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Extremes of Multidimensional Stationary Diffusion Processes and Applications in Finance

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Nach einer unvollendeten Mathematikarbeit *

Alles
durchdringe die mathematik, sagt
der lehrer: medizin
psychologie
sprachen

Er vergißt
meine träume

In ihnen rechne ich unablässig
das unberechenbare

Und ich schrecke auf wenn es klingelt
wie du

Reiner Kunze

*aus Zimmerlautstärke, S. Fischer Verlag

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Abstract

The extreme behavior of one-dimensional, stationary diffusion processes is well understood. In the multidimensional case however, only a few special processes have been studied so far. In this thesis, the extremes of a general multidimensional, uniformly elliptic, reversible diffusion process $(X_t)_{t\geq 0}$ are characterized. We consider the partial maxima $M_T := \max_{0\leq t\leq T} q(X_t)$, $T > 0$, where q is some distance function. The fine tail asymptotics of M_T for fixed $T > 0$ is evaluated, i.e., the asymptotic behavior of the probability $P(M_T > R)$ as $R \to \infty$. Also the long time behavior of M_T as $T \to \infty$ is analyzed, in the spirit of classical extreme value theory.

The key idea is that, under general and tractable assumptions, $P(M_T \leq R)$ can be expressed in terms of the principal eigenvalue λ_R of the generator of the process $(X_t)_{t\geq 0}$, subject to Dirichlet boundary conditions on some bounded domain O_R, $R > 0$. Hence, it is sufficient to evaluate the asymptotic behavior of λ_R as $R \to \infty$ in terms of the drift and diffusion coefficient of the process $(X_t)_{t\geq 0}$. With the aid of suitable test-functions, we give conditions for obtaining sharp upper and lower bounds on λ_R. Additionally, the asymptotics of λ_R as $R \to \infty$ is evaluated also via singular perturbation methods.

Stationary diffusion processes are used in mathematical finance to model the term structure of interest rates, for instance. From the point of view of risk management, it is important to know about large fluctuations of these processes. We present some multivariate short-rate models that incorporate also spatial dependence, and analyze explicitly their extreme behavior. The theoretical results are corroborated by examination of both simulated and real life financial data, for which suitable goodness-of-fit tests are developed.

Chapter 1

Introduction

Extreme value theory is mainly concerned with investigating the probability of very rare extreme events. The interest in this theory has increased continuously during the last decades, partly due to the fact that catastrophic events are often followed by large pecuniary claims. Hence, extreme value theory has found applications in many different fields, among others in

- financial risk management to investigate the riskiness of portfolios,
- insurance mathematics to quantify the probability of huge claims,
- engineering sciences to study the reliability of mechanical structures,
- environmental engineering to determine the height of dams or dikes,
- environmental statistics to establish limit values for ground-level ozone.

Many models for the above applications describe the development of observations continuously in time. In financial mathematics for instance, continuous-time Markov processes cover the time dependence in asset price models. Univariate models are well investigated in this framework. On the other hand, financial risk in practice does not depend only on one parameter, but is influenced by several correlated factors.

In this thesis, we evaluate the extreme behavior of multidimensional diffusion processes. We combine methods of classical extreme value theory with techniques from spectral analysis for differential operators. The theoretical results are applied to real life financial data.

1.1 From Classical Extreme Value Theory to Extremes of Multidimensional Diffusions

We introduce some basic notations of classical extreme value theory that will be used in Section 1.2 to give a formal description of the topic of the present work. In addition, a short overview of extensions to continuous-time processes is given.

The part of extreme value theory which nowadays is called classical extreme value theory was developed in the late twenties of the last century and deals with the maxima of independent, identically distributed (i.i.d.) real random variables. The work of Fisher and Tippett [FT28] and Gumbel [Gum58] have been the milestones of this theory. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. real random variables distributed according to a distribution function F, and consider the partial maxima

$$M_n := \max\{X_i : i = 1, \ldots, n\} \qquad n \in \mathbb{N}.$$

Motivated by the central limit theorem which describes the limit distribution of normalized sums of i.i.d. random variables, the objective of extreme value theory is to derive an analogous result for the maxima, i.e., to determine under which conditions there exist norming constants $c_n > 0$ and $d_n \in \mathbb{R}$, $n \in \mathbb{N}$, and a non-degenerated limit distribution function H such that

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} H \qquad (n \to \infty)\,, \tag{1.1}$$

where the symbol $\xrightarrow{d}$ denotes convergence in distribution. The central result of extreme value theory, often referred to as the *Extremal Types Theorem*, was derived for the first time by Fisher and Tippett [FT28] and was generalized and formally proved by Gnedenko [Gne43]. It states that there are only three possible non-degenerated limit distributions, i.e., the distribution function H is one of the following so-called *extreme value distributions*:

$$\text{Fréchet:} \quad \Phi_\alpha(x) = \begin{cases} 0 & x > 0 \\ \exp\{-x^{-\alpha}\} & x \le 0 \end{cases} \qquad \alpha > 0\,, \tag{1.2}$$

$$\text{Weibull:} \quad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & x \le 0 \\ 1 & x > 0 \end{cases} \qquad \alpha > 0\,,$$

$$\text{Gumbel:} \qquad \Lambda(x) = \exp\{-e^{-x}\} \qquad x \in \mathbb{R}\,.$$

Relation (1.1) can also be expressed in terms of the distribution function F using the following simple calculation

$$P(c_n^{-1}(M_n - d_n) \le x) = P\{X_i \le c_n x + d_n \ \forall i = 1, \dots, n\} = F(c_n x + d_n)^n\,.$$

Hence (1.1) is equivalent to

$$\lim_{n\to\infty} F(c_n x + d_n)^n = H(x) \qquad x \in \mathbb{R}\,. \tag{1.3}$$

If a distribution function F satisfies (1.3) for an extreme value distribution $H \in \{\Phi_\alpha, \Psi_\alpha, \Lambda\}$ and norming constants $c_n > 0$ and $d_n \in \mathbb{R}$, $n \in \mathbb{N}$, we say that F is *in the domain of attraction of* H ($F \in \mathrm{DA}(H)$). Classical extreme value theory gives a characterization of the domain of attraction $\mathrm{DA}(H)$ for each $H \in \{\Phi_\alpha, \Psi_\alpha, \Lambda\}$ and provides methods to calculate the norming constants c_n, d_n in (1.1) and (1.3). For an introduction to classical extreme value theory we refer to Embrechts et al. [EKM97], Leadbetter et al. [LLR83], and Resnick [Res87].

In most applications, the risky factors are highly dependent. This was the motivation to weaken the i.i.d. assumption and to extend the classical extreme value theory to characterize the extreme behavior of dependent random sequences and continuous parameter processes.

In this thesis we concentrate on extreme value theory for continuous-time processes with some time dependence structure. The pioneering work in that area was achieved for one-dimensional Gaussian processes, i.e., processes for which all finite dimensional distributions are multivariate normal. Recall that a one-dimensional, stationary, standardized Gaussian process $(X_t)_{t\ge 0}$ (with zero mean and unit variance) is uniquely determined by its covariance function $r(t) = E(X_t X_0)$, $t \ge 0$. Rice [Ric39] showed that for such a Gaussian process with differentiable sample paths the number of upcrossings over a high threshold form a stationary point process and evaluated the intensity. For the level u, the process $(X_t)_{t\ge 0}$ has an upcrossing at t_0 if for some $\varepsilon > 0$, $X_t \le u$ in $(t_0 - \varepsilon, t_0)$ and $X_t > u$ in $(t_0, t_0 + \varepsilon)$. From this level-crossing property, a characterization of the extreme behavior of the process has been obtained. This result has been extended by Qualls and Watanabe [QW72], Pickands [Pic69b, Pic69a], Slepian [Sle61], Berman [Ber71] and many others. Nowadays we consider that the crucial assumption to control the time dependence in the framework of Gaussian processes is Berman's condition on the covariance function r, namely that for some $C > 0$ and $\alpha \in (0, 2]$

$$(1.4) \qquad r(t)\ln t \to 0 \quad (t \to \infty) \qquad \text{and} \qquad r(t) = 1 - C\,|t|^\alpha + o(|t|^\alpha) \quad (t \to 0)\,.$$

For surveys on extremes of one-dimensional Gaussian processes, we refer to Leadbetter et al. [LLR83], Leadbetter and Rootzén [LR88], Berman [Ber92], Adler [Adl90], and Albin [Alb90].

The extreme behavior of more general one-dimensional stationary diffusion processes has been investigated by Newell [New62], Berman [Ber64], Mandl [Man68], and Davis [Dav82]. A univariate diffusion process $(X_t)_{t\geq 0}$ can be described as the solution of a stochastic differential equation (SDE) in the sense of Itô

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \qquad t \geq 0\,,$$

where $b : \mathbb{R} \to \mathbb{R}$ denotes the drift, $\sigma : \mathbb{R} \to \mathbb{R}^+$ is the diffusion coefficient, and $(B_t)_{t\geq 0}$ is a standard Brownian motion. To guarantee the stationarity of $(X_t)_{t\geq 0}$, several conditions have to be imposed on the drift term b and the diffusion term σ. The partial maxima of $(X_t)_{t\geq 0}$ are defined as $M_T := \max_{0\leq t\leq T} X_t$, $T > 0$. Note that this definition is related to the first hitting time $T_a := \inf\{t > 0 : X_t = a\}$ of the level $a \in \mathbb{R}$, since $P(M_T \leq a) = P(T_a > T)$ for every $T > 0$ and $a \in \mathbb{R}$. To apply the methods of classical extreme value theory, one has to find a distribution function F such that

$$(1.5) \qquad |P(M_T \leq b) - F(a)^T| \to 0 \qquad (T, a \to \infty)\,.$$

This reduces the asymptotic behavior of the maximum of the process to that of the maximum of i.i.d. random variables distributed according to F in the sense of (1.3). Note that the distribution function F is in general not related to the stationary distribution of the diffusion process.

This result has been proved by several authors using different techniques. An analytic approach was chosen by Newell [New62]. He assumed the existence of a smooth transition probability density that satisfies the Focker-Planck equation. He then expressed the probability $P(M_T \leq b)$ in terms of a formal eigenfunction expansion of the associated Sturm-Liouville problem and gave asymptotically sharp estimates on the eigenvalues and eigenfunctions. Berman [Ber64] presented a probabilistic proof using discrete approximation techniques and the regenerative property of the diffusion process $(X_t)_{t\geq 0}$. Mandl [Man68] showed that the Laplace transform of the properly scaled first hitting time T_a of the level a converges to the Laplace transform of the function e^{-x}, $x \geq 0$, as $a \to \infty$. This convergence result can then be transformed into (1.5). A very elegant approach by Davis [Dav82] is the reduction of the asymptotic distribution of the maximum of a general one-dimensional stationary diffusion process to that of an Ornstein-Uhlenbeck process (linear drift $b(x) = -\alpha x$ and constant diffusion coefficient $\sigma(x) = \sigma$, $x \in \mathbb{R}$). The transformation of the general process was achieved by the technique of random time change using the scale function and the speed measure of the diffusion process.

Borkovec and Klüppelberg [BK98] showed that under appropriate conditions the point process of ε-upcrossings of a one-dimensional diffusion process $(X_t)_{t\geq 0}$ converges in distribution to a homogeneous Poisson process. For $\varepsilon > 0$, the process $(X_t)_{t\geq 0}$ has an ε-upcrossing for the level u at t_0, if $X_t < u$ in $(t_0 - \varepsilon, t_0)$ and $X_{t_0} = u$. In addition, the extreme behavior of term structure models is explicitly studied, such as for the Vašiček model [Vaš77], the Cox-Ingersoll-Ross model [CIR85] and the generalized hyperbolic diffusion, introduced by Bibby and Sørensen [BS97] and by Eberlein and Keller [EU95].

Whereas the extreme behavior of one-dimensional processes seems to be well understood, the multidimensional case still offers mostly open problems. In the framework of Gaussian processes, some research has been done on the extremes of vector-valued Gaussian processes, see Sharpe [Sha78], Lindgren [Lin80a, Lin80b], Aronowich and Adler [AA86], Berman [Ber84], and Albin [Alb90], for instance. A central aspect of their work is the analysis of large fluctuations of the χ^2-process: let $(\xi_t^i)_{t\geq 0}$, $i = 1, \dots, d$, be independent, standardized Gaussian processes; then the χ^2-process with d degrees of freedom is given by $\chi_t^2 := \sum_{i=1}^d (\xi_t^i)^2$, $t \geq 0$. Note that the maximum of the χ^2-process corresponds to the maximum in Euclidean norm of the vector process $(\xi_t^1, \dots, \xi_t^d)$, $t \geq 0$. An extension of Berman's condition (1.4) is crucial in this context. Lindgren [Lin80a, Lin80b] investigated extremes and level crossing properties of the χ^2-process and other functionals of the vector process $(\xi_t^1, \dots, \xi_t^d)$, $t \geq 0$. Berman [Ber84] analyzed the sojourn times of the χ^2-process outside large spheres. For a characterization of the extreme behavior of infinite sequences of independent Gaussian processes and of Gaussian fields, we refer to Berman [Ber80] and Piterbarg [Pit96].

Iscoe and McDonald [IM92, IM89] characterized the asymptotics of the maximum of a multidimensional Ornstein-Uhlenbeck process in Euclidean norm (also the l^2-valued case was considered). Their approach is similar to that of Newell [New62], i.e., they derived the eigenvalue asymptotics for the generator of the process. Aldous [Ald89] investigated also for multidimensional diffusion processes the exponential rate of the probability that the process stays outside a large domain using the heuristic method of Poisson clumping.

The characterization of the extremes of general multidimensional diffusion processes is still an open problem. In this thesis, the extreme behavior of the class of uniformly elliptic, reversible diffusions is investigated.

1.2 Objective and Setup

We introduce the necessary notations and basic facts of the theory. An outline of the most important techniques is given in Section 1.3. A diffusion process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}^n$, $n \in \mathbb{N}$, can be specified by a multidimensional Itô SDE

$$dX_t^i = b^i(X_t)dt + \sum_{j=1}^{n} \sigma^{ij}(X_t)dB_t^j \qquad i = 1, \dots, n\,, \tag{1.6}$$

where $b^i, \sigma^{ij} : \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \dots, n$, are the drift and diffusion coefficients, respectively, and $(B_t^j)_{t\geq 0}$, $j = 1, \dots, n$, are independent one-dimensional standard Brownian motions. Under appropriate conditions on the drift vector b and the diffusion matrix $(\sigma^{ij})_{ij}$, the process $(X_t)_{t\geq 0}$ is stationary and the stationary measure μ is finite.

In a multidimensional framework it is not immediately clear how to measure extremes of the diffusion process $(X_t)_{t\geq 0}$. We say that an extreme event occurs, if $(X_t)_{t\geq 0}$ exits a very large domain of $\mathbb{R}^n$. This corresponds to intuition, since in this case the process moves far away from the center of its stationary distribution. We try to capture such an event in the following definition:

Definition 1.1 An increasing family $(O_R)_{R>R_0}$ of open, bounded subsets of $\mathbb{R}^n$ with smooth boundary which satisfies $\bigcup_{R>R_0} O_R = \mathbb{R}^n$, is called *exhausting family* of $\mathbb{R}^n$. The *distance function* $q : \mathbb{R}^n \to \mathbb{R}$ associated to an exhausting family $(O_R)_{R>R_0}$ is defined by

$$q(x) = \inf\{R > R_0 : x \in O_R\} \qquad x \in \mathbb{R}^n \, . \tag{1.7}$$

Note that the set $\{R > R_0 : x \in O_R\}$ is not empty for every $x \in \mathbb{R}^n$ and hence q is well defined. We have chosen the terminology 'distance function' and not 'semi-norm', since the sets O_R, $R > R_0$, of an exhausting family need not be convex. The simplest example of an exhausting family of $\mathbb{R}^n$ are the open balls $B_R := \{x \in \mathbb{R}^n : |x| < R\}$, $R > 0$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. The associated distance function q then coincides with the Euclidean norm.

We consider the partial maxima of the diffusion process $(X_t)_{t\geq 0}$ in the sense

$$M_T := \max_{0\leq t\leq T} q(X_t) \qquad T \geq 0 \, , \tag{1.8}$$

where q is the distance function associated to some exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$. Note that this definition is related to the *first exit time* of the process $(X_t)_{t\geq 0}$

$$\tau_R := \inf\{s > 0 : X_s \in \mathbb{R}^n \setminus O_R\} \qquad R > R_0 \tag{1.9}$$

of the sets O_R, since we obtain by (1.7)

$$\begin{aligned} P(M_T \leq R) &= P\left(q(X_t) \leq R \;\forall\, t \in [0,T]\right) \\ &= P\left(X_t \in O_R \;\forall\, t \in [0,T]\right) \\ &= P\left(\tau_R > T\right) \qquad R > R_0,\, T > 0 \, . \end{aligned} \tag{1.10}$$

Generalizing results mentioned in the previous section, we do not assume that the diffusion process is Gaussian. The approach of Davis [Dav82] to analyze extremes of one-dimensional diffusion processes has no obvious generalization to the multidimensional setting, since there is no multivariate analogue of the concept of scale function and speed measure, and hence the technique of random time change cannot be used. The method of Newell [New62], i.e., expressing $P(M_T \leq R)$ in terms of spectral properties of the generator of the process and analyzing the eigenvalue asymptotics, is more promising for an extension to the multidimensional framework. For the multivariate Ornstein-Uhlenbeck process, this techniques have already been successfully used by Iscoe and McDonald [IM92, IM89]. The approach takes advantage of the very elaborate machinery of spectral analysis of differential operators which has been developed in mathematical physics with the focus on multidimensional Schrödinger operators. This holds particularly, if the generator of the process has an interpretation as a self-adjoint operator on some Hilbert space, because an even more refined spectral theory is available, see Reed and Simon [RS78] for instance.

We generalize the approach of Iscoe and McDonald [IM92, IM89] for the Ornstein-Uhlenbeck process to the class of diffusion processes of *gradient field type*, i.e., the drift in the SDE (1.6) is given by the gradient of a potential function Φ and the diffusion term is a constant $\sigma > 0$. Such a process is specified by a SDE of the form

$$dX_t^i = -\partial_{x_i}\Phi(X_t)dt + \sigma dB_t^i \qquad i = 1, \ldots, n\,. \tag{1.11}$$

These processes constitute a rich class of diffusions for physical applications but also for financial models. Diffusion processes of gradient field type are reversible and stationary and the stationary measure μ has a Lebesgue density formally given by $e^{-2\Phi(x)/\sigma^2}$, $x \in \mathbb{R}^n$. The generator of such a process admits a self-adjoint extension on some L^2-space. Moreover, there is a very natural extension to the class of uniformly elliptic, reversible diffusion processes, see Section 2.3.

The objective of this thesis is the following: for a uniformly elliptic, reversible diffusion process $(X_t)_{t\geq 0}$, we consider the maxima M_T as defined in (1.8) for a distance function q associated to some exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$. The asymptotic behavior of M_T is characterized in two different manners:

- we derive the tail asymptotics of M_T for fixed $T > 0$, i.e., evaluate the *fine* asymptotics of the convergence $P_\mu(M_T > R) \to 0$ as $R \to \infty$. Here P_μ denotes the law of the process $(X_t)_{t\geq 0}$ starting with its stationary measure μ,
- we evaluate the long term behavior of M_T as $T \to \infty$ in the sense of classical extreme value theory. In particular, a multidimensional analogue to (1.5) is established.

To this aim, we focus on two concrete distance functions q and give conditions to obtain the asymptotic characterization of the maximum M_T. Firstly, we consider the case, where the distance function q is given by the Euclidean norm and hence the exhausting family consists of the open balls B_R, $R > 0$. Secondly, we assume that the exhausting family consists of the level sets of the potential, which is more adapted to the geometry of the problem.

To conclude this section, we give some intuitive explanations of large fluctuations for a diffusion process of gradient field type. The assumption that the stationary measure μ is finite, i.e., that the function $e^{-2\Phi(x)/\sigma^2}$, $x \in \mathbb{R}^n$, is integrable, implies for the potential Φ that $\Phi(x) \nearrow \infty$ as $|x| \to \infty$ (at least almost surely). Intuitively, large fluctuations of a diffusion process $(X_t)_{t\geq 0}$ of gradient field type should occur where $|\nabla\Phi|$ is small (here ∇ denotes the gradient) and hence the effect of the drift $-\nabla\Phi$ that repulses the process to the center of the stationary distribution should not be very strong. This means that extremes of the process $(X_t)_{t\geq 0}$ can be expected in regions where the potential Φ is flat.

Observe the connection with the exit problem of Freidlin-Wentzell and the associated large deviation principle, see also Freidlin and Wentzell [FW84] and Dembo and Zeitouni [DZ98]. For a diffusion process whose diffusion matrix vanishes with order ε, this theory evaluates asymptotically the exit probability of the process of a fixed domain as $\varepsilon \to 0$. The interplay with our situation can be conveniently illustrated for the Ornstein-Uhlenbeck process, i.e., the process $(X_t)_{t\geq 0}$ specified by the SDE (1.11) with potential given by $\Phi^{OU}(x) := (1/2)\sum_{i=1}^n \alpha_i x_i^2$, $x \in \mathbb{R}^n$, with $\alpha_i > 0$, $i = 1, \ldots, n$ (see also Section 4.2.4). Note that Φ^{OU} is of parabolic shape. Consider the maximum of $(X_t)_{t\geq 0}$ in Euclidean norm, i.e., the exhausting family is given by the balls B_R, $R > 0$. Instead of evaluating the first exit time τ_R of $(X_t)_{t\geq 0}$ of the ball B_R, one can also rescale the process to the unit ball B_1 and analyze the first exit time of the unit ball. The rescaled process is defined by $X_{R,t} := R^{-1}X_t$, $t \geq 0$, $R > 0$, and we set $\widehat{\tau}_R := \{t > 0 : X_{R,t} \in \mathbb{R}^n \setminus B_1\}$. By (1.10), we have

$$P(M_T > R) = P(\tau_R \leq T) = P(\widehat{\tau}_R \leq T) \qquad T, R > 0\,.$$

An application of Itô's rule yields that the scaled process $(X_{R,t})_{t\geq 0}$ satisfies the SDE

$$dX^i_{R,t} = \frac{1}{R}\left(-\partial_{x_i}\Phi^{OU}(RX_{R,t})dt + \sigma dB^i_t\right) = -\partial_{x_i}\Phi^{OU}(X_{R,t})dt + \frac{\sigma}{R}dB^i_t \qquad i = 1, \ldots, n\,.$$

Note that the potential Φ^{OU} remains unchanged under this rescaling procedure. For the scaled process $(X_{R,t})_{t\geq 0}$ the Freidlin-Wentzell theory is applicable and we obtain

$$\lim_{R\to\infty} R^{-2}\ln\left(T^{-1}P(\widehat{\tau}_R \leq T)\right) = -\frac{2}{\sigma^2}\min_{|x|=1}\Phi(x) = -\frac{2}{\sigma^2}\min_{1\leq i\leq d}\alpha_i =: c\,. \tag{1.12}$$

This means that $T^{-1}P(\widehat{\tau}_R \leq T)$ behaves on a logarithmic scale asymptotically like e^{-cR^2} as $R\to\infty$. Note that the constant c in (1.12) corresponds to the direction, where the slope of Φ^{OU} is minimal, justifying the intuition mentioned above.

With our approach, we are able to derive the *fine* asymptotics of the probability $P(\tau_R \leq T) = P(M_T > R)$ as $R\to\infty$, not only on a logarithmic scale as in (1.12).

1.3 Main Results and Outline of the Thesis

The purpose of the present work is to analyze for uniformly elliptic, reversible diffusion processes the asymptotic behavior of the maximum M_T defined in (1.8) w.r.t. a distance function q associated to some exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$.

The thesis is divided into three parts. In the first part (Chapters 2 and 3) we explain how the probability $P_\mu(M_T \leq R)$ can be expressed in terms of the principal eigenvalue of the generator of the process. We show how the tail asymptotics and the long term behavior of M_T can be derived if the eigenvalue asymptotics is already known. In the second part (Chapters 4, 5 and 6), the asymptotic behavior of the principal eigenvalue is evaluated in terms of the drift and diffusion coefficient of the process for special exhausting families of $\mathbb{R}^n$. Chapter 7 is devoted to financial applications.

Chapter 2 and 3: A transformation procedure of the problem into the language of operator theory is presented. In Chapter 2 we introduce the main ideas, technical details are deferred to Chapter 3. In order to express $P_\mu(M_T \leq R)$ in terms of spectral properties of some differential operator, we combine the following techniques, see Sections 2.1, 3.1, and 3.2.

The generator L of a diffusion process of gradient field type specified by the SDE (1.11) is a second order differential operator (given formally by (2.6)). Using the theory of Dirichlet forms (see e.g. Fukushima et al. [FOT94] and Ma and Röckner [MR92]), the generator L can be extended to a self-adjoint operator L_∞ on the L^2-space $L^2(\mathbb{R}^n,\mu)$

weighted with the stationary measure μ. In Proposition 3.1, we cite a result of Meyer and Zheng [MZ85] that guarantees also for potentials Φ with singularities the existence of a stationary, weak solution $(X_t)_{t\geq 0}$ of the SDE (1.11). In addition, $(X_t)_{t\geq 0}$ is associated to the operator L_∞ in the sense, that its extended backward semigroup coincides with the semigroup on $L^2(\mathbb{R}^n, \mu)$ generated by L_∞. A result of the theory of Dirichlet forms (see Lemma 3.3) states that the part of the process $(X_t)_{t\geq 0}$ on the set O_R, i.e., the process $(X_t)_{t\geq 0}$ that is killed when it leaves O_R, is associated to the self-adjoint extension L_R on $L^2(O_R, \mu)$ of the generator L with Dirichlet boundary conditions on the set O_R.

This enables us to express $P_\mu(M_T \leq R)$ in terms of the operator L_R. We use an estimation result of Iscoe and McDonald [IM94] on the semigroup generated by the operator L_R (see Proposition 2.1) to derive the fundamental inequality

$$(1 - \lambda_R/\Lambda_{sg})e^{-\lambda_R T} \leq P_\mu(M_T \leq R) \leq e^{-\lambda_R T}\,, \tag{1.13}$$

where λ_R is the bottom eigenvalue of the operator $-L_R$, and Λ_{sg} denotes the spectral gap, see (2.7).

By (1.13), the asymptotic behavior of λ_R as $R \to \infty$ determines that of the probability $P_\mu(M_T \leq R)$. Assuming that an asymptotic expression $l(R)$ for λ_R as $R \to \infty$ has already been derived in terms of σ and Φ appearing in the SDE (1.11), we evaluate in Section 2.2 the tail asymptotics of M_T (Theorem 2.3) by replacing in (1.13) λ_R by the asymptotic expression $l(R)$. Also the long term behavior of M_T is analyzed passing in (1.13) formally to the limit $T \to \infty$, and we derive a multivariate analogue to (1.5), see Theorem 2.5. This leads to limit results for the properly normalized maxima M_T as $T \to \infty$ in the sense of classical extreme value theory, see Corollary 2.6. In Section 3.2, we recall also a standard result to get upper and lower bound on λ_R using suitable test-functions, see Proposition 3.5. Upper bounds are obtained by the variational principle and lower bounds by Temple's inequality. Sections 2.3 and 3.3 are devoted to extensions to uniformly elliptic, reversible diffusions.

Chapter 4: In this chapter, we summarize the results of Kunz [Kun02c]. For a diffusion process of gradient field type, the asymptotics of the bottom eigenvalue λ_R as $R \to \infty$ is evaluated, where the exhausting family is given by the open balls B_R, $R > 0$, see

Theorem 4.1. Hence the maximum M_T defined in (1.8) of the process is considered in Euclidean norm. The idea is to find suitable test-functions $(v_R)_{R>0}$ such that the upper and lower bounds on λ_R in Proposition 3.5 get sharp in the limit $R \to \infty$.

Since the test-functions $(v_R)_{R>0}$ must satisfy Dirichlet boundary conditions on the balls B_R, it is appropriate to use rotationally symmetric functions, which is obviously the correct choice if the potential Φ is spherically symmetric. General non-symmetric potentials Φ are approximated by a rotationally symmetric test-potential ϕ; in many cases, ϕ can be chosen as the spherical minimum of Φ as expected from the exit problem of Freidlin-Wentzell, see (1.12). We show the remarkable fact that also for highly non-symmetric potentials Φ the asymptotics of λ_R can be derived using rotationally symmetric test-functions. The main assumption is an asymptotic growth condition on the asymmetric part of the potential Φ (see Condition (4.5)).

Some examples are presented in Section 4.2. Besides the rotationally symmetric case, also a non-symmetric potential Φ is considered whose asymmetric part factorizes into radial and spherical component (see Section 4.2.2), and the extremes of a diffusion process with a bivariate gamma distribution are evaluated (see Section 4.2.3).

Chapter 5: The results of Kunz [Kun02b] are presented in this chapter. We consider the situation, where the asymptotics of the bottom eigenvalue λ_R cannot be obtained using rotationally symmetric test-functions. This occurs for instance when the exhausting family is different from the balls B_R, $R > 0$, or when the conditions of Chapter 4 are not satisfied. In the latter case, we propose a new exhausting family which is more adapted to the geometry of the problem, namely the level sets of the potential Φ in the SDE (1.11). Note that this exhausting family focusses on regions where large fluctuations of the diffusion process of gradient field type are expected, since the level sets are more extended in the area where the potential Φ is flat.

The eigenvalue asymptotics w.r.t. these not necessarily Euclidean level sets is derived in Theorem 5.2. The test-functions used to obtain sharp upper and lower bounds on λ_R (see Proposition 3.5) must satisfy Dirichlet boundary conditions on the level sets of Φ. Hence, we choose test-functions which are constant on the iso-level sets, i.e., which are of the form $f \circ \Phi$, where f is a real function. Section 5.2 presents some concluding examples.

Chapter 6: The topics of this chapter correspond to those of Kunz [Kun01]. We present an additional, fundamentally different approach to evaluate the asymptotics of the bottom eigenvalue λ_R via singular perturbation methods, see Theorem 6.1. The idea is to derive an asymptotic expansion of the principal eigenfunction corresponding to the bottom eigenvalue λ_R as $R \to \infty$. Singular perturbation techniques, though based on heuristics (the existence of an asymptotic expansion of the principal eigenfunction is a priori assumed), provide a method to obtain the eigenvalue asymptotics that is more intuitive and often easier to implement than techniques of the previous chapters.

This approach works for all exhausting families satisfying a scaling property, see (6.1). We illustrate the techniques for the open balls B_R, $R > 0$, and hence the maximum M_T of the process is considered in Euclidean norm. It is convenient to scale the eigenvalue problem for the bottom eigenvalue λ_R to a fixed domain (here the unit ball B_1). This enables us to evaluate asymptotically the rate of decay of the principle eigenfunctions near the boundary of B_1 enforced by the Dirichlet boundary conditions. It turns out that this decay is not isotropic, but depends on the growth of the potential Φ in different directions.

We show for the examples of Chapter 4, that the eigenvalue asymptotics derived by singular perturbation techniques indeed coincides with that evaluated by the methods of Chapter 4.

Chapter 7: This chapter exhibits financial applications and is an adapted version of Kunz [Kun02a]. We describe three diffusion processes of gradient field type serving as multivariate interest rate modes and evaluate explicitly their extreme behavior, applying the results of the previous chapters.

Besides a multivariate Vašiček model, we present in Section 7.3 a diffusion process with a symmetric exponential distribution as stationary measure, designed to model the semi-heavy tails observed in financial data. A further diffusion process of this kind is introduced whose stationary measure is a bivariate gamma distribution. It allows for spatial dependence that is obtained by copula techniques. This model is proposed as an alternative to the multivariate Cox-Ingersoll-Ross model.

The parameter estimation for these models is fairly easy using maximum likelihood techniques, see Section 7.4. Since the theoretical extreme behavior is known, tests can be constructed to asses the goodness-of-fit of these models to a discrete data set in the extremes, see Section 7.5. Results of parameter estimation and of the goodness-of-fit tests for simulated and real world financial data are presented in Section 7.6.

Chapter 2

Large Fluctuations and Eigenvalue Asymptotics

Let $(O_R)_{R>R_0}$ be an arbitrary exhausting family of $\mathbb{R}^n$ and q the associated distance function according to (1.7). We will show in this chapter how for a stationary reversible diffusion process $(X_t)_{t\geq 0}$ specified in the sequel the asymptotic behavior of the maximum $M_T := \max_{0\leq t\leq T} q(X_t)$ can be characterized in terms of spectral properties of the generator of the process $(X_t)_{t\geq 0}$. Here we want to focus on the main ideas. In order to avoid too many technical details in this chapter, a rigorous treatment of the results about Markov processes and operator theory necessary for the proofs is deferred to Chapter 3.

First we restrict ourselves to the case of diffusion processes of gradient field type. In Section 2.1 we give conditions that guarantee the existence of a weak solution $(X_t)_{t\geq 0}$ of the SDE (1.11) in a certain sense for a quite general class of potentials Φ. Moreover, this process is symmetric w.r.t. to the stationary measure μ and its generator admits a representation in a Hilbert space setting. We state a result giving upper and lower bounds on the probability $P_\mu(M_T \leq R)$ in terms of the first and second eigenvalue of the generator of process $(X_t)_{t\geq 0}$ subject to Dirichlet boundary conditions on the sets O_R, $R > R_0$. Section 2.2 exhibits the asymptotic characterization of M_T, if the behavior of the bottom eigenvalue of the generator is only known asymptotically. A generalization of these results to uniformly elliptic, reversible diffusion processes is given in Section 2.3.

2.1 Weak Solutions and Embedding in L^2-Spaces

A diffusion process of gradient field type specified by the SDE (1.11) is known to be stationary and reversible and the stationary measure μ has a Lebesgue density $\widetilde{\mu}$ on $\mathbb{R}^n$. This density is related formally to the potential Φ appearing in the SDE (1.11) by

$$\widetilde{\mu}(x) = e^{-2\Phi(x)/\sigma^2} \qquad x \in \mathbb{R}^n \,.$$

We will allow that the zero set of $\widetilde{\mu}$ is not empty. This implies that the potential Φ can become singular and take the value $+\infty$.

We formulate conditions on the potential Φ that guarantee the existence of a weak solution of the SDE (1.11). Set

$$\mathcal{Z} := \{x \in \mathbb{R}^n : \Phi(x) = +\infty\}\,, \qquad \mathcal{Z}^c := \mathbb{R}^n \setminus \mathcal{Z}\,. \tag{2.1}$$

Note that $\mathcal{Z}$ coincides with the zero set of the stationary density, i.e., $\mathcal{Z} = \{x \in \mathbb{R}^n : \widetilde{\mu}(x) = 0\}$. Assume the regularity conditions

$$\Phi \in C(\mathbb{R}^n, \mathbb{R} \cup \{+\infty\})\,, \qquad \Phi_{|\mathcal{Z}^c} \in C^1(\mathcal{Z}^c, \mathbb{R})\,, \tag{2.2}$$

where $\Phi_{|\mathcal{Z}^c}$ denotes the restriction of Φ to the set $\mathcal{Z}^c$. These conditions imply that $\widetilde{\mu}$ is continuous, i.e., $\widetilde{\mu} \in C(\mathbb{R}^n, [0,\infty)$. Further assume the integrability condition

$$\int_{\mathcal{Z}^c} e^{-4\Phi(x)/\sigma^2} |\nabla\Phi(x)|^2 dx < \infty\,, \tag{2.3}$$

where ∇ denotes the gradient. A result of Meyer and Zheng [MZ85] cited in Proposition 3.1 states the following: under these conditions there exists a process $(X_t)_{t\geq 0}$ that is symmetric w.r.t. the measure μ with Lebesgue density

$$\widetilde{\mu}(x) = \begin{cases} e^{-2\Phi(x)/\sigma^2} & x \in \mathcal{Z}^c\,, \\ 0 & x \in \mathcal{Z}\,, \end{cases} \tag{2.4}$$

and is a weak solution of the SDE (1.11) up to a terminal time T_∞ (the limit of an increasing sequence of stopping times), where $T_\infty < \infty$ with P_μ probability zero. Here P_μ denotes again the law of $(X_t)_{t\geq 0}$ starting with the stationary measure μ.

Assume further that the stationary measure μ is finite, i.e.,

$$|\mu| := \int_{\mathbb{R}^n} e^{-2\Phi(x)/\sigma^2} dx < \infty \,. \tag{2.5}$$

The infinitesimal generator of the process $(X_t)_{t\geq 0}$ reads formally

$$Lu = \frac{\sigma^2}{2}\Delta u - \sum_{i=1}^{n} \partial_{x_i}\Phi \, \partial_{x_i} u = \frac{\sigma^2}{2} e^{2\Phi/\sigma^2} \sum_{i=1}^{n} \partial_{x_i} \left(e^{-2\Phi/\sigma^2} \partial_{x_i} u \right) \,. \tag{2.6}$$

In Section 3.1 it is shown how the operator L can be rigorously defined in the Hilbert space $L^2(\mathbb{R}^n, \mu)$ by the standard techniques of Dirichlet forms. For $R \in (R_0, \infty]$, there exists a self-adjoint extension L_R acting on $L^2(O_R, \mu)$ of the operator L subject to Dirichlet boundary conditions on O_R (set $O_\infty := \mathbb{R}^n$ and no boundary conditions are present); the operator L_R generates a strongly continuous contraction semigroup $(e^{L_R t})_{t\geq 0}$ on $L^2(O_R, \mu)$. Assume that $-L_\infty$ enjoys the spectral gap property in the sense that

$$\Lambda_{sg} := \inf \Sigma(-L_\infty) \cap (0, \infty) > 0 \,, \tag{2.7}$$

where Σ denotes the spectrum of the operator. This condition is the standard one to ensure ergodicity of the process $(X_t)_{t\geq 0}$. In Proposition 3.7 we state a sufficient condition for (2.7) to hold.

For $R \in (R_0, \infty]$, the bottom of the spectrum of the operator $-L_R$ is denoted by

$$\lambda_R := \inf \Sigma(-L_R) \qquad R > R_0 \,. \tag{2.8}$$

It turns out that λ_R is a simple eigenvalue for $R \in (R_0, \infty)$, see Section 3.2. Proposition 3.1 shows that the process $(X_t)_{t\geq 0}$ is related to the operator L_∞ in the following sense: the backward semigroup of $(X_t)_{t\geq 0}$ admits an extension to the space $L^2(\mathbb{R}^n, \mu)$ which coincides with the semigroup $(e^{L_\infty t})_{t\geq 0}$. Further for $R \in (R_0, \infty)$, the probability $P_\mu(M_T \leq R)$ can be expressed in terms of the semigroup $(e^{L_R t})_{t\geq 0}$, see Lemma 3.3.

The following proposition provides the fundamental estimate, which plays the central role in this thesis. Upper and lower bounds for the probability $P_\mu(M_T \leq R)$ are given in terms of the bottom eigenvalue λ_R and the spectral gap Λ_{sg}. The proof, which is defered to Section 3.2, is based on Theorem 2.13 of Iscoe and McDonald [IM94].

Proposition 2.1 *Let $(X_t)_{t\geq 0}$ be a weak solution of the SDE (1.11) in the sense of Proposition 3.1. Assume that (2.5) and (2.7) hold. Then for every $T > 0$ and sufficiently large $R > 0$*

$$(1 - \lambda_R/\Lambda_{sg})e^{-\lambda_R T} \leq P_\mu(M_T \leq R) \leq e^{-\lambda_R T}\,.$$

Remark 2.2 (1) Without loss of generality is suffices to prove Proposition 2.1 under the following additional assumptions: $\sigma = \sqrt{2}$ and the potential Φ is normalized, so that the stationary density $\widetilde{\mu}$ defined in (2.4) is a probability density on $\mathbb{R}^n$. Given a general $\sigma > 0$ and potential Φ, the result for normalized potentials has to be applied to the modified potential $\Phi_\sigma := (2/\sigma^2)\Phi + \ln|\mu|$. Further the bottom eigenvalue λ_R for the normalized problem has to be multiplied by $\sigma^2/2$, since also the infinitesimal generator L defined in (2.6) is multiplied by this constant.

(2) In Iscoe and McDonald [IM94] the spectral gap property (2.7) plays an important role for the lower bounds on $P_\mu(M_T \leq R) = P_\mu(\tau_R > T)$. The advantage of this approach is that the lower bound can be written asymptotically for small λ_R in the form $(T + \text{const})\lambda_R$, see also Theorem 2.3. The constant becomes unimportant if T is large and this effect allows to analyze the long time behavior of M_T as $T \to \infty$, see also Theorem 2.5. Lower bounds without assuming spectral gap can be obtained using capacity inequalities as in Iscoe and McDonald [IM90]. More precisely, for every $R, T > 0$ and arbitrary $\theta > 0$, setting $\theta_1 := \theta/T$

$$P_\mu(\tau_R > T) \geq 1 - \theta^{-1}e^\theta T\,\mathrm{Cap}_{\theta_1}(\mathbb{R}^n \setminus O_R)\,,$$

where $\mathrm{Cap}_{\theta_1}(\mathbb{R}^n \setminus O_R)$ is the θ_1-capacity of the set $\mathbb{R}^n \setminus O_R$ w.r.t. the process $(X_t)_{t\geq 0}$ (see e.g. Fukushima et al. [FOT94] for a definition). But this lower bound is not sharp enough to evaluate the long time behavior of M_T.

2.2 Exploiting the Eigenvalue Asymptotics

Proposition 2.1 tells us that the asymptotics of $P_\mu(M_T \leq R)$ as $R \to \infty$ is given by the behavior of the bottom eigenvalue λ_R in the limit $R \to \infty$. Evidently $\lambda_R \to 0$ as $R \to \infty$.

Unfortunately λ_R is not directly available in general. It suffices however to have an explicit expression for the convergence $\lambda_R \to 0$ as $R \to \infty$.

We make use of the following *asymptotic notations*: given two real functions a and b, we write $a(t) \sim b(t)$ and $a(t) \lesssim b(t)$ as $t \to t_0 \in \mathbb{R} \cup \{\pm\infty\}$ if $\lim_{t\to t_0} a(t)/b(t) = 1$ and $\limsup_{t\to t_0} a(t)/b(t) \le 1$, respectively. By $a(t) \gtrsim b(t)$ we mean that $b(t) \lesssim a(t)$ as $t \to t_0$ and we write further $a(t) = o(b(t))$ as $t \to t_0$ if $\lim_{t\to t_0} |a(t)/b(t)| = 0$.

> **Aim:** *Find a simple function $l : \mathbb{R}^+ \to \mathbb{R}^+$, given in terms of the potential Φ and the diffusion coefficient σ, such that*
>
> $$\lambda_R \sim l(R) \qquad (R \to \infty)\,. \tag{2.9}$$

The main part of this thesis (Chapter 4–6) consists of giving conditions which allow to find a function l satisfying (2.9) for particular choices of distance functions q, i.e. for particular exhausting families $(O_R)_{R>R_0}$.

Assume for the moment that a function l as required is already given. Replacing in Proposition 2.1 λ_R by the asymptotic expression $l(R)$, sharp asymptotic upper and lower bounds can be obtained for the tail of the maximum M_T for fixed T.

Theorem 2.3 *Assume the situation of Proposition 2.1. Let l be a function satisfying (2.9). Then for every $T > 0$*

$$T\, l(R) \lesssim P_\mu(M_T > R) \lesssim (T + 1/\Lambda_{sg})\, l(R) \quad (R \to \infty)\,.$$

PROOF. Fix $T > 0$. Using Proposition 2.1 and the inequality $1 - x \le e^{-x}$ for every $x \in \mathbb{R}$, we get for sufficiently large $R > 0$

$$1 - e^{-\lambda_R T} \le P_\mu(M_T > R) \;\le\; 1 - (1 - \lambda_R/\Lambda_{sg})e^{-\lambda_R T} \tag{2.10}$$

$$\le\; 1 - (1 - \lambda_R/\Lambda_{sg})(1 - T\lambda_R)$$

$$=\; (T + 1/\Lambda_{sg})\lambda_R + (T/\Lambda_{sg})\lambda_R^2\,. \tag{2.11}$$

We deduce that

$$T\,\lambda_R \lesssim P_\mu(M_T > R) \lesssim (T + 1/\Lambda_{sg})\lambda_R \quad (R \to \infty)\,. \tag{2.12}$$

The left asymptotic inequality follows since $T\lambda_R P_\mu(M_T > R)^{-1} \leq T\lambda_R(1 - e^{-\lambda_R T})^{-1}$ by (2.10). The latter converges to 1 as $R \to \infty$. Dividing inequality (2.11) by the term $(T + 1/\Lambda_{sg})\lambda_R$ and passing to the limit yields the right asymptotic inequality. Since by assumption $\lim_{R\to\infty} \lambda_R/l(R) = 1$, the result follows. □

Theorem 2.3 provides a possibility to compare the maximum M_T with the maximum $\widetilde{M}_T := \max_{0\leq t\leq T} \widetilde{q}(X_t)$ w.r.t. a distance function $\widetilde{q}$ associated to a different exhausting family $(\widetilde{O}_r)_{r>r_0}$ of $\mathbb{R}^n$. The exhausting family $(\widetilde{O}_r)_{r>r_0}$ is called *compatible* to the exhausting family $(O_R)_{R>R_0}$ if the set $\{R > R_0 : \widetilde{O}_r \subset O_R\}$ is not empty for every $r > r_0$. In this case we set

$$R_r := \inf\{R > R_0 : \widetilde{O}_r \subset O_R\} < \infty \qquad r > r_0 . \tag{2.13}$$

The next corollary describes how asymptotic lower bounds for the bottom eigenvalue $\widetilde{\lambda}_r$ associated to $(\widetilde{O}_r)_{r>r_0}$ and hence also for the tail of the maximum $\widetilde{M}_T$ can be evaluated asymptotically.

Corollary 2.4 *Assume that there exists a function l satisfying (2.9) with λ_R associated to $(O_R)_{R>R_0}$. Let $(\widetilde{O}_r)_{r>r_0}$ be an exhausting family of $\mathbb{R}^n$ compatible to $(O_R)_{R>R_0}$. Set $\widetilde{l}(r) := l(R_r)$, $r > r_0$. Then $\widetilde{\lambda}_r \gtrsim \widetilde{l}_r$ as $r \to \infty$ and hence for every $T > 0$*

$$T\,\widetilde{l}(r) \lesssim P_\mu(\widetilde{M}_T > r) \qquad (r \to \infty) .$$

PROOF. The proof is based on a comparison result for eigenvalue, see (3.5). Since $\widetilde{O}_r \subset O_{R_r}$ we have $\widetilde{\lambda}_r \geq \lambda_{R_r}$ for every $r > r_0$. Hence we get for the first exit times $\{\tau_{O_{R_r}} \geq T\} \subset \{\tau_{\widetilde{O}_r} \geq T\}$ and weobtain invoking the relation (1.10) that

$$P_\mu(M_T > R_r) \leq P_\mu(\widetilde{M}_T > r) \qquad r > r_0 .$$

Further $R_r \to \infty$ as $r \to \infty$ since $(O_R)_{R>R_0}$ and $(\widetilde{O}_r)_{r>r_0}$ are exhausting families of $\mathbb{R}^n$. Hence the result follows from assumption (2.9) on the function l and from the left asymptotic inequality of Theorem 2.3. □

We present here a new approach to characterize the long time behavior of M_T as $T \to \infty$ using the fundamental inequality in Proposition 2.1. More precisely, the possibly non-degenerated limit distribution of the properly normalized maximum M_T can be obtained in the limit $T \to \infty$ in the spirit of classical extreme value theory, see Section 1.1. Passing in Theorem 2.3 to the limit $T \to \infty$, the difference between asymptotic upper and lower bound tends to zero, since the term $1/\Lambda_{sg}$ vanishes in this limit. We obtain the following theorem, which is a multi-dimensional analog to (1.5).

Theorem 2.5 *Assume the situation of Proposition 2.1. Let l be a function satisfying (2.9). Then for every sequence $(R_T)_{T>0}$, $R_T \in \mathbb{R}$, with $R_T \nearrow \infty$ as $T \to \infty$*

$$|P_\mu(M_T \le R_T) - e^{-l(R_T)T}| \to 0 \qquad (T \to \infty)\,.$$

PROOF. By assumption $\lambda_R \sim l(R)$ as $R \to \infty$ and hence $\lambda_R = l(R) + \epsilon(R)$ where $\epsilon(R) = o(l(R))$ as $R \to \infty$. Using Proposition 2.1 and the inequality $|e^x - 1| \le |x|(1+e^{|x|})$ for every $x \in \mathbb{R}$, we estimate for fixed $R > R_0$ and $T > 0$, having in mind that $\lambda_R \ge 0$ for every $R > R_0$:

$$\begin{aligned}
\left|P_\mu(M_T \le R) - e^{-l(R)T}\right| &\le e^{-l(R)T} \max\left\{\left|e^{-\epsilon(R)T} - 1\right|, \left|\left(1 - \frac{\lambda_R}{\Lambda_{sg}}\right) e^{-\epsilon(R)T} - 1\right|\right\} \\
&\le e^{-l(R)T}\left(|e^{-\epsilon(R)T} - 1| + \frac{\lambda_R}{\Lambda_{sg}} e^{-\epsilon(R)T}\right) \\
&= e^{-\lambda_R T}\left(|1 - e^{\epsilon(R)T}| + \frac{\lambda_R}{\Lambda_{sg}}\right) \\
&\le e^{-\lambda_R T}\, |\epsilon(R)| T \left|1 + e^{|\epsilon(R)|T}\right| + \frac{\lambda_R}{\Lambda_{sg}} =: I(R) + \frac{\lambda_R}{\Lambda_{sg}}\,.
\end{aligned}$$

Now let $(R_T)_{T\ge 0}$ be an arbitrary sequence with $R_T \nearrow \infty$ as $R \to \infty$. Replace in the above estimations R by R_T. Since $\lim_{R\to\infty} \lambda_R = 0$ we also have $\lim_{T\to\infty} \lambda_{R_T} = 0$.

It remains to show that also $\lim_{T\to\infty} I(R_T) = 0$. We choose an arbitrary sequence $(T_i)_{i\in\mathbb{N}}$ with $T_i \nearrow \infty$ as $i \to \infty$ and we write for short $\lambda_i := \lambda_{R_{T_i}}$, $\epsilon_i := \epsilon_{R_{T_i}}$, and $I_i := I(R_{T_i})$. Assume for the moment that $\{\lambda_i T_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ is bounded. Then $\lim_{i\to\infty} \epsilon_i T_i = 0$, since $\lim_{i\to\infty} \lambda_i^{-1}\epsilon_i = 0$ by definition of $\epsilon(R)$. Since $\lambda_i T_i \ge 0$, it follows

$$I_i \le |\epsilon_i T_i|(1 + e^{|\epsilon_i T_i|}) \to 0 \quad (i \to \infty)\,.$$

If $\{\lambda_i T_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ is unbounded, then $\lambda_i T_i \nearrow \infty$ as $i \to \infty$ after extraction of a subsequence. Since $\lim_{i\to\infty} \lambda_i^{-1}\epsilon_i = 0$, we have $|\epsilon_i| \le \lambda_i/2$ for large i. Hence for large i

$$I_i \le (\lambda_i/2)T_i(e^{-\lambda_i T_i} + e^{-(\lambda_i/2)T_i}) \le \lambda_i T_i e^{-(\lambda_i/2)T_i} \to 0 \quad (i \to \infty)\,.$$

Since the choice of the sequence $(T_i)_{i\in\mathbb{N}}$ was arbitrary, the result follows. □

We want to use Theorem 2.5 to derive the the possibly non-degenerated limit distribution of the properly normalized maximum M_T as $T \to \infty$. We use methods from classical extreme value theory as stated in Section 1.1 (see also Chapter 3 of Embrechts et al. [EKM97]).

Recall from (1.3) that a cumulative distribution function F of some real random variable is in the domain of attraction of an extreme value distribution function H ($F \in \mathrm{DA}(H)$), if there exist norming sequences $(c_T)_{T>0}$ and $(d_T)_{T>0}$ with $c_T > 0$, $d_T \in \mathbb{R}$, such that

$$\lim_{T\to\infty} F(c_T x + d_T)^T = H(x) \qquad x \in \mathbb{R}. \tag{2.14}$$

The extreme value distribution function H is one of the following probability laws, see (1.2): the Gumbel law Λ, the Fréchet law Φ_α, and the Weibull law Ψ_α. This definition characterizes the long term behavior of the running maxima of an i.i.d. sequence of real random variables distributed according to F after appropriate normalization, see (1.1). By Theorem 2.5, the long term behavior of the maxima $M_T := \max_{0\le t\le T} q(X_t)$, $T > 0$, of the process $(X_t)_{t\ge 0}$ can be reduced to that of the maxima of an i.i.d. sequence.

Corollary 2.6 *Assume the situation of Theorem 2.5. Set $F(R) := e^{-l(R)}$, $R > 0$. If $F \in \mathrm{DA}(H)$ for an extreme value distribution H with norming constants $(c_T)_{T>0}$, $(d_T)_{T>0}$ according to (2.14), then for every $x \in \mathbb{R}$*

$$P_\mu\left(c_T^{-1}(M_T - d_T) \le x\right) \to H(x) \qquad (T \to \infty).$$

Proof. Assume that $F := e^{-l} \in \mathrm{DA}(H)$ for an extreme value distribution H with norming constants $(c_T)_{T>0}$, $(d_T)_{T>0}$ according to (2.14). For every $x \in \mathbb{R}$ we set $R_T := c_T x + d_T$. Then,

$$\begin{aligned} &\left|P_\mu\left(c_T^{-1}(M_T - d_T) \le x\right) - H(x)\right| \\ &\quad \le \left|P_\mu(M_T \le R_T) - F(R_T)^T\right| + \left|F(R_T)^T - H(x)\right| . \end{aligned}$$

The first term vanishes by Theorem 2.5 and the second by (2.14) as $T \to \infty$. □

2.3 Uniformly Elliptic Reversible Diffusions

Up to now we have restricted ourselves to diffusion processes of gradient field type. Most of the results obtained in the previous sections can be generalized to uniformly elliptic reversible diffusions.

Consider the general multidimensional SDE (1.6) and define

$$a^{ij}(x) := \frac{1}{2}(\sigma(x)\sigma(x)^T)^{ij} \qquad x \in \mathbb{R}^n,\ i,j = 1,\dots,n\,,$$

where $(\sigma_{ij})^{ij}$ is the diffusion matrix in the SDE (1.6). Assume that $a^{ij} \in C^1(\mathbb{R}^n)$, $i,j = 1,\dots,n$ and that the following uniform ellipticity condition: there exist constants $0 < \alpha_* < \alpha^*$ such that

$$\alpha_*|\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \le \alpha^*|\xi|^2 \qquad x,\xi \in \mathbb{R}^n\,. \tag{2.15}$$

A necessary condition for reversibility is that there exists a positive function $\widetilde{\mu}$ such that the drift in the SDE (1.6) reads formally for $i = 1,\dots,n$

$$b^i(x) = \frac{1}{\widetilde{\mu}(x)}\sum_{j=1}^{n}\partial_{x_j}\left(a^{ij}(x)\widetilde{\mu}(x)\right) \qquad x \in \{x \in \mathbb{R}^n : \widetilde{\mu}(x) > 0\}\,. \tag{2.16}$$

To any uniformly elliptic reversible diffusion we associate a diffusion process of gradient field type specified by the SDE (1.11) with $\Phi = -\ln\widetilde{\mu}$ and $\sigma = \sqrt{2}$. Assume that Φ satisfies Conditions (2.2)-(2.5) and the spectral gap condition (2.7). An extension of Proposition 3.1 guarantees the existence of a weak solution of the SDE (1.6) which is reversible w.r.t. the measure μ with Lebesgue density $\widetilde{\mu}$ on $\mathbb{R}^n$ (see Section 3.3). The following corollary is a generalization of Theorem 2.3 to the class of uniformly elliptic reversible diffusions; the proof is deferred to Section 3.3.

Corollary 2.7 *Let $(X_t)_{t\ge 0}$ be a uniformly elliptic reversible diffusion process. Suppose there exists a function l satisfying $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R corresponds to the associated process of gradient field type. Then for every $T > 0$*

$$\alpha_* T\, l(R) \lesssim P_\mu(M_T > R) \lesssim \alpha^*(T + 1/\Lambda_{sg})\, l(R) \qquad (R \to \infty)\,.$$

Remark 2.8 For Theorem 2.5, which allows to analyze the long term behavior of the maximum M_T, there is no straightforward generalization to uniformly elliptic reversible diffusions. This is due to the fact that in the limit $T \to \infty$, the difference between asymptotic upper and lower bound in Corollary 2.7 does not tend to zero.

Chapter 3

Preliminaries: Markov Processes and Operator Theory

We present here a rigorous treatment of the results of the theory of Markov processes, operator theory, and spectral analysis necessary for the understanding of the present work. We first concentrate on diffusion processes of gradient field type specified by the SDE

$$dX_t^i = -\partial_{x_i}\Phi(X_t)dt + \sigma dB_t^i \qquad i = 1, \dots, n\,,$$

see also (1.11). Assuming that Φ satisfies Condition (2.2), we denote by μ the measure on $\mathbb{R}^n$ with Lebesgue density

$$\widetilde{\mu}(x) = \begin{cases} e^{-2\Phi(x)/\sigma^2} & x \in \mathcal{Z}^c\,, \\ 0 & x \in \mathcal{Z}\,, \end{cases}$$

see also (2.4). We assume without loss of generality according to Remark 2.2.(1) that the potential Φ is normalized, so that

$$\sigma = \sqrt{2} \quad \text{and} \quad \int_{\mathbb{R}^n} \widetilde{\mu}(x)\, dx = 1\,. \tag{3.1}$$

Let $(O_R)_{R>R_0}$ be an arbitrary exhausting family of $\mathbb{R}^n$. We use the following notations: for $R \in (R_0, \infty]$ we denote by μ_R the restriction of μ to the set O_R (where we set $O_\infty := \mathbb{R}^n$). We write for short $L^2_{\mu_R}$ for $L^2(O_R, \mu_R)$ and $\|\cdot\|_{2,R}$ and $(\cdot,\cdot)_R$ for norm and scalar product in $L^2_{\mu_R}$, respectively. Further the indicator function of a set A is denoted by I_A.

In Section 3.1 we describe how a μ-symmetric weak solution of the SDE (1.11) can be constructed, such that the generator of the process and the generator of the part of the process on the set O_R, $R > R_0$, extends to a self-adjoint operator on some L^2-space. Section 3.2 provides some standard results on spectral analysis for self-adjoint operators. Some extensions to uniformly elliptic reversible diffusions are given in Section 3.3.

3.1 Weak Solutions and Embedding in L^2-Spaces

We construct a weak solution of the SDE (1.11) which is symmetric (and hence reversible) w.r.t. the measure μ. We denote by $C_c(\mathbb{R}^n)$ the continuous functions on $\mathbb{R}^n$ with compact support. Recall that for a μ-symmetric process $(X_t)_{t\geq 0}$ the associated backward semigroup $(P_t)_{t\geq 0}$ with $P_t f(x) := E_x[f(X_t)]$, $f \in C_c(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $t \geq 0$, extends to a strongly contraction semigroup on $L^2_{\mu_\infty}$. The generator of this semigroup stands in one-to-one correspondence with a Dirichlet form, see e.g. Chapter 1.4 of Fukushima et al. [FOT94].

We define the following operators and quadratic forms: for $R \in (R_0, \infty]$ set

$$\mathcal{E}'_R(u,v) := \sum_{i=1}^{n} \int_{O_R} \partial_{x_i} u(x)\, \partial_{x_i} v(x)\, \widetilde{\mu}(x)\, dx \qquad u, v \in C_0^2(O_R)\,, \tag{3.2}$$

where $C_0^2(O_R)$ is the set of two times continuously differentiable functions which can be extended continuously by 0 to the boundary of O_R. Note that on the set $\mathcal{Z}^c = \{x \in \mathbb{R}^n : \widetilde{\mu}(x) > 0\}$ the function $\widetilde{\mu}^{-1}$ is continuous by Condition (2.2) and hence is an element of $L^1_{loc}(\mathcal{Z}^c)$. Thus for every $R \in (R_0, \infty]$, the quadratic form $(\mathcal{E}'_R, C_0^2(O_R))$ is closable in $L^2_{\mu_R}$ and its closure $(\mathcal{E}_R, \mathcal{D}(\mathcal{E}_R))$ is a symmetric Dirichlet form, see e.g. Section II.2.(a) in Ma and Röckner [MR92]. Let $(-L_R, \mathcal{D}(L_R))$ be the positive, self-adjoint operator on $L^2_{\mu_R}$ associated to $(\mathcal{E}_R, \mathcal{D}(\mathcal{E}_R))$. Note that L_R is the self-adjoint extension of the operator L defined in (2.6) with Dirichlet boundary conditions on O_R.

We want to show that the space $C_0^2(O_R)$ is a subset of the domain $\mathcal{D}(L_R)$ of the operator L_R. For every $u \in C_0^2(O_R)$, the term $(\sum_i \partial_{x_i}\Phi\, \partial_{x_i} u) I_{\mathcal{Z}^c}$ is an element of $L^2_{\mu_R}$ by means of the Cauchy-Schwartz inequality and the integrability condition (2.3). Hence also $Lu \cdot I_{\mathcal{Z}^c} \in L^2_{\mu_R}$, where $L = (\sigma^2/2)\Delta u - \sum_{i=1}^n \partial_{x_i}\Phi\, \partial_{x_i} u$ is defined in (2.6). Using the continuity of $\widetilde{\mu}$ in $\mathbb{R}^n$ and the differentiability in $\mathcal{Z}^c$ (stated in Condition (2.2)) we obtain that $\mathcal{E}_R(u,v) = (-Lu,v)_R$ for every $v \in C_0^2(O_R)$. This implies (see e.g. Proposition 2.16 of Ma and Röckner [MR92])

$$C_0^2(O_R) \subset \mathcal{D}(L_R) \qquad R \in (R_0, \infty]\,. \tag{3.3}$$

Since the operator $-L_R$ is positive for every $R \in (R_0, \infty]$, the operator L_R induces a strongly continuous contraction semigroup $(e^{L_R t})_{t\geq 0}$ on $L^2_{\mu_R}$.

The connection between the semigroup $(e^{L_\infty t})_{t\geq 0}$ and weak solutions of the SDE (1.11) is provided in the following proposition. The conditions on the potential Φ are the regularity assumption (2.2) and the integrability condition (2.3) on the gradient of Φ. A proof can be found in Meyer and Zheng [MZ85], see also Section 6.3 of Fukushima et al. [FOT94]. Let us define the measurable space $\Omega := C([0,\infty), \mathbb{R}^n)$, equipped with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the canonical projections $(X_t)_{t\geq 0}$.

Proposition 3.1 *Assume that the potential Φ satisfies (2.2) and (2.3). There exists a μ-symmetric diffusion process $\mathbf{X} := (\Omega, (\mathcal{F}_t), (X_t), (P_x)_{x\in\mathbb{R}^n})$ with life time ζ satisfying $P_\mu(\zeta < \infty) = 0$, where μ is the measure with density $\widetilde{\mu}$ defined in (2.4) and $P_\mu := \int_{\mathbb{R}^n} P_x \widetilde{\mu}(x)dx$. Moreover $\mathbf{X}$ never hits the set $\mathcal{Z}$ defined in (2.1) in the sense that $P_\mu(\dot{\sigma}_\mathcal{Z} < \infty) = 0$, where $\dot{\sigma}_\mathcal{Z} := \inf\{t \geq 0 : X_t \in \mathcal{Z}\}$. $\mathbf{X}$ is associated to the Dirichlet form $\mathcal{E}_\infty$, i.e., the $L^2_{\mu_\infty}$-extension of its backward semigroup $(P_t)_{t\geq 0}$ coincides with the semigroup $(e^{L_\infty t})_{t\geq 0}$. $\mathbf{X}$ solves the SDE (1.11) in the following sense: there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times with $T_\infty := \lim_{n\to\infty} T_n$ such that $P_\mu(T_\infty < \infty) = 0$ and that $\mathbf{X}$ is a weak solution of the SDE (1.11) on $[0, T_n)$ for every $n \in \mathbb{N}$.*

Remark 3.2 (1) The basic idea for the construction of the process $\mathbf{X}$ is as follows: starting with the standard Brownian motion $\mathbf{W} := (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x^W)_{x\in\mathbb{R}^n})$ on $\mathbb{R}^n$, the law $(P_x)_{x\in\mathbb{R}^n}$ of the process $\mathbf{X}$ is obtained by a change of measure generated by a multiplicative functional.

(2) Section II of Meyer and Zheng [MZ85] provides a different formulation for $\mathbf{X}$ being a weak solution of the SDE (1.11) without restriction of the time horizon. In this setting however the process can not start in an arbitrary point $x \in \mathbb{R}^n$, but a small set of starting points has to be excluded. Recall that a set $N \subset \mathbb{R}^n$ is *polar* if there exists

a Borel set $N' \subset \mathbb{R}^n$ with $N \subset N'$ such that $P_x(\sigma_{N'} < \infty) = 0$ for every $x \in \mathbb{R}^n$, where $\sigma_{N'} := \inf\{t > 0 : X_t \in N'\}$. Meyer and Zheng have shown that there exists a polar set N such that for every $x \notin N$ the process $(X_t - X_0 + \int_0^t \nabla\Phi(X_s)ds)_{t\geq 0}$ is a standard Brownian motion for the law P_x.

Let $(X_t)_{t\geq 0}$ be the process of gradient field type obtained by Proposition 3.1 and we keep this process fixed for the end of this chapter. For $R \in (R_0, \infty)$, we denote by $(X_t^R)_{t\geq 0}$ the part of $(X_t)_{t\geq 0}$ on O_R, i.e., the process $(X_t)_{t\geq 0}$ killed when it hits the set $\mathbb{R}^n \setminus O_R$. $(X_t^R)_{t\geq 0}$ is μ_R-symmetric and the backward semigroup $(P_t^R)_{t\geq 0}$ of $(X_t^R)_{t\geq 0}$ is given by $P_t^R f(x) := E_x[f(X_t)I_{\{\tau_R > t\}}]$, $f \in C_c(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $t \geq 0$, where τ_R is the first exit time of the set O_R defined in (1.9). We denote by $\mathbf{1}$ the constant function of value 1.

Lemma 3.3 *Let $R \in (R_0, \infty)$.*
(i) $(X_t^R)_{t\geq 0}$ is associated to the Dirichlet form $\mathcal{E}_R$ in the sense that the $L^2_{\mu_R}$-extension of the backward semigroup $(P_t^R)_{t\geq 0}$ of $(X_t^R)_{t\geq 0}$ coincides with the semigroup $(e^{L_R t})_{t\geq 0}$.
(ii) $P_\mu(\tau_R > T) = (e^{L_R T}\mathbf{1}, \mathbf{1})_R$ for every $T > 0$.

Proof. *(i)* Since O_R is open, this follows e.g. from the Theorems 4.4.2 and 4.4.3(i) of Fukushima et al. [FOT94]. *(ii)* Obviously $\mathbf{1} \in L^2_{\mu_R}$. Hence

$$\begin{aligned} P_\mu(\tau_R > T) &= \int_{O_R} P_x(\tau_R > T)\, \widetilde{\mu}(x)dx = \int_{O_R} E_x[I_{\{\tau_R > T\}}]\, \widetilde{\mu}(x)dx \\ &= \int_{O_R} P_T^R \mathbf{1}(x)\, \widetilde{\mu}(x)dx = (e^{L_R T}\mathbf{1}, \mathbf{1})_R . \quad \square \end{aligned}$$

3.2 Some Topics of Spectral Analysis

We describe some spectral properties of the operators L_R, $R \in (R_0, \infty]$. These results are standard in the theory of self-adjoint operators on Hilbert spaces, we refer e.g. to Reed and Simon [RS78]. Recall Definition (2.8) of $\lambda_R := \inf \Sigma(-L_R)$, $R \in (R_0, \infty]$, where $\Sigma(-L_R)$ is the spectrum of the operator $-L_R$ (w.r.t. the space $L^2_{\mu_R}$). Obviously $\Sigma(-L_R) \subset [0, \infty)$ for every $R \in (R_0, \infty]$ and $\lambda_\infty = 0$. Further we set

(3.4) $$\lambda_{R,2} := \inf \Sigma(-L_R) \cap (\lambda_R, \infty) \qquad R \in (R_0, \infty] .$$

For $R \in (R_0, \infty)$, the operator $-L_R$ has discrete spectrum, since the domain O_R is bounded, and hence λ_R is an eigenvalue and is known to be simple. Further $\lambda_{R,2} > \lambda_R \geq 0$ for every $R \in (R_0, \infty]$.

Let us mention a comparison result for eigenvalues: since $O_{R_1} \subset O_{R_2}$ for $R_1, R_2 \in (R_0, \infty]$ with $R_1 < R_2$, there is an obvious embedding of the form domain $\mathcal{D}(\mathcal{E}_{R_1})$ in the form domain $\mathcal{D}(\mathcal{E}_{R_2})$. Hence by the min-max principle

$$\lambda_{R_1} \geq \lambda_{R_2} \qquad R_1, R_2 \in (R_0, \infty],\ R_1 < R_2\,. \tag{3.5}$$

Similarly we obtain that $\lambda_{R,2} \geq \lambda_{\infty,2}$ for $R \in (R_0, \infty]$ and $\lambda_R \searrow 0$ as $R \to \infty$. Note that the spectral gap property (2.7) just states that $\Lambda_{sg} = \lambda_{\infty,2} > 0$.

We cite Theorem 2.13 of Iscoe and McDonalds [IM94] giving upper and lower bounds for the quantity $(e^{L_R T}\mathbf{1}, \mathbf{1})_R$ (appearing in Lemma 3.3.(ii)). Since it is of some importance here, a sketch of the proof is given.

Lemma 3.4 *Suppose that the spectral gap property (2.7) and the simplifying assumption (3.1) hold. Then for every $T > 0$ and sufficiently large $R > 0$*

$$(1 - \lambda_R/\Lambda_{sg})e^{-\lambda_R T} \leq (e^{L_R T}\mathbf{1}, \mathbf{1})_R \leq e^{-\lambda_R T}\,.$$

Sketch of the Proof. For the upper bound we use Cauchy-Schwartz inequality and the fact that μ is a probability measure on $\mathbb{R}^n$. Then,

$$(e^{L_R T}\mathbf{1}, \mathbf{1})_R \leq \|e^{L_R T}\mathbf{1}\|_{2,R}\, \|\mathbf{1}\|_{2,R} \leq \|e^{L_R T}\|_{L^2_{\mu_R} \to L^2_{\mu_R}}\, \|\mathbf{1}\|^2_{2,\infty} \leq e^{-\lambda_R T}\,.$$

For the last inequality note that the norm of the operator $e^{L_R T}$ can be estimated by $\sup\{e^{-\lambda T} : \lambda \in \Sigma(-L_R)\} = e^{-\lambda_R T}$ using the spectral theorem.

To obtain the lower bound, let $\phi \in L^2_{\mu_R}$ be an eigenfunction of $-L_R$ corresponding to the simple eigenvalue λ_R (extended to be 0 outside O_R). $\mathbf{1} \in L^2_{\mu_\infty}$ and hence there exists $\psi \in L^2_{\mu_\infty}$ such that $\mathbf{1} = \phi + \psi$ and $(\phi, \psi)_\infty = 0$. Since $e^{L_R T}$ is a positive operator we obtain

$$\begin{aligned} (e^{L_R T}\mathbf{1}, \mathbf{1})_R &= \|\phi\|^2_{2,R} e^{-\lambda_R T} + (e^{L_R T}\psi, \psi)_R \\ &\geq \|\phi\|^2_{2,\infty} e^{-\lambda_R T} = (1 - \|\psi\|^2_{2,\infty})e^{-\lambda_R T}\,. \end{aligned}$$

We need an upper bound for $\|\psi\|_{2,\infty}^2$. Let $\{E_\lambda : \lambda \in \Sigma(-L_\infty)\}$ be the family of spectral projections associated to $-L_\infty$ and set $\mu_\psi(d\lambda) = d(E_\lambda\psi,\psi)_\infty$. With the simplifying assumption that $-L_\infty$ is bounded (which can be abandoned) we obtain

$$\begin{aligned} \|\psi\|_{2,\infty}^2 &= \int_{\Sigma(-L_\infty)} \mu_\psi(d\lambda) = (E_{\{0\}}\psi,\psi)_\infty + \int_{\Lambda_{sg}}^\infty \mu_\psi(d\lambda) \\ &\leq (\psi,\mathbf{1})_\infty^2 + \frac{1}{\Lambda_{sg}}\int_{\Lambda_{sg}}^\infty \lambda\,\mu_\psi(d\lambda) = \|\psi\|_{2,\infty}^4 + \frac{1}{\Lambda_{sg}}\mathcal{E}_\infty(\psi,\psi)\,. \end{aligned} \tag{3.6}$$

Further $\mathcal{E}_\infty(\psi,\psi) = \mathcal{E}_\infty(\phi,\phi) = \lambda_R\|\phi\|_{2,R}^2 = \lambda_R(1-\|\psi\|_{2,R}^2)$. Plugging this in (3.6), we obtain a quadratic inequality in $\|\psi\|_{2,\infty}^2$, which yields $\|\psi\|_{2,\infty}^2 \leq \min(\lambda_R/\Lambda_{sg},1)$. □

Proof of Proposition 2.1. For the process $(X_t)_{t\geq 0}$ obtained by Proposition 3.1 we have to show the inequality $(1-\lambda_R/\Lambda_{sg})e^{-\lambda_R T} \leq P_\mu(M_T \leq R) \leq e^{-\lambda_R T}$, $T>0$ and $R>0$ sufficiently large. Having (1.10) in mind, the probability $P_\mu(M_T \leq R) = P_\mu(\tau_R > T)$ can be expressed as $(e^{L_R T}\mathbf{1},\mathbf{1})_R$ by Lemma 3.3.(ii). Since the spectral gap condition (2.7) holds, Lemma 3.4 is applicable and this finishes the proof. □

To estimate the bottom eigenvalue λ_R, we use the variational principle for upper bounds and Temple's inequality for lower bounds. For $R \in (R_0,\infty]$ and a function $v \in \mathcal{D}(L_R)$ we define

$$\rho_R(v) := \|v\|_{2,R}^{-2}\mathcal{E}_R(v,v)\,, \quad l_R(v) := \|v\|_{2,R}^{-2}\|L_R v\|_{2,R}^2\,. \tag{3.7}$$

Note that ρ_R is the Rayleigh quotient. We summarize the bounds on λ_R in the following proposition (for a proof see e.g. Theorems XIII.2 and XIII.5 of Reed and Simon [RS78]).

Proposition 3.5 *Let $R \in (R_0,\infty]$. Then for every $v \in \mathcal{D}(L_R)$ with $\rho_R(v) < \lambda_{R,2}$*

$$\rho_R(v) - \frac{l_R(v)-\rho_R(v)^2}{\lambda_{R,2}-\rho_R(v)} \leq \lambda_R \leq \rho_R(v)\,.$$

Remark 3.6 For the lower bound we need to show that $\rho_R(v) < \lambda_{R,2}$. The situation simplifies if we assume the spectral gap property (2.7). Since $\lambda_{R,2} \geq \lambda_{\infty,2} = \Lambda_{sg}$ for every $R \in (R_0,\infty)$, we can replace $\lambda_{R,2}$ by Λ_{sg} in Proposition 3.5 and obtain the following: if $\rho_R(v) < \Lambda_{sg}$ then

$$\rho_R(v) - \frac{l_R(v)-\rho_R(v)^2}{\Lambda_{sg}-\rho_R(v)} \leq \lambda_R\,.$$

Next we state a condition on the potential Φ, so that the spectral gap assumption (2.7) holds. Provided that $\Phi \in C^2(\mathbb{R}^n, \mathbb{R})$, we make use of the fact, that the operator $-L_\infty$ is unitarily equivalent to the Schrödinger operator $-\Delta + V_\Phi$ on $\mathbb{R}^n$ with potential

$$V_\Phi(x) := \tfrac{1}{4}|\nabla\Phi(x)|^2 - \tfrac{1}{2}\Delta\Phi(x) \qquad x \in \mathbb{R}^n . \tag{3.8}$$

This technique is known as ground state transformation. For a function $V : \mathbb{R}^n \to \mathbb{R}$ we use the notation $\liminf_{|x|\to\infty} V(x) := \lim_{R\to\infty} \inf_{|x|>R} V(x)$.

Proposition 3.7 *Suppose* $\Phi \in C^2(\mathbb{R}^n, \mathbb{R})$ *and* $\liminf_{|x|\to\infty} V_\Phi(x) > 0$. *Then the spectral gap property (2.7) holds.*

Proof. Consider the unitary transform

$$U : L^2(\mathbb{R}^n, dx) \to L^2_{\mu_\infty},\ f \mapsto e^{\Phi/2} f ,$$

where dx is the Lebesgue measure on $\mathbb{R}^n$. For functions $u, v \in C_0^2(\mathbb{R}^n)$ we get after some simple calculations using the integration by parts theorem

$$\mathcal{E}_\infty(Uu, Uv) = \sum_{i=1}^n \int_{\mathbb{R}^n} (\partial_{x_i} u\, \partial_{x_i} v + V_\Phi uv)\, dx =: Q_\Phi(u, v) . \tag{3.9}$$

Q_Φ is the quadratic form of the Schrödinger operator $H_\Phi := -\Delta + V_\Phi$ on $L^2(\mathbb{R}^n, dx)$. A standard result in the theory of Schrödinger operators (see e.g. Theorem 3.1 of Berezin and Shubin [BS91]) tells us that $c := \liminf_{|x|\to\infty} V_\Phi(x) > 0$ implies that H_Φ has discrete spectrum in $(-\infty, c)$. Since the transform U is unitary we deduce from (3.9) that the spectra of H_Φ on $L^2(\mathbb{R}^n, dx)$ and of $-L_\infty$ on $L^2_{\mu_\infty}$ coincide. Hence $-L_\infty$ also has a discrete spectrum in $(-\infty, c)$. Thus $\lambda_\infty = 0$ is an eigenvalue and since $c > 0$, the result follows. □

3.3 Uniformly Elliptic Reversible Diffusions

Most of the techniques presented in the previous sections are also working for the class of uniformly elliptic, reversible diffusion processes. Let us construct these class of processes in the same spirit as the diffusion processes of gradient field type.

Consider the general multidimensional SDE (1.6) and assume that $a^{ij} \in C^1(\mathbb{R}^n)$, $i,j = 1,\dots,n$, where $a^{ij}(x) = (1/2)(\sigma(x)\sigma^T(x))^{ij}$ and $(\sigma^{ij})_{ij}$ is the diffusion matrix in the SDE (1.6). We suppose that the uniform ellipticity condition (2.15) holds for the matrix $(a^{ij})_{ij}$. Further assume that the positive function $\widetilde{\mu}$ appearing in the formal definition (2.16) of the drift in the SDE (1.6) is continuous on $\mathbb{R}^n$. We denote by μ the measure on $\mathbb{R}^n$ with Lebesgue density $\widetilde{\mu}$ and we assume that μ is finite. Let us define the associated operators and quadratic forms. Set for $R \in (R_0, \infty]$

$$\mathcal{E}'_{a,R}(u,v) := \sum_{i,j=1}^n \int_{O_R} a^{ij}(x)\, \partial_{x_i} u(x)\, \partial_{x_j} v(x)\, \widetilde{\mu}(x)\ dx\,, \quad u,v \in C_0^2(O_R)\,.$$

This quadratic form is also closable in $L^2_{\mu_R}$ (see e.g. Section II.2.(b) in Ma and Röckner [MR92]) and extends to a symmetric Dirichlet form $(\mathcal{E}_{a,R}, \mathcal{D}(\mathcal{E}_{a,R}))$ with associated positive, self-adjoint operator $(-L_{a,R}, \mathcal{D}(L_{a,R}))$ and strongly continuous contraction semigroup $(e^{L_{a,R}t})_{t\geq 0}$ on $L^2_{\mu_R}$.

Recall the definition of the associated diffusion process of gradient field type specified by the SDE (1.11) with $\Phi = -\ln \widetilde{\mu}$ and $\sigma = \sqrt{2}$. Assume that Φ satisfies the regularity condition (2.2), which implies that $\widetilde{\mu}$ is differentiable in the set $\mathcal{Z}^c = \{\widetilde{\mu} > 0\}$, and the integrabitily condition (2.3). By a modification of Proposition 3.1 it can be shown that there exists a μ-symmetric weak solution of the SDE (1.6) in the sense of Proposition 3.1, where the drift of the SDE (1.6) is given by $b^i(x) = \widetilde{\mu}(x)^{-1} \sum_{j=1}^n \partial_{x_j} (a^{ij}(x)\widetilde{\mu}(x))$, $i = 1,\dots,n$, $x \in \mathcal{Z}^c$, see (2.16). Further the $L^2_{\mu_\infty}$-extension of the associated backward semigroup coincides with the semigroup $(e^{L_{a,\infty}t})_{t\geq 0}$. In the proof of Proposition 3.1, see Remark 3.2.(1), the Brownian motion $\mathbf{W}$ has to be replaced by the strong solution $\mathbf{V}$ of the SDE $dV_t^i = \sum_j \sigma^{ij}(V_t) dW_t^i$, $i = 1,\dots,n$. The existence of the process $\mathbf{V}$ is guaranteed by the uniform ellipticity condition (2.15).

Assume that the spectral gap condition (2.7) is satisfied for the associated diffusion process of gradient field type. Then Lemma 3.3, Lemma 3.4, and hence Proposition 2.1 remain valid if we replace the operator L_R by $L_{a,R}$ and λ_R by $\lambda_R^a := \inf \Sigma(-L_{a,R})$. Comparing $\mathcal{E}_{a,R}$ with the Dirichlet form $\mathcal{E}_R$ (defined in (3.2)) of the associated diffusion process of gradient field type, we obtain

$$\alpha_* \mathcal{E}_R(u,v) \leq \mathcal{E}_{a,R}(u,v) \leq \alpha^* \mathcal{E}_R(u,v) \quad u,v \in \mathcal{D}(\mathcal{E}_R),\ R \in (R_0, \infty]\,.$$

Hence we deduce that

$$\alpha_* \lambda_R \le \lambda_R^a \le \alpha^* \lambda_R \qquad R \in (R_0, \infty]\,, \tag{3.10}$$

where $\lambda_R := \inf \Sigma(-L_R)$ and $-L_R$ is the self-adjoint operator associated to $\mathcal{E}_R$.

Proof of Corollary 2.7. For the associated diffusion process of gradient field type we have derived in the proof of Theorem 2.3 from the fundamental inequality of Proposition 2.1 the asymptotic estimation for the tail of the maximum M_T in the form $T\,\lambda_R \lesssim P_\mu(M_T > R) \lesssim (T + 1/\Lambda_{sg})\lambda_R$ as $R \to \infty$, see (2.12). Since the fundamental inequality of Proposition 2.1 remains valid for uniformly elliptic reversible diffusions replacing λ_R by λ_R^a, we obtain by the same techniques as in the proof of Theorem 2.3

$$T\,\lambda_R^a \lesssim P_\mu(M_T > R) \lesssim (T + 1/\Lambda_{sg})\lambda_R^a \quad (R \to \infty)\,.$$

Using (3.10) and replacing λ_R by its asymptotic expression $l(R)$, since by assumption $\lambda_R \sim l(R)$ as $R \to \infty$, we obtain the desired result

$$\alpha_* T\, l(R) \lesssim P_\mu(M_T > R) \lesssim \alpha^*(T + 1/\Lambda_{sg})\, l(R) \qquad (R \to \infty)\,.$$

□

Chapter 4

The Euclidean Case

In this chapter we analyze for a diffusion process $(X_t)_{t\geq 0}$ of gradient field type the asymptotics of the maximum $M_T := \max_{0\leq t\leq T} |X_t|$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. In this case the exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$ is given by the open balls around the origin with radius R, i.e., $O_R = B_R := \{x \in \mathbb{R}^n : |x| < R\}$, $R > 0$, see ?? in the introduction.

We concentrate on the evaluation of the eigenvalue asymptotics in the spirit of (2.9). We will give conditions, when an asymptotic expression l in terms of the parameters of the process $(X_t)_{t\geq 0}$ can be found, such that $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R is the bottom eigenvalue of the operator $-L_R$ associated to the ball B_R defined in Section 3.1. The tail asymptotics of the maximum M_T for fixed $T > 0$ as well as the long time behavior are then determined by the methods developed in Section 2.2.

The idea is to approximate the potential Φ by a rotationally symmetric potential $x \mapsto \phi(|x|)$, where $\phi \in C^1(\mathbb{R}^+, \mathbb{R})$. If the potential Φ is already rotationally symmetric, the process $(X_t)_{t\geq 0}$ can be identified with a one-dimensional process for which the eigenvalue asymptotics is known, see e.g. Newell [New62]. The influence caused by the asymmetric part of the potential

$$\Phi_{as}(x) := \Phi(x) - \phi(|x|) \quad (\in \mathbb{R} \cup \{\infty\}) \qquad x \in \mathbb{R}^n \tag{4.1}$$

needs to be small (as expressed in the crucial condition (4.5)). The evaluation of the eigenvalue asymptotics is thus reduced to find a rotationally symmetric potential ϕ satisfying certain conditions.

The main tool for the evaluation of the eigenvalue asymptotics is Proposition 3.5. We have to find test-functions $v_R \in \mathcal{D}(L_R)$, $R > 0$ (hence satisfying Dirichlet boundary conditions on B_R), such that the upper and lower bounds in Proposition 3.5 become asymptotically sharp in the limit $R \to \infty$. If the potential Φ is spherically invariant, the test-functions should be rotationally symmetric. We analyze which non-symmetric potentials allow an evaluation of the eigenvalue asymptotics using rotationally symmetric test-functions.

4.1 Main Result and Proof

Let us describe the conditions on the rotationally symmetric test-potential $x \mapsto \phi(|x|)$, where $\phi \in C^1(\mathbb{R}^+, \mathbb{R})$, such that the terms caused by the asymmetric part the potential Φ_{as} of defined in (4.1) can be controlled and admit the evaluation of the eigenvalue asymptotics. To do this, we introduce the following spherical integral: for a function $f : \mathbb{R}^n \to \mathbb{R}$ set

$$m_R[f] := \int_{S^{n-1}} f(R\xi) d\sigma(\xi) \,,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $d\sigma$ the surface measure of S^{n-1}. Note that this corresponds to the integral over the sphere with radius R normalized to the volume γ_n of the unit sphere S^{n-1}. We define two terms measuring the asymmetry of the potential Φ w.r.t. the rotationally symmetric test-potential ϕ. Recall the definition of the density $\widetilde{\mu}$ of the stationary measure μ in (2.4). Set

$$\delta_{as}(R) := e^{2\phi(R)/\sigma^2} m_R[\widetilde{\mu}] = m_R[e^{-2\Phi_{as}/\sigma^2}] \qquad R > 0 \,, \tag{4.2}$$

where $e^{-2\Phi_{as}/\sigma^2} \equiv 0$ on the set $\mathcal{Z}$ defined in (2.1). Note that $\delta_{as}(R)$ is finite for every $R > 0$, since $\widetilde{\mu}$ is continuous. The term $\delta_{as}(R)$ can be interpreted as the spherical mean of the stationary measure of $(X_t)_{t \geq 0}$ relative to the stationary measure of the rotationally symmetric test-potential. If Φ is already rotationally symmetric, then $\delta_{as}(R) \equiv \gamma_n$, $R > 0$. Also the derivative of the asymmetric part Φ_{as} must be small. To make this more precise, we set

$$\Delta_{as}(x) := \begin{cases} \frac{1}{|x|}\sum_{i=1}^{n} x_i\,\partial_{x_i}\Phi(x) - \phi'(|x|) & x \in \mathcal{Z}^c \setminus \{0\}\,, \\ 0 & x \in \mathcal{Z} \setminus \{0\}\,, \end{cases} \tag{4.3}$$

$$D_{as}(R) := e^{2\phi(R)/\sigma^2} m_R[\widetilde{\mu}\,\Delta_{as}^2] = m_R[e^{-2\Phi_{as}/\sigma^2}\Delta_{as}^2] \qquad R > 0\,. \tag{4.4}$$

Note that Δ_{as} is essentially the derivative of Φ_{as} in radial direction. Further, if Φ is already rotationally symmetric, Δ_{as} and hence D_{as} vanish identically. The term D_{as} can be interpreted as the spherical mean of the square of the radial derivative of the asymmetric part Φ_{as} of the potential Φ weighted with the stationary measure corresponding to Φ_{as}.

The crucial condition that we pose on ϕ is the asymptotic relation

$$D_{as}(R) = o\left(\delta_{as}(R)\right) \quad (R \to \infty)\,. \tag{4.5}$$

If Φ is already rotationally symmetric, this condition is trivially satisfied, since $D_{as}(R) \equiv 0$, $R > 0$. For special non-symmetric potentials, Condition (4.5) can often be shown to hold with the help of Laplace's method, see Lemma 4.3 and the examples in Section 4.2.

Further, some weak growth conditions need to be imposed on ϕ. To this aim we define for a measurable real function g

$$\nu[g](R) := \int_1^R r^{1-n} g(r) e^{2\phi(r)/\sigma^2}\,dr\,, \tag{4.6}$$

whenever the integral exists. We set $\nu(R) := \nu[\mathbf{1}](R)$, where $\mathbf{1}$ is the constant function with value 1. Note that the integrand is essentially the reciprocal of the stationary measure associated to the rotationally symmetric potential integrated over the ball B_r. The growth conditions on ϕ are as follows:

$$\nu(R), \nu[\delta_{as}](R) \nearrow \infty\,, \quad \nu[\delta_{as}](R) = o\left(\nu(R)^2\right) \qquad (R \to \infty)\,. \tag{4.7}$$

The second condition implies that δ_{as} does not decay so fast that the growth of $e^{2\phi/\sigma^2}$, the reciprocal of the stationary measure, is destroyed, whereas the third condition forbids δ_{as} to grow faster than $e^{2\phi/\sigma^2}$.

Theorem 4.1 *Assume that Condition (2.5), the finiteness of the stationary measure, and the spectral gap condition (2.7) hold. Further suppose that there exists a function*

$\phi \in C^1(\mathbb{R}^+, \mathbb{R})$ satisfying (4.5) and (4.7). Set

$$l(R) := \frac{\sigma^2}{2|\mu|}\, \delta_{as}(R)\, \nu(R)^{-1} \qquad R > 0\,,$$

where $|\mu|$ the total mass of the stationary measure. Then the function l satisfies (2.9), i.e., $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R is the bottom eigenvalue of the operator $-L_R$ defined in Section 3.1 associated to the ball B_R.

Remark 4.2 (1) As explained in Section 1.2 in the introduction, large fluctuations of a process of gradient field type are expected in regions where the slope of the potential Φ is small. This fact should have an impact on the asymptotic behavior of the bottom eigenvalue λ_R as $R \to \infty$, which determines the asymptotics of the probability $P_\mu(M_T \leq R)$. Thus, a natural candidate for the rotationally symmetric test-potential ϕ is the spherical minimum of the potential Φ, i.e.,

$$\phi(R) := \min\{\Phi(y) : |y| = R\} \qquad R > 0\,.$$

However this is not always the correct choice for ϕ. In Section 4.2.5 we present an example where ϕ can not be chosen as the spherical minimum of the potential Φ.

(2) If ϕ is the spherical minimum of Φ as in Part (1), the condition $\nu[\delta_{as}](R) \nearrow \infty$ as $R \to \infty$ also implies both other conditions in (4.7). This is due to the fact that $0 \leq \delta_{as}(R) \leq 1$ for every $R > 0$. Hence also $0 \leq \nu[\delta_{as}](R) \leq \nu(R)$ for every $R > 0$.

(3) The condition $\phi \in C^1(\mathbb{R}^+, \mathbb{R})$ of Theorem 4.1 is not necessarily satisfied if ϕ is the spherical minimum of Φ as in Part (1). A counterexample can be constructed by means of the potential $\Phi(x_1, x_2) := 1 + x_1(x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2 + 1)^{-1}$. Obviously $\Phi \in C^1(\mathbb{R}^2, \mathbb{R})$ but $\phi(R) = \min\{\Phi(y) : |y| = R\}$ is not differentiable at $R = 1$.

(4) Since $\nu(R) \nearrow \infty$ as $R \to \infty$ by (4.7), the definition of ν in (4.6) is independent of the lower limit of the integral (here chosen to be 1). This is an easy consequence of L'Hôpital's rule.

(5) We use the spectral gap assumption (2.7) to simplify the lower bound on λ_R given by Temple's inequality in Proposition 3.5, see Remark 3.6. If the spectral gap assumption does not hold, one has to show that the Rayleigh quotient $\rho_R(v_R)$ for

the test-function v_R satisfies $\rho_R(v_R) < \lambda_{R,2}$ for large R, where $\lambda_{R,2}$ is the second eigenvalue of the operator $-L_R$, see (3.4). In particular the asymptotic relation $\lambda_R = o(\lambda_{R,2})$ as $R \to \infty$ must be established.

PROOF. According to Remark 2.2.(1) we can make w.l.o.g. the simplifying assumption (3.1), i.e., $\sigma = \sqrt{2}$ and $\int_{\mathbb{R}^n} \widetilde{\mu}(x)\, dx = 1$. The quadratic forms $\mathcal{E}_R$ and the operators L_R defined in Section 3.1 are associated here to the balls B_R, $R > 0$. For a function $v \in \mathcal{D}(\mathcal{E}_R)$, we write $\mathcal{E}_R(v)$ instead of $\mathcal{E}_R(v, v)$ and the norm in $L^2_{\mu_R} := L^2(B_R, \mu_R)$ is again denoted by $\|\cdot\|_{2,R}$.

Step 1: Construction of the test-functions. By assumption, $\phi \in C^1(\mathbb{R}^+, \mathbb{R})$. We deduce from Definition (4.6) that $\nu \in C^2((1,\infty), \mathbb{R}^+)$. ν can be extended to a function $\widetilde{\nu} \in C^2_c((0,\infty), \mathbb{R}^+)$ with compact support in $(0,\infty)$. For $R > 1$ we define test-functions by

$$v_R(x) := 1 - \widetilde{\nu}(|x|)/\nu(R) \qquad |x| \leq R\,.$$

Note that $\nu(R) > 0$ for $R > 1$ and v_R is therefore well defined. By the above construction we have obviously $v_R \in C^2_0(B_R)$ and hence $v_R \in \mathcal{D}(L_R)$ for every $R > 1$, see (3.3).

Step 2: We show that $\mathcal{E}_R(v_R) \sim \nu[\delta_{as}](R)\, \nu(R)^{-2} \to 0$ as $R \to \infty$. Using Step 1 and Definition (4.6), we compute for $R > 1$

$$\begin{aligned} \nu(R)^2 \mathcal{E}_R(v_R) &= \nu(R)^2 \sum_{i=1}^{n} \int_{B_R} |\partial_{x_i} v_R|^2 e^{-\Phi} dx \\ &= \int_0^R r^{n-1} \widetilde{\nu}'(r)^2 m_r[e^{-\Phi}]\, dr \\ &= \left(\kappa_1 + \int_1^R r^{1-n} e^{2\phi(r)} m_r[e^{-\Phi}]\, dr \right) \\ &= (\kappa_1 + \nu[\delta_{as}](R))\;, \end{aligned}$$

where $\kappa_1 = \int_0^1 r^{n-1} \widetilde{\nu}'(r)^2 m_r[e^{-\Phi}] dr < \infty$. Further, $\kappa_1 + \nu[\delta_{as}](R) \sim \nu[\delta_{as}](R)$ as $R \to \infty$ since $\nu[\delta_{as}](R) \nearrow \infty$ as $R \to \infty$ by Assumption (4.7). This proves the asymptotic equivalence, and the convergence $\nu[\delta_{as}](R)\, \nu(R)^{-2} \to 0$ as $R \to \infty$ also follows from Assumption (4.7).

Step 3: The theorem is proved, if the following condition holds:

(4.8) $$\|Lv_R\|^2_{2,R} = o\left(\mathcal{E}_R(v_R)\right) \quad (R \to \infty)\,.$$

Since the spectral gap condition (2.7) holds, Proposition 3.5 is applicable for large R, if we can prove that

$$\rho_R(v_R) = \|v\|_{2,R}^{-2}\mathcal{E}_R(v,v) \to 0 \qquad (R\to\infty)\,, \tag{4.9}$$

where $\rho_R(v_R)$ is the Rayleigh quotient defined in (3.7). Thus, we can deduce that $\lambda_R \sim \rho_R(v_R)$ as $R\to\infty$ if we show (using again Remark 3.6) that

$$\frac{l_R(v_R)-\rho_R(v_R)^2}{\Lambda_{sg}-\rho_R(v_R)} = o\left(\rho_R(v_R)\right) \quad (R\to\infty)\,. \tag{4.10}$$

Let us establish Condition (4.9). Since $\nu(R)\nearrow\infty$ as $R\to\infty$ by Assumption (4.7), we deduce that $v_R\nearrow 1$ μ-a.s. as $R\to\infty$ (where v_R is extended to a function on $\mathbb{R}^n$ by setting it to 0 on $\mathbb{R}^n\setminus B_R$). Since μ is a probability measure on $\mathbb{R}^n$, we have

$$\|v_R\|_{2,R}^2 \to 1 \qquad (R\to\infty)\,. \tag{4.11}$$

Hence (4.9) follows immediately from Step 2. Further, (4.11) allows to replace the terms $l_R(v_R)$ and $\rho_R(v_R)$ in (4.10) in the limit $R\to\infty$ by $\|Lv_R\|_{2,R}^2$ and $\mathcal{E}_R(v_R)$, respectively. Moreover, we can ignore for the asymptotic evaluation the denominator in (4.10), since it converges to the constant Λ_{sg} as $R\to\infty$ by (4.9). But then (4.10) follows from (4.8) using again (4.9) and we obtain $\lambda_R\sim\mathcal{E}_R(v_R)$ as $R\to\infty$. In order to obtain the asymptotic expression $l(R)$ for λ_R we use Step 2 and the fact that $\nu[\delta_{as}](R)\,\nu(R)^{-1}\sim\delta_{as}(R)$ as $R\to\infty$ (which is a consequence of L'Hôpital's rule, applicable since $\nu(R)\nearrow\infty$ as $R\to\infty$ by Assumption (4.7)). To obtain the general asymptotic expression $l(R)$ for arbitrary $\sigma>0$ and not normalized potential Φ, we refer to Remark 2.2.(1).

Step 4: Condition (4.8) holds. We evaluate the term $\|Lv_R\|_{2,R}^2$. Recalling the alternative form of the operator L defined in (2.6), we calculate for $x\in\mathcal{Z}^c$ with $1<|x|<R$

$$\begin{aligned}
\nu(R)Lv_R(x) &= \nu(R)e^{\Phi(x)}\sum_{i=1}^n \partial_{x_i}\left(e^{-\Phi}\partial_{x_i}v_R\right)(x)\\
&= e^{\Phi(x)}\sum_{i=1}^n \partial_{x_i}\left(e^{-\Phi(x)}|x|^{-n}e^{\phi(|x|)}x_i\right)\\
&= e^{\phi(|x|)}\Big\{\frac{n}{|x|^n}+\sum_{i=1}^n x_i\Big[-\frac{n}{|x|^{n+1}}\frac{x_i}{|x|}+\frac{1}{|x|^n}\Big(\phi'(|x|)\frac{x_i}{|x|}-\partial_{x_i}\Phi(x)\Big)\Big]\Big\}\\
&= -|x|^{1-n}e^{\phi(|x|)}\Delta_{as}(x)\,.
\end{aligned}$$

Setting $\kappa_2 := \|L\widetilde{\nu}\|_{2,1}^2 < \infty$ (the function $x \mapsto \widetilde{\nu}(|x|)$ is also denoted by $\widetilde{\nu}$), we have for $R > 1$

$$\begin{aligned}\nu(R)^2 \|Lv_R I_{\mathcal{Z}^c}\|_{2,R}^2 &= \kappa_2 + \int_{B_R \setminus B_1} |x|^{2(1-n)} e^{2\phi(|x|)-\Phi(x)} \Delta_{as}(x)^2 dx \\ &= \kappa_2 + \int_1^R r^{1-n} e^{2\phi(r)} m_r[e^{-\Phi}\Delta_{as}^2]\, dr \\ &= \kappa_2 + \nu[D_{as}](R)\,.\end{aligned}$$

In particular, $Lv_R I_{\mathcal{Z}^c} \in L^2_{\mu_R}$. Using Step 2, we see that Condition (4.8) is satisfied if $\nu[D_{as}](R) = o\,(\nu[\delta_{as}](R))$ as $R \to \infty$. Since $\nu[\delta_{as}](R) \nearrow \infty$ as $R \to \infty$ by Assumption (4.7), this follows immediately from the crucial condition (4.5) by an application of L'Hôpital's rule. $\square$

4.2 Examples

We give some examples of diffusion processes of gradient field type for which the sharp eigenvalue asymptotics can be evaluated by Theorem 4.1. Apart from the rotationally symmetric case, we consider the situation of non-symmetric processes where the asymmetric part of the potential factorizes in radial and spherical components. Further a process is presented whose stationary measure is a bivariate gamma distribution. The long term behavior of the normalized maximum is explicitly derived for the Ornstein-Uhlenbeck process. We also describe a situation where the eigenvalue asymptotics is not determined by the spherical minimum of the potential, see Remark 4.2.(1).

For some special potentials, the crucial condition (4.5) of Theorem 4.1 can be shown by means of Laplace's method stated in the next lemma (for a proof see e.g. Theorem 7.1 of Olver [Olv74])

Lemma 4.3 (Laplace's method) *Let $I \subset \mathbb{R}$ be an open interval containing 0 and $p \in C^1(I)$, $q \in C(I)$, where p attains its minimum only at 0. Assume further that there exist constants $P, \varpi, \eta > 0$ and $Q \in \mathbb{R}$ such that $p(\theta) - p(0) \sim P\theta^{\varpi}$, $p'(\theta) \sim \varpi P \theta^{\varpi-1}$, and $q(\theta) \sim Q\theta^{\eta-1}$ as $\theta \to \pm 0$. If $J(x) := \int_I e^{-xp(\theta)} q(\theta)\, d\theta$ converges absolutely for large x, then $J(x) \sim 2Q\varpi^{-1}\Gamma(\eta/\varpi)(Px)^{-\eta/\varpi} e^{-xp(0)}$ as $x \to \infty$.*

Remark 4.4 Laplace's method is also applicable if the only minimum of p occurs at an endpoint of the interval I. Then the function J has to be multiplied by 1/2.

The following lemma is used for the asymptotic evaluation of integrals over exponential terms appearing in the function ν defined in (4.6).

Lemma 4.5 *Let $A, \gamma > 0$ and $\delta \in \mathbb{R}$. Then*

$$\int_1^R r^\delta e^{Ar^\gamma} dr \sim (\gamma A)^{-1} R^{\delta-\gamma+1} e^{AR^\gamma} \quad (R \to \infty) .$$

Proof. Apply L'Hôpital's rule to the quotient. □

4.2.1 The Rotationally Symmetric Case

Assume that the potential Φ in the SDE (1.11) has the property that there exist $R_0 > 0$ and $\phi \in C^2([R_0, \infty), \mathbb{R})$ such that

$$\Phi(x) = \phi(|x|) \qquad |x| > R_0 . \tag{4.12}$$

Note that a diffusion process of gradient field type with rotationally symmetric potential can be reduced to a one-dimensional process. As mentioned in Section 1.1, the asymptotic behavior of the running maxima for one-dimensional diffusions has been studied by many authors using different techniques. Here we only refer to Newell [New62] who derived the eigenvalue asymptotics for the one-dimensional problem in a similar way as in our approach. We describe here how the techniques developed in the preceding sections work in the situation of rotationally symmetric potentials.

Suppose that Φ satisfies (2.2) and (2.3). Then Proposition 3.1 guarantees the existence of a weak solution $(X_t)_{t \geq 0}$ of the SDE (1.11). Set

$$V(R) := \frac{1}{4}\phi'(R)^2 - \frac{1}{2}\left(\phi''(R) + \frac{n-1}{R}\phi'(R)\right) \qquad R > R_0 .$$

To satisfy the spectral gap condition, we assume

$$\liminf_{R \to \infty} V(R) > 0 . \tag{4.13}$$

Note that this condition is satisfied for instance if ϕ has polynomial form $\phi(R) = R^\alpha$ where $\alpha \geq 1$. The volume of the unit sphere S^{n-1} in $\mathbb{R}^n$ is denoted by γ_n.

Theorem 4.6 *Let $(X_t)_{t\geq 0}$ be a diffusion process of gradient field type specified by the SDE (1.11) with potential Φ of the form (4.12). Assume Condition (2.5), the finiteness of the stationary measure, and Condition (4.13). Then the result of Theorem 4.1 holds with ϕ defined in (4.12) and $\delta_{as}(R) \equiv \gamma_n$ for every $R > R_0$.*

Remark 4.7 (a) The lower integration limit in the definition of ν in (4.6) is replaced by R_0 (see Remark 4.2.(4)).

(b) In the article of Newell [New62], the spectral gap condition (4.13) was not needed, since the decay of the second eigenvalue $\lambda_{R,2}$ could be controlled, see Remark 4.2.(5). For this aim, a suitable approximation of the second eigenfunction has been derived. This approximation however can not be transfered to the multidimensional setting in an obvious manner.

Proof. We have to show that Conditions (2.7), (4.7), and (4.5) are satisfied. The spectral gap condition (2.7) holds by Proposition 3.7 and Assumption (4.13). As stated already in Section 4.1, $\delta_{as}(R) \equiv \gamma_n$ and $D_{as}(R) \equiv 0$ for $R > R_0$. Hence the crucial condition (4.5) holds immediately. Since $r^{n-1}e^{-2\phi(r)/\sigma^2} \to 0$ as $r \to \infty$ by (2.5), the growth condition (4.7) is also satisfied (see Remark 4.2.(2)). □

4.2.2 Non-Symmetric Processes

We now turn to consider the situation where the potential Φ in the SDE (1.11) is not rotationally symmetric. For notational convenience we restrict ourselves to the two-dimensional case and use polar coordinates writing $\mathbb{R}^2 \setminus \{0\} \ni x = Re_\theta$, where $R = |x| > 0$ and $e_\theta = (\cos\theta, \sin\theta)$, $\theta \in [\theta_0, \theta_0 + 2\pi)$, with $\theta_0 \in [0, 2\pi)$ (the same symbol is used for functions in Cartesian as well as in polar coordinates). The extension to the multidimensional case is straightforward. Assume $\Phi \in C^2(\mathbb{R}^2, \mathbb{R})$. If Φ also satisfies (2.2) and (2.3), Proposition 3.1 guarantees the existence of a weak solution $(X_t)_{t\geq 0}$ of the SDE (1.11).

We set $\phi(R) := \min\{\Phi(y) : |y| = R\}$ as in Remark 4.2.(1). In order to show the crucial condition (4.5) by means of Laplace's method, we assume further that the asymmetric part of the potential factorizes in radial and spherical component, i.e., that there exist

$R > R_0$ and functions $\psi \in C^2((R_0,\infty),\mathbb{R}^+)$ and $p \in C^2(S^1,\mathbb{R}^+)$ such that

$$\Phi_{as}(R,\theta) = \Phi(R,\theta) - \phi(R) = p(\theta)\psi(R) \qquad R > R_0,\ \theta \in [\theta_0, \theta_0 + 2\pi)\,. \tag{4.14}$$

Note that $\Phi_{as} \geq 0$ by definition and that the minimum in the definition of ϕ is attained, since Φ is continuous and the minimum is taken over a compact set. Hence $\psi \geq 0$, $p \geq 0$ and the zero-set $\mathcal{N}(p) := \{\theta : p(\theta) = 0\}$ is not empty. We assume that $\mathcal{N}(p)$ is finite, i.e.,

$$\mathcal{N}(p) = \{\theta_1, \ldots, \theta_N\}\,, \tag{4.15}$$

where w.l.o.g. $\theta_0 < \theta_1 < \ldots < \theta_N < \theta_0 + 2\pi$. By Assumption (4.15) we can find open disjoint intervals I_i containing θ_i such that $[\theta_0, \theta_0 + 2\pi] = \bigcup_{i=1}^N \overline{I_i}$, where $\overline{I_i}$ denotes the closure of I_i. Further we assume that for every $i = 1, \ldots, N$

$$\begin{aligned} &p(\cdot + \theta_i) \text{ satisfies the conditions of Lemma 4.3 on } \{\theta : \theta + \theta_i \in I_i\} \\ &\text{with corresponding constants } P_i, \varpi_i > 0. \end{aligned} \tag{4.16}$$

We set $\varpi_* := \max\{\varpi_i : i = 1, \ldots, N\}$ and $J_* := \{i : \varpi_* = \varpi_i\}$. Further we need that

$$\psi(R) \to \infty \qquad (R \to \infty)\,. \tag{4.17}$$

To enforce the spectral gap condition (2.7) we assume that

$$\begin{gathered} \liminf_{R\to\infty} \phi'(R) > 0\,, \\ \{\phi''(R), \psi''(R), R^{-1}\psi'(R), R^{-2}\psi(R)\} \text{ are } o\left(\phi'(R)^2\right) \quad (R \to \infty)\,. \end{gathered} \tag{4.18}$$

In this setting, the crucial condition (4.5) of Theorem 4.1 takes the form of a regularity condition on ψ

$$\psi'(R) = o\left(\psi(R)\right) \quad (R \to \infty)\,. \tag{4.19}$$

The growth conditions (4.7) then read

$$\int_{R_0}^R r^{-1}\psi(r)^{-1/\varpi_*} e^{2\phi(r)/\sigma^2}\, dr \nearrow \infty \quad (R \to \infty)\,. \tag{4.20}$$

Theorem 4.8 *Let $(X_t)_{t\geq 0}$ be a diffusion process of gradient field type solving the SDE (1.11), where Φ is of the form (4.14). Assume (2.5) and (4.15)-(4.19). Set*

$$l(R) \;:=\; C\psi(R)^{-1/\varpi_*} \left(\int_{R_0}^R r^{-1} e^{2\phi(r)/\sigma^2}\, dr\right)^{-1} \qquad R > R_0\,,$$

where $C := 2(|\mu|\varpi_*)^{-1}(\sigma^2/2)^{1+1/\varpi_*}\Gamma(1/\varpi_*)\sum_{i\in J_*}P_i^{-1/\varpi_*}$. *Then* l *satisfies (2.9), i.e.,* $\lambda_R \sim l(R)$ *as* $R\to\infty$*, where* λ_R *is the bottom eigenvalue of the operator* $-L_R$ *defined in Section 3.1 associated to the ball* B_R.

Remark 4.9 (a) Assume that ϕ,ψ are of polynomial form, i.e., $\phi(R)=R^\alpha$, $\psi(R)=R^\beta$ for large R. Then Conditions (4.18) and (4.20) are satisfied if $\alpha\geq 1$ and $\beta\in(0,2\alpha)$.

(b) If ψ is regularly varying with index $\gamma>0$ as $R\to\infty$, i.e., $\lim_{R\to\infty}\psi(R)^{-1}\psi(tR)=t^\gamma$ for every $t>0$, then ψ satisfies Condition (4.19), since by Karamata's theorem

$$R\psi'(R)\psi(R)^{-1}\to 1+\gamma \qquad (R\to\infty)\,,$$

see e.g. Theorem A3.6 in Embrechts et al. [EKM97]. Especially (4.19) holds if ψ is of polynomial form.

Proof. We have to show that Conditions (2.7), (4.7), and (4.5) of Theorem 4.1 are satisfied. First we evaluate the term δ_{as}. Invoking (4.16) and Laplace's method (Lemma 4.3), where by (4.17) the limes $x\to\infty$ can be replaced by $\psi(R)\to\infty$ as $R\to\infty$, we calculate

$$\begin{aligned}\delta_{as}(R) &= m_R[e^{-2\Phi_{as}/\sigma^2}]\\ &= \sum_{i=1}^N\int_{I_i} e^{-2p(\theta+\theta_i)\psi(R)/\sigma^2}\,d\theta\\ &\sim \sum_{i=1}^N \frac{2}{\varpi_i}\Gamma\left(\frac{1}{\varpi_i}\right)\left(\frac{2P_i}{\sigma^2}\psi(R)\right)^{-1/\varpi_i}\\ &\sim \frac{1}{\varpi_*}\left(\frac{\sigma^2}{2}\right)^{1/\varpi_*}\Gamma\left(\frac{1}{\varpi_i}\right)\left(\sum_{i\in J_*}P_i^{-1/\varpi_*}\right)\psi(R)^{-1/\varpi_*}\quad (R\to\infty)\,. \end{aligned}\tag{4.21}$$

The growth condition (4.7) follows immediately from Assumption (4.20) and Remark 4.2.(2). To show the crucial condition (4.5), we need to evaluate asymptotically the term $D_{as}(R)$. Δ_{as} reads in polar coordinates

$$\Delta_{as}(R,\theta)=\partial_R\Phi(R,\theta)-\phi'(R)=p(\theta)\psi'(R)\,.$$

Using Laplace's method (Lemma 4.3) and (4.17) we calculate

$$D_{as}(R) \;=\; m_R[e^{-2\Phi_{as}/\sigma^2}\Delta_{as}^2]$$

$$
\begin{aligned}
&= \psi'(R)^2 \sum_{i=1}^{N} \int_{I_i} e^{-2p(\theta+\theta_i)\psi(R)/\sigma^2} p(\theta+\theta_i)^2 d\theta \\
&\sim \psi'(R)^2 \sum_{i=1}^{N} K_i \psi(R)^{-(2\varpi_i+1)/\varpi_i} \\
&\sim K\left((\psi'/\psi)^2 \psi^{-1/\varpi_*}\right)(R) \quad (R \to \infty)\,,
\end{aligned}
\tag{4.22}
$$

where K and K_i are positive constants. From (4.21) and (4.22) we obtain that

$$
D_{as}(R)\delta_{as}(R)^{-1} \sim K'(\psi'/\psi)^2(R) \qquad (R \to \infty)\,,
$$

where $K' > 0$ is a further constant. Hence the crucial condition (4.5) holds by the regularity condition (4.19) on the function ψ. It remains to show that the spectral gap condition (2.7) holds. We will do this with the help of Proposition 3.7. To obtain lower bonds on the function V_Φ defined in (3.8), we can estimate $|\nabla\Phi(R,\theta)| \geq \phi'(R)^2$ uniformly in θ. Using the fact that p and p'' are bounded on S^1, we can find a constant $K > 0$ such that the following estimation holds uniformly in θ for every $R > 0$

$$
|\Delta\Phi(R,\theta)| \leq K \left\{ |\phi''(R)| + |\psi''(R)| + \frac{|\phi'(R)| + |\psi'(R)|}{R} + \frac{|\psi(R)|}{R^2} \right\}.
$$

By Condition (4.18) every term between the braces is $o\left(\phi'(R)^2\right)$ as $R \to \infty$. Since by Condition (4.18) also $\liminf_{R\to\infty} \phi'(R) > 0$, we have $\liminf_{|x|\to\infty} V_\Phi(x) > 0$ and the spectral gap condition (2.7) holds by Proposition 3.7. Thus the result of Theorem 4.1 holds and by (4.21) we also obtain the desired form of the asymptotic expression $l(R)$ for the eigenvalue asymptotics. □

4.2.3 A Diffusion Process with Gamma Distribution

We present here the method to generate a diffusion process of gradient field type by specifying first the stationary measure. To illustrate this method, we construct a two-dimensional stationary diffusion process living only in the positive quadrant. The stationary measure μ of this process is given by the product measure of two gamma distributions. For this new process, the eigenvalue asymptotics can be evaluated by Theorem 4.1.

The density $\widetilde{\mu}$ we choose for the stationary measure μ shall be given by

$$\widetilde{\mu}(x) := \begin{cases} \prod_{i=1}^{2} \left(\beta_i^{\alpha_i}\Gamma(\alpha_i)\right)^{-1} x_i^{\alpha_i-1} e^{-x_i/\beta_i} & x_1, x_2 > 0\,, \\ 0 & \text{otherwise}, \end{cases} \tag{4.23}$$

where $\alpha_1, \alpha_2 \geq 1$ and $0 < \beta_2 \leq \beta_1$. In order to construct a stationary diffusion process of gradient field type with $\sigma = \sqrt{2}$ having the stationary measure μ as above, the potential Φ needs to be set to

$$\Phi(x_1, x_2) := \begin{cases} \sum_{i=1}^{2} x_i/\beta_i - (\alpha_i - 1)\ln x_i & x_1, x_2 > 0, \\ \infty & \text{otherwise}. \end{cases} \tag{4.24}$$

If $\alpha_1, \alpha_2 > 3/2$, then Φ satisfies (2.2) and (2.3). By Proposition 3.1 there exists a weak solution $(X_t)_{t\geq 0}$ of the SDE (1.11) with Φ as above and $\sigma = \sqrt{2}$. In polar coordinates Φ reads for $R > 0$

$$\Phi(R,\theta) = R\left(\frac{\cos\theta}{\beta_1} + \frac{\sin\theta}{\beta_2}\right) - (\alpha_1 + \alpha_2 - 2)\ln R - \ln\left((\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1}\right)$$

for $\theta \in (0, \pi/2)$ and $\Phi(R,\theta) = +\infty$ for $\theta \in [\pi/2, 2\pi]$. We choose ϕ as the spherical minimum of the non-logarithmic term of Φ, i.e.,

$$\phi(R) := R/\beta_1 \qquad R > 0\,. \tag{4.25}$$

Note that ϕ coincides at least asymptotically with the spherical minimum of Φ, since the logarithmic term vanishes against the linear term in the limit $R \to \infty$, see also Remark 4.2.(1).

Theorem 4.10 *Let $(X_t)_{t\geq 0}$ be the two-dimensional stationary diffusion process of gradient field type specified by the SDE (1.11) with $\sigma = \sqrt{2}$ and Φ defined in (4.24), where $0 < \beta_2 \leq \beta_1$ and $\alpha_1, \alpha_2 > 3$. Set*

$$l(R) := \beta_1^{-(\alpha_1+1)}\left(\frac{R^{\alpha_1-1}}{\Gamma(\alpha_1)} + \delta_{\beta_1\beta_2}\beta_1^{\alpha_1-\alpha_2}\frac{R^{\alpha_2-1}}{\Gamma(\alpha_2)}\right)e^{-R/\beta_1}\,,$$

where $\delta_{\beta_1\beta_2} = 1$ if $\beta_1 = \beta_2$ and $= 0$ otherwise. Then l satisfies (2.9), i.e., $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R is the bottom eigenvalue of the operator $-L_R$ defined in Section 3.1 associated to the ball B_R.

Remark 4.11 (a) The conditions $\alpha_1, \alpha_2 > 3$ ensure the spectral gap property (2.7). This condition may be relaxed, see also Remarks 2.2.(2) and 4.2.(5).

(b) The stationary measure of this process is the product measure of two independent gamma distributions. This measure can be replaced by a bivariate distribution with gamma distributed marginals implementing spatial depencence, see Section 7.3.3. Such a distribution can be created by means of copula techniques, we refer to Joe [Joe97]. In this case, the crucial condition (4.5) is shown to hold by numerical methods.

PROOF. We show that Conditions (2.5), (2.7), (4.7), and (4.5) of Theorem 4.1 are satisfied. The total mass of the stationary measure is $|\mu| = \prod_{i=1}^{2} \beta_i^{\alpha_i}\Gamma(\alpha_i)$, see (4.23), and hence Condition (2.5) holds. First we evaluate the term δ_{as}.

$$\begin{aligned} \delta_{as}(R) &= \int_0^{\pi/2} R^{\alpha_1+\alpha_2-2}(\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1}e^{-R\left(\frac{\cos\theta}{\beta_1}+\frac{\sin\theta}{\beta_2}-\frac{1}{\beta_1}\right)}d\theta \\ &= R^{\alpha_1+\alpha_2-2}\int_0^{\pi/2}(\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1}e^{-Rp(\theta)}d\theta\,, \end{aligned} \tag{4.26}$$

where $p(\theta) := \beta_1^{-1}(\cos\theta - 1) + \beta_2^{-1}\sin\theta$, $\theta \in [0, \pi/2]$. To evaluate this integral asymptotically using Laplace's method (Lemma 4.3), we need to know the zero points $\mathcal{N}(p) = \{\theta : p(\theta) = 0\}$. It can be seen that $\mathcal{N}(p) := \{0, \pi/2\}$ if $\beta_1 = \beta_2$ and $\mathcal{N}(p) = \{0\}$ if $\beta_1 > \beta_2$. Note that as $\theta \searrow 0$

$$\begin{aligned} &p(\theta) \sim \theta/\beta_2\,, \quad (\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1} \sim \theta^{\alpha_2-1}\,, \\ &p(\tfrac{\pi}{2}-\theta) \sim \theta/\beta_1\,, \quad \left(\cos(\tfrac{\pi}{2}-\theta)\right)^{\alpha_1-1}\left(\sin(\tfrac{\pi}{2}-\theta)\right)^{\alpha_2-1} \sim \theta^{\alpha_1-1}\,, \end{aligned}$$

if $\beta_1 = \beta_2$ is the latter case. We obtain invoking Laplace's method (Lemma 4.3 and Remark 4.4)

$$\begin{aligned} \delta_{as}(R) &\sim R^{\alpha_1+\alpha_2-2}\left(\Gamma(\alpha_2)\left(\frac{R}{\beta_2}\right)^{-\alpha_2} + \delta_{\beta_1\beta_2}\Gamma(\alpha_1)\left(\frac{R}{\beta_1}\right)^{-\alpha_1}\right) \\ &= \beta_2^{\alpha_2}\Gamma(\alpha_2)R^{\alpha_1-2} + \delta_{\beta_1\beta_2}\beta_1^{\alpha_1}\Gamma(\alpha_1)R^{\alpha_2-2} \quad (R \to \infty)\,. \end{aligned} \tag{4.27}$$

It is easily seen by the definition of ϕ that the growth condition (4.7) holds, see also Remark 4.2.(2). To show the crucial condition (4.5), the term $D_{as}(R)$ must be evaluated asymptotically. Δ_{as} reads in polar coordinates

$$\Delta_{as}(R,\theta) = \frac{\cos\theta}{\beta_1} + \frac{\sin\theta}{\beta_2} - \frac{\alpha_1+\alpha_2-2}{R} - \frac{1}{\beta_1} = p(\theta) - \frac{\alpha_1+\alpha_2-2}{R}\,.$$

Hence we obtain, analogously to the calculation in (4.27)

$$
\begin{aligned}
D_{as}(R) &= \int_0^{\pi/2} R^{\alpha_1+\alpha_2-2}(\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1}\times \\
&\quad \times e^{-Rp(\theta)}\left(p(\theta)-\frac{\alpha_1+\alpha_2-2}{R}\right)^2 d\theta \\
&\leq \kappa_1\left\{\frac{\delta_{as}(R)}{R^2}+R^{\alpha_1+\alpha_2-2}\int_0^{\pi/2}(\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1}p(\theta)^2e^{-Rp(\theta)}d\theta\right\} \\
&\lesssim \kappa_2\left\{\frac{\delta_{as}(R)}{R^2}+R^{\alpha_1+\alpha_2-2}\left(R^{-(\alpha_2+2)}+\delta_{\beta_1\beta_2}R^{-(\alpha_1+2)}\right)\right\} \\
&= \kappa_2\left\{\frac{\delta_{as}(R)}{R^2}+R^{\alpha_1-4}+\delta_{\beta_1\beta_2}R^{\alpha_2-4}\right\} \qquad (R\to\infty)\,, \qquad (4.28)
\end{aligned}
$$

where κ_1 and κ_2 are positive constants. Comparing (4.28) with (4.27) we see that the crucial condition (4.5) is satisfied. It remains to show that the spectral gap condition (2.7) holds. Since the two components of $(X_t)_{t\geq 0}$ are independent, it suffices to prove the spectral gap condition for the generator $L^{(1)}$ of the first component of the process. As in the proof of Proposition 3.7, $-L^{(1)}$ is unitarily equivalent to the Schrödinger operator $Hu := -u'' + Vu$ on $\mathbb{R}$ with

$$
V(x) = \begin{cases} \frac{1}{4}(\frac{1}{\beta_1}-\frac{\alpha_1-1}{x})^2-\frac{\alpha_1-1}{2x^2} = \frac{1}{4\beta_1^2}-\frac{\alpha_1-1}{2x}+(\frac{\alpha_1-1}{2}-1)\frac{\alpha_1-1}{2x^2} & x>0, \\ \infty & x\leq 0. \end{cases}
$$

Note that $\lim_{x\to\infty}V(x) = (2\beta_1)^{-2} > 0$ and $\lim_{x\searrow 0}V(x)=\infty$ since $\alpha_1 > 3$. Hence H and also $-L^{(1)}$ have spectral gap by Proposition 3.7. Thus Theorem 4.1 is applicable and we obtain the eigenvalue asymptotics using (4.27) and Lemma 4.5

$$
\begin{aligned}
\lambda_R &\sim |\mu|^{-1}\delta_{as}(R)\left(\int_1^R r^{-1}e^{r/\beta_1}dr\right)^{-1} \\
&\sim \left(\frac{R^{\alpha_1-2}}{\beta_1^{\alpha_1}\Gamma(\alpha_1)}+\delta_{\beta_1\beta_2}\frac{R^{\alpha_2-2}}{\beta_2^{\alpha_2}\Gamma(\alpha_2)}\right)\left(\beta_1R^{-1}e^{R/\beta_1}\right)^{-1} \\
&= \beta_1^{-(\alpha_1+1)}\left(\frac{R^{\alpha_1-1}}{\Gamma(\alpha_1)}+\delta_{\beta_1\beta_2}\beta_1^{\alpha_1-\alpha_2}\frac{R^{\alpha_2-1}}{\Gamma(\alpha_2)}\right)e^{-R/\beta_1} \qquad (R\to\infty)\,. \qquad \square
\end{aligned}
$$

4.2.4 The Ornstein-Uhlenbeck Process

We analyze here explicitly the eigenvalue asymptotics for the Ornstein-Uhlenbeck process (OU process) in one and two dimensions. It turns out that the eigenvalue asymptotics differs significantly in the rotationally symmetric case and in the non-symmetric case. In particular, the behavior in the extremes of the two-dimensional non-symmetric process is very similar to the behavior of the one-dimensional process. Further, e discuss the effects of symmety breaking. In addition, the long term behavior of the normalized maxima in Euclidean norm is presented (see Corollary 2.6).

The OU process is of gradient field type specified by the SDE (1.11), where the diffusion coefficient $\sigma > 0$ is arbitrary and the potential Φ in two dimensions is given by

$$\Phi(x_1, x_2) = \tfrac{1}{2}(\alpha x_1^2 + \beta x_2^2) \qquad x_1, x_2 \in \mathbb{R},\ 0 < \alpha \le \beta\,,$$

and in one dimension by $\Phi(x) = (\alpha/2)x^2$, $x \in \mathbb{R}$.

• *The symmetric case*, i.e., the one-dimensional and the two-dimensional case for $\alpha = \beta$. Setting $\phi(R) = (\alpha/2)R^2$, the conditions of Theorem 4.6 are obviously satisfied (for dimension $n = 1, 2$). For $n = 1$ we obtain from Theorem 4.6 using Lemma 4.5 (here $\gamma_1 = 2$, $|\mu| = \sqrt{\sigma^2\pi\alpha^{-1}}$)

$$\begin{aligned}\lambda_R &\sim \sigma^2\sqrt{\frac{\alpha}{\sigma^2\pi}}\left(\int_1^R e^{\alpha r^2/\sigma^2}\,dr\right)^{-1} \\ &\sim \sqrt{\frac{\sigma^2\alpha}{\pi}}\left(\frac{\sigma^2}{2\alpha}R^{-1}e^{\alpha R^2/\sigma^2}\right)^{-1} = 2\sqrt{\frac{\alpha^3}{\sigma^2\pi}}Re^{-\alpha R^2/\sigma^2} \quad (R \to \infty)\,.\end{aligned}$$

Hence the asymptotic expression l_α for the bottom eigenvalue satisfying (2.9) can be chosen for $\alpha > 0$

$$l_\alpha(R) = C_\alpha R e^{-\alpha R^2/\sigma^2}\,, \qquad \text{where } C_\alpha = 2\sqrt{\frac{\alpha^3}{\sigma^2\pi}}\,. \tag{4.29}$$

Note that this characterizes the maximum of the *absolute value* of a one-dimensional OU process. Similarly, for $n = 2$ and $\alpha = \beta$ (here $\gamma_2 = 2\pi$, $|\mu| = \sigma^2\pi\alpha^{-1}$)

$$\begin{aligned}\lambda_R &\sim 2\pi\frac{\sigma^2}{2}\frac{\alpha}{\sigma^2\pi}\left(\int_1^R r^{-1}e^{\alpha r^2/\sigma^2}\,dr\right)^{-1} \\ &\sim \alpha\left(\frac{\sigma^2}{2\alpha}R^{-2}e^{\alpha R^2/\sigma^2}\right)^{-1} = \frac{2\alpha^2}{\sigma^2}R^2e^{-\alpha R^2/\sigma^2} \quad (R \to \infty)\,.\end{aligned} \tag{4.30}$$

Hence the asymptotic expression $l_{\alpha\alpha}$ for the bottom eigenvalue satisfying (2.9) can be chosen

$$l_{\alpha\alpha}(R) = C_{\alpha\alpha}R^2e^{-\alpha R^2/\sigma^2}\,, \qquad \text{where } C_{\alpha\alpha} = \frac{2\alpha^2}{\sigma^2}\,. \tag{4.31}$$

• *The non-symmetric case*, i.e., the two-dimensional case for $\alpha < \beta$. This process is a special example of the class of non-symmetric processes introduced in Section 4.2.2. The potential Φ in the present setting, written in polar coordinates, is of the form (4.14) with

$$\phi(R) = \frac{\alpha}{2}R^2,\ \psi(R) = \frac{\beta-\alpha}{2}R^2,\ p(\theta) = \sin^2\theta \qquad R > 0,\ \theta \in \left[-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right) .$$

We show that the conditions for Theorem 4.8 are satisfied. Condition (2.5) holds with $|\mu| = \sigma^2\pi(\alpha\beta)^{-1/2}$ and (4.15) is also satisfied since $\mathcal{N}(p) = \{0,\pi\}$. (4.16) holds with $\varpi_i = 2$, $P_i = 1$ for $i = 1,2$ and hence $\varpi_* = 2$. (4.17) is obvious and (4.18)-(4.20) also hold, since ϕ and ψ are polynomials, see Remark 4.9. Thus we get from Theorem 4.8 together with Lemma 4.5

$$\begin{aligned}
\lambda_R &\sim \pi^{-1}\sqrt{\frac{\sigma^2\alpha\beta}{2}}\,\Gamma\left(\frac{1}{2}\right)\left(\frac{\beta-\alpha}{2}R^2\right)^{-1/2}\left(\int_1^R r^{-1}e^{\alpha r^2/\sigma^2}dr\right)^{-1} \\
&\sim \sqrt{\frac{\sigma^2\alpha\beta}{\pi(\beta-\alpha)}}R^{-1}\left(\frac{\sigma^2}{2\alpha}R^{-2}e^{\alpha R^2/\sigma^2}\right)^{-1} \\
&= 2\sqrt{\frac{\alpha^3\beta}{\sigma^2\pi(\beta-\alpha)}}Re^{-\alpha R^2/\sigma^2} \qquad (R\to\infty)\,.
\end{aligned}$$

Hence the function $l_{\alpha\beta}$ satisfying (2.9) can be chosen

(4.32) $$l_{\alpha\beta}(R) = C_{\alpha\beta}Re^{-\alpha R^2/\sigma^2}\,, \qquad \text{where } C_{\alpha\beta} = 2\sqrt{\frac{\alpha^3\beta}{\sigma^2\pi(\beta-\alpha)}}\,.$$

• *Effects of symmetry breaking:* A comparison of the two-dimensional rotationally symmetric and the non-symmetric OU process shows that the asymptotic expressions $l_{\alpha\alpha}$ and $l_{\alpha\beta}$ for the bottom eigenvalue differ both in the pre-exponential factor and the constant. Most notably, the pre-exponential factor is reduced from R^2 to R. On the other hand, the extreme fluctuations of the two-dimensional non-symmetric OU process behave like that of the one-dimensional OU process (compare (4.32) with (4.29)), only the constant is different. If we let the steeper direction of the potential Φ (here the x_2-direction with growing parameter $\beta > \alpha$) become infinitely steep, i.e., if $\beta \to \infty$, we expect that the two-dimensional process converges in law to the one-dimensional projection. Indeed, we observe for the constants (4.32) and (4.29) in the eigenvalue asymptotics that $C_{\alpha\beta} \to C_\alpha$ as $\beta \to \infty$. To the contrary, if the asymmetric potential tends to the symmetric one, i.e., $\beta \searrow \alpha$, there is no convergence $l_{\alpha\alpha} \to l_{\alpha\beta}$ as $\beta \searrow \alpha$.

A similar effect of symmetry breaking can be observed if we look at the tail of the Euclidean norm of a bivariate normally distributed random variable. More precisely, given a bivariate normal random variable $X \sim N(0,\Sigma)$, where $\Sigma = \mathrm{diag}(\alpha,\beta)$ with $0 < \alpha \leq \beta$, the asymptotic behavior of $P(|X| > R)$ as $R \to \infty$ is different in the symmetric case ($\alpha = \beta$) and in the non-symmetric case ($\alpha < \beta$).

• *Long term behavior of normalized maxima:* In view of Corollary 2.6 we show that $F := e^{-l}$, $l \in \{l_\alpha, l_{\alpha\beta}, l_{\alpha\alpha}\}$, is in the domain of attraction of the Gumbel distribution Λ ($F \in \mathrm{DA}(\Lambda)$). Note that every $l \in \{l_\alpha, l_{\alpha\beta}, l_{\alpha\alpha}\}$ is of the form $l(R) = CR^\gamma e^{-\alpha R^2/\sigma^2}$, where

$$(4.33) \qquad \begin{cases} C = C_\alpha,\ \gamma = 1 & \\ C = C_{\alpha\beta},\ \gamma = 1 & \text{if} \quad l = l_{\alpha\beta}, \\ C = C_{\alpha\alpha},\ \gamma = 2 & \end{cases} \begin{matrix} l = l_\alpha, \\ \\ l = l_{\alpha\alpha}. \end{matrix}$$

For results of classical extreme value theory used in the sequel we refer to §3.3.3 of Embrechts et al. [EKM97]. It can be shown that F is a so called *Von Mises* function (this follows e.g. from $\lim_{R\to\infty}(1 - F(R))F''(R)/(F'(R))^2 = -1$) and hence $F \in \mathrm{DA}(\Lambda)$. The norming constants in (2.14) can be obtained from the relations

$$F(d_T) = 1 - 1/T\,, \quad c_T = (1 - F(d_T))/F'(d_T)\,,$$

see e.g. Theorem 3.3.26 of [EKM97]. A careful asymptotic expansion of these relations as $T \to \infty$ leads to the following choice of the norming constants

$$(4.34) \quad c_T = \frac{1}{2}\sqrt{\frac{\sigma^2}{\alpha \ln T}}\,, \quad d_T = \sqrt{\frac{\sigma^2 \ln T}{\alpha}} + \frac{\gamma}{4}\sqrt{\frac{\sigma^2}{\alpha \ln T}}\left(\ln\ln T + \ln(C^{2/\gamma}\sigma^2/\alpha)\right)\,.$$

Hence we obtain from Corollary 2.6 that $P_\mu(c_T^{-1}(M_T - d_T) \leq x) \to \Lambda(x)$ for every $x \in \mathbb{R}$ as $T \to \infty$ with c_T and d_T as above, where in d_T the correct values have to be plugged in for the the constants γ and C as in (4.33) depending on $l \in \{l_\alpha, l_{\alpha\beta}, l_{\alpha\alpha}\}$.

• *Spectral gap:* For the OU process, the spectral gap Λ_{sg} defined in (2.7) can be evaluated explicitly. The generator $L_\infty^{(1)}$ for the one-dimensional OU process with state space $\mathbb{R}$, restricted to functions in $C_0^2(\mathbb{R}) \subset \mathcal{D}(L_\infty^{(1)})$, has the form $L_\infty^{(1)}u = (\sigma^2/2)u'' - \alpha u'$, $u \in C_0^2(\mathbb{R})$. The spectrum of $-L_\infty^{(1)}$ is given by (see e.g. Section 5.5.1 of Risken [Ris89])

$$\Sigma(-L_\infty^{(1)}) = \{\alpha n : n = 0, 1, 2, \ldots\}\,.$$

Hence we obtain for the spectral gap

$$\Lambda_{sg}^{(1)} = \inf \Sigma(-L_\infty^{(1)}) \cap (0,\infty) = \alpha > 0\,.$$

For the two-dimensional OU process with state space $\mathbb{R}^2$, the generator $L_\infty^{(2)}$ restricted to functions in $C_0^2(\mathbb{R}^2) \subset \mathcal{D}(L_\infty^{(2)})$ reads $L_\infty^{(2)} u = (\sigma^2/2)\Delta u - \sum_{i=1}^2 \alpha_i\, \partial_{x_i} u$, $u \in C_0^2(\mathbb{R}^2)$. Since $L_\infty^{(2)}$ is the tensor product of two one-dimensional operators of the form $L_\infty^{(1)}$, the spectrum of $-L_\infty^{(2)}$ is given by

$$\Sigma(-L_\infty^{(2)}) = \{\alpha_1 n_1 + \alpha_2 n_2 : n_1, n_2 = 0, 1, 2, \ldots\}\,.$$

This implies for the spectral gap

$$\Lambda_{sg}^{(2)} = \inf \Sigma(-L_\infty^{(2)}) \cap (0,\infty) = \min\{\alpha_1, \alpha_2\} > 0\,. \tag{4.35}$$

4.2.5 A Counterexample

In the preceding examples it turned out, that the spherical minimum of the potential Φ was a suitable choice for the rotationally symmetric test-potential ϕ, which is used for the evaluation of the eigenvalue asymptotics, see Remark 4.2.(1). But this is not always the correct choice. We present here a diffusion process of gradient field type with potential Φ, where the eigenvalue asymptotics can be obtained, if we choose for the test-potential the spherical *maximum* of Φ, i.e., $\phi(R) := \max\{\Phi(y) : |y| = R\}$, $R > 0$. We construct the potential Φ in such a way that the gap, where the minimum of Φ occurs, becomes too narrow as $R \to \infty$ to influence the large fluctuations of the process.

Set $\sigma = \sqrt{2}$ in the SDE (1.11) and the potential Φ is as follows: let $g \in C_c^\infty((-1,1),[0,1])$ be a function with $g(\theta) = 1 - \theta^2$ in a neighborhood of 0 and the maximum of g is only attained in 0. Using polar coordinates, we set for $R > 0$

$$p(R,\theta) := \begin{cases} g(e^{R^2}\theta) & |\theta| < e^{-R^2}\,, \\ 0 & e^{-R^2} \le |\theta| \le \pi\,. \end{cases}$$

Note that $p(R,\cdot) \in C^\infty(S^1)$ for every $R > 0$. We define the potential Φ by

$$\Phi(R,\theta) := R^2 - (R^2 - R)p(R,\theta) \qquad R > 1, \theta \in [-\pi,\pi)\,,$$

and Φ can be extended to $C^\infty(\mathbb{R}^2, \mathbb{R})$. This means that Φ is essentially the potential of a two-dimensional symmetric OU process but with a gap occurring near the angle $\phi = 0$. Observe that

$$\min_{\theta\in[-\pi,\pi]} \Phi(R,\theta) = R \qquad \max_{\theta\in[-\pi,\pi]} \Phi(R,\theta) = R^2 \qquad R > 1\,,$$

and we choose $\phi(R) = R^2$, $R > 1$, for the rotationally symmetric test-potential.

Obviously, the stationary measure is finite, since $\int_{|x|>1} e^{-\Phi} dx \le \int_1^\infty re^{-r} dr < \infty$. Hence Condition (2.5) is satisfied. We evaluate the term $\delta_{as}(R)$ using Laplace's method (Lemma 4.3)

$$\begin{aligned} \delta_{as}(R) &= \int_{-\pi}^{\pi} e^{(R^2-R)p(R,\theta)} d\theta \\ &= (2\pi - 2e^{-R^2}) + \int_{-e^{-R^2}}^{e^{-R^2}} e^{(R^2-R)g(e^{R^2}\theta)} d\theta \\ &= (2\pi - 2e^{-R^2}) + e^{-R^2} \int_{-1}^{1} e^{(R^2-R)g(\eta)} d\eta \\ &= (2\pi - 2e^{-R^2}) + e^{-R} \int_{-1}^{1} e^{-(R^2-R)(1-g(\eta))} d\eta \\ &\sim 2\pi + e^{-R}\sqrt{\frac{\pi}{R^2-R}} \sim 2\pi \qquad (R\to\infty)\,. \end{aligned} \tag{4.36}$$

Hence the growth condition (4.7) holds since we obviously have $\nu(R) \nearrow \infty$ as $R \to \infty$ by the choice of ϕ. Further, we have to evaluate the term $D_{as}(R)$. Note that

$$\Delta_{as}(R,\theta) = \begin{cases} -(2R-1)g(e^{R^2}\theta) - 2R(R^2-R)e^{R^2} g'(e^{R^2}\theta)\theta & |\theta| < e^{-R^2} \\ 0 & e^{-R^2} \le |\theta| \le \pi \end{cases}$$

Hence we can calculate

$$\begin{aligned} D_{as}(R) &= \int_{-e^{-R^2}}^{e^{-R^2}} e^{(R^2-R)g(e^{R^2}\theta)} \Delta_{as}(R,\theta)^2 d\theta \\ &\le K_1 e^{-R^2} \int_{-1}^{1} e^{(R^2-R)g(\theta)} \Big[(2R-1)^2 g(\eta)^2 + R^2(R^2-R)^2 e^{2R^2} g'(\eta)^2 e^{-2R^2}\eta^2\Big] d\eta \\ &\le K_2 R^6 e^{-R} \int_{-1}^{1} e^{-(R^2-R)(1-g(\theta))} \left[g(\eta)^2 + g'(\eta)^2\eta^2\right] d\eta\,, \end{aligned}$$

where K_1, K_2 are suitable positive constants. By Laplace's method (Lemma 4.3), the last integral is asymptotically equivalent to $\Gamma(5/2)(R^2-R)^{-5/2}$ as $R \to \infty$. Hence $D_{as}(R) \to 0$ as $R \to \infty$ and invoking (4.36), the crucial condition (4.5) of Theorem 4.1 is satisfied.

Thus, (4.36) and Theorem 4.1 imply together with Lemma 4.5 that the eigenvalue asymptotics is given by

$$\lambda_R \sim \frac{2\pi}{|\mu|}\left(\int_1^R r^{-1}e^{r^2}dr\right)^{-1} \sim \frac{4\pi}{|\mu|}R^2e^{-R^2} \qquad (R \to \infty)\,,$$

where $|\mu|$ is the total mass of the stationary measure. This eigenvalue asymptotics corresponds to that of the two-dimensional symmetric OU process, see (4.30).

Chapter 5

The Level Set Case

In Chapter 4 we evaluated for a diffusion process of gradient field type the asymptotics as $R \to \infty$ of the bottom eigenvalue λ_R of the operator $-L_R$, $R > 0$, associated to the balls B_R around the origin with radius $R > 0$. Hence, the maximum of the corresponding diffusion process was considered w.r.t the Euclidean norm. Conditions were given, when sharp eigenvalue asymptotics can be obtained using rotationally symmetric test-functions. If these conditions fail or if we replace the balls $(B_R)_{R>0}$ by an arbitrary exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$, we can no longer use rotationally symmetric test-functions for the evaluation of the eigenvalue asymptotics, since the test-functions must satisfy Dirichlet boundary conditions.

It is not a realistic task to determine for a diffusion process of gradient field type the eigenvalue asymptotics for any arbitrary exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$. Therefore, we choose an exhausting family of $\mathbb{R}^n$ which is adapted to the geometry of the problem, namely the level sets of the potential Φ in the SDE (1.11) itself, i.e.,

$$O_R^\Phi := \{x \in \mathbb{R}^n : \Phi(x) < R\} \qquad R > R_0 := \inf_{x \in \mathbb{R}^n} \Phi(x) . \tag{5.1}$$

Conditions that guarantee that the sets O_R^Φ, $R > R_0$, are an exhausting family of $\mathbb{R}^n$ in the sense of Definition 1.1 are given in Section 5.1.

The choice of the level sets as exhausting family has the following advantage: large fluctuations of the process $(X_t)_{t \geq 0}$ of gradient field type solving the SDE (1.11) are expected in regions where the potential Φ is flat. The level sets of Φ are more extended in this regions and hence, the sets O_R^Φ, $R > R_0$, stress the directions of large fluctuations of $(X_t)_{t\geq 0}$.

Note that in this situation, the distance function q appearing in the definition of the maximum M_T coincides with the potential itself, see (1.7) and (1.8). Hence, we analyze the maximum of the process $(X_t)_{t\geq 0}$ in the form $M_T := \max_{0\leq t\leq T} \Phi(X_t)$.

The test-functions that we need for the evaluation of the eigenvalue asymptotics have to satisfy Dirichlet boundary conditions on the boundary of the sets O_R^Φ, $R > R_0$. This suggests, that we should use test-functions which are constant on the iso-level sets of Φ, i.e., that are of the form $f \circ \Phi$, where f is a real function. We give conditions when the sharp eigenvalue asymptotics can be obtained by means of test-functions of the shape described above.

5.1 Main Result and Proof

In order to apply the generator L, the second order differential operator defined in (2.6), to the test-functions described above, we assume

$$\Phi \in C^2(\mathbb{R}^n, \mathbb{R})\,. \tag{5.2}$$

In particular, we do not allow Φ to take the value $+\infty$. This implies that $\bigcup_{R>R_0} O_R^\Phi = \mathbb{R}^n$. To guarantee that the level sets form an exhausting family of $\mathbb{R}^n$, we suppose that

$$\Phi(x) \to \infty \qquad (|x| \to \infty)\,. \tag{5.3}$$

Hence R_0 in Definition (5.1) is finite. Moreover the sets O_R^Φ, $R > R_0$, are open and bounded (by Condition (5.3)), and have a C^2-boundary (by Assumption (5.2)). Hence $(O_R^\Phi)_{R>R_0}$ is an exhausting family of $\mathbb{R}^n$ in the sense of Definition 1.1.

It is convenient to split up integrals over the level sets O_R^Φ, $R \in (R_0, \infty)$, into integrals w.r.t. the iso-level sets of Φ. To this aim assume that there exists $R_1 \geq R_0$ such that

$$\nabla\Phi(x) \neq 0 \qquad x \in \mathbb{R}^n \setminus O_{R_1}^\Phi\,. \tag{5.4}$$

Hence for a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and $R > R_1$ we can define the following weighted integral over the iso-level set $\partial O_R^\Phi := \{x : \Phi(x) = R\}$

$$m_{\Phi,R}[f] := \int_{\partial O_R^\Phi} \frac{f(\xi)}{|\nabla\Phi(\xi)|}\, d\sigma_{\Phi,R}(\xi)\,, \tag{5.5}$$

where $d\sigma_{\Phi,R}$ is the surface measure on ∂O_R^Φ. The proof of the following lemma is deferred to the end of this section.

Lemma 5.1 *Assume (5.3) and (5.4). Then for every $f \in C(\mathbb{R}^n, \mathbb{R})$ and $R > R_1$*

$$\int_{O_R^\Phi \setminus O_{R_1}^\Phi} f\, dx = \int_{R_1}^R m_{\Phi,r}[f]\, dr\,.$$

The crucial condition to obtain sharp eigenvalue asymptotics is the relation

$$m_{\Phi,R}[(\Delta\Phi)^2] = o\left(m_{\Phi,R}[|\nabla\Phi|^2]\right) \qquad (R \to \infty)\,. \tag{5.6}$$

If Φ is of polynomial form in x, then $\Delta\Phi$ (as a second order derivative term) is of lower order than the first order derivative term $|\nabla\Phi|$ in the limit $|x| \to \infty$. Hence in this case, Condition (5.6) has a good chance to hold. As in Chapter 4, we need some growth conditions on Φ. Set

$$I(R) := \int_{O_R^\Phi} |\nabla\Phi(x)|^2 e^{2\Phi(x)/\sigma^2}\, dx \qquad R > R_0\,. \tag{5.7}$$

Assume

$$I(R) \nearrow \infty\,, \quad I(R) = o\left(e^{4R/\sigma^2}\right) \qquad (R \to \infty)\,. \tag{5.8}$$

The interpretation of the first condition is that $|\nabla\Phi(x)|$ must not decay too fast to zero as $|x| \to \infty$. By L'Hôpital's rule and Lemma 5.1, the second condition also reads as a growth condition on $|\nabla\Phi|$ in the form $m_{\Phi,R}[|\nabla\Phi|^2] = o(e^{2R/\sigma^2})$ as $R \to \infty$. In Lemma 5.4, we state explicit growth conditions on Φ for (5.8) to hold.

Theorem 5.2 *Assume that Condition (2.5), the finiteness of the stationary measure, and the spectral gap condition (2.7) hold. Further suppose that Assumptions (5.3), (5.2), (5.4), (5.6), and (5.8) hold. Set*

$$l(R) := \frac{2}{\sigma^2|\mu|} e^{-4R/\sigma^2} I(R) \qquad R > R_0\,,$$

where $|\mu|$ is the total mass of μ and $I(R)$ is defined in (5.7). Then the function l satisfies (2.9), i.e., $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R is the bottom eigenvalue of the operator $-L_R$ defined in Section 3.1 associated to the level set O_R^Φ, $R \in (R_0, \infty)$.

Remark 5.3 (1) By L'Hôpital's rule and Lemma 5.1, the function l can be replaced by

$$l(R) := \frac{2}{\sigma^2|\mu|} e^{-4R/\sigma^2} \int_{R_1}^{R} e^{2r/\sigma^2} m_{\Phi,r}[|\nabla\Phi|^2]\,dr \qquad R > R_1\,.$$

(2) The proof of Theorem 5.2 is very similar to the proof of Theorem 4.1. The main difference is that we use test-functions of the form $v_R = f_R \circ \Phi$, $R > R_0$, where f_R is a real function, instead of rotationally symmetric test-functions.

(3) Corollary 2.4 allows to compare the eigenvalue asymptotics corresponding to the level sets $(O_R^\Phi)_{R>R_0}$ with the eigenvalue asymptotics w.r.t. a different exhausting family $(\widetilde{O}_r)_{r>r_0}$ of $\mathbb{R}^n$.

PROOF. The quadratic forms $\mathcal{E}_R$ and the operators L_R defined in Section 3.1 are associated here to the level sets O_R^Φ, $R > R_0$. For a function $v \in \mathcal{D}(\mathcal{E}_R)$, we write $\mathcal{E}_R(v)$ instead of $\mathcal{E}_R(v,v)$ and the norm in $L^2_{\mu_R} := L^2(O_R^\Phi, \mu_R)$ is again denoted by $\|\cdot\|_{2,R}$. According to Remark 2.2.(1) we first assume that $\sigma = \sqrt{2}$ and that the potential Φ is normalized such that $\int_{\mathbb{R}^n} e^{-\Phi(x)}\,dx = 1$.

Let us define the test-function as follows:

$$v_R(x) := 1 - e^{\Phi(x)-R} \qquad x \in O_R^\Phi,\ R \in (R_0, \infty)\,.$$

By Assumption (5.2), we have $v_R \in C^2(O_R^\Phi, \mathbb{R})$. Further, $v_R \equiv 0$ on the set $\{x : \Phi(x) = R\}$. Hence $v_R \in C_0^2(O_R^\Phi, \mathbb{R}) \subset \mathcal{D}(L_R)$ for every $R \in (R_0, \infty)$, see (3.3). By construction, $v_R \nearrow 1$ μ-a.s. as $R \to \infty$ (where v_R is extended to a function on $\mathbb{R}^n$ by setting it to 0 on $\mathbb{R}^n \setminus O_R^\Phi$). Since μ is a probability measure on $\mathbb{R}^n$, we have

$$\|v_R\|_{2,R}^2 \to 1 \qquad (R \to \infty)\,. \tag{5.9}$$

The gradient of v_R reads $\nabla v_R(x) = -e^{\Phi(x)-R}\nabla\Phi(x)$ for every $x \in O_R^\Phi$. Plugging this into the quadratic form $\mathcal{E}_R$, we obtain for $R \in (R_0, \infty)$

$$\mathcal{E}_R(v_R) = e^{-2R} \int_{O_R^\Phi} |\nabla\Phi|^2 e^\Phi\,dx = e^{-2R} I(R)\,. \tag{5.10}$$

The second growth condition in (5.8) implies that

$$\mathcal{E}_R(v_R) \searrow 0 \qquad (R \to \infty)\,. \tag{5.11}$$

Using the alternative representation of L defined in (2.6), one obtains for $R \in (R_0, \infty)$

$$\|Lv_R\|_{2,R}^2 = e^{-2R} \int_{O_R^\Phi} (\Delta\Phi)^2 e^\Phi \, dx \,. \tag{5.12}$$

As in Step 2 of the proof of Theorem 4.1 (using the convergence results (5.9) and (5.11)), one can show the following: suppose that

$$\|Lv_R\|_{2,R}^2 = o\left(\mathcal{E}_R(v_R)\right) \quad (R \to \infty) \,. \tag{5.13}$$

Then the upper and lower bounds in Proposition 3.5 get sharp in the limit $R \to \infty$ and we obtain the eigenvalue asymptotics

$$\lambda_R \sim \mathcal{E}_R(v_R) = e^{-2R} \int_{O_R^\Phi} |\nabla\Phi|^2 e^\Phi \, dx = e^{-2R} I(R) \qquad (R \to \infty) \,. \tag{5.14}$$

The result for arbitrary $\sigma > 0$ and not normalized potentials Φ is obtained according to Remark 2.2.(1) by multiplying the RHS of (5.14) by $\sigma^2/2$ and plugging in the potential $\Phi_\sigma := (2/\sigma^2)\Phi + \ln|\mu|$. Hence

$$\lambda_R^\sigma \sim \frac{\sigma^2}{2} e^{-2R} \int_{O_R^{\Phi_\sigma}} |\nabla\Phi_\sigma|^2 e^{\Phi_\sigma} \, dx = \frac{2|\mu|}{\sigma^2} e^{-2R} \int_{O_R^{\Phi_\sigma}} |\nabla\Phi|^2 e^\Phi \, dx \quad (R \to \infty) \,, \tag{5.15}$$

where λ_R^σ is the bottom eigenvalue of the operator $-L_R$ corresponding to the level sets of Φ_σ. To obtain the eigenvalue asymptotics for the bottom eigenvalue λ_R corresponding to $(O_R^\Phi)_{R>R_0}$, we have to replace R by $(2/\sigma^2)R + \ln|\mu|$ in the RHS of (5.15), since $O_R^\Phi = O_{(2/\sigma^2)R+\ln|\mu|}^{\Phi_\sigma}$. This yields the asymptotic expression for λ_R as stated in the theorem.

It remains to show that Condition (5.13) holds. This means

$$q_R := \mathcal{E}_R(v_R)^{-1} \|Lv_R\|_{2,R}^2 = \left(\int_{O_R^\Phi} |\nabla\Phi|^2 e^\Phi \, dx \right)^{-1} \int_{O_R^\Phi} (\Delta\Phi)^2 e^\Phi \, dx \to 0 \qquad (R \to \infty) \,,$$

where we used the representations (5.10) and (5.12). The growth condition (5.8) implies that $\int_{O_R^\Phi} |\nabla\Phi|^2 e^\Phi \, dx \nearrow \infty$ as $R \to \infty$ and hence L'Hôpital's rule can be applied to the quotient q_R. Thus, we use Lemma 5.1 to obtain

$$\lim_{R\to\infty} q_R = \lim_{R\to\infty} \left(e^R m_{\Phi,R}[|\nabla\Phi|^2] \right)^{-1} e^R m_{\Phi,R}[(\Delta\Phi)^2] = \lim_{R\to\infty} \frac{e^R m_{\Phi,R}[(\Delta\Phi)^2]}{m_{\Phi,R}[|\nabla\Phi|^2]} \,.$$

But the last limit is 0 by the crucial condition (5.6). Hence Condition (5.13) holds. $\square$

Proof of Lemma 5.1. We fix $R > R_1$ and set for $\delta > 0$

$$\Gamma_{R,\delta} := \{x : R \leq \Phi(x) \leq R + \delta\}\,.$$

It suffices to show that

$$\frac{1}{\delta}\int_{\Gamma_{R,\delta}} f\,dx = m_{\Phi,R}[f] \qquad (\delta \searrow 0)\,.$$

Since $\Phi \in C^1(\mathbb{R}^n, \mathbb{R})$, we obtain with (5.4) and the implicit function theorem, that $\partial O_R^\Phi = \{x : \Phi(x) = R\}$ is a $n-1$-dimensional C^1-surface, which is orthogonal to the gradient field $\nabla\Phi$. Let $(\xi)_{\xi\in\Xi}$ be a smooth parametrization of ∂O_R^Φ. For every $\xi \in \partial O_R^\Phi$ we define the flow $[0, s^*) \ni s \mapsto T_s\xi \in \mathbb{R}^n$ as the maximal solution of the system of ODEs

$$\dot{z}(s) = |\nabla\Phi(z(s))|^{-1}\nabla\Phi(z(s)) \qquad z(0) = \xi\,.$$

Note that this is well defined by (5.4) and that the flow $s \mapsto T_s\xi$ has unit speed. Set

$$\phi_\xi(s) := \Phi(T_s\xi) \qquad s \in [0, s^*),\ \xi \in \partial O_R^\Phi\,.$$

Obviously ϕ_ξ is differentiable at $s = 0$ with $\phi_\xi'(0) = |\nabla\Phi(\xi)| > 0$ by (5.4). Hence we can find for every $\xi \in \partial O_R^\Phi$ and small $\delta > 0$ a constant $S_{\xi,\delta} > 0$ such that $\phi_\xi(S_{\xi,\delta}) = R + \delta$. Since ϕ_ξ is locally invertible near $s = 0$, $S_{\xi,\delta}$ is differentiable w.r.t. δ at $\delta = 0$ with

$$\lim_{\delta\searrow 0}\frac{S_{\xi,\delta}}{\delta} = (\phi_\xi^{-1})'(R) = \frac{1}{\phi_\xi'(\phi_\xi^{-1}(R))} = \frac{1}{\phi_\xi'(0)} = \frac{1}{|\nabla\Phi(\xi)|}\,. \tag{5.16}$$

Since $\Phi \in C^1(\mathbb{R}^n, \mathbb{R})$, the mapping $T : (s,\xi) \mapsto T_s\xi$ is a local diffeomorphism. From Assumption (5.3) we deduce that ∂O_R^Φ is compact. Hence by shrinking δ if necessary, T is reduced to a global diffeomorphism $T : \Gamma'_{R,\delta} \to \Gamma_{R,\delta}$ (again denoted by T), where $\Gamma'_{R,\delta} := \{(s,\xi) : \xi \in \partial O_R^\Phi, s \in [0, S_{\xi,\delta}]\}$. Further the limit in (5.16) is uniform in $\xi \in \partial O_R^\Phi$, since also $(s,\xi) \mapsto \phi_\xi(s) = \Phi(T_s\xi)$ is a local diffeomorphism and since ∂O_R^Φ is compact. Using the transformation rule for integrals having in mind that the flow $s \mapsto T_s\xi$ has unit speed we get

$$\frac{1}{\delta}\int_{\Gamma_{R,\delta}} f\,dx = \frac{1}{\delta}\int_{\partial O_R^\Phi}\left(\int_0^{S_{\xi,\delta}} f(T_s\xi)\,ds\right)d\sigma_{\Phi,R}(\xi)\,, \tag{5.17}$$

where $d\sigma_{\Phi,R}$ is the surface measure on ∂O_R^Φ. Using (5.16) we compute by means of the chain rule

$$\lim_{\delta\searrow 0}\frac{1}{\delta}\int_0^{S_{\xi,\delta}} f(T_s\xi)\,ds = f(T_0\xi)\lim_{\delta\searrow 0}\frac{S_{\xi,\delta}}{\delta} = \frac{f(\xi)}{|\nabla\Phi(\xi)|}\,.$$

Using the uniform continuity of f on the compact set $\Gamma_{R,\delta}$ and the fact that the limit in (5.16) is uniform in $\xi \in \partial O_R^\Phi$, also the above limit is uniform in $\xi \in \partial O_R^\Phi$. Hence in (5.17), the limit $\delta \searrow 0$ can be interchanged with the integration over ∂O_R^Φ and the result follows. □

5.2 Examples

We give some examples of diffusion processes of gradient field type, for which the sharp eigenvalue asymptotics can be evaluated by Theorem 5.2.

The following lemma provides a method to check whether the growth conditions (5.8) hold. Assume that the open balls $(B_\rho)_{\rho>0}$ around the origin and the level sets $(O_R^\Phi)_{R>R_0}$ are compatible to each other in the sense of (2.13). For sufficiently large $R > 0$ set

$$\rho_*(R) := \sup\{\rho > 0 : B_\rho \subset O_R^\Phi\}\,, \qquad \rho^*(R) := \inf\{\rho > 0 : O_R^\Phi \subset B_\rho\}\,.$$

Note that $\rho_*(R) \nearrow \infty$ as $\rho \to \infty$ by Condition (5.3) and that $\rho^*(R) < \infty$ for every $R > R_0$. Further, we define for a continuous function $f : \mathbb{R}^n \to \mathbb{R}$

$$f_*(\rho) := \min_{|x|=\rho} f(x)\,, \qquad f^*(\rho) := \max_{|x|=\rho} f(x)\,.$$

Lemma 5.4 *Assume $n \geq 2$.*
(a) The first growth condition in (5.8) holds if there exist constants $C > 0$ and $\rho_0 > 0$ such that

$$|\nabla\Phi|_*(\rho) \geq C\rho^{-n/2} \qquad \rho > \rho_0\,. \tag{5.18}$$

(b) Assume that Φ_* is differentiable and that there exists a constant $\overline{\rho} > 0$ such that $\Phi_*'(\rho) > 0$ for every $\rho > \overline{\rho}$. The second growth condition in (5.8) holds if Φ satisfies

$$\rho^{n-1}\frac{|\Delta\Phi|^*(\rho)}{(\Phi_*)'(\rho)} = o(e^{\Phi_*(\rho)}) \qquad (\rho \to \infty)\,. \tag{5.19}$$

Proof. We set w.l.o.g. $\sigma = \sqrt{2}$ and hence the term $I(R)$ defined in (5.7) reads $I(R) = \int_{O_R^\Phi} |\nabla\Phi|^2 e^\Phi dx$.

(a) Choose $\rho_1 > \rho_0$ such that $\Phi(x) > 0$ for every $|x| \geq \rho_1$ (possible by Assumption (5.3)). Denoting by γ_n the volume of the unit sphere in $\mathbb{R}^n$, we estimate for R with $\rho_*(R) > \rho_1$

$$I(R) \geq \int_{B_{\rho_*(R)}} |\nabla\Phi|^2 e^{\Phi}\, dx \geq \gamma_n \int_{\rho_1}^{\rho_*(R)} r^{n-1} |\nabla\Phi|_*^2(r)\, dr \geq C\gamma_n \int_{\rho_1}^{\rho_*(R)} r^{-1}\, dr\,.$$

The last expression tends to infinity as $R \to \infty$, since also $\rho_*(R) \nearrow \infty$ as $R \to \infty$.

(b) To prove that $I(R) = o(e^{2R})$ as $R \to \infty$, we apply Lemma 5.1 to the term $I(R)$ and hence it suffices by L'Hôpital's rule to show

$$e^{-R} m_{\Phi,R}[|\nabla\Phi|^2] \to 0 \qquad (R \to \infty)\,. \tag{5.20}$$

Since O_R^Φ has C^1 boundary by Condition (5.2), we are allowed to apply Stoke's formula (having in mind that the outer normal to ∂O_R^Φ is given by $|\nabla\Phi|^{-1}\nabla\Phi$)

$$m_{\Phi,R}[|\nabla\Phi|^2] = \int_{\partial O_R^\Phi} \nabla\Phi(\xi) \cdot \tfrac{\nabla\Phi(\xi)}{|\nabla\Phi(\xi)|}\, d\sigma_{\Phi,R}(\xi) = \int_{O_R^\Phi} \operatorname{div}(\nabla\Phi(x))\, dx = \int_{O_R^\Phi} \Delta\Phi(x)\, dx\,.$$

Estimating further we obtain

$$\left| \int_{O_R^\Phi} \Delta\Phi(x)\, dx \right| \leq \int_{B_{\rho^*(R)}} |\Delta\Phi(x)|\, dx \leq \gamma_n \int_0^{\rho^*(R)} r^{n-1} |\Delta\Phi|^*(r)\, dr\,.$$

Using the fact that $\rho^*(\Phi_*(\rho)) = \rho$ and that $\Phi_*^{-1}(\rho) \nearrow \infty$ as $\rho \to \infty$, we can replace R by $\Phi_*(\rho)$ in the limit in (5.20). Using L'Hôpital's rule and Assumption (5.19) we obtain

$$\limsup_{R\to\infty} \frac{m_{\Phi,R}[|\nabla\Phi|^2]}{e^R} \leq \gamma_n \limsup_{\rho\to\infty} \frac{\int_0^\rho r^{n-1}|\Delta\Phi|^*(r)\, dr}{e^{\Phi_*(\rho)}} = \limsup_{\rho\to\infty} \frac{\rho^{n-1}|\Delta\Phi|^*(\rho)}{\Phi_*'(\rho)e^{\Phi_*(\rho)}} = 0\,.$$

Hence Condition (5.20) is proved. □

5.2.1 The Rotationally Symmetric Case

As in Section 4.2.1, we assume that the potential Φ in the SDE (1.11) has the property that there exist $\rho_0 > 0$ and $\phi \in C^2([\rho_0, \infty), \mathbb{R})$ such that

$$\Phi(x) = \phi(|x|) \qquad |x| > \rho_0\,. \tag{5.21}$$

Assume further that

$$\liminf_{\rho\to\infty} \phi'(\rho) > 0\,. \tag{5.22}$$

The crucial condition (5.6) in this context has the form of a regularity condition

$$\phi''(\rho) = o(\phi'(\rho)) \quad (\rho \to \infty)\,. \tag{5.23}$$

Note that $\phi(\rho) \nearrow \infty$ as $\rho \to \infty$ at least linearly by (5.22). Moreover, ϕ^{-1} exists on $[R_2, \infty)$ for some $R_2 > 0$ large enough, and also $\phi^{-1}(R) \nearrow \infty$ as $R \to \infty$. Further, we obtain for the level sets

$$O_R^\Phi = B_{\phi^{-1}(R)} = \{x : |x| < \phi^{-1}(R)\}\,. \tag{5.24}$$

A simple calculation yields

$$|\nabla\Phi(x)| = \phi'(|x|)\,, \quad \Delta\Phi(x) = \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|) \qquad |x| > \rho_0\,. \tag{5.25}$$

In the rotationally symmetric case, Theorem 5.2 leads to the following corollary.

Corollary 5.5 *Let $n \geq 2$. Assume that the potential Φ in the SDE (1.11) is of the form (5.21). Suppose that (5.22) and (5.23) holds.*
(a) Set

$$l(R) := \frac{2\gamma_n}{\sigma^2|\mu|} e^{-4R/\sigma^2} \int_{\phi^{-1}(R_0)}^{\phi^{-1}(R)} e^{2\phi(t)/\sigma^2} t^{n-1} \phi'(t)^2\, dt \qquad R > R_2\,.$$

Then l satisfies (2.9), i.e., $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R is the bottom eigenvalue of the operator $-L_R$ associated to the level set $(O_R^\Phi)_{R>R_0}$.
(b) Consider the exhausting family $(B_\rho)_{\rho>0}$ of $\mathbb{R}^n$. Set $\widetilde{l}(\rho) := l(\phi(\rho))$, $\rho > \rho_0$. Then $\widetilde{l}(\rho) \sim \widetilde{\lambda}_\rho$ as $\rho \to \infty$, where $\widetilde{\lambda}_\rho$ is the bottom eigenvalue corresponding to $(B_\rho)_{\rho>0}$.

Remark 5.6 Assume the situation of Part (b) of the above theorem. In Theorem 4.6, we could show under slightly weaker conditions that $\widetilde{l}_1(\rho) \sim \widetilde{\lambda}_\rho$ as $\rho \to \infty$ where

$$\widetilde{l}_1(\rho) := \frac{\sigma^2\gamma_n}{2|\mu|}\left(\int_{\rho_0}^{\rho} t^{1-n} e^{2\phi(t)/\sigma^2}\, dt\right)^{-1} \qquad \rho > \rho_0\,.$$

The only assumption was that $\liminf_{|x|\to\infty} V_\Phi(x) > 0$ with V_Φ defined in (5.26). In particular, we did not need to assume the regularity condition (5.23), because, for the proof of Theorem 4.6, we used test-functions which are more adapted to the exhausting family $(B_\rho)_{\rho>0}$. If $\phi(\rho) = \rho^\alpha$, $\rho > \rho_0$, $\alpha \geq 1$, then the two asymptotic expressions $\widetilde{l}(\rho)$ and $\widetilde{l}_1(\rho)$

coincide. Using Lemma 4.5, we get as $\rho \to \infty$

$$\begin{aligned}\widetilde{l}(\rho) &= \frac{2\gamma_n}{\sigma^2|\mu|}e^{-4\rho^\alpha/\sigma^2}\int_{\rho_0}^{\rho} e^{2t^\alpha/\sigma^2}t^{n-1}(\alpha t^{\alpha-1})^2\,dt \sim \frac{\gamma_n\alpha}{|\mu|}\rho^{n+\alpha-2}e^{-2\rho^\alpha/\sigma^2} \quad \text{and}\\ \widetilde{l}_1(\rho) &\sim \frac{\sigma^2\gamma_n}{2|\mu|}\left(\frac{\sigma^2}{2\alpha}\rho^{-n-\alpha+2}e^{2\rho^\alpha/\sigma^2}\right)^{-1} \sim \widetilde{l}(\rho)\,.\end{aligned}$$

PROOF. We have to check that the conditions of Theorem 5.2 are satisfied. Since $\phi(\rho) \nearrow \infty$ as $\rho \to \infty$ at least linearly by (5.22), Conditions (5.3) and (5.4) hold implying also that the stationary measure is finite (Condition (2.5)). We show the spectral gap condition (2.7) by means of Proposition 3.7. Using the representations of (5.25), the function V_Φ defined in (3.8) reads here setting ($\rho = |x|$)

$$V_\Phi(x) = \frac{\phi'(\rho)^2}{4} - \frac{1}{2}\left(\phi''(\rho) + \frac{n-1}{\rho}\phi'(\rho)\right) = \phi'(\rho)\left(\frac{\phi'(\rho)}{4} - \frac{\phi''(\rho)}{2\phi'(\rho)} - \frac{n-1}{2\rho}\right). \tag{5.26}$$

Using (5.22) and (5.23), it can be seen that $\liminf_{|x|\to\infty} V_\Phi(x) > 0$, and hence the spectral gap condition (2.7) holds by Proposition 3.7. To show the crucial condition (5.8) we compute

$$m_{\Phi,R}[|\nabla\Phi|^2] = \int_{|\xi|=\phi^{-1}(R)} |\phi'(\xi)|\,d\sigma(\xi) = \gamma_n(t^{n-1}\phi'(t))_{t=\phi^{-1}(R)}\,, \tag{5.27}$$

$$\begin{aligned} m_{\Phi,R}[(\Delta\Phi)^2] &= \int_{|\xi|=\phi^{-1}(R)} \frac{(\phi''(|\xi|) + \frac{n-1}{|\xi|}\phi'(|\xi|))^2}{|\phi'(\xi)|}\,d\sigma(\xi)\\ &= \gamma_n\left[t^{n-1}\left(\tfrac{\phi''(t)^2}{\phi'(t)} + \tfrac{2(n-1)}{t}\phi''(t) + \tfrac{(n-1)^2}{t^2}\phi'(t)\right)\right]_{t=\phi^{-1}(R)}. \end{aligned} \tag{5.28}$$

Since $\phi^{-1}(R) \nearrow \infty$ as $R \to \infty$ we have using (5.23)

$$\lim_{R\to\infty}\frac{m_{\Phi,R}[(\Delta\Phi)^2]}{m_{\Phi,R}[|\nabla\Phi|^2]} = \lim_{R\to\infty}\left((\tfrac{\phi''(t)}{\phi'(t)})^2 + \tfrac{2(n-1)}{t}\tfrac{\phi''(t)}{\phi'(t)} + (\tfrac{n-1}{t})^2\right)_{t=\phi^{-1}(R)} = 0\,.$$

To check the growth conditions (5.8), it suffices to prove Conditions (5.18) and (5.19) of Lemma 5.4. Obviously $\phi'(\rho) > \rho^{-n/s}$ for sufficiently large $\rho > 0$ by (5.22), and hence Condition (5.18) holds. Using that $\phi(\rho) \nearrow \infty$ as $\rho \to \infty$ at least linearly by (5.22) and Condition (5.23), we obtain

$$\rho^{n-1}\left|\frac{\phi''(\rho) + \frac{n-1}{\rho}\phi'(\rho)}{\phi'(\rho)}\right| e^{-\phi(\rho)} = \rho^{n-1}\left|\frac{\phi''(\rho)}{\phi'(\rho)} + \frac{n-1}{\rho}\right| e^{-\phi(\rho)} \to 0 \qquad (\rho\to\infty)\,.$$

Hence also Condition (5.19) is satisfied, and the growth conditions (5.8) holds by Lemma 5.4. This finishes the proof of Part (a). Part (b) is obvious, noting (5.24), see also Corollary 2.4. □

5.2.2 Non-Symmetric Processes

We consider the class of non-symmetric potentials appearing in the SDE (1.11) introduced in Section 4.2.2. To avoid trivialities, we assume $n \geq 2$. We use polar coordinates writing $\mathbb{R}^n \setminus \{0\} \ni x = \rho e_\theta$ where $\rho = |x| > 0$ and $(e_\theta)_{\theta \in \Theta}$ is a smooth parametrization of the unit sphere S^{n-1} in $\mathbb{R}^n$.

We illustrate the methods developed in the preceding section for the following specific potential Φ. Suppose that there exists $\rho_0 > 1$ and a function $p \in C^2(S^{n-1}, [0, \infty))$ with $\min_{\theta \in \Theta} p(\theta) = 0$ such that

$$\Phi(\rho, \theta) = \rho^\alpha + p(\theta)\rho^\beta \qquad \rho > \rho_0,\, \theta \in \Theta\,, \tag{5.29}$$

where $\alpha \geq 1$ and $\beta \in \mathbb{R}$.

As in Section 4.2.2, the essential feature in the definition of Φ is that the asymmetric part factorizes into radial and spherical component. It is possible to replace in Definition (5.29) of the potential Φ the terms ρ^α and ρ^β by functions $\phi(\rho)$ and $\psi(\rho)$, see (4.14). In this case, quite technical compatibility and asymptotic growth Conditions have to be imposed. We omit these cumbersome calculations in this thesis.

In Section 4.2.2, further requirements were needed for the spherical function p. The zero set of p was assumed to be finite (see (4.15)) and conditions have been imposed on the curvature of p around its zero-points (see (4.16)). These conditions allowed the application of Laplace's method. Here we do not need any further condition on the function p. This fact emphasizes that the choice of the level sets is more adapted than the balls $(B_\rho)_{\rho>0}$ to the geometry of the potential Φ.

To evaluate the crucial condition (5.8), we express $|\nabla\Phi|$ and $\Delta\Phi$ in polar coordinates:

$$|\nabla\Phi|^2(\rho, \theta) = (\alpha\rho^{\alpha-1} + \beta p(\theta)\rho^{\beta-1})^2 + |\nabla_\theta p(\theta)|^2 \rho^{2\beta-2}\,, \tag{5.30}$$

$$\Delta\Phi(\rho, \theta) = \alpha(\alpha + n - 2)\rho^{\alpha-2} + \beta(\beta + n - 2)p(\theta)\rho^{\beta-2} + \Delta_\theta p(\theta)\rho^{\beta-2}\,. \tag{5.31}$$

Here ∇_θ and Δ_θ denote the gradient and Laplace-Beltrami operator w.r.t. the angular coordinates θ, respectively. We obtain the following estimates

$$|\nabla\Phi|(\rho,\theta) \geq \alpha\rho^{\alpha-1} \qquad \rho > \rho_0,\, \theta\in\Theta\,. \tag{5.32}$$

Since $p \in C^2(S^{n-1})$ and S^{n-1} is compact, there exists a constant $\kappa > 0$ such that

$$|\Delta\Phi|(\rho,\theta) \leq \kappa\rho^{\max\{\alpha,\beta\}-2} \qquad \rho > \rho_0,\, \theta\in\Theta\,. \tag{5.33}$$

Theorem 5.7 *Assume that the potential Φ in the SDE (1.11) is of the form (5.29). The assertion of Theorem 5.2 holds in the following situations:*
(i) $\alpha \in [1,2)$ and $\beta < 2$,
(ii) $\alpha \geq 2$ and $\beta < 1 + \sqrt{\alpha(\alpha-1)+1}$.

Remark 5.8 (a) Let us compare this result with Theorem 4.8 in the Euclidean case. We can use Corollary 2.4 to obtain from the eigenvalue asymptotics corresponding to the level sets lower asymptotic bounds on the bottom eigenvalue $\widetilde{\lambda}_\rho$ associated to the balls $(B_\rho)_{\rho>0}$. Set

$$p^* := \max_{\theta\in\Theta} p(\theta)\,, \qquad \overline{\phi}(\rho) := \rho^\alpha + p^*\rho^\beta \quad \rho > \rho_0\,. \tag{5.34}$$

Invoking (2.13), we get $R_\rho := \inf\{R > R_0 : B_\rho \subset O_R^\Phi\} = \overline{\phi}(\rho)$ and obtain that $\widetilde{l}(\rho) \lesssim \widetilde{\lambda}_\rho$ as $\rho\to\infty$ where for $\rho > \rho_0$

$$\begin{aligned}
\widetilde{l}(\rho) := l(R_\rho) &= \frac{2}{\sigma^2|\mu|}e^{-4\overline{\phi}(\rho)/\sigma^2}\int_{O^\Phi_{\overline{\phi}(\rho)}} |\nabla\Phi|^2 e^{2\Phi/\sigma^2}\,dx \\
&\geq \frac{2}{\sigma^2|\mu|}e^{-4\overline{\phi}(\rho)/\sigma^2}\int_{B_\rho} |\nabla\Phi|^2 e^{2\Phi/\sigma^2}\,dx \\
&= \frac{2}{\sigma^2|\mu|}e^{-2\overline{\phi}(\rho)/\sigma^2}\int_0^\rho s^{n-1}\left(\int_\Theta |\nabla\Phi|^2(s,\theta)e^{-2(p^*-p(\theta))s^\beta/\sigma^2}\,d\sigma(\theta)\right)ds\,.
\end{aligned}$$

It can be seen by Laplace's method that the last integral decays polynomially to zero as $\rho\to\infty$. Hence we obtain that

$$\ln\widetilde{l}(\rho) \gtrsim -\frac{2}{\sigma^2}\overline{\phi}(\rho) = -\frac{2}{\sigma^2}(\rho^\alpha + p^*\rho^\beta) \qquad \rho\to\infty\,.$$

In Theorem 4.8, the fine eigenvalue asymptotics of $\widetilde{\lambda}_\rho$ was evaluated for a potential of the form (5.29) in the two-dimensional case for $\alpha \geq 1$ and $\beta \in (0, 2\alpha)$ in the sense that

$\widetilde{\lambda}_\rho \sim \widetilde{l}_1(\rho)$ as $\rho \to \infty$ with

$$\widetilde{l}_1(\rho) := C\rho^{-\beta/\varpi^*}\left(\int_{\rho_0}^{\rho} r^{-1}e^{2r^\alpha/\sigma^2}\,dr\right)^{-1} \sim \frac{\alpha C}{\sigma^2}\rho^{\alpha-\beta/\varpi^*}e^{-2\rho^\alpha/\sigma^2} \qquad (\rho \to \infty)\,,$$

where the constants $C, \varpi^* > 0$ depend on the curvature of p in its zero points. We used Lemma 4.5 for the last asymptotic evaluation. It can be seen that the exponential decay of $\widetilde{l}(\rho)$ and $\widetilde{l}_1(\rho)$ differ by the factor $e^{-2p^*\rho^\beta/\sigma^2}$ as $\rho \to \infty$. This effect is due to the fact that the ball B_ρ is compared to the domain $O^\Phi_{\overline{\phi}(\rho)}$ which is in general much bigger.

(b) The upper bounds on β in situation (i) and (ii) of Theorem 5.7 are used to ensure the crucial condition (5.8) by means of crude estimates, see (5.32)-(5.40). These upper bounds do not seem to be crucial and may be tightened by a more careful analysis for specific expressions of the function p.

Proof. Set $m := \max\{\alpha, \beta\}$. We have to check the conditions of Theorem 5.2. (5.3) and (5.4) obviously hold by inequality (5.32), since $\alpha \geq 1$. The finiteness of the stationary measure, Condition (2.5), and the spectral gap condition (2.7) have already been shown to hold in the proof of Theorem 4.8, see also Remark 4.9.(b). To show the growth conditions (5.8), we prove Conditions (5.18) and (5.19) of Lemma 5.4. Condition (5.18) holds by the inequality in (5.32), since $\alpha \geq 1$. To establish Condition (5.19), we also use the estimations in (5.32) and (5.33)

$$\rho^{n-1}\frac{|\Delta\Phi|^*(\rho)}{(\Phi_*)'(\rho)}\,e^{-\Phi_*(\rho)} \leq \frac{\kappa}{\alpha}\rho^{n-1}\rho^{m-2}\rho^{-(\alpha-1)}\,e^{-\rho^\alpha} \to 0 \qquad (\rho \to \infty)\,.$$

It remains to show the crucial condition (5.8). We need to parametrize the iso-level sets $\partial O^\Phi_R = \{x : \Phi(x) = R\}$. Recall the Definition of $\overline{\phi}(\rho)$ in (5.34). Note that $\rho \mapsto \Phi(\rho, \theta)$ is strictly monotonously increasing for every $\theta \in \Theta$. Thus

$$\gamma_R(\theta) := \Phi(\cdot, \theta)^{-1}(R) \qquad R > \overline{\phi}(\rho_0),\, \theta \in \Theta$$

exists. From the inequality $\phi(\rho) \leq \Phi(\rho, \theta) \leq \overline{\phi}(\rho)$, $\rho > \rho_0$, $\theta \in \Theta$, we deduce for the inverse functions

$$\overline{\phi}^{-1}(R) \leq \gamma_R(\theta) \leq R^{1/\alpha} \qquad R > \overline{\phi}(\rho_0),\, \theta \in \Theta\,. \tag{5.35}$$

Recalling Definition (5.5) of $m_{\Phi,R}[\cdot]$ we estimate

$$m_{\Phi,R}[|\nabla\Phi|^2] \geq \min_{\theta\in\Theta} |\nabla\Phi|(\gamma_R(\theta),\theta)\cdot \mathrm{Vol}(\partial O_R^\Phi)\,, \tag{5.36}$$

$$m_{\Phi,R}[(\Delta\Phi)^2] \leq \frac{\max_{\theta\in\Theta}(\Delta\Phi)^2(\gamma_R(\theta),\theta)}{\min_{\theta\in\Theta}|\nabla\Phi|(\gamma_R(\theta),\theta)}\cdot \mathrm{Vol}(\partial O_R^\Phi)\,. \tag{5.37}$$

Hence the crucial condition (5.8) holds if we can show that

$$J(R) := \frac{\max_{\theta\in\Theta}|\Delta\Phi|(\gamma_R(\theta),\theta)}{\min_{\theta\in\Theta}|\nabla\Phi|(\gamma_R(\theta),\theta)} \to 0 \qquad (R\to\infty)\,. \tag{5.38}$$

Using (5.32), (5.33), and (5.35), we obtain that for every $\theta\in\Theta$

$$|\nabla\Phi|(\gamma_R(\theta),\theta) \geq \alpha\gamma_R(\theta)^{\alpha-1} \geq \alpha\overline{\phi}^{-1}(R)^{\alpha-1}\,, \tag{5.39}$$

$$|\Delta\Phi|(\gamma_R(\theta),\theta) \leq \kappa\gamma_R(\theta)^{m-2} \leq \kappa\begin{cases} R^{(m-2)/\alpha} & m\geq 2\,,\\ \overline{\phi}^{-1}(R)^{m-2} & m<2\,.\end{cases} \tag{5.40}$$

In the case $m<2$, corresponding to Situation (i), the term $J(R)$ in (5.38) can be further analyzed:

$$J_R \leq \frac{\kappa}{\alpha}\left(\overline{\phi}^{-1}(R)\right)^{m-2-(\alpha-1)} \to 0 \qquad (R\to\infty)\,,$$

since $\overline{\phi}^{-1}(R)\nearrow\infty$ as $R\to\infty$ and $m-2-(\alpha-1)<1-\alpha\leq 0$. Hence the crucial condition (5.8) holds in this case.

In the case $m\geq 2$, corresponding to Situation (ii), the term $J(R)$ is estimated using (5.39) and (5.40)

$$J_R \leq (\kappa/\alpha)R^{(m-2)/\alpha}\left(\overline{\phi}^{-1}(R)\right)^{-(\alpha-1)} \qquad R>\overline{\phi}(\rho_0)\,.$$

Instead of showing $\lim_{R\to\infty} J_R = 0$, it suffices to prove $\lim_{\rho\to\infty} J_{\overline{\phi}(\rho)}$, since also $\overline{\phi}(\rho)\nearrow\infty$ as $\rho\to\infty$. We estimate

$$\begin{aligned} J_{\overline{\phi}(\rho)} &\leq \frac{\kappa}{\alpha}(\rho^\alpha + p^*\rho^\beta)^{(m-2)/\alpha}\rho^{-(\alpha-1)}\\ &= \frac{\kappa}{\alpha}\rho^{m(m-2)/\alpha+1-\alpha}(\rho^{\alpha-m}+p^*\rho^{\beta-m})^{(m-2)/\alpha} \to 0 \qquad (\rho\to 0)\,. \end{aligned}$$

This convergence follows, since the last term between parentheses is bounded, and since the upper bound on β in Situation (ii) ensures that $m(m-2)/\alpha+1-\alpha<0$. Hence the crucial condition (5.8) holds also in this case. □

5.2.3 A Potential of Tetragonal Shape

We present here another non-symmetric potential Φ and analyze the eigenvalue asymptotics w.r.t. the level sets. One can evaluate explicitly for this potential the terms $m_{\Phi,R}[|\nabla\Phi|^2]$ and $m_{\Phi,R}[(\Delta\Phi)^2]$ appearing in the crucial condition (5.8) of Theorem 5.2. For the non-symmetric potential in Section 5.2.2, growth restrictions on the asymmetric part of the potential (growing with r^β) have to be imposed (see Theorem 5.7), which result from crude estimates, see Remark 5.8.(b). For the potential presented here, there are no growth restrictions on the asymmetric part.

We consider the following two-dimensional example where the potential Φ appearing in the SDE (1.11) satisfies the relation

$$|x_1|\Phi(x_1,x_2)^{-1/\beta_1} + |x_2|\Phi(x_1,x_2)^{-1/\beta_2} = 1 \qquad x_1, x_2 \in \mathbb{R}\setminus\{0\}\,, \tag{5.41}$$

where $0 < \beta_1 \le \beta_2$. Obviously $\Phi(x_1,0) = |x_1|^{\beta_1}$ and $\Phi(0,x_2) = |x_2|^{\beta_2}$ for $x_1, x_2 \in \mathbb{R}\setminus\{0\}$. The iso-level set $\partial O_R^\Phi = \{x : \Phi(x) = R\}$ for large $R > 0$ is a tetragon with edges $(0, \pm R^{1/\beta_2})$ and $(\pm R^{1/\beta_1}, 0)$, see Figure 5.1.

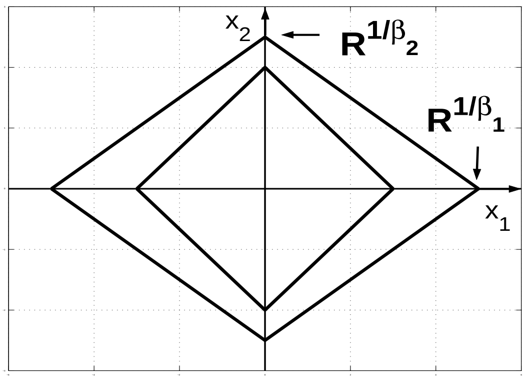

Figure 5.1: Contour plot of the two-dimensional potential Φ defined by the relation (5.41)

To overcome the problem that $\Phi \notin C^2(\mathbb{R}^2,\mathbb{R})$, i.e., Condition (5.2) is not fulfilled, it is possible to smooth the edges of Φ in such a way that the terms resulting from this smoothing procedure do not matter in the limit $R \to \infty$. We will check the remaining conditions of Theorem 5.2. By the definition of Φ, the density $e^{-2\Phi(x)/\sigma^2}$, $x \in \mathbb{R}^n$, of the stationary measure is integrable and hence Condition (2.5) holds. Further $\Phi(x) \nearrow \infty$ as $|x| \to \infty$, i.e., Condition (5.3) is satisfied. We compute $|\nabla\Phi|$ and $\Delta\Phi$. By the symmetry of the potential Φ, it is sufficient to restrict the calculations to the positive quadrant. Set

$$\delta := 1/\beta_1 - 1/\beta_2 > 0\,, \qquad h(R,x) := 1/\beta_1 - \delta R^{-1/\beta_2} x\,, \quad R, x > 0\,.$$

The relation (5.41) can be rewritten $x_1 + x_2\Phi^\delta = \Phi^{1/\beta_1}$, $x_1, x_2 > 0$, in the positive quadrant. Applying the partial derivatives w.r.t. x_1 and x_2 to this equation we obtain, after some obvious transformations,

$$\partial_{x_1}\Phi = \Phi^{1-1/\beta_1} h(\Phi, x_2)^{-1}\,, \quad \partial_{x_2}\Phi = \Phi^{1-1/\beta_2} h(\Phi, x_2)^{-1}\,. \tag{5.42}$$

Hence

$$|\nabla\Phi| = \frac{\Phi}{h(\Phi,x_2)}\sqrt{\Phi^{-2/\beta_1} + \Phi^{-2/\beta_2}} = \frac{\Phi^{1-1/\beta_1-1/\beta_2}}{h(\Phi,x_2)}\sqrt{\Phi^{2/\beta_1} + \Phi^{2/\beta_2}}\,. \tag{5.43}$$

Applying again the partial derivative w.r.t. x_1 to the first equation in (5.42) and w.r.t. x_2 to the second equation, respectively, and substituting the arising terms $\partial_{x_i}\Phi$, $i = 1, 2$, according to (5.42), we obtain

$$\partial^2_{x_i}\Phi = \Phi^{1-2/\beta_i} h(\Phi, x_2)^{-3}(\kappa_i - K_i\Phi^{-1/\beta_2}x_2) \qquad i = 1, 2\,,$$

where $\kappa_1 = 1/\beta_1 - 1/\beta_1^2$, $K_1 = \delta - \delta^2$, $\kappa_2 = 1/\beta_1(1 + 1/\beta_1 - 2/\beta_2)$, and $K_2 = \delta + \delta^2$. Hence

$$(\Delta\Phi)^2 = \frac{\Phi^{2-4/\beta_2}}{h(\Phi,x_2)^6}\Big\{\Phi^{-2\delta}(\kappa_1 - K_1\Phi^{-1/\beta_2}x_2) + (\kappa_2 - K_2\Phi^{-1/\beta_2}x_2)\Big\}^2\,. \tag{5.44}$$

We need to parametrize the iso-level sets of Φ. Recall that the iso-level set ∂O_R^Φ in the positive quadrant is the line joining the points $(0, R^{1/\beta_2})$ and $(R^{1/\beta_1}, 0)$. This line can be parametrized by

$$\gamma_R(x_2) = (R^{1/\beta_1} - R^\delta x_2, x_2)\,, \quad x_2 \in [0, R^{1/\beta_2}]\,. \tag{5.45}$$

The following estimates are needed: since $1/\beta_2 \le h(R, x_2) \le 1/\beta_1$ for $x_2 \in [0, R^{1/\beta_2}]$, we deduce from (5.43) and (5.44) using $x_2 \in [0, R^{1/\beta_2}]$ to parametrize ∂O_R^Φ as in (5.45)

$$\min_{\partial O_R^\Phi} |\nabla\Phi|^2 \gtrsim \beta_1^2 R^{2-2/\beta_2}\,, \quad \max_{\partial O_R^\Phi} |\Delta\Phi| \lesssim \beta_2^3\kappa_2 R^{1-2/\beta_2} \qquad (R \to \infty)\,. \tag{5.46}$$

From the left estimation we deduce for sufficiently large $R_1 > 0$ that $|\nabla\Phi(x)| > 0$, $x \in \mathbb{R}^n \setminus O_{R_1}^\Phi$, and hence that Condition (5.4) is satisfied. We claim that the spectral gap property (2.7) holds if $\beta_2 \ge 1$. To this aim we estimate the function V_Φ defined in (3.8) on the iso-level set ∂O_R^Φ using (5.46)

$$\min_{\partial O_R^\Phi} V_\Phi \gtrsim \frac{\beta_1^2}{4} R^{2(1-1/\beta_2)} - \frac{\beta_2^3\kappa_2}{2} R^{1-2/\beta_2} = R^{2(1-1/\beta_2)}\left(\frac{\beta_1^2}{4} - \frac{\beta_2^3\kappa_2}{2R}\right) \qquad (R \to \infty)\,.$$

Since $\beta_2 \geq 1$ and $\Phi(x) \nearrow \infty$ as $|x| \to \infty$, we obtain that $\liminf_{|x|\to\infty} V_\Phi(x) > 0$ and hence the spectral gap property (2.7) holds by Proposition 3.7.

In order to show the crucial condition (5.8), we compute $m_{\Phi,R}[|\nabla\Phi|^2]$ and $m_{\Phi,R}[(\Delta\Phi)^2]$. Fix $R > 0$ sufficiently large. Setting $d_R := \sqrt{R^{2/\beta_1} + R^{2/\beta_2}}$ we obtain for the infinitesimal curvature of the parametrization γ_R defined in (5.45) $|\gamma_R'(x_2)|dx_2 = (d_R/R^{1/\beta_2})dx_2$. To obtain the values of $|\nabla\Phi|$ and $(\Delta\Phi)^2$ on the iso-level set ∂O_R^Φ in the positive quadrant we simply have to set $\Phi \equiv R$ constant in (5.43) and (5.44) using x_2 to parametrize ∂O_R^Φ. Recalling Definition (5.5) of $m_{\Phi,R}[\cdot]$, we obtain, invoking (5.43) and (5.44)

$$\begin{aligned}
m_{\Phi,R}[|\nabla\Phi|^2] &= \int_{\partial O_R^\Phi} |\nabla\Phi(\xi)|\, d\sigma_{\Phi,R}(\xi) \\
&= 4\, R^{1-1/\beta_1-1/\beta_2} \sqrt{R^{2/\beta_1} + R^{2/\beta_2}} \int_0^{R^{1/\beta_2}} \frac{1}{h(R,x_2)} \frac{d_R}{R^{1/\beta_2}} dx_2 \\
&= 4\, R^{1-1/\beta_1-2/\beta_2}\, d_R^2\, R^{1/\beta_2} \int_0^1 \frac{dz}{1/\beta_1 - \delta z} \\
&= \frac{4}{\delta} \ln\left(\frac{\beta_2}{\beta_1}\right) R^{1-1/\beta_1-1/\beta_2} (R^{2/\beta_1} + R^{2/\beta_2}) \\
&\sim \frac{4}{\delta} \ln\left(\frac{\beta_2}{\beta_1}\right) R^{1+\delta} \qquad (R \to \infty)\,.
\end{aligned} \tag{5.47}$$

Similarly,

$$\begin{aligned}
m_{\Phi,R}[(\Delta\Phi)^2] &= \int_{\partial O_R^\Phi} \frac{(\Delta\Phi)^2}{|\nabla\Phi|}(\xi)\, d\sigma_{\Phi,R}(\xi) \\
&\sim 4 \frac{R^{2-4/\beta_2}}{R^{1-1/\beta_1-1/\beta_2}\, d_R} \int_0^{R^{1/\beta_2}} \frac{(\kappa_2 - K_2 R^{-1/\beta_2} x_2)^2}{h(R,x_2)^5} \frac{d_R}{R^{1/\beta_2}} dx_2 \\
&= 4\, R^{1+1/\beta_1-4/\beta_2}\, R^{1/\beta_2} \int_0^1 \frac{(\kappa_2 - K_2 z)^2}{(1/\beta_1 - \delta z)^5} dz \\
&= K\, R^{1+\delta-2/\beta_2} \qquad (R \to \infty)\,,
\end{aligned} \tag{5.48}$$

where $K > 0$ is a constant. From (5.47) and (5.48) we see that the crucial condition (5.6) holds for every choice of $0 < \beta_1 \leq \beta_2$.

The term $I(R)$ defined in (5.7) reads in our situation using (5.47) and Lemma 5.1

$$\begin{aligned}
I(R) &= \frac{4}{\delta} \ln\left(\frac{\beta_2}{\beta_1}\right) \int_0^R e^{2r/\sigma^2} r^{1+\delta} (1 + r^{-2\delta})\, dr \\
&\sim \frac{2\sigma^2}{\delta} \ln\left(\frac{\beta_2}{\beta_1}\right) R^{1+\delta} e^{2R/\sigma^2} \qquad (R \to \infty)\,.
\end{aligned} \tag{5.49}$$

The last step follows from Lemma 4.5. Hence the growth conditions (5.8) are obviously satisfied. Setting

$$l(R) := \frac{4}{\delta} \ln\left(\frac{\beta_2}{\beta_1}\right) R^{1+\delta} e^{-2R/\sigma^2} \qquad R > 0\,,$$

we obtain by Theorem 5.2 and (5.49), that $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R is the bottom eigenvalue associated to the exhausting family $(O_R^\Phi)_{R>0}$ of $\mathbb{R}^2$.

Chapter 6

Singular Perturbations Methods

In Chapter 4 and 5, we evaluated for a diffusion process of gradient field type the asymptotics as $R \to \infty$ of the bottom eigenvalues λ_R of the operators $-L_R$ associated to two different exhausting families of $\mathbb{R}^n$. The main idea was to find suitable test-functions (satisfying Dirichlet boundary conditions) such that the upper and lower bounds in Proposition 3.5 get sharp in the limit $R \to \infty$. This means in particular, that the test-functions represent a reasonable approximation for the principal eigenfunction corresponding to the bottom eigenvalue λ_R in the limit $R \to \infty$.

This chapter is devoted to the examination of the eigenvalue asymptotics by singular perturbation techniques. The basic idea is to derive an asymptotic expansion of the principal eigenfunction as $R \to \infty$. Since the bottom eigenvalue λ_R can be expressed in terms of the principal eigenfunction, the asymptotics of λ_R as $R \to \infty$ is obtained by plugging the asymptotic expansion of the principal eigenfunction into this expression.

It must be mentioned that the singular perturbation techniques have more or less heuristic character since the existence of an asymptotic expansion of the principal eigenfunction is assumed a priori. Moreover the convergence of this expansion to the exact solution in any norm can not be proved in general. The advantage of these techniques is that they provide a simple and efficient method to obtain the eigenvalue asymptotics for most cases. In addition, these methods provide a good intuition for the shape of the principal eigenfunction at least asymptotically, which is hard to evaluate rigorously in general.

The evaluation of the eigenvalue asymptotics by singular perturbation methods is not restricted to a special exhausting family of $\mathbb{R}^n$. It works for exhausting families $(O_R)_{R>R_0}$ satisfying the following scaling property: there exists a constant $\alpha > 0$ such that the sets

$$R^{-\alpha}O_R := \{R^{-\alpha}x : x \in O_R\} \quad \text{are independent of } R > R_0. \tag{6.1}$$

We illustrate these methods in the situation of the open balls $(B_R)_{R>0}$ around the origin.

To apply singular perturbation methods, it is necessary to scale the operator $-L_R$ to the unit ball B_1, and to derive an asymptotic expansion of the principal eigenfunction q^R corresponding to the bottom eigenvalue of the scaled operator. This eigenvalue coincides with λ_R (as shown in Section 6.1). The eigenvalue λ_R can be expressed by the principal eigenfunction q^R and for the eigenvalue asymptotics we use the leading term in the asymptotic expansion of q^R. It turns out that the rate of decay of the principal eigenfunction q^R near the boundary of B_1 depends on the spherical variables. It has to be adjusted to the slope of the potential Φ in each particular direction.

In the first section we explain the scaling procedure and give a short description of the singular perturbation techniques. The main result is stated and proved in the second section. In the last section we show that the eigenvalue asymptotics obtained by singular perturbation techniques for the examples treated in Section 4.2 coincides with the eigenvalue asymptotics evaluated by the methods of Chapter 4.

6.1 Rescaling and Introduction to Singular Perturbation Methods

The bottom eigenvalue of the operator $-L_R$ is defined as $\lambda_R := \inf \Sigma(-L_R)$, $R > 0$, where Σ denotes the spectrum of $-L_R$ in the space $L^2(B_R, \mu)$, see Section 3.1. We derive an expression of the operator L_R scaled to the unit ball and show that the bottom eigenvalue of the scaled operator coincides with λ_R.

Less formally, λ_R is the smallest $\lambda \in \mathbb{R}$ such that there exists a sufficiently smooth function $u \neq 0$ satisfying

$$Lu = -\lambda u \quad \text{on } B_R\,, \qquad u \equiv 0 \quad \text{on } \partial B_R = \{x : |x| = R\}\,. \tag{6.2}$$

Instead of letting the radius R of the ball B_R tend to infinity, another approach is to rescale the eigenvalue problem (6.2) to the unit ball B_1. For a function $u : \mathbb{R}^n \to \mathbb{R}$, the scaled function is defined by

$$u^R(x) := u(Rx) \qquad x \in \mathbb{R}^n,\ R > 0\,. \tag{6.3}$$

The scaled operator L^R is obtained by the relation $Lu = L^R u^R$, $R > 0$, for every sufficiently smooth function u (see (2.6)). L^R reads

$$L^R v = \frac{\sigma^2}{2R^2}\Delta v - \frac{1}{R^2}\sum_{i=1}^{n} \partial_{x_i}\Phi^R \partial_{x_i} v = \frac{\sigma^2}{2R^2} e^{2\Phi^R/\sigma^2} \sum_{i=1}^{n} \partial_{x_i}\left(e^{-2\Phi^R/\sigma^2}\partial_{x_i} v\right)\,, \tag{6.4}$$

where the rescaled potential Φ^R is defined in (6.3). An alternative method to derive the scaled operator L^R is as follows: let $(X_t)_{t\geq 0}$ be the diffusion process of gradient field type solving the SDE (1.11) and define the scaled process $(X_t^R)_{t\geq 0}$ by $X_t^R := R^{-1}X_t$, $t \geq 0$. Then an application of Itô's rule shows that L^R is the generator of $(X_t^R)_{t\geq 0}$.

Applying the scaling procedure to the eigenvalue problem (6.2), λ_R can also be expressed in terms of the scaled operator L^R as the smallest $\lambda \in \mathbb{R}$ such that there exists a sufficiently smooth function $u \neq 0$ satisfying

$$L^R u = -\lambda u \quad \text{on } B_1\,, \qquad u \equiv 0 \quad \text{on } S^{n-1}\,, \tag{6.5}$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

The scaled operator L^R with Dirichlet boundary conditions on B_1 can be constructed more formally in suitable L^2–spaces as in Section 3.1. This enables us to analyze the spectral properties of L^R in a simpler form. Assume that the potential Φ does not take the value $+\infty$, i.e., the set $\mathcal{Z}$ defined in (2.1) is empty. By μ^R we denote the scaled stationary measure having Lebesgue density $e^{-2\Phi^R(x)/\sigma^2}$, $x \in \mathbb{R}^n$. Following the approach in Section 3.1, the scaled operator L^R defined in (6.4) applied to functions in $C_0^2(B_1)$ has a self-adjoint extension (also denoted by L^R) on the space $L^2(B_1, \mu^R)$. Note that this operator corresponds to the part of the scaled process $(X_t^R)_{t\geq 0}$ on the unit ball B_1, i.e., to the process $(X_t^R)_{t\geq 0}$ killed when it leaves B_1, see Lemma 3.3 for details.

Let us mention some spectral properties of the scaled operator L^R. From (6.2) and (6.5) it is seen that the bottom eigenvalue for the operators $-L_R$ and $-L^R$ coincide, i.e.,

$\lambda_R = \inf \Sigma(-L_R) = \inf \Sigma(-L^R)$, $R > 0$. Since B_1 is bounded, $-L^R$ has discrete spectrum and the eigenspace corresponding to the bottom eigenvalue λ_R is known to be simple, i.e., it is of the form $\{\alpha q^R : \alpha \in \mathbb{R}\}$ where q^R is a function in the domain of L^R. If the potential Φ is regular enough, i.e., if $\Phi \in C^{5+\left[\frac{n}{2}\right]}(\mathbb{R}^n, \mathbb{R})$, where $\left[\frac{n}{2}\right]$ is the largest integer smaller than $n/2$, regularity theory for elliptic differential operator tells us that $q^R \in C_0^2(B_1)$, see e.g. Theorem 6 in §5.6 and Theorem 5 in §6.3 of Evans [Eva98].

Let us express the bottom eigenvalue λ_R by means of the eigenfunction q^R, plugged into the eigenvalue problem (6.5). Since the potential Φ is supposed to be differentiable, multiplication by $e^{-2\Phi^R/\sigma^2}$ and integration over B_1 yields together with the representation (6.4) of L^R and Stoke's formula

$$\text{(6.6)} \qquad \lambda_R \int_{B_1} q^R e^{-2\Phi^R/\sigma^2}\, dx \;=\; -\frac{\sigma^2}{2R^2} \sum_{i=1}^{n} \int_{B_1} \partial_{x_i} \left(e^{-2\Phi^R/\sigma^2} \partial_{x_i} q^R \right) dx$$
$$= \frac{\sigma^2}{2R^2} \int_{S^{n-1}} e^{-2\Phi^R/\sigma^2} \partial_r q^R \, d\sigma \,,$$

where $d\sigma$ is the surface measure of the unit sphere S^{n-1} and ∂_r denotes the derivative in radial direction.

The idea of our approach in this chapter is to use singular perturbation techniques to derive a formal expansion of the eigenfunction q^R. We give a short introduction to these methods, for details we refer to the book of Kevorkian and Cole [KC81]. We make the ansatz that the principal eigenfunction q^R is given by the asymptotic expansion

$$\text{(6.7)} \qquad q^R \sim \sum_{i=0}^{\infty} q_i R^{-i} \qquad (R \to \infty)\,,$$

meaning that $q^R(x) - \sum_{i=0}^{k} q_i(x) R^{-i} = o(R^{-k})$ as $R \to \infty$ uniformly in $x \in B_1$ for every $k = 1, 2, \ldots$. This expansion is plugged into the eigenvalue equation $L^R q^R = -\lambda_R q^R$ and terms of the same order in R are grouped together and solved recursively. This leads to the so called outer expansion and is usually quite accurate in the interior of B_1. However the outer expansion is not capable to gather the boundary condition $q^R \equiv 0$ on S^{n-1}. In order to ensure its validity we introduce a boundary layer. This means that we choose suitable new variables depending on R which are concentrated at the boundary of B_1, see (6.17). We denote by Q^R the function q^R expressed in the boundary layer variables and assume again that Q^R admits an expansion

$$Q^R \sim \sum_{i=0}^{\infty} Q_i R^{-i} \qquad (R \to \infty)\,. \tag{6.8}$$

Expressing also the eigenvalue equation $L^R q^R = -\lambda_R q^R$ in the new variables in terms of the function Q^R, we repeat the above described expansion procedure, i.e., we group the terms of the same order in R together and solve recursively. This yields a good approximation near the boundary of B_1. To obtain a uniform expansion

$$q^R \sim \sum_{i=0}^{\infty} q_i^u R^{-i} \qquad (R \to \infty)\,, \tag{6.9}$$

which covers the outer expansion as well as the boundary layer expansion, the terms q_i and Q_i, $i = 0, 1, \ldots$, have to be matched, avoiding multiple counting of intermediate terms. The leading term of the asymptotics of λ_R is obtained by plugging the leading term q_0^u of the uniform expansion into Equation (6.6).

Similar methods are used in Naeh et al. [NKMS90] for the exit problem of Freidlin and Wentzell, see Section 1.2. In their work, the scaled potential $R^{-2}\Phi^R$ in Definition (6.4) of L^R is replaced by a potential independent of R. The new feature of our approach is that the boundary layer variable does not only depend on R but also on the spherical variables. This enables us to adjust the decay of the principal eigenfunction q^R near the boundary of B_1 asymptotically in the limit $R \to \infty$ to the slope of the potential in the particular direction.

6.2 Main Result and Proof

Since the domain of the scaled operator L^R is the unit ball B_1, it is appropriate to state the results in polar coordinates. We write $\mathbb{R}^n \setminus \{0\} \ni x = r e_\theta$ where $r = |x| > 0$ and $(e_\theta)_{\theta\in\Theta}$ is a smooth parameterization of the unit sphere S^{n-1} in $\mathbb{R}^n$. The defining equation for the principal eigenfunction q^R (see (6.4) and (6.5)) reads

$$\frac{\sigma^2}{2}\left(\partial_r^2 q^R + \frac{n-1}{r}\,\partial_r q^R + \frac{1}{r^2}\Delta_\theta q^R\right) - \left(\partial_r \Phi^R \partial_r q^R + \frac{1}{r^2}\nabla_\theta \Phi^R \cdot \nabla_\theta q^R\right) = R^2 \lambda_R\, q^R\,. \tag{6.10}$$

Here ∇_θ and Δ_θ denotes the gradient and Laplace operator w.r.t. the spherical coordinates θ, respectively. We introduce the following uniform asymptotic relation: for functions

$a_R, b_R : I \to \mathbb{R}$, where I is an arbitrary set, we say $a_R(t) = o(b_R(t))$ as $R \to \infty$ uniformly in $t \in I$ if $\lim_{R\to\infty} \sup_{t\in I} |a_R(t)/b_R(t)| = 0$.

Let us formulate the assumptions on the potential Φ. Suppose that $\Phi \in C^2(\mathbb{R}^n, \mathbb{R})$ for the potential in the SDE (1.11). Further assume that

$$\liminf_{r\to\infty} \min_{\theta\in\Theta} \partial_r \Phi(r,\theta) > 0\,. \tag{6.11}$$

Note that this type of assumption was also needed in the examples of Section 4.2 to ensure the spectral gap condition, see especially Assumption (4.18) in Section 4.2.2. Further, (6.11) implies that the stationary measure with Lebesgue density $e^{-2\Phi(x)/\sigma^2}dx$, $x \in \mathbb{R}^n$, is finite with total mass $|\mu|$, see (2.5). For the scaled potential Φ^R defined in (6.3) we obtain

$$\partial_r \Phi^R(r,\theta) = R\,\partial_r \Phi(Rr,\theta) \qquad R > 0, r \in [0,1], \theta \in \Theta\,.$$

Hence for every $\delta \in (0,1)$ there exists a constant $\kappa > 0$ by (6.11) such that

$$\partial_r \Phi^R(r,\theta) \geq R\kappa \qquad r \in (\delta,1], \theta \in \Theta\,. \tag{6.12}$$

To control the derivative of Φ w.r.t. the spherical variables θ we assume that for every $\delta \in (0,1)$

$$\nabla_\theta \Phi^R(r,\theta) = o\left(\partial_r \Phi^R(r,\theta)\right) \quad \text{as } R \to \infty \text{ uniformly in } (r,\theta) \in [\delta,1] \times \Theta. \tag{6.13}$$

We write for short

$$f_R(\theta) := \partial_r \Phi^R(1,\theta)\,, \qquad f_{2,R}(\theta) := \partial_r^2 \Phi^R(1,\theta)\,, \tag{6.14}$$

and assume that

$$\left|\frac{\nabla_\theta f_R}{f_R}\right|^2, \ \frac{\Delta_\theta f_R}{f_R} = o(f_R^2)\,, \quad f_{2,R} = o(f_R) \qquad \text{as } R \to \infty \text{ uniformly in } \theta \in \Theta. \tag{6.15}$$

Note that $f_R(\theta) \to \infty$ as $R \to \infty$ at least linearly uniformly in $\theta \in \Theta$ by (6.12). The first part of Condition (6.15) ensures that the growth of the potential Φ in radial direction dominates the behavior in the spherical components. In particular, a spherical oscillation of Φ increasing too fast with the radius is forbidden. The second part of Condition (6.15) is a regularity condition on the radial growth of the potential Φ (compare this with Condition (4.19) for the example of Section 4.2.2).

Theorem 6.1 *Assume (6.11), (6.13), and (6.15). Set*

$$(6.16)\, l(R) := \frac{R^{n-2}}{|\mu|}\int_{S^{n-1}} \partial_r \Phi^R e^{-2\Phi^R/\sigma^2} d\sigma = \frac{R^{n-2}}{|\mu|}\int_{\Theta} \partial_r \Phi^R(1,\theta)\, e^{-2\Phi^R(1,\theta)/\sigma^2} d\sigma(\theta)\,,$$

where $|\mu|$ the total mass of the stationary measure. Then the function l satisfies (2.9), i.e., $\lambda_R \sim l(R)$ as $R \to \infty$.

Proof. In the sequel we drop the index R of the eigenfunction q^R.

Step 1: We show that $R^2\lambda_R \searrow 0$ as $R \to \infty$. We use the variational principle, see Proposition 3.5. Set $\phi(r) := \min_{\theta\in\Theta} \Phi(r,\theta)$. As in Step 1 of the proof of Theorem 4.1 we obtain (see Remark 4.2.(1)) that there exists a constant $k > 0$ such that

$$\lambda_R \lesssim k \left(\int_1^R t^{1-n} e^{2\phi(t)/\sigma^2} dt \right)^{-1} \qquad (R \to \infty)\,.$$

Condition (6.11) implies that $\phi(r)$ grows at least linearly as $r \to \infty$ and this implies the result.

Step 2: Outer expansion. We expand the eigenvalue equation (6.10) in powers of R. Using Condition (6.13) and the relation (6.12) together with Step 1, it turns out that the term of highest order in R is the term $\partial_r\Phi^R$. Hence the leading term q_0 of the outer expansion (6.7) satisfies $\partial_r\Phi^R\partial_r q_0 = 0$. Since $\partial_r\Phi^R > 0$ on the complement of an arbitrary small neighborhood of the origin by (6.12), the function $q_0(\cdot,\theta)$ is constant for every $\theta \in \Theta$. But q_0 is continuous at the origin and hence constant on B_1. Since q_0 is specified up to a multiplicative constant, we scale q_0 to 1.

Step 3: Boundary layer expansion. We define the boundary layer variable

$$(6.17) \qquad \rho = \rho(r,\theta) := f_R(\theta)(r-1)\ (\le 0) \qquad r \in [0,1], \theta \in \Theta\,,$$

where f_R is defined in (6.14). We denote by $Q(\rho,\theta) = q(r,\theta)$ the function q in the new variables ρ, θ. The eigenfunction q satisfies the boundary condition $q(1,\theta) = 0$, $\theta \in \Theta$. This induces

$$(6.18) \qquad Q(0,\theta) = 0 \qquad \theta \in \Theta\,.$$

A function $h = h(r,\theta)$ expressed in the new variable ρ is denoted by $h_\rho = h_\rho(\rho,\theta) = h(1 + \rho/f_R(\theta), \theta)$. To express the derivatives of q in terms of Q, we use that $q(r,\theta) =$

$Q(f_R(\theta)(r-1),\theta)$ to obtain

$$\begin{aligned}
(\partial_r q)_\rho &= f_R\,\partial_\rho Q\,,\\
(\partial_r^2 q)_\rho &= f_R^2\,\partial_\rho^2 Q\,,\\
(\nabla_\theta q)_\rho &= \nabla_\theta Q + (r-1)\nabla_\theta f_R\,\partial_\rho Q = \nabla_\theta Q + \rho\frac{\nabla_\theta f_R}{f_R}\partial_\rho Q\,,\\
(\Delta_\theta q)_\rho &= \Delta_\theta Q + 2(r-1)\nabla_\theta f_R\cdot\nabla_\theta Q + (r-1)^2|\nabla_\theta f_R|^2\partial_\rho^2 Q + (r-1)\Delta_\theta f_R\,\partial_\rho Q\\
&= \Delta_\theta Q + 2\rho\frac{\nabla_\theta f_R}{f_R}\cdot\nabla_\theta Q + \rho^2\left|\frac{\nabla_\theta f_R}{f_R}\right|^2\partial_\rho^2 Q + \rho\frac{\Delta_\theta f_R}{f_R}\partial_\rho Q\,.
\end{aligned}$$

Let us express the eigenvalue equation (6.10) in the new variable Q. We replace r by $1+\rho/f_R(\theta)$ and we obtain

$$\begin{aligned}
\frac{\sigma^2}{2}\Big\{&[f_R^2]\,\partial_\rho^2 Q + (n-1)\Big[\frac{f_R^2}{\rho+f_R}\Big]\partial_\rho Q\\
&+\Big[\frac{f_R}{\rho+f_R}\Big]^2\Big(\Delta_\theta Q + 2\rho\Big[\frac{\nabla_\theta f_R}{f_R}\Big]\cdot\nabla_\theta Q + \rho^2\Big[\frac{|\nabla_\theta f_R|}{f_R}\Big]^2\partial_\rho^2 Q + \rho\Big[\frac{\Delta_\theta f_R}{f_R}\Big]\partial_\rho Q\Big)\Big\}\\
&-\Big\{f_R(\partial_r\Phi^R)_\rho\partial_\rho Q + \Big[\frac{f_R}{\rho+f_R}\Big]^2(\nabla_\theta\Phi^R)_\rho\cdot\nabla_\theta Q\Big\} = R^2\lambda_1(R)\,Q\,.
\end{aligned}\tag{6.19}$$

We expand the terms $(\partial_r\Phi^R)_\rho$ and $(\nabla_\theta\Phi^R)_\rho$ asymptotically as $R\to\infty$

$$\begin{aligned}
(\partial_r\Phi^R)_\rho(\rho,\theta) &= \partial_r\Phi^R(1+\rho/f_R(\theta),\theta)\\
&\sim \partial_r\Phi^R(1,\theta) + \partial_r^2\Phi^R(1,\theta)\frac{\rho}{f_R(\theta)}\\
&= f_R\left(1+\rho\frac{f_{2,R}}{f_R}\right)\,,\\
(\nabla_\theta\Phi^R)_\rho(\rho,\theta) &= \nabla_\theta\Phi^R(1+\rho/f_R(\theta),\theta)\\
&\sim \nabla_\theta\Phi^R(1,\theta) + \partial_r\nabla_\theta\Phi^R(1,\theta)\frac{\rho}{f_R(\theta)}\\
&= f_R\frac{\nabla_\theta\Phi^R(1,\theta)}{\partial_r\Phi^R(1,\theta)} + \rho\frac{\nabla_\theta f_R}{f_R}\,.
\end{aligned}$$

By Condition (6.15), the leading term of $(\partial_r\Phi^R)_\rho$ is f_R. Further, using Assumption (6.13) and again Condition (6.15) we see that $(\nabla_\theta\Phi^R)_\rho = o(f_R) = o(f_R^2)$ as $R\to\infty$ uniformly in ρ,θ. Looking at the remaining terms depending on R in Equation (6.19) (the terms in square brackets), it is seen by Condition (6.15) and Step 1 that the leading term in R is the term f_R^2 (see (6.12)). Hence the leading term Q_0 of the boundary layer expansion (6.8) has to satisfy the equation

$$\frac{\sigma^2}{2}\partial_\rho^2 Q_0 - \partial_\rho Q_0 = 0\,.\tag{6.20}$$

In order to guarantee that the leading term Q_0 of the boundary layer expansion matches with the first order outer expansion $q_0 \equiv 1$, we need to assume that for every $\theta \in \Theta$ (having Definition (6.17) of the boundary layer variable ρ in mind)

$$Q_0(\rho, \theta) \to 1 \qquad (\rho \to -\infty)\,. \tag{6.21}$$

Equation (6.20) together with the boundary conditions (6.18) and (6.21) is uniquely solved by

$$Q_0(\rho, \theta) = 1 - e^{2\rho/\sigma^2} \qquad \rho \in (-\infty, 0], \theta \in \Theta\,.$$

Note that Q_0 is independent of θ.

Step 4: Uniform expansion and eigenvalue asymptotics. Since the leading term of the outer expansion $q_0 \equiv 1$ and the leading term of the boundary layer expansion Q_0 satisfy the matching condition (6.21), the leading term q_0^u of the uniform expansion (6.9) coincides with Q_0. Writing this in the original variables we obtain

$$q_0^u(r, \theta) = 1 - e^{-2f_R(\theta)(1-r)/\sigma^2} \qquad r \in [0, 1],\ \theta \in \Theta\,.$$

To obtain the leading term of the asymptotics for λ_R, we plug q_0^u into Equation (6.6). For the integral on the LHS of Equation (6.6), we can use the outer expansion $q_0 \equiv 1$ and obtain

$$\int_{B_1} q^R e^{-2\Phi^R/\sigma^2} dx \sim \int_{B_1} e^{-2\Phi^R/\sigma^2} dx = \frac{1}{R^n} \int_{B_R} e^{-2\Phi/\sigma^2} dx \sim \frac{|\mu|}{R^n} \quad (R \to \infty)\,. \tag{6.22}$$

For the integral on the RHS of Equation (6.6) we obtain

$$\begin{aligned} \int_{S^{n-1}} e^{-2\Phi^R/\sigma^2} \partial_r q^R \, d\sigma \ &\sim \int_{S^{n-1}} e^{-2\Phi^R/\sigma^2} \partial_r q_0^R \, d\sigma \\ &= \frac{2}{\sigma^2} \int_{\Theta} e^{-2\Phi^R(1,\theta)/\sigma^2} f_R(\theta)\, d\sigma(\theta) \\ &= \frac{2}{\sigma^2} \int_{S^{n-1}} e^{-2\Phi^R/\sigma^2} \partial_r \Phi^R \, d\sigma \quad (R \to \infty)\,. \end{aligned} \tag{6.23}$$

From Equation (6.6) we obtain together with (6.22) and (6.23)

$$\begin{aligned} \lambda_R &= \frac{\sigma^2}{2R^2} \int_{S^{n-1}} e^{-2\Phi^R/\sigma^2} \partial_r q^R \, d\sigma \left(\int_{B_1} q^R e^{-2\Phi^R/\sigma^2} dx \right)^{-1} \\ &\sim \frac{R^{n-2}}{|\mu|} \int_{S^{n-1}} e^{-2\Phi^R/\sigma^2} \partial_r \Phi^R \, d\sigma \qquad (R \to \infty)\,. \end{aligned}$$

□

6.3 Examples

In this section we show that the asymptotics of the bottom eigenvalue λ_R as $R \to \infty$ evaluated by means of singular perturbation techniques (Theorem 6.1) applied to the examples in Section 4.2 coincide with the eigenvalue asymptotics obtained using the methods of Chapter 4. We omit here to show explicitly that the asymptotic growth conditions of Theorem 6.1 are satisfied, which are needed for the singular perturbation approach. This inaccurateness is justified, since we have shown for the examples of Section 4.2 that the assumptions of Theorem 4.1 are satisfied, which is a more profound result than Theorem 6.1 derived via the heuristically based singular perturbation techniques.

6.3.1 The Rotationally Symmetric Case

Assume as in Section 4.2.1 that the potential Φ in the SDE (1.11) has the property that there exist $r_0 > 0$ and $\phi \in C^2([r_0, \infty), \mathbb{R})$ such that

$$\Phi(r, \theta) = \phi(r) \qquad r > r_0, \theta \in \Theta\,.$$

Note that in this case Conditions (6.13) and (6.15) on the spherical derivatives of Φ are obvious since $\nabla_\theta \Phi \equiv 0$. Further, the asymptotic growth condition (6.11) on the radial derivative of Φ corresponds to Assumption (4.13) in Section 4.2.1 which is used to guarantee the spectral gap property by Proposition 3.7.

To evaluate the asymptotics of λ_R by means of Theorem 6.1, we note that

$$f_R(\theta) = R\phi'(R) \qquad R > r_0,\ \theta \in \Theta\,.$$

Plugging this into the asymptotic expression (6.16) for the eigenvalue asymptotics, we obtain

$$\begin{aligned} \lambda_R &\sim \frac{R^{n-2}}{|\mu|} \int_\Theta R\phi'(R) e^{-2\phi(R)/\sigma^2}\, d\sigma(\theta) \\ &= \frac{\gamma_n}{|\mu|} R^{n-1} \phi'(R) e^{-2\phi(R)/\sigma^2} \qquad (R \to \infty)\,. \end{aligned} \tag{6.24}$$

In Theorem 4.6 we derived the asymptotic expression l_1 for the eigenvalue asymptotics satisfying $\lambda_R \sim l_1(R)$ as $R \to \infty$, given by

$$l_1(R) := \frac{\sigma^2 \gamma_n}{2|\mu|} \left(\int_{r_0}^R r^{1-n} e^{2\phi(r)/\sigma^2}\, dr \right)^{-1} \qquad R > 0\,.$$

If ϕ is of polynomial form, i.e., $\phi(r) = r^\alpha$, $\alpha \geq 1$, the asymptotic expression l_1 coincides with the asymptotics in (6.24), since by virtue of Lemma 4.5

$$\begin{aligned} l_1(R) &\sim \frac{\sigma^2\gamma_n}{2|\mu|}\left(\frac{\sigma^2}{2\alpha}R^{1-n-\alpha-1}e^{2R^\alpha/\sigma^2}\right)^{-1} \\ &= \frac{\gamma_n\alpha}{|\mu|}R^{n-1}R^{\alpha-1}e^{-2R^\alpha/\sigma^2} \qquad (R\to\infty)\,. \end{aligned}$$

6.3.2 Non-Symmetric Processes

We consider the two-dimensional example of Section 4.2.2 with a potential Φ in the SDE (1.11) of the following form: there exist $r_0 > 0$ and functions $\phi \in C^2([r_0,\infty),\mathbb{R})$, $\psi \in C^2([r_0,\infty),\mathbb{R}^+)$, and $p \in C^2(S^1,\mathbb{R}^+)$ such that

$$\Phi(r,\theta) = \phi(r) + p(\theta)\psi(r) \qquad r > r_0, \theta \in [\theta_0, \theta_0 + 2\pi)\,.$$

We assume the situation of Section 4.2.2, i.e., that Conditions (4.15) and (4.16) on the spherical function p and the growth conditions (4.17) and (4.19) on the asymmetric factor ψ hold. Note that the asymptotic growth condition (6.11) on the radial derivative of Φ coincides with the first condition of (4.18) ensuring the spectral gap property. Conditions (6.13) and (6.15) can easily be shown to hold if ϕ and ψ are of polynomial form.

To evaluate the asymptotics of λ_R by means of Theorem 6.1, we calculate

$$f_R(\theta) = \partial_r\Phi^R(1,\theta) = R\left(\phi'(R) + p(\theta)\psi'(R)\right) \qquad R > r_0,\ \theta \in [\theta_0,\theta_0+2\pi)\,.$$

Plugging this into the asymptotic expression (6.16) for the eigenvalue, we obtain

$$\lambda_R \sim \frac{R}{|\mu|}e^{-2\phi(R)/\sigma^2}\Big(\phi'(R)I_1(R) + \psi'(R)I_2(R)\Big) \qquad (R\to\infty)\,,$$

where

$$I_1(R) := \int_{\theta_0}^{\theta_0+2\pi} e^{-2p(\theta)\psi(R)/\sigma^2}d\theta\,, \qquad I_2(R) := \int_{\theta_0}^{\theta_0+2\pi} p(\theta)e^{-2p(\theta)\psi(R)/\sigma^2}d\theta\,.$$

The term $I_1(R)$ was evaluated asymptotically in (4.21) by $I_1(R) \sim C_1\psi(R)^{-1/\varpi_*}$ as $R\to\infty$, where

$$C_1 := (2/\varpi_*)(\sigma^2/2)^{1/\varpi_*}\Gamma(1/\varpi_*)\sum_{i\in J_*}P_i^{-1/\varpi_*}\,,$$

and ϖ_*, J_* are defined in (4.16). Using Laplace's method (Lemma 4.3), the term $I_2(R)$ can be evaluated similarly to the calculations in (4.22) as follows: there exists a constant $\kappa > 0$ such that $I_2(R) \sim \kappa\psi(R)^{-(1+1/\varpi_*)}$ as $R \to \infty$. Hence

$$\Big(\phi'(R)I_1(R) + \psi'(R)I_2(R)\Big) \sim \psi(R)^{-1/\varpi_*}\left(C_1\phi'(R) + \kappa\frac{\psi'(R)}{\psi(R)}\right) \qquad (R \to \infty)\,.$$

Since $\psi'(R) = o(\psi(R))$ as $R \to \infty$ by Condition (4.19) we obtain

$$\lambda_R \sim \frac{C_1}{|\mu|}R\psi(R)^{-1/\varpi_*}\phi'(R)e^{-2\phi(R)/\sigma^2} \qquad (R \to \infty)\,. \tag{6.25}$$

In Theorem 4.8 we derived the asymptotic expression l_1 for the eigenvalue asymptotics satisfying $\lambda_R \sim l_1(R)$ as $R \to \infty$ given by

$$l_1(R) := \frac{\sigma^2 C_1}{2|\mu|}\psi(R)^{-1/\varpi_*}\left(\int_{r_0}^{R} r^{-1}e^{2\phi(r)/\sigma^2}\,dr\right)^{-1} \qquad R > 0\,.$$

If ϕ is again of polynomial form, i.e., $\phi(r) = r^\alpha$ with $\alpha \geq 1$, the asymptotic expression l_1 coincides with the asymptotics in (6.25), since by virtue of Lemma 4.5

$$\begin{aligned} l_1(R) &\sim \frac{\sigma^2 C_1}{2|\mu|}\psi(R)^{-1/\varpi_*}\left(\frac{\sigma^2}{2\alpha}R^{-1-\alpha-1}e^{2R^\alpha/\sigma^2}\right)^{-1} \\ &= \frac{C_1\alpha}{|\mu|}R\psi(R)^{-1/\varpi_*}R^{\alpha-1}e^{-2R^\alpha/\sigma^2} \qquad (R \to \infty)\,. \end{aligned}$$

6.3.3 A Diffusion Process with Gamma Distribution

We consider the situation of Section 4.2.3 of a diffusion process of gradient field type having a bivariate gamma distribution as stationary measure. We choose $\sigma = \sqrt{2}$ in the SDE (1.11) and the potential Φ is given in polar coordinates for $r > 0$ by

$$\Phi(r,\theta) = \begin{cases} r\left(\frac{\cos\theta}{\beta_1} + \frac{\sin\theta}{\beta_2}\right) - \ln\left((r\cos\theta)^{\alpha_1-1}(r\sin\theta)^{\alpha_2-1}\right) & \theta \in (0,\pi/2)\,, \\ \infty & \text{otherwise,} \end{cases}$$

where $\alpha_1, \alpha_2 \geq 1$ and $0 < \beta_2 \leq \beta_1$. Note that in this case the potential Φ takes the value $+\infty$ and hence the set $\mathcal{Z}$ defined in (2.1) is not empty. In Section 4.2.3 however it is shown that there exists a weak solution of the SDE (1.11) with the above defined potential Φ in the sense of Proposition 3.1. For the evaluation of the eigenvalue asymptotics it suffices as in Section 4.2.3 to restrict the attention to the positive quadrant, i.e., to the parameter set $\{(r,\theta) : r > 0, \theta \in (0,\pi/2)\}$. We calculate in the positive quadrant

$$\partial_r \Phi^R = R\left(\frac{\cos\theta}{\beta_1} + \frac{\sin\theta}{\beta_2}\right) - \frac{\alpha_1+\alpha_2-2}{r}, \qquad \partial_r^2 \Phi^R = \frac{\alpha_1+\alpha_2-2}{r^2}.$$

Note that the asymptotic growth conditions (6.11) and (6.15) on the spherical derivative of Φ hold. However, Condition (6.13) on the radial derivative of Φ is not satisfied uniformly in $\theta \in (0,\pi/2)$, it holds only locally uniformly in θ.

Set $p(\theta) := \beta_1^{-1}(\cos\theta - 1) + \beta_2^{-1}\sin\theta$, $\theta \in (0,\pi/2)$. In order to evaluate the eigenvalue asymptotics by Theorem 6.1, we calculate

$$f_R(\theta) = R(\beta_1^{-1} + p(\theta)) - \alpha_1 + \alpha_2 - 2 \qquad R > 0,\ \theta \in (0,\pi/2).$$

Plugging this into (6.16), we obtain

$$\lambda_R \sim \frac{R^{\alpha_1+\alpha_2-1}}{|\mu|} e^{-R/\beta_1} \int_0^{\pi/2} \left(\beta_1^{-1} I_1(R) + I_2(R) + \frac{\alpha_1+\alpha_2-2}{R} I_1(R)\right) \tag{6.26}$$

as $R \to \infty$, where

$$\begin{aligned} I_1(R) &:= \int_0^{\pi/2} (\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1} e^{-Rp(\theta)} d\theta, \\ I_2(R) &:= \int_0^{\pi/2} p(\theta)(\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1} e^{-Rp(\theta)} d\theta. \end{aligned}$$

The term $I_1(R)$ is asymptotically evaluated in (4.27)

$$I_1(R) \sim \beta_2^{\alpha_2}\Gamma(\alpha_2)R^{-\alpha_2} + \delta_{\beta_1\beta_2}\beta_1^{\alpha_1}\Gamma(\alpha_1)R^{-\alpha_1} \qquad (R \to \infty), \tag{6.27}$$

where $\delta_{\beta_1\beta_2} = 1$ if $\beta_1 = \beta_2$ and $= 0$ otherwise. The term $I_2(R)$ can be evaluated similarly to the calculations in (4.28) using Laplace's method (Lemma 4.3) as follows: there exist constants $\kappa_1, \kappa_2 > 0$ such that $I_2(R) \sim \kappa_1 R^{-(\alpha_2+1)} + \delta_{\beta_1\beta_2}\kappa_2 R^{-(\alpha_1+1)}$ as $R \to \infty$. Hence $I_2(R) = o(I_1(R))$ as $R \to \infty$ and we obtain from (6.26) and (6.27) noting that $|\mu| = \prod_{i=1}^2 \beta_i^{\alpha_i}\Gamma(\alpha_i)$

$$\lambda_R \sim \frac{1}{\beta_1}\left(\frac{R^{\alpha_1-1}}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} + \delta_{\beta_1\beta_2}\frac{R^{\alpha_2-1}}{\Gamma(\alpha_2)\beta_2^{\alpha_2}}\right) e^{-R/\beta_1} \qquad (R \to \infty).$$

This expression coincides with the asymptotic expression of Theorem 4.10.

Chapter 7

Applications to Finance

7.1 Introduction

Multivariate stationary diffusions play an important role as models for the dynamics of a portfolio of stocks, exchange rates, or bonds with different maturities. An important feature of these models is the possibility of incorporating reasonable dependence structures between the different risk factors. From the point of view of risk management it is important to know about the extreme behavior of these models.

One-dimensional diffusion models have been proved useful for the modelling of financial data; stationary models such as those of Vašiček [Vaš77] or Cox-Ingersoll-Ross [CIR85] are prominent for modelling interest rates. Their extreme behavior is well-understood, see e.g. Borkovec and Klüppelberg [BK98]. For the investigation of more than one risk factor, as for the joint modelling of the term structure of interest rates for different currencies, the dependence structure is of high importance and triggered the present investigation.

Assume that for a given discrete multivariate data set a continuous time multivariate diffusion model is chosen and fitted to the data. Note that the fit of parameters is mainly based on the center of the distribution, i.e., where most data points are available. From the point of view of risk management, the following question arises: can the fitted model also explain the large fluctuations in the data? To illustrate this problem, suppose that a stationary Gaussian diffusion model such as the Vašiček model (see Section 7.3.1) has been fitted to a data set. It is important to know whether the postulation of a multivariate normal stationary distribution, having very little probability mass far away from its mean, is compatible with the extremes in the data. If this is not the case one should use a diffusion model whose stationary distribution has heavier than normal tails.

In this chapter, we show that multivariate diffusion processes of gradient field type represent a suitable class of models for financial applications. These processes are known to be stationary and time-reversible and the stationary distribution is explicitly available. Moreover for a fairly arbitrary continuous probability density on $\mathbb{R}^n$, a diffusion model of gradient field type can be constructed having this particular probability density as stationary density. In the preceding chapters, methods have been provided to analyze the large fluctuations of multivariate diffusion models of gradient field type. Moreover the fitting procedure of these continuous time multivariate diffusion models to discrete data is fairly straightforward.

We present some multivariate short-rate diffusion models of gradient field type and analyze explicitly their extreme behavior. For simplicity we restrict ourselves to the two-dimensional case. The most prominent example is the bivariate Vašiček model (see Section 7.3.1), a shifted Ornstein-Uhlenbeck process. The stationary measure of this model is a bivariate normal. In order to replace the stationary normal distribution by a more realistic distribution with heavier tails, we introduce a bivariate diffusion process having a symmetric bivariate exponential distribution as stationary measure (see Section 7.3.2). Further, we present a bivariate stationary diffusion model having a bivariate distribution with gamma distributed margins as stationary measure (see Section 7.3.3). The spatial dependence in the stationary measure is modeled using copula techniques. In particular the state space of this new model is the positive cone in $\mathbb{R}^n$, which is ideal for interest rate modelling. Further, the model has the nice feature of a stationary distribution with exponential tails in contrast to the Vašiček model.

As an application of the characterization of the large fluctuations for diffusion models of gradient field type, we develop methods for assessing the goodness-of-fit of such models in the extremes. The idea for the goodness-of-fit tests is to compare the sample maximum in Euclidean norm of the multivariate data set with its theoretical asymptotics. This approach has the disadvantage that it is very crude. An advantage is however, that it is fast and that there are in principal no restrictions on dimensionality. An important fact is that our tests respect the spatial dependence structure, since the asymptotics of the maximum in Euclidean norm of the diffusion is affected by this dependence.

We will present some simulation results for the bivariate Vašiček model and the exponential diffusion model together with the test results. The results of the goodness-of-fit tests are established when fitting these models to short term interest rates of different currencies (30-days Libor rates for Euro, British Pound, and US Dollar). It turns out that both models fitted to the data explain reasonably the extremes in the data set. The results for the exponential model are slightly better as expected.

In Section 7.2, the class of diffusion models of gradient field type is presented together with a characterization of the large fluctuations of these models. The above mentioned multivariate short-rate diffusion models are introduced in Section 7.3. The parameter estimation methods for the fit of diffusion models of gradient field type to discrete data is discussed in Section 7.4. In Section 7.5, the goodness-of-fit tests are developed and estimation and test results for simulated and real financial data are presented in Section 7.6.

7.2 Gradient Field Models and Their Large Fluctuations

For $n \in \mathbb{N}$, a n-dimensional diffusion process $(X_t)_{t\geq 0}$ can be defined in general form as the solution of a multivariate Itô SDE (see also (1.6))

$$dX_t^i = b^i(X_t)dt + \sum_{j=1}^{n} \sigma^{ij}(X_t)dB_t^j \qquad i = 1, \dots, n\,, \tag{7.1}$$

where $b^i, \sigma^{ij} : \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \dots, n$, and $(B_t^j)_{t\geq 0}$, $j = 1, \dots, n$, are independent one-dimensional standard Brownian motions. In such a general situation it is however difficult to answer immediately emerging important questions such as:

- is the process stationary?
- what is the shape of the stationary measure?
- how can the model be fitted to discrete data?
- what is the behavior of the large fluctuations of the process?

Instead of considering the general model (7.1), we suggest to restrict to diffusion processes of gradient field type. Let us recall the definition: the diffusion coefficient is constant in the sense that $\sigma^{ij}(x) = \sigma\delta_{ij}$, $x \in \mathbb{R}^n$, $i, j = 1, \ldots, n$, where $\sigma > 0$ and δ_{ij} is the Kronecker symbol. Further, the drift is given by the gradient of a potential function Φ, i.e., there exists a differentiable function Φ such that $b^i(x) = -\partial_{x_i}\Phi(x)$, $x \in \mathbb{R}^n$, $i = 1, \ldots, n$. Hence $(X_t)_{t\geq 0}$ solves a SDE of the form (see also (1.11))

$$dX_t^i = -\partial_{x_i}\Phi(X_t)dt + \sigma dB_t^i \qquad i = 1, \ldots, n\,. \tag{7.2}$$

This class of diffusion processes may also be compared with the *reducible diffusion processes* introduced in Aït-Sahalia [AS02], which can be transformed in a diffusion process whose diffusion matrix is the identity. The drift however may be arbitrary. Reducible diffusion processes show many advantages in statistical estimation.

It is known that a diffusion process of gradient field type is stationary and time reversible. The stationary measure μ on $\mathbb{R}^n$ has a simple structure; its Lebesgue density is given by

$$\widetilde{\mu}(x) = e^{-2\Phi(x)/\sigma^2} \qquad x \in \mathbb{R}^n\,. \tag{7.3}$$

Some applications require that the state space of the diffusion process is reduced from $\mathbb{R}^n$ to some open set $O \subset \mathbb{R}^n$. In this case, it must be guaranteed that the process does not leave the set O. For the class of diffusion processes of gradient field type, this can be done under quite general conditions. The requirement that the process does not leave the set O implies that the density $\widetilde{\mu}$ of the stationary measure equals 0 on the set $\mathbb{R}^n \setminus O$. Relation (7.3) suggests that the potential Φ must take the value $+\infty$ on $\mathbb{R}^n \setminus O$. Meyer and Zheng [MZ85] have shown the following existence theorem, see also Proposition 3.1: assume that the potential Φ satisfies Conditions (2.2) and (2.3). This means that $\Phi \in C(\mathbb{R}^n, \mathbb{R}\cup\{+\infty\})$ and $\Phi_{|O} \in C^1(O, \mathbb{R})$, where $O := \{x \in \mathbb{R}^n : \Phi(x) < \infty\}$. Further suppose that the integrability condition

$$\int_O |\nabla\Phi(x)|^2\, e^{-4\Phi(x)/\sigma^2}\, dx < \infty \tag{7.4}$$

holds, where ∇ denotes the gradient. Then there exists a weak solution $(X_t)_{t\geq 0}$ of the SDE (7.2), which is stationary and reversible w.r.t. the stationary measure μ having Lebesgue density

$$\widetilde{\mu}(x) = \begin{cases} e^{-2\Phi(x)/\sigma^2} & x \in O\,, \\ 0 & x \in \mathbb{R}^n \setminus O\,. \end{cases}$$

Further, $(X_t)_{t\geq 0}$ exits the set O with P_μ probability zero, where P_μ denotes the law of the process $(X_t)_{t\geq 0}$ starting with its stationary measure μ.

This result shows that the class of diffusion models of gradient field type is a rather flexible class. We can start with an arbitrary probability density $\widetilde{\mu} \in C(\mathbb{R}^n, [0,\infty))$ which is differentiable on the set $O := \{x \in \mathbb{R}^n : \widetilde{\mu}(x) > 0\}$. Extract the potential Φ from the relation (7.3), i.e., $\Phi(x) = -(\sigma^2/s)\ln\widetilde{\mu}(x)$, $x \in \mathbb{R}^n$ (setting $\ln 0 := +\infty$). If Φ satisfies the integrability condition (7.4), we obtain a stationary diffusion process where $\widetilde{\mu}$ is the density of the stationary measure. This approach can be seen in the tradition of the Markov Chain Monte Carlo method, where a given spatial distribution is put into a dynamic framework in the sense that an ergodic Markov chain is constructed having the given distribution as stationary distribution.

A further advantage of the diffusion models of gradient field type is that the fit to discrete data and parameter estimation can be performed in a relatively simple way. Especially the diffusion coefficient σ can be easily estimated with high precision, see Section 7.4.

The most interesting point for risk management is to quantify the extreme behavior of these models. In Chapter 2 and 4, the partial maxima of the process in Euclidean norm have been analyzed, i.e., the random variable

$$M_T := \max_{0\leq t\leq T} |X_t| \qquad T \geq 0\,, \tag{7.5}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. An asymptotic characterization of the tail behavior for M_T for fixed $T > 0$ could be provided and the long term behavior of M_T as $T \to \infty$ was determined. Let us recall here the two main results. Theorem 2.3 and 4.1 state that under certain conditions on Φ there exist a positive real function l and a constant $k > 0$ (both depending only on Φ and σ) such that for every fixed $T > 0$

$$T\,l(R) \lesssim P_\mu(M_T > R) \lesssim (T+k)\,l(R) \qquad (R \to \infty)\,. \tag{7.6}$$

Here $a(R) \lesssim b(R)$ as $R \to \infty$ means that $\limsup_{R\to\infty} a(R)/b(R) \leq 1$, where a, b are real functions. Note that $k = 1/\Lambda_{sg}$, where Λ_{sg} is the spectral gap defined in (2.7).

Further, the long time behavior of M_T as $T \to \infty$ is characterized in the sense of classical extreme value theory, see Section 1.1. Assume that the function

$$F(R) := e^{-l(R)} \qquad R > 0 \tag{7.7}$$

is in the domain of attraction of an extreme value distribution function H ($F \in \mathrm{DA}(H)$) in the sense that there exist norming constants $c_T > 0$ and $d_T \in \mathbb{R}$, $T > 0$ (given in terms of Φ and σ), such that

$$\lim_{T\to\infty} F(c_T x + d_T)^T = H(x) \qquad x \in \mathbb{R}, \tag{7.8}$$

see also (1.3) and (2.14). We then obtain the following long term limit result for the renormalized maximum (see Theorem 2.5 and Corollary 2.6)

$$c_T^{-1}(M_T - d_T) \xrightarrow{d} H \qquad (T \to \infty), \tag{7.9}$$

where $\xrightarrow{d}$ denotes convergence in distribution. In our concrete examples it turns out that $F \in \mathrm{DA}(\Lambda)$, where Λ is the Gumbel distribution, see (1.2).

7.3 Multivariate Short-Rate Models

We describe in this section the multivariate short-rate models mentioned in the introduction and evaluate the large fluctuations of these models. We give an explicit expression for the function l appearing in the tail asymptotics (7.6) of the maximum M_T. Further, the norming constants $(c_T)_{T>0}$ and $(d_T)_{T>0}$ in the long time limit (7.9) of the renormalized maximum are stated explicitly. For simplicity we restrict to the bivariate case.

7.3.1 Vašiček Model

The bivariate Vašiček model is the diffusion process $(X_t^V)_{t\geq 0}$ of gradient field type solving the SDE (7.2) with potential

$$\Phi(x) = \frac{1}{2}\sum_{i,j=1}^{2} \alpha^{ij}(x_i - m_i)(x_j - m_j) \qquad x \in \mathbb{R}^2, \tag{7.10}$$

where the matrix $A = (\alpha^{ij})_{i,j=1,2}$ is symmetric and strictly positive definite and $(m_1, m_2) \in \mathbb{R}^2$ corresponds to the mean of the process. Note that this potential generates a linear drift of the form

$$b^i(x) = -\sum_{j=1}^{2} \alpha^{ij}(x_j - m_j) \qquad x \in \mathbb{R}^2,\ i = 1, 2\,.$$

We suggest a parameterization of the class of possible matrices A (which also leads to a parameterization of the potentials Φ) in terms of its eigenvalues α_1, α_2 and the rotation part expressed by a matrix R^ϕ as follows:

$$A_\theta = R^\phi \mathrm{diag}(\alpha_1, \alpha_2) R^{-\phi} \qquad R^\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \tag{7.11}$$
$$\theta := (\alpha_1, \alpha_2, \phi) \in \Theta := (0,\infty)^2 \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

Here $\mathrm{diag}(\alpha_1, \alpha_2)$ denotes the diagonal matrix in $\mathbb{R}^{2\times 2}$ with entries α_1, α_2. The stationary measure of $(X_t^V)_{t\geq 0}$ is a bivariate normal $N(m, \Sigma)$ with covariance matrix

$$\Sigma = \left(2A_\theta/\sigma^2\right)^{-1} = \frac{\sigma^2}{2} R^{-\phi} \mathrm{diag}(\alpha_1^{-1}, \alpha_2^{-1}) R^{-\phi}\,. \tag{7.12}$$

Hence the spatial dependence structure of this process is induced by the rotation matrix R^ϕ.

If $m_1 = m_2 = 0$, the Vašiček model reduces to an Ornstein-Uhlenbeck process (OU process). This process is denoted by $(X_t^{OU})_{t\geq 0}$ and solves the SDE (7.2) with potential

$$\Phi(x) = \frac{1}{2}\sum_{i,j=1}^{2} \alpha^{ij} x_i x_j \qquad x \in \mathbb{R}^2\,. \tag{7.13}$$

In particular, the centered Vašiček model coincides with the OU process in the sense that

$$(X_t^V - m)_{t\geq 0} = (X_t^{OU})_{t\geq 0} \quad \text{in law.} \tag{7.14}$$

We analyze the large fluctuation of the centered Vašiček model $(X_t^V)_{t\geq 0}$ assuming that $m_1 = m_2 = 0$. Set $\alpha^* := \max\{\alpha_1, \alpha_2\}$ and $\alpha_* := \min\{\alpha_1, \alpha_2\}$. In Section 4.2.4, the function l satisfying (7.6) was evaluated by (see (4.31) and (4.32))

$$l(R) = \begin{cases} \dfrac{2\alpha_*^2}{\sigma^2} R^2 e^{-\alpha_* R^2/\sigma^2} & \alpha_* = \alpha^*\,, \\[2ex] 2\sqrt{\dfrac{\alpha_*^3 \alpha^*}{\sigma^2 \pi (\alpha^* - \alpha_*)}} R e^{-\alpha_* R^2/\sigma^2} & \alpha_* < \alpha^*\,. \end{cases} \tag{7.15}$$

The constant k appearing in (7.6) reads in our situation, see (4.35)

$$k = 1/\Lambda_{sg} = 1/\min\{\alpha_1, \alpha_2\} = \alpha_*^{-1}. \tag{7.16}$$

Further, $F \in \mathrm{DA}(\Lambda)$, where $F := e^{-l}$ is defined as in (7.7) and Λ is the Gumbel distribution. The long time limit (7.9) then holds with norming constants (see (4.34))

$$c_T = \frac{1}{2}\sqrt{\frac{\sigma^2}{\alpha_* \ln T}},$$

$$d_T = \sqrt{\frac{\sigma^2 \ln T}{\alpha_*}} + \begin{cases} \frac{1}{2}\sqrt{\frac{\sigma^2}{\alpha_* \ln T}} (\ln\ln T + \ln(2\alpha_*)) & \alpha_* = \alpha^*, \\ \frac{1}{4}\sqrt{\frac{\sigma^2}{\alpha_* \ln T}} \left(\ln\ln T + \ln\left(\frac{4\alpha_*^2\alpha^*}{\pi(\alpha^* - \alpha_*)}\right)\right) & \alpha_* < \alpha^*. \end{cases} \tag{7.17}$$

7.3.2 Exponential Process

We present a diffusion model of gradient field type having a bivariate symmetric double-exponential distribution as stationary measure. This means that the potential is of the form $\Phi(x) = \sum_{j=1}^2 \alpha_j |x_j|$, $x \in \mathbb{R}^2$, with $\alpha_j > 0$, $j = 1, 2$. Comparing this with the OU process having a bivariate normal as stationary measure, this model is expected to explain better larger values in the data as seen in most financial data sets. We present here the structure of the model and defer tedious calculations to Appendix A.1.

In order to allow for spatial dependence and for the mean to be different from zero, the potential Φ in the SDE (7.2) is parameterized as follows:

$$\begin{aligned} \Phi_\theta(x) &:= \sum_{i=1}^2 \alpha_i \left| \sum_{j=1}^2 R^\phi_{ij}(x_j - m_j) \right| \qquad x \in \mathbb{R}^2, \\ &\theta := (\alpha_1, \alpha_2, m_1, m_2, \phi) \in \Theta := (0,\infty)^2 \times \mathbb{R}^2 \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \end{aligned} \tag{7.18}$$

where R^ϕ is the rotation matrix defined in (7.11) and $m = (m_1, m_2) \in \mathbb{R}^2$ corresponds to the mean of the process. Let $(X_t^E)_{t\geq 0}$ be the diffusion process of gradient field type solving the SDE (7.2) with potential defined above. Note that the potential Φ_θ corresponds to a drift of the form

$$b^i(x) = \sum_{j=1}^2 R^{-\phi}_{ij} \alpha_j \operatorname{sign}\left(\sum_{k=1}^2 R^\phi_{jk}(x_k - m_k)\right) \qquad x \in \mathbb{R}^2,\ i = 1, 2.$$

As for the OU process, the spatial dependence is modeled by the rotation matrix R^{ϕ}. The covariance matrix Σ of the stationary measure of $(X_t^E)_{t\geq 0}$ reads in this situation

$$\Sigma = \frac{\sigma^4}{4} R^{-\phi} \mathrm{diag}(\alpha_1^{-2}, \alpha_2^{-2}) R^{\phi} \,. \tag{7.19}$$

For the evaluation of the large fluctuations, we assume $m_1 = m_2 = 0$, i.e., the process is centered at the origin. Set $\alpha^* := \max\{\alpha_1, \alpha_2\}$ and $\alpha_* := \min\{\alpha_1, \alpha_2\}$. The function l satisfying (7.6) is given in this situation by (see Appendix A.1 for a derivation)

$$l(R) = \frac{2\alpha_*^2}{\sigma^2}(1 + \delta_{\alpha_1\alpha_2}) e^{-2\alpha_* R/\sigma^2} \qquad R > 0\,, \tag{7.20}$$

where $\delta_{\alpha_1\alpha_2} = 1$ if $\alpha_1 = \alpha_2$ and $= 0$ otherwise. Note that in the two-dimensional case the function l does not depend on α^*; this changes in higher dimensions.

Further, $F := e^{-l} \in \mathrm{DA}(\Lambda)$ and the long time limit (7.9) then holds with norming constants (see Appendix A.1)

$$c_T = \frac{\sigma^2}{2\alpha_*}\,, \quad d_T = \frac{\sigma^2}{2\alpha_*} \ln\left(\frac{2\alpha_*^2}{\sigma^2}(1 + \delta_{\alpha_1\alpha_2}) T\right) \qquad T > 0\,. \tag{7.21}$$

7.3.3 Gamma Process

In Section 4.2.3, a bivariate stationary diffusion process of gradient field type was presented having the product measure of two independent gamma distributions as stationary measure. This process was constructed assuming $\sigma = \sqrt{2}$. We want to generalize the model implementing a spatial dependence structure. Further, an arbitrary $\sigma > 0$ shall be allowed. As for the exponential process, we defer tedious calculations to Appendix A.2.

This diffusion model can be seen as a contribution to the efforts to present a tractable multivariate extension of the one-dimensional Cox-Ingersoll-Ross model (CIR model) first established in Cox et al. [CIR85]. A multivariate generalization was suggested by Duffie and Kan [DK96]. Jacobson [Jac01] showed that this model is not reversible and presented a different model, which though also not reversible has some nice analytical properties. Since our diffusion model is of gradient field type, it does not possess the square root diffusion term, which is characteristic for the CIR model. But it has the advantage of being reversible w.r.t. an easily accessible measure.

Let us introduce some notation. For $\alpha \geq 1$, $\beta > 0$, the density of the one-dimensional $\Gamma(\alpha, \beta)$-distribution is given by $g_{\alpha,\beta}(x) = (\beta^\alpha \Gamma(\alpha))^{-1} x^{\alpha-1} e^{-x/\beta}$, $x \geq 0$, and the cumulative distribution function is denoted by $G_{\alpha,\beta}$.

To create a bivariate gamma distribution with dependence, we use the techniques of copulas. We refer to the book of Joe [Joe97] for details. A copula $C = C(u,v)$ is the distribution function of a multivariate (here bivariate) random variable with all univariate margins being uniformly distributed on the interval $[0,1]$. In combination with arbitrarily given marginals it defines a multivariate distribution function uniquely. The density is given by $c(u,v) := \partial_u \partial_v C(u,v)$.

As an example we use the following one-parameter family of copulas, classified as Family B5 in Joe [Joe97]:

$$C_\eta(u,v) := 1 - (\overline{u}^\eta + \overline{v}^\eta - \overline{u}^\eta \overline{v}^\eta)^{1/\eta} \qquad u, v \in [0,1],\ \eta \in [1, \infty)\,,$$

where $\overline{u} := 1 - u$, $\overline{v} := 1 - v$. The density reads

$$c_\eta(u,v) = (\overline{u}^\eta + \overline{v}^\eta - \overline{u}^\eta \overline{v}^\eta)^{-2+1/\eta} (\overline{u}\overline{v})^{\eta-1} (\eta - 1 + \overline{u}^\eta + \overline{v}^\eta - \overline{u}^\eta \overline{v}^\eta)\,.$$

We have chosen this particular family of copulas due to its simple shape, since we also need to differentiate the density c_η. This copula family is constructed as a mixture of powers via the Laplace transform. It is stochastically increasing and shows an upper tail dependence. Note that for $\eta = 1$ we obtain the independent copula $C_1(u,v) = uv$, and in the limit $\eta \to \infty$ the Fréchet upper bound $C_\infty(u,v,) = \min(u,v)$ is reached. Moreover this one-parameter symmetric copula family has multi-parameter asymmetric extensions. We could have taken any parameterized family of copulas leading to a differentiable density.

Let us define the parameter set

$$\Theta := \Big\{ (\alpha_1, \alpha_2, \beta_1, \beta_2, \eta) : \alpha_i > 3, \beta_i > 0, \eta \geq 1;\ i = 1, 2 \Big\}\,. \tag{7.22}$$

For $(\alpha_1, \alpha_2, \beta_1, \beta_2, \eta) \in \Theta$ an application of Sklar's theorem then yields that

$$G_\eta(x_1, x_2) := C_\eta\Big(G_{\alpha_1,\beta_1}(x_1), G_{\alpha_2,\beta_2}(x_2)\Big) \qquad x_1, x_2 > 0$$

is the cumulative distribution function of a bivariate random variable with $\Gamma(\alpha_i, \beta_i)$-distributed margins, $i = 1, 2$, with density

$$g_\eta(x_1, x_2) := \widetilde{c}_\eta(x_1, x_2) \prod_{i=1}^{2} g_{\alpha_i,\beta_i}(x_i) \qquad x_1, x_2 > 0 .$$

Here we used the abbreviation $\widetilde{c}_\eta(x_1, x_2) := c_\eta(G_{\alpha_1,\beta_1}(x_1), G_{\alpha_2,\beta_2}(x_2))$. To create a stationary diffusion process of gradient field type, we define the potential by

$$\Phi_\theta(x) := \left(\sum_{i=1}^{2} x_i/\beta_i - (\alpha_i - 1) \ln x_i \right) - \ln \widetilde{c}_\eta(x_1, x_2) \qquad x_1, x_2 > 0,\ \theta \in \Theta . \tag{7.23}$$

Note that the stationary measure μ of $(X_t^G)_{t\geq 0}$ has Lebesgue density

$$\widetilde{\mu}(x) := \widetilde{c}_\eta(x_1, x_2)^{2/\sigma^2} \prod_{i=1}^{2} x_i^{a_i - 1} e^{-x_i/b_i} \qquad x_1, x_2 > 0 , \tag{7.24}$$

where $a_i := 2(\alpha_i - 1)/\sigma^2 + 1$, $b_i := \sigma^2 \beta_i/2$, $i = 1, 2$.

Due to the singularity in the potential Φ_θ, $\theta \in \Theta$, for $x_1, x_2 \to 0$, the existence of a solution of the SDE (7.2) is not straightforward. The density $\widetilde{\mu}$ is strictly positive and differentiable on the set $\{x_1, x_2 > 0\}$ and can be continuously extended to a function on $\mathbb{R}^n$ by setting it to 0 on the set $\mathbb{R}^2 \setminus \{x_1, x_2 > 0\}$. Further, Φ_θ defined in (7.23) satisfies the integrability condition (7.4). Hence by Proposition 3.1, there exists a weak solution $(X_t^G)_{t\in[0,T]}$ of the SDE (7.2) with potential Φ_θ, $\theta \in \Theta$, see also Section 7.2.

Let us proceed to the evaluation of the asymptotics of the maximum M_T defined in (7.5) for the process $(X_t^G)_{t\geq 0}$. Set $b^* := \max\{b_1, b_2\}$ and a^* being the a_i corresponding to b^*; analogously for $b_* := \min\{b_1, b_2\}$. In Theorem 4.10, the function l satisfying (7.6) is given for the process $(X_t^G)_{t\in[0,T]}$ in the independent case $\eta = 1$ by

$$l_1(R) := \begin{cases} \dfrac{\sigma^2}{2b^*} g_{a^*,b^*}(R) & b_* < b^* , \\[2ex] \dfrac{\sigma^2}{2b} \left(\dfrac{R^{a_1-1}}{b^{a_1}\Gamma(a_1)} + \dfrac{R^{a_2-1}}{b^{a_2}\Gamma(a_2)} \right) e^{-R/b} & b_* = b^* =: b . \end{cases} \tag{7.25}$$

To treat the general case $\eta \geq 1$ with spatial dependence we define

$$\delta_{as}(R; \eta) := R^{a_1+a_2-2} \int_0^{\pi/2} (\cos\gamma)^{a_1-1} (\sin\gamma)^{a_2-1} \widetilde{c}_\eta(R, \gamma)^{2/\sigma^2} e^{-Rp(\gamma)} d\gamma , \tag{7.26}$$

where $p(\gamma) := (\cos\gamma)/b_1 + (\sin\gamma)/b_2 - 1/b^*$ and $\widetilde{c}_\eta$ is written in polar coordinates (compare with (4.26) in the independent case $\eta = 1$). Further, we denote by $|\mu|$ the total mass of the stationary measure μ in (7.24). In applications, these expressions have be to calculated numerically. The procedure of the evaluation of the function l satisfying (7.6) is deferred to Appendix A.2. We obtain

$$l_\eta(R) := \frac{\sigma^2}{2|\mu| b^*}\,\delta_{as}(R;\eta) R e^{-R/b^*} \qquad R > 0, \eta \geq 1\,, \tag{7.27}$$

To evaluate the long time behavior of M_T for the process $(X_t^G)_{t\geq 0}$, we set $F := e^{-l_\eta}$ as in (7.7). In Appendix A.2 it is shown that $F \in \mathrm{DA}(\Lambda)$, where Λ is the Gumbel distribution. Moreover the long time limit (7.9) holds in the general case $\eta \geq 1$ with the following norming constants:

$$\begin{aligned} c_T &:= b^*\,, \\ d_T &:= b^* \left[\ln T + \ln\ln T + \ln\left(\delta_{as}(b^* \ln T;\eta)\right) + \ln\left(\frac{\sigma^2}{2|\mu|}\right)\right]. \end{aligned} \tag{7.28}$$

In the independent case $\eta = 1$ we obtain the explicit expression

$$d_T := \begin{cases} b^* \left[\ln T + (a^* - 1)\ln\ln T + \ln\left(\dfrac{\sigma^2}{2b^{*2}\Gamma(a^*)}\right)\right] & b_* < b^*\,, \\[2ex] b^* \left[\ln T + \ln\left(\dfrac{(\ln T)^{a_1 - 1}}{\Gamma(a_1)} + \dfrac{(\ln T)^{a_2-1}}{\Gamma(a_2)}\right) + \ln\left(\dfrac{\sigma^2}{2b^{*2}}\right)\right] & b_* = b^*\,. \end{cases} \tag{7.29}$$

7.4 Parameter Estimation

The aim of this section is to fit a diffusion model of gradient field type to discrete data in the following sense: assume that we are given a multivariate data set

$$(x_0, \ldots, x_N) \text{ with } x_j \in O \subset \mathbb{R}^n \text{ open, } j = 1, \ldots, N. \tag{7.30}$$

We consider this data set as a discrete realization of a stationary diffusion process $(X_t)_{t\in[0,T]}$ of gradient field type solving the SDE (7.2) for some fixed time horizon $T > 0$. This means that there exists a grid

$$(t_j)_{j=0,\ldots,N} \text{ with } 0 = t_0 < t_1 < \ldots < t_N = T, \tag{7.31}$$

such that $(x_0, \dots, x_N)$ is a realization of $(X_{t_0}, \dots, X_{t_N})$. We describe how point estimates for the diffusion coefficient σ and the potential Φ appearing in the SDE (7.2) can be obtained, where Φ has a general parameterization $(\Phi_\theta)_{\theta\in\Theta}$. We consider in particular the models of Section 7.3. For simplicity we assume that the grid (7.31) is equidistant, i.e., $t_j = js$, $j = 0, \dots, N$, with step-size $s := T/N$.

7.4.1 Estimation of σ

A diffusion process $(X_t)_{t\in[0,T]}$ of gradient field type specified by the SDE (7.2) has additive noise, and the diffusion coefficient $\sigma > 0$ just multiplies the n–dimensional Brownian motion. In this situation, σ can be estimated independently of the drift coefficients by a quadratic variation-like formula, see Florens-Zmirou [FZ89].

We denote the Euclidean norm mapping by $f(x) := |x|$, $x \in \mathbb{R}^n$. The idea is to estimate σ via the covariance process of the process $(f(X_t))_{t\in[0,T]} = (|X_t|)_{t\in[0,T]}$. In order to show that $(|X_t|)_{t\in[0,T]}$ is a semi-martingale, we apply Itô's formula to $(X_t)_{t\in[0,T]}$ for the function f. The difficulty that f is not differentiable can be solved by smoothing f near the origin, see for example the derivation of the integral representation of the Bessel process in Prop. 3.2.1 of Karatzas and Shreve [KS91]. We obtain for $t > 0$

$$|X_t| = |X_0| + \int_0^t g_s\, ds + \sigma W_t\,, \qquad W_t := \sum_{i=1}^n \int_0^t \frac{X_s^i}{|X_s|} dB_s^i\,,$$

where $(g_s)_{s\geq 0}$ is some process depending on $(X_t)_{t\in[0,T]}$. Note that $(|X_t|)_{t\in[0,T]}$ is a one-dimensional object, but not a one-dimensional diffusion, since the process $(g_s)_{s\geq 0}$ does in general not only depend on $(|X_t|)_{t\in[0,T]}$. By Levy's theorem, $(W_t)_{t\geq 0}$ is again a one-dimensional Brownian motion. From the standard construction of the covariance process $< |X_t| >_{t\in[0,T]}$ for the semimartingale $(|X_t|)_{t\in[0,T]}$ (see e.g. section 2.3 and Thm 8.6 of Durrett [Dur96]) we get

$$\sum_{i=1}^N \left(|X_{t_i}| - |X_{t_{i-1}}|\right)^2 \xrightarrow{P} \int_0^T < |X| >_s ds = \sigma^2 T \qquad (N \to \infty)\,,$$

where $\xrightarrow{P}$ demotes convergence in probability. Hence we obtain the following standard estimator

$$\widehat{\sigma} = \sqrt{\frac{1}{T}\sum_{i=1}^{N}\left(|X_{t_i}| - |X_{t_{i-1}}|\right)^2}\,. \tag{7.32}$$

This estimator is consistent, see Florens-Zmirou [FZ89].

7.4.2 Estimation of Φ

Let $(X_t)_{t\in[0,T]}$ be a diffusion process of gradient field type where the potential has a general parameterization $(\Phi_\theta)_{\theta\in\Theta}$. The correct parameter $\widehat{\theta} \in \Theta$ needs to be estimated from the data. The problem of estimating the drift for one-dimensional diffusion processes has been treated in many articles. We refer to Bibby and Sørensen [BS95, BS96] using martingale estimation functions and to Pederson [Ped95a, Ped95b] for simulated likelihood inference by approximating the Markov transition densities, see also Aït-Sahalia [AS02] for a multivariate extension. Further, references can be found in this articles. We present here a maximum likelihood approach for the estimation based on a continuous ansatz. This procedure is a multivariate generalization of section 6.4 and 13.2. of Kloeden and Platen [KP92] and works for a general parameterization $(\Phi_\theta)_{\theta\in\Theta}$ of the potential.

Note that the mean m of the stationary measure of the process $(X_t)_{t\in[0,T]}$ fitted to a data set (7.30) can be easily obtained. An unbiased estimator of m is given by the sample mean (see §11.2 of Brockwell and Davis [BD98])

$$\widehat{m} = \frac{1}{N+1}\sum_{j=0}^{N} x_j\,. \tag{7.33}$$

In the case of diffusion models with linear drift (as for the Vašiček model), we can also work with the discrete likelihood of a multivariate AR(1)–process, which corresponds to the discrete skeleton of these diffusion processes. This is not possible for the exponential process and the gamma process (see Section 7.3.2 and 7.3.3), since the drift component is no longer linear.

Continuous Ansatz

Let $(X_t)_{t\in[0,T]}$ be a diffusion process solving the SDE (7.2) where the potential has a general parameterization $(\Phi_\theta)_{\theta\in\Theta}$. Set $Y_t := \sigma^{-1}X_t$, $t \in [0,T]$. By an application of Itô's lemma it is seen that the process $(Y_t)_{t\in[0,T]}$ satisfies the SDE

$$dY_t^i = b_\theta^i(Y_t)dt + dB_t^i\,, \quad i=1,\ldots,n\,, \qquad \text{where} \quad b_\theta^i(y) := -\sigma^{-1}(\partial_{x_i}\Phi_\theta)(\sigma y)\,.$$

By P_Y and P_B we denote the law of the process $(Y_t)_{t\in[0,T]}$ and of the Brownian motion $(B_t)_{t\geq 0}$ on the space $C([0,T],\mathbb{R}^n)$, respectively; dP_Y/dP_B is the Radon-Nikodym derivative. The log-likelihood ratio is defined by $l_T(\theta) := \ln(dP_Y/dP_B)$ and from Girsanov's theorem we obtain

$$\begin{aligned} l_T(\theta) &= \int_0^T b_\theta(Y_t)\cdot dY_t - \frac{1}{2}\int_0^T |b_\theta(Y_t)|^2 dt \\ &= -\frac{1}{\sigma^2}\left(\int_0^T \nabla\Phi_\theta(X_t)\cdot dX_t + \frac{1}{2}\int_0^T |\nabla\Phi_\theta(X_t)|^2 dt\right)\,, \end{aligned}$$

where ∇ denotes the gradient and $\cdot$ is the scalar product in $\mathbb{R}^n$. Since we are dealing with discrete data, we use the following discrete version:

$$l_T^d(\theta) := -\frac{1}{\sigma^2}\sum_{j=0}^{N-1}\left(\nabla\Phi_\theta(X_{t_j})\cdot(X_{t_{j+1}} - X_{t_j}) + \frac{s}{2}|\nabla\Phi_\theta(X_{t_j})|^2\right)\,.$$

The estimated parameter $\widehat{\theta}$ is obtained by maximizing l_T^d over $\theta\in\Theta$, plugging in the data points, i.e., $X_{t_j} = x_j$, $j=0,\ldots,N$. Hence

$$\widehat{\theta} = \arg\min_{\theta\in\Theta}\left\{\sum_{j=0}^{N-1}\left(\nabla\Phi_\theta(x_j)\cdot(x_{j+1}-x_j) + \frac{s}{2}|\nabla\Phi_\theta(x_j)|^2\right)\right\}\,. \tag{7.34}$$

Discrete Ansatz

For diffusion models with linear drift and diffusion coefficients, the parameter estimation can be evaluated by means of the likelihood of the discretized process. The only process of this type solving the SDE (7.2) is the Vašiček model $(X_t^V)_{t\in[0,T]}$ introduced in section 7.3.1 with parameterization $(A_\theta)_{\theta\in\Theta}$ of the drift matix as in (7.11). Recall from (7.14) that the centered Vašiček model $(Y_t)_{t\in[0,T]} := (X_t^V - m)_{t\in[0,T]}$ equals in law the OU process defined in (7.13), where m is the mean of $(X_t^V)_{t\in[0,T]}$ and can be estimated according to (7.33). Writing for short $Y_j := X_{t_j}$, $j=0,\ldots,N$, for the discretized version, we obtain, using the solution formula for OU processes,

$$\begin{aligned} Y_{j+1} &= F_\theta Y_j + \xi_j \qquad j=0,\ldots,N \\ &\text{where} \quad F_\theta := e^{-A_\theta s}\,, \quad \xi_j := \sigma\int_{t_j}^{t_{j+1}} e^{-A_\theta(t_{j+1}-u)}dB_u\,. \end{aligned} \tag{7.35}$$

Note that the ξ_j, $j = 0, \ldots, N$, are independent by definition and

$$\xi_j = \sigma \int_0^s e^{-A_\theta(s-v)} dB_{t_j+v} \stackrel{\text{law}}{=} \sigma \int_0^s e^{-A_\theta(s-v)} dB_v \qquad j = 0, \ldots, N .$$

Hence $(Y_j)_{j=0,\ldots,N}$ is a bivariate AR(1)–process. Further, ξ_1 has a $N(0, \Sigma_\theta)$-normal distribution with covariance

$$\begin{aligned}
\Sigma_\theta &:= E_0(\xi_1 \xi_1^t) \\
&= \sigma^2 \int_0^s e^{-A_\theta(s-v)} (e^{-A_\theta(s-v)})^t dv \\
&= \sigma^2 R^\phi \left(\int_0^s e^{-2(s-v)\mathrm{diag}(\alpha_1,\alpha_2)} dv \right) R^{-\phi} \\
&= \sigma^2 R^\phi \,\mathrm{diag}\Big((2\alpha_1)^{-1}(1 - e^{-2\alpha_1 s}), (2\alpha_2)^{-1}(1 - e^{-2\alpha_2 s}) \Big) R^{-\phi} ,
\end{aligned} \tag{7.36}$$

where R^ϕ is the rotation matrix defined in (7.11). Similarly we obtain

$$F_\theta = R^\phi \,\mathrm{diag}(e^{-\alpha_1 s}, e^{-\alpha_2 s}) R^{-\phi} . \tag{7.37}$$

The likelihood for the AR(1)–process $(Y_i)_{i=0,\ldots,N}$ is given by (see §11.5 of Brockwell and Davis [BD98])

$$L(\theta, \sigma) = \Big((2\pi)^n \det(\Sigma_\theta) \Big)^{-N/2} \exp\left(-\frac{1}{2} \sum_{j=1}^N (Y_j - F_\theta Y_j)^t \Sigma_\theta^{-1} (Y_j - F_\theta Y_j) \right) .$$

We obtain the estimated parameter by maximizing the logarithm of $L(\theta, \sigma)$ over $\theta \in \Theta$, $\sigma \geq 0$, plugging in the data points from (7.30) Hence

$$(\widehat{\theta}, \widehat{\sigma}) = \arg \min_{\theta \in \Theta, \sigma \geq 0} \left(\sum_{j=1}^N (y_j - F_\theta y_j)^t \Sigma_\theta^{-1} (y_j - F_\theta y_j) + N \ln\left(\det(\Sigma_\theta)\right) \right) , \tag{7.38}$$

where $y_j := x_j - \widehat{m}$, $j = 0, \ldots, N$, and F_θ and Σ_θ is defined in (7.37) and (7.36), respectively.

7.5 The Goodness-of-Fit Tests

We present a method using the characterization of the large fluctuations of diffusion models of gradient field type presented in section 7.2 to answer the following question:

does a diffusion model of gradient field type, which is fitted to a data set, also explain the extremes in the data?

More precisely assume that for a diffusion model $(X_t)_{t\in[0,T]}$ solving the SDE (7.2) for a fixed time horizon $T > 0$ the potential Φ and the diffusion coefficient σ are estimated from a data set $(x_0, \dots, x_N)$ as in (7.30) and point estimates $\widehat{\Phi}$ and $\widehat{\sigma}$ have been obtained (e.g. with the estimators derived in Section 7.4). We test the null-hypothesis

> H_0: $(X_t)_{t\in[0,T]}$ defined in (7.2) with parameters $\widehat{\Phi}$, $\widehat{\sigma}$ describes correctly the extremes of the data set $(x_0, \dots, x_N)$.

Assuming a grid of the form (7.31), we use as test-statistic the maximum of the discretization of $(X_t)_{t\in[0,T]}$ in Euclidean norm

$$M := \max\{|X_{t_0}|, \dots, |X_{t_N}|\} .$$

The realization $\widehat{M}$ of M is just the sample maximum in Euclidean norm, i.e.,

$$\widehat{M} := \max\{|x_0|, \dots, |x_N|\} . \tag{7.39}$$

From the point of view of risk management H_0 should be rejected to be on the safe side if $\widehat{M}$ is too large. Hence we suggest a one-sided test. For a significance level $\alpha > 0$, we reject H_0 if $\widehat{M} > \kappa_\alpha$, where $\kappa_\alpha > 0$ is given in terms of the estimated parameters $\widehat{\Phi}$ and $\widehat{\sigma}$. Assuming the distribution of M to be continuous, κ_α has to be chosen in such a way that we have for the error of first kind (i.e. rejecting H_0 though it is true)

$$P_0(M > \kappa_\alpha) = \alpha\,, \tag{7.40}$$

where P_0 is the probability under H_0 that the fitted model is correct in the extremes.

The idea to calculate the rejection level κ_α is to compare M with the continuous time maximum $M_T := \max_{0\le t\le T} |X_t|$ defined in (7.5) of the process $(X_t)_{t\in[0,T]}$ up to the time horizon T in Euclidean norm. We use the heuristic fact that

$$P_0(M > \kappa_\alpha) \approx P_\mu(M_T > \kappa_\alpha)\,, \tag{7.41}$$

provided that the grid is sufficiently fine. Here P_μ denotes the law of the process $(X_t)_{t\in[0,T]}$ starting with its stationary measure μ. For a treatment of this discretization problem see e.g. Jacod and Protter [JP98]. Denoting by $(U_t^N)_{t\in[0,T]}$ and $(\overline{U}_t^N)_{t\in[0,T]}$ the error process of $(X_t)_{t\in[0,T]}$ w.r.t. the discrete continuous and discontinuous Euler approximation of order N respectively, they have shown that $\sup_{t\in[0,T]}|U_t^N|$ and $\sup_{t\in[0,T]}|\overline{U}_t^N|$ tends to zero in probability as $N\to\infty$ in a quite general semimartingale framework.

We use that characterization of the tail asymptotics of the maximum M_T for fixed $T>0$ in (7.6) and the characterization of the long term behavior of M_T as $T\to\infty$ in (7.9) to suggest some tests and to determine the rejection level κ_α in (7.40).

TEST A: Assume that κ_α is large enough such that the asymptotics in (7.6) is reasonably sharp. Further, suppose that T is much larger than the constant k appearing in the right hand side of the asymptotic inequality of (7.6). Hence k can be neglected and (7.6) reads in sloppy notation $P_\mu(M_T>R)\sim T\,l(R)$ as $R\to\infty$. Combining this relation with (7.40) using (7.41), we see that κ_α should be chosen such that

$$\alpha = T\,\widehat{l}(\kappa_\alpha)\,, \tag{7.42}$$

where $\widehat{l}$ is an estimate of the function l from (7.6), which is obtained by plugging in the point estimates $\widehat{\Phi}$ and $\widehat{\sigma}$.

Alternatively, the test can be expressed in terms of its p-value, which is given for a realization $\widehat{M}$ of M by $p_{\widehat{M}} := \inf\{\alpha : \widehat{M} > \kappa_\alpha\}$, i.e., the smallest significance level α such that H_0 is rejected based on the realization $\widehat{M}$. Presuming that $\alpha\mapsto\kappa_\alpha$ is strictly increasing, we get for the p-value together with (7.42)

$$p_{\widehat{M}} = T\,\widehat{l}(\widehat{M})\,. \tag{7.43}$$

TEST B: Assume that T is large such that the convergence in (7.9) has reached a reasonable level of precision. Denoting by $(\widehat{c}_T)_{T>0}$, $(\widehat{d}_T)_{T>0}$ the estimated norming constants in (7.9) obtained by plugging in the point estimates $\widehat{\Phi}$ and $\widehat{\sigma}$, we obtain from (7.9) using (7.41)

$$\begin{aligned} P_0(M>\kappa_\alpha) &\approx P_\mu\left(\widehat{c}_T^{-1}(M_T-\widehat{d}_T) > \widehat{c}_T^{-1}(\kappa_\alpha-\widehat{d}_T)\right) \\ &\overset{T\text{ large}}{\approx} 1-H\left(\widehat{c}_T^{-1}(\kappa_\alpha-\widehat{d}_T)\right)\,. \end{aligned}$$

Hence κ_α should be chosen

$$\kappa_\alpha = \widehat{c}_T q_\alpha + \widehat{d}_T\,, \tag{7.44}$$

where q_α is the α-quantile of the extreme value distribution H, i.e., $1 - H(q_\alpha) = \alpha$.

7.6 Application to Simulated and Financial Data

We demonstrate how the goodness-of-fit tests developed in Section 7.5 behave in applications to simulated and real financial data. In Section 7.3, the large fluctuations have been explicitly characterized for some multivariate short-rate models, namely the bivariate Vašiček model (Section 7.3.1), the bivariate double-exponential process (Section 7.3.2), and the bivariate gamma process (Section 7.3.3). This enables us to perform explicitly the goodness-of-fit tests for these models and we will do this for the Vašiček model and the double-exponential process.

The financial data we use are the 30-days Libor rates for Euro, British Pound, and US Dollar (September 21, 1999 until September 5, 2000), which are commonly used as a proxy for the short-rate. These data set was kindly provided by Risklab Germany. In particular, I thank Prof. Zagst and Dr. Hessenberger for this purpose.

7.6.1 Vašiček Model

Maximum Asymptotics and Simulation Results

We consider here the centered bivariate Vašiček model introduced in section 7.3.1, which coincides with the bivariate OU process, see (7.14). To verify the theoretical tail asymptotics (7.6) of the maximum M_T and the long time limit (7.9) of the renormalized maximum for this process, we simulate 20,000 sample paths for a large time horizon $T = 50$. We use for the simulation an order 1.5 strong Taylor scheme, see section 10.4 of Kloeden and Platen [KP92]. Evaluating for each path the maximum w.r.t. Euclidean norm, an empirical distribution of the random variable M_T (defined in (7.5)) is obtained. In Figure 7.1 we present the histogram and compare the empirical tail behavior with the theoretical tail asymptotics (7.6). Further, the convergence in distribution in the long time limit $T \to \infty$ of the normalized empirical maxima (see (7.9)) to the Gumbel distribution is demonstrated.

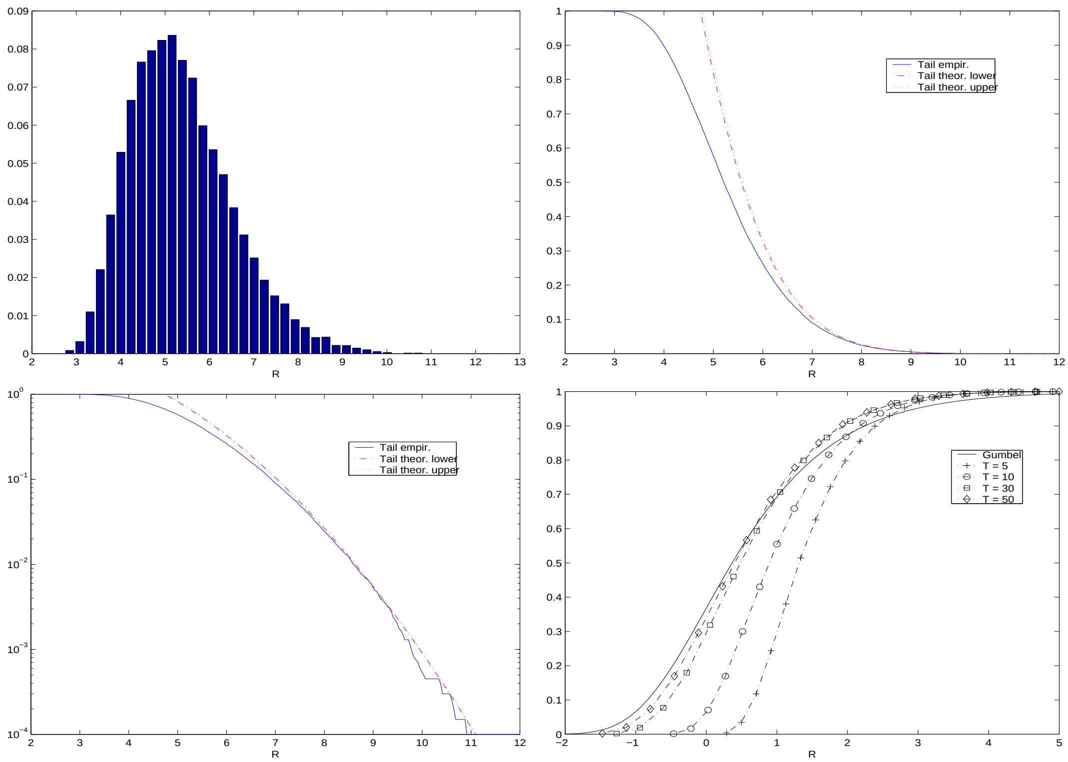

Figure 7.1: Simulation of 20,000 sample paths of the OU process $(X_t^{OU})_{t\in[0,T]}$ introduced in section 7.3.1 with time horizon $T = 50$ and parameters $\sigma = 1$ and $\theta = (\alpha_1, \alpha_2, \phi)$ for the matrix A_θ defined in (7.11) given by $\alpha_1 = 0.3$, $\alpha_2 = 0.5$, $\phi = 0$. The step-size for the simulation is $s = T/N = 0.005$, where N is the sample size. [Top Left]: Histogram of the random variable $M_T = \max_{0\le t\le 50} |X_t^{OU}|$. [Top Right]: Empirical tail of M_T together with the theoretical asymptotics from (7.6), where the function l is defined in (7.15). Here the constant k appearing in the upper asymptotic bound in (7.6) is given for this parameter setting by $k = 1/\min\{\alpha_1, \alpha_2\} = 10/3$ according to (7.16); It is seen that the upper and lower asymptotic bound in (7.6) nearly coincide, since $T >> k$. [Bottom Left]: The same on logarithmic scale. [Bottom Right]: Cumulative distribution function of normalized empirical maximum $c_T^{-1}(M_T - d_T)$ as in (7.9) approaching Gumbel distribution for increasing T.

These simulations show that for this parameter setting the tail asymptotics in (7.6) is reasonably sharp for values $R > 6$. The long time limit in (7.9) attains a reasonable level for $T > 30$.

Parameter Estimation and Test Performance for Simulated Data

In Table B.1 in Appendix B we show a simulation study for the bivariate OU process and present the results of the parameter estimation. In particular, the estimator (7.34) developed from the continuous ansatz is compared with the discrete likelihood estimator (7.38). For the optimization procedure in the maximum likelihood estimation, we use the simplex search method of Lagarias et al. [LRWW99] implemented in the Matlab package. It can be seen that the estimation of the parameter θ for the drift matrix A_θ

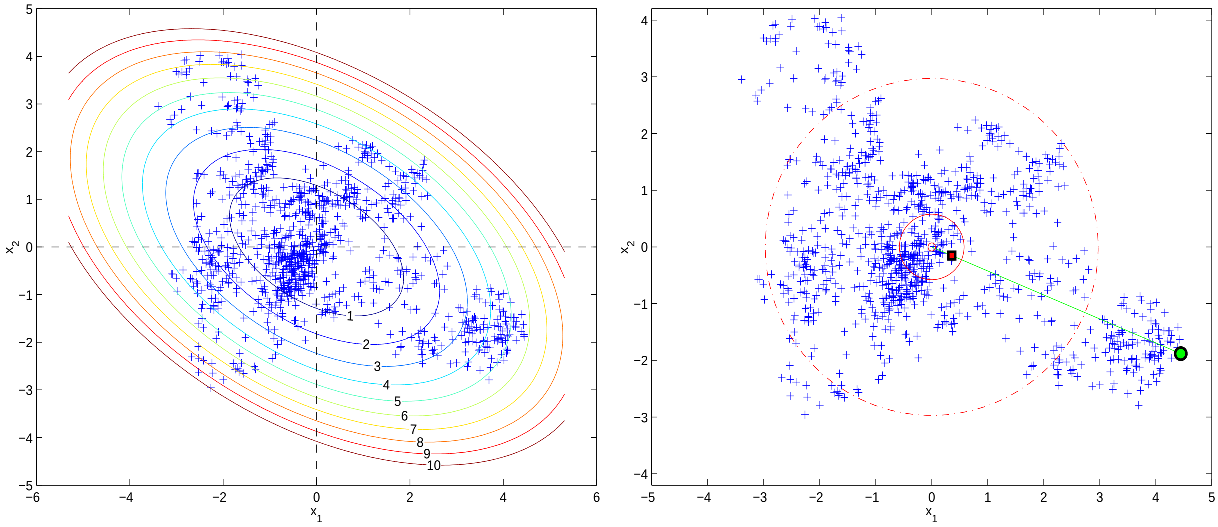

Figure 7.2: We simulated a sample path of the OU process introduced in section 7.3.1 with parameters $\sigma = 1$ and $\theta = (\alpha_1, \alpha_2, \phi)$ for the matrix A_θ defined in (7.11) given by $\alpha_1 = 0.5$, $\alpha_2 = 0.1$, $\phi = -\pi/3$; time horizon $T = 50$ and step-size $s = T/N = 0.05$, where N is the sample size. The stationary measure indicates for negative correlation and is given by a bivariate normal $N(0, \Sigma)$, where the covariance matrix reads in this situation $\Sigma = \begin{pmatrix} 4 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}$, see (7.12). [Left]: Plotted are the points visited by this sample path together with the contour plot of the potential Φ_θ defined in (7.13) and (7.11). The parameters σ and θ are estimated from the sample. [Right]: Plotted are again the points visited by this sample path. The sample maximum $\widehat{M}$ in Euclidean norm is marked by a circle. The normalized sample maximum $\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$ as in (7.9) is indicated by a square on the line joining $\widehat{M}$ and the origin. The radius of the solid circle is the mean of the Gumbel distribution $m^\Lambda = 0.5772$, the radius of the dashed circle corresponds to the 5%-quantile $q^\Lambda_{0.05} = 2.9702$. If the renormalized sample maximum lies inside the dashed circle, then the fitted OU process model is not rejected at the significance level of 5%.

defined in (7.11) yields accurate values only for large data sizes (with sample size $N \approx 10,000$), which is a known fact when dealing with estimation of the drift of diffusion processes from discrete or continuous data. The estimator $\widehat{\sigma}$ defined in (7.32) of the diffusion coefficient however gives good results also for smaller data sets (with sample size $N \approx 1,000$). Further, the results of the simulation study show that there is a high level of consistency between the continuous ansatz estimator (7.34) and the discrete likelihood estimator (7.38).

We apply the goodness-of-fit tests described in Section 7.5 to simulated sample paths of the OU process. The parameter estimation for these sample paths is reported in Table B.2 in Appendix B and in Table B.3 in Appendix B we present the results of the goodness-of-fit tests. For all but one simulated sample paths the goodness-of-fit tests do not reject as expected the null-hypothesis that the fitted OU process is the right model.

Figure 7.2 shows the shape of the stationary density of the bivariate OU process with parameters fitted from a simulated sample path. Further, a graphical demonstration of Test B is presented.

Parameter Estimation and Test Performance for Financial Data

We present the results of the parameter estimation and the goodness-of-fit tests when fitting the bivariate Vašiček short-rate model to 30-days Libor rates of the currencies Euro, British Pound, and US Dollar (September 21, 1999 until September 5, 2000).

The time horizon $T > 0$ can be chosen arbitrarily. For the parameter estimation the grid size $s := T/N$ and hence T should be small, where N denotes the sample size. For the Test B however, exploiting the long time limit of the maximum M_T, the time horizon T needs to be large. Since the data sets cover the period of one year, we bridge this trade-off by measuring T in monthly units, i.e., $T = 12$.

The results of the parameter estimation are presented in Table B.7 in Appendix B. In contrast to the estimation of simulated data, the difference between the continuous ansatz estimator (7.34) and the discrete likelihood estimator (7.38) is much larger. In Table 7.1 we give the results of Test A and Test B, the goodness-of-fit tests developed in Section 7.5. The estimated values of the parameters are taken from Table B.7 in Appendix B. The outcome of Test A is that the fit to the bivariate Vašiček model is in principal not rejected for the 30-days Libor rates data set of Euro, British Pound, and US Dollar. In Test B the fit to the bivariate Vašiček model is not rejected for all tested financial data sets. This may be due to the fact that the convergence in (7.9), the long time limit for the maximum M_T, does not reach a sufficiently high level for the chosen time horizon $T = 12$.

			Test A	Test B		
	$\widehat{M}$		$p_{\widehat{M}}$	$\kappa_{0.05}$	$\kappa_{0.01}$	$\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$
EUR-GBP	1.3832	cont	0.0751	2.6291	3.7365	1.1365
		disc	0.1145	2.4081	3.3183	1.1349
EUR-USD	1.4577	cont	0.0730	2.3155	3.2552	1.4824
		disc	0.0431*	2.9898	4.4135	1.2162
USD-GBP	1.0436	cont	0.1193	1.8744	2.5833	1.0598
		disc	0.0845	2.2278	3.1553	0.8890

Table 7.1: Results of the goodness-of-fit tests for the bivariate Vašiček short-rate model applied to 30-days Libor rates of the currencies Euro [EUR], British Pound [GBP], and US Dollar [USD] (September 21, 1999 until September 5, 2000). We used the parameters estimated in Table B.7 in Appendix B. $\widehat{M}$ is the sample maximum defined in (7.39) and $p_{\widehat{M}}$ defined in (7.43) is the p-value for Test A. The quantities $\kappa_{0.05}$ and $\kappa_{0.01}$ correspond to the constants (7.44) in Test B. The term $\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$ is the renormalized sample maximum as in (7.9) and should be compared with the mean of the Gumbel distribution $m^\Lambda = 0.5772$ and the quantiles $q^\Lambda_{0.05} = 2.9702$, $q^\Lambda_{0.01} = 4.6001$. The superscript * denotes that the fit to the bivariate Vašiček model is rejected by Test A at the significance level of 5%, i.e., if $p_{\widehat{M}} < 0.05$. Note that no rejection occurs at the significance level of 1%. Test B does not reject the fit to the bivariate Vašiček model for any of the tested samples at the significance levels of 5% and 1%.

In Figure 7.3 we show for the USD-GBP data set the shape of the stationary density and the visualization of the Test B as in Figure 7.2 using the estimated parameters of the continuous ansatz in Table B.7 in Appendix B.

7.6.2 Exponential Process

Maximum Asymptotics and Simulation Results

As for the OU process, we will compare the theoretical tail asymptotics (7.6) of the maximum M_T and the long time limit (7.9) of the renormalized maximum to simulations for the exponential process introduced in Section 7.3.2. We simulate 5,000 sample paths for a large time horizon $T = 50$ using the Euler scheme.

In Figure 7.4 we present the histogram for the random variable M_T (defined in (7.5))

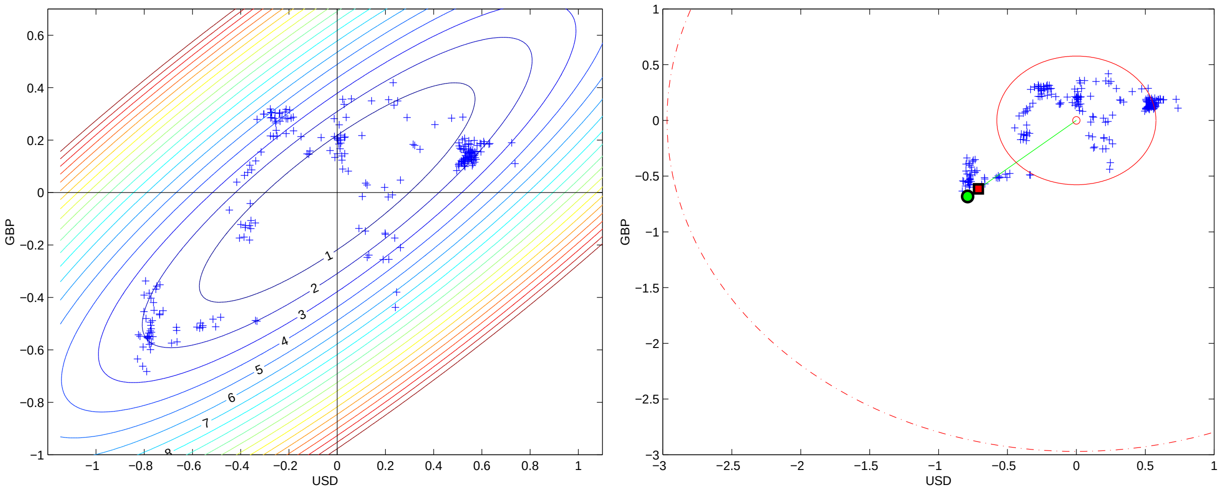

Figure 7.3: Stationary density of the centered bivariate Vašiček model fitted to the USD-GBP data set (30-days Libor rates of British Pound and US Dollar) and visualization of Test B in analogy to Figure 7.2; the estimated parameters $\widehat{\theta}$ and $\widehat{\sigma}$ are taken from the continuous ansatz estimators in Table B.7 in Appendix B. [Left]: Points visited by the centered sample together with the contour plot of the estimated potential Φ_θ defined in (7.13) and (7.11). [Right]: Points visited by the centered sample together with the sample maximum $\widehat{M}$ (circle) and normalized sample maximum (square). The fit is not rejected at a significance level of 5%, since the normalized sample maximum is smaller than the 5%-quantile $q^{\Lambda}_{0.05} = 2.9702$ of the Gumbel distibution (dashed circle).

and compare the empirical tail behavior with the theoretical tail asymptotics (7.6). Further, the convergence in distribution in the long time limit of the normalized empirical maxima according to (7.9) to the Gumbel distribution is demonstrated.

Parameter Estimation and Test Performance for Simulated Data

A simulation study for the exponential process together with results of the parameter estimation is shown in Table B.4 in Appendix B. We used the maximum likelihood estimator (7.34) developed from the continuous ansatz. Recall that the discrete maximum likelihood estimator (7.38) can not be used for this process, since its drift is not linear.

These estimation results show as for the OU process (see Table B.2 in Appendix B) that the estimator (7.34) for the parameter θ of the potential Φ_θ defined in (7.18) yields accurate values only for large data sizes (with sample size $N \approx 10,000$). The estimator $\widehat{\sigma}$ (defined in (7.32)) of the diffusion coefficient gives again good results also for smaller

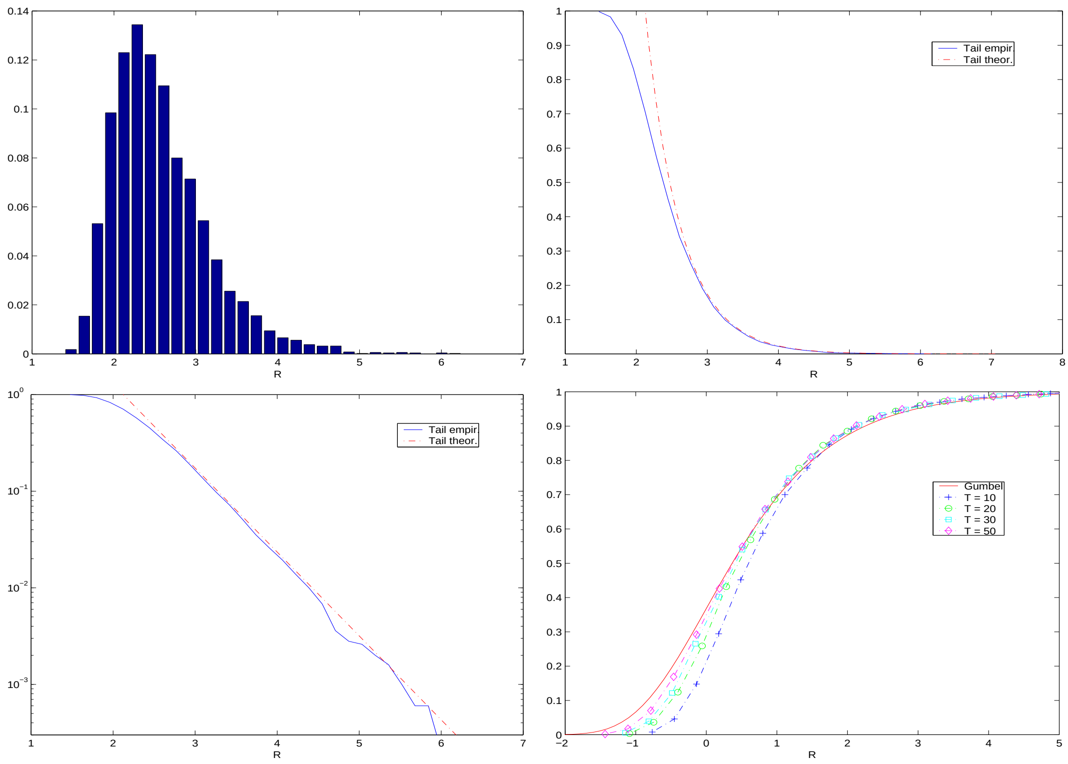

Figure 7.4: Simulation of 5,000 sample paths of the process $(X_t)_{t\in[0,T]}$ with exponential distribution introduced in Section 7.3.2; time horizon $T = 50$ with parameters $\sigma = 1$ and $\theta = (\alpha_1, \alpha_2, \phi)$ for the potential Φ_θ defined in (7.18) given by $\alpha_1 = 1$, $\alpha_2 = 2$, $\phi = 0$. The step-size for the simulation is $s = T/N = 0.1$, where N is the sample size. The figures show the following. [Top Left]: Histogram of $M_T = \max_{0\le t\le 50} |X_t|$. [Top Right]: Empirical tail of M_T together with theoretical asymptotics from (7.6) with function l defined in (7.20). For this process, there is no explicite expression for the constant k appearing in the upper asymptotic bound in (7.6). [Bottom Left]: The same on logarithmic scale. [Lower Right]: Cumulative distribution function of normalized empirical maximum $c_T^{-1}(M_T - d_T)$ as in (7.9) approaching Gumbel distribution for increasing T.

data set (with sample size $N \approx 1,000$).

We apply the goodness-of-fit tests to simulated sample paths of the exponential process. The parameter estimation for these sample paths is reported in Table B.5 in Appendix B and in Table B.6 in Appendix B we present the results of the goodness-of-fit tests.

As for the OU process (see Table B.3 in Appendix B), the goodness-of-fit tests accept for all but one simulated sample paths as expected the null-hypothesis that the fitted process with exponential distribution is the right model also in the extremes. A graph-

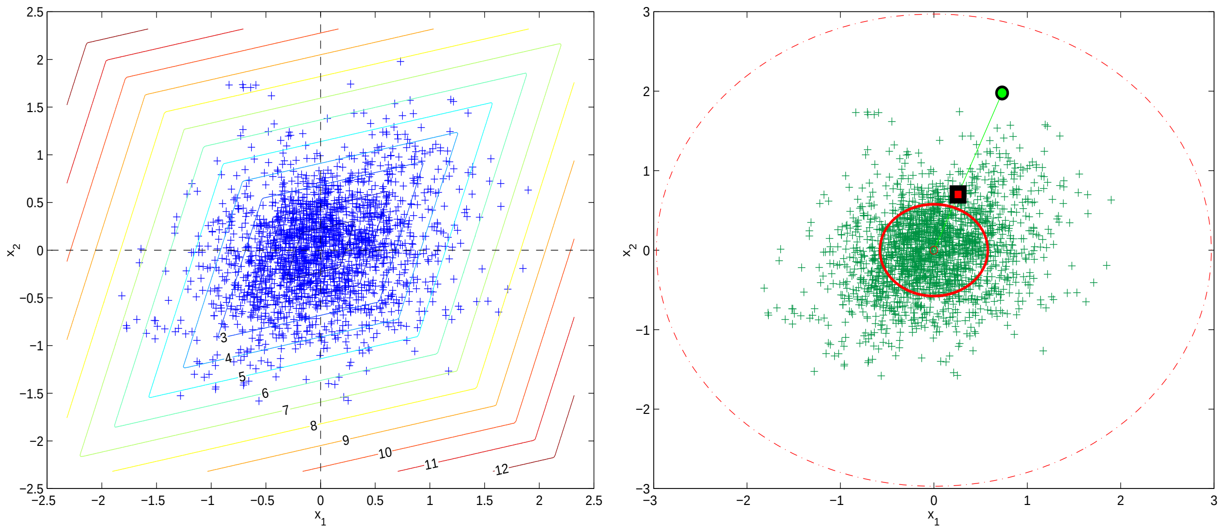

Figure 7.5: We simulated a sample path of the bivariate exponential process introduced in Section 7.3.2 with parameters $\sigma = 1$ and $\theta = (\alpha_1, \alpha_2, \phi)$ for the potential Φ_θ defined in (7.18) given by $\alpha_1 = 1$, $\alpha_2 = 2$, $\phi = -\pi/4$; time horizon $T = 200$ and step-size $s = T/N = 0.1$, where N is the sample size. The stationary measure indicates for positive correlation; the covariance matrix Σ reads in this situation $\Sigma = (1/32)\binom{5\ \ 3}{3\ \ 5}$, see (7.19). [Left]: Plotted are the points visited by this sample path together with the contour plot of the potential Φ_θ with parameter θ estimated from the sample. [Right]: Plotted are again the points visited by this sample path. The sample maximum $\widehat{M}$ in Euclidean norm is marked by a circle. The normalized sample maximum $\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$ as in (7.9) is indicated by a square on the line joining $\widehat{M}$ and the origin. The radius of the solid circle is the mean of the Gumbel distribution $m^\Lambda = 0.5772$, the radius of the dashed circle corresponds to the 5%-quantile $q^\Lambda_{0.05} = 2.9702$. If the renormalized sample maximum lies inside the dashed circle, then the fitted bivariate process with exponential distribution is not rejected at the significance level of 5%.

ical demonstration of Test B is presented in Figure 7.5 together with the shape of the stationary distribution of the fitted process with exponential distribution.

Parameter Estimation and Test Performance for Financial Data

We present the results of the parameter estimation and the goodness-of-fit tests when fitting the bivariate short-rate model with exponential distribution introduced in Section 7.3.2 to the 30-days Libor rates of the currencies Euro, British Pound, and US Dollar (September 21, 1999 until September 5, 2000). Again we measure the time horizon in monthly units and hence $T = 12$.

		Test A	Test B		
	$\widehat{M}$	$p_{\widehat{M}}$	$\kappa_{0.05}$	$\kappa_{0.01}$	$\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$
EUR-GBP	1.3832	0.0268*	−0.4490*	4.1467	3.6200
EUR-USD	1.4577	0.0816	1.8859	3.3885	2.5057
USD-GBP	1.0436	0.1388	1.6292	2.5879	1.9747

Table 7.2: Results of the goodness-of-fit tests for the bivariate process with exponential distribution introduced in Section 7.3.2 applied to 30-days Libor rates of the currencies Euro [EUR], British Pound [GBP], and US Dollar [USD] (September 21, 1999 until September 5, 2000). The parameter are estimated in Table B.8 in Appendix B. For the description of the presented values see the annotations of Table B.3. The superscript * denotes that the fit to the bivariate exponential process is rejected at the significance level of 5%, i.e., if $p_{\widehat{M}} < 0.05$. Note that no rejection occurs at the significance level of 1%.

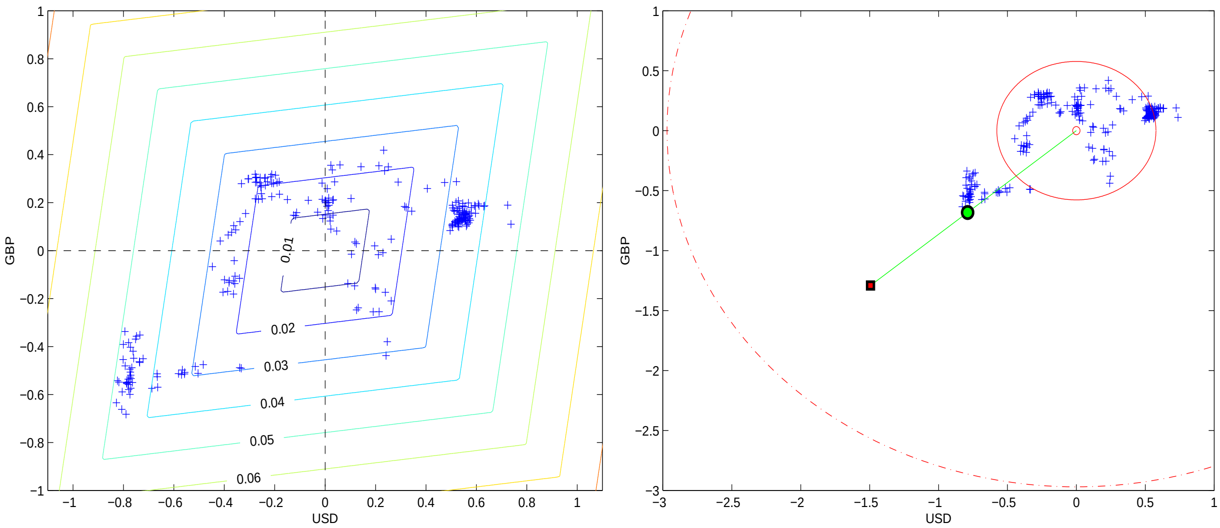

Figure 7.6: Stationary density of the centered bivariate process with exponential distribution introduced in Section 7.3.2 fitted to the USD-GBP data set (30-day Libor rates of British Pound and US Dollar) [left] and visualization of Test B [right] in analogy to Figure 7.5; the estimated parameters are taken from Table B.8. [Left]: Points visited by the centered sample together with the contour plot of the estimated Φ_θ defined in (7.18). [Right]: Points visited by the centered sample together with the sample maximum $\widehat{M}$ (circle) and normalized sample maximum (square). The fit is not rejected since the renormalized maximum is less than the 5%-quantile $q^{\Lambda}_{0.05} = 2.9702$ of the Gumbel distibution (dashed circle).

The results of the parameter estimation are presented in Table B.8 in Appendix B and Table 7.2 shows the results of the goodness-of-fit tests developed in Section 7.5. In

Figure 7.6 we show for the USD-GBP data set the shape of the stationary density and the visualization of Test B in analogy to Figure 7.5.

In Test B, the value of the constants $\kappa_{0.05}$ appearing in (7.44) can even become negative. The reason for this is again that the convergence in (7.9) of the normalized maximum M_T as $T \to \infty$ does not reach a sufficiently high level for the time horizon $T = 12$.

Let us compare these results of the goodness-of-fit tests with those of Table 7.1 for the bivariate Vašiček model. We expect that the fit of the exponential process explains better the extremes in the data than the fit to the bivarate Vašiček model with a normal distribution as stationary measure. Indeed for the EUR-USD and the USD-GBP data set the p-value of Test A corresponding to the exponential process is higher than the p-value for the Vašiček model. However this is not true for the EUR-GBP data set where the fit to the exponential process is even rejected for the significance level of 5%. This may be due to the fact that the estimation method (7.34) of the parameters of the potential Φ_θ defined in (7.18) is quite rough and the calculation of the p-value is very sensitive for slight changes of these parameters.

Appendix A

Calculations for the Maximum Asymptotics of Financial Models

A.1 Exponential Process

We use Theorem 4.1 to derive the function l satisfying (7.6) for the diffusion process $(X_t^E)_{t\in[0,T]}$ of gradient field type with exponential distribution introduced in Section 7.3.2. The norming constants in the long time limit (7.9) are also exhibited.

The diffusion process $(X_t^E)_{t\in[0,T]}$ is of gradient field type specified by the SDE (7.2) with potential Φ_θ, $\theta = (\alpha_1, \alpha_2, m_1, m_2, \phi) \in \Theta$, defined in (7.18). Assume that $m_1 = m_2 = 0$, i.e., the process is centered at the origin. Since we are interested in the maximum of the process in Euclidean norm, we may w.l.o.g. assume that the rotation matrix R^ϕ in Definition (7.18) of the potential Φ_θ is the identity (i.e.the rotation parameter $\phi = 0$) and that the potential Φ_θ has the form

$$\Phi(x) = \alpha_*|x_1| + \alpha^*|x_2| \qquad x \in \mathbb{R}^2\,,$$

where $\alpha^* := \max\{\alpha_1, \alpha_2\}$ and $\alpha_* := \min\{\alpha_1, \alpha_2\}$.

Note that $\Phi \notin C^1(\mathbb{R}^2, \mathbb{R})$. To overcome this problem one can smooth the edges in such a way that the asymptotics of the function l is not affected by the smoothing terms.

To evaluate the function l satisfying (7.6), the conditions of Theorem 4.1 must be shown to hold. The stationary measure μ with density $e^{-2\Phi(x)/\sigma^2}$, $x \in \mathbb{R}^2$, has total mass

$|\mu| = \sigma^4/(\alpha_*\alpha^*)$. In particular, Condition (2.5) holds. We restrict ourselves to show the crucial condition (4.5); the validation of the growth condition (4.7) and the spectral gap condition (2.7) is straightforward.

For the rotationally symmetric test-potential ϕ in Theorem 4.1, we choose the spherical minimum of Φ, i.e., $\phi(R) := \alpha_* R$, $R > 0$. Setting $p(\theta) := \alpha_*(\cos\theta - 1) + \alpha^* \sin\theta$, $\theta \in [0, \pi/2]$, we obtain for the term δ_{as} defined in (4.2), writing the potential Φ in polar coordinates

$$\delta_{as}(R) = \int_{-\pi}^{\pi} e^{-2R(\alpha_*(|\cos\theta|-1)+\alpha^*|\sin\theta|)/\sigma^2} d\theta = 4\int_0^{\pi/2} e^{-2Rp(\theta)/\sigma^2} d\theta\,.$$

Note that $p \geq 0$. For the zero set $\mathcal{N}(p) := \{\theta : p(\theta) = 0\}$ we have $\mathcal{N}(p) = \{0\}$ if $\alpha_* < \alpha^*$ and $\mathcal{N}(p) = \{0, \pi/2\}$ if $\alpha_* = \alpha^*$. Moreover $p(\theta) \sim \alpha^*\theta$ as $\theta \searrow 0$. An application of Laplace's method (Lemma 4.3) then yields

$$\delta_{as}(R) \sim \frac{2\sigma^2}{\alpha^*}(1 + \delta_{\alpha_1\alpha_2})R^{-1} \qquad (R \to \infty)\,.$$

To evaluate the term D_{as} defined in (4.4), we note that Δ_{as} defined in (4.3) reads in polar coordinates

$$\Delta_{as}(r,\theta) = \alpha_*(|\cos\theta| - 1) + \alpha^*|\sin\theta| \qquad r > 0,\ \theta \in [0, 2\pi)\,.$$

Similar to the above calculation we obtain

$$D_{as}(R) = 4\int_0^{\pi/2} p(\theta)^2 e^{-2Rp(\theta)/\sigma^2} d\theta\,.$$

A further application of Laplace's method then yields that there exists a constant $\kappa > 0$ such that $D_{as}(R) \sim \kappa R^{-3}$ as $R \to \infty$. Hence, $D_{as}(R) = o(\delta_{as}(R))$ as $R \to \infty$ and the crucial condition (4.5) is satisfied.

From Theorem 4.1 we obtain together with Lemma 4.5, that the function l satisfying (7.6) has the form

$$\begin{aligned} l(R) &= \frac{\sigma^2}{2|\mu|}\,\delta_{as}(R)\left(\int_1^R r^{-1} e^{2\alpha_* r/\sigma^2}\,dr\right)^{-1} \\ &\sim \frac{\alpha^*}{R}(1 + \delta_{\alpha_1\alpha_2})\left(\frac{\sigma^2}{2\alpha_*}R^{-1}e^{2\alpha_* R/\sigma^2}\right)^{-1} \\ &= \frac{2\alpha_*^2}{\sigma^2}(1 + \delta_{\alpha_1\alpha_2})e^{-2\alpha_* R/\sigma^2} \qquad (R \to \infty)\,. \end{aligned}$$

This expression coincides with l defined in (7.20).

It remains to shown that $F := e^{-l}$ is in the domain of attraction of the Gumbel distribution Λ ($F \in \mathrm{DA}(\Lambda)$) and to determine the norming constants according to (7.8). Denoting the tail of F by $\overline{F} := 1 - F$, we see that $\overline{F}(R) \sim l(R) = Ke^{-\lambda R}$ as $R \to \infty$, where $K = 2\alpha_*^2(1 + \delta_{\alpha_1\alpha_2})/\sigma^2$ and $\lambda = (2\alpha_*)/\sigma^2$. Hence $\overline{F}$ is exponential like and thus $F \in \mathrm{DA}(\Lambda)$ with norming constants given by $c_T = \lambda^{-1}$ and $d_T = \lambda^{-1}\ln(KT)$, $T > 0$, see Table 3.4.4 of Embrechts et al. [EKM97]. These expressions coincide with (7.21).

A.2 Gamma Process

We perform here the calculations to derive the function l_η appearing in the tail asymptotics (7.6) of the maximum for the gamma process $(X_t^G)_{t\in[0,T]}$ with spatial dependence ($\eta \geq 1$) introduced in Section 7.3.3. Further, it is shown that $F := e^{-l_\eta}$ is in the domain of attraction of the Gumbel distribution Λ ($F \in \mathrm{DA}(\Lambda)$), and the norming constants in the long term limit (7.9) are exhibited.

The diffusion process $(X_t^G)_{t\in[0,T]}$ is of gradient field type specified by the SDE (7.2) with potential Φ_θ, $\theta = (\alpha_1, \alpha_2, \beta_1, \beta_2, \eta) \in \Theta$, defined in (7.22) and (7.23). We evaluate the function l_η satisfying (7.6) by Theorem 4.1 in the general case $\eta \geq 1$ allowing for spatial dependence. The stationary measure μ with density $\widetilde{\mu}$ defined in (7.24) is finite by construction. Hence Condition (2.5) holds. Noting that $\alpha_1, \alpha_2 > 3$ by (7.22), the growth condition (4.7) and the spectral gap condition (2.7) can be shown to hold as in the proof of Theorem 4.10, where the independent case $\eta = 1$ is treated. It remains to show the crucial condition (4.5).

Let us recall some definitions: set $a_i := 2(\alpha_i - 1)/\sigma^2 + 1$ and $b_i := \sigma^2\beta_i/2$, $i = 1, 2$. Further, $b^* := \max\{b_1, b_2\}$ and a^* is the a_i corresponding to b^*; analogously for $b_* := \min\{b_1, b_2\}$.

We define the rotationally symmetric test-potential ϕ in Theorem 4.1 by $\phi(R) := R/\beta^*$, $R > 0$ (compare with (4.25) in the independent case $\eta = 1$). The term δ_{as} defined in (4.2) is given in the present situation by (7.26), i.e.,

$$\delta_{as}(R;\eta) = R^{a_1+a_2-2}\int_0^{\pi/2}(\cos\gamma)^{a_1-1}(\sin\gamma)^{a_2-1}\widetilde{c}_\eta(R,\gamma)^{2/\sigma^2}e^{-Rp(\gamma)}d\gamma \qquad R > 0\,,$$

where $p(\gamma) := (\cos\gamma)/b_1 + (\sin\gamma)/b_2 - 1/b^*$. To evaluate the term D_{as} defined in (4.4), we note that Δ_{as} defined in (4.3) reads here in polar coordinates for $R > 0$ and $\gamma \in (0, \pi/2)$

$$\Delta_{as}(R,\gamma;\eta) = \left(\frac{\cos\gamma}{\beta_1} + \frac{\sin\gamma}{\beta_2} - \frac{1}{\beta^*}\right) - \frac{\alpha_1+\alpha_2-2}{R} - \frac{\partial_r\widetilde{c}_\eta}{\widetilde{c}_\eta}(R,\gamma)\,. \tag{1.1}$$

Hence D_{as} is given by

$$D_{as}(R;\eta) = \int_0^{\pi/2} (R\cos\gamma)^{a_1-1}(R\sin\gamma)^{a_2-1}\widetilde{c}_\eta(R,\gamma)^{2/\sigma^2}e^{-Rp(\gamma)}\Delta_{as}(R,\gamma;\eta)^2\,d\gamma\,. \tag{1.2}$$

In the independent case ($\eta = 1$), it was shown in (4.27) and (4.28) in the proof of Theorem 4.10 that as $R \to \infty$

$$\begin{aligned} \delta_{as}(R;1) &\sim b_*^{a_*}\Gamma(a_*)R^{a^*-2} + \delta_{b_1b_2}b_*^{a^*}\Gamma(a^*)R^{a_*-2}\,, \\ D_{as}(R;1) &\gtrsim \kappa_2(R^{-2}\delta_{as}(R;1) + R^{a_1-4} + \delta_{b_1b_2}R^{a_2-4})\,, \end{aligned} \tag{1.3}$$

where $\delta_{b_1b_2} = 1$ if $b_1 = b_2$ and $= 0$ otherwise and κ_2 is a positive constant. Hence, $D_{as}(R) = o(\delta_{as}(R))$ as $R \to \infty$, i.e., the crucial condition (4.5) is satisfied in this case. The function l_1 satisfying (7.6) is then obtained by Theorem 4.10, see also (7.25).

In the case with spatial dependence ($\eta > 1$) however, the crucial condition (4.5) can only be shown numerically, see Figure A.1 for a concrete setting. Hence we obtain from Theorem 4.1 that in the case with spatial dependence ($\eta > 1$) the function l_η (the asymptotic expression of the bottom eigenvalue) has the form

$$\begin{aligned} l_\eta(R) &= \frac{\sigma^2}{2|\mu|}\,\delta_{as}(R;\eta)\left(\int_1^R r^{-1}e^{r/b^*}\,dr\right)^{-1} \\ &\sim \frac{\sigma^2}{2|\mu|}\,\delta_{as}(R;\eta)\left(b^*R^{-1}e^{R/b^*}\right)^{-1} \qquad R > 0\,. \end{aligned} \tag{1.4}$$

For the last step, we used Lemma 4.5. This expression coincides with (7.27).

It remains to shown in the general case with spatial dependence ($\eta \geq 1$) that $F := e^{-l_\eta} \in \mathrm{DA}(\Lambda)$ and to determine the norming constants. We adapt the methods of Example 3 in Chapter 1.5 of Resnick [Res87], where this problem is treated for the gamma distribution. The tail of a distribution function F_1 is denoted by $\overline{F}_1 := 1 - F_1$. Assume that $\delta_{as}(\,\cdot\,;\eta)$ is differentiable for every $\eta \geq 1$ and

$$\delta'_{as}(R;\eta) = o\left(\delta_{as}(R;\eta)\right) \qquad (R \to \infty)\,. \tag{1.5}$$

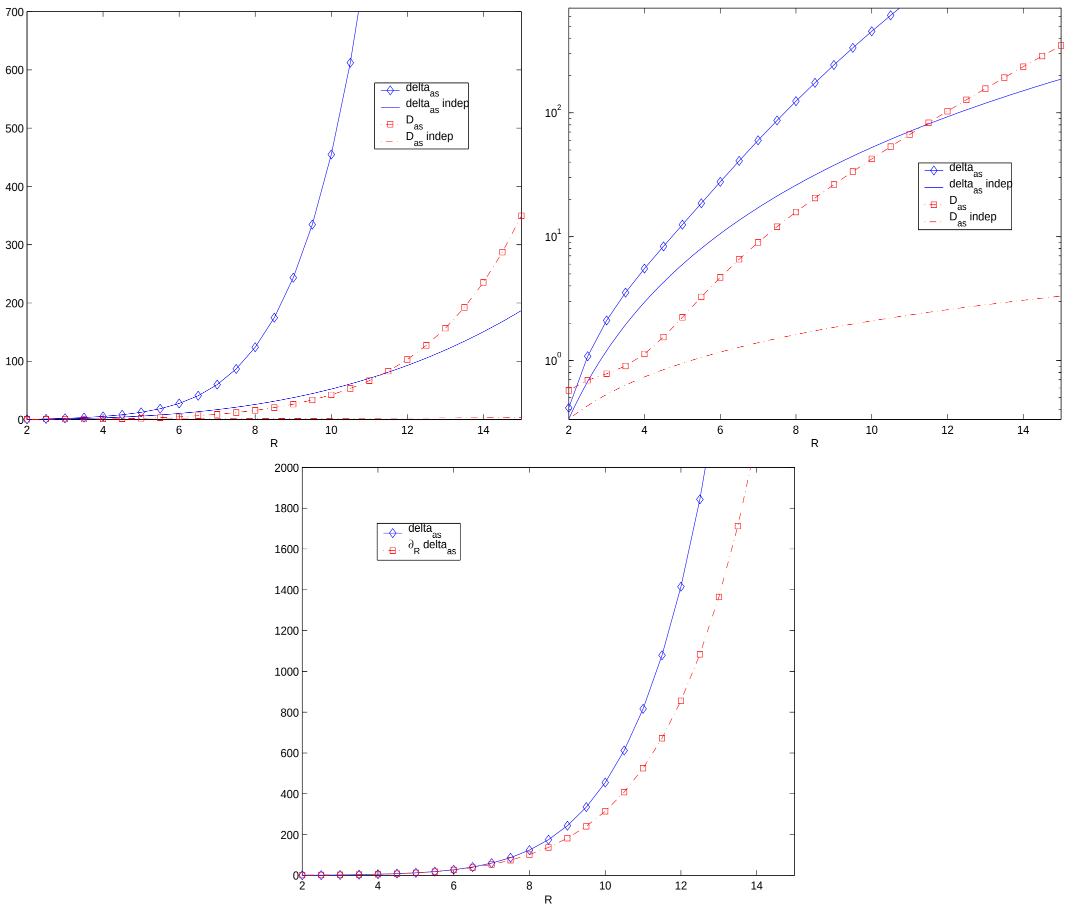

Figure A.1: In this figure, the crucial condition (4.5) is evaluated by numerical methods for the gamma process $(X_t^G)_{t\in[0,T]}$ with spatial dependence $(\eta \geq 1)$ introduced in Section 7.3.3. $(X_t^G)_{t\in[0,T]}$ is a diffusion process of gradient field type specified by the SDE (7.2) with potential Φ_θ, $\theta = (\alpha_1, \alpha_2, \beta_1, \beta_2, \eta) \in \Theta$, defined in (7.22) and (7.23). For the parameter setting $\sigma = 1$ and $\alpha_1 = 3$, $\alpha_2 = 4$, $\beta_1 = 1$, $\beta_2 = 0.5$ we plotted for case with spatial dependence $(\eta = 2)$ the functions $R \mapsto \delta_{as}(R; \eta = 2)$ defined in (7.26) and $R \mapsto D_{as}(R; \eta = 2)$, see (1.1) and (1.2) [top left]. These functions have been evaluated using a quadratic integration scheme. In addition the functions $R \mapsto \delta_{as}(R; 1)$ and $R \mapsto D_{as}(R; 1)$ are plotted in the independent case $\eta = 1$, compare with (1.3). The same is plotted on a logarithmic scale [top right]. It is seen that the crucial condition (4.5) is satisfied for $\eta = 2$ in this setting. For $\eta = 1$, the crucial condition (4.5) holds by (1.3). Further, the functions $R \mapsto \partial_R \delta_{as}(R; \eta = 2)$ and $R \mapsto \delta_{as}(R; \eta = 2)$ are plotted [bottom]. This proves that also Condition (1.5) is satisfied in this setting.

See Figure A.1 for a validation with numerical methods for a concrete setting. We obtain for l_η defined in (7.27) (see also (1.4))

$$(1.6) \qquad l'_\eta(R) = -\frac{l_\eta(R)}{b^*}\left(1 - b^*\left(\frac{1}{R} + \frac{\delta'_{as}(R;\eta)}{\delta_{as}(R;\eta)}\right)\right) \sim -\frac{l_\eta(R)}{b^*} \qquad (R \to \infty)\,.$$

Let F_1 be the cumulative distribution function with $F_1'(R) = l_\eta(R)/b^*$. Using (1.6) and the fact that $\overline{F}(R) \sim l_\eta(R)$ as $R \to \infty$, we see with the help of L'Hôpital's rule that F and F_1 are tail equivalent, i.e.,

$$\overline{F}(R) \sim \overline{F}_1(R) \; (\sim l_\eta(R)) \qquad (R \to \infty)\,.$$

Using again (1.6), it follows easily that

$$\lim_{R\to\infty} \frac{F_1''(R)\overline{F}_1(R)}{F_1'(R)^2} = \lim_{R\to\infty} \frac{(l_\eta'(R)/b^*)\, l_\eta(R)}{(l_\eta(R)/b^*)^2} = -1\,.$$

This implies that F_1 is a Von Mises function and hence $F_1 \in \mathrm{DA}(\Lambda)$, see Definition 3.3.18, Example 3.3.23, and Proposition 3.3.25 of Embrechts et al. [EKM97].

Let us turn to the calculation of the norming constants. The tail equivalence implies that also $F \in \mathrm{DA}(\Lambda)$ having the same norming constants as F_1, see Proposition 3.3.28 of Embrechts et al. [EKM97]. Setting $f := \overline{F}_1/F_1'$, the norming constants can be obtained by a careful asymptotic expansion as $T \to \infty$ of the relations

$$\overline{F}_1(d_T) = 1/T\,, \qquad c_T = f(d_T)\,,$$

see Proposition 1.1 of Resnick [Res87]. Note that also $\overline{F}_1(R) \sim l_\eta(R)$ as $R \to \infty$. Using (1.6), we get

$$\lim_{R\to\infty} f(R) = \lim_{R\to\infty} \frac{\overline{F}_1(R)}{F_1'(R)} = \lim_{R\to\infty} \frac{l_\eta(R)}{l_\eta(R)/b^*} = b^*\,.$$

Hence the norming constants $(c_T)_{T>0}$ can be chosen $c_T := b^*$, $T > 0$.

Using again that $\overline{F}_1(R) \sim l_\eta(R)$ as $R \to \infty$, the norming constants $(d_T)_{T>0}$ can alternatively be exhibited by a careful asymptotic expansion of the relation $l_\eta(d_T) = 1/T$ as $T \to \infty$. Doing this for the general case $\eta \geq 1$ with $l_\eta(R) = \sigma^2(2|\mu|b^*)^{-1}\delta_{as}(R;\eta)Re^{-R/b^*}$, $R > 0$, defined in (7.27) (see also (1.4)), we obtain

$$d_T = b^* \left[\ln T + \ln\ln T + \ln\Big(\delta_{as}(b^*\ln T;\eta)\Big) + \ln\left(\frac{\sigma^2}{2|\mu|}\right)\right] \qquad T > 0\,,$$

which coincides with (7.28).

For the independent case $\eta = 1$, we can use the function l_1 defined in (7.25), and obtain

$$d_T := \begin{cases} b^* \left[\ln T + (a^* - 1) \ln \ln T + \ln \left(\dfrac{\sigma^2}{2b^{*2}\Gamma(a^*)} \right) \right] & b_* < b^* \, , \\ \\ b \Big[\ln T + \ln \Big(\dfrac{(\ln T)^{a_1 - 1}}{\Gamma(a_1)} + \dfrac{(\ln T)^{a_2 - 1}}{\Gamma(a_2)} \Big) + \ln \left(\dfrac{\sigma^2}{2b^2} \right) \Big] & b_* = b^* = b* \ . \end{cases}$$

This expression coincides with (7.29).

Appendix B

Tables for Estimation and Test Results

OU Process: Simulation Study and Estimation Results

$B = \widehat{\theta} - \theta_{true}$	m_1	m_2		α_1	α_2	ϕ	σ
θ_{true}	0	0		0.5	0.1	0	1
N = 1000, T = 50, s = 0.05							
$\overline{B}$ = mean(B)	0.0793	0.0576	cont	0.0966	0.0507	-0.0342	-0.0056
			disc	0.0922	0.0543	-0.0362	0.0142
std($\overline{B}$)	0.0701	0.2808	cont	0.0434	0.0196	0.0479	0.0044
			disc	0.0409	0.0196	0.0500	0.0037
MSE	0.0995	1.5010	cont	0.0451	0.0099	0.0448	4.0635e-04
			disc	0.0403	0.0102	0.0488	4.5722e-04
N = 10000, T = 500, s = 0.05							
$\overline{B}$ = mean(B)	0.0361	0.0242	cont	-3.7405e-05	0.0106	0.0148	0.0005
			disc	-0.0031	0.0131	0.0163	0.0154
std($\overline{B}$)	0.0195	0.1058	cont	0.0114	0.0052	0.0138	0.0014
			disc	0.0114	0.0053	0.0145	0.0010
MSE	0.0085	0.2135	cont	0.0025	0.0006	0.0039	3.9615e-05
			disc	0.0025	0.0007	0.0043	2.5740e-04

Table B.1: This table reports the quality of the parameter estimation of a simulation study for the bivariate OU process introduced in section 7.3.1. We used two sets of simulated sample paths with different sample size N and time horizon T but with constant step-size $s = T/N$. Each set consists of 20 sample paths which are simulated with the parameters reported in θ_{true}. The estimation of the mean (m_1, m_2) is obtained by the estimator (7.33). The results of the parameter estimation of $\theta = (\alpha_1, \alpha_2, \phi)$ for the drift matrix A_θ defined in (7.11), which are obtained by the continuous ansatz maximum likelihood estimator (7.34), are indicated by (cont); the diffusion coefficient σ is obtained in this case by the estimator (7.32). By (disc) we denote the results of the simultaneous estimation of $\theta = (\alpha_1, \alpha_2, \phi)$ and σ using the discrete likelihood estimator (7.38) for the discretized process. We report the following quantities: by B_j we denote the bias defined as the difference between the estimate and the true value for each simulated sample path $j = 1, \ldots, 20$. For each parameter we present the mean $\overline{B} = N^{-1} \sum_{j=1}^{20} B_j$ of the bias and the standard deviation $\text{std}(\overline{B}) = \sqrt{(N(N-1))^{-1} \sum_{j=1}^{20} (B_j - \overline{B})^2}$. In addition the mean square error $\text{MSE} = N^{-1} \sum_{j=1}^{20} B_j^2$ is reported.

OU Process: Parameter Estimation for Simulated Sample Paths used for Goodness-of-Fit Tests

	$\widehat{m}$			$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\phi}$	$\widehat{\sigma}$	k
N = 1000, T = 50, s = 0.05								
1	0.4007	0.3741	cont	0.3766	0.1241	0.0047	1.0201	8.0580
			disc	0.3731	0.1271	0.0014	1.0154	7.8678
2	-0.0319	-0.0121	cont	0.6166	0.2218	0.0001	1.0110	4.5086
			disc	0.6229	0.2217	0.0033	1.0177	4.5106
3	0.4103	-0.1023	cont	0.4639	0.3398	-0.0751	0.9634	2.9429
			disc	0.4405	0.3131	0.0029	1.0050	3.1939
N = 11000, T = 55, s=0.005								
1	0.3408	-0.0025	cont	0.4270	0.2640	-0.0001	0.9959	3.7879
			disc	0.4342	0.2572	-0.0236	1.0017	3.8880
2	-0.0589	0.0062	cont	0.5270	0.1251	0.0001	0.9990	7.9936
			disc	0.5306	0.1247	0.0264	0.9997	8.0192
3	0.1423	0.2480	cont	0.7344	0.1267	-0.0027	0.9958	7.8927
			disc	0.7231	0.1288	0.0018	1.0067	7.7640
N = 20000, T = 200, s=0.01								
1	-0.0182	-0.2729	cont	0.4890	0.0788	0.0000	0.9997	12.6904
			disc	0.4983	0.0775	0.0080	1.0027	12.9032
2	0.1452	0.1319	cont	0.5552	0.1048	0.0034	0.9957	9.5420
			disc	0.5577	0.1048	0.0060	1.0027	9.5420
3	-0.0933	0.0352	cont	0.6581	0.1073	-0.0799	0.9975	9.3197
			disc	0.6594	0.1075	-0.0791	1.0012	9.3023

Table B.2: Estimation results of simulated sample paths for the bivariate OU process introduced in section 7.3.1 with parameters $\sigma = 1$ and $\theta = (\alpha_1, \alpha_2, \phi)$ for the matrix A_θ defined in (7.11) given by $\alpha_1 = 0.5$, $\alpha_2 = 0.1$, $\phi = 0$, using different time horizons T and step-sizes $s = T/N$, where N is the sample size. The estimator $\widehat{m}$ for the mean is given by (7.33). The estimates for θ obtained by the continuous ansatz estimator (7.34) are indicated by (cont); $\widehat{\sigma}$ is obtained in this case by the estimator (7.32). The values for θ and σ from the discrete likelihood estimator (7.38) are denoted by (disc). In addition, the constant k appearing in the upper asymptotic bound in (7.6) is reported, which has for the OU process the form $k = 1/\min(\alpha_1, \alpha_2)$ according to (7.16). As stated in Test A, this constant should be small compared with the time horizon T.

OU Process: Test Results for Simulated Data

			Test A	Test B		
	$\widehat{M}$		$p_{\widehat{M}}$	$\kappa_{0.05}$	$\kappa_{0.01}$	$\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$
N = 1000, T = 50, s = 0.05						
1	4.8287	cont	0.4422	7.5037	9.1917	0.3873
		disc	0.4319	7.4356	9.1030	0.4217
2	4.0809	cont	0.4008	5.8886	7.1396	0.6147
		disc	0.3998	5.8881	7.1393	0.6159
3	5.0272	cont	0.0054*	4.8941*	5.8572	3.1955
		disc	0.0095*	5.0459	6.0492	2.9398
N = 11000, T = 55, s=0.005						
1	4.5142	cont	0.1361	5.5609	6.6769	1.4414
		disc	0.1457	5.6050	6.7357	1.3976
2	5.3249	cont	0.2397	7.3325	8.9586	0.9580
		disc	0.2409	7.3409	8.9696	0.9528
3	5.2112	cont	0.2506	7.2432	8.8539	0.9140
		disc	0.2534	7.2570	8.8709	0.9040
N = 20000, T = 200, s=0.01						
1	7.2459	cont	0.3150	9.7514	11.5346	0.6800
		disc	0.3274	9.8173	11.6153	0.6393
2	6.6784	cont	0.2556	8.6240	10.1642	0.9111
		disc	0.2677	8.6774	10.2290	0.8701
3	5.8439	cont	0.6389	8.5434	10.0682	0.0846
		disc	0.6463	8.5626	10.0915	0.0718

Table B.3: Results of the goodness-of-fit tests for the simulated sample paths reported in Table B.2 of the bivariate OU process introduced in section 7.3.1. The value for σ is taken from the estimator (7.32). $\widehat{M}$ is the sample maximum defined in (7.39) and $p_{\widehat{M}}$ defined in (7.43) is the p-value for the Test A. The quantities $\kappa_{0.05}$ and $\kappa_{0.01}$ correspond to the constants (7.44) in Test B. The term $\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$ is the renormalized sample maximum as in (7.9) and should be compared with the mean of the Gumbel distribution $m^{\Lambda} = 0.5772$ and the quantiles $q^{\Lambda}_{0.05} = 2.9702$, $q^{\Lambda}_{0.01} = 4.6001$. The superscript * denotes that the null-hypothesis, that the underlying model is a OU process, is rejected at the significance level of 5%, i.e., if $p_{\widehat{M}} < 0.05$. Note that no rejection occurs at the significance level of 1%.

Exponential Model: Simulation Study and Estimation Results

$B = \widehat{\theta} - \theta_{true}$	m_1	m_2	α_1	α_2	ϕ	σ
true	0	0	2	1	0	1
N = 1000, T = 50, s = 0.05						
$\overline{B}$ = mean(B)	-0.0383	-0.0097	-0.0792	-0.1455	-6.2527e-05	-0.0632
std($\overline{B}$)	0.0264	0.0114	0.0392	0.0458	0.0014	0.0059
MSE	0.0140	0.0024	0.0340	0.0589	3.3172e-05	0.0046
N = 10000, T = 500, s = 0.05						
$\overline{B}$ = mean(B)	0.0070	0.0077	-0.0518	-0.0551	-0.0013	-0.0577
std($\overline{B}$)	0.0087	0.0033	0.0114	0.0125	0.0004	0.0016
MSE	0.0015	0.0003	0.0051	0.0060	4.9428e-06	0.0034

Table B.4: This table reports the results of the parameter estimation of a simulation study for the bivariate exponential process introduced in Section 7.3.2. We used two sets of simulated sample paths with different sample size N and time horizon T but with constant step-size $s = T/N$. Each set consists of 20 sample paths which are simulated with the parameters reported in θ_{true}. The estimation of the mean (m_1, m_2) is obtained by the estimator (7.33). The parameter $\theta = (\alpha_1, \alpha_2, \phi)$ of the potential Φ_θ defined in (7.18) are estimated by the continuous ansatz maximum likelihood estimator (7.34). The diffusion coefficient σ is obtained by the estimator (7.32). For the description of the reported values we refer to Table B.1.

Exponential Model: Parameter Estimation for Simulated Sample Paths used for Goodness-of-Fit Tests

	$\widehat{m}$		$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\phi}$	$\widehat{\sigma}$
N = 2000, T = 200, s = 0.1						
1	-0.0136	-0.0293	1.0789	1.8876	0.0017	0.9026
2	0.1447	-0.0153	0.7476	2.0152	-0.0015	0.9244
3	-0.0269	-0.0008	1.1437	1.8467	-0.0012	0.9367
N = 10000, T = 100, s = 0.01						
1	0.0678	-0.0476	1.0070	1.6795	0.0000	0.9647
2	-0.1552	0.0073	0.6978	2.2101	-0.0010	0.9670
3	0.0925	0.0281	0.7650	1.8802	-0.0009	0.9936
4	-0.0222	0.0287	0.8775	1.8479	-0.0017	0.9822
N = 8000, T = 400, s = 0.05						
1	-0.0069	0.0040	1.0248	2.0198	-0.0008	0.9270
2	-0.0430	0.0084	1.0031	1.9204	-0.0006	0.9509
3	0.0554	0.0219	0.9678	1.9615	-0.0019	0.9528

Table B.5: Estimation results of simulated sample paths for the bivariate exponential process introduced in Section 7.3.2 with parameters $\sigma = 1$ and $\theta = (\alpha_1, \alpha_2, \phi)$ for the potential Φ_θ defined in (7.18) given by $\alpha_1 = 1$, $\alpha_2 = 2$, $\phi = 0$, using different time horizons T and step-sizes s. The parameter $\theta = (\alpha_1, \alpha_2, \phi)$ is obtained by the estimator (7.34) developed from the continuous ansatz; σ is estimated using (7.32) and the mean m by (7.33).

Exponential Model: Test Results for Simulated Data

		Test A	Test B		
	$\widehat{M}$	$p_{\widehat{M}}$	$\kappa_{0.05}$	$\kappa_{0.01}$	$\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$
N = 2000, T = 200, s = 0.1					
1	2.1086	0.8830	3.5182	4.1336	-0.7634
2	3.7622	0.3037	4.8790	5.8105	1.0162
3	2.3062	0.7676	3.5903	4.2154	-0.3780
N = 10000, T = 100, s = 0.01					
1	3.9849	0.0385*	3.8608*	4.6140	3.238
2	3.2076	0.5803	5.1031	6.1953	0.1413
3	2.7286	0.8223	4.9979	6.0496	-0.5467
4	2.9556	0.5218	4.4208	5.3167	0.3043
N = 8000, T = 400, s = 0.05					
1	3.0371	0.5028	4.1322	4.8156	0.3585
2	3.3095	0.4381	4.4000	5.1347	0.5509
3	3.1839	0.6054	4.5428	5.3073	0.0727

Table B.6: Results of the goodness-of-fit tests for the simulated sample paths of the bivariate exponential process reported in Table B.5. $\widehat{M}$ is the sample maximum defined in (7.39) and $p_{\widehat{M}}$ defined in (7.43) is the p-value for the Test A. The quantities $\kappa_{0.05}$ and $\kappa_{0.01}$ correspond to the constants (7.44) in Test B. The term $\widehat{c}_T^{-1}(\widehat{M} - \widehat{d}_T)$ is the renormalized sample maximum as in (7.9) and should be compared with the mean of the Gumbel distribution $m^\Lambda = 0.5772$ and the quantiles $q^\Lambda_{0.05} = 2.9702$, $q^\Lambda_{0.01} = 4.6001$. The superscript * denotes that the null-hypothesis, that the underlying model is the bivariate exponential process introduced in Section 7.3.2, is rejected at the significance level of 5%, i.e., if $p_{\widehat{M}} < 0.05$. Note that no rejection occurs at the significance level of 1%.

Vašiček Model: Parameter Estimation for Financial Data

	$\widehat{m}$		$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\phi}$	$\widehat{\sigma}$
EUR-GBP	3.7805	cont	0.0254	0.1387	0.0071	0.2412
	6.1051	disc	0.0376	0.0770	0.0314	0.2325
EUR-USD	3.7805	cont	1.5814	0.0342	0.9612	0.2375
	6.3691	disc	2.7062	0.0149	0.9337	0.2660
USD-GBP	6.3691	cont	0.7117	0.0505	0.9632	0.2177
	6.1051	disc	0.0295	0.9351	2.5927	0.2765

Table B.7: Results of parameter estimation of the fit of 30-days Libor rates of the currencies Euro [EUR], British Pound [GBP], and US Dollar [USD] (September 21, 1999 until September 5, 2000) to the bivariate Vašiček short-rate model introduced in section 7.3.1: The variables and estimators correspond to those of Table B.2; time horizon $T = 12$, sample size $N = 241$ and step-size $s = T/N = 0.05$.

Exponential Model: Parameter Estimation for Financial Data

	$\widehat{m}$		$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\phi}$	$\widehat{\sigma}$
EUR-GBP	3.7805	6.1051	0.0103	0.1354	-0.3783	0.2412
EUR-USD	3.7805	6.3691	0.0306	0.1719	0.5644	0.2375
USD-GBP	6.3691	6.1051	0.0403	0.0528	-0.7784	0.2177

Table B.8: Results of parameter estimation of the fit of 30-day Libor rates of the currencies Euro [EUR], British Pound [GBP], and US Dollar [USD] (September 21, 1999 until September 5, 2000) to the bivariate exponential process introduced in Section 7.3.2: pairs of Sample size $N = 244$, time horizon $T = 12$, step-size $s = 0.05$. The variables and estimators correspond to those of Table B.5.

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7.6 Stationary density of the centered bivariate process with exponential distribution introduced in Section 7.3.2 fitted to the USD-GBP data set (30-day Libor rates of British Pound and US Dollar) [left] and visualization of Test B [right] in analogy to Figure 7.5; the estimated parameters are taken from Table B.8. [Left]: Points visited by the centered sample together with the contour plot of the estimated Φ_θ defined in (7.18). [Right]: Points visited by the centered sample together with the sample maximum $\widehat{M}$ (circle) and normalized sample maximum (square). The fit is not rejected since the renormalized maximum is less than the 5%-quantile $q^{\Lambda}_{0.05} = 2.9702$ of the Gumbel distibution (dashed circle). 115

A.1 In this figure, the crucial condition (4.5) is evaluated by numerical methods for the gamma process $(X^G_t)_{t\in[0,T]}$ with spatial dependence ($\eta \geq 1$) introduced in Section 7.3.3. $(X^G_t)_{t\in[0,T]}$ is a diffusion process of gradient field type specified by the SDE (7.2) with potential Φ_θ, $\theta = (\alpha_1, \alpha_2, \beta_1, \beta_2, \eta) \in \Theta$, defined in (7.22) and (7.23). For the parameter setting $\sigma = 1$ and $\alpha_1 = 3$, $\alpha_2 = 4$, $\beta_1 = 1$, $\beta_2 = 0.5$ we plotted for case with spatial dependence ($\eta = 2$) the functions $R \mapsto \delta_{as}(R;\eta = 2)$ defined in (7.26) and $R \mapsto D_{as}(R;\eta = 2)$, see (1.1) and (1.2) [top left]. These functions have been evaluated using a quadratic integration scheme. In addition the functions $R \mapsto \delta_{as}(R;1)$ and $R \mapsto D_{as}(R;1)$ are plotted in the independent case $\eta = 1$, compare with (1.3). The same is plotted on a logarithmic scale [top right]. It is seen that the crucial condition (4.5) is satisfied for $\eta = 2$ in this setting. For $\eta = 1$, the crucial condition (4.5) holds by (1.3). Further, the functions $R \mapsto \partial_R \delta_{as}(R;\eta = 2)$ and $R \mapsto \delta_{as}(R;\eta = 2)$ are plotted [bottom]. This proves that also Condition (1.5) is satisfied in this setting. 121

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B.1 This table reports the quality of the parameter estimation of a simulation study for the bivariate OU process introduced in section 7.3.1. We used two sets of simulated sample paths with different sample size N and time horizon T but with constant step-size $s = T/N$. Each set consists of 20 sample paths which are simulated with the parameters reported in θ_{true}. The estimation of the mean (m_1, m_2) is obtained by the estimator (7.33). The results of the parameter estimation of $\theta = (\alpha_1, \alpha_2, \phi)$ for the drift matrix A_θ defined in (7.11), which are obtained by the continuous ansatz maximum likelihood estimator (7.34), are indicated by (cont); the diffusion coefficient σ is obtained in this case by the estimator (7.32). By (disc) we denote the results of the simultaneous estimation of $\theta = (\alpha_1, \alpha_2, \phi)$ and σ using the discrete likelihood estimator (7.38) for the discretized process. We report the following quantities: by B_j we denote the bias defined as the difference between the estimate and the true value for each simulated sample path $j = 1, \ldots, 20$. For each parameter we present the mean $\overline{B} = N^{-1} \sum_{j=1}^{20} B_j$ of the bias and the standard deviation $\mathrm{std}(\overline{B}) = \sqrt{(N(N-1))^{-1} \sum_{j=1}^{20} (B_j - \overline{B})^2}$. In addition the mean square error $\mathrm{MSE} = N^{-1} \sum_{j=1}^{20} B_j^2$ is reported. 126

B.2 Estimation results of simulated sample paths for the bivariate OU process introduced in section 7.3.1 with parameters $\sigma = 1$ and $\theta = (\alpha_1, \alpha_2, \phi)$ for the matrix A_θ defined in (7.11) given by $\alpha_1 = 0.5$, $\alpha_2 = 0.1$, $\phi = 0$, using different time horizons T and step-sizes $s = T/N$, where N is the sample size. The estimator $\widehat{m}$ for the mean is given by (7.33). The estimates for θ obtained by the continuous ansatz estimator (7.34) are indicated by (cont); $\widehat{\sigma}$ is obtained in this case by the estimator (7.32). The values for θ and σ from the discrete likelihood estimator (7.38) are denoted by (disc). In addition, the constant k appearing in the upper asymptotic bound in (7.6) is reported, which has for the OU process the form $k = 1/\min(\alpha_1, \alpha_2)$ according to (7.16). As stated in Test A, this constant should be small compared with the time horizon T. 127

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B.4 This table reports the results of the parameter estimation of a simulation study for the bivariate exponential process introduced in Section 7.3.2. We used two sets of simulated sample paths with different sample size N and time horizon T but with constant step-size $s = T/N$. Each set consists of 20 sample paths which are simulated with the parameters reported in θ_{true}. The estimation of the mean (m_1, m_2) is obtained by the estimator (7.33). The parameter $\theta = (\alpha_1, \alpha_2, \phi)$ of the potential Φ_θ defined in (7.18) are estimated by the continuous ansatz maximum likelihood estimator (7.34). The diffusion coefficient σ is obtained by the estimator (7.32). For the description of the reported values we refer to Table B.1. 129

B.5 Estimation results of simulated sample paths for the bivariate exponential process introduced in Section 7.3.2 with parameters $\sigma = 1$ and $\theta = (\alpha_1, \alpha_2, \phi)$ for the potential Φ_θ defined in (7.18) given by $\alpha_1 = 1$, $\alpha_2 = 2$, $\phi = 0$, using different time horizons T and step-sizes s. The parameter $\theta = (\alpha_1, \alpha_2, \phi)$ is obtained by the estimator (7.34) developed from the continuous ansatz; σ is estimated using (7.32) and the mean m by (7.33). 130

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B.7 Results of parameter estimation of the fit of 30-days Libor rates of the currencies Euro [EUR], British Pound [GBP], and US Dollar [USD] (September 21, 1999 until September 5, 2000) to the bivariate Vašiček short-rate model introduced in section 7.3.1: The variables and estimators correspond to those of Table B.2; time horizon $T = 12$, sample size $N = 241$ and step-size $s = T/N = 0.05$. 132

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List of Symbols

b, b^i	drift coefficient	
B_R	open balls $\{x \in \mathbb{R}^n : \|x\| < R\}$	
$(B_t)_{t\geq 0}$	standard Brownian motion	
$C(O)$	class of continuous functions on O	
$C^i(O)$	class of i-times continuously differentiable functions on O	
$C_c(O)$	class of continuous functions on O with compact support	
$C_0(O)$	class of continuous functions on O which can be extended continuously by 0 to ∂O	
c_n, c_T	norming constants	2, 22
$\mathcal{D}(\mathcal{E}_R)$	domain of quadratic form $\mathcal{E}_R$	26
$\mathcal{D}(L_R)$	domain of the operator L_R	26
$D_{as}(R)$	asymmetric term $m_R[e^{-2\Phi_{as}/\sigma^2}\Delta_{as}^2]$	37
$\mathrm{DA}(H)$	domain of attraction of an extreme value distribution H	3, 22
$d\sigma$	surface measure of unit sphere S^{n-1}	
d_n, d_T	norming constants	2, 22
Δ	Laplace operator	
Δ_θ	Laplace-Beltrami operator w.r.t. spherical coordinates	
$\Delta_{as}(x)$	$\frac{1}{\|x\|}\sum_{i=1}^n x_i\,\partial_{x_i}\Phi(x) - \phi'(\|x\|)$	37
$\delta_{as}(R)$	asymmetric term $m_R[e^{-2\Phi_{as}/\sigma^2}]$	36
∂O	boundary of the set O	
$(e^{L_R t})_{t\geq 0}$	semigroup on $L^2(O_R,\mu)$ generated by L_R	27
$\mathcal{E}_R$	quadratic (Dirichlet) form	26
F	distribution function of a real random variable	

Γ	the Gamma function	
γ_n	volume of the unit sphere S^{n-1}	
H	extreme value distribution	2
I_A	indicator function of the set A	
i.i.d.	independent, identically distributed	
κ_α	rejection level for goodness-of-fit test	105
L	infinitesimal generator (formally)	17
L_R	self-adjoint extension of generator L on $L^2(O_R, \mu)$	26
$L^1_{loc}(O)$	class of locally integrable functions on O	
$L^2(O, \mu)$	class of μ-square integrable functions on O	
$L^2_{\mu_R}$	$L^2(O_R, \mu_R)$	
$l(R)$	asymptotic expression for λ_R	19
$l_R(v)$	$\|v\|^{-2}_{2,R} \, \|L_R v\|^2_{2,R}$	30
$\liminf_{\|x\|\to\infty} V(x)$	$\lim_{R\to\infty} \inf_{\|x\|>R} V(x)$	
Λ	Gumbel distribution function	2
Λ_{sg}	spectral gap	17
λ_R	bottom eigenvalue of operator $-L_R$	17
$\lambda_{R,2}$	$\inf \Sigma(-L_R) \cap (\lambda_R, \infty)$	28
$\widehat{M}$	sample maximum in Euclidean norm	105
M_n	maximum of i.i.d. real random variables	2
M_T	maximum of a diffusion process $\max_{0\le t\le T} q(X_t)$	7
m	mean of a stationary process	102
$m_R[f]$	spherical mean $\int_{S^{n-1}} f(R\xi) d\sigma(\xi)$	
$m_{\Phi,R}[f]$	$\int_{\partial O^\Phi_R} \|\nabla\Phi(\xi)\|^{-1} f(\xi)\, d\sigma_{\Phi,R}(\xi)$	59
μ	stationary measure	
$\widetilde{\mu}$	density of the stationary measure μ	16
$\|\mu\|$	total mass of the stationary measure μ	17
μ_R	restriction of stationary measure μ to the set O_R	
$\mathbb{N}$	set of positive integers	
$\mathcal{N}(p)$	zero set of a function p	

$\nu[g](R)$	$\int_1^R r^{1-n} g(r) e^{2\phi(r)/\sigma^2}\, dr$	37
$\nu(R)$	$\nu[\mathbf{1}](R)$	37
$(O_R)_{R>R_0}$	exhausting family of $\mathbb{R}^n$	7
O_R^Φ	$\{x \in \mathbb{R}^n : \Phi(x) < R\}$, level set of Φ	57
$o(1)$	$a(t) = o(b(t))$ as $t \to t_0$ means that $\lim_{t\to t_0} \lvert a(t)/b(t) \rvert = 0$	19
P_μ	law of a diffusion process starting with its stationary measure μ	
$p_{\widehat{M}}$	p-value for realization $\widehat{M}$	106
Φ	potential function	8
Φ_{as}	asymmetric part of potential Φ	35
Φ_σ	modified potential $(2/\sigma^2)\Phi + \ln \lvert\mu\rvert$	
Φ_α	Fréchet distribution function	2
Ψ_α	Weibull distribution function	2
ϕ	rotationally symmetric test-potential	35
q	$\inf\{R > R_0 : x \in O_R\}$, distance function	7
$\mathbb{R}$	real numbers	
R^ϕ	rotation matrix	95
$\rho_R(v)$	Rayleigh quotient $\lVert v\rVert_{2,R}^{-2}\, \mathcal{E}_R(v,v)$	30
S^{n-1}	unit sphere in $\mathbb{R}^n$	
SDE	stochastic differential equation	
σ, $(\sigma^{ij})_{ij}$	diffusion coefficient/matrix	
$\Sigma(A)$	spectrum of the operator A	
τ_R	$\inf\{s > 0 : X_s \in \mathbb{R}^n \setminus O_R\}$, first exit time	7
V_Φ	Schrödinger potential associated to Φ	31
v_R	test-function	39, 60
$(X_i)_{i\in\mathbb{N}}$	sequence of i.i.d. real random variables	
$(X_t)_{t\geq 0}$	diffusion process	
$\mathcal{Z}$	$\{x \in \mathbb{R}^n : \Phi(x) = +\infty\}$	16
$\mathcal{Z}^c$	complement of set $\mathcal{Z}$	16
$\xrightarrow{d}$	convergence in distribution	
$\lvert\cdot\rvert$	Euclidean norm	

∇	the gradient	
∇_θ	gradient w.r.t. spherical coordinates	
$\sim$	$a(t) \sim b(t)$ as $t \to t_0$ means that $\lim_{t\to t_0} a(t)/b(t) = 1$	19
$\lesssim$	$a(t) \lesssim b(t)$ as $t \to t_0$ means that $\limsup_{t\to t_0} a(t)/b(t) \leq 1$	19
$\gtrsim$	$a(t) \gtrsim b(t)$ as $t \to t_0$ means that $b(t) \lesssim a(t)$ as $t \to t_0$	19
$\\|\cdot\\|_{2,R}$	norm in $L^2(O_R, \mu_R)$	
$(\cdot,\cdot)_R$	scalar product in $L^2(O_R, \mu_R)$	
$\mathbf{1}$	constant function of value 1	
u^R	scaled function $u^R(x) := u(Rx)$	77
$\widehat{\alpha}$	estimator of the parameter α	

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Conferences attended

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- 2nd Stochastic Day of Erlangen and Munich, Munich, July 5, 2002

Publications

- On Extremes of Multidimensional Stationary Diffusion Processes in Euclidean Norm, Sumitted for publication, 2002
- Maxima of Diffusion Processes of Gradient Field Type with Respect to the Level Sets of the Potential, Sumitted for publication, 2002

- Extremes of Multivariate Stationary Diffusions in Finance: A Data Analysis, Preprint, 2002
- Asymptotics of the Euclidean Maximum of Diffusion Processes associated to Gradient Fields in Singular Perturbations, Working paper