal{C}_1 \otimes \mathcal{C}_2$ as set of lattice sets. Codewords in $\mathcal{C}_1 \otimes \mathcal{C}_2$ are also called **weaving patterns**.*
In the following we will restrict our considerations to regular $p \times q$-weaving patterns.

Definition 7.1.3 (regular $p \times q$-weaving patterns)
*For $p_1, \ldots, p_k \in \mathbb{N}$ we define $\mathcal{C}(p_1, \ldots, p_k)$ as the linear $(\sum_{i=1}^k p_i, k, \min_{1 \leq i \leq k} p_i)_2$-code $G(p_1, \ldots, p_k)(\mathbb{F}_2)^k$ with **generator matrix***

$$G(p_1, \ldots, p_k) := \begin{bmatrix} \mathbf{1}_{p_1} & & \\ & \ddots & \\ & & \mathbf{1}_{p_k} \end{bmatrix} \quad (7.2)$$

and $\mathbf{1}_{p_i} := (1, \ldots, 1)^t \in \mathbb{F}_2^{p_i}$.
*The set $\mathcal{C}_1 \otimes \mathcal{C}_2$ is called a set of **regular** $p \times q$**-weaving patterns** if $\mathcal{C}_1 = \mathcal{C}(p, \ldots, p)$ and $\mathcal{C}_2 = \mathcal{C}(q, \ldots, q)$.*

7.2 Algebraic aspects of switching components

In [69] a complete characterization of switching components within finite lattice sets is given. By taking the special structure of regular $p \times q$-weaving patterns into account we get adequate assertions for our case.
First of all we introduce some definitions already used in [69] as well as some further or modified definitions.

Definition 7.2.1

1. *Let $(r,s) \in \mathbb{Z}^2 \backslash \{0\}$ for $\gcd(r,s) = 1$ and $r \geq 0$. We define $\tilde{f}_{(r,s)}(x,y) := f_{(\lambda_{rs} r, \lambda_{rs} s)}(x,y)$ for*

$$f_{(a,b)}(x,y) := \begin{cases} x^a y^b - 1, & \text{if } a,b \geq 0 \\ x^a - y^{|b|}, & \text{if } a \geq 0, b < 0 \end{cases} \tag{7.3}$$

and $\lambda_{rs} := \mathrm{lcm}(\frac{p}{\gcd(p,r)}, \frac{q}{\gcd(q,s)}) > 0$.

2. *For a set S of lattice directions we define $\tilde{F}_S$ by*

$$\tilde{F}_S(x,y) := \prod_{(r,s) \in S} \tilde{f}_{(r,s)}(x,y) \tag{7.4}$$

and $\tilde{F}_{(u,v;S)}$ by

$$\tilde{F}_{(u,v;S)}(x,y) := x^u y^v \tilde{F}_S(x,y). \tag{7.5}$$

3. *We further define $M_{(u_0,v_0;S)}$ by*

$$M_{(u_0,v_0;S)}(i,j) := \mathrm{coeff}_{i,j}(\sum_{u=0}^{p-1} \sum_{v=0}^{q-1} \tilde{F}_{(u_0+u, v_0+v;S)}) \tag{7.6}$$

for $p|u_0, q|v_0$ and $(i,j) \in \mathbb{Z}^2$.

Remark 7.2.2 The parameter λ_{rs} is defined as the smallest parameter λ which fulfills $p|\lambda r$ and $q|\lambda s$, the polynomial $\tilde{f}_{(r,s)}(x,y)$ describes the periodicity in direction (r,s).
The product of the periodicities over all lattice directions within the set S is described by the polynomial $\tilde{F}_S(x,y)$, its translation in direction (u,v) by $\tilde{F}_{(u,v;S)}(x,y)$.
The value $M_{(u_0,v_0;S)}(i,j)$ represents the coefficient of $x^i y^j$ within the elementary switching component with respect to the regular $p \times q$-weaving pattern structure which is characterized by (u_0, v_0).
Using the following lemma we will characterize switching components within regular $p \times q$-weaving patterns subsequent to the lemma.

Lemma 7.2.3 *Let (r, s) be a lattice direction. Assuming regular $p \times q$-weaving pattern structure, the function $g : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ with finite support has zero line sums along the lines in direction (r, s) if and only if $\tilde{f}_{(r,s)}(x, y)$ divides $h(x, y) := \sum_{(i,j)\in\mathbb{Z}^2} g(i, j)x^i y^j$ in $\mathbb{Z}[x, y]$.*

Proof

For further considerations let us assume that the support of the function g lies within the reference area

$$G = \{(i, j) \in \mathbb{Z}^2 | 0 \leq i < n_1, 0 \leq j < n_2\}. \tag{7.7}$$

If $x^p - 1 = (x - 1)(x^{p-1} + x^{p-2} + \cdots + 1)$ divides $h(x, y)$, then also $x - 1$ divides $h(x, y)$. The same is true for $y^q - 1$ and $y - 1$. As we further calculate that

$$x^{\lambda_{rs} r} y^{\lambda_{rs} s} - 1 = (x^r y^s - 1)((x^r y^s)^{\lambda_{rs}-1} + (x^r y^s)^{\lambda_{rs}-2} + \cdots + 1) \tag{7.8}$$

and that

$$x^{\lambda_{rs} r} - y^{\lambda_{rs}|s|} = (x^r - y^{|s|})((x^r)^{\lambda_{rs}-1} + (x^r)^{\lambda_{rs}-2} y^{|s|} + \cdots + (y^{|s|})^{\lambda_{rs}-1}), \tag{7.9}$$

one direction of the assertion is clear because of the equivalence without weaving pattern structure in [69].

Let us now assume that all horizontal line sums equal zero. Taking the regular $p \times q$-weaving pattern structure into account we calculate that

$$h(x, y) = \sum_{(i,j)\in G} g(i, j)x^i y^j = \tag{7.10}$$

$$= \sum_{j=0}^{n_2-1} \sum_{p|i} g(i, j)x^i y^j \sum_{k=0}^{p-1} x^k = \tag{7.11}$$

$$= \sum_{k=0}^{p-1} x^k [\sum_{j=0}^{n_2-1} \sum_{p|i} g(i, j)(x^i - 1)y^j + \sum_{j=0}^{n_2-1} y^j \sum_{p|i} g(i, j)] = \tag{7.12}$$

$$= \sum_{k=0}^{p-1} x^k [\sum_{j=0}^{n_2-1} y^j (x^p - 1) \sum_{p|i} g(i, j) \frac{x^i - 1}{x^p - 1}], \tag{7.13}$$

as $\sum_{p|i} g(i, j) = \frac{1}{p} \sum_{i=0}^{n_1-1} g(i, j) = 0$ for $j = 1, \ldots, n_2 - 1$. Therefore, $x^p - 1$ divides $h(x, y)$. Analogously, $y^q - 1$ divides h(x,y) if the vertical line sums equal zero.

To treat the non-horizontal and non-vertical lattice directions (r, s) let us assume that $r, s > 0$ without loss of generality. Let us look at the left-upper most line in direction (r, s) passing through at least one non-zero value and let us fix one of these positions (u_0, v_0).

By the choice of the line (see Figure 7.1) all non-zero positions $(\bar{u}, \bar{v})$ on that line satisfy

$$(\bar{u}, \bar{v}) = (u_0, v_0) + k\lambda_{rs}(r, s) \text{ for some } k \in \mathbb{Z}. \tag{7.14}$$

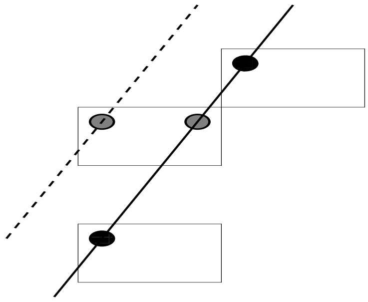

Figure 7.1: Left-upper most line in direction (r, s)

Therefore, the polynomial

$$\sum_{(i,j)\in(u_0,v_0)+\mathbb{Z}\cdot(r,s)} g(i,j)x^i y^j \tag{7.15}$$

and also

$$h^1(x,y) := \sum_{(i,j)\in(u_0,v_0)+\mathbb{Z}\cdot(r,s)} g(i,j)x^i y^j (1+x+\cdots+x^{p-1})(1+y^{-1}+\cdots+y^{-(q-1)}) \tag{7.16}$$

are divided by $(x^{\lambda_{rs}r}y^{\lambda_{rs}s}-1)$. *Using inductive arguments for* $h(x,y)-h^1(x,y)$, *we get a finite sequence* $(h^\mu)_{\mu=1,\dots,L}$ *so that* $h(x,y) = \sum_{\mu=1}^{L} h^\mu(x,y)$ *and* $(x^{\lambda_{rs}r}y^{\lambda_{rs}s}-1)$ *divides each* $h^\mu(x,y)$.
Thus, also $h(x,y)$ *is divided by* $(x^{\lambda_{rs}r}y^{\lambda_{rs}s}-1)$. □

Theorem 7.2.4
Assuming regular $p \times q$*-weaving pattern structure, the function* $g : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ *with finite support has zero line sums along the lines corresponding to the directions in the set* S *if and only if* g *can uniquely be written as*

$$g = \sum_{p|u}\sum_{q|v} c_{u,v} M_{(u,v;S)}. \tag{7.17}$$

Proof
According to the definition of $M_{(u,v;S)}$ *and Lemma 7.2.3 every function (7.17) has zero line sums.*
Thus, it is left to show analogously to [69] that every function g *with zero line sums can be written in the mentioned form and that the* $M_{(u,v;S)}$ *are linearly independent. The second is absolutely clear by the same arguments as in [69].*
Now let us take any function $g : \mathbb{Z}^2 \to \mathbb{Z}$ *which has zero line sums and let* $h(x,y) = \sum_{(i,j)\in\mathbb{Z}^2} g(i,j)x^i y^j$ *be its polynomial representation. By Lemma 7.2.3 we know that* $\tilde{F}_S(x,y)$ *divides* $h(x,y)$, *i. e. there is a polynomial* $t(x,y) =$

$\sum_{u,v} c_{u,v} x^u y^v$ *satisfying* $t(x,y)\tilde{F}_S(x,y) = h(x,y)$. *By taking the regular* $p \times q$-*weaving pattern structure of the function* g *into account, we get that*

$$t(x,y) = \sum_{p|u_0,q|v_0} c_{u_0,v_0} x^{u_0} y^{v_0} \sum_{u=0}^{p-1} \sum_{v=0}^{q-1} x^u y^v \tag{7.18}$$

and conclude that the function g *can be written in the mentioned form.* □

Example 7.2.5 Figure 7.2 shows an elementary switching component, which is described by $M_{(u_0,v_0;S)}$ for some $u_0 \in 2{\cdot}\mathbb{Z}$, $v_0 \in 1{\cdot}\mathbb{Z}$ and $S = \{(1,0),(1,1),(1,2)\}$.

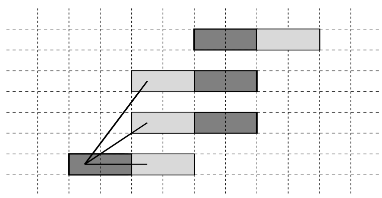

Figure 7.2: Elementary switching component in regular 2×1-weaving patterns with respect to the direction set $S = \{(1,0),(1,1),(1,2)\}$

7.3 Uniqueness results

If we know how large an elementary switching component gets for a fixed set of lattice directions, we have an upper bound on the size of a reference area

$$G = \{(i,j) \in \mathbb{Z}^2 | 0 \leq i < n_1, 0 \leq j < n_2\} \tag{7.19}$$

we can guarantee uniqueness on.

Theorem 7.3.1
Let $C_1 \otimes C_2$ be a set of regular $p \times q$-weaving patterns. Any kind of nonuniqueness is suppressed if one of the following conditions is satisfied for the number n_1 of columns and the number n_2 of rows:

$$n_1 < \sum_{i=1}^{m} \operatorname{lcm}(\frac{p}{\gcd(p,r_i)}, \frac{q}{\gcd(q,s_i)})r_i + p \tag{7.20}$$

$$n_2 < \sum_{i=1}^{m} \operatorname{lcm}(\frac{p}{\gcd(p,r_i)}, \frac{q}{\gcd(q,s_i)})|s_i| + q \tag{7.21}$$

Proof
According to Theorem 7.2.4 the elementary switching components within regular $p \times q$-weaving patterns are represented by $M_{u_0,v_0;S}$. Their width and height are equal to the right hand side data in (7.20) and (7.21). □

Remark 7.3.2 If $\gcd(p, r_i) = \gcd(q, s_i) = \gcd(p, q) = 1$, we get that

$$n_1 < pq \sum_{i=1}^{m} r_i + p, \tag{7.22}$$

$$n_2 < pq \sum_{i=1}^{m} |s_i| + q. \tag{7.23}$$

Therefore, the size of the elementary switching components quadratically grows within the factor pq.

Example 7.3.3 Let the parameters p, q be chosen by $p = 2, q = 3$ and let us restrict to a quadratic reference area G. With respect to the direction set $S = \{(3,2),\ (5,2),\ (3,4)\}$ we calculate that $(pq)(3+5+3)+2 = 68$ and $(pq)(2+2+4)+3 = 51$. Thus, the choice $n_1 = n_2 = 66 \in p \cdot \mathbb{Z} \cap q \cdot \mathbb{Z}$ guarantees uniqueness.
Regular $p \times q$-weaving patterns can be interpreted within a process of scaling and rescaling.

Corollary 7.3.4 *Within the process of scaling and rescaling by the scaling factors* p, q*, uniqueness with respect to the direction set* $S := \{(r_1, s_1), \ldots, (r_m, s_m)\}$ *is improved by the factor*

$$\frac{\sum_{i=1}^{m} \operatorname{lcm}(\frac{p}{\gcd(p,r_i)}, \frac{q}{\gcd(q,s_i)}) r_i}{p \cdot \sum_{i=1}^{m} r_i} \tag{7.24}$$

for the horizontal lines and by the factor

$$\frac{\sum_{i=1}^{m} \operatorname{lcm}(\frac{p}{\gcd(p,r_i)}, \frac{q}{\gcd(q,s_i)}) |s_i|}{q \cdot \sum_{i=1}^{m} |s_i|} \tag{7.25}$$

for the vertical lines.

Proof

The assertion is implied by Theorem 7.3.1 after division by p *resp. by* q *for the rescaling process and by* $\sum_{i=1}^{m} r_i$ *resp. by* $\sum_{i=1}^{m} |s_i|$. □

Remark 7.3.5 In the case that $\gcd(p, r_i) = \gcd(q, s_i) = \gcd(p, q) = 1$ the process of scaling and rescaling provides improvement by the factor q for the horizontal lines and by the factor p for the vertical lines.

7.4 Absorption and weaving patterns

In [71] Hajdu and Tijdeman characterize those functions $g : G \longrightarrow \mathbb{Z}$ defined on the reference area G (as given in (7.7)) which have zero line sums under absorption. We want to transfer the results to regular $p \times q$-weaving patterns. For that purpose let us recapitulate some algebraic knowledge about pure polynomials.

Lemma 7.4.1 *Let $X^l - a \in k[X]$ be a pure polynomial over the field k and let $\alpha := \sqrt[l]{a}$ define the lth root of a.*

1. *The splitting field K of $X^l - a$ contains the splitting field k_l of $X^l - 1$.*
2. *$\beta^l - a = 0$ if and only if $\beta = \alpha\zeta$ for some ζ satisfying $\zeta^l - 1 = 0$.*
3. *$K = k_l(\alpha) = k(\zeta, \alpha)$.*

Proof
For the proof we refer to [93]. □

Remark 7.4.2 There exists some $\beta \in \mathbb{R}, \beta \geq 0$ satisfying $\beta^l - a = 0$, i. e. β is a root of the polynomial $X^l - a$, if and only if $a \geq 0$, in particular $\beta = \sqrt[l]{a}$.
By taking in mind that the absorption rate β is a real value greater than or equal to 1, i. e. $\beta \in \mathbb{R}, \beta \geq 1$, we characterize those absorption rates which lead to nonuniqueness in regular $p \times q$-weaving patterns.

Lemma 7.4.3 (relevant absorption rates) *Nonuniqueness is possible for the absorption rate $\beta_{rs} \in \mathbb{R}, \beta_{rs} \geq 1$ along the X-ray direction (r, s) if and only if there exists a polynomial P_{rs} having coefficients in $\{0, \pm 1\}$ so that $P_{rs}(\beta_{rs}^{\lambda_{rs}}) = 0$ for $\lambda_{rs} := \mathrm{lcm}(\frac{p}{\gcd(p,r)}, \frac{q}{\gcd(q,s)})$. Those absorption rates are given by $\beta_{rs} = \sqrt[\lambda_{rs}]{\mu}$ for any root $\mu \in \mathbb{R} \backslash \mathbb{Q} \cup \{1\}$, $\mu \geq 1$ of some polynomial with coefficients in $\{0, \pm 1\}$.*

Proof

1. *If β_{rs} is transcendent, then there exists no polynomial P_{rs} so that $P_{rs}(\beta_{rs}) = 0$ and thus uniqueness is guaranteed.*
2. *The binarity within discrete tomography restricts us to polynomials with coefficients in $\{0, \pm 1\}$. Thus, let $P_{rs} = \sum_{i=0}^{k} c_i X^i \in \{0, \pm 1\}[X]$ for $c_k \neq 0$ and let $\beta_{rs} = \frac{p}{q}$ for $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Furthermore, let l be defined by*

$$l := \min\{i | c_i \neq 0\}, \tag{7.26}$$

which is strictly smaller than k for $P_{rs}(\beta_{rs}) = 0$ and $\beta_{rs} \geq 1$. We calculate *that*

$$P_{rs}(\frac{p}{q}) = 0 \Leftrightarrow \sum_{i=0}^{k} c_i p^i q^{k-i} = 0 \Leftrightarrow c_k p^k = -\sum_{i=0}^{k-1} c_i p^i q^{k-i} \qquad (7.27)$$

$$\Leftrightarrow c_l p^l q^{k-l} = -\sum_{i=l+1}^{k} c_i p^i q^{k-i}. \qquad (7.28)$$

As $q \neq 1$ *divides* $\sum_{i=0}^{k-1} c_i p^i q^{k-i}$, *but not* $c_k p^k$ *and as the term* p^{l+1} *divides* $\sum_{i=l+1}^{k} c_i p^i q^{k-i}$, *but not* $c_l p^l q^{k-l}$ *for* $p \neq 1$, *all rational absorption rates greater than 1 guarantee uniqueness.*

3. *The case* $\beta_{rs} = 1$ *(no absorption) is already treated at the beginning of this chapter.*

4. *For the horizontal direction the regular* $p \times q$*-weaving pattern structure implies that any absorption rate* $\beta_{1,0}$ *which does not guarantee uniqueness is a root of some polynomial*

$$P_{1,0}(X) = (1 + X + \cdots + X^{p-1}) \sum c_i (X^p)^i. \qquad (7.29)$$

As the term $(1 + X + \cdots + X^{p-1})$ *has no positive root, it remains to look at the polynomials* $\sum c_i (X^p)^i$ *and their roots.*
Similar arguments work for any lattice direction (r, s) *by applying the arguments in Lemma 7.2.3 using the left-upper most line (if* $r, s \geq 0$*) for the non-horizontal and the non-vertical directions again. Thus, we only have to consider the polynomials* $\sum c_i (X^{\lambda_{rs}})^i$ *for* $\lambda_{rs} := \mathrm{lcm}(\frac{p}{\gcd(p,r)}, \frac{q}{\gcd(q,s)})$.

5. *Let* $\sum c_i (X^{\lambda_{rs}})^i = \prod_i (X^{\lambda_{rs}} - \mu_i) \in \{0, \pm 1\}[X]$ *be the factorization of some polynomial* $\sum c_i Y^i$ *for* $Y := X^{\lambda_{rs}}$ *within 4. over its splitting field. Thus, any relevant absorption rate is given by*

$$\beta_i := \sqrt[\lambda_{rs}]{\mu_i} \qquad (7.30)$$

for some $\mu_i \in \mathbb{R} \backslash \mathbb{Q}$, $\mu_i \geq 1$. □

□

The results before help us to give an upper bound on the size of the reference area we can guarantee uniqueness on in the case of absorption.

Definition 7.4.4 (degree of the absorption rate β)
The ***degree*** $\deg(\beta)$ ***of the absorption rate*** $\beta \geq 1$ *is defined by*

$$\deg(\beta) := \min\{\deg(P) | 0 \neq P \in \{0, \pm 1\}[X] \text{ and } P(\beta) = 0\}. \qquad (7.31)$$

If none polynomial exists at all, we define $\deg(\beta) := \infty$.

Theorem 7.4.5 (size of switching components)
Any kind of nonuniqueness is suppressed on the reference area $G := \{(i,j) \in \mathbb{Z}^2 | 0 \leq i < n_1, 0 \leq j < n_2\}$ *if one of the following conditions is satisfied for* $\lambda_{r_i s_i} := \mathrm{lcm}(\frac{p}{\gcd(p,r_i)}, \frac{q}{\gcd(q,s_i)}))$ *and the absorption rate* $\beta_{r_i s_i}$ *in direction* (r_i, s_i), $i = 1, \ldots, m$*:*

$$n_1 \leq \sum_{i=1}^{m} \lambda_{r_i s_i} deg(\beta_{r_i s_i}^{\lambda_{r_i s_i}}) r_i \tag{7.32}$$

$$n_2 \leq \sum_{i=1}^{m} \lambda_{r_i s_i} deg(\beta_{r_i s_i}^{\lambda_{r_i s_i}}) |s_i| \tag{7.33}$$

Proof
The assertion is implied by Lemma 7.4.3. □

Lemma 7.4.6 *The absorption rates of small degree are characterized by*

1. $\deg(\beta) = 1 \Longleftrightarrow \beta = 1$,
2. $\deg(\beta) = 2 \Longleftrightarrow \beta = \frac{1+\sqrt{5}}{2}$.

Proof

1. *The polynomial* $X - 1$ *is the only one of degree* 1 *which has coefficients in* $\{0, \pm 1\}$ *and some root at least of value* 1.
2. (a) *The polynomial* $X^2 + aX + b$ *of degree* 2 *has the roots* $\beta_{1/2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$ *in* $\mathbb{C}$. *As* β *has to be real and as* $a, b \in \{0, \pm 1\}$, *the condition* $a^2 - 4b \geq 0$ *implies that* $b \in \{0, -1\}$.
   (b) *The setting* $b = 0$ *implies that the degree of any root is less than* 2. *Thus, we conclude that* $b = -1$.
   (c) *For* $a = 0$ *we get that* $X^2 - 1 = (X+1)(X-1)$ *and thus also in that case the degree of any root is less than* 2.
   (d) *Because of the condition* $\beta \geq 1$ *we conclude that* $a = -1$ *and thus* $\beta = \frac{1+\sqrt{5}}{2}$. □

7.5 Some remarks about stability and instability

Even if we completely exclude nonuniqueness by choosing the reference area not too large, it is not clear at all whether instabilities are suppressed. Thus, let us take a look at instabilities within regular $p \times q$*-weaving patterns and their right hand side data in the following.*

For two finite lattice sets of equal cardinality and m *different lattice directions the smallest difference not equal to zero between their right hand side data is* $2(m-1)$*, see [6], [3]. But this result cannot be transferred to the case of regular* $p \times q$*-weaving patterns by simply multiplying by the factor* $p \cdot q$*, see Figure 7.3.*

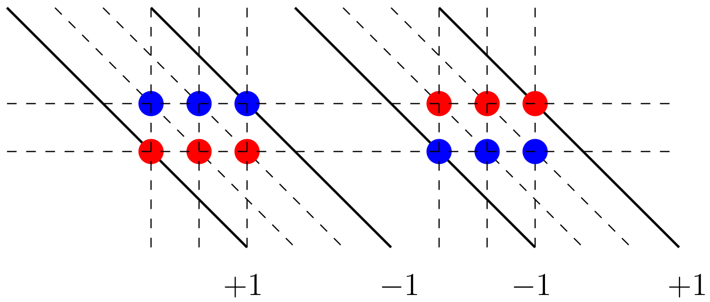

Figure 7.3: Two lattice sets F_1 (blue), F_2 (red) within regular 3×1-weaving patterns having right hand side difference 4 with respect to the direction set $S := \{(1,0), (0,1), (1,-1)\}$

If $|F_1| \neq |F_2|$ *the regular* $p \times q$*-weaving pattern structure implies that* $|F_1 \triangle F_2| \geq pq$ *and thus the difference in the right hand side data is bounded from below by the value* $m \cdot pq$*.*

In the case of horizontal or vertical discrepancy for $|F_1| = |F_2|$ *the difference in the right hand side data is at least* $2 \cdot pq$ *if only the horizontal or the vertical direction is concerned resp.* $4 \cdot pq$ *if both the horizontal and the vertical direction are concerned.*

For later use let us formulate the following two lemmata.

Lemma 7.5.1 *Let us consider regular* $p \times q$*-weaving patterns and the lattice direction* (r, s) *(without loss of generality* $r, s \geq 0$*) satisfying*

$$\gcd(p, q) = \gcd(p, r) = \gcd(q, s) = 1. \tag{7.34}$$

Every line in direction (r, s) *passes through at least one lattice point* $(kp, lq-1)$*, i. e. through the left-upper corner of at least one* $p \times q$*-rectangle.*

Proof

Let us look at the equation system

$$\lambda \cdot \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \mu_1 \cdot p \\ \mu_2 \cdot q \end{pmatrix} \text{ for } \lambda, \mu_1, \mu_2 \in \mathbb{Z}. \tag{7.35}$$

Condition (7.34) implies that

$$\lambda \in p \cdot \mathbb{Z}, \tag{7.36}$$
$$\lambda \in q \cdot \mathbb{Z} \tag{7.37}$$

and therefore

$$\lambda \in p \cdot \mathbb{Z} \cap q \cdot \mathbb{Z} = \mathrm{lcm}(p, q) \cdot \mathbb{Z} = pq \cdot \mathbb{Z}. \tag{7.38}$$

Now let us assume that there is a line in direction (r, s) *which does not pass through any lattice point* $(kp, lq - 1)$*. But then there are two lattice points* $(k_1p + i, l_1q + j)$*,* $(k_2p + i, l_2q + j)$ *so that*

$$(k_1p + i, l_1q + j) + \lambda(r, s) = (k_2p + i, l_2q + j), \tag{7.39}$$
$$\Longleftrightarrow (k_1p, l_1q) + \lambda(r, s) = (k_2p, l_2q) \tag{7.40}$$

for some $0 < \lambda < pq$ *in contradiction to (7.38).* □

Lemma 7.5.2 *Let the distance of two lines be defined by the number of lines which have to be passed from one line to the other one. Then, the lines along direction* (r, s) *which are incident to the lattice points* $(0, 0)$ *and* $(p, 0)$ *resp.* $(0, q)$ *for* $p, q \in \mathbb{Z}$ *have distance* ps *resp.* qr *to each other.*

Proof

Because of

$$t_{(i,j)} = j - (\frac{s}{r})i = \frac{1}{r}(rj - si) \tag{7.41}$$

and as $\gcd(r, s) = 1$*, the smallest ordinate distance equals* $\frac{1}{r}$*. The assertion is then implied by*

$$t_{(p,0)} = 0 - \frac{s}{r}p = -ps \cdot \frac{1}{r}, \tag{7.42}$$
$$t_{(0,q)} = q - \frac{s}{r}0 = qr \cdot \frac{1}{r}. \tag{7.43}$$

□

Theorem 7.5.3

In the case that $\gcd(p, r) = \gcd(q, s) = \gcd(p, q) = 1$ *there are two finite lattice sets* F_1, F_2 *of same cardinality within the regular* $p \times q$*-weaving patterns so that the right hand side difference with respect to the direction* (r, s) *is given by the value* 2 *and the error lines have distance*

$$\lambda \cdot \mathrm{lcm}(ps, qr) = \lambda \cdot psqr \tag{7.44}$$

to each other for some $\lambda \in \mathbb{N}\backslash\{0\}$*.*

Proof

Using Lemma 7.5.2 each $p \times q$-rectangle is represented by

$$(1 + x^s + \cdots + (x^s)^{p-1})(1 + x^r + \ldots (x^r)^{q-1}) = \frac{x^{ps} - 1}{x^s - 1} \frac{x^{qr} - 1}{x^r - 1} \tag{7.45}$$

in that way that the exponents correspond to a successive numeration of the lines in direction (r, s). Because of divisibility in number theory we get that

$$\frac{x^{ps} - 1}{x^s - 1} \frac{x^{qr} - 1}{x^r - 1} | x^n - 1 \Leftrightarrow n \in psqr \cdot \mathbb{N}, \tag{7.46}$$

which implies the assertion of the theorem by keeping Lemma 7.5.1 in mind. □

Remark 7.5.4 Because of $\gcd(r, s) = 1$ the polynomials $x^s - 1$ and $x^r - 1$ have no common factor besides the term $x - 1$. Thus, if $\gcd(ps, qr) \neq 1$ the polynomials $1 + x^r + \cdots + x^{(q-1)r}$ and $1 + x^s + \cdots + x^{(p-1)s}$ have common factors, which implies that the polynomial $g = \frac{x^{qr}-1}{x^r-1} \frac{x^{ps}-1}{x^s-1}$ does not divide $x^n - 1$ for any $n \in \mathbb{N}$. Therefore, the smallest difference value within the right hand side data not equal to zero is at least of value 4.

The following lemma generalizes the result of Lemma 7.5.2 to the case that $\gcd(ps, qr) \neq 1$ and helps us to extend Theorem 7.5.3 afterwards.

Lemma 7.5.5 *The distance (defined as in Lemma 7.5.2) of two lattice points $(\lambda_1 p, \mu_1 q - 1), (\lambda_2 p, \mu_2 q - 1) \in \mathbb{Z}^2$ lies within $\gcd(ps, qr) \cdot \mathbb{Z}$.*

Proof

The usage of Lemma 7.5.2 gives us that two lattice points $(0, -1)$, $(\lambda p, \mu q - 1) \in \mathbb{Z}^2$ have distance $|\lambda ps - \mu qr|$ to each other. Therefore, some knowledge about divisibility in number theory implies that the smallest distance equals the value $\gcd(ps, qr)$ and thus every distance value has to lie within $\gcd(ps, qr) \cdot \mathbb{Z}$. □

Corollary 7.5.6 *In the case that*

$$\gcd(ps, qr) = \gcd(p, r) \cdot \gcd(q, s) \tag{7.47}$$

there are two finite lattice sets F_1, F_2 of same cardinality within the regular $p \times q$-weaving patterns so that the right hand side difference with respect to the direction (r, s) is given by the value $2 \cdot \gcd(ps, qr)$, which is the smallest error value not equal to zero.

Proof

We bundle $\gcd(p, r) \times \gcd(q, s)$-rectangles and consider only the lines passing through the left-upper corners of those rectangles (if $r, s \geq 0$). Thus, the situation is reduced to the case in Theorem 7.5.3 and therefore the smallest error value is given by the value $2 \cdot gcd(ps, qr)$. □

Now let us consider the case that $\gcd(ps, qr) \neq \gcd(p, r) \cdot \gcd(q, s)$*. Because of the representation (7.45) of any* $p \times q$*-rectangle and because of Lemma 7.5.5 we calulate that*

$$(x^{(q-1)r} + \cdots + x^r + 1)(x^{(p-1)s} + \cdots + x^s + 1) \cdot h(x^{\gcd(ps,qr)}) =$$

$$= \frac{x^{qr} - 1}{x^r - 1} \cdot \frac{x^{ps} - 1}{x^s - 1} \cdot h(x^{\gcd(ps,qr)}) = \tag{7.48}$$

$$= \frac{(x^r)^{\frac{\gcd(ps,qr)}{\gcd(p,r)}} - 1}{x^r - 1} \cdot \frac{(x^s)^{\frac{\gcd(ps,qr)}{\gcd(q,s)}} - 1}{x^s - 1} \cdot \frac{y^{\frac{qr}{\gcd(ps,qr)}} - 1}{y^{\frac{r}{\gcd(p,r)}} - 1} \cdot \frac{y^{\frac{ps}{\gcd(ps,qr)}} - 1}{y^{\frac{s}{\gcd(q,s)}} - 1} \cdot h(y)$$

for $y := x^{\gcd(ps,qr)}$

$$= \frac{(x^r)^{\frac{\gcd(ps,qr)}{\gcd(p,r)}} - 1}{x^r - 1} \cdot \frac{(x^s)^{\frac{\gcd(ps,qr)}{\gcd(q,s)}} - 1}{x^s - 1} \cdot (y^{\mathrm{lcm}(\frac{qr}{\gcd(ps,qr)}, \frac{ps}{\gcd(ps,qr)})} - 1) \tag{7.49}$$

for some polynomial h

$$= \frac{(x^r)^{\frac{\gcd(ps,qr)}{\gcd(p,r)}} - 1}{x^r - 1} \cdot \frac{(x^s)^{\frac{\gcd(ps,qr)}{\gcd(q,s)}} - 1}{x^s - 1} \cdot (x^{\mathrm{lcm}(ps,qr)} - 1). \tag{7.50}$$

Thus, there are two finite lattice sets F_1, F_2 *of same cardinality which have right hand side difference of value*

$$2 \cdot \frac{\gcd(ps, qr)}{\gcd(p, r)} \frac{\gcd(ps, qr)}{\gcd(q, s)} = 2 \cdot \frac{(\gcd(ps, qr))^2}{\gcd(p, r) \gcd(q, s)}, \tag{7.51}$$

as the first factor within (7.50) represents a sum of $\frac{\gcd(ps,qr)}{\gcd(p,r)}$ *monomials and the second factor a sum of* $\frac{\gcd(ps,qr)}{\gcd(q,s)}$ *monomials. But that value is not always the smallest right hand side difference besides the value* 0 *as the following example shows.*

Example 7.5.7 Let the polynomial h be given by

$$h(x) := x^4 - 1 \tag{7.52}$$

and the parameters p, q, r, s by $p = q = 2$, $r = 3$ and $s = 5$, in particular it yields that $\gcd(ps, qr) = \gcd(p, q) = 2$. We calculate that

$$\frac{x^{ps} - 1}{x^s - 1} \frac{x^{qr} - 1}{x^r - 1} h(x^{\gcd(ps,qr)}) = (x^5 + 1)(x^3 + 1)h(x^2) =$$

$$= (1 + x^3 + x^5 + x^8)(x^8 - 1) = -1 - x^3 - x^5 + x^{11} + x^{13} + x^{16}.$$

Therefore, there exist two finite lattice sets F_1, F_2 within the regular 2×2-weaving patterns which have right hand side difference of value

$$2 \cdot 3 = 6 < 2 \cdot \frac{(\gcd(ps, qr))^2}{\gcd(p, r) \gcd(q, s)} = 2 \cdot 4 = 8.$$

The next two lemmata take a closer look at some special situations in the case that $\gcd(ps, qr) \neq \gcd(p, r) \cdot \gcd(q, s)$.

Lemma 7.5.8 *Let* $r + s \equiv 0 \mod \gcd(ps, qr)$. *Then, there exist two finite lattice sets* F_1, F_2 *of same cardinality within the regular* $p \times q$*-weaving patterns which have right hand side difference equal to the value*

$$2 \cdot [pq - (p-1)(q-1)] = 2 \cdot (p + q - 1). \tag{7.53}$$

In the case that $p = q = 2$ *and* $\gcd(p, r) = \gcd(q, s) = 1$ *the value* $2 \cdot (p+q-1) = 6$ *is minimal among all non-zero right hand side differences.*

Proof

The first assertion follows by the calculation

$$\begin{aligned}
&(1 + x^r + \cdots + (x^r)^{q-1})(1 + x^s + \cdots + (x^s)^{p-1})(x^{r+s} - 1) = \\
&= (x^r + \cdots + (x^r)^q)(x^s + \cdots + (x^s)^p) - \\
&\quad -(1 + x^r + \cdots + (x^r)^{q-1})(1 + x^s + \cdots + (x^s)^{p-1}) = \\
&= (x^r(x^s)^q + \cdots + (x^r)^p(x^s)^q + (x^r)^p(x^s)^{q-1} + \cdots + (x^r)^p x^s) - \\
&\quad -(1 + x^r + \cdots + (x^r)^{q-1} + x^s + \cdots + (x^s)^{p-1}).
\end{aligned} \tag{7.54}$$

To show the second assertion by regarding Remark 7.5.4 let us assume that there exist two finite lattice sets F_1, F_2 *of same cardinality which have right hand side difference of value* 4. *Without loss of generality let* $0 < r \leq s$ *and let the left-upper most error line be incident to the* 2×2*-rectangle* $\{(0,0), (0,1), (1,0), (1,1)\}$. *In the case that* $r < s$ *the lines incident to the lattice points* $(0,1)$, $(0,0)$ *have to be afflicted with error, but the line incident to the lattice point* $(1,0)$ *must not, and in the case* $r = s = 1$ *the line incident to both lattice points* $(1,0)$ *and* $(0,1)$ *may only assume error value* 1. *That is implied by the fact that the sum of same signed error values may not extend the value* 2. *But because of* $\gcd(q, s) = \gcd(2, s) = 1$ *that is not possible as* $s \notin gcd(ps, qr) \cdot \mathbb{N} = 2 \cdot \mathbb{N}$ *in contradiction to our assumption.* □

Remark 7.5.9 Because of $\gcd(r, s) = 1$ neither the parameter r nor the parameter s has common prime factors with the value $gcd(ps, qr)$ in Lemma 7.5.8. Thus, we calculate that $\gcd(ps, qr) = \gcd(p, q)$ and $2 \cdot (p+q-1) > 2 \cdot \gcd(ps, qr)$ for $p = q \geq 2$.

Lemma 7.5.10 *Let* $\gcd(ps, qr) = \gcd(p, q) = p = q$ *and let*

$$r, s \equiv 1 \mod p \ (\textit{or } r, s \equiv -1 \mod p). \tag{7.55}$$

The right hand side difference of two finite lattice sets of same cardinality within regular $p \times q$*-weaving patterns which are not tomographically equivalent is bounded from below by the value*

$$2p + 2 > 2 \cdot \gcd(ps, qr) = 2 \cdot p. \tag{7.56}$$

Proof
Let us assume that there exist two finite lattice sets F_1, F_2 which are not tomographically equivalent and which have right hand side difference at most of value $2p$. Without loss of generality let $r \leq s$. Because of condition (7.55) the equation

$$kr = tp + l_1 r + l_2 s \tag{7.57}$$

cannot be fulfilled for $0 \leq k < p$, $0 \leq l_1 + l_2 < k$, $l_1, l_2 \geq 0$ and any nonnegative parameter $t \in \mathbb{N}$ as equality is also not reached by taking both sides modulo p, i. e. every lattice point $(0, k)$ for $k = 0, \ldots, p-1$ causes right hand side error if the lattice set $\{(i,j)|0 \leq i, j \leq p-1\}$ represents the left-upper most rectangle within the lattice set $F_1 \cup F_2$. Similar arguments work for the right-down most $p \times q$-rectangle. Thus, it remains to take a closer look at the possibility that the error value $2p$ occurs:

$$(1 + x^r + \ldots x^{(q-1)r})(1 + x^s + \ldots x^{(p-1)s})h(x^p) = \tag{7.58}$$
$$= (1 + x^r + \ldots x^{(q-1)r})(x^l - 1)$$
$$\Longleftrightarrow (1 + x^s + \ldots x^{(p-1)s})h(x^p) = (x^l - 1) \tag{7.59}$$

The monomial x^s within the first factor on the left hand side in equation (7.59) is canceled if and only if

$$pt + 0 = s \Longleftrightarrow s \equiv 0 \mod p \tag{7.60}$$

in contradiction to (7.55). □

Remark 7.5.11 If we consider each direction separately the choice of a good direction for uniqueness aspects and the guarantee of large differences in the right hand side data are in general contrary aims. Taking the last results into account we can summarize that we get rather good results for both error treating and uniqueness aspects if $\gcd(ps, qr) = \gcd(p, q)$.
Actually, we need assertions which depend on the complete right hand side data instead of only those with respect to one direction.

Lemma 7.5.12 *Let the direction set be given by $S := \{(r_1, s_1), \ldots, (r_m, s_m)\}$ so that $\gcd(ps_1, qr_1) = 1$. Then, there exist two finite lattice sets F_1, F_2 of same cardinality within the regular $p \times q$-weaving patterns which have right hand side difference at most of value 2^m resp. of value 2^{m-1} in the case that $(1, 0), (0, 1) \in S$.*
If $S = \{(r, s), (1, 0), (0, 1))\}$ and $gcd(ps, qr) = 1$, the value $2^{m-1} = 4$ is minimal among all non-zero right hand side differences.

Proof
In the same manner as in the proof of Theorem 7.5.3 any elementary switching component within the regular $p \times q$-weaving patterns for the direction set

$S\backslash\{(r_1,s_1)\}$ is represented by the polynomial

$$\frac{x^{ps_1-1}}{x^{s_1}-1}\frac{x^{qr_1-1}}{x^{r_1}-1}\prod_{j=2}^{m}(x^{\lambda_{r_js_j}\cdot(r_js_1-s_jr_1)}-1) \tag{7.61}$$

for $\lambda_{rs} := \mathrm{lcm}(\frac{p}{\gcd(p,r)}, \frac{q}{\gcd(q,s)})$, which divides the polynomial

$$(x^{pqr_1s_1}-1)^2 \prod_{(r,s)\in S\backslash\{(r_1,s_1),(1,0),(0,1)\}} (x^{\lambda_{rs}\cdot(rs_1-sr_1)}-1) \tag{7.62}$$

in the case that $(0,1),(1,0)\in S$ resp. the polynomial

$$(x^{pqr_1s_1}-1) \prod_{(r,s)\in S\backslash\{(r_1,s_1)\}} (x^{\lambda_{rs}\cdot(rs_1-sr_1)}-1) \tag{7.63}$$

in the other case.

If binarity is violated for the elementary switching component, we will apply larger translations in the directions $(r_j,s_j)\in S\backslash\{(r_1,s_1),(1,0),(0,1)\}$. The switching component is again represented by (7.61) if we replace $\lambda_{r_js_j}$ by $k_j\cdot\lambda_{r_js_j}$ for those directions (r_j,s_j) and sufficiently large $k_j\in\mathbb{N}$. Thus, the assertion is implied.

The minimality of the value 4 is given by [6], [3]. □

It is an open question how to get a complete characterization of possible right hand side differences. In particular, the smallest data error is not known in general for the complete right hand side data.

List of Figures

Table of symbols

$\mathbb{N}$	set of the natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Q}$	set of the rational numbers
$\mathbb{R}$	set of the real numbers
$\mathbb{Z}$	set of the integral numbers
$[.,.]$	closed interval of real numbers
$\gcd(.,.)$	greatest common divisor
$\text{lcm}(.,.)$	lowest common multiplier
$\deg(.)$	degree of a polynomial
$[.,.,.,.]$	cross ratio of four points
$< . >$	the group generated by the arguments
$\vert . \vert$	cardinality
$\vert\vert . \vert\vert$	Euclidean norm
$\vert\vert . \vert\vert_1$	1-norm
$\text{dist}(.,.)$	distance
$\angle(.,.)$	angle between two lines
$F_1 \triangle F_2$	symmetric difference $(F_1 \backslash F_2) \cup (F_2 \backslash F_1)$
$(.)_i$	ith component of a vector
e_i	ith elementary vector
$\mathcal{G}$	candidate grid
$\text{im}(.)$	range of an operator
$\oplus$	direct sum
$L^2_0(.)$	Lebesgue space of the 2-integrable functions with local support
$\text{coeff}_{i,j}(.)$	coefficient of the monomial $x^i y^j$ within a polynomial
$\sim$	equivalence relation
$u^\perp$	orthogonal space of the vector u
$\text{aff}(.)$	affine hull
$\text{span}(.)$	linear span
$\text{bd}(.)$	boundary of a set
$\text{clos}(.)$	closure of a set
$\text{int}(.)$	interior of a set
$O(.)$	Landau symbol
$\text{sgn}(.)$	signum function

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