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Optimal Portfolios with Bounded Downside Risks

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Contents

Abstract

In this thesis we optimize portfolios of one riskless bond and several risky assets in the Black-Scholes model as well as in more complicated models. As an alternative to the classical mean-variance portfolio selection we take up an idea going back to Fishburn (1977) and Harlow (1991). They introduced so-called lower partial moments as risk measures. The lower partial moment of order $n \in \mathbb{N}_0$ is defined as

$$\mathrm{LPM}_n(x) = \int_{-\infty}^{x} (x-r)^n dF(r)\,, \quad x \in \mathbb{R}\,,$$

where F is the distribution function of the portfolio return. The advantage of such lower partial moments is that they are only based on negative deviations. Here we replace the variance by risk measures defined by lower partial moments. The lower partial moment of order 0 is the probability that the terminal wealth of a portfolio is below a certain benchmark, e.g. the DAX or the Dow Jones index. The Capital-at-Risk with respect to the expected shortfall (CaR^S) is based on a lower partial moment of order 1, where we measure risk as the difference between riskless wealth and the expected shortfall. Another downside risk measure we consider is the Capital-at-Risk (CaR) with respect to the quantile, which is defined in the same way as the (CaR^S), but the expected shortfall is replaced by a low quantile. We think of the (CaR) and the (CaR^S) as some capital reserve in equity.

This thesis is organized in five parts.

After an introductory chapter we derive explicit closed form solutions for the mean-CaR problem in a Black-Scholes market. Then we move to more general price processes like the Black-Scholes market with jumps and the generalized inverse Gaussian diffusion, where we develop an algorithm for the numerical solution of the mean-CaR problem, since these problems are not analytically tractable any more.

In the third chapter we consider first the CaR^S as risk measure. Since an analytic solution cannot be found neither in the Black-Scholes case nor in the Black-Scholes case with jumps, we work out upper and lower bounds for the optimal strategy and solve the problems for some examples numerically. Then we replace the CaR^S by the shortfall probability with respect to a certain benchmark, for example a market index, and maximize the expected relative wealth of the portfolio, i.e. the expected ratio of the portfolio's wealth and the benchmark, instead of the expected terminal wealth of the portfolio. It seems to be useful to maximize the wealth of the portfolio relative to the benchmark, i.e. the ratio of the wealth of the portfolio and the benchmark, since we measure risk depending on this benchmark. In this case we derive an explicit closed form solution.

The fourth chapter is devoted to the study of optimal portfolios when stock prices follow an exponential Lévy process. First we calculate the moments and find out that the optimal strategy in the mean-variance problem has a similar structure as in the Black-Scholes world. For the mean-CaR optimization we approximate the CaR in this exponential Lévy model using a method introduced by Asmussen and Rosinski (2000): under certain assumptions one can approximate the small jumps of a Lévy process by a Brownian motion with the same variance.

In the fifth chapter we model asset prices by an SDE driven by a Lévy process. There the problem of negative asset prices occurs if the Lévy process has jumps of size lower than -1. Taking up an idea of Eberlein and Keller (1995) we interpret such a jump as a crash and set the price of the concerning asset equal to zero after this crash. We calculate moments in this crash scenario and derive optimal portfolios under a variance constraint.

Zusammenfassung

In dieser Arbeit werden Portfolios aus einem risikolosen Bond und mehreren Aktien sowohl im Black-Scholes-Modell als auch in komplizierteren Modellen optimiert. Als Alternative zum klassischen Erwartungswert-Varianz-Ansatz wird eine Idee aufgegriffen, die auf Fishburn (1977) und Harlow (1991) zurückgeht. Sie haben sogenannte ,,Lower Partial Moments" als Risikomaße eingeführt. Das Lower Partial Moment der Ordnung $n \in \mathbb{N}_0$ ist definiert als

$$\mathrm{LPM}_n(x) = \int_{-\infty}^{x} (x-r)^n dF(r)\,, \quad x \in \mathbb{R}\,,$$

wobei F die Verteilungsfunktion des Portfoliovermögens ist. Der Vorteil solcher Lower Partial Moments ist, daß sie nur auf negativen Abweichungen basieren. Die Varianz wird hier durch Risikomaße ersetzt, die durch Lower Partial Moments definiert werden. Das Lower Partial Moment der Ordnung 0 ist die Wahrscheinlichkeit, daß das Portfoliovermögen zum Endzeitpunkt unter einer bestimmten Benchmark, z.B. dem DAX oder dem Dow Jones Index, liegt. Der Capital-at-Risk bezüglich des erwarteten Shortfalls (CaR^S) basiert auf einem Lower Partial Moment der Ordnung 1, wobei wir das Risiko als die Differenz zwischen risikolosem Vermögen und erwartetem Shortfall messen. Ein anderes Downside Risikomaß, das wir betrachten, ist der Capital-at-Risk (CaR) bezüglich des Quantils. Dieses Risikomaß ist auf die gleiche Art wie der (CaR^S) definiert, wobei der erwartete Shortfall durch ein kleines Quantil ersetzt wird. (CaR) und (CaR^S) kann man als Kapitalreserve interpretieren.

Diese Arbeit besteht aus fünf Teilen.

Nach einem einführenden Kapitel werden explizite geschlossene Lösungen für das Erwartungswert-CaR-Problem im Black-Scholes-Modell hergeleitet. Dann werden allgemeinere

Preisprozesse wie das Black-Scholes-Modell mit Sprüngen und die verallgemeinerte invers Gaußsche Diffusion behandelt, wo ein Algorithmus zur numerischen Lösung des Erwartungswert-CaR-Problems entwickelt wird, da diese Probleme analytisch nicht mehr lösbar sind.
Im dritten Kapitel wird zuerst der CaR^S als Risikomaß betrachtet. Da eine analytische Lösung weder im Black-Scholes-Modell noch im Black-Scholes-Modell mit Sprüngen gefunden werden kann, werden Ober- und Untergrenzen für die optimale Strategie erarbeitet und das Optimierungsproblem für einige Beispiele numerisch gelöst. Dann wird der CaR^S durch die Shortfallwahrscheinlichkeit bezüglich einer Benchmark, z.B. eines Marktindexes, ersetzt und das erwartete relative Portfoliovermögen, d.h. das erwartete Verhältnis aus Portfoliovermögen und Benchmark, anstelle des erwarteten Portfoliovermögens maximiert. Es scheint sinnvoll, das Portfoliovermögen bezüglich der Benchmark, d.h. das Verhältnis aus Portfoliovermögen und Benchmark, zu maximieren, da das Risiko in Abhängigkeit von dieser Benchmark gemessen wird. In diesem Fall werden explizite geschlossene Lösungen hergeleitet.
Das vierte Kapitel behandelt die Optimierung von Portfolios, wenn die Aktienpreise exponentiellen Lévyprozessen folgen. Zunächst werden Momente berechnet und gezeigt, daß die optimale Strategie im Erwartungswert-Varianz-Problem eine ähnliche Struktur hat wie im Black-Scholes-Modell. Für die Erwartungswert-CaR-Optimierung wird der CaR im exponentiellen Lévy-Modell mit Hilfe einer Idee von Asmussen und Rosinski (2000) approximiert: unter bestimmten Voraussetzungen kann man die kleinen Sprünge eines Lévyprozesses durch eine Brownsche Bewegung mit der gleichen Varianz approximieren.
Im fünften Kapitel werden Aktienpreise durch eine von einem Lévyprozeß getriebene SDE modelliert. Hier tritt das Problem von negativen Aktienkursen auf, wenn der Lévyprozeß Sprünge niedriger als -1 hat. Nach einer Idee von Eberlein und Keller (1995) wird so ein Sprung als Crash interpretiert und der Preis der betreffenden Aktie nach dem Crash auf Null gesetzt. Es werden in diesem Crash-Szenario Momente berechnet und optimale Portfolios hergeleitet.

Chapter 1

Introduction

During the last 20 years daily business at stock market exchanges has been vastly growing. So the question of the optimal investment has become more and more important over the last years.

The traditional method of portfolio selection was introduced by Markowitz (1959) and Sharpe (1964) and is based on a mean-variance optimization in the classical Black-Scholes model. Still nowadays it is very popular in risk departments of banks, since it can be applied with basic knowledge on stochastic models. For his ideas on the mean-variance approach Markowitz received the Nobel prize in economic sciences in 1990. The principle can be summarized in two basic formulations of this approach:

- maximization of the expected terminal wealth of a portfolio under a constraint on the upper bound of its variance.

- minimization of the variance given a lower bound on the expected terminal wealth.

Since the first optimization seems to be the more natural one we take up the idea for some optimization problems in this thesis.

Another common approach to portfolio optimization is the maximization of expected utility of wealth. Depending on the choice of the risk measure there exists an equivalent utility maximization approach for certain mean-risk optimization problems (see Fishburn (1977) and Harlow (1991)). Since for the mean-risk optimization approach the interpretation is much easier and for a better comparability to the Markowitz approach we do not work with utility functions in this thesis. Thus we restrict ourselves to mean-risk-optimization problems and modify the mean-variance principle of Markowitz such that it becomes more realistic. The variance as a risk measure shows several deficiencies. It leads to a decreasing proportion of risky assets, when the time horizon increases, whereas it is a well-known fact, that long term stock investment leads to an almost sure gain over locally riskless bond investment and hence the longer the planning horizon, the more one should invest in risky assets. This contradiction cannot be solved using the variance as risk measure. Besides that the variance takes into account positive deviations as well as negative ones. But for asset prices positive deviations are gains which cannot be interpreted as risk. So better alternatives to the variance are non-symmetric risk measures, e.g. downside risk measures which are only based on negative deviations. Wellknown examples are the so called *lower partial moments*, which are investigated by Fishburn (1977) and Harlow (1991). The lower partial moment of order n is defined as

$$\mathrm{LPM}_n(x) = \int_{-\infty}^{x} (x-r)^n dF(r)\,, \quad x \in \mathbb{R}\,,$$

where F is the distribution function of the portfolio return.

In this thesis we consider three different downside risk measures for optimization in the Black-Scholes model. We start with the Capital-at-Risk with respect to a quantile (CaR), which is defined as the difference between the riskless wealth attained by a pure bond strategy and some low quantile (typically the 5%- or 1%-quantile) of the wealth of the portfolio; see e.g. Jorion (1997). The CaR can be interpreted as some capital reserve in equity, which is required by the Basle accord. This risk measure provides the advantage that one can derive explicit closed form solutions for our portfolio problem at least in a Gaussian world.

The CaR also shows several disadvantages. Artzner, Delbaen, Eber, and Heath (1999) argue that for the effective regulation and management of risk any risk measure should be coherent, i.e. translation invariant, positive homogeneous, monotone, and subadditive. But the CaR fails to be coherent, since it is not subadditive. Another deficiency of the CaR is that it does not take into account the shape of the profit-loss distribution on the left side of the quantile. Thus as another risk measure we investigate the Capital-at-Risk with respect to the expected shortfall (CaR^S), which is based on a lower partial moment of order 1. The expected shortfall is defined as the conditional expectation of the terminal wealth under the condition that the terminal wealth is below a low quantile (again typically the 5%- or 1%-quantile). Analogously to the CaR the CaR^S is then defined as the difference between the riskless wealth and the expected shortfall. In comparison to the CaR the CaR^S has the advantage to take also into account how large losses are to be expected, if the portfolio's wealth falls below the quantile. Unfortunately, it is not possible to derive explicit closed form solutions for a mean-CaR^S optimization even in the Black-Scholes model. Hence we work out upper and lower bounds for the optimal strategy and solve the problem numerically in the Black Scholes model, possibly enriched with jumps. As we show in Chapter 2 and Chapter 3 and is also demonstrated in several figures the replacement of the variance by the CaR or the CaR^S resolves the above mentioned contradiction between theory and empirical facts, since the CaR and the CaR^S lead to a higher investment in risky assets for very large time horizons. These two risk measures, the CaR and the CaR^S, only look at absolute losses of the portfolio and do not take into account the performance of the portfolio relative to the whole situation on the capital market. Therefore we consider a third non-symmetric risk measure, the shortfall probability, which is based on a lower partial moment of order 0. It is defined as the probability that the terminal wealth of a portfolio is below a certain benchmark, e.g. the DAX or the Dow Jones index. Thus this definition gives us the opportunity to measure risk relative to the market on which the assets are traded. In this case it seems to be useful also to maximize wealth with respect to the benchmark, i.e. the expected ratio of the wealth of the portfolio and the benchmark. This provides also the possibility to derive explicit closed form solutions in a Gaussian world.

In the early years of portfolio opimization most approaches proceeded from the assumption of the Black-Scholes model, i.e. lognormally distributed stock prices and stationary, independent increments of their logarithms. The basic idea for this continuous time model was already found in 1900 by Louis Ferdinand Bachelier. He modelled stock prices as Brownian motions with drift. This, however, leads to a positive probability for negative asset prices, which does not correspond to reality. In the Black-Scholes model this problem is solved, since asset prices are modelled by geometric Brownian motions, which cannot attain negative values. Because of the normal distribution the Black-Scholes model is mathematically easily tractable in many cases, e.g. for the mean-variance optimization and the mean-CaR optimization as we see in Chapter 2. Nowadays it is well-known that the normal distribution is not a realistic model for the returns of most financial assets. One can often observe leptocurtic data, i.e. asset returns have semi-heavy tails, such that the curtosis is higher than the curtosis of the normal distribution. Consequently, one can improve the classical Black-Scholes model dropping the normal assumption and replacing the Brownian motion by a general stochastic process with stationary, independent increments, i.e. a Lévy process. For example Eberlein and Keller (1995) proposed generalized hyperbolic Lévy processes or certain subclasses as a model for the logarithmic asset price processes and examined statistically their fit in a very convincing way. These generalized hyperbolic distributions which model the increments of the logarithmic asset price, are a normal mean variance mixture and were first introduced by Barndorff-Nielsen (1977), who applied them to model grain size distributions of wind blown sands. Typical examples for these normal mixture models which play an increasing role also in the financial industry are the normal inverse Gaussian and the variance gamma model.

In this thesis we optimize portfolios for general exponential Lévy processes under variance constraints as well as under CaR constraints and illustrate the results by examples, i.e. the exponential normal inverse Gaussian Lévy processes which are a subclass of the exponential generalized hyperbolic Lévy processes, the exponential Meixner Lévy process, and the exponential variance gamma Lévy process. Calculating moments and the CaR one can see that these models are mathematically less tractable than the Black-Scholes model. In most cases the CaR can not be calculated explicitly. Here we use an idea of Asmussen and Rosinski (2000) to approximate the small jumps of a Lévy process by a Brownian motion or some other limit process. This leads to the replacement of the Lévy process by the sum of a drift term, a simpler Lévy process, and a compound Poisson process. We derive certain relations between a Lévy process and its stochastic exponential to apply this result for the calculation of quantiles of the wealth process.

Besides this exponential Lévy-Black-Scholes model which is a first step in extending geometric Brownian motion there are as well interesting alternatives dropping even the assumption of stationary, independent increments. In Chapter 2 we investigate the generalized inverse Gaussian diffusion model, which was introduced by Borkovec and Klüppelberg (1998) and is a formal extension of the Black-Scholes model in a different direction. This model contains the generalized Cox-Ingersoll-Ross model as a special case.

In this thesis we also discuss another model which can be seen as a generalization of the classical Black-Scholes model. Asset prices in the Black-Scholes model can also be written as stochastic differential equations (SDE) driven by Brownian motion equivalently to the approach using geometric Brownian motion. Defining a model by replacing the Brownian motion in the SDE by a general Lévy process leads to a positive probability for negative asset prices. The reason for this are possibly negative jumps of the driving Lévy process with absolute size greater than one. To solve this problem we take up an idea of Eberlein and Keller (1995) who interpret such a jump of size lower than -1 as crash and, after this bankruptcy, all wealth invested in the crash asset is lost and its asset price is zero afterwards. Thus, in this model asset prices are not exponential Lévy processes, but stopped exponential Lévy processes, since the solution to the SDE is an exponential Lévy process until crash time. Because of the independent, stationary increments of the Lévy process such a crash appears always with the same probability independent of the actual asset prize. This can be used for example as a realistic approach to model new economy asset prices. Mean and variance in this model with a possible crash have very complicated forms because of the stopping times. Hence even for the variance as risk measure, it is not possible to solve the optimization problem explicitly. We optimize such portfolios numerically and compare the results to those of a Gaussian world.

Chapter 2

Optimal portfolios with bounded Capital-at-Risk

It seems to be common wisdom that long term stock investment leads to an almost sure gain over locally riskless bond investments. In the long run stock indices are growing faster than riskless rates, despite the repeated occurrence of stock market declines. The conventional wisdom therefore holds that the more distant the planning horizon, the greater should be one's wealth in risky assets. One of our main findings presented in this chapter will be the demonstration that there is indeed a reasonable portfolio problem with a solution that supports this empirical observation.

Traditional portfolio selection as introduced by Markowitz (1959) and Sharpe (1964) is based on a mean-variance analysis. This approach cannot explain the above phenomenon: the use of the variance as a risk measure of an investment leads to a decreasing proportion of risky assets in a portfolio, when the planning horizon increases (see Example 2.1.11).

In recent years certain variants of the classical Markowitz mean-variance portfolio selection criterion have been suggested. Such alternatives are typically based on the notion of downside risk concepts such as *lower partial moments*. The lower partial moment of order n is defined as

$$\mathrm{LPM}_n(x) = \int_{-\infty}^{x} (x-r)^n dF(r)\,, \quad x \in \mathbb{R}\,, \tag{2.0.1}$$

where F is the distribution function of the portfolio return. Examples can be found in

Fishburn (1977) or Harlow (1991), who suggested for instance the shortfall probability ($n = 0$), the expected target shortfall ($n = 1$), the target semi-variance ($n = 2$), and target semi-skewness ($n = 3$). Harlow (1991) also discusses some practical consequences of various downside risk measures.

In this chapter we concentrate on the Capital-at-Risk (CaR) as a replacement of the variance in portfolio selection problems. We think of the CaR as the capital reserve in equity. The CaR is defined via the Value-at-Risk; i.e. a low quantile (typically the 5%- or 1%-quantile) of the profit-loss distribution of a portfolio; see e.g. Jorion (1997). The CaR of a portfolio is then commonly defined as the difference between the mean of the profit-loss distribution and the VaR. VaR has become the most prominent risk measure during recent years. Even more, the importance of VaR models continues to grow since regulators accept these models as a basis for setting capital requirements for market risk exposure. If the profit-loss distribution of a portfolio is normal with mean μ and variance σ^2, then the CaR of the portfolio based on the α-quantile (e.g., $\alpha = 0.05$ or $\alpha = 0.01$) is

$$\mathrm{CaR} = \mu - (\mu - \sigma z_\alpha)\,, \tag{2.0.2}$$

where z_α is the α-quantile of the standard normal distribution and σ is positive. In this chapter we will use another definition of the CaR.

The crucial point in the application of CaR models for setting capital requirement is the determination of reliable and accurate figures for the VaR, especially for non-normal cases. Consequently, VaR has attracted attention from a statistical point of view; e.g., see Embrechts, Klüppelberg and Mikosch (1997) for estimation via extreme value methods and further references, see Emmer, Klüppelberg and Trüstedt (1998) for an example.

In the context of hedging, VaR has been considered as a risk measure by Föllmer and Leukert (1999); see also Cvitanic and Karatzas (1999). They replace the traditional "hedge without risk" (perfect hedge) which typically only works in a complete market setting by a "hedge with small remaining risk" (so-called quantile-hedging). This concept can also deal with incomplete markets. In contrast to our problem, their main task consists of approximating a given claim. Surprisingly, the existence of that target wealth makes their problem more tractable than ours.

In a discrete world Zagst and Kehrbaum (1998) investigate the problem of optimizing portfolios under a limited CaR from a practical point of view, they solve the problem by numerical approximation, and they present a case study. This work is continued in Scheuenstuhl and Zagst (1998). Under a mean-variance and shortfall preference structure for the investor, they obtain optimal portfolios consisting of stocks and options via an approximation method.

One aim of this chapter is to show that a replacement of the variance by the CaR in a continuous-time Markowitz-type model resolves exactly the above-mentioned contradiction between theory and empirical facts. Furthermore, we aim at closed form solutions and an economic interpretation of our results. In a Gaussian world, represented by a Black-Scholes market, possibly enriched with a jump component, the mean-CaR selection procedure leads to rather explicit solutions for the optimal portfolio. It is, however, not surprising that as soon as we move away from the Gaussian world, the optimization problem becomes analytically untractable. This chapter is organized as follows. In Section 2 we highlight the consequences of the introduction of the CaR as risk measure in a simple Black-Scholes market where we can obtain explicit closed form solutions. We also examine consequences for the investor when introducing CaR in a portfolio optimization problem. This approach indeed supports the above-mentioned market strategy that one should always invest in stocks for long-term investment.

Section 3 is devoted to the study of the portfolio problem for more general models of the stock price. As prototypes of models to allow for larger fluctuations than pure Gaussian models, we study jump diffusions and generalized inverse Gaussian diffusion processes. This also shows how the solution of the problem becomes much more involved when the Black-Scholes assumptions are abandoned. In particular, we show how the optimal portfolio under a CaR constraint reacts to the possibility of jumps. In the generalized inverse Gaussian diffusion setting even the problem formulation becomes questionable as we cannot ensure a finite expected terminal wealth of the optimal portfolio. We give an approximate solution, which allows for some interpretation, and also a numerical algorithm. The optimization problems and the solution methods discussed in this chapter are based on an idea of Ralf Korn.

2.1 Optimal portfolios and Capital-at-Risk in the Black-Scholes setting

In this section, we consider a standard Black-Scholes type market consisting of one *riskless bond* and several *risky stocks*. Their respective prices $(P_0(t))_{t\geq 0}$ and $(P_i(t))_{t\geq 0}$ for $i = 1,\ldots,d$ evolve according to the equations

$$\begin{aligned} dP_0(t) &= P_0(t) r dt\,, & P_0(0) &= 1\,,\\ dP_i(t) &= P_i(t)\left(b_i dt + \textstyle\sum_{j=1}^d \sigma_{ij} dW_j(t)\right)\,, & P_i(0) &= p_i\,, \quad i = 1,\ldots,d\,. \end{aligned}$$

Here $W(t) = (W_1(t),\ldots,W_d(t))'$ is a standard d-dimensional Brownian motion, $r \in \mathbb{R}$ is the *riskless interest rate*, $b = (b_1,\ldots,b_d)'$ the vector of *stock-appreciation rates* and $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ is the matrix of *stock-volatilities*. For simplicity, we assume that σ is invertible and that $b_i \geq r$ for $i = 1,\ldots,d$.

Let $\pi(t) = (\pi_1(t),\ldots,\pi_d(t))' \in \mathbb{R}^d$ be an admissible portfolio process, i.e. $\pi_i(t)$ is the fraction of the wealth $X^\pi(t)$, which is invested in asset i (see Korn (1997), Section 2.1 for relevant definitions). Denoting by $(X^\pi(t))_{t\geq 0}$ the *wealth process*, it follows the dynamic

$$dX^\pi(t) = X^\pi(t)\left\{((1-\pi(t)'\underline{1})r + \pi(t)'b)dt + \pi(t)'\sigma dW(t)\right\}\,, \quad X^\pi(0) = x\,, \qquad (2.1.1)$$

where $x \in \mathbb{R}$ denotes the initial capital of the investor and $\underline{1} = (1,\ldots,1)'$ denotes the vector (of appropriate dimension) having unit components. The fraction of the investment in the bond is $\pi_0(t) = 1-\pi(t)'\underline{1}$. Throughout the chapter, we restrict ourselves to constant portfolios $\pi(t) = \pi = (\pi_1,\ldots,\pi_d)$ for all $t \in [0,T]$. This means that the fractions in the different stocks and the bond remain constant on $[0,T]$. The advantage of this is two-fold: first we obtain, at least in a Gaussian setting, explicit results; and furthermore, the economic interpretation of the mathematical results is comparably easy. Finally, let us mention that for many other portfolio problems the optimal portfolios are constant ones (see Sections 3.3. and 3.4 of Korn (1997)). It is also important to point out that following a constant portfolio process does not mean that there is no trading. As the stock prices evolve randomly one has to trade at every time instant to keep the fractions of wealth invested in the different securities constant. Thus, following a constant portfolio process still means one must follow a dynamic trading strategy.

Standard Itô integration and the fact that $Ee^{sW(1)} = e^{s^2/2}$, $s \in \mathbb{R}$, yield the following explicit formulae for the wealth process for all $t \in [0,T]$ (see e.g. Korn and Korn (2000)).

$$\begin{aligned} X^\pi(t) &= x\exp\left((\pi'(b - r\underline{1}) + r - \|\pi'\sigma\|^2/2)t + \pi'\sigma W(t)\right) , && (2.1.2)\\ E(X^\pi(t)) &= x\exp\left((\pi'(b - r\underline{1}) + r)t\right) , && (2.1.3)\\ \mathrm{var}(X^\pi(t)) &= x^2\exp\left(2(\pi'(b - r\underline{1}) + r)t\right)\left(\exp(\|\pi'\sigma\|^2 t) - 1\right) . && (2.1.4)\end{aligned}$$

The norm $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Definition 2.1.1 (Capital-at-Risk)
Let x be the initial capital and T a given time horizon. Let z_α be the α-quantile of the standard normal distribution. For some portfolio $\pi \in \mathbb{R}^d$ and the corresponding terminal wealth $X^\pi(T)$, the α-quantile of $X^\pi(T)$ is given by

$$\rho(x,\pi,T) = x\exp\left((\pi'(b - r\underline{1}) + r - \|\pi'\sigma\|^2/2)T + z_\alpha\|\pi'\sigma\|\sqrt{T}\right) ,$$

i.e., $\rho(x,\pi,T) = \inf\{z \in \mathbb{R} : P(X^\pi(T) \le z) \ge \alpha\}$. Then we define

$$\begin{aligned} \mathrm{CaR}(x,\pi,T) &= x\exp(rT) - \rho(x,\pi,T) \\ &= x\exp(rT) && (2.1.5)\\ &\quad\times\left(1 - \exp((\pi'(b - r\underline{1}) - \|\pi'\sigma\|^2/2)T + z_\alpha\|\pi'\sigma\|\sqrt{T})\right) \end{aligned}$$

the Capital-at-Risk *of the portfolio π (with initial capital x and time horizon T).* □

Assumption 2.1.2 *To avoid (non-relevant) subcases in some of the following results we always assume $\alpha < 0.5$ which leads to $z_\alpha < 0$.*

Remark 2.1.3 (i) Our definition of the Capital-at-Risk limits the possibility of excess losses over the riskless investment.

(ii) We typically want to have a positive CaR (although it can be negative in our definition as the examples below will show) as the upper bound for the "likely losses" (in the sense that $(1-\alpha)\times 100\%$ of occurring "losses" are smaller than $\mathrm{CaR}(x,\pi,T)$) compared to the pure bond investment. Further, we concentrate on the actual amount of losses appearing at the time horizon T. This is in line with the mean-variance selection procedure enabling us to directly compare the results of the two approaches; see below.

In the following it will be convenient to introduce the function $f(\pi)$ for the exponent in (2.1.5), that is

$$f(\pi) := z_\alpha \|\pi'\sigma\|\sqrt{T} - \|\pi'\sigma\|^2 T/2 + \pi'(b - r\underline{1})T\,, \quad \pi \in \mathbb{R}^d\,. \tag{2.1.6}$$

By the obvious fact that

$$f(\pi) \overset{\|\pi'\sigma\|\to\infty}{\longrightarrow} -\infty$$

we have

$$\sup_{\pi\in\mathbb{R}^d} \mathrm{CaR}(x, \pi, T) = x\exp(rT)\,;$$

i.e., the use of extremely risky strategies (in the sense of a high norm $\|\pi'\sigma\|$) can lead to a CaR which is close to the total capital. The computation of the minimal CaR is done in the following proposition.

(iii) Note how crucial the definition of CaR depends on the assumption of a constant portfolio process. Moving away from this assumption makes the problem untractable. In particular, $\rho(x, \pi, T)$ is nearly impossible to obtain for a general random $\pi(.)$. $\square$

Proposition 2.1.4 *Let* $\theta = \|\sigma^{-1}(b - r1)\|$.
(a) If $b_i = r$ *for all* $i = 1, \dots, d$, *then* $f(\pi)$ *attains its maximum for* $\pi^* = 0$ *leading to a minimum Capital-at-Risk of* $\mathrm{CaR}(x, \pi^*, T) = 0$.
(b) If $b_i \neq r$ *for some* $i \in \{1, \dots, d\}$ *and*

$$\theta\sqrt{T} < |z_\alpha|\,, \tag{2.1.7}$$

then the minimal CaR *equals zero and is only attained for the pure bond strategy.*
(c) If $b_i \neq r$ *for some* $i \in \{1, \dots, d\}$ *and*

$$\theta\sqrt{T} \geq |z_\alpha|\,, \tag{2.1.8}$$

then the minimal CaR *is attained for*

$$\pi^* = \left(\theta - \frac{|z_\alpha|}{\sqrt{T}}\right) \frac{(\sigma\sigma)^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} \tag{2.1.9}$$

with

$$\mathrm{CaR}(x, \pi^*, T) = x \exp(rT) \left(1 - \exp\left(\frac{1}{2}(\sqrt{T}\theta - |z_\alpha|)^2\right)\right) < 0. \tag{2.1.10}$$

Proof (a) follows directly from the explicit form of $f(\pi)$ under the assumption of $b_i = r$ for all $i = 1, \ldots, d$ and the fact that σ is invertible.
(b),(c) Consider the problem of maximizing $f(\pi)$ over all π which satisfy

$$\|\pi'\sigma\| = \varepsilon \tag{2.1.11}$$

for a fixed positive ε. Over the (boundary of the) ellipsoid defined by (2.1.11) $f(\pi)$ equals

$$f(\pi) = z_\alpha \varepsilon \sqrt{T} - \varepsilon^2 T/2 + \pi'(b - r\underline{1})T \ .$$

Thus, the problem is just to maximize a linear function (in π) over the boundary of an ellipsoid. Such a problem has the explicit solution

$$\pi_\varepsilon^* = \varepsilon \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} \tag{2.1.12}$$

with

$$f(\pi_\varepsilon^*) = -\varepsilon^2 T/2 + \varepsilon \left(\theta T - |z_\alpha|\sqrt{T}\right). \tag{2.1.13}$$

As every $\pi \in \mathbb{R}^d$ satisfies relation (2.1.11) with a suitable value of ε (due to the fact that σ is regular), we obtain the minimum CaR strategy π^* by maximizing $f(\pi_\varepsilon^*)$ over all non-negative ε. Due to the form of $f(\pi_\varepsilon^*)$ the optimal ε is positive if and only if the multiplier of ε in representation (2.1.13) is positive. Thus, condition (2.1.7) implies assertion (b). Under assumption (2.1.8) the optimal ε is given as

$$\varepsilon = \theta - \frac{|z_\alpha|}{\sqrt{T}}.$$

Inserting this into equations (2.1.12) and (2.1.13) yields the assertions (2.1.9) and (2.1.10) (with the help of equations (2.1.5) and (2.1.6)). □

Remark 2.1.5 (i) Part (a) of the proposition states that in a risk-neutral market the CaR of every strategy containing stock investment is bigger than the CaR of the pure bond strategy.
(ii) Part (c) states the (at first sight surprising) fact that the existence of at least one stock with a mean rate of return different from the riskless rate implies the existence of a stock and bond strategy with a negative CaR as soon as the time horizon T is large. Thus, even if the CaR would be the only criterion to judge an investment strategy the pure bond investment would not be optimal if the time horizon is far away. On one hand this fact is in line with empirical results on stock and bond markets. On the other hand this shows a remarkable difference between the behaviour of the CaR and the variance as risk measures. Independent of the time horizon and the market coefficients, pure bond investment would always be optimal with respect to the variance of the corresponding wealth process.
(iii) The decomposition method to solve the optimization problem in the proof of parts (b) and (c) of Proposition 2.1.4 will be crucial for some of the proofs later in this chapter. Note how we use it to overcome the problem that $f(\pi)$ is not differentiable in $\pi = 0$. □

The rest of this section is devoted to setting up a Markowitz mean-variance type optimization problem where we replace the variance constraint by a constraint on the CaR of the terminal wealth. More precisely, we solve the following problem:

$$\boxed{\max_{\pi \in \mathbb{R}^d} E(X^\pi(T)) \quad \text{subject to} \quad \mathrm{CaR}(x, \pi, T) \leq C\,, \qquad (2.1.14)}$$

where C is a given constant of which we assume that it satisfies

$$C \leq x \exp(rT)\,. \qquad (2.1.15)$$

Due to the explicit representations (2.1.4), (2.1.5) and a variant of the decomposition method as applied in the proof of Proposition 2.1.4 we can solve problem (2.1.14) explicitly.

Proposition 2.1.6 *Let* $\theta = \|\sigma^{-1}(b - r\underline{1})\|$ *and assume that* $b_i \neq r$ *for at least one* $i \in \{1, \ldots, d\}$*. Assume furthermore that* C *satisfies*

$$0 \leq C \leq x \exp(rT) \ \textit{if} \ \theta\sqrt{T} < |z_\alpha|, \tag{2.1.16}$$

$$x \exp(rT) \left(1 - \exp\left(\frac{1}{2}(\sqrt{T}\theta - |z_\alpha|)^2\right)\right) \leq C \leq x \exp(rT) \ \textit{if} \ \theta\sqrt{T} \geq |z_\alpha| \,. \tag{2.1.17}$$

Then problem (2.1.14) will be solved by

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|}$$

with

$$\varepsilon^* = (\theta + z_\alpha/\sqrt{T}) + \sqrt{(\theta + z_\alpha/\sqrt{T})^2 - 2c/T}\,,$$

where $c = \ln\left(1 - \frac{C}{x}\exp(-rT)\right)$*. The corresponding maximal expected terminal wealth under the* CaR *constraint equals*

$$E\left(X^{\pi^*}(T)\right) = x \exp\left(\left(r + \varepsilon^* \|\sigma^{-1}(b - r\underline{1})\|\right) T\right) \ . \tag{2.1.18}$$

Proof The requirements (2.1.16) and (2.1.17) on C ensure that the CaR constraint in problem (2.1.14) cannot be ignored: in both cases C lies between the minimum and the maximum value that CaR can attain (see also Proposition 2.1.4). Every admissible π for problem (2.1.14) with $\|\pi'\sigma\| = \varepsilon$ satisfies the relation

$$(b - r\underline{1})'\pi T \geq c + \frac{1}{2}\varepsilon^2 T - z_\alpha \varepsilon \sqrt{T} \tag{2.1.19}$$

which is in this case equivalent to the CaR constraint in (2.1.14). But again, on the set given by $\|\pi'\sigma\| = \varepsilon$ the linear function $(b - r\underline{1})'\pi T$ is maximized by

$$\pi_\varepsilon = \varepsilon \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|}\,. \tag{2.1.20}$$

Hence, if there is an admissible π for problem (2.1.14) with $\|\pi'\sigma\| = \varepsilon$ then π_ε must also be admissible. Further, due to the explicit form (2.1.3) of the expected terminal wealth, π_ε also maximizes the expected terminal wealth over the ellipsoid. Consequently, to obtain π for problem (2.1.14) it suffices to consider all vectors of the form π_ε for all positive

ε such that requirement (2.1.19) is satisfied. Inserting (2.1.20) into the left-hand side of inequality (2.1.19) results in

$$(b - r\underline{1})'\pi_\varepsilon T = \varepsilon \|\sigma^{-1}(b - r\underline{1})\| T \,, \tag{2.1.21}$$

which is an increasing linear function in ε equalling zero in $\varepsilon = 0$. Therefore, we obtain the solution of problem (2.1.14) by determining the biggest positive ε such that (2.1.19) is still valid. But the right-hand side of (2.1.21) stays above the right-hand side of (2.1.19) until their largest positive point of intersection which is given by

$$\varepsilon^* = (\theta + z_\alpha/\sqrt{T}) + \sqrt{(\theta + z_\alpha/\sqrt{T})^2 - 2c/T} \,,$$

The remaining assertion (2.1.18) can be verified by inserting π^* into equation (2.1.3). □

Remark 2.1.7 The principle of this proof follows an idea of Ralf Korn to optimize first over the boundary of an ellipsoid and then to determine the optimal ellipsoid by the condition on the risk measure. In the following chapters of this thesis we will take up this method.

Remark 2.1.8 (i) Note that the optimal expected value only depends on the stocks via the norm $\|\sigma^{-1}(b - r\underline{1})\|$. There is no explicit dependence on the number of stocks. We therefore interpret Proposition 2.1.4 as a kind of *mutual fund theorem* as there is no difference between investment in our multi-stock market and a market consisting of the bond and just one stock with appropriate market coefficients b and σ.

(ii) Consider for a general utility function $U(x)$ the problem of

$$\max_{\pi \in \mathbb{R}^d} E(U(X^\pi(T))) \quad \text{subject to} \quad \mathrm{CaR}(x, \pi, T) \le C.$$

The above method of solving the mean-CaR problem would still work as long as $E(U(X^\pi(T)))$ is of the form $f(x)\exp(h(\pi))$ with h a linear function. This is e.g. the case for the choice of the HARA function $U(x) = x^\gamma/\gamma$. It would also work for the log-utility case; i.e. $U(x) = \ln x$ as then we would have

$$E(U(X^\pi(T))) = \ln x + rT + (b - r\underline{1})'\pi T - \pi'\sigma\sigma'\pi T/2 \,.$$

Here, instead of looking at the exponent, we can also look at

$$\ln x + rT - (b - r\underline{1})'\pi t - \varepsilon^2 T/2\,,$$

which for all π with $\|\pi'\sigma\| = \varepsilon$ is a linear function in π. However, for reasons of comparison to the Markowitz type problems below we restrict ourselves to the mean-CaR problem.

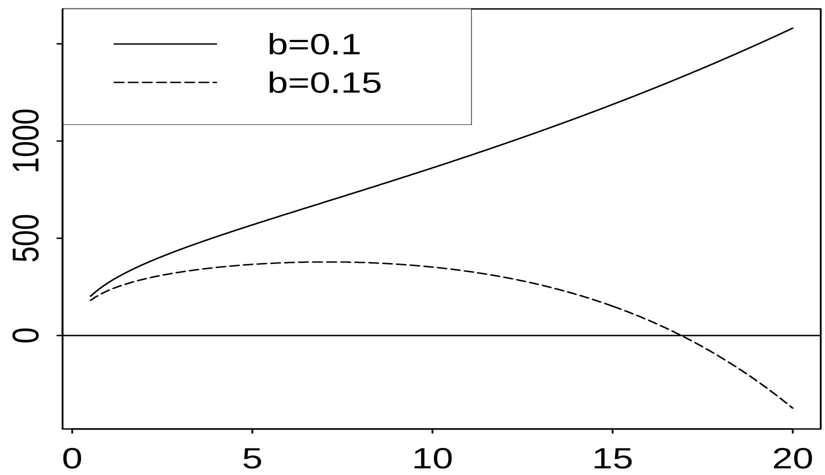


Figure 2.1: CaR$(1\,000, 1, T)$ of the pure stock portfolio (one risky asset only) for different appreciation rates as a function of the planning horizon T; $0 < T \leq 20$. The volatility is $\sigma = 0.2$. The riskless rate is $r = 0.05$.

Example 2.1.9 Figure 2.1 shows the dependence of CaR on the time horizon illustrated by CaR(1 000,1,T). Note that the CaR first increases and then decreases with time, a behaviour which was already indicated by Proposition 2.1.4. It differs substantially from the behaviour of the variance of the pure stock strategy, which increases with T. Figures 2.2 and 2.3 illustrate the behaviour of the optimal expected terminal wealth with varying time horizon corresponding to the pure bond strategy and the pure stock strategy as functions of the time horizon T. The expected terminal wealth of the optimal portfolio even exceeds the pure stock investment. The reason for this becomes clear if we look at the corresponding portfolios. The optimal portfolio always contains a short position in the bond as long as this is tolerated by the CaR constraint. This is shown in Figure 2.4 where we have plotted the optimal portfolio together with the pure stock portfolio as function of the time horizon. For $b = 0.15$ the optimal portfolio always contains a short position in the bond. For $b = 0.1$ and $T > 5$ the optimal portfolio (with the same CaR constraint as in Figures 2.2 and 2.3) again contains a long position in both bond and stock (with decreasing tendency of π as time increases!). This is an immediate consequence of the increasing CaR of the stock price. For the smaller appreciation rate of the stock it is simply not attractive enough to take the risk of a large stock investment. Figure 2.5 shows the mean-CaR efficient frontier for the above parameters with $b = 0.1$ and fixed time horizon $T = 5$. As expected it has a similar form as a typical mean-variance efficient frontier.

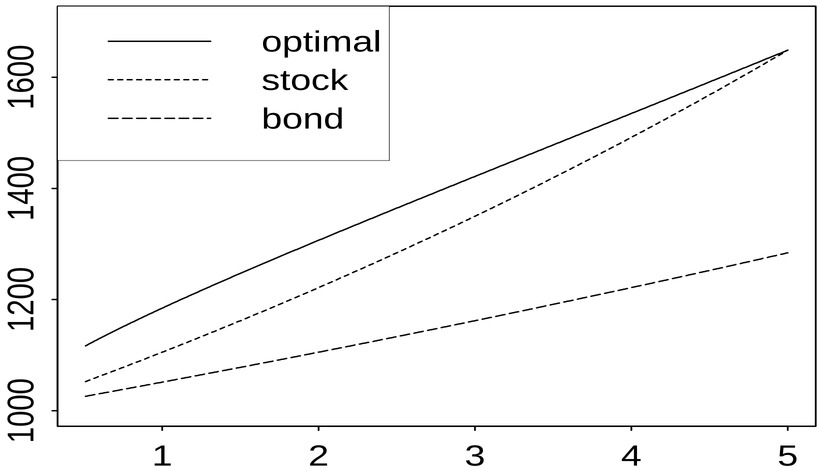


Figure 2.2: Expected terminal wealth of different investment strategies depending on the time horizon T, $0 < T \leq 5$. The parameters are $d = 1$, $r = 0.05$, $b = 0.1$, $\sigma = 0.2$, and $\alpha = 0.05$. As the upper bound C of the CaR we used $\mathrm{CaR}(1\,000, 1, 5)$, the CaR of the pure stock strategy with time horizon $T = 5$.

We will now compare the behaviour of the optimal portfolios for the mean-CaR with solutions of a corresponding mean-variance problem. To this end we consider the following simpler optimization problem:

$$\boxed{\max_{\pi \in \mathbb{R}^d} E(X^\pi(T)) \quad \text{subject to} \quad \mathrm{var}(X^\pi(T)) \leq C \, . \qquad (2.1.22)}$$

By using the explicit form (2.1.4) of the variance of the terminal wealth, we can rewrite

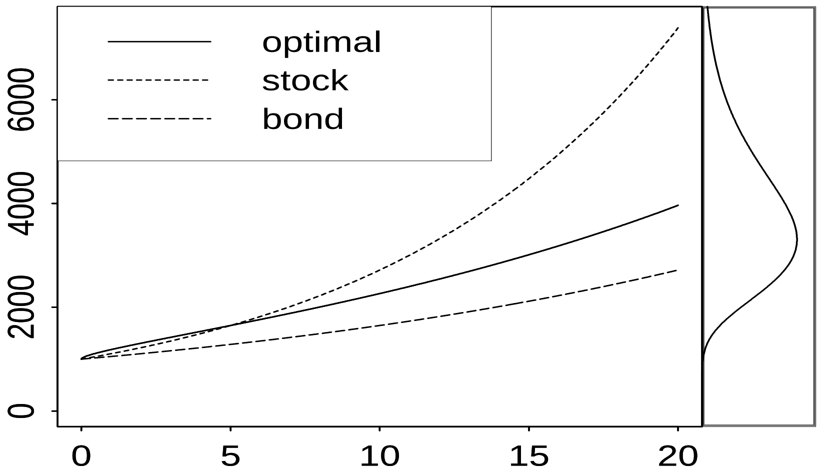


Figure 2.3: Expected terminal wealth of different investment strategies depending on the time horizon T, $0 \leq T \leq 20$. The parameters are $d = 1$, $r = 0.05$, $b = 0.1$, $\sigma = 0.2$, and $\alpha = 0.05$. As the upper bound C of the CaR we used $\mathrm{CaR}(1\,000, 1, 5)$, the CaR of the pure stock strategy with time horizon $T = 5$. On the right border we have plotted the density function of the wealth for the optimal portfolio. It is always between 0 and 0.0004.

the variance constraint in problem (2.1.22) as

$$(b - r\underline{1})'\pi T \leq \frac{1}{2}\ln\left(\frac{C}{x^2(\exp(\varepsilon^2 T) - 1)}\right) - rT =: h(\varepsilon), \qquad \|\pi'\sigma\| = \varepsilon \qquad (2.1.23)$$

for $\varepsilon > 0$. More precisely, if $\pi \in \mathbb{R}^d$ satisfies the constraints in (2.1.23) for one $\varepsilon > 0$ then it also satisfies the variance constraint in (2.1.22) and vice versa. Noting that $h(\varepsilon)$ is strictly decreasing in $\varepsilon > 0$ with

$$\lim_{\varepsilon \downarrow 0} h(\varepsilon) = \infty \qquad \lim_{\varepsilon \to \infty} h(\varepsilon) = -\infty$$

we see that left-hand side of (2.1.23) must be smaller than the right-hand one for small values of $\varepsilon > 0$ if we plug in π_ε as given by equation (2.1.20). Recall that this was the portfolio with the highest expected terminal wealth of all portfolios π satisfying $\|\pi'\sigma\| = \varepsilon$. It even maximizes $(b - r\underline{1})'\pi T$ over the set given by $\|\pi'\sigma\| \leq \varepsilon$. If we have equality

$$(b - r\underline{1})'\pi_{\widehat{\varepsilon}} T = h(\widehat{\varepsilon}) \qquad (2.1.24)$$

for the first time with increasing $\varepsilon > 0$ then this determines the optimal $\widehat{\varepsilon} > 0$. To see this, note that we have

$$E(X^\pi(T)) \leq E(X^{\pi_{\widehat{\varepsilon}}}(T)) \quad \text{for all } \pi \text{ with } \quad \|\pi'\sigma\| \leq \widehat{\varepsilon}\,,$$

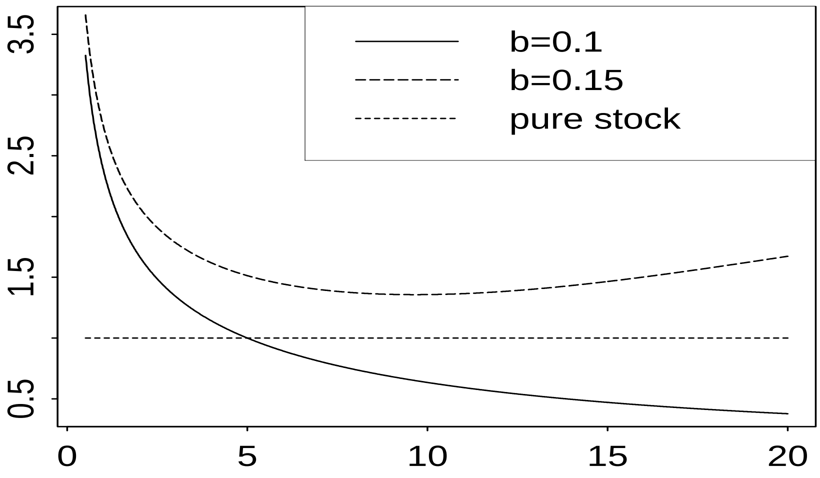


Figure 2.4: For the same parameters as in Figure 2.2 and different appreciation rates the figure shows the optimal portfolio and the pure stock portfolio.

and for all admissible π with $\varepsilon = \|\pi'\sigma\| > \widehat{\varepsilon}$ we obtain

$$(b - r\underline{1})'\pi T \leq h(\varepsilon) < h(\widehat{\varepsilon}) = (b - r\underline{1})'\pi_{\widehat{\varepsilon}} T \ .$$

By solving the non-linear equation (2.1.24) for $\widehat{\varepsilon}$ we have thus completely determined the solution of problem (2.1.22):

Proposition 2.1.10 *If $b_i \neq r$ for at least one $i \in \{1, \dots, d\}$, then the optimal solution of the mean-variance problem (2.1.22) is given by*

$$\widehat{\pi} = \widehat{\varepsilon} \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} ,$$

where $\widehat{\varepsilon}$ is the unique positive solution of the non-linear equation

$$\|\sigma^{-1}(b - r\underline{1})\|\varepsilon T - \frac{1}{2}\ln\left(\frac{C}{x^2(\exp(\varepsilon^2 T) - 1)}\right) + rT = 0 \ .$$

The corresponding maximal expected terminal wealth under the variance constraint equals

$$E(X^{\widehat{\pi}}(T)) = x \exp\left((r + \widehat{\varepsilon}\|\sigma^{-1}(b - r\underline{1})\|)T\right) \ . \qquad \square$$

Example 2.1.11 Figure 2.6 below compares the behaviour of $\widehat{\varepsilon}$ and ε^* as functions of the time horizon. We have used the same data as in Example 2.1.9. To make the solutions of

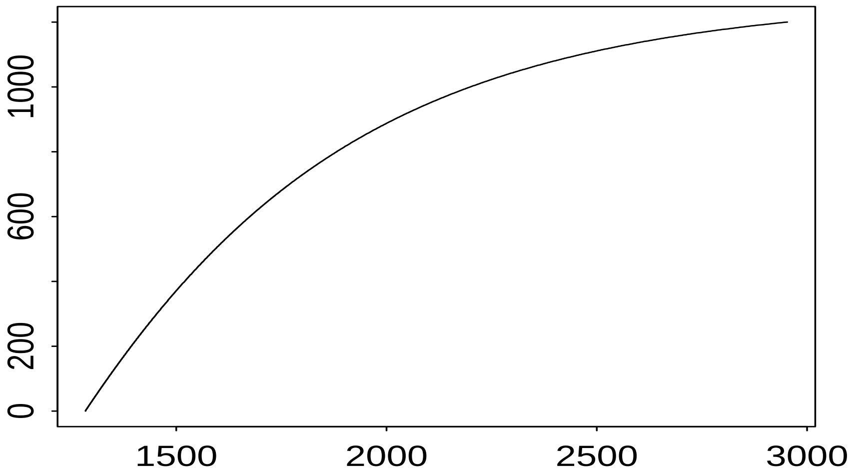


Figure 2.5: Mean-CaR efficient frontier with the mean on the horizontal axis and the CaR on the vertical axis. The parameters are the same as in Figure 2.2.

problems (2.1.14) and (2.1.22) comparable we have chosen C differently for the variance and the CaR risk measures in such a way that $\widehat{\varepsilon}$ and ε^* concide for $T = 5$. Notice that C for the variance problem is roughly the square of C for the CaR problem taking into account that the variance measures an L^2-distance, whereas CaR measures an L^1-distance. The (of course expected) bottom line of Figure 2.6 is that with increasing time the variance constraint demands a smaller fraction of risky securities in the portfolio. This is also true for the CaR constraint for small time horizons. For larger time horizon T ($T \geq 20$) ε^* increases again due to the fact that the CaR decreases. In contrast to that, $\widehat{\varepsilon}$ decreases to 0, since the variance increases. $\square$

2.2 Capital-at-Risk portfolios and more general price processes

In this section we consider again the mean-CaR problem (2.1.14) but drop the assumption of log-normality of the stock price process. The self-financing condition, however, will still

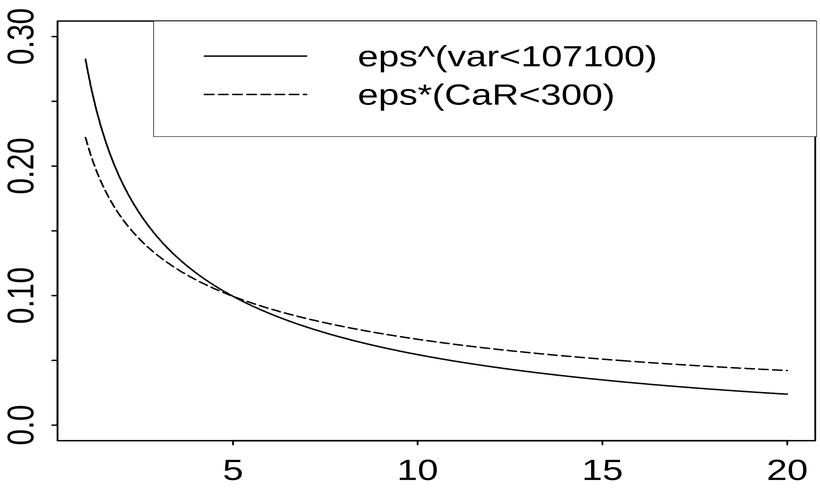


Figure 2.6: $\widehat{\varepsilon}$ and ε^* as functions of the time horizon; $0 < T \leq 20$. The parameters are the same as in Figure 2.2.

manifest itself in the form of the wealth equation

$$\frac{dX^\pi(t)}{X^\pi(t-)} = (1 - \pi'\underline{1})\frac{dP_0(t)}{P_0(t-)} + \sum_{i=1}^{d} \pi_i \frac{dP_i(t)}{P_i(t-)}, \quad t > 0, \qquad X^\pi(0) = x,$$

where P_i is the price process for stock i. Of course, the explicit form of the stochastic process P_i is crucial for the computability of the expected terminal wealth $X^\pi(T)$. To concentrate on these tasks we simplify the model in assuming $d = 1$, a bond price given by $P_0(t) = e^{rt}$, $t \geq 0$, as before, and a risky asset price satisfying

$$\frac{dP(t)}{P(t-)} = bdt + dY(t), \quad t > 0, \qquad P(0) = p, \tag{2.2.1}$$

where $b \in \mathbb{R}$ and Y is a semimartingale with $Y(0) = 0$. Under these assumptions the choice of the portfolio π leads to the following explicit formula for the wealth process

$$\begin{aligned} X^\pi(t) &= x \exp((r + \pi(b - r))t)\mathcal{E}(\pi Y(t)) \\ &= x \exp((r + \pi(b - r))t) \exp\left(\pi Y^c(t) - \tfrac{1}{2}\pi^2 \langle Y^c \rangle_t\right) \\ &\quad \times \prod_{0 < s \leq t} (1 + \pi \Delta Y(s)), \; t \geq 0, \end{aligned} \tag{2.2.2}$$

where Y^c denotes the continuous part and ΔY the jump part of the process Y (more precisely, $\Delta Y(t)$ is the height of a (possible) jump at time t). This means that the wealth process is a product of a deterministic process and the stochastic exponential $\mathcal{E}(\pi Y)$ of

πY (see Protter (1990)). Analogously to Definition 2.1.1 we define the CaR in this more general context.

Definition 2.2.1 *Consider the market given by a riskless bond with price $P_0(t) = e^{rt}$, $t \geq 0$, for $r \in \mathbb{R}$ and one stock with price process P satisfying (2.2.1) for $b \in \mathbb{R}$ and a semimartingale Y with $Y(0) = 0$. Let x be the initial capital and T a given time horizon. For some portfolio $\pi \in \mathbb{R}$ and the corresponding terminal wealth $X^\pi(T)$ the α-quantile of $X^\pi(T)$ is given by*

$$\widetilde{\rho}(x, \pi, T) = x \exp((\pi(b - r) + r)T) \cdot \widetilde{z}_\alpha \,,$$

where $\widetilde{z}_\alpha$ is the α-quantile of $\mathcal{E}(\pi Y(T))$, i.e. $\widetilde{z}_\alpha = \inf\{z \in \mathbb{R} : P(\mathcal{E}(\pi Y(T)) \leq z) \geq \alpha\}$. Then we call

$$\mathrm{CaR}(x, \pi, T) = x \exp(rT)(1 - \exp(\pi(b - r)T) \cdot \widetilde{z}_\alpha) \tag{2.2.3}$$

the Capital-at-Risk *of the portfolio π (with initial capital x and time horizon T).* □

One of our aims of this section is to explore the behaviour of the solutions to the mean-CaR problem (2.1.14) if we model the returns of the price process by processes having heavier tails than the Brownian motion. We present some specific examples in the following subsections.

2.2.1 The Black-Scholes model with jumps

We consider a stock price process P, where the random fluctuations are generated by both a Brownian motion and a compound jump process, i.e., we consider the model (2.2.1) with

$$dY(t) = \sigma dW(t) + \sum_{i=1}^{n} (\beta_i dN_i(t) - \beta_i \lambda_i dt) \;, \quad t > 0 \,, \quad Y(0) = 0 \,, \tag{2.2.4}$$

where $n \in \mathbb{N}$, and for $i = 1, \ldots, n$ the process N_i is a homogeneous Poisson process with intensity λ_i. It counts the number of jumps of height β_i of Y. In order to avoid negative stock prices we assume

$$-1 < \beta_1 < \cdots < \beta_n < \infty \,.$$

An application of Itô's formula results for $t \geq 0$ in the explicit form

$$P(t) = p\exp\left(\left(b - \frac{1}{2}\sigma^2 - \sum_{i=1}^{n}\beta_i\lambda_i\right)t + \sigma W(t) + \sum_{i=1}^{n}\left(N_i(t)\ln(1+\beta_i)\right)\right). \quad (2.2.5)$$

In order to avoid the possibility of negative wealth after an "unpleasant" jump we have to restrict the portfolio π as follows

$$\pi \in \begin{cases} \left[-\frac{1}{\beta_n}, -\frac{1}{\beta_1}\right) & \text{if} \quad \beta_n > 0 > \beta_1\,, \\ \left(-\infty, -\frac{1}{\beta_1}\right] & \text{if} \quad \beta_n < 0\,, \\ \left[-\frac{1}{\beta_n}, \infty\right) & \text{if} \quad \beta_1 > 0\,. \end{cases} \quad (2.2.6)$$

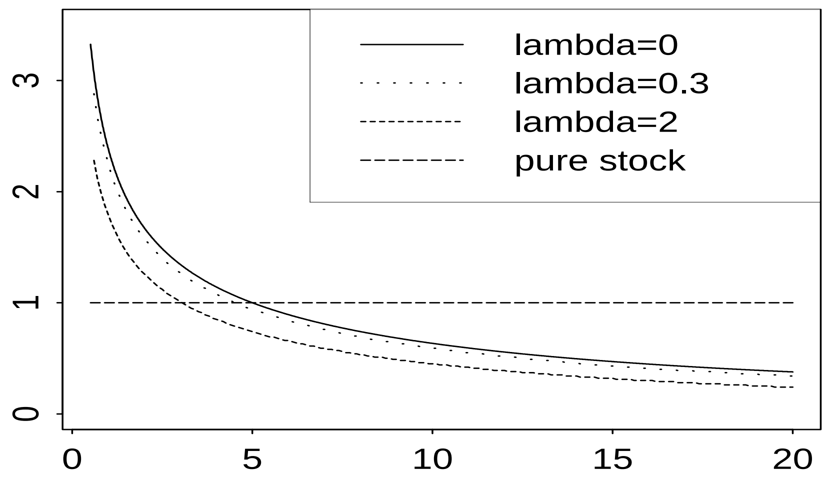


Figure 2.7: Optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 < T \leq 20$. The basic parameters are the same as in Figure 2.2. The possible jump size is $\beta = -0.1$.

Under these preliminary conditions we obtain explicit representations of the expected terminal wealth and the CaR corresponding to a portfolio π similar to the equations (2.1.3) and (2.1.5).

Lemma 2.2.2 *With a stock price given by equation (2.2.5) let X^π be the wealth process corresponding to the portfolio π satisfying (2.2.6). Then for initial capital x and finite time horizon T,*

$$
\begin{aligned}
X^\pi(T) &= x\exp((r+\pi(b-r)-\sum_{i=1}^n \pi\beta_i\lambda_i - \frac{1}{2}\pi^2\sigma^2)T) \\
&\quad \times \exp(\pi\sigma W(T) + \sum_{i=1}^n N_i(T)\ln(1+\pi\beta_i))\,, \\
E(X^\pi(T)) &= x\exp((r+\pi(b-r))T), \\
\mathrm{CaR}(x,\pi,T) &= x\exp(rT)\left(1-\exp\left(\left(\pi(b-r)-\sum_{i=1}^n \pi\beta_i\lambda_i - \frac{1}{2}\pi^2\sigma^2\right)T+\widetilde{z}_\alpha\right)\right)\,,
\end{aligned}
$$

where $\widetilde{z}_\alpha$ *is the* α*-quantile of*

$$
\pi\sigma W(T) + \sum_{i=1}^n (N_i(T)\ln(1+\pi\beta_i))\,,
$$

i.e. the real number $\widetilde{z}_\alpha$ *satisfying*

$$
\begin{aligned}
\alpha &= P\left(\pi\sigma W(T) + \sum_{i=1}^n (N_i(T)\ln(1+\pi\beta_i)) \le \widetilde{z}_\alpha\right) \\
&= \sum_{n_1,\ldots,n_n=0}^{\infty}\left(\Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}\left(\widetilde{z}_\alpha - \sum_{i=1}^n (n_i\ln(1+\pi\beta_i))\right)\right)\right. \qquad (2.2.7) \\
&\qquad \left.\times\exp\left(-T\sum_{i=1}^n \lambda_i\right)\prod_{i=1}^n \frac{(T\lambda_i)^{n_i}}{n_i!}\right)\,.
\end{aligned}
$$

Proof $X^\pi(T)$ is a result of an application of Itô's formula. To obtain the expected value simply note that the two processes

$$
\exp\left(-\frac{1}{2}\sigma^2 t + \sigma W(t)\right) \quad \text{and} \quad \exp\left(-\sum_{i=1}^n \beta_i\lambda_i t + \sum_{i=1}^n \sum_{j=1}^{N_i(t)} \ln(1+\beta_i)\right)
$$

are both martingales with unit expectation and that they are independent. Regarding the representation of the CaR, only equation (2.2.7) has to be commented on. But this is a consequence of conditioning on the number of jumps of the different jump heights in $[0,T]$. □

Unfortunately, $\widetilde{z}_\alpha$ cannot be represented in such an explicit form as in the case without jumps. However, due to the explicit form of $E(X^\pi(T))$, it is obvious that the corresponding mean-CaR problem (2.1.14) will be solved by the largest π that satisfies both the CaR constraint and requirement (2.2.6). Thus for an explicit example we obtain the optimal mean-CaR portfolio by a simple numerical iteration procedure, where we approximated the infinite sum in (2.2.7) by the finite sum of its first $2[\lambda T] + 1$ summands, if we set $n = 1$ and $\lambda = \lambda_1$. Comparisons of the solutions for the Brownian motion with and without jumps are given in Figure 2.7.

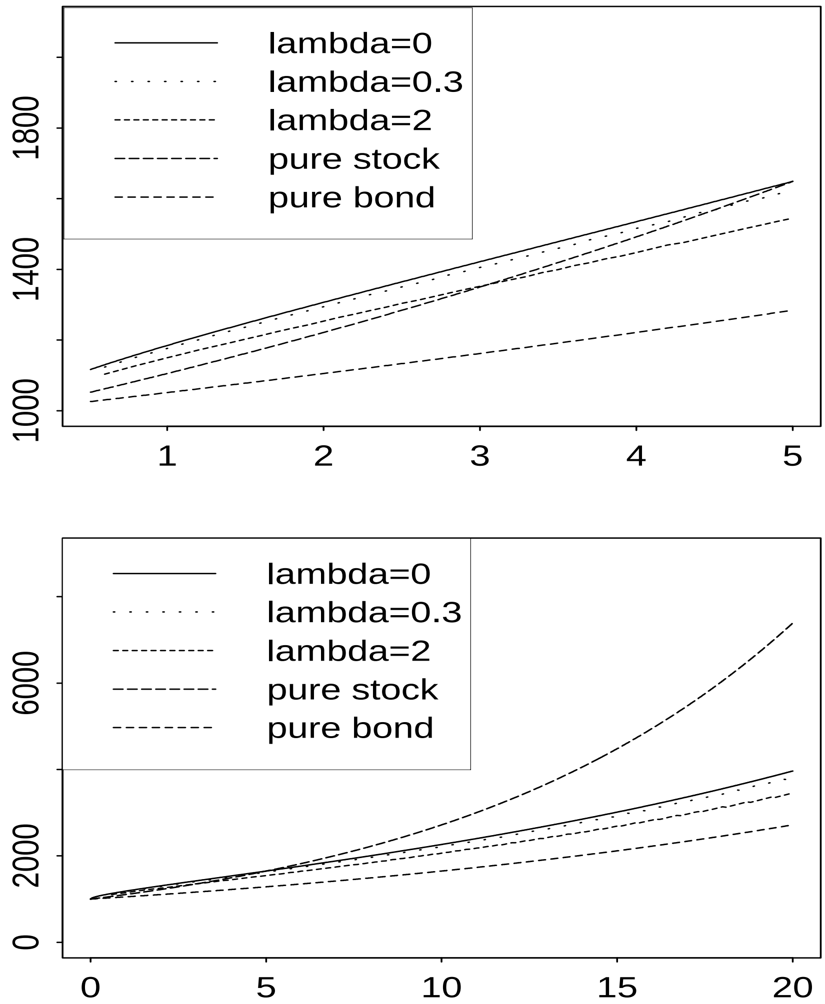


Figure 2.8: Wealth corresponding to the optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 < T \leq 5$ (top) and $0 < T \leq 20$ (bottom). The parameters are the same as in Figure 2.7. The possible jump size is again $\beta = -0.1$.

We have used the same parameters as in the examples of Section 2.1, but have included the possibility of a jump of height $\beta = -0.1$, occuring with different intensities. For $\lambda = 0.3$ one would expect a jump approximately every three years, for $\lambda = 2$ even two jumps per year. Notice that the stock has the same expected terminal value in both cases! To explain this we rewrite equation (2.2.5) as follows:

$$\frac{dP(t)}{P(t-)} = \left(b - \sum_{i=1}^{n} \beta_i \lambda_i\right) dt + \sigma W(t) + \sum_{i=1}^{n} \beta_i dN_i(t)\,, \quad t > 0\,, \qquad P(0) = p\,.$$

Whereas a jump occurs for instance for $\lambda = 0.3$ on average only every three years, meaning that with rather high probability there may be no jump within two years, the drift has a permanent influence on the dynamic of the price process. Despite this additional stock drift of $-\beta'\lambda$ the optimal portfolio for stock prices following a geometric Brownian motion with jumps is always below the optimal portfolio of the geometric Brownian motion (solid line). This means that the threat of a downwards jump of 10% leads an investor to a less risky behaviour, and the higher λ is, the less risky is the investor's behaviour.

2.2.2 Generalized inverse Gaussian diffusion

Moving away from the Black-Scholes model towards more general diffusion models is a rather obvious generalization. It is also desirable, since marginal distributions of the log-returns of stock prices are often heavier tailed than normal. This has been shown very convincingly, for instance, by a data analysis in Eberlein and Keller (1995). Various models have been suggested: a simple hyperbolic model has been investigated by Bibby and Sørensen (1997); a more general class of models has been suggested by Barndorff-Nielsen (1998).

We consider a generalized inverse Gaussian diffusion model (for brevity we write GIG diffusion) for the log-returns of stock prices. This class of diffusions has been introduced in Borkovec and Klüppelberg (1998) and we refer to this source for details.

The following equations determine a general diffusion market.

$$\begin{aligned} dP_0(t) &= P_0(t) r dt\,, & P_0(0) &= 1\,, \\ dP(t) &= P(t)(bdt + dY(t))\,, & P(0) &= p\,, \\ Y(t) &= U(t) - u\,, & Y(0) &= 0\,, \end{aligned} \tag{2.2.8}$$

In our case we now choose U as a GIG diffusion given by the SDE

$$\begin{aligned} dU(t) &= \tfrac{1}{4}\sigma^2 U^{2\gamma-2}(t)\left(\psi + 2(2\gamma+\lambda-1)U(t) - \chi U^2(t)\right) dt \\ &\quad + \sigma U^{\gamma}(t) dW(t),\ U(0) = u > 0\,, \end{aligned} \tag{2.2.9}$$

where W is standard Brownian motion. The parameter space is given by $\sigma > 0$, $\gamma \geq 1/2$, $\chi, \psi \geq 0$, $\max(\chi, \psi) > 0$, and

$$\begin{aligned}
&\lambda \in \mathbb{R} && \text{if} && \chi, \psi > 0\,, \\
&\lambda \leq \min(0, 2(1-\gamma)) && \text{if} && \chi = 0, \psi > 0\,, \\
&\lambda \geq \min(0, 2(1-\gamma)) && \text{if} && \chi > 0, \psi = 0\,.
\end{aligned} \tag{2.2.10}$$

The GIG model is a formal extension of the Black-Scholes model, which corresponds to the choice of parameters $\gamma = \psi = 0$, $\lambda = 1, \chi = 0$. It also contains the (generalized) Cox-Ingersoll-Ross model as a special case. The advantage of our construction lies in the structural resemblance of the resulting price process to the geometric Brownian motion model. We can decompose the stock price into a drift term multiplied by a local martingale:

$$\begin{aligned}
P(t) &= p \exp\left(bt + \frac{1}{4}\sigma^2 \int_0^t U^{2\gamma-2}(s)\left(\psi + 2(2\gamma+\lambda-1)U(s) - \chi U^2(s)\right) ds\right) \\
&\quad \times \exp\left(\sigma \int_0^t U^\gamma(s) dW(s) - \frac{1}{2}\sigma^2 \int_0^t U^{2\gamma}(s) ds\right), \quad t \geq 0\,.
\end{aligned}$$

The following lemma shows another property of the process U that is useful, when describing the wealth process.

Lemma 2.2.3 *Let U be the GIG diffusion given by (2.2.9) and $\pi > 0$. Then the process $\widetilde{U} = \pi U$ is again a GIG diffusion with $\widetilde{U}(0) = \pi U(0)$ and parameters*

$$\widetilde{\sigma} = \sigma \pi^{1-\gamma}\,, \quad \widetilde{\psi} = \psi\pi\,, \quad \widetilde{\chi} = \chi/\pi\,. \tag{2.2.11}$$

The parameters γ and λ remain the same.

Proof Notice first that all parameters of $\widetilde{U}$ satisfy the necessary non-negativity assumptions and (2.2.9). The assertion now follows by calculating $d\widetilde{U}(t) = d(\pi U(t)) = \pi dU(t)$, $t \geq 0$. □

Remark 2.2.4 As a consequence of Lemma 2.2.3 the wealth process X^π has a very nice explicit form. Indeed it is of a similar form as the stock price process P:

$$X^\pi(t) = x \exp\left((1-\pi)rt + \widetilde{b}t + \widetilde{Y}(t) - \frac{1}{2}\langle \widetilde{Y} \rangle_t\right), \quad t \geq 0\,, \tag{2.2.12}$$

where

$$\widetilde{b} = \pi b \quad \text{and} \quad \widetilde{Y}(t) = \widetilde{U}(t) - \pi u\,, \quad t \geq 0\,,$$

for any positive portfolio π. $\square$

According to Definition 2.2.1 for the $\mathrm{CaR}(x, \pi, T)$ we have to determine the α-quantile of $\widetilde{Y}(T) - \frac{1}{2}\langle \widetilde{Y} \rangle_T$. Here we see one of the big advantages of the CaR as a risk measure: it does not depend on the existence of moments. Even for an infinite mean it is well-defined.

However, if we want to solve the mean-CaR problem, we have to ensure that $X^\pi(T)$ has a finite mean. In general, it is not always possible to easily decide if this is the case. A natural assumption is to assume $U(T)$ or $\widetilde{U}(T)$ to have the stationary distribution of the process U or $\widetilde{U}$ respectively. This is certainly justified if the time horizon T is chosen sufficiently large. As in Bibby and Sørensen (1998) we therefore make this simplifying assumption which helps us to give a result about the existence of $E(X^\pi(T))$.

Proposition 2.2.5 *Assume that $U(T)$ and $\widetilde{U}(T)$ are GIG distributed with parameters ψ, χ, λ and $\widetilde{\psi}$, $\widetilde{\chi}$, λ respectively, i.e. they have the stationary distributions of the processes $U(\cdot)$ and $\widetilde{U}(\cdot)$ respectively. Assume that π is a positive portfolio. Then $X^\pi(T)$ has a finite mean if $\widetilde{\chi} = \chi/\pi > 2$.*

Proof As $\widetilde{U}$ is always positive, we estimate

$$X^\pi(T) \leq x \exp\Big((1-\pi)rT + \widetilde{b}T + \widetilde{U}(T) - \pi u\Big)\,.$$

If $E\exp(\widetilde{U}(T)) < \infty$, then $EX^\pi(T) < \infty$. By Jørgensen (1982) we know the explicit form of the moment generating function of the GIG distribution leading to

$$E\Big(\exp(\widetilde{U}(T))\Big) = \frac{K_\lambda\left(\sqrt{\chi\psi}(1-2/\widetilde{\chi})\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)(1-2/\widetilde{\chi})^{\lambda/2}}\,, \tag{2.2.13}$$

where $K_\lambda(\cdot)$ denotes the generalized Bessel function of the third kind. The rhs of equation (2.2.13) is finite for $\widetilde{\chi} > 2$. $\square$

Thus if the original parameters satisfy $\chi > 2$ and $\pi \in [0,1]$, then also $\widetilde{\chi} > 2$ and in this case $X^\pi(T)$ has a finite mean. In this case the mean-CaR problem is well-defined and can be solved, however one cannot hope for an analytic solution. In the following example we show how the mean-CaR problem can be solved using analytic properties of the process as far as possible, and then present a simple simulation procedure to solve the problem numerically.

Example 2.2.6 (Generalized Cox-Ingersoll-Ross model (GCIR))
As an example we consider the *generalized Cox-Ingersoll-Ross model*, i.e., the GIG market model with parameters $\gamma = 1$, $\chi = 0$. This results in the following explicit form for U:

$$U(t) = \exp\left(\frac{1}{2}\sigma^2\lambda t + \sigma W(t)\right)\left\{u + \frac{1}{4}\sigma^2\psi\int_0^t \exp\left(-\frac{1}{2}\sigma^2\lambda s - \sigma W(s)\right)ds\right\},\ t \geq 0\,,$$

which has mean

$$EU(t) = \begin{cases} \exp\left((\lambda+1)\dfrac{\sigma^2}{2}t\right)\left(u + \dfrac{\psi}{2(\lambda+1)}\left(1 - \exp\left(-(\lambda+1)\dfrac{\sigma^2}{2}t\right)\right)\right) & \text{if } \lambda \neq -1\,, \\ u + \frac{1}{2}\sigma^2\psi t & \text{if } \lambda = -1\,, \end{cases}$$

(see e.g. Borkovec and Klüppelberg (1998)). Further, note that we have

$$Y(t) = U(t) - u = \frac{1}{4}\sigma^2\psi t + \frac{1}{2}(1+\lambda)\sigma^2\int_0^t U(s)ds + \sigma\int_0^t U(s)dW(s) \qquad (2.2.14)$$

and we obtain the same representations for $\widetilde{U}(t)$ and $\widetilde{Y}(t)$ if we substitute ψ by $\widetilde{\psi} = \pi\psi$. An explicit solution of the mean-CaR problem does not seem to be possible. What remains are Monte-Carlo simulations and numerical approximations.

A simple algorithm to solve the mean-CaR problem would be the following:
For large N and $i = 1, \ldots, N$:

- Simulate sample paths $(W_i(t))_{t\in[0,T]}$ of the Brownian motion $(W(t))_{t\in[0,T]}$.
- Compute realisations $U_i(T)$ and $\int_0^T U_i^2(t)dt$ of $U(T)$ and $\int_0^T U^2(t)dt$, respectively, from the simulated sample paths of $(W_i(t))_{t\in[0,T]}$.
- For "all" $\pi \in \mathbb{R}$ compute

$$\widetilde{Z}_i^\pi(T) = \pi U_i(T) - \frac{1}{2}\pi^2\sigma^2\int_0^T U_i^2(t)dt - \pi u.$$

- Get estimators $\widehat{\mu}(\pi)$ for $E(X^\pi(T))$ and $\widehat{\nu}(x,\pi,T)$ for $\mathrm{CaR}(x,\pi,T)$:

$$\begin{aligned}
\widehat{\mu}(\pi) &:= \frac{x}{N}\sum_{i=1}^{N}\exp\left((r+(b-r)\pi)T+\widetilde{Z}_i^\pi(T)\right)\\
\widehat{\nu}(x,\pi,T) &:= x\exp(rT)\left(1-\exp\left(\pi(b-r)T+\widehat{z}_\alpha(\pi)\right)\right),
\end{aligned}$$

  where $\widehat{z}_\alpha(\pi)$ is the α-quantile of the empirical distribution of the $\widetilde{Z}_i^\pi(T)$ with the convention we already used in Definition 2.2.1.

- Choose the portfolio π with the largest value of $\widehat{\mu}(\pi)$ such that $\widehat{\nu}(x,\pi,T)$ is below the upper bound C for the CaR.

Of course, it is not possible to compute the quantities $\widehat{\mu}(\pi)$ and $\widehat{z}_\alpha(\pi)$ for all $\pi\in\mathbb{R}$ explicitly. A practical method consists in choosing $K=100$ values of π in a bounded interval of interest and derive functions $\mu(\pi), z_\alpha(\pi)$ via interpolation. One then chooses that value of π that solves the mean-CaR problem corresponding to these functions.

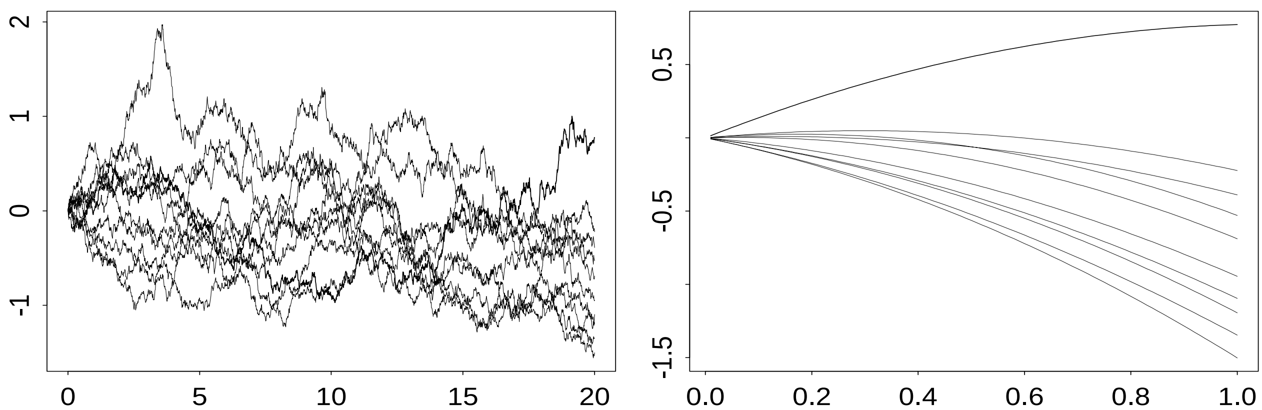


Figure 2.9: Ten sample paths of $(\widetilde{Z}(t))_{0\le t\le 20}$ for $\pi=1$ (left) and ten sample paths of $(\widetilde{Z}^\pi(20))_{\pi\in(0,1)}$ (right) for parameter values $x=1000, r=0.05, b=0.10, \psi=4, \lambda=0, \sigma=0.05$ and $u=5$.

To give an impression of the behaviour of $\widetilde{Z}(t)$ the first diagram in Figure 2.9 shows ten sample paths for the parameter values $x=1000, r=0.05, b=0.10, \psi=4, \lambda=0, \sigma=0.05$ and $u=5$. The second diagram depicts the behaviour of $\widetilde{Z}(20)$ as a function of π. Figure 2.10 shows a result of the simulation algorithm described above. It is the result of

$N = 100$ simulations for $T = 20$ and the remaining parameters chosen as those of Figure 2.9. As expected, both the mean terminal wealth and the CaR increase with π. Therefore the problem can be solved by identifying that portfolio π in the right side diagram that corresponds to the given upper bound C for the CaR.

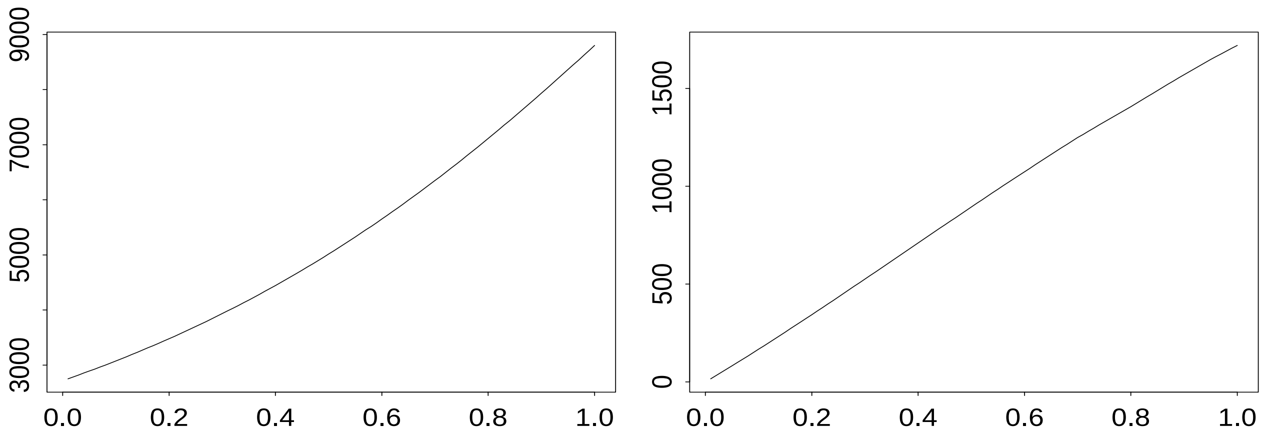


Figure 2.10: Estimated expected terminal wealth (left) and the corresponding CaR (right) as functions of the portfolio π for the GCIR model for T=20 and the same parameters as in Figure 2.9 (based on N=100 simulations). The expected terminal wealth and the CaR for the GCIR model increase for all $\pi \in (0, 1)$.

2.3 Conclusion

We have investigated some simple portfolio problems containing an upper bound on the CaR as an additional constraint. As long as we were able to calculate expectations and quantiles of the stock prices in explicit form we could also solve the problems explicitly. This can be done within a Gaussian world, but very little beyond. The Black-Scholes model with jumps is just feasible and easily understood. As soon as one moves away from such simple models the solution of the mean-CaR problems becomes less tractable and Monte Carlo simulation and numerical solutions are called for. As an example we treated the generalized Cox-Ingersoll-Ross model, which gave us a first impression of the complexity of the problem.

In this sense this chapter should be understood as the starting point of a larger research project. We indicate some of the problems we want to deal with in future work:

– A deeper analysis should investigate the influence of the parameters of the generalized inverse Gaussian; also other models should be investigated as for instance hyperbolic and normal inverse Gaussian models (see Eberlein, Keller and Prause (1998) and Barndorff-Nielsen (1998)).

– Investigate the optimization problem for other downside risk measures; replace for instance the quantile in Definition 2.1.1 by the expected shortfall. Comparisons of results for the CaR with respect to the quantile and the shortfall can be found in Emmer, Klüppelberg and Korn (2000).

– Replace the constant portfolio by a general portfolio process. Then we have to bring in much more sophisticated techniques to deal with the quantiles of the wealth process, and our method of solving the optimization problem explicitly will no longer work.

Chapter 3

Optimal portfolios with bounded lower partial moments

Lower partial moments as in (2.0.1) describe the downside risk of a portfolio, where the concept has to be adapted to our situation and the benchmark has to be chosen appropriately. In chapter 2 we considered a low quantile of the terminal wealth $X^\pi(T)$ to define the risk of a portfolio by its Capital-at-Risk (CaR). The Capital-at-Risk with respect to the quantile has several deficiencies, e.g. it is not coherent and it does not take into account the shape of the distribution function on the left side of the quantile. Hence we shall also consider lower partial moments order 0 and 1. In this chapter we discuss some portfolio optimization under a constraint on the Capital-at-Risk with respect to the expected shortfall (CaR^S) and under a constraint on the shortfall probability below a certain benchmark.

3.1 Expected shortfall portfolios in the Black-Scholes setting

Definition 3.1.1 (Risk measures)

Let ρ be the quantile as defined in 2.1.1. For a portfolio $\pi \in \mathbb{R}^d$, initial capital $x > 0$ and time horizon $T > 0$ we define the following risk measures.

(a) The *expected shortfall* of $X^\pi(T)$:

$$\rho_1(x,\pi,T) = E(X^\pi(T)|X^\pi(T) \le \rho(x,\pi,T)) .$$

(b) The *semi-standard deviation* of $X^\pi(T)$:

$$\rho_2(x,\pi,T) = \sqrt{E((X^\pi(T))^2|X^\pi(T) \le \rho(x,\pi,T))} . \qquad \square$$

Next we define the Capital-at-Risk (CaR^S) with respect to the expected shortfall as its difference to the pure bond strategy. This is different to some authors who take the difference to the mean terminal wealth $EX^\pi(T)$ of exactly this portfolio, a quantity which is called *Earnings at Risk*. Our definition has the advantage that different portfolios can be compared with respect to their market risks.

Definition 3.1.2 (Capital-at-Risk)
We define the difference between the terminal wealth of the pure bond strategy and the expected shortfall of $X^\pi(T)$ as the *Capital-at-Risk (CaR^S) of the portfolio π with respect to the expected shortfall (with initial capital x and time horizon T).* It is given by

$$\mathrm{CaR}^S(x,\pi,T) = xe^{rT} - \rho_1(x,\pi,T) . \qquad \square$$

Next we calculate the expected shortfall and the semi-standard deviation explicitly.

Proposition 3.1.3 *Let $(X^\pi(t))$ be the wealth process of a portfolio π in the Black-Scholes market and $\rho = \rho(x,\pi,t)$ be defined as in Definition 2.1.1. Denote by φ the density and by Φ the distribution function of a standard normal random variable $N(0,1)$. Let T be a fixed time horizon. Set*

$$\alpha^* = \Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T}) \quad \text{and} \quad \alpha^{**} = \Phi(z_\alpha - 2\|\pi'\sigma\|\sqrt{T}) . \tag{3.1.1}$$

and

$$a(x,\pi,T) = x\exp\{(\pi'(b - r\underline{1}) + r - \|\pi'\sigma\|^2/2)T\} .$$

Then

$$\alpha^{**} < \alpha^* < \alpha \tag{3.1.2}$$

and

$$\rho_1(x,\pi,T) = a(x,\pi,T)\frac{\alpha^*}{\alpha}\exp\left\{\frac{\|\pi'\sigma\|^2}{2}T\right\}, \tag{3.1.3}$$

$$\rho_2(x,\pi,T) = a(x,\pi,T)\sqrt{\frac{\alpha^{**}}{\alpha}}\exp\{\|\pi'\sigma\|^2T\}. \tag{3.1.4}$$

Proof Recall the following identity in law

$$\frac{\pi'\sigma}{\|\pi'\sigma\|}\frac{W(t)}{\sqrt{t}} \stackrel{d}{=} N(0,1), \quad t>0, \tag{3.1.5}$$

which implies

$$\begin{aligned} X^\pi(T) &= a(x,\pi,T)\exp\{\pi'\sigma W(T)\} \\ &\stackrel{d}{=} a(x,\pi,T)\exp\{N(0,1)\|\pi'\sigma\|\sqrt{T}\}. \end{aligned} \tag{3.1.6}$$

Furthermore, by definition, $P(X^\pi(T)\leq\rho_0) = P(N(0,1)\leq z_\alpha) = \alpha$. Hence, for the shortfall we obtain

$$\begin{aligned} \rho_1(x,\pi,T) &= \frac{E(X^\pi(T)I(X^\pi(T)\leq\rho_0(x,\pi,T)))}{P(X^\pi(T)\leq\rho_0(x,\pi,T))} \\ &= \frac{a(x,\pi,T)}{\alpha}\int_{-\infty}^{z_\alpha}\exp\{x\|\pi'\sigma\|\sqrt{T}\}\varphi(x)dx, \end{aligned}$$

where $I(A)$ is the indicator function of the set A. We calculate the integral by change of variables and obtain:

$$\rho_1(x,\pi,T) = \frac{a(x,\pi,T)}{\alpha}\exp\{\|\pi'\sigma\|^2T/2\}\Phi(z_\alpha-\|\pi'\sigma\|\sqrt{T}).$$

For the semi-standard deviation we obtain

$$\begin{aligned} \rho_2(x,\pi,T) &= \sqrt{\frac{E((X^\pi(T))^2I(X^\pi(T)\leq\rho_0(x,\pi,T)))}{P(X^\pi(T)\leq\rho_0(x,\pi,T))}} \\ &= \sqrt{\frac{a^2(x,\pi,T)}{\alpha}\int_{-\infty}^{z_\alpha}\exp\{2x\|\pi'\sigma\|\sqrt{T}\}\varphi(x)dx} \\ &= \sqrt{\frac{a^2(x,\pi,T)}{\alpha}\exp\{2\|\pi'\sigma\|^2T\}\Phi(z_\alpha-2\|\pi'\sigma\|\sqrt{T})} \\ &= a(x,\pi,T)\sqrt{\frac{\alpha^{**}}{\alpha}}\exp\{\|\pi'\sigma\|^2T\} \end{aligned}$$

□

Corollary 3.1.4 $\rho_1(x,\pi,T) \le \rho_2(x,\pi,T) \le \rho(x,\pi,T)$.

Proof

$$\rho_2(x,\pi,T)^2 \;=\; E((X^\pi(T))^2|X^\pi(T) \le \rho(x,\pi,T)) \;\le\; \rho_0(x,\pi,T)^2,$$

which implies $\rho_2(x,\pi,T) \le \rho(x,\pi,T)$, since $\rho(x,\pi,T) > 0$ and $\rho_2(x,\pi,T) > 0$.

$$\begin{aligned}
&\rho_2(x,\pi,T)^2 - \rho_1(x,\pi,T)^2 \\
&\quad = \; E((X^\pi(T))^2|X^\pi(T) \le \rho(x,\pi,T)) - (E(X^\pi(T)|X^\pi(T) \le \rho(x,\pi,T)))^2 \\
&\quad = \; E((X^\pi(T) - E(X^\pi(T)|X^\pi(T) \le \rho(x,\pi,T)))^2|X^\pi(T) \le \rho(x,\pi,T)) \\
&\quad \ge \; 0,
\end{aligned}$$

which implies $\rho_2(x,\pi,T) \ge \rho_1(x,\pi,T)$, since $\rho_1(x,\pi,T) > 0$ and $\rho_2(x,\pi,T) > 0$. □

Now we want to analyse the behaviour of CaR^S depending on the strategy π. Therefore it will be convenient to introduce the function

$$f(\pi) = \pi'(b - r\underline{1})T + \ln(\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})/\alpha)\,, \tag{3.1.7}$$

i.e. $\mathrm{CaR}^S(x,\pi,T) = xe^{rT}(1 - e^{f(\pi)})$. Notice that

$$\lim_{\|\pi'\sigma\|\to\infty} f(\pi) = -\infty\,,$$

hence the use of extremely risky strategies can lead to a risk which is close to the total capital. The same is true for the measure ρ as was shown in chapter 2.

We shall frequently use the following estimate for the standard normal distribution; see e.g. Gänssler and Stute (1977).

Lemma 3.1.5 *Let $x > 0$. Then*

$$(x^{-1} - x^{-3})(2\pi)^{-1/2}\exp\{-x^2/2\} \le 1 - \Phi(x) \le x^{-1}(2\pi)^{-1/2}\exp\{-x^2/2\}$$

and

$$\frac{x\overline{\Phi}(x)}{\varphi(x)} \to 1, \quad x \to \infty$$

Proposition 3.1.6 *Set* $\theta = \|\sigma^{-1}(b - r\underline{1})\|, \varepsilon = \|\pi'\sigma\|$ *and* $\alpha^* = \Phi(z_\alpha - \varepsilon\sqrt{T})$.

(*a*) *If* $b_i = r$ *for all* $i = 1, \ldots, d$*, then* $f(\pi)$ *attains its unique maximum for* $\pi^* = 0$*, i.e.* $\varepsilon = 0$ *and* $\mathrm{CaR}^S(x, 0, T) = 0$*. Moreover, for arbitrary* $\varepsilon > 0$ *and all* π *with*

$$\|\pi'\sigma\| = \varepsilon \tag{3.1.8}$$

we have

$$f(\pi) = \ln(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha) = \ln(\alpha^*/\alpha) \tag{3.1.9}$$

and

$$0 < \mathrm{CaR}^S(x, \pi, T) = xe^{rT}(1 - \alpha^*/\alpha) < xe^{rT}.$$

(*b*) *If* $b_i \neq r$ *for some* $i \in \{1, \ldots, d\}$ *and if* $\sqrt{T} \leq \frac{\varphi(z_\alpha)}{\alpha\theta}$*, then* $f(\pi)$ *attains its unique maximum only for* $\pi^* = 0$*, i.e.* $\varepsilon = 0$ *and* $\mathrm{CaR}^S(x, 0, T) = 0$.

(*c*) *If* $b_i \neq r$ *for some* $i \in \{1, \ldots, d\}$ *and if* $\sqrt{T} > \frac{\varphi(z_\alpha)}{\alpha\theta}$ *and* $\alpha < 0.15$*, i.e.* $z_\alpha < -1.1$*, then* $f(\pi)$ *attains its unique maximum for a strategy*

$$\pi^* = \varepsilon\frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|}$$

such that

$$\left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+ < \varepsilon < \theta + z_\alpha/\sqrt{T}. \tag{3.1.10}$$

Denote by $a \vee b = \max\{a, b\}$ *and by* $a \wedge b = \min\{a, b\}$*. Then*

$$\left(\left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+ \theta T + \ln\left(\Phi(-\frac{2}{3}\theta\sqrt{T} \wedge z_\alpha)/\alpha\right)\right) \vee \left(\left(\theta + z_\alpha/\sqrt{T}\right)\theta T + \ln\left(\Phi(-\theta\sqrt{T})/\alpha\right)\right) \leq$$

$$f(\pi^*) \quad \leq \quad (\theta + z_\alpha/\sqrt{T})\theta T + \ln\left(\Phi(-\frac{2}{3}\theta\sqrt{T} \wedge z_\alpha)/\alpha\right)$$

Let $\pi^*_\varepsilon = \mathrm{argmax}_{\{\pi\in\mathbb{R}^d:\|\pi'\sigma\|=\varepsilon\}} f(\pi)$.

If $\varepsilon = 0$*, then* $f(\pi^*_0) = 0$ *and hence* $\mathrm{CaR}^S(x, 0, T) = 0$.

If $\varepsilon > 0$*, then*

$$\mathrm{CaR}^S(x, \pi^*_\varepsilon, T) \quad \begin{cases} > \; 0 & T < \dfrac{\ln(\alpha/\alpha^*)}{\varepsilon\theta} \\ < \; 0 & T > \dfrac{\ln(\alpha/\alpha^*)}{\varepsilon\theta} \end{cases} \tag{3.1.11}$$

Proof (a) If $b_i = r$ for all $i = 1, \ldots, d$, then

$$f(\pi) = \ln \Phi((z_\alpha - \varepsilon\sqrt{T})/\alpha)$$

with $\varepsilon = \|\pi'\sigma\| \geq 0$. Then the maximum over all non-negative ε is attained for $\varepsilon = 0$. Due to the regularity of σ this is equivalent to π equalling zero.

(b)(c) Consider the problem of maximizing $f(\pi)$ over all π which satisfy the requirement (3.1.8) for a fixed positive ε. Over the (boundary of the) ellipsoid defined by (3.1.8) $f(\pi)$ equals

$$f(\pi) = \pi'(b - r\underline{1})T + \ln(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha)$$

Thus the problem is just to maximise a linear function (in π) over the boundary of an ellipsoid. This problem has the explicit solution

$$\pi_\varepsilon^* = \varepsilon \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} \tag{3.1.12}$$

with

$$f(\pi_\varepsilon^*) = \varepsilon\theta T + \ln(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha)\,.$$

As every $\pi \in \mathbb{R}^d$ satisfies relation (3.1.8) with a suitable value of ε (due to the fact that σ is regular), we obtain the minimum strategy π^* by maximizing $f(\pi_\varepsilon^*)$ over all non-negative ε. Since

$$\frac{df(\pi_\varepsilon^*)}{d\varepsilon} = \theta T - \sqrt{T}\frac{\varphi(z_\alpha - \varepsilon\sqrt{T})}{\Phi(z_\alpha - \varepsilon\sqrt{T})}$$

$\frac{df(\pi_\varepsilon^*)}{d\varepsilon}(0) < 0$ if and only if $\sqrt{T} < \frac{\varphi(z_\alpha)}{\alpha\theta}$. Furthermore, using Lemma 3.1.5 we obtain

$$\begin{aligned}
\frac{d^2 f(\pi_\varepsilon^*)}{d\varepsilon^2} &= T\frac{\varphi(z_\alpha - \varepsilon\sqrt{T})}{(\Phi(z_\alpha - \varepsilon\sqrt{T}))^2}(\Phi(z_\alpha - \varepsilon\sqrt{T})(\varepsilon\sqrt{T} - z_\alpha) - \varphi(z_\alpha - \varepsilon\sqrt{T})) \\
&\leq T\frac{\varphi(z_\alpha - \varepsilon\sqrt{T})}{(\Phi(z_\alpha - \varepsilon\sqrt{T}))^2}(\varphi(z_\alpha - \varepsilon\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T})) = 0\,.
\end{aligned} \tag{3.1.13}$$

This implies that $\frac{df(\pi_\varepsilon^*)}{d\varepsilon}$ decreases in ε on $(0,\infty)$. Then the optimal ε is positive if and only if $\sqrt{T} > \frac{\varphi(z_\alpha)}{\alpha\theta}$. Thus, $\sqrt{T} \leq \frac{\varphi(z_\alpha)}{\alpha\theta}$ implies assertion (b).

Now take $\sqrt{T} > \frac{\varphi(z_\alpha)}{\alpha\theta}$. Then $\frac{df(\pi_\varepsilon^*)}{d\varepsilon}(0) > 0$ and $\frac{d^2 f(\pi_\varepsilon^*)}{d\varepsilon^2} < 0\,\forall \varepsilon > 0$ implies the uniqueness of an optimal ε. We shall derive bounds for this optimal ε. Notice that

$$\begin{aligned} f \text{ increases in } \varepsilon \quad \Leftrightarrow \quad & \frac{df(\pi^*_\varepsilon)}{d\varepsilon} = \theta T - \sqrt{T}\frac{\varphi(z_\alpha - \varepsilon\sqrt{T})}{\Phi(z_\alpha - \varepsilon\sqrt{T})} \geq 0 \\ \Leftrightarrow \quad & \theta\sqrt{T}\Phi(z_\alpha - \varepsilon\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T}) \geq 0. \end{aligned}$$

Set $\varepsilon_1 = \frac{2}{3}\theta + z_\alpha/\sqrt{T}$, then

$$\theta\sqrt{T}\Phi(z_\alpha - \varepsilon_1\sqrt{T}) - \varphi(z_\alpha - \varepsilon_1\sqrt{T}) = \theta\sqrt{T}\Phi(-\frac{2}{3}\theta\sqrt{T}) - \varphi(-\frac{2}{3}\theta\sqrt{T})\,.$$

Now define

$$P(y) = \frac{3}{2}y\Phi(-y) - \varphi(-y) = \frac{3}{2}y\overline{\Phi}(y) - \varphi(y)\,, \quad y > 0\,,$$

where we used the symmetry of the standard normal distribution. Taking the first derivative and using the fact that $\varphi'(y) = -y\varphi(y)$ we find that $P(y)$ is increasing if and only if $y\varphi(y)/\overline{\Phi}(y) < 3$. Since the hazard rate $\varphi(y)/\overline{\Phi}(y)$ of the standardnormal distribution is increasing (see e.g. Gaede (1977)), $y\varphi(y)/\overline{\Phi}(y)$ is increasing in $y > 0$. Thus $P(y)$ is increasing till its unique maximum (where $3 = y\varphi(y)/\overline{\Phi}(y)$) and then always decreasing. Furthermore, by l'Hospital, $P(y)$ converges to 0 for $y \to \infty$. Therefore, if $P(y_0) \geq 0$ for some $y_0 > 0$, then $P(y) > 0$ for all $y > y_0$. But $P(y) = 0$ for $y = 1.04$. This implies that

$$P(\frac{2}{3}\theta\sqrt{T}) = \theta\sqrt{T}\Phi(-\frac{2}{3}\theta\sqrt{T}) - \varphi(-\frac{2}{3}\theta\sqrt{T}) > 0 \quad \text{for} \quad \theta\sqrt{T} > 1.5 \cdot 1.04 = 1.56\,.$$

But $\theta\sqrt{T} \geq 1.56$ is satisfied by condition $\theta\sqrt{T} \geq \dfrac{\varphi(z_\alpha)}{\alpha}$ for $\alpha < 0.15$, i.e. $z_\alpha < -1.1$. This gives a lower bound ε_1^+ for the optimal ε.

Next we derive an upper bound. We know that

$$f \text{ decreases in } \varepsilon \quad \Leftrightarrow \quad \theta\sqrt{T}\Phi(z_\alpha - \varepsilon\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T}) \leq 0. \tag{3.1.14}$$

Since by Lemma 3.1.5

$$\theta\sqrt{T}\Phi(z_\alpha - \varepsilon\sqrt{T}) - \varphi(z_\alpha - \varepsilon\sqrt{T}) \leq \varphi(z_\alpha - \varepsilon\sqrt{T})\left(\frac{\theta\sqrt{T}}{\varepsilon\sqrt{T} - z_\alpha} - 1\right)$$

and $\varphi(z_\alpha - \varepsilon\sqrt{T}) > 0$, f decreases in ε if

$$\frac{\theta\sqrt{T}}{\varepsilon\sqrt{T} - z_\alpha} - 1 \leq 0\,.$$

Thus f decreases for $\varepsilon \geq \varepsilon_2 := \theta + \dfrac{z_\alpha}{\sqrt{T}}$. Then

$$f(\pi_{\varepsilon_1^+}) \vee f(\pi_{\varepsilon_2}) \leq f(\pi^*) \leq \varepsilon_2\theta T + \ln(\Phi(z_\alpha - \varepsilon_1^+\sqrt{T})/\alpha)\,,$$

since

$$\max_{[\varepsilon_1^+,\varepsilon_2]} \varepsilon\theta T = \varepsilon_2\theta T \quad \text{and} \quad \max_{[\varepsilon_1^+,\varepsilon_2]} \ln(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha) = \ln(\Phi(z_\alpha - \varepsilon_1^+\sqrt{T})/\alpha)\,.$$

The estimate (3.1.11) for the CaR^S follows from the fact that $f(\pi_\varepsilon^*) < 0$ or $f(\pi_\varepsilon^*) > 0$ according as $T < \ln(\alpha/\alpha^*)/(\varepsilon\theta)$ or $T > \ln(\alpha/\alpha^*)/(\varepsilon\theta)$. □

Now we look at the problem:

$$\boxed{\max_{\pi\in\mathbb{R}^d} E(X^\pi(T)) \quad \text{subject to} \quad \mathrm{CaR}^S \leq C.} \tag{3.1.15}$$

Proposition 3.1.7 *Assume that C satisfies*

$$0 \leq C \leq x\exp\{rT\}.$$

If $b_i \neq r$ for some $i \in \{1,\ldots,d\}$ then problem (3.1.15) will be solved by

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|} \tag{3.1.16}$$

with ε^* between

$$\frac{z_\alpha - \Phi^{-1}(\alpha\exp(c - (\frac{z_\alpha}{\sqrt{T}} + \frac{2}{3}\theta)^+\theta T))}{\sqrt{T}} \vee \left(\frac{2}{3}\theta + \frac{z_\alpha}{\sqrt{T}}\right)^+ \tag{3.1.17}$$

and

$$\theta + \frac{z_\alpha}{\sqrt{T}} + \sqrt{(\theta + \frac{z_\alpha}{\sqrt{T}})^2 - \frac{1}{T}(z_\alpha^2 + 2c + 2\ln(\theta\sqrt{2\pi T}\alpha))}, \tag{3.1.18}$$

where $\theta = \|\sigma^{-1}(b - r\underline{1})\|$ and $c = \ln(1 - \frac{C}{x}e^{-rT})$.
The corresponding maximal expected terminal wealth under the CaR^S constraint (3.1.15) equals

$$E(X^\pi(T)) = x\exp\{(r + \varepsilon^*\|\sigma^{-1}(b - r\underline{1})\|)T\} \tag{3.1.19}$$

Proof Every admissible π for problem (3.1.15) with $\|\pi'\sigma\| = \varepsilon$ satisfies the relation

$$\mathrm{CaR}^S(x,\pi,T) = xe^{rT}(1 - e^{f(\pi)}) \leq C \tag{3.1.20}$$

which is equivalent to

$$f(\pi) \geq c$$

with $c = \ln\left(1 - \frac{C}{x}\exp(-rT)\right)$. On the set of portfolios given by $\|\pi'\sigma\| = \varepsilon$ the linear function $(b - r\underline{1})'\pi T$ is maximised by

$$\pi_\varepsilon = \varepsilon \frac{(\sigma\sigma')^{-1}(b - r\underline{1})}{\|\sigma^{-1}(b - r\underline{1})\|}. \tag{3.1.21}$$

Hence, if there is an admissible π for problem (3.1.15) with $\|\pi'\sigma\| = \varepsilon$ then π_ε must also be admissible. Further, due to the explicit form (2.1.3) of the expected terminal wealth, π_ε also maximizes the expected terminal wealth over the ellipsoid. Consequently, to obtain an optimal π for problem (3.1.15) it is enough to consider all vectors of the form π_ε for all positive ε such that requirement (3.1.20) is satisfied. Inserting (3.1.21) into the left-hand side of inequality (3.1.20) results in

$$(b - r\underline{1})'\pi_\varepsilon T = \varepsilon\|\sigma^{-1}(b - r\underline{1})\|T \tag{3.1.22}$$

which is an increasing linear function in ε equalling zero in $\varepsilon = 0$. Therefore, we obtain the solution of problem (3.1.15) by determining the biggest positive ε such that (3.1.20) is still valid.

We shall derive bounds for this optimal ε.

Notice that for $\pi = \pi_\varepsilon$ by (3.1.22)

$$(3.1.20) \Leftrightarrow f(\pi_\varepsilon^*) = \varepsilon\theta T + \ln\left(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha\right) \geq c\,.$$

Since $c < \max\limits_{\varepsilon > o} f(\pi_\varepsilon^*)$, by (3.1.10) we have

$$\varepsilon > \mathrm{argmax}_{\varepsilon>0} f(\pi_\varepsilon^*) > \left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+ .$$

By (3.1.10) $f(\pi_\varepsilon^*) \geq c$ is satisfied, when

$$\left(\frac{2}{3}\theta + z_\alpha/\sqrt{T}\right)^+ \theta T + \ln\left(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha\right) \geq c\,.$$

But this is equivalent to

$$\varepsilon \leq \left(z_\alpha - \Phi^{-1} \left(\alpha \exp \left(c - \left(\frac{2}{3}\theta + z_\alpha/\sqrt{T} \right)^+ \theta T \right) \right) \right) /\sqrt{T}.$$

Thus $f(\pi_\varepsilon^*) \geq c$ holds for all ε with

$$\mathrm{argmax}_{\varepsilon>0} f(\pi_\varepsilon^*) < \varepsilon \leq \left(z_\alpha - \Phi^{-1} \left(\alpha \exp \left(c - \left(z_\alpha/\sqrt{T} + \frac{2}{3}\theta \right)^+ \theta T \right) \right) \right) /\sqrt{T}.$$

In (3.1.13) we have shown that $f(\pi_\varepsilon)$ is increasing till its unique maximum and then decreasing. Hence we have to determine an $\varepsilon > (z_\alpha - \Phi^{-1}(\alpha \exp(c - (z_\alpha/\sqrt{T} + \frac{2}{3}\theta)^+ \theta T)))/\sqrt{T}$ as small as possible such that $f(\pi_\varepsilon^*) < c$ to find an upper bound for the optimal ε. Since $\varepsilon\theta T + \ln\left(\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha\right)$ is decreasing for all ε greater than the optimal ε, we know that

$$\Phi(z_\alpha - \varepsilon\sqrt{T}) \leq \varphi(z_\alpha - \varepsilon\sqrt{T})/(\theta\sqrt{T})$$

by (3.1.14). Notice that

$$f(\pi_\varepsilon^*) < c \Leftrightarrow e^{\varepsilon\theta T}\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha < e^c.$$

Since this implies that

$$e^{\varepsilon\theta T}\Phi(z_\alpha - \varepsilon\sqrt{T})/\alpha \leq e^{\varepsilon\theta T}\varphi(z_\alpha - \varepsilon\sqrt{T})/(\theta\sqrt{T}\alpha),$$

we need to determine an ε with

$$\exp(\varepsilon\theta T - \frac{1}{2}(z_\alpha - \varepsilon\sqrt{T})^2)/(\theta\sqrt{2\pi T}\alpha) \leq e^c.$$

But this is equivalent to

$$-\varepsilon^2 T/2 + \varepsilon(\theta T + z_\alpha\sqrt{T}) - z_\alpha^2/2 - c - \ln(\theta\sqrt{2\pi T}\alpha) \leq 0$$

This inequality is satisfied for all

$$\varepsilon \geq \theta + z_\alpha/\sqrt{T} + \sqrt{(\theta + z_\alpha/\sqrt{T})^2 - (z_\alpha^2 + 2c + 2\ln(\theta\sqrt{2\pi T}\alpha))/T}.$$

Thus the optimal $\varepsilon < \theta + z_\alpha/\sqrt{T} + \sqrt{(\theta + z_\alpha/\sqrt{T})^2 - (z_\alpha^2 + 2c + 2\ln(\theta\sqrt{2\pi T}\alpha))/T}$. □

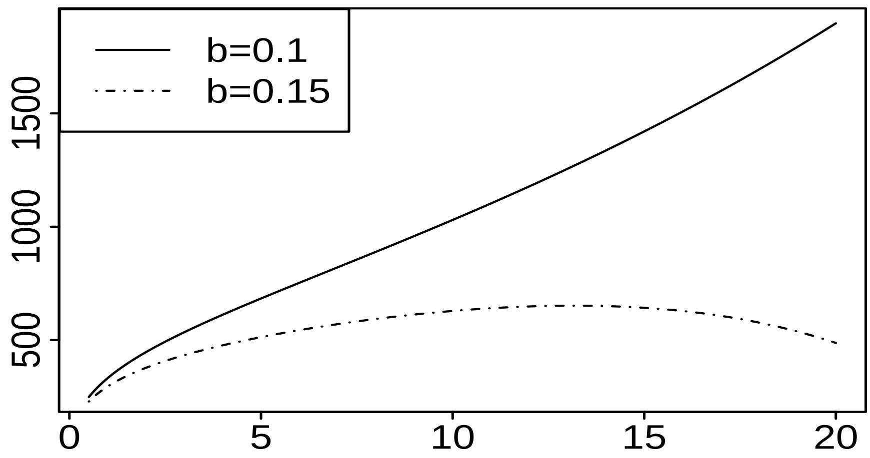


Figure 3.1: $\mathrm{CaR}^S(1000, 1, T)$ of the pure stock portfolio for different stock appreciation rates for $0 \leq T \leq 20$. The parameters are $d = 1$, $r = 0.05$, $\sigma = 0.2, \alpha = 0.05$.

Example 3.1.8 Figure 3.1 describes the dependence of $\mathrm{CaR}^S(x, \pi, T)$ on time as illustrated by $\tilde{\rho}_1(1000, 1, T)$ for $b = 0.1$ and $b = 0.15$. Note that for $b = 0.15$ the CaR^S first increases and then decreases with time, while for $b = 0.1$ the CaR^S increases with time for $T < 20$ and decreases only for very large T. The following figures illustrate the behaviour of the optimal strategy and the maximal expected terminal wealth for varying planning horizon T. In Figures 3.3 and 3.4 we have plotted the expected terminal wealth corresponding to the different strategies as functions of the planning horizon T. For a planning horizon $T < 5$ the expected terminal wealth of the optimal portfolio even exceeds the pure stock investment. The reason for this becomes clear if we look at the corresponding portfolios. The optimal portfolio always contains a short position in the bond as long as this is tolerated by the CaR^S constraint (see Figure 3.2). After 5 years the optimal portfolio contains a long position in both bond and stock for $b = 0.10$. For $b = 0.15$ the optimal portfolio contains a short position in the bond for all planning horizons. This is due to the behaviour of CaR^S of the stock price. For $b = 0.10$ CaR^S is always much larger than for $b = 0.15$ (see Figure 3.1). This leads to a smaller strategy for $b = 0.10$. Figure 3.5 shows the mean-CaR^S efficient frontier for the above parameters with fixed tim $T = 5$. As expected it has a similar form as a mean-variance efficient frontier.

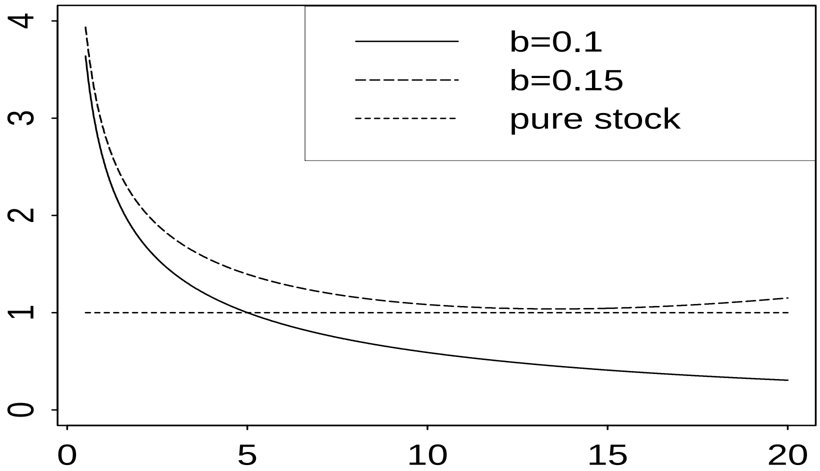


Figure 3.2: Optimal portfolios and pure stock portfolio for different stock appreciation rates. As upper bound of the $\mathrm{CaR}^S(x, \pi, T)$ we took $\mathrm{CaR}^S(1000, 1, 5, b = 0.1)$, the CaR^S of the pure stock strategy with time horizon T=5. All other parameters are chosen as in Figure 3.1.

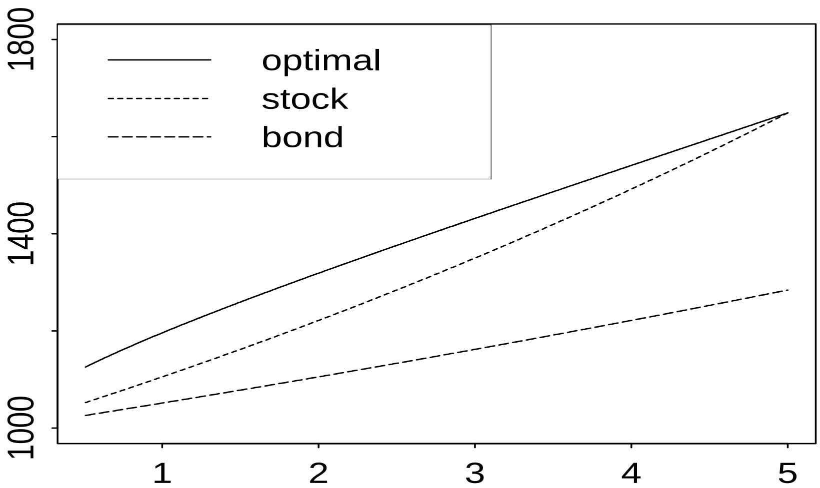


Figure 3.3: Expected terminal wealth of the optimal portfolio for $b = 0.1$ in comparison to the wealth of a pure bond and a pure stock portfolio depending on the time horizon T, $0 < T \leq 5$. All other parameters are chosen as in Figure 3.2.

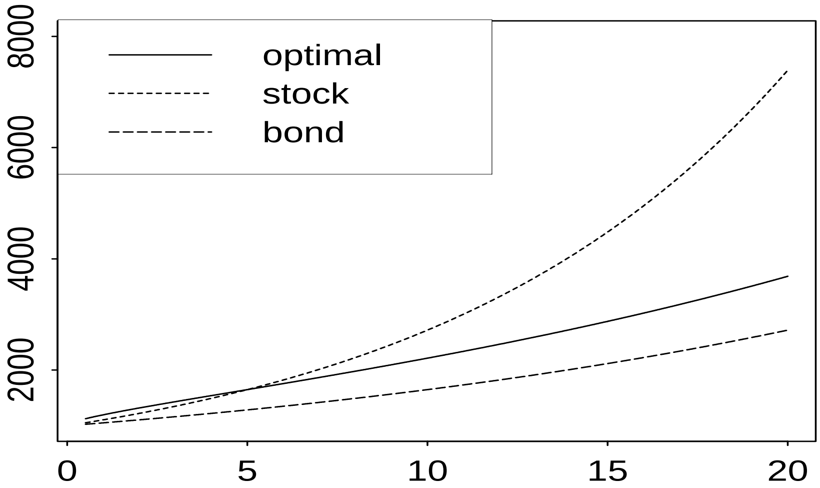


Figure 3.4: Expected terminal wealth of the optimal portfolio for $b = 0.1$ in comparison to the wealth of a pure bond and a pure stock portfolio depending on the time horizon T, $0 < T \leq 20$. All other parameters are chosen as in Figure 3.2.

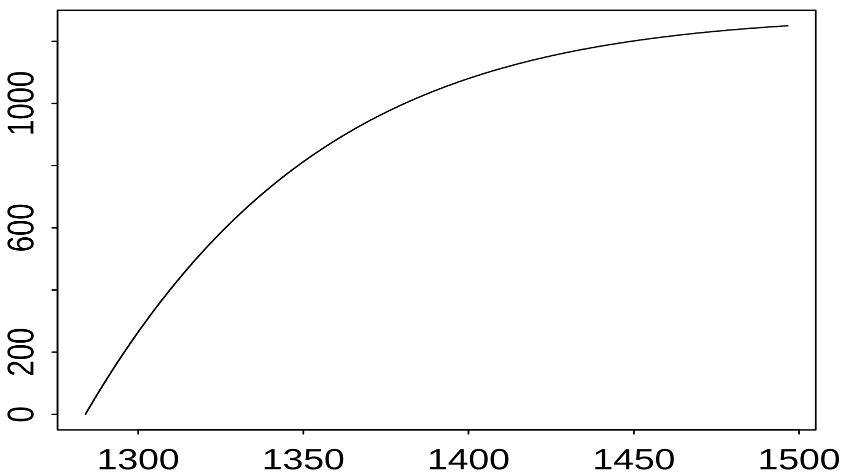


Figure 3.5: Mean-CaR^S efficient frontier. The parameters are the same as in Figure 3.3.

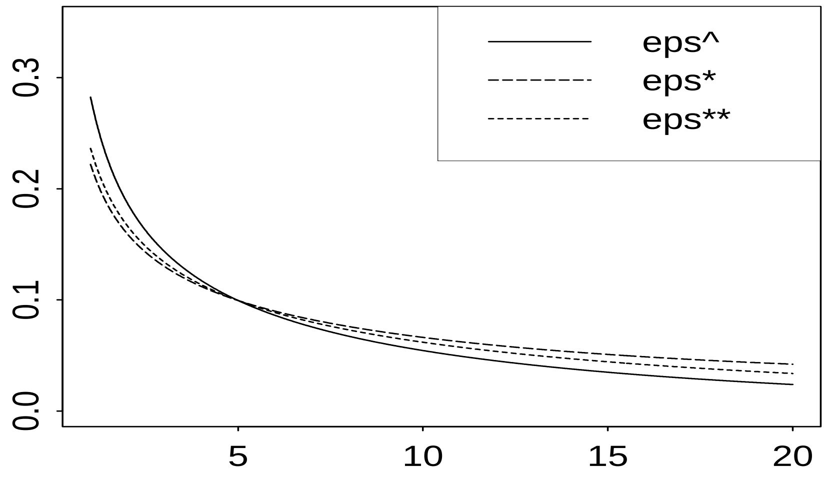


Figure 3.6: $\hat{\varepsilon}$, ε^* and ε^{**} as functions of the time horizon for $0 \leq T \leq 20$ and $\widehat{C} = 107100$, $C^{**} = 300$ and $C^* = 384$.

We will now compare the behaviour of the optimal portfolios for the mean-CaR^S problem with solutions of a corresponding mean-variance problem and with solutions of a corresponding mean-CaR problem. These two corresponding problems are discussed in chapter 2.

Example 3.1.9 Figure 3.6 compares the behaviour of $\widehat{\varepsilon}$, ε^{**} and ε^* as functions of the time horizon, where $\hat{\varepsilon}$ is the optimal ε for the mean-variance problem, ε^{**} for the mean-CaR problem and ε^* for the mean-CaR^S problem. We have used the same data as in the foregoing example. To make the solutions of the three problems comparable we have chosen C in such a way that $\hat{\varepsilon}$, ε^{**} and ε^* coincide for T=5, i.e. for the variance $C = 107100$, for the CaR of the quantile $C = 384$ and for the CaR^S of the expected shortfall $C = 300$.

3.2 Expected shortfall portfolios and the Black-Scholes model with jumps

In this section we consider again the mean-CaR^S problem (3.1.15), but drop the assumption of log-normality of the stock price process. We work with the Black-Scholes model with jumps which we already introduced in section 2.2.

Definition 3.2.1 *Consider the market given by a riskless bond with price $P_0(t) = e^{rt}$, $t \geq 0$, for $r \in \mathbb{R}$ and one stock with price process P satisfying (2.2.1) for $b \in \mathbb{R}$ and a semimartingale Y with $Y(0) = 0$. Assume that the dynamic of the wealth process is given by (2.2.2).*

Let x be the initial capital and T a given time horizon. For some portfolio $\pi \in \mathbb{R}$ and the corresponding terminal wealth $X^\pi(T)$ the α-quantile ρ of $X^\pi(T)$ is given by Definition 2.2.1. Then we call

$$\mathrm{CaR}^S(x,\pi,T) = x\exp\{rT\} - E(X^\pi(T)|X^\pi(T) \leq \rho(x,\pi,T)) \tag{3.2.1}$$

the Capital-at-Risk (CaR^S) with respect to the expected shortfall *of the portfolio π (with initial capital x and time horizon T).* □

The aim of this section is to explore the behaviour of the solutions to the mean-CaR^S problem (3.1.15) if we model the returns of the price process by a Brownian motion with jumps. We present some specific examples.

Lemma 3.2.2 *With a stock price given by equation (2.2.5) let $(X^\pi(t))_{t\geq 0}$ be the wealth process corresponding to the portfolio π satisfying (2.2.6). Let $\rho(x,\pi,T)$ be the α-quantile of $X^\pi(T)$. Set*

$$B(x,\pi,T) = \exp\{(\pi(b-r) - \sum_{i=1}^{n} \pi\beta_i\lambda_i)T\}.$$

Then we have for some finite time horizon T:

$$E(X^\pi(T)) = \exp\{(r + \pi(b-r))T\} \tag{3.2.2}$$

and

$$\begin{aligned}
\mathrm{CaR}^S(x,\pi,T) &= xe^{rT} - E(X^\pi(T)|X^\pi(T) \leq \rho(x,\pi,T)) \\
&= xe^{rT}\left(1 - \frac{B(x,\pi,T)}{\alpha} \sum_{n_1,\ldots,n_n=0}^{\infty} \exp\left\{\sum_{i=1}^{n} \ln(1+\pi\beta_i)n_i - \lambda_i T\right\}\right. \\
&\quad \left.\times \prod_{i=1}^{n} \frac{(\lambda_i T)^{n_i}}{n_i!} \Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}(\tilde{z}_\alpha - \sum_{i=1}^{n} \ln(1+\pi\beta_i)n_i - |\pi\sigma|^2 T)\right)\right).
\end{aligned}$$

Here, $\tilde{z}_\alpha$ is the α-quantile of

$$\pi\sigma W(T) + \sum_{i=1}^{n} \ln(1+\pi\beta_i)N_i(T),$$

i.e. the real number $\tilde{z}_\alpha$ satisfying

$$\begin{aligned}
\alpha &= P\left(\pi\sigma W(T) + \sum_{i=1}^{n}(\ln(1+\pi\beta_i)N_i(T)) \leq \tilde{z}_\alpha\right) \\
&= \sum_{n_1,\dots,n_n=0}^{\infty}\left(\Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}\left(\tilde{z}_\alpha - \sum_{i=1}^{n}(\ln(1+\pi\beta_i)n_i)\right)\right)e^{-T\sum_{i=1}^{n}\lambda_i}\prod_{i=1}^{n}\frac{(\lambda_i T)^{n_i}}{n_i!}\right)
\end{aligned}$$

Proof To obtain the expected value see proof of Lemma 2.2.2. For the CaR^S recall (3.1.5). Hence for the shortfall we obtain

$$\begin{aligned}
&E(X^\pi(T)|X^\pi(T) \leq \rho(x,\pi,T)) \\
&= \frac{E(X^\pi(T)I(X^\pi(T) \leq \rho(x,\pi,T)))}{P(X^\pi(T) \leq \rho(x,\pi,T))} \\
&= \frac{B(x,\pi,T)}{\alpha}\exp\left\{-\frac{1}{2}\pi^2\sigma^2 T + rT\right\} \times \\
&\quad E\left(\exp\left\{\pi\sigma W(T) + \sum_{i=1}^{n}(N_i(T)\ln(1+\pi\beta_i))\right\} I\left(\pi\sigma W(T) + \sum_{i=1}^{n}(N_i(T)\ln(1+\pi\beta_i)) \leq \tilde{z}_\alpha\right)\right) \\
&= \frac{B(x,\pi,T)}{\alpha}\exp\left\{-\frac{1}{2}\pi^2\sigma^2 T + rT\right\}\sum_{n_1,\dots,n_n=0}^{\infty}\prod_{i=1}^{n}\frac{(\lambda_i T)^{n_i}}{n_i!}\exp\left\{\sum_{i=1}^{n} n_i\ln(1+\pi\beta_i) - \lambda_i T\right\} \times \\
&\quad \int_{-\infty}^{\frac{1}{|\pi\sigma|\sqrt{T}}(\tilde{z}_\alpha - \sum_{i=1}^{n} n_i \ln(1+\pi\beta_i))} \exp\{\sqrt{T}|\pi\sigma|x\}\varphi(x)dx \\
&= \frac{B(x,\pi,T)}{\alpha}\exp\{rT\}\sum_{n_1,\dots,n_n=0}^{\infty}\prod_{i=1}^{n}\frac{(\lambda_i T)^{n_i}}{n_i!}\exp\left\{\sum_{i=1}^{n} n_i\ln(1+\pi\beta_i) - \lambda_i T\right\} \times \\
&\quad \Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}(\tilde{z}_\alpha - \sum_{i=1}^{n} n_i\ln(1+\pi\beta_i)) - |\pi\sigma|\sqrt{T}\right) \\
&= \frac{B(x,\pi,T)}{\alpha}\exp\{rT\}\sum_{n_1,\dots,n_n=0}^{\infty}\exp\left\{\sum_{i=1}^{n} n_i\ln(1+\pi\beta_i) - \lambda_i T\right\} \times \\
&\quad \Phi\left(\frac{1}{|\pi\sigma|\sqrt{T}}\left(\tilde{z}_\alpha - \sum_{i=1}^{n}(n_i\ln(1+\pi\beta_i))\right) - |\pi\sigma|\sqrt{T}\right)\prod_{i=1}^{n}\frac{(\lambda_i T)^{n_i}}{n_i!}.
\end{aligned}$$

□

Unfortunately, $\widetilde{z}_\alpha$ cannot be represented in such an explicit form as in the case without

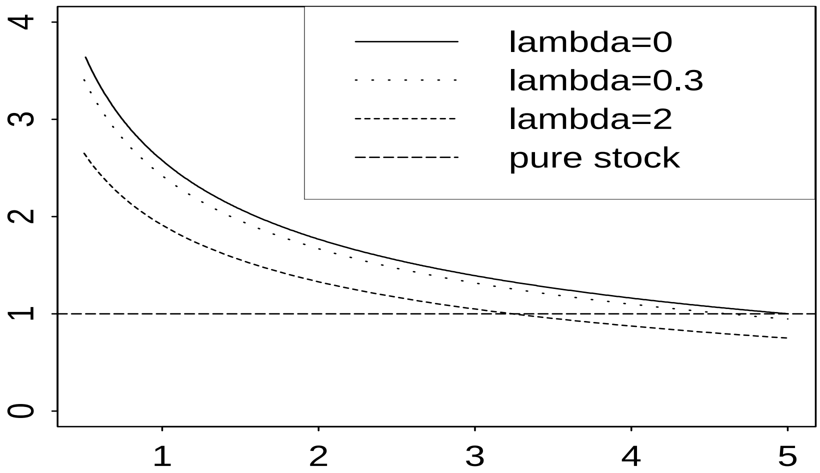


Figure 3.7: Optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 \leq T \leq 5$ for different jump parameters $\beta = -0.1$ and $\lambda = 0.3$ and $\lambda = 2$. The basic parameters are the same as in Figure 3.3.

jumps. However, due to the explicit form of $E(X^{\pi}(T))$, it is obvious that the corresponding mean-CaR^S problem will be solved by the largest π that satisfies both the CaR^S constraint and requirement (2.2.6). Thus for an explicit example we obtain the optimal mean-CaR^S portfolio by a simple numerical iteration procedure. Comparisons of the solutions for the Brownian motion with and without jumps are given in Figure 3.7 and Figure 3.8.

Example 3.2.3 We have used the same parameters as in the examples of Section 3.1, but have included the possibility of a jump of height $\beta = -0.1$, occuring with intensity $\lambda = 0.3$, i.e. one would expect a jump approximately every three years, and with intensity $\lambda = 2$, i.e. one would expect a jump twice a year. An optimal portfolio for stock prices following a geometric Brownian motion with jumps is always below the optimal portfolio of the geometric Brownian motion (solid line) and the higher the intensity λ the lower is the portfolio. The reason for this is that the threat of a downwards jump of 10% leads an investor to a less risky behaviour.

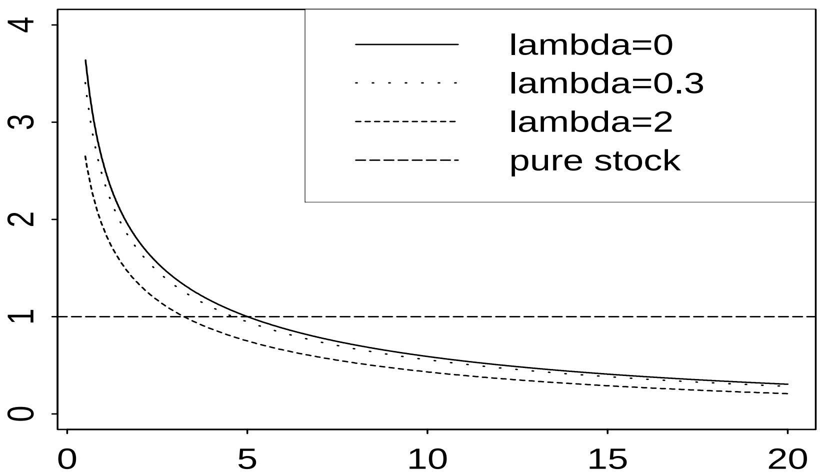


Figure 3.8: Optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 \leq T \leq 20$ for different jump parameters $\beta = -0.1$ and $\lambda = 0.3$ and $\lambda = 2$. The basic parameters are the same as in Figure 3.3.

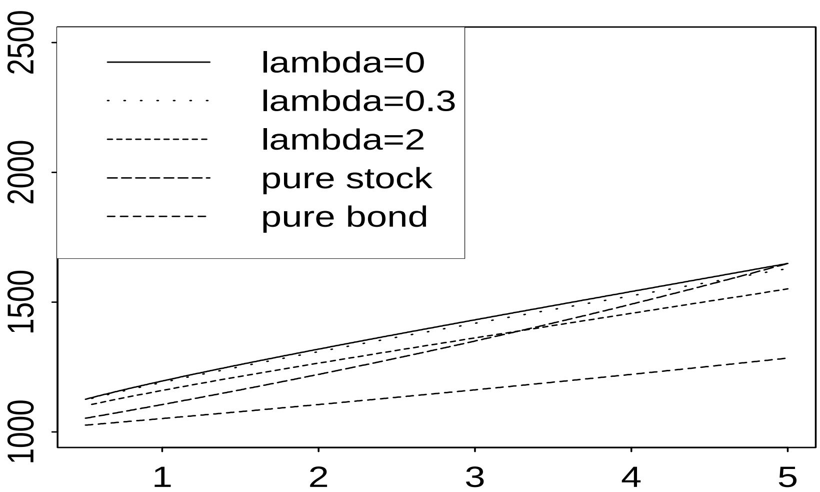


Figure 3.9: Expected terminal wealth corresponding to the optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 \leq T \leq 5$. The parameters are the same as in Figure 3.7.

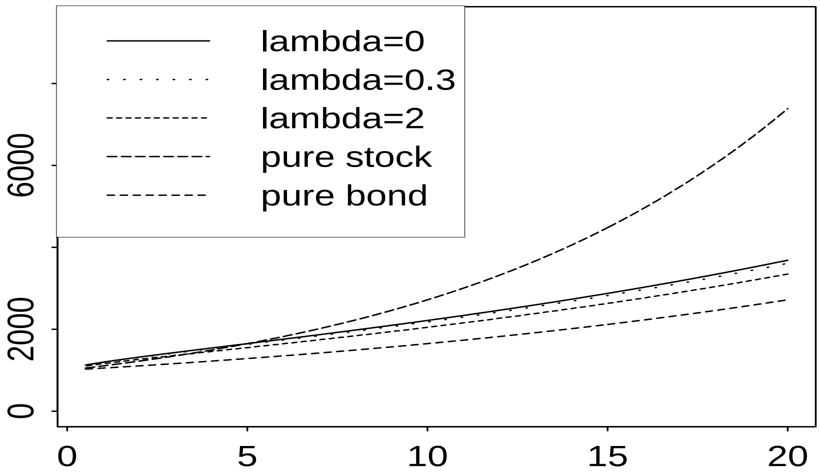


Figure 3.10: Expected terminal wealth corresponding to the optimal portfolios for Brownian motion with and without jumps depending on the time horizon T, $0 \leq T \leq 20$. The parameters are the same as in Figure 3.7.

3.3 Shortfall probability portfolios in the Black-Scholes setting

In Section 3.1 and 3.2 we considered lower partial moments of order one as risk measures in different models. In this Section we consider again the Black-Scholes market explained in Section 2.1, but move to lower partial moments of order 0 in this section, i.e. the shortfall probability below a certain benchmark $Z = Z(t)_{t\geq 0}$, which is defined by a stochastic differential equation. We can think of this benchmark process as a market index, e.g. the DAX or the Dow Jones index. This idea has the advantage that we can measure risk relative to the behaviour of the market. Since we measure risk depending on this benchmark, it seems to be useful to compare also the portfolio's wealth to the benchmark. Thus we maximize the expected ratio of the wealth of the portfolio and the benchmark under a constraint on the shortfall probability below this benchmark. This has also the advantage that we obtain an explicit closed form solution.

Definition 3.3.1 (Shortfall probability) Let a benchmark Z be defined as the solution to the SDE

$$dZ(t) = Z(t)(\mu dt + \nu dW(t)),\ t \geq 0, \quad Z(0) = x,$$

where $\mu > 0$ is the appreciation rate, $\nu \in \mathbb{R}^d$ is the volatility vector of the benchmark and $(W(t))_{t\geq 0}$ is the same d-dimensional Brownian motion as in $(X^\pi(t))_{t\geq 0}$. Then for a portfolio $\pi \in \mathbb{R}^d$, initial capital $x > 0$ and time horizon $T > 0$ we define the *shortfall probability* of $X^\pi(T)$ by

$$P(X^\pi(T) \leq Z(T)).$$

Note that we modeled the correlation structure of the assets and the benchmark by linear combinations of the same Brownian motion.

Next we calculate this shortfall probability explicitly.

Proposition 3.3.2 *Let $X = (X^\pi(t))_{t\geq 0}$ be the wealth process of a portfolio $\pi \in \mathbb{R}^d$ in the Black-Scholes market and $(Z(t))_{t\geq 0}$ and the shortfall probability as in Definition 3.3.1. Denote by Φ the df of a standard normal rv N(0,1). Let $\|\pi'\sigma - \nu\| > 0$ and T be a fixed* *time horizon and set*

$$A := \frac{1}{2}\|\pi'\sigma - \nu\|^2 - \pi'(b - r\underline{1} - \sigma\nu) - r + \mu - \nu'\nu.$$

Then

$$P(X^\pi(T) \leq Z(T)) = \Phi\left(\frac{A\sqrt{T}}{\|\pi'\sigma - \nu\|}\right) \tag{3.3.1}$$

Proof Recall the following identity in law

$$\frac{(\pi'\sigma - \nu)W(t)}{\|\pi'\sigma - \nu\|\sqrt{t}} \stackrel{d}{=} N(0,1), \quad t > 0,$$

and that Z and X^π are driven by the same Brownian motion, which implies

$$\begin{aligned}\frac{Z(T)}{X^\pi(T)} &= \exp(AT + (\nu - \pi'\sigma)W(T)) \\ &\stackrel{d}{=} \exp\{AT\}\exp(\|\pi'\sigma - \nu\|\sqrt{T}N(0,1))\end{aligned} \tag{3.3.2}$$

and hence by taking logarithms,

$$\begin{aligned}P(X^\pi(T) \leq Z(T)) &= P\left(\frac{Z(T)}{X^\pi(T)} \geq 1\right) \\ &= P(AT + \|\pi'\sigma - \nu\|\sqrt{T}N(0,1) \geq 0) \\ &= \Phi\left(\frac{A\sqrt{T}}{\|\pi'\sigma - \nu\|}\right)\end{aligned}$$

□

Now we want to analyze the behaviour of the shortfall probability depending on the strategy π. Therefore it will be convenient to introduce the function $f(\pi)$ for the argument of Φ in (3.3.1)

$$f(\pi) = \left(\frac{\frac{1}{2}\|\pi'\sigma - \nu\|^2 - \pi'(b - r\underline{1} - \sigma\nu) - r + \mu - \nu'\nu}{\|\pi'\sigma - \nu\|}\right)\sqrt{T} \tag{3.3.3}$$

Since

$$f(\pi) \stackrel{\|\pi'\sigma - \nu\| \to \infty}{\longrightarrow} \infty,$$

we have

$$\sup_{\pi \in \mathbb{R}^d} P(X^\pi(T) \leq Z(T)) = 1$$

In the following proposition we calculate the minimum shortfall probability.

Proposition 3.3.3 *Set $\theta = \|(b - r\underline{1} - \sigma\nu)\sigma^{-1}\|$ and let $\|\pi'\sigma - \nu\| > 0$.*
(a) If $\sigma'(\mu - r) > \nu'(b - r\underline{1})$, then $P(X^\pi(T) \leq Z(T))$ attains its minimum for

$$\pi^* = \sqrt{2(\sigma^{-1}\nu(b - r\underline{1}) - r + \mu)}\frac{(\sigma\sigma')^{-1}(b - r\underline{1} - \sigma\nu)}{\|(b - r1 - \sigma\nu)\sigma^{-1}\|} + \sigma'^{-1}\nu \tag{3.3.4}$$

with

$$P(X^{\pi^*}(T) \leq Z(T)) = \Phi\left(\left(\sqrt{2(\sigma^{-1}\nu(b - r\underline{1}) - r + \mu)} - \theta\right)\sqrt{T}\right) \tag{3.3.5}$$

(b) If $\sigma'(\mu - r) < \nu(b - r\underline{1})$, then $P(X^\pi(T) \leq Z(T))$ attains its minimum for $\pi^* = \sigma'^{-1}\nu$ with $P(X^{\pi^*}(T) \leq Z(T)) = 0$.

Proof (a) Consider the problem of minimizing $f(\pi)$ over all π which satisfy

$$\|\pi'\sigma - \nu\| = \varepsilon \tag{3.3.6}$$

for a fixed positive ε. Over the (boundary of the) ellipsoid defined by (3.3.6) $f(\pi)$ equals

$$\frac{\sqrt{T}}{\varepsilon}(\frac{1}{2}\varepsilon^2 - \pi'(b - r\underline{1} - \sigma\nu) - r + \mu - \nu'\nu).$$

Thus the problem is to maximize a linear function (in π) over the boundary of an ellipsoid. Such a problem has the explicit solution

$$\pi^*_\varepsilon = \sigma'\nu + \varepsilon\frac{(\sigma\sigma')^{-1}(b - r\underline{1} - \sigma\nu)}{\|(b - r\underline{1} - \sigma\nu)\sigma^{-1}\|} \tag{3.3.7}$$

and

$$f(\pi^*_\varepsilon) = \left(\frac{1}{2}\varepsilon - \theta + \frac{\mu - r - \sigma'^{-1}\nu(b - r\underline{1})}{\varepsilon}\right)\sqrt{T}. \tag{3.3.8}$$

As every $\pi \in \mathbb{R}^d$ satisfies relation (3.3.6) with a suitable value of ε (due to the fact that σ is regular), we obtain the minimum shortfall probability strategy π^* by minimizing $f(\pi^*_\varepsilon)$ over all non-negative ε. Due to the form of $f(\pi^*_\varepsilon)$ there is only a solution if $\mu - r - \sigma'^{-1}\nu(b - r\underline{1})$ is positive. Under the condition $\sigma'(\mu - r) > \nu'(b - r\underline{1})$ the optimal ε is given as

$$\varepsilon = \sqrt{2(\mu - r - \sigma'^{-1}\nu(b - r\underline{1}))}. \tag{3.3.9}$$

Inserting this into equations (3.3.7) and (3.3.8) yields the assertions (3.3.4) and (3.3.5) (with the help of equations (3.3.1) and (3.3.3)).

(b) Assertion (b) follows from fact that under the condition $\sigma'(\mu - r) < \nu(b - r\underline{1})$ $f(\pi_\varepsilon^*)$ tends to $-\infty$ as ε tends to zero and hence the shortfall probability tends to zero. □

Since $E(e^{sW(1)}) = e^{s^2/2}$, $s \in \mathbb{R}$, immediately by (3.3.2) we obtain the following explicit formula for the expected relative terminal wealth for all $T > 0$.

$$E\left(\frac{X^\pi(T)}{Z(T)}\right) = \exp((\pi'(b - r1 - \sigma\nu) + r - \mu + \nu'\nu)T) \tag{3.3.10}$$

Now we consider the following optimization problem:

$$\boxed{\max_{\pi\in\mathbb{R}^d} E\left(\frac{X^\pi(T)}{Z(T)}\right) \quad \text{subject to} \quad P(X^\pi(T) \le Z(T)) \le \alpha \in [0,1]\,, \qquad (3.3.11)}$$

Due to the explicit representations (3.3.1) and (3.3.10) and we can solve the problem explicitly.

Proposition 3.3.4 *Let $\theta = \|(b - r1 - \sigma\nu)\sigma^{-1}\| > 0$. Assume that α satisfies*

$$0 \le \alpha \le 1 \quad \textit{if } \sigma'(\mu - r) < \nu(b - r\underline{1}), \tag{3.3.12}$$

$$\Phi\left(\left(\sqrt{2(\sigma^{-1}\nu(b - r\underline{1}) - r + \mu)} - \theta\right)\sqrt{T}\right) \le \alpha \le 1 \quad \textit{if } \sigma'(\mu - r) > \nu(b - r\underline{1})\,. \tag{3.3.13}$$

Then problem (3.3.11) has the unique solution

$$\pi_\varepsilon^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r1 - \sigma\nu)}{\|(b - r1 - \sigma\nu)\sigma^{-1}\|} + \sigma'^{-1}\nu, \tag{3.3.14}$$

where

$$\varepsilon^* = \theta + \frac{1}{\sqrt{T}}\left(z_\alpha + \sqrt{(z_\alpha + \theta\sqrt{T})^2 + 2T(\sigma'^{-1}\nu(b - r1) + r - \mu)}\right) > 0,$$

where z_α is the α-quantile of the standard normal distribution.

The corresponding maximal expected relative terminal wealth under the shortfall probability constraint equals

$$E\left(\frac{X^\pi(T)}{Z(T)}\right) = \exp((\varepsilon^*\theta + \sigma^{-1}\nu'(b - r1) + r - \mu)T). \tag{3.3.15}$$

Proof Requirements (3.3.12) and (3.3.13) ensure that the shortfall probability constraint in problem (3.3.11) cannot be ignored: in both cases α lies between the minimum and the maximum value the shortfall probability can attain see also Proposition 3.3.3. Every admissible π for problem (3.3.11) with $\|\pi'\sigma - \nu\| = \varepsilon$ satisfies the relation

$$(b - r\underline{1} - \sigma\nu)'\pi T \geq (-\nu'\nu - r + \mu)T + \frac{1}{2}\varepsilon^2 T - \varepsilon z_\alpha \sqrt{T} \tag{3.3.16}$$

which is in this case equivalent to the shortfall probability constraint in 3.3.11. But again, on the set given by $\|\pi'\sigma - \nu\| = \varepsilon$ the linear function $(b - r\underline{1} - \sigma\nu)'\pi T$ is maximized by

$$\pi_\varepsilon = \sigma'^{-1}\nu + \varepsilon \frac{(\sigma\sigma')^{-1}(b - r\underline{1} - \sigma'\nu)}{\|(b - r1 - \sigma\nu)\sigma^{-1}\|}. \tag{3.3.17}$$

Hence, if there is an admissible π for problem (3.3.11) with $\|\pi'\sigma - \nu\| = \varepsilon$ then π_ε must also be admissible. Further, due to the explicit form of the expected relative terminal wealth (3.3.10), π_ε also maximizes the expected relative terminal wealth over the ellipsoid. Consequently, to find π for problem (3.3.11) it suffices to consider all vectors of the form π_ε for all positive ε such that requirement (3.3.16) is satisfied. Inserting (3.3.17) into the left-hand side of inequality (3.3.16) results in

$$(b - r\underline{1} - \sigma\nu)'\pi T = \sigma'^{-1}\nu(b - r\underline{1} - \sigma\nu)T + \varepsilon\|(b - r\underline{1} - \sigma\nu)\sigma^{-1}\|T \tag{3.3.18}$$

which is an increasing function in ε equalling $\sigma'^{-1}\nu(b - r\underline{1} - \sigma\nu)T$ in $\varepsilon = 0$. Therefore we obtain the solution of problem (3.3.11) by determining the largest positive ε such that (3.3.16) is still valid. But the right hand side of (3.3.18) stays above the right hand side of (3.3.16) until their largest point of intersection which is given by

$$\varepsilon^* = \theta + \frac{1}{\sqrt{T}}\left(z_\alpha + \sqrt{(z_\alpha + \theta\sqrt{T})^2 + 2T(\sigma'^{-1}\nu(b - r1) + r - \mu)}\right). \tag{3.3.19}$$

$\varepsilon^* > 0$, since, if $\sigma'^{-1}\nu(b-r1)+r-\mu < 0$, we have $z_\alpha \geq \left(\sqrt{2(\sigma^{-1}\nu(b - r\underline{1}) - r + \mu)} - \theta\right)\sqrt{T}$ by (3.3.13) and hence $\frac{z_\alpha}{\sqrt{T}} + \theta > 0$. The remaining assertion (3.3.15) can be verified by inserting π^* into equation (3.3.10). □

Example 3.3.5 Figure 3.11 shows the dependence of the shortfall probability on the time horizon. Note that the behaviour of the shortfall probability depends essentially on the choice of the stock parameters relative to the benchmark parameters. For $d = 1$ and $\pi = 1$ the factor A in Proposition 3.3.2 reduces to $\frac{1}{2}(\sigma^2 - \nu^2) + \mu - b$. Whether the shortfall probability is increasing or decreasing depends on the sign of A. For $b = 0.7$ and $\sigma = 0.15$ the factor A is positive and hence the shortfall probability is increasing and converges to 1 for large time horizons T. For $b = 0.15$ and $\sigma = 0.25$ A is negative and so the shortfall probability is decreasing and tends to 0 if the time horizon T tends to ∞. In Figure 3.12 we have plotted the expected terminal wealth as a function of the time horizon for the optimal, the pure stock, and the pure bond strategy for $b = 0.15$ and $\sigma = 0.25$. Even the pure stock investment leads to a lower expected terminal wealth than the optimal portfolio. The reason for this can be seen in Figure 3.13, which illustrates the optimal portfolio with varying time horizon corresponding to the pure stock strategy as a function of the time horizon. The optimal portfolio always contains a short position in the bond as long as this is allowed by the shortfall probability constraint. In Figure 3.14 we have plotted the mean probability efficient frontier for the above parameters and a fixed time horizon $T = 5$. As expected it has a similar form as a typical mean-variance efficient frontier and as the mean-CaR and the mean-CaR^S efficient frontiers.

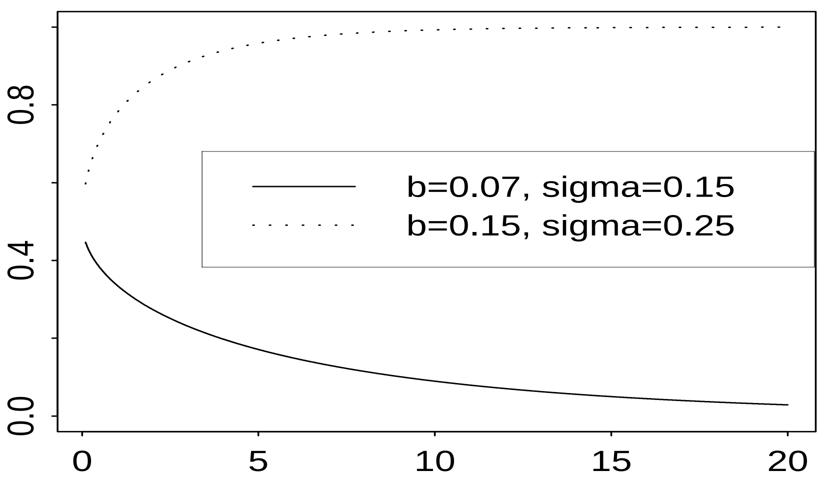


Figure 3.11: Shortfall probability of the pure stock portfolio (one risky asset only) for different appreciation rates b and volatilities σ as a function of the planning horizon T; $0 < T \leq 20$. The volatility of the benchmark is $\nu = 0.2$, its appreciation rate $\mu = 0.1$.

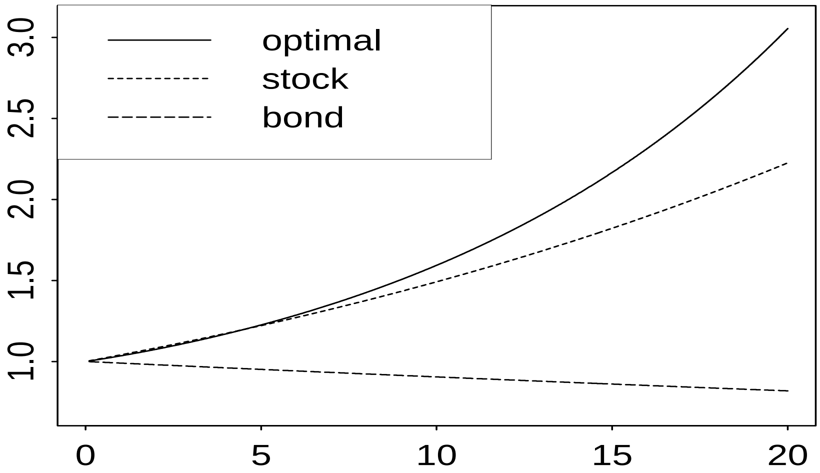


Figure 3.12: Expected ratio of the terminal wealth of the portfolio and the benchmark for different investment strategies depending on the time horizon T, $0 < T \leq 20$. The parameters are $d = 1$, $r = 0.05$, $\mu = 0.1$, $\nu = 0.2$, $b = 0.15$, $\sigma = 0.25$. As upper bound for the shortfall probability we have chosen $\alpha = 0.05$.

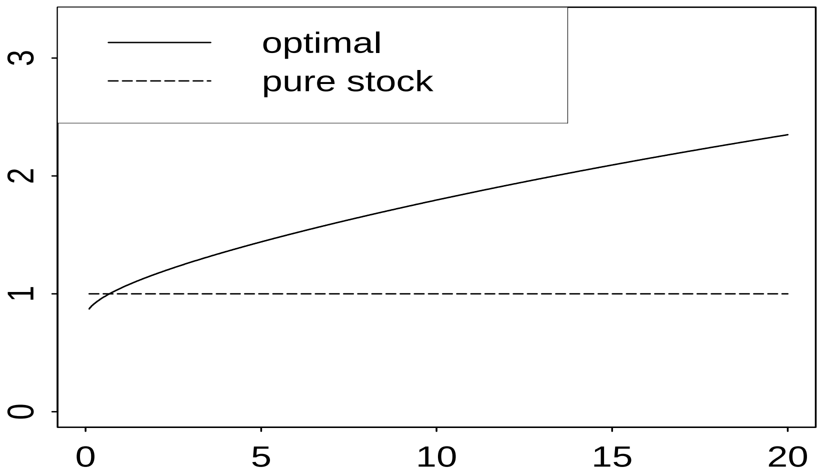


Figure 3.13: For the same parameters as in Figure 3.12 the figure shows the optimal portfolio and the pure stock portfolio.

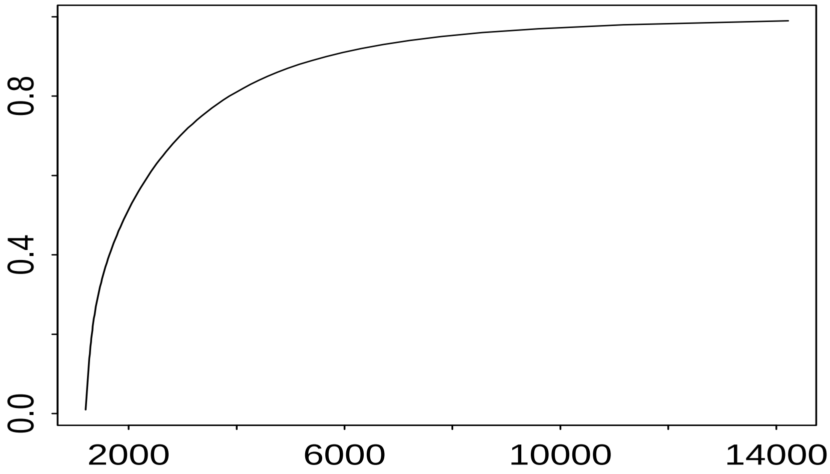


Figure 3.14: Mean shortfall probability efficient frontier with the mean on the horizontal axis and the shortfall probability on the vertical axis. The parameters are the same as in Figure 3.12 the planning horizon is $T = 5$.

Chapter 4

Optimal portfolios when stock prices follow an exponential Lévy process

It is well-known that the normal distribution does not describe the behaviour of asset returns in a very realistic way. One reason for this is that the distribution of real data is often leptokurtic, i.e. it exhibits more small values than a normal law and has often semi-heavy tails, in other words its curtosis is higher than the curtosis of the normal distribution. Eberlein and Keller (1995) showed for instance the fit of the generalized hyperbolic distribution to financial data in a very convincing way. Normal mixture models like the normal inverse Gaussian and the variance gamma model play an increasing role also in the financial industry. Consequently, to replace in the classical geometric Brownian motion the Wiener process by some general Lévy process is an important improvement of the classical Black-Scholes model.

Also certain changes to the classical Markowitz approach are called for. The traditional risk measure has been the variance; however, it does not capture high risk sufficiently. This has also been acknowledged by the regulatory authorities and financial institutions: the Value-at-Risk (VaR) has been accepted as benchmark risk measure. The VaR is a low quantile (typically the 5% or 1% quantile) of the profit-loss-distribution of a portfolio; see e.g. Jorion (2000) for a textbook treatment.

Another deficiency of the variance is the well-known fact that the variance as a risk measure is for exponential Lévy processes increasing with the time horizon. This is in contrast to the common wisdom of asset managers that in the long run stock investment leads to an almost sure gain over riskless bond investment and hence the larger the planning horizon, the greater should be the investment in risky stocks. For this reason we also concentrate on portfolio optimization under a timely and more realistic risk constraint based on the VaR. We replace the variance by the Capital-at-Risk (CaR) which is defined via the Value-at-Risk (VaR). We define the CaR as the difference between the riskless wealth and the VaR. We think of the CaR as the capital reserve in equity; see Emmer, Klüppelberg, and Korn (2001).

In this paper we investigate some portfolio optimization problems, when the price processes are governed by exponential Lévy processes. This large class of stochastic processes includes the Brownian motion, but also processes with jumps. We explain some basic theory of Lévy processes and refer to Bertoin (1996), Protter (1990) and, in particular, Sato (1999) for relevant background.

Each infinitely divisible distribution function F on $\mathbb{R}^d$ generates a d-dimensional Lévy process L by choosing F as distribution function of $L(1)$. This can be seen immediately, since the characteristic function of $L(t)$ is for each $t > 0$ given by

$$E\exp(isL(t)) = \exp(t\Psi(s)), \quad s \in \mathbb{R}^d, \tag{4.0.1}$$

where Ψ has Lévy-Khintchine representation

$$\Psi(s) = ia_L's - \frac{1}{2}s'\beta_L'\beta_L s + \int_{-\infty}^{\infty}(e^{is'x} - 1 - is'x1_{\{|x|\le 1\}})\nu_L(dx)\,, \tag{4.0.2}$$

with $a_L \in \mathbb{R}^d$, $\beta_L'\beta_L$ is a non-negative definite symmetric $d\times d$-matrix, and ν_L is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2\wedge 1)\nu(dx) < \infty$. The term corresponding to $xI_{\{|x|\le 1\}}$ represents a centering without which the integral may not converge, i.e. $\int_{-\infty}^{\infty}(e^{is'x} - 1)\nu_L(dx)$ may not be finite. The characteristic triplet $(a_L, \beta_L'\beta_L, \nu_L)$ characterizes the Lévy process. We often write $(a, \beta'\beta, \nu)$ instead of $(a_L, \beta_L'\beta_L, \nu_L)$, if it is clear which Lévy process is concerned.

According to Sato (1999), Chapter 4, the following holds: for each ω in the probability space, define $\Delta L(t,\omega) = L(t,\omega) - L(t-,\omega)$. For each Borel set $B \subset [0,\infty)\times\mathbb{R}^{d*}$ ($\mathbb{R}^{d*} = \mathbb{R}^d \setminus \{0\}$) set

$$M(B,\omega) = \#\{(t, \Delta L(t,\omega)) \in B\}\,. \tag{4.0.3}$$

Lévy's theory says that M is a Poisson random measure with intensity

$$m(dt, dx) = dt\nu(dx)\,, \tag{4.0.4}$$

where ν is the Lévy measure of the process L. Notice that m is σ-finite and $M(B,\cdot) = \infty$ a.s. when $m(B) = \infty$.

Take $B = [a,b] \times A$, $0 \le a < b < \infty$, A a Borel set in $\mathbb{R}^{d*}$ then

$$M(B,\omega) = \#\{(t, \Delta L(t,\omega)) : a \le t \le b, \Delta L(t,\omega) \in A\}$$

counts jumps of size in A which happen in the time interval $[a,b]$. According to the above, this is a Poisson random variable with mean $(b-a)\nu(A)$.

With this notation, the Lévy-Khintchine representation corresponds to the representation

$$\begin{aligned} L(t) &= at + \beta W(t) + \sum_{0<s\le t} \Delta L(s) 1_{\{|\Delta L(s)|>1\}} \\ &\quad + \int_0^t \int_{|x|\le 1} x(M(dx,ds) - \nu(dx)ds)\,, t \ge 0\,. \end{aligned} \tag{4.0.5}$$

This means that $L(t)$ has a Brownian component $\beta W(t)$ and a pure jump part with Lévy measure ν, having the interpretation that a jump of size x occurs at rate $\nu(dx)$. This representation reduces in the finite variation case to

$$L(t) = \gamma t + \beta W(t) + \sum_{0<s\le t} \Delta L(s)\,, \quad t \ge 0\,,$$

where $\gamma = a - \int_{|x|\le 1} x\nu(dx)$; i.e. $L(t)$ is the independent sum of a drift term, a Brownian component and a pure jump part.

The paper is organized as follows. In Section 2 we introduce the Lévy Black-Scholes model and calculate the terminal wealth of a portfolio and its moments provided they exist. In Section 3 we use these results for a portfolio optimization that consists of maximizing the expected terminal wealth of a portfolio under some constraint on the variance. We show different examples and demonstrate the solutions in various plots. In Section 4 we introduce the CaR, which is defined via a low quantile of the wealth process, and discuss methods for its calculation and approximation. In Section 5 we optimize portfolios where we replace the variance by the CaR. We work out real life examples as the normal inverse Gaussian and variance gamma model. Here we do not obtain closed form analytic solutions, but solve the optimization problem by numerical algorithms. We also compare the optimal portfolios for the mean-variance and the mean-CaR criterion. Section 6 is devoted to the proof of the weak limit theorem which we need for the approximation of the quantile of the wealth process. It involves some new results on the stochastic exponential of a Lévy process.

4.1 The market model

We consider a standard Black-Scholes type market consisting of a *riskless bond* and several *risky stocks*, which follow exponential Lévy processes. Their respective prices $(P_0(t))_{t\geq 0}$ and $(P_i(t))_{t\geq 0}$, $i = 1, \ldots, d$, evolve according to the equations

$$P_0(t) = e^{rt} \quad \text{and} \quad P_i(t) = p_i \exp(b_i t + \sum_{j=1}^{d} \sigma_{ij} L_j(t)), \quad t \geq 0. \tag{4.1.1}$$

Here $(L(t))_{t\geq 0} = (L_1(t), \ldots, L_d(t))_{t\geq 0}$ is a d-dimensional Lévy process (stationary independent increments with cadlag sample paths). We assume the $L_i, i = 1, \ldots, d$, to be independent. L has characteristic triplet $(a, \beta'\beta, \nu)$, where $a \in \mathbb{R}^d$, β is an arbitrary d-dimensional diagonal matrix. We introduce β as a diagonal matrix into the model to allow for some extra flexibility apart from the $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$. This also includes the possibility of a pure jump process (for $\beta_i = 0$). Since the components of βW are independent Wiener processes with different variances possible, we allow for different scaling factors for the Wiener process and the non-Gaussian components. By the independence of the components we obtain for the Lévy measure ν of L and a d-dimensional rectangle $A = \times_{i=1}^{d}(a_i, b_i] \subset \mathbb{R}^d$ that $\nu(A) = \sum_{i=1}^{d} \nu_i(a_i, b_i]$, where ν_i is the Lévy measure of L_i for $i = 1, \ldots, d$; i.e. the Lévy measure is supported on the union of the coordinate axes (see Sato (1999), E12.10, p. 67). Thus the probability that two components have a jump at the same time point is zero; i.e. jumps of different components occur a.s. at different times.

The quantity $r \in \mathbb{R}$ is the *riskless interest rate* and $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ is an invertible matrix, $b \in \mathbb{R}^d$ can be chosen such that each stock has the desired appreciation rate. Since the assets are on the same market, they show some dependence structure which we model by a linear combination of the same Lévy processes $L_1, \ldots, L_d$ for each asset price. This means the dependence structure of the market is the same as that of the Black-Scholes market in Emmer, Klüppelberg, and Korn (2001).

We need the corresponding SDE in order to derive the wealth process. By Itô's formula, P_i is the solution to the SDE

$$\begin{aligned} dP_i(t) &= P_i(t-)(b_i dt + d\widehat{L}_i(t)) \\ &= P_i(t-)\left(\left(b_i + \frac{1}{2}\sum_{j=1}^d (\sigma_{ij}\beta_{jj})^2\right) dt + \sum_{j=1}^d \sigma_{ij} dL_j(t)\right. \\ &\quad \left. + \exp(\sum_{j=1}^d \sigma_{ij}\Delta L_j(t)) - 1 - \sum_{j=1}^d \sigma_{ij}\Delta L_j(t)\right), \quad t > 0, \quad P_i(0) = p_i, \end{aligned} \tag{4.1.2}$$

i.e. $\widehat{L}_i$ is such that $\exp(\sum_{j=1}^d \sigma_{ij} L_j(t)) = \mathcal{E}(\widehat{L}_i)$, where $\mathcal{E}$ denotes the stochastic exponential of a process (see Protter (1990) for background on stochastic analysis).

Remark 4.1.1 We see that the formulae show some similarity to the classical Black-Scholes model, in particular we have an additional Itô term in the drift component. But we also recognize a big difference to the Black-Scholes model. First of all, it has jumps. The jumps of $\widehat{L}_i$ occur at the same time as those of $(\sigma L)_i = \sum_{j=1}^d \sigma_{ij} L_j$, but they have another size. A jump of size $\sum_{j=1}^d \sigma_{ij}\Delta L_j$ is replaced by one of size $\exp(\sum_{j=1}^d \sigma_{ij}\Delta L_j) - 1$ leading to the term $\exp(\sum_{j=1}^d \sigma_{ij}\Delta L_j) - 1 - \sum_{j=1}^d \sigma_{ij}\Delta L_j$ in formula (4.1.2), whereas the Brownian component remains the same as in $(\sigma L)_i$.

The following Lemma describes the relation between the characteristic triplets of a Lévy process and its stochastic exponential, which we need in the sequel.

Lemma 4.1.2 (Goll and Kallsen (2000))
If L is a real-valued Lévy process with characteristic triplet (a, β^2, ν), then also $\widehat{L}$ defined by $e^L = \mathcal{E}(\widehat{L})$ is a Lévy process with characteristic triplet $(\widehat{a}, \widehat{\beta}^2, \widehat{\nu})$ given by

$$\begin{aligned} \widehat{a} - a &= \frac{1}{2}\beta^2 + \int ((e^x - 1)1_{\{(|e^x-1|<1\}} - x1_{\{|x|<1\}})\nu(dx) \\ \widehat{\beta}^2 &= \beta^2 \\ \widehat{\nu}(\Lambda) &= \nu(\{x | e^x - 1 \in \Lambda\}) \textit{ for any Borel set } \Lambda \subset \mathbb{R}^* . \end{aligned} \tag{4.1.3}$$

In the following Lemma the relation between the characteristic triplets of a d-dimensional Lévy process L and its linear transformation $\pi' L$ is given for $\pi \in \mathbb{R}^d$.

Lemma 4.1.3 (Sato (1999), Prop. 11.10)
If L is a d-dimensional Lévy process with characteristic triplet $(a, \beta'\beta, \nu)$, then $\pi' L$ is for $\pi \in \mathbb{R}^d$ a one-dimensional Lévy process with characteristic triplet $(a_\pi, \beta_\pi^2, \nu_\pi)$ given by

$$\begin{aligned} a_\pi &= \pi' a + \int \pi x' (1_{\{|\pi' x|<1\}} - 1_{\{|x|<1\}}) \nu(dx) \\ \beta_\pi^2 &= \|\pi'\beta\|^2 \\ \nu_\pi(\Lambda) &= \nu(\{x | \pi' x \in \Lambda\}) \text{ for any Borel set } \Lambda \subset \mathbb{R}^* . \end{aligned} \tag{4.1.4}$$

Let $\pi(t) = (\pi_1(t) \dots \pi_d(t))' \in \mathbb{R}^d$ be an admissible portfolio process, i.e. $\pi(t)$ is the fraction of the wealth $X^\pi(t)$, which is invested in asset i (see Korn (1997), Section 2.1 for relevant definitions). The fraction of the investment in the bond is $\pi_0(t) = 1 - \pi(t)\underline{1}$, where $\underline{1} = (1, \dots, 1)'$ denotes the vector (of appropriate dimension) having unit components. Throughout the paper, we restrict ourselves to constant portfolios; i.e. $\pi(t) = \pi$, $t \in [0, T]$, for some fixed *planning horizon* T. This means that the fractions in the different stocks and the bond remain constant on $[0, T]$. The advantages of this restriction are discussed in Emmer, Klüppelberg, and Korn (2001) and Sections 3.3 and 3.4 of Korn (1997). In order to avoid negative wealth we require that $\pi \in [0, 1]^d$, hence shortselling is not allowed in this model. We also require $\pi'\underline{1} \le 1$; see Remark 4.1.4 below.

We want to indicate that it is at least not obvious how to derive a dynamic portfolio optimization strategy. Schweizer (1984) determines a dynamic optimal portfolio for some mean-variance optimization using a utility optimization approach. Kallsen (2000) optimizes portfolios for exponential Lévy processes for different utility functions and obtains constant optimal portfolios for power and logarithmic utility, but not for exponential utility. By Fishburn (1977) and Harlow (1991), however, the mean-CaR optimization cannot be solved using utility functions, since the corresponding utility function is not concave.

Denoting by $(X^\pi(t))_{t \ge 0}$ the *wealth process*, it follows the dynamic

$$dX^\pi(t) = X^\pi(t-) \left(((1 - \pi'\underline{1})r + \pi' b)dt + \pi' d\widehat{L}(t) \right) , \quad t > 0 , \quad X^\pi(0) = x ,$$

where $x \in \mathbb{R}$ denotes the *initial capital* of the investor. Using Itô's formula, this SDE has solution

$$\begin{aligned} X^\pi(t) &= x\exp(t(r+\pi'(b-r\underline{1})))\mathcal{E}(\pi'\widehat{L}(t)) && (4.1.5)\\ &= x\exp(a_X t+\pi'\sigma\beta W(t))\widetilde{X}^\pi(t)\,, \quad t\geq 0\,, && (4.1.6) \end{aligned}$$

where a_X is as in Lemma 4.1.5 and

$$\begin{aligned} \ln \widetilde{X}^\pi(t) &= \int_0^t\int_{\mathbb{R}^d} \ln(1+\pi'(e^{\sigma x}-\underline{1}))1_{\{|\ln(1+\pi'(e^{\sigma x}-\underline{1}))|>1\}}M_L(ds,dx)\\ &+\int_0^t\int_{\mathbb{R}^d} \ln(1+\pi'(e^{\sigma x}-\underline{1}))1_{\{|\ln(1+\pi'(e^{\sigma x}-\underline{1}))|\leq 1\}}(M_L(ds,dx)-ds\nu_L(dx))\,, \quad t\geq 0\,. \end{aligned}$$

Remark 4.1.4 Note that a jump $\Delta L(t)$ of L leads to a jump $\Delta\ln X^\pi(t)$ of $\ln X^\pi$ of size $\ln(1+\pi'(e^{\sigma\Delta L(t)}-\underline{1}))$ and hence $\Delta\ln X^\pi(t)>\ln(1-\pi'\underline{1})$, hence $\pi'\underline{1}\leq 1$.

The wealth process is again an exponential Lévy process. We calculate the characteristic triplet of its logarithm in the following Lemma.

Lemma 4.1.5 *Consider model (4.1.1) with Lévy process L and characteristic triplet $(a,\beta'\beta,\nu)$. Define for the $d\times d$-matrix $\sigma\beta$ the vector $[\sigma\beta]^2$ with components*

$$[\sigma\beta]_i^2=\sum_{j=1}^d(\sigma_{ij}\beta_{jj})^2, \quad i=1,\dots,d.$$

The process $\ln(X^\pi/x)$ is a Lévy process with characteristic triplet (a_X,β_X^2,ν_X) given by

$$\begin{aligned} a_X &= r+\pi'(b+[\sigma\beta]^2/2-r\underline{1}+\sigma a)-\|\pi'\sigma\beta\|^2/2\\ &\quad+\int(\ln(1+\pi'(e^{\sigma x}-1))1_{\{|\ln(1+\pi(e^{\sigma x}-1))|\leq 1\}}-\pi'\sigma x1_{\{|x|\leq 1\}})\nu(dx)\,,\\ \beta_X^2 &= \|\pi'\sigma\beta\|^2\,,\\ \nu_X(A) &= \nu(\{x|\ln(1+\pi(e^{\sigma x}-1))\in A\}) \text{ for any Borel set } A\subset\mathbb{R}^*\,. \end{aligned}$$

Proof The calculation of the characteristic triplet of $\mathcal{E}(\pi'\widehat{L}(t))$ is an application of Lemma 4.1.2 and Lemma 4.1.3. Then we obtain the characteristic triplet (a_X,β_X^2,ν_X) by equation (4.1.5). □

For the calculation of moments of the wealth process we need the existence of the moment generating function in some neighbourhood of 0. This corresponds to an analytic extension of the characteristic function. The following lemma gives some condition when this is possible.

Lemma 4.1.6 (Sato (1999), Theorem 25.17)
Let $(X(t))_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic triplet $(a,\beta'\beta,\nu)$. Let

$$C=\{c\in\mathbb{R}^d:\int_{|x|>1}e^{c'x}\nu(dx)<\infty\}.$$

(a) The set C is convex and contains the origin.
(b) $c\in C$ if and only if $Ee^{c'X(t)}<\infty$ for some $t>0$ or, equivalently, for every $t>0$.
(c) If $w\in\mathbb{C}^d$ is such that $\operatorname{Re} w\in C$, then

$$\Psi(w)=a'w+\frac{1}{2}w'\beta'\beta w+\int_{\mathbb{R}^d}(e^{w'x}-1-w'x1_{\{|x|<1\}})\nu(dx)$$

is definable, $E|e^{w'X(t)}|<\infty$, and $E[e^{w'X(t)}]=e^{t\Psi(w)}$.

Extending the characteristic function of $\ln X^\pi(t)$ on $\mathbb{C}$ as in Lemma 4.1.6 we obtain for all $k\in\mathbb{N}$, such that the k-th moment exists,

$$E[(X^\pi(t))^k] \quad=\quad x^k\exp((ka_X+k^2\beta_X^2/2)T)E[(\widetilde{X}^\pi(t))^k],\quad t\geq 0$$

and

$$E[(\widetilde{X}^\pi(t))^k] \quad=\quad \exp(\widetilde{\mu}_k t)\,,\quad t\geq 0\,, \tag{4.1.7}$$

where

$$\widetilde{\mu}_k=\int_{\mathbb{R}^d}\big((1+\pi'(e^{\sigma x}-\underline{1}))^k-1-k\ln(1+\pi'(e^{\sigma x}-\underline{1}))1_{\{|\ln(1+\pi'(e^{\sigma x}-\underline{1}))|\leq 1\}}\big)\,\nu(dx)$$

and ν is the Lévy measure of L. In particular,

$$E[\widetilde{X}^\pi(t)]=\exp\left(t\int_{\mathbb{R}^d}(\pi'(e^{\sigma x}-\underline{1})-\ln(1+\pi'(e^{\sigma x}-\underline{1}))1_{\{|\ln(1+\pi'(e^{\sigma x}-\underline{1}))|\leq 1\}})\nu(dx)\right)\,,t\geq 0\,.$$

Proposition 4.1.7 *Assume in the situation of equation (4.1.1) that $L(1)$ has moment generating function $\widehat{f}(s) = E\exp(s'L(1))$ such that $\widehat{f}(e_i'\sigma) < \infty$ for $i = 1, \ldots, d$, where e_i is the i-th d-dimensional unit vector. Let $X^\pi(t)$ be as in equation (4.1.5). Then*

$$E\left[X^\pi(t)\right] = x\exp(t(r + \pi'(b - r\underline{1} + \ln\widehat{f}(\sigma)))), \quad t \geq 0, \tag{4.1.8}$$

and

$$\begin{aligned}\operatorname{var}\left[X^\pi(t)\right] &= x^2\exp(2t(r + \pi'(b - r\underline{1} + \ln\widehat{f}(\sigma)))) \\ &\quad \times (\exp(t\pi' A\pi) - 1), \quad t \geq 0,\end{aligned} \tag{4.1.9}$$

where $\ln\widehat{f}(\sigma) = (\ln\widehat{f}(e_1'\sigma), \ldots, \ln\widehat{f}(e_d'\sigma))'$ *and* $A = (A_{ij})_{1\leq i,j\leq d}$ *with*

$$A_{ij} = \ln\widehat{f}((e_i + e_j)'\sigma) - \ln\widehat{f}(e_i'\sigma) - \ln\widehat{f}(e_j'\sigma), \quad 1 \leq i, j \leq d.$$

Proof Recall that $(a, \beta'\beta, \nu)$ is the characteristic triplet of L. By equation (4.1.7) and Lemma 4.1.5 we obtain for $t \geq 0$:

$$\begin{aligned}&E\left[X^\pi(t)\right] = \\ &\quad x\exp(t(r + \pi'(b - r\underline{1} + \frac{1}{2}[\sigma\beta]^2 + \sigma a + \int\limits_{\mathbb{R}^d}(e^{\sigma x} - 1 - \sigma x 1_{\{|x|<1\}})\nu(dx))))\end{aligned} \tag{4.1.10}$$

$$\begin{aligned}\operatorname{var}(X^\pi(t)) &= x^2\exp\left(2t(r + \pi'(b - r\underline{1} + \frac{1}{2}[\sigma\beta]^2 + \sigma a + \int_{\mathbb{R}^d}(e^{\sigma x} - 1 - \sigma x 1_{\{|x|<1\}})\nu(dx)))\right) \\ &\quad \times \left(\exp\left(t(\|\pi'\sigma\beta\|^2 + \int_{\mathbb{R}^d}(\pi'(e^{\sigma x} - \underline{1}))^2\nu(dx))\right) - 1\right),\end{aligned} \tag{4.1.11}$$

On the other hand we calculate

$$\widehat{f}(e_i'\sigma) = E\exp(e_i'\sigma L(1)) = \exp\left((\sigma a + [\sigma\beta]^2/2 + \int(e^{\sigma x} - 1 - \sigma x 1_{\{|x|<1\}})\nu(dx))_i\right)$$

and

$$\pi' A\pi = \|\pi'\sigma\beta\|^2 + \int_{\mathbb{R}^d}(\pi'(e^{\sigma x} - \underline{1}))^2\nu(dx)).$$

Plugging this into (4.1.10) and (4.1.11) we obtain (4.1.8) and (4.1.9). □

Remark 4.1.8 Note that for $l = 1, \ldots, d\,(i = \sqrt{-1})$

$$\ln \widehat{f}(e_l'\sigma) = \ln(E\exp(\sum_{j=1}^{d} \sigma_{lj} L_j(1))) = \sum_{j=1}^{d} \ln \widehat{f}_j(\sigma_{lj}) = \ln(E[\mathcal{E}(\widehat{L}_l)(1)]) = \sum_{j=1}^{d} \Psi_j(-i\sigma_{lj})$$

by the independence of $L_1, \ldots, L_d$. This implies in particular

$$E\mathcal{E}(\pi'\widehat{L}(t)) = \prod_{l=1}^{d} (E[\mathcal{E}(\widehat{L}_l(t))])^{\pi_l}.$$

Remark 4.1.9 For $d = 1$ our portfolio consists of one bond and one stock only.

(a) Formula (4.1.9) reduces to

$$\begin{aligned} \operatorname{var}(X^\pi(t)) &= x^2 \exp(2t(r + \pi(b - r + \ln(\widehat{f}(\sigma))))) \\ &\quad \times \left(\exp(\pi^2 t(\ln(\widehat{f}(2\sigma)) - 2\ln(\widehat{f}(\sigma)))) - 1\right). \end{aligned} \tag{4.1.12}$$

Moreover, we can set w.l.o.g. $\sigma = 1$. In this case the Lévy density f_X of the process $\ln X^\pi$ can be calculated from the Lévy density f_L of ν_L as

$$f_X(x) = f_L\left(\ln\left(\frac{e^x - 1}{\pi} + 1\right)\right) \frac{e^x}{e^x - (1-\pi)} 1_{\{x > \ln(1-\pi)\}}, \quad x \in \mathbb{R}.$$

(b) In the case of a jump part of finite variation we obtain for $t \geq 0$,

$$E\left[X^\pi(t)\right] = x\exp(t(r + \pi(b - r + \frac{1}{2}\beta^2 + \gamma + \widehat{\mu}))), \tag{4.1.13}$$

$$\begin{aligned} \operatorname{var}\left(X^\pi(t)\right) &= x^2 \exp\left(2t(r + \pi(b - r + \gamma + \widehat{\mu} + \frac{1}{2}\beta^2))\right) \\ &\quad \times \left(\exp\left(\pi^2 t(\beta^2 + \widehat{\mu}_2 - 2\widehat{\mu})\right) - 1\right), \end{aligned} \tag{4.1.14}$$

for $\widehat{\mu} = \int (e^x - 1)\nu(dx)$, $\widehat{\mu}_2 = \int (e^{2x} - 1)\nu(dx)$, and $\gamma = a - \int_{|x|<1} x\nu(dx)$.

4.2 Optimal portfolios under variance constraints

In this section we consider the following optimization problem using the variance as risk measure

$$\max_{\{\pi\in[0,1]^d|\pi'\underline{1}\leq 1\}} E[X^\pi(T)] \quad \text{subject to} \quad \operatorname{var}(X^\pi(T)) \leq C\,, \tag{4.2.1}$$

where T is some given planning horizon and C is a given bound for the risk.

Theorem 4.2.1 *Let L be a Lévy process with representation (4.0.5). Then the optimal solution of problem (4.2.1) is given by*

$$\pi^* = \varepsilon^*(A^{-1}(b - r\underline{1} + \ln\widehat{f}(\sigma)))/\sqrt{\widetilde{a}} \tag{4.2.2}$$

where

$$\widetilde{a} = (b - r\underline{1} + \ln\widehat{f}(\sigma))'A^{-1}(b - r\underline{1} + \ln\widehat{f}(\sigma))$$

(provided $\pi^* \in [0,1]^d$ and $\pi^*\underline{1} \leq 1$) where $\ln\widehat{f}(\sigma)$ and the matrix A are defined in Proposition 4.1.7 and ε^* is the unique positive solution of

$$rT + \sqrt{\widetilde{a}}\varepsilon T + \frac{1}{2}\ln\left(\frac{x^2}{C}\left(\exp(T\varepsilon^2) - 1\right)\right) = 0\,. \tag{4.2.3}$$

Remark 4.2.2 If the solution to (4.2.3) does not satisfy $\pi^* \in [0,1]^d$ and $\pi^*\underline{1} \leq 1$, then the problem can be solved by the Lagrange method using some numerical optimization algorithm, for example the SQP method (sequential quadratic programming) (see e.g. Nocedal and Wright (1999) and Boggs and Tolle (1995)). If for $d = 1$ the solution of (4.2.3) leads to $\pi^* > 1$, the optimal $\pi^* = 1$.

Proof of Theorem 4.2.1. Following the proof of Proposition 2.9 of Emmer, Klüppelberg, and Korn (2001), where the same optimization problem has been solved for geometric Brownian motion, we obtain (4.2.2) as the portfolio with the highest terminal wealth over all portfolios satisfying $\pi'A\pi = \varepsilon^2$. Plugging (4.2.2) into the explicit form (4.1.11) of the variance of the terminal wealth the constraint has the same form as in Proposition 2.9 of Emmer, Klüppelberg, and Korn (2001). Hence the result follows from a comparison of constants. The only difference to the optimization problem in Emmer, Klüppelberg, and Korn (2001) is the constraint $\pi^* \in [0,1]^d$ and $\pi^*\underline{1} \leq 1$, which we took care of. $\square$

Remark 4.2.3 In the finite variation case and for $d = 1$ where we choose w.l.o.g. $\sigma = 1$ (4.2.3) can be rewritten in the form

$$rT + \pi\left(b - r + \gamma + \widehat{\mu} + \frac{1}{2}\beta^2\right)T + \frac{1}{2}\ln\left(\frac{x^2}{C}\left(\exp\left(\pi^2\left(\beta^2 + \widehat{\mu}_2 - 2\widehat{\mu}\right)T\right) - 1\right)\right) = 0$$

with $\widehat{\mu}$, $\widehat{\mu}_2$ and γ as in Remark 4.1.9(b).

Remark 4.2.4 One can also start with a general d-dimensional Lévy process L_g with arbitrary characteristic triplet (a_g, c_g, ν_g) and consider the model

$$P_i(t) = p_i \exp((L_g)_i(t)), \quad t \geq 0.$$

Using Lemmata 4.1.2 and 4.1.3 the characteristic triplet of $\ln(X^\pi(t)/x)$ is then

$$\begin{aligned} a_{Xg} &= r + \pi(a_g - r + (c_g^\Delta - c_g\pi)/2) + \int (l(x)1_{\{|l(x)|\leq 1\}} - \pi x 1_{\{|x|\leq 1\}})\nu_g(dx) \\ c_{Xg} &= \pi' c_g \pi \\ \nu_{Xg}(\Lambda) &= \nu_g(\{x \in \mathbb{R}^d : l(x) \in \Lambda\}), \end{aligned}$$

where $c_g^\Delta = (c_{g11}, \ldots, c_{gdd})$ and $l(x) = \ln(1 + \pi'(e^x - 1))$. Then using the same argumentation as above we obtain for $t \geq 0$

$$E(X^\pi(t)) = x\exp(t(r + \pi(a_g - r\underline{1} + c_g^\Delta/2 + \int (e^x - 1 - x1_{\{|x|\leq 1\}})\nu_g(dx))))$$

and

$$\begin{aligned} \operatorname{var}(X^\pi(t)) &= x^2 \exp\left(2t(r + \pi(a_g - r\underline{1} + c_g^\Delta/2 + \int (e^x - 1 - x1_{\{|x|\leq 1\}})\nu_g(dx)))\right) \\ &\quad \times \left(\exp\left(t(\pi' c_g \pi + \int (\pi'(e^x - 1))^2 \nu_g(dx))\right) - 1\right). \end{aligned}$$

The optimal portfolio for the mean-variance optimization is then

$$\pi^* = \varepsilon^* \overline{c}^{-1}\overline{a}/\sqrt{\overline{a}'\overline{c}^{-1}\overline{a}},$$

with $\overline{a} := a_g - r\underline{1} + c_g^\Delta/2 + \int (e^x - 1 - x1_{\{|x|\leq 1\}})\nu_g(dx)$
and $\overline{c}_{ij} := c_{gij} + \int (e^{x_i} - 1)(e^{x_j} - 1)\nu_g(dx)$,
where ε^* is the unique positive solution of

$$2T\sqrt{\overline{ac}^{-1}\overline{a}}\varepsilon + \ln\left(\frac{x^2}{C}(e^{T\varepsilon^2} - 1)\right) = 0.$$

This solution is derived analogously to Theorem 4.2.1. We prefer, however, to work with a linear dependence structure, since it allows for nice formulae and can also be interpreted easily. It is a special case of the general model such that

$$
\begin{aligned}
a_g &= b + \sigma a + \int \sigma x (1_{\{|\sigma x| \le 1\}} - 1_{\{|x| \le 1\}}) \nu(dx \\
c_g &= (\sigma\beta)'(\sigma\beta) \\
\nu_g(\Lambda) &= \nu(\{x \in \mathbb{R}^d : \sigma x \in \Lambda\}) \, \forall \Lambda \subset \mathbb{R}^{d*}
\end{aligned}
$$

and

$$
(L_g)_i(t) = b_i t + \sum_j \sigma_{ij} L_j(t) = b_i t + (\sigma L)_i(t), \; t \ge 0.
$$

In the following we consider some examples in order to understand the influence of the jumps on the choice of the optimal portfolio. For simplicity we take $d = 1$ in these examples and hence we choose w.l.o.g. $\sigma = 1$. In the case of jumps of finite variation we choose γ such that the expected wealth processes are equal to make the results comparable. Then the influence of the jumps is shown in the risk measure, here the variance.

Example 4.2.5 *(Exponential Brownian motion with jumps)*
Let $Y_1, Y_2, \ldots$ be iid random variables with distribution p on $\mathbb{R}^*$ and $(N(t))_{t \ge 0}$ a Poisson process with intensity $c > 0$, independent of the Y_i. Then $\overline{L}(t) := \sum_{i=1}^{N(t)} Y_i$, $t \ge 0$, defines a compound Poisson process with Lévy measure $\nu(dx) = cp(dx)$. The Lévy process $(L(t))_{t \ge 0}$ is taken as the sum of a Brownian motion with drift $(\beta W(t) + \gamma t)_{t \ge 0}$, and the compound Poisson process $(\overline{L}(t))_{t \ge 0}$.

If $\widehat{g}(s) = Ee^{sY} < \infty$, then

$$
\widehat{f}(s) = E \exp(s\overline{L}(1)) = \exp(c(\widehat{g}(s) - 1)) \, .
$$

If $\widehat{g}(1)$ resp. $\widehat{g}(2)$ exists, then we obtain the corresponding $\widehat{\mu}$ resp. $\widehat{\mu}_2$ in Remark 4.1.9(b) as

$$
\widehat{\mu} = c(\widehat{g}(1) - 1) \text{ and } \widehat{\mu}_2 = c(\widehat{g}(2) - 1) \, .
$$

The drift $\gamma = -\frac{1}{2}\beta^2 - \widehat{\mu}$ is chosen such that the asset price has the same expectation as in the Black-Scholes model in Emmer, Klüppelberg, and Korn (2001), Section 2. By (4.1.13) and (4.1.14) we obtain for $t \geq 0$

$$\begin{aligned}
X^\pi(t) &= x \exp\left(t(r + \pi(b - \widehat{\mu} - r) - \frac{1}{2}\pi^2\beta^2) + \pi\beta W(t) \right) \prod_{i=1}^{N(t)} (1 + \pi(e^{Y_i} - 1)) \,, \\
E[X^\pi(t)] &= x \exp(t(r + \pi(b - r))) \,, \\
\mathrm{var}(X^\pi(t)) &= x^2 \exp(2t(r + \pi(b - r))) \left(\exp(\pi^2 t(\beta^2 + c(\widehat{g}(2) - 2\widehat{g}(1) + 1))) - 1 \right) \,.
\end{aligned}$$

The exponential compound Poisson process ($\beta = 0$) and the exponential Brownian motion ($c = 0$) are special cases of this example. Figure 4.1 shows sample paths for a jump scenario, namely possible jumps of height -0.1, i.e. a downwards jump of 10% of the Lévy process L, with intensity 2; i.e. we expect 2 jumps per year.

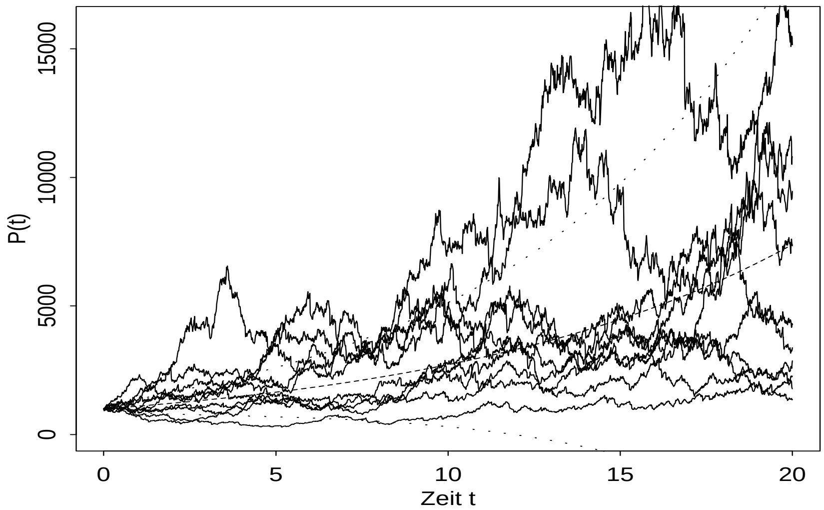


Figure 4.1: Ten sample paths of an asset in the exponential Black-Scholes model with compensated jumps of height -0.1 and intensity 2, its expectation (dashed line) and standard deviation (dotted lines). The parameters are $x = 1\,000$, $b = 0.1$ and $r = 0.05$.

Example 4.2.6 *(Exponential normal inverse Gaussian (NIG) Lévy process)*
The normal inverse Gaussian Lévy process has been introduced by Barndorff-Nielsen (1977) and investigated further in Barndorff-Nielsen and Shephard (2001). It belongs to the class of generalized hyperbolic Lévy processes. The applicability of this class of Lévy processes to finance is also discussed in Eberlein and Raible (2000). Their fit is empirically convincing; see Eberlein and Keller (1995). The normal inverse Gaussian Lévy model is a

normal variance-mean mixture model such that

$$L(t) \quad = \quad \rho + \lambda\zeta^2(t) + W(\zeta^2(t)) \,, \quad t \geq 0$$

where $\zeta^2(t) \sim IG(t^2\delta^2, \xi^2 - \lambda^2)$, W is a standard Brownian motion and $\xi \geq |\lambda| \geq 0$, $\delta > 0$, $\rho \in \mathbb{R}$. This process is uniquely determined by the distribution of the increment $L(1)$ whose density is given by

$$nig(x;\xi,\lambda,\rho,\delta) := \frac{\xi}{\pi}\exp\left(\delta\sqrt{\xi^2-\lambda^2}+\lambda(x-\rho)\right)\frac{K_1(\delta\xi g(x-\rho))}{g(x-\rho)}\,, \quad x \in \mathbb{R}\,,$$

where $g(x) = \sqrt{\delta^2 + x^2}$ and $K_1(x) = \frac{1}{2}\int_0^\infty \exp(-x(y+y^{-1})/2)dy$, $x > 0$, is the modified Bessel function of the third kind of order one. Note that for $s > 0$ the density of $L(t+s) - L(t)$, $t \geq 0$, is given by $nig(x,\xi,\lambda,s\rho,s\delta)$. The parameter ξ is a steepness parameter, i.e. for larger ξ we get less large and small jumps and more jumps of middle size, δ is a scale parameter, λ is a symmetry parameter and ρ a location parameter. For $\rho = 0$ and $\lambda = 0$ (symmetry around 0) the characteristic triplet $(0,0,\nu)$ of a NIG Lévy process is given by

$$\nu(dx) = \frac{\delta\xi}{\pi}|x|^{-1}K_1(\xi|x|)dx\,, \quad x \in \mathbb{R}^*\,.$$

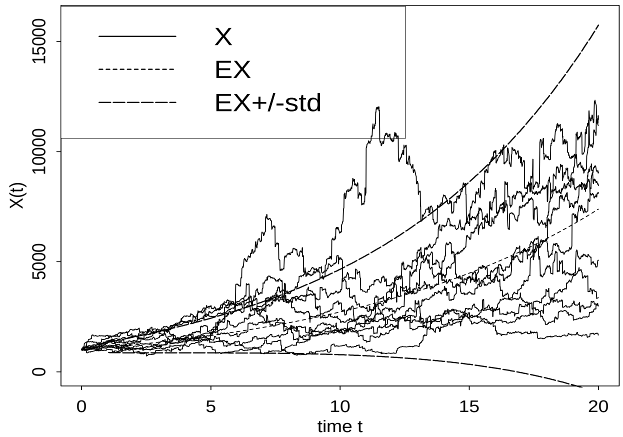
Since $\int_{|x|\leq 1}|x|\nu(dx) = \infty$ the sample paths of L are a.s. of infinite variation in any finite interval. The moment generating function of $L(1)$ is for the NIG distribution given by

$$\widehat{f}(s) = E\exp(sL(1)) = \exp(\delta(\xi - \sqrt{\xi^2 - s^2}))\,,$$

(see e.g. Raible (2000), Example 1.6). We use (4.1.5), (4.1.8), and (4.1.9) to obtain for $t \geq 0$

$$\begin{aligned}
X^\pi(t) &= x\exp(t(r+\pi(b-r)))\mathcal{E}(\pi\widehat{L}(t))\,, \\
E\left[X^\pi(t)\right] &= x\exp(t(r+\pi(b-r+\delta(\xi-\sqrt{\xi^2-1}))))\,, \\
\operatorname{var}(X^\pi(t)) &= x^2\exp(2t(r+\pi(b-r+\delta(\xi-\sqrt{\xi^2-1})))) \\
&\quad \times\left(\exp\left(\pi^2 t(2\sqrt{\xi^2-1}-\xi-\sqrt{\xi^2-4})\delta\right)-1\right)\,.
\end{aligned}$$

To obtain the same expected wealth as in Example 4.2.5 we have to choose b such that $b = b_{BS} - \delta(\xi - \sqrt{\xi^2-1})$, where b_{BS} is b as chosen in Example 4.2.5. Figures 4.2 show sample paths for a geometric NIG-Lévy process with certain parameter values. For comparison to the CaR-optimization the Figure showing the optimal portfolio can be found in Section 4.3



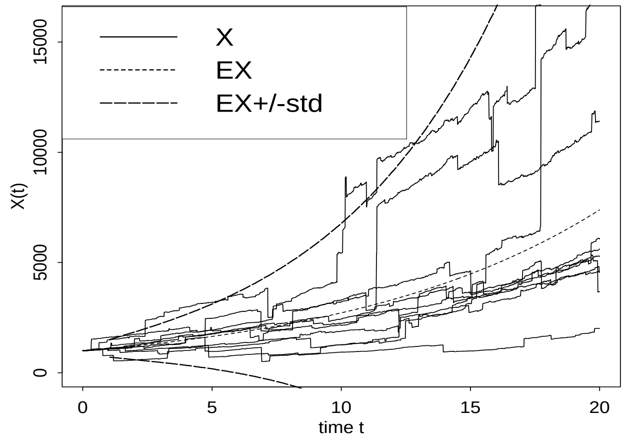


Figure 4.2: Ten sample paths of the exponential NIG-Lévy process with $\xi = 8$ and $\delta = 0.32$ (left) and with $\xi = 2$ and $\delta = 0.08$ (right), its expectation $E(\exp L(T))$ (dotted line) and expectation±standard deviation (dashed lines) for $x = 1\,000$, $b_{BS} = 0.1$, and $r = 0.05$.

Example 4.2.7 *(Exponential variance gamma (VG) Lévy process)*

This normal-mean mixture model is of the same structure as the NIG model and has been suggested by Madan and Seneta (1990). Its non-symmetric version can be found in Madan, Carr and Chang (1998):

$$L(t) = \mu - \delta\zeta^2(t) + W(\zeta^2(t)), \quad t \geq 0,$$

where $\mu, \delta \in \mathbb{R}$, W is a standard Brownian motion and $\zeta^2(t)$ is a Γ-Lévy process, i.e. $\zeta^2(t+s) - \zeta^2(t) \sim \Gamma(\xi s, \theta)$ for parameters $\xi, \theta > 0$; i.e. $\zeta^2(1)$ has density

$$h(x; \xi, \theta) = \frac{x^{\xi-1}}{\Gamma(\xi)\theta^\xi} e^{-x/\theta}, \quad x > 0.$$

By conditioning on $\zeta^2(t)$ we obtain the characteristic function

$$\begin{aligned} E\exp(isL(t)) &= \exp(is\mu t)E[\exp(-(is\delta - s^2/2)\zeta^2(t)] \\ &= \frac{\exp(is\mu t)}{(1 - is\theta\delta + s^2\theta/2)^{\xi t}} \quad = \quad e^{t\Psi(s)}, \quad t \geq 0, \end{aligned}$$

where $\Psi(s) = i\mu s - \xi \ln(1 - is\theta\delta + s^2\theta/2)$. Thus $\mu = \gamma$, $\beta = 0$, hence L is a pure jump process with Lévy density

$$\nu(dx) = \frac{\xi}{|x|} \exp\left(-\sqrt{\frac{2}{\theta} + \delta^2}\, |x| - \delta x \right) dx, \quad x \in \mathbb{R}^*.$$

Since $\int_{|x|\le 1}|x|\nu(dx) < \infty$, the sample paths of L are a.s. of finite variation in any finite interval; furthermore, those jumps are dense in $[0,\infty)$, since $\nu(\mathbb{R}) = \infty$; see Sato (1999). The properties of this model are similar to those of the NIG model, since both are normal-mean variance mixture models and their Lévy measures have similar properties. An interesting empirical investigation has been conducted by Carr et al. (2001).

In order to calculate the wealth process and its mean and variance we use (4.1.5) and Remark 4.1.9(b). We observe that $E\exp(isL(1)) = e^{\Psi(s)}$ is analytic around 0, hence

$$\begin{aligned}
\ln \widehat{f}(1) &= \Psi(-i) = \mu - \xi\ln(1-\theta\delta-\theta/2) < \infty\,,\\
\ln \widehat{f}(2) &= \Psi(-2i) = 2\mu - \xi\ln(1-2\theta\delta-2\theta) < \infty\,.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
a &= \gamma + \int_{|x|\le 1} x\frac{\xi}{|x|}\exp\left(-\sqrt{\frac{2}{\theta}+\delta^2}\,|x| - \delta x\right)dx\\
&= \mu - \xi\theta\delta + \xi\theta\frac{c_2}{2}e^{c_1} - \xi\theta\frac{c_1}{2}e^{c_2}\,,
\end{aligned}$$

where

$$c_1 = -\left(\sqrt{\frac{2}{\theta}+\delta^2}+\delta\right) \text{ and } c_2 = -\left(\sqrt{\frac{2}{\theta}-\delta^2}+\delta\right)$$

We obtain for $t \ge 0$

$$\begin{aligned}
X^\pi(t) &= x\exp(t(r+\pi(b-r+\mu)))\prod_{s\le t}(1+\pi(e^{\Delta L(s)}-1))\\
E[X^\pi(t)] &= x\exp(t(r+\pi(b-r-\xi\ln(1-\theta\delta-\theta/2)+\mu)))\\
\operatorname{var}(X^\pi(t)) &= x^2\exp(2t(r+\pi(b-r-\xi\ln(1-\theta\delta-\theta/2)+\mu)))\\
&\quad\times\left(\exp\left(\xi\pi^2 t\,(2\ln(1-\theta\delta-\theta/2)-\ln(1-2\theta\delta-2\theta))\right)-1\right).
\end{aligned}$$

There are different possible choices of parameters such that the expected wealth is the same as in Example 4.2.5. The simplest one is to choose b as in Example 4.2.5 and $\mu = \xi\ln(1-\theta\delta-\theta/2)$. Alternatively, set $\mu = \delta = 0$ such that the process L is symmetric around 0 and choose $b = b_{BS} + \xi\ln(1-\theta/2)$, where b_{BS} is b as chosen in Example 4.2.5.

Remark 4.2.8 Since Examples 4.2.6 and 4.2.7 have so many parameters, we can always attain the same expectation and variance for all three examples. But the shape of the distributions differs as can be seen in Figure 4.8. The variance gamma distribution is also leptokurtic as the NIG. For illustration, Figure 4.3 shows ten sample paths of the exponential variance gamma Lévy process, its expectation and standard deviation as a function of the planning horizon $0 < T \leq 20$ for different parameters. Expectation and standard deviation are increasing with the planning horizon T. This leads to a decreasing optimal portfolio in Figure 4.4, where we use the same parameters as in Figure 4.3 (left) and the constraint $\mathrm{var}(X^\pi(T)) \leq 100\,000$. Note that the optimal portfolio is the same for all Lévy processes with the same mean and variance.

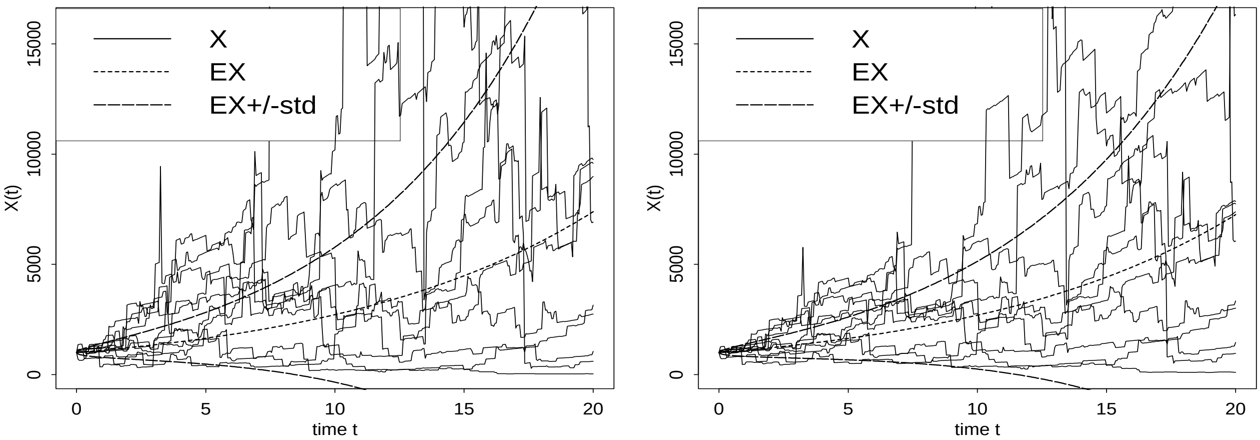


Figure 4.3: Ten sample paths of the exponential variance gamma Lévy process with $\xi = 0.1$, $\delta = 0$, $\theta = 0.35$ and $\mu = -0.019$ (left) and with $\xi = 0.2$, $\delta = 0$, $\theta = 0.2$ and $\mu = -0.022$ (right), its expectation $E(\exp L(T))$ and expectation±standard deviation for $x = 1\,000$, $b = 0.1$, and $r = 0.05$.

Example 4.2.9 *(Meixner model)*
The Meixner model was introduced by Grigelionis (1999) and discussed for applications in finance by Schoutens (2001). The distribution of the increment $L(1)$ of the Meixner process is the Meixner distribution (see e.g. Schoutens (2001)) given by the density

$$meixner(x;\xi,\theta,m,\delta) = \frac{(2\cos(\theta/2))^{2\delta}}{2\xi\pi\Gamma(2\delta)}\exp\left(\frac{\theta(x-m)}{\xi}\right)\left|\Gamma\left(\delta+\frac{i(x-m)}{\xi}\right)\right|^2, x \in \mathbb{R},$$

where $\xi > 0$, $-\pi < \theta < \pi$, $\delta > 0$, and $m \in \mathbb{R}$. The Meixner process has no Brownian component and $\int_{|x|\leq 1}|x|\nu(dx) = \infty$, i.e. its paths are of infinite variation in any finite interval, and γ does not exist. Its characteristic triplet is $(a, 0, \nu)$, where

$$a = m + \xi\delta\tan(\theta/2) - 2\delta\int_1^\infty \frac{\sinh(\theta x/\xi)}{\sinh(\pi x/\xi)}dx$$

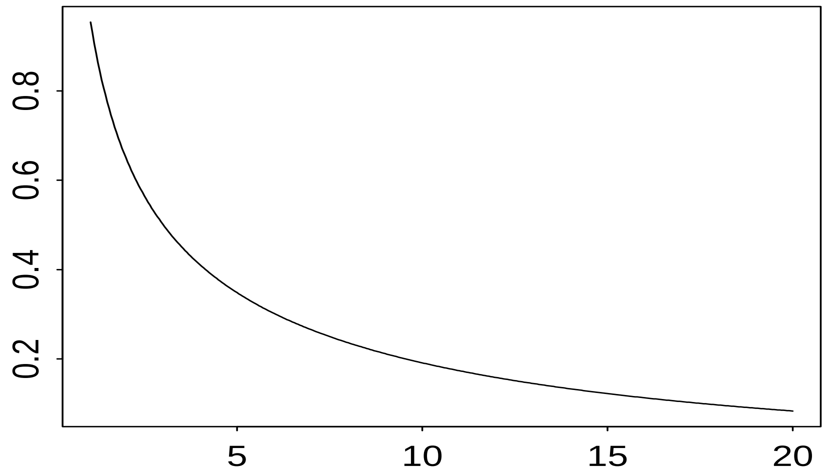


Figure 4.4: Optimal portfolio in the exponential variance gamma Lévy model with $\xi = 0.1$, $\delta = 0$, $\theta = 0.35$ and $\mu = -0.019$ for the same parameters as in Figure 4.3 (left) under the constraint $\mathrm{var}(X^\pi(t)) \leq 100\,000$.

and

$$\nu(dx) = \delta \frac{\exp(\theta x/\xi)}{x \sinh(\pi x/\xi)} dx$$

(see e.g. Grigelionis (1999)). By (4.1.8) and (4.1.12) we can calculate the expectation of the wealth of the portfolio and its variance via the moment generating function of $L(1)$, which is for the Meixner distribution given by

$$\widehat{f}(s) = E \exp(sL(1)) = \left(\frac{\cos(\theta/2)}{\cos(-(s\xi + \theta)/2)} \right)^{2\delta} e^{sm}$$

for $s \neq -((2k+1)\pi + \theta)/\xi$ for all $k \in \mathbb{Z}$. Hence we obtain

$$\begin{aligned} \ln \widehat{f}(1) &= 2\delta \ln \left(\frac{\cos(\theta/2)}{\cos(-(\xi + \theta)/2)} \right) + m, \\ \ln \widehat{f}(2) &= 2\delta \ln \left(\frac{\cos(\theta/2)}{\cos(-(2\xi + \theta)/2)} \right) + 2m. \end{aligned}$$

Plugging these results into (4.1.8) and (4.1.12) we obtain for $t \geq 0$

$$\begin{aligned} E[X^\pi(t)] &= x \exp(t(r + \pi(b - r + \left(2\delta \ln \left(\frac{\cos(\theta/2)}{\cos(-(\xi + \theta)/2)} \right) + m \right)))) \\ &= x \left(\frac{\cos(\theta/2)}{\cos(-(\xi + \theta)/2)} \right)^{2\delta\pi t} \exp(t(r + \pi(b - r + m))) \\ \mathrm{var}(X^\pi(t)) &= x^2 \left(\frac{\cos(\theta/2)}{\cos(-(\xi + \theta)/2)} \right)^{4\delta\pi t} \exp\left(2t(r + \pi(b - r + m))\right) \times \end{aligned}$$

$$\times \left(\left(\frac{(\cos(-(\xi+\theta)/2))^2}{\cos(\theta/2)\cos(-(2\xi+\theta)/2)} \right)^{2\delta\pi^2 t} - 1 \right)$$

Figures 4.5 and 4.6 show the expectation ($\pm$ standard deviation) of a Meixner Process and the optimal portfolio in the Meixner model for certain parameters. In our examples we have chosen

$$m = -2\delta \ln \left(\frac{\cos(\theta/2)}{\cos(-(\xi+\theta)/2)} \right)$$

such that

$$E[X^\pi(t)] \quad = \quad x \exp((r + \pi(b-r))t), \quad t \geq 0.$$

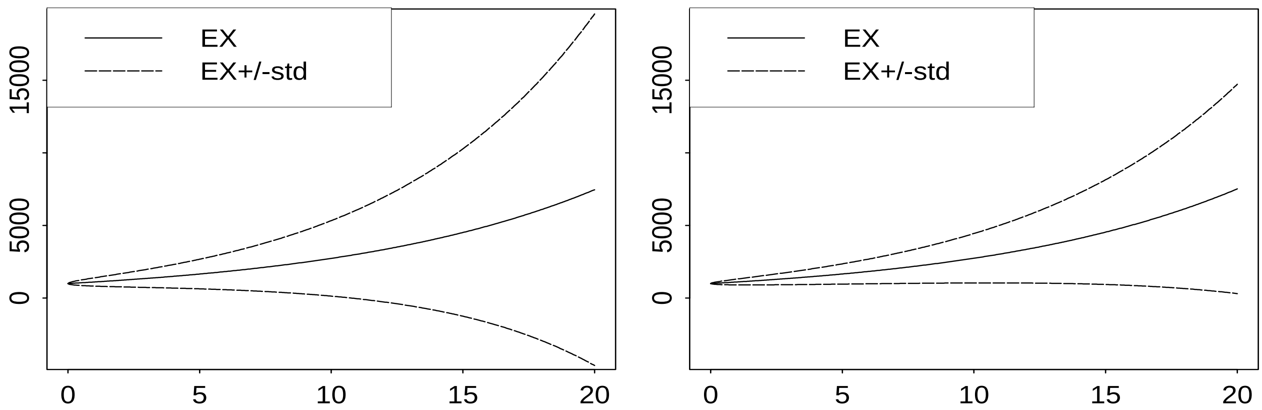


Figure 4.5: Expectation $E(\exp(L(T)))$ and expectation$\pm$standard deviation of the exponential Meixner process with $\xi = 0.03$, $\theta = 0.13$, $\delta = 142.5$ and $m = -0.31$ (left) and with $\xi = 0.015$, $\theta = -0.014$, $\delta = 290$ and $m = 0.015$ (right), for $x = 1\,000$, $b = 0.1$, and $r = 0.05$.

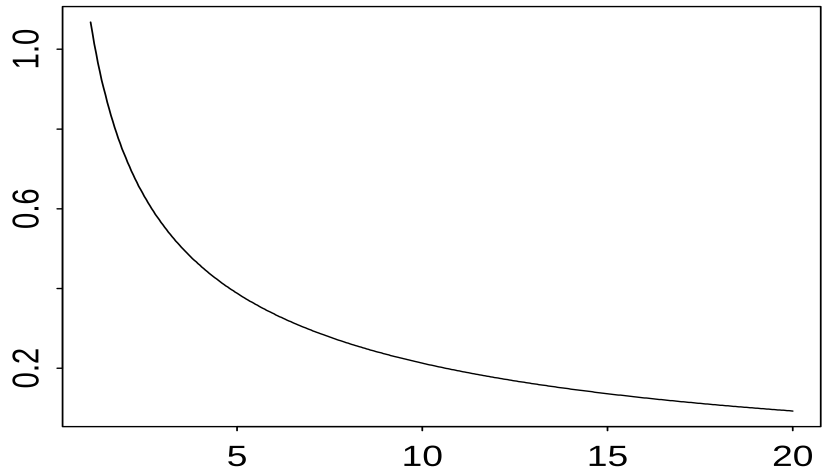


Figure 4.6: Optimal portfolio in the exponential Meixner Lévy model with $\xi = 0.03$, $\beta = 0.13$, $\delta = 142.5$ and $m = -0.31$ for the same parameters as in Figure 4.5 under the constraint $\mathrm{var}(X^\pi(t)) \leq 100\,000$.

4.3 The Capital-at-Risk - calculation and approximation

In this section we replace the variance by the Capital-at-Risk (CaR). Before we pose and solve the mean-CaR optimization problem, we define the CaR and indicate some properties. We further show how it can be determined (approximated) in the case of a general Lévy process.

Definition 4.3.1 *Let x be the initial capital and T a given planning horizon. Let furthermore z_α be the α-quantile of the distribution of $\mathcal{E}(\pi\widehat{L}(T))$ for some portfolio $\pi \in [0,1]^d$, $\pi'\underline{1} \leq 1$, and $X^\pi(T)$ the corresponding terminal wealth. Then the Value-at-Risk (VaR) is given by*

$$\mathrm{VaR}(x,\pi,T) = \inf\{z \in \mathbb{R} : P(X^\pi(T) \leq z) \geq \alpha\} = x z_\alpha \exp((\pi'(b - r\underline{1}) + r)T)\,.$$

We define

$$\mathrm{CaR}(x,\pi,T) = x\exp(rT) - \mathrm{VaR}(x,\pi,T) = x\exp(rT)\,(1 - z_\alpha \exp(\pi'(b - r\underline{1})T)) \qquad (4.3.1)$$

the Capital-at-Risk (CaR) of the portfolio π (with initial capital x and time horizon T).

The calculation of the CaR involves the quantile z_α of $\mathcal{E}(\pi\widehat{L}(T))$, which is quite a complicated object as we have seen in Lemma 4.1.5. To calculate its distribution explicitly is certainly not possible for Examples 4.2.6 and 4.2.7. One possibility would be to calculate the characteristic function of $\mathcal{E}(\pi\widehat{L}(T))$ using its characteristic triplet as given in Lemma 4.1.2. From this then one could approximate its density using the inverse Fast Fourier transform method, which is explained later in this section. However, the complicated expressions of its characteristic triplet in combination with the complicated integral in the Lévy-Khinchine formula seems to advise a different approach. As an alternative method we suggest an approximation method based on a weak limit theorem.

For simplicity we restrict ourselves to $d = 1$ and invoke an idea used for instance by Bondesson (1982) and Rydberg (1997) for simulation purposes and made mathematically precise by Asmussen and Rosinski (2000). The intuition behind is to approximate small jumps of absolute size smaller than ε by a simpler stochastic process, often by Brownian motion, such that the stochastic part of the Lévy process is approximated by an independent sum of a Brownian motion and a compound Poisson process. Before we study the applicability of their results to approximate quantiles of the wealth process, we explain the idea.

In a first step the small jumps with absolute size smaller than some $\varepsilon > 0$ are replaced by their expectation. This leads to the process

$$L_\varepsilon(t) = \mu(\varepsilon)t + \beta W(t) + N^\varepsilon(t)\,, \quad t \geq 0, \tag{4.3.2}$$

where $\mu(\varepsilon)$ is defined below, and

$$L(t) - L_\varepsilon(t) = \int_0^t \int_{|x|<\varepsilon} x(M(dx,ds) - \nu(dx)ds)\,, \quad t \geq 0\,.$$

In a second step the contribution from the variation of small jumps is also incorporated. To this end we use the following representation

$$\begin{aligned} L(t) \;&=\; t\left(a - \int_{\varepsilon<|x|\leq 1} x\nu(dx)\right) + \beta W(t) \\ &\quad + \sum_{0<s\leq t} \Delta L(s) 1_{\{|\Delta L(s)|\geq\varepsilon\}} + \int_0^t \int_{|x|<\varepsilon} x(M(dx,ds) - \nu(dx)ds, \quad t \geq 0. \end{aligned}$$

Set

$$\mu(\varepsilon) = a - \int_{\varepsilon\le|x|\le 1} x\nu(dx) \quad \text{and} \quad N^\varepsilon(t) \;=\; \sum_{s\le t} \Delta L(s) 1_{\{|\Delta L(s)|\ge\varepsilon\}}\,.$$

In order to replace the small jumps by some Gaussian term, we need that for $\varepsilon \to 0$

$$\sigma(\varepsilon)^{-1}(L(t) - (\mu(\varepsilon)t + \beta W(t) + N^\varepsilon(t))) = \sigma(\varepsilon)^{-1}(L(t) - L_\varepsilon(t)) \stackrel{d}{\to} W'(t)\,, t \ge 0, \tag{4.3.3}$$

for some Brownian motion W', where

$$\sigma^2(\varepsilon) \;=\; \int_{|x|<\varepsilon} x^2\nu(dx), \quad \varepsilon > 0\,. \tag{4.3.4}$$

We denote by $\stackrel{d}{\to}$ weak convergence in $D[0,\infty)$ uniformly on compacta; see Pollard (1984). In the finite variation case (4.3.3) can be rewritten to

$$\sigma(\varepsilon)^{-1}\left(\sum_{0<s\le t} \Delta L(s) I(|\Delta L(s)| < \varepsilon) - E\left[\sum_{0<s\le t} \Delta L(s) I(|\Delta L(s)| < \varepsilon)\right]\right) \stackrel{d}{\to} W'(t),\ t \ge 0.$$

This reminds of the classical central limit theorem and the Brownian motion as limit process. Here we can see that the standardized process of the small jumps converges to Brownian motion as the jump size ε tends to 0. In fact, since Gaussian part and jump part are independent, the Brownian motion W' is independent of W, and this justifies the approximation in distribution

$$L(t) \approx \mu(\varepsilon)t + (\beta^2 + \sigma^2(\varepsilon))^{\frac{1}{2}} W(t) + N^\varepsilon(t)\,, \quad t \ge 0\,.$$

Proposition 4.3.2 *[Asmussen and Rosinski (2000)]*
(a) A necessary and sufficient condition for (4.3.3) to hold is

$$\lim_{\varepsilon\to 0} \frac{\sigma(h\sigma(\varepsilon) \wedge \varepsilon)}{\sigma(\varepsilon)} = 1 \quad \forall h > 0\,. \tag{4.3.5}$$

(b) $\lim_{\varepsilon\to 0} \sigma(\varepsilon)/\varepsilon = \infty$ implies (4.3.5). If the Lévy measure does not have atoms in some neighbourhood of 0, then condition (4.3.5) is equivalent to $\lim_{\varepsilon\to 0} \sigma(\varepsilon)/\varepsilon = \infty$.

We want to invoke this result to approximate quantiles of $\mathcal{E}(\pi\widehat{L}(T))$. We do this in two steps: firstly, we approximate $\mathcal{E}(\pi\widehat{L}(T))$, secondly, we use that convergence of distribution functions implies also convergence of their generalized inverses; see Proposition 0.1 of Resnick (1987). This gives us the approximation of the quantiles.

Lemma 4.3.3 *Recall model (4.1.1) and (4.1.2) for $d = 1$ and $\sigma = 1$; i.e. $L = \ln \mathcal{E}(\widehat{L})$ and $\widehat{L}$ are Lévy processes with Lévy measures ν and $\widehat{\nu}$ respectively. Then*

$$\begin{aligned} \sigma^2(\varepsilon) &= \int_{(-\varepsilon,\varepsilon)} x^2 \nu(dx) = \int_{(e^{-\varepsilon}-1, e^{\varepsilon}-1)} (\ln(1+x))^2 \widehat{\nu}(dx)\,, \\ \widehat{\sigma}^2(\varepsilon) &= \int_{(-\varepsilon,\varepsilon)} x^2 \widehat{\nu}(dx) = \int_{(\ln(1-\varepsilon), \ln(1+\varepsilon))} (e^x - 1)^2 \nu(dx)\,,\,. \end{aligned} \tag{4.3.6}$$

Proof The transformation from L to $\widehat{L}$ only affects the jumps, which are related by $\Delta L(s) = \ln(1 + \Delta \widehat{L}(s))$ for $s \geq 0$. We calculate

$$\begin{aligned} \sigma^2(\varepsilon) &= E\left[\sum_{s\leq 1} (\Delta L(s))^2 1_{\{|\Delta L(s)|<\varepsilon\}}\right] \\ &= E\left[\sum_{s\leq 1} (\ln(1+\Delta \widehat{L}(s)))^2 1_{\{e^{-\varepsilon}-1<\Delta\widehat{L}(s)<e^{\varepsilon}-1\}}\right] \\ &= \int_{(e^{-\varepsilon}-1, e^{\varepsilon}-1)} (\ln(1+x))^2 \widehat{\nu}(dx)\,. \end{aligned}$$

The calculation of $\widehat{\sigma}^2$ is analogous.

We formulate the following main result of this section. The proof is postponed to Section 4.5.

Theorem 4.3.4 *Let Z^ε, $\varepsilon > 0$, be Lévy processes without Gaussian component and $Y^\varepsilon = \ln \mathcal{E}(Z^\varepsilon)$ their logarithmic stochastic exponentials with characteristic triplets $(a_Z, 0, \nu_Z)$ and $(a_Y, 0, \nu_Y)$ as defined in Lemma 4.1.2; for notational convenience we suppress ε. Let $g : \mathbb{R} \to \mathbb{R}^+$ with $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let V be a Lévy process. Then equivalent are as $\varepsilon \to 0$,*

$$\frac{Z^\varepsilon(t)}{g(\varepsilon)} \stackrel{d}{\to} V(t)\,, \quad t \geq 0\,, \tag{4.3.7}$$

$$\frac{Y^\varepsilon(t)}{g(\varepsilon)} \stackrel{d}{\to} V(t)\,, \quad t \geq 0\,. \tag{4.3.8}$$

We apply this result to approximate $\ln \mathcal{E}(\pi \widehat{L})$ for $\pi \in (0, 1]$ as follows:

Corollary 4.3.5 *Let L be a Lévy process and L_ε the process given in (4.3.2). Let furthermore $\mathcal{E}^{\leftarrow}(\exp(L)) = \widehat{L}$ be such that $\mathcal{E}\widehat{L} = \exp(L)$ with characteristic triplet given in Lemma 4.1.2. Then*

$$\sigma(\varepsilon)^{-1}(L(t) - L_\varepsilon(t)) \stackrel{d}{\to} V(t)\,, \quad t \geq 0 \tag{4.3.9}$$

is equivalent to

$$(\pi\sigma(\varepsilon))^{-1}(\ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp(L(t)))) - \ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp(L_\varepsilon(t))))) \xrightarrow{d} V(t)\,, \quad t \geq 0. \tag{4.3.10}$$

For the proof of this Corollary we need the following Lemma.

Lemma 4.3.6 *Let L be a Lévy process and L_ε as defined in (4.3.2). Then*

$$\ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp(L(t)-L_\varepsilon(t)))) = \ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp(L(t)))) - \ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp L_\varepsilon(t))), \quad t \geq 0.$$

Proof Since

$$L(t) - L_\varepsilon(t) = \int_0^t \int_{|x|<\varepsilon} x(M(dx,ds) - \nu(dx)ds)\,, \quad t \geq 0\,,$$

we obtain by Itô's formula

$$\begin{aligned}\ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp(L(t)-L_\varepsilon(t)))) &= \int_0^t \int_{|x|<\varepsilon} \ln(1+\pi(e^x-1))(M(dx,ds)-\nu(dx)ds) \\ &\quad + \int_0^t \int_{|x|<\varepsilon} (\ln(1+\pi(e^x-1)) - \pi x)\nu(dx)ds,\ t \geq 0.\end{aligned}$$

Again using Itô's formula we calculate

$$\begin{aligned}&\ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp(L(t)))) \\ &= \pi(a + \frac{1}{2}(1-\pi)\beta^2)t + \pi\beta W(t) + \int_0^t \int_{|x|>1} \ln(1+\pi(e^x-1))M(dx,ds) \\ &\quad + \int_0^t \int_{|x|\leq 1} \ln(1+\pi(e^x-1))(M(dx,ds)-\nu(dx)ds) \\ &\quad + \int_0^t \int_{|x|\leq 1} (\ln(1+\pi(e^x-1)) - \pi x)\nu(dx)ds\,,\ t \geq 0\,,\end{aligned}$$

and

$$\begin{aligned}\ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp L_\varepsilon(t)))) &= \pi(a - \int_{\varepsilon<|x|\leq 1} x\nu(dx) + \frac{1}{2}(1-\pi)\beta^2)t + \pi\beta W(t) \\ &\quad + \int_0^t \int_{|x|>\varepsilon} \ln(1+\pi(e^x-1))M(dx,ds)\,,\ t \geq 0\,.\end{aligned} \tag{4.3.11}$$

Calculating the difference of the last two terms leads to the assertion. $\square$

Proof of Corollary 4.3.5 Setting $g(\varepsilon) := \sigma(\varepsilon)$ and $Y^\varepsilon := L - L_\varepsilon$ in Theorem 4.3.4 we obtain that (4.3.9) holds if and only if

$$\sigma(\varepsilon)^{-1}\mathcal{E}^{\leftarrow}(\exp(L(t) - L_\varepsilon(t))) \xrightarrow{d} V(t), \quad t \geq 0 \tag{4.3.12}$$

where $\mathcal{E}^{\leftarrow}(\exp(L) = \widehat{L}$ is such that $\mathcal{E}\widehat{L} = \exp(L)$. Applying Theorem 4.3.4 to $g(\varepsilon) := \pi\sigma(\varepsilon)$ and $Z_\varepsilon(t) := \pi\mathcal{E}^{\leftarrow}(\exp(L(t) - L_\varepsilon(t)))$ leads to the equivalence of (4.3.12) and

$$(\pi\sigma(\varepsilon))^{-1}\ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(\exp(L(t) - L_\varepsilon(t)))) \xrightarrow{d} V(t), \quad t \geq 0.$$

Lemma 4.3.6 leads to the assertion of the Corollary. □

From this corollary and (4.3.11) we conclude the following approximation for $\ln\mathcal{E}(\pi\widehat{L}(t))$, which is needed for the calculation of the CaR in Definition 4.3.1.

Proposition 4.3.7

$$\begin{aligned}\ln\mathcal{E}(\pi\widehat{L}(t)) &\approx \ln\mathcal{E}(\pi\mathcal{E}^{\leftarrow}(L_\varepsilon(t)))) + \pi\sigma(\varepsilon)V(t) \\ &= \gamma_\pi^\varepsilon t + \pi\beta W(t) + M_\pi^\varepsilon(t) + \pi\sigma(\varepsilon)V(t)\,, \quad t \geq 0.\end{aligned}$$

If V is a Brownian motion, then

$$\ln\mathcal{E}(\pi\widehat{L}(t))) \approx \gamma_\pi^\varepsilon t + \pi(\beta^2 + \sigma^2(\varepsilon))^{1/2}W(t) + M_\pi^\varepsilon(t)\,, \quad t \geq 0.$$

We have the following representations

$$\begin{aligned}\gamma_\pi^\varepsilon &= \pi(\mu(\varepsilon) + \frac{1}{2}\beta^2(1-\pi))\,, \\ M_\pi^\varepsilon(t) &= \sum_{s\leq t}\ln(1 + \pi(e^{\Delta L(s)1_{\{|\Delta L(s)|>\varepsilon\}}} - 1))\,;\end{aligned}$$

i.e. M_π^ε is a compound Poisson process with jump measure

$$\nu_{M_\pi^\varepsilon}(\Lambda) = \nu_L(\{x : \ln(1 + \pi(e^x - 1)) \in \Lambda\}\backslash(-\varepsilon, \varepsilon))$$

for any Borel set $\Lambda \subset \mathbb{R}$. Moreover, if the Lévy measure ν_L has a density ν_L', the density of the Lévy measure ν_M of the process M_π^ε is given by

$$\nu_M'(x) = \nu_L'\left(\ln\left(\frac{e^x - 1}{\pi} + 1\right)\right)\frac{e^x}{e^x - (1-\pi)}1_{\{x>\ln(1-\pi)\}}1_{\{|\ln((e^x-1)/\pi+1)|>\varepsilon\}}$$

and thus the Poisson intensity of M_π^ε is $\int_{\mathbb{R}} \nu'_M(x)dx$. The density of the jump sizes of M_π^ε is given by $\nu'_M(x)/\int \nu'_M(y)dy, x \in \mathbb{R}$.

By Proposition 0.1 of Resnick (1987) we obtain the corresponding approximation for the α-quantile z_α of $\mathcal{E}(\pi\widehat{L}(T))$, where T is some fixed planning horizon.

Proposition 4.3.8 *With the quantities as defined in Proposition 4.3.7 we obtain*

$$z_\alpha \approx z_\alpha^\varepsilon(\pi) = \inf\{z \in \mathbb{R} : P(\gamma_\pi^\varepsilon T + M_\pi^\varepsilon(T) + \pi\beta W(T) + \pi\sigma_L(\varepsilon))V(T) \le \ln z) \ge \alpha\}.$$

Moreover, if V is a Brownian motion, then

$$z_\alpha \approx z_\alpha^\varepsilon(\pi) = \inf\{z \in \mathbb{R} : P(\gamma_\pi^\varepsilon T + M_\pi^\varepsilon(T) + \pi(\beta^2 + \sigma_L^2(\varepsilon))^{1/2} W(T) \le \ln z) \ge \alpha\}.$$

We obtain

$$\begin{aligned} \mathrm{VaR}(x,\pi,T) &\approx x z_\alpha^\varepsilon(\pi)\exp((\pi(b-r)+r)T), && (4.3.13)\\ \mathrm{CaR}(x,\pi,T) &\approx x\exp(rT)(1 - z_\alpha^\varepsilon(\pi)\exp(\pi(b-r)T)). && (4.3.14) \end{aligned}$$

We have now reduced the problem of the calculation of a low quantile of $\ln\mathcal{E}(\pi\widehat{L}(T))$ and only have to determine a low quantile of the sum of the compound Poisson variable $M_\pi^\varepsilon(T)$, the normal distributed variable $\pi\beta W(T)$, and the limit variable $\pi\sigma(\varepsilon)V(T)$. Therefore we calculate first the density f_T of $M_\pi^\varepsilon(T) + \pi\beta W(T) + \pi\sigma(\varepsilon)V(T)$ using the Fast Fourier transform method, henceforth abbreviated as FFT. If h_M is the Lévy density of M_π^ε we have for the characteristic function of $M_\pi^\varepsilon(1) + \pi\beta W(1) + \pi\sigma(\varepsilon)V(1)$

$$\phi_{M+\pi\beta W+\pi\sigma(\varepsilon)V}(u) = \int_{-\infty}^{\infty} e^{iux} f_1(x)dx = \phi_M(u)\phi_{\pi\beta W}(u)\phi_{\pi\sigma(\varepsilon)V}(u),$$

where

$$\begin{aligned} \phi_M(u) &= \exp(\nu_{M_\pi^\varepsilon}(\mathbb{R})(\phi_Y(u)-1)),\\ \phi_Y(u) &= \int e^{iux}\nu_{M_\pi^\varepsilon}(\mathbb{R})^{-1}\nu_{M_\pi^\varepsilon}(dx) = \int e^{iux}\nu_{M_\pi^\varepsilon}(\mathbb{R})^{-1}h_M(x)dx,\\ \phi_{\pi\beta W}(u) &= \exp\left(-u^2\pi^2\beta^2/2\right) \end{aligned}$$

and $\phi_{\pi\sigma(\varepsilon)V}(u)$ is given by Lemma 4.1.3 and the Lévy-Khintchine formula. For $g(x) = h_M(x)/\nu_{M_\pi^\varepsilon}(\mathbb{R})$ and $g(x) = f_1(x)$ respectively we approximate the integrals in the following way:

choose a number $n \in \{2^d | d \in \mathbb{N}\}$ of intervals and a step size Δx; then we truncate the integral at the points $(n/2-1)\Delta x$ and $-(n/2)\Delta x$ and obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{iux} g(x)dx &\approx \int_{-(n/2)\Delta x}^{(n/2-1)\Delta x} e^{iux} g(x)dx \\
&\approx \sum_{-(n/2)}^{n/2-1} e^{iuk\Delta x} g(k\Delta x)\Delta x \\
&= \sum_{0}^{n-1} e^{iu(k-n/2)\Delta x} g((k-n/2)\Delta x)\Delta x \\
&= \Delta x e^{-iun\Delta x/2} \sum_{0}^{n-1} e^{iuk\Delta x} g((k-n/2)\Delta x)
\end{aligned}$$

For $g_k := g((k-n/2)\Delta x)$, $k = 0, \ldots, n-1$, this is the discrete Fourier transform of the complex numbers g_k and can be calculated by the FFT algorithm for $u_k = 2\pi k/(n\Delta x)$, $k = 0, \ldots, n-1$, simultaneously (see e.g. Brigham (1974), Chapter 10) and we obtain an approximation for $\phi_{M+\pi\beta W+\pi\sigma(\varepsilon)V}$. By the inverse FFT we obtain the density f_1 and hence we can calculate quantiles. Because of the infinite divisibility we have $\phi_{M(T)+\pi\beta W(T)+\pi\sigma(\varepsilon)V(T)}(u) = \phi^T_{M+\pi\beta W+\pi\sigma(\varepsilon)V}(u)$ and hence we obtain f_T for any $T > 0$.
In the normal approximation case the procedure simplifies. There we only have to determine quantiles of the sum of the compound Poisson variable $M^\varepsilon_\pi(T)$ and the normal distributed variable $\widetilde{W} := \pi(\beta^2 + \sigma^2_L(\varepsilon))^{1/2} W(T)$. The characteristic function simplifies to

$$\phi_{M+\widetilde{W}}(u) = \int_{-\infty}^{\infty} e^{iux} f_1(x)dx = \phi_M(u)\phi_{\widetilde{W}}(u),$$

where

$$\phi_{\widetilde{W}}(u) = \exp\left(-\frac{u^2}{2}\pi^2(\beta^2 + \sigma^2_L(\varepsilon))\right).$$

4.4 Optimal portfolios under CaR constraints

We consider now the following optimization problem using the Capital-at-Risk as risk measure.

$$\max_{\pi\in[0,1]} E[X^\pi(T)] \quad \text{subject to} \quad \mathrm{CaR}(x,\pi,T) \le C\,, \tag{4.4.1}$$

where T is some given planning horizon and C is a given bound for the risk.

Unfortunately, there is no analogue of Theorem 4.2.1. Due to the fact that, immediately by (4.1.8), the mean wealth $E[X^\pi(T)]$ is increasing in π, the optimal solution of (4.4.1) is the largest $\pi \in [0,1]$ that satisfies the CaR constraint.

We investigate some examples.

Example 4.4.1 *(Exponential normal inverse Gaussian Lévy process)*
Recall the model as defined in Example 4.2.6, where we set again $\lambda = \rho = 0$. For the calculation of the CaR we use the approximation of Proposition 4.3.8. Setting $f_L(x) = f_{nig}(x) = \xi\delta K_1(\xi|x|)/(\pi|x|)$, $x \in \mathbb{R}$, the Lévy density of the NIG Lévy process, the intensity of the compound Poisson process M_π^ε and the density of its jump sizes can be calculated as explained in Proposition 4.3.7 . Plugging f_{nig} into definition (4.3.4) we obtain

$$\sigma^2(\varepsilon) = \frac{\xi\delta}{\pi}\int_{|x|<\varepsilon} |x|K_1(\xi|x|)dx\,, \quad \varepsilon > 0\,.$$

As shown in Asmussen and Rosinski (2000) for the normal inverse Gaussian Lévy process the normal approximation for small jumps is allowed since $\sigma(\varepsilon) \sim (2\delta/\pi)^{1/2}\varepsilon^{1/2}$ as $\varepsilon \to 0$. Since $\beta = 0$ the approximating Lévy process has a Gaussian component with variance $\sigma^2(\varepsilon)$. Moreover, $a = 0$, hence

$$\mu(\varepsilon) = -\int_{\varepsilon\le|x|\le 1} \frac{\xi\delta x K_1(\xi|x|)}{\pi|x|}dx.$$

For the calculation of these integrals we use a polynomial approximation for the modified Bessel function of the third kind (see Abramowitz and Stegun (1968), pp. 378-379). For the FFT we use $n = 2^10$ and $\Delta x = 0.002$.[2mm] Figure 4.7 shows the dependence of CaR on the time horizon T illustrated by CaR(1 000,1,T) for $0 < T \le 22$. For short planning horizons the CaR increases, whereas for very large planning horizons the CaR is decreasing with T. Comparison with Figure 1 of Emmer, Klüppelberg, and Korn (2001) shows that the CaR is smaller than in the Black-Scholes case with the same variance. The reason can be seen in Figure 4.8: the 5%-quantile is larger than the 5%-quantile for the normal distribution with the same variance. Since the 1%-quantile is lower than the 1%-quantile for the normal distribution with the same variance, the CaR with respect to the 1%-quantile here would be larger than in the Black-Scholes case. Here we get into the heavier tails of the NIG density. The increasing CaR for the time horizons $0 < T \leq 5$ in Figure 4.7 leads to a decreasing optimal portfolio in Figure 4.9 (left).

For comparison we have plotted the optimal portfolio under the contraint var $\leq 100\,000$ in Figure 4.9 (right). For small planning horizons the strategies look very similar, but since the CaR in decreasing for large planning horizons, we then obtain an increasing π, which is in contrast to the results for the variance. Figure 4.10 illustrates the behaviour of the optimal expected terminal wealth and terminal wealth of the pure bond and of the pure stock strategy with varying time horizon T under a constraint on the CaR (left) and on the variance (right).

As is obvious from Figure 4.8 for the 1%-quantile investment in stock would be more cautious for the exponential NIG Lévy process than for the exponential Brownian motion.

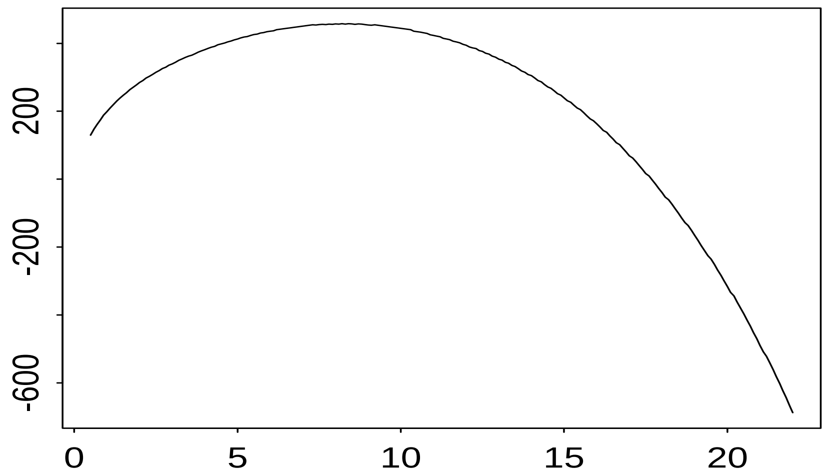


Figure 4.7: CaR(1000,1,T) of a pure stock portfolio in the exponential normal inverse Gaussian Lévy model as a function of the time horizon T, $0 < T \leq 22$. The parameters are $\xi = 2$, $\delta = 0.08$, $\lambda = \rho = 0$, $x = 1\,000$, $b = 0.1$ and $r = 0.05$.

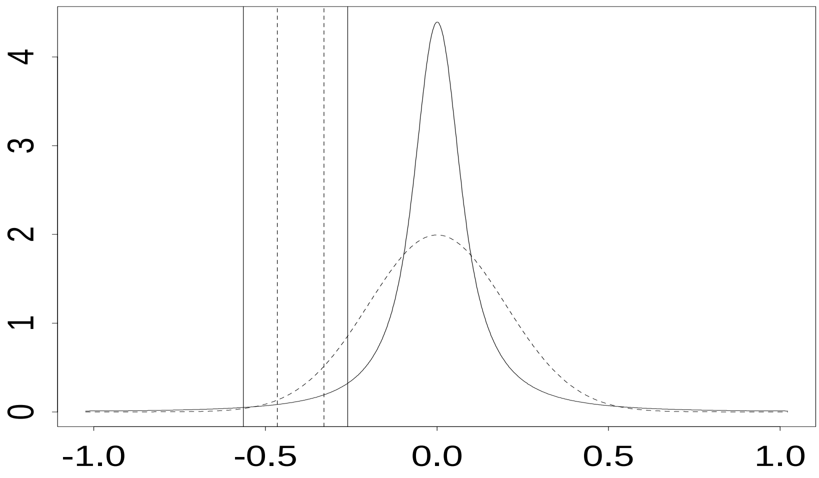


Figure 4.8: Density of $L(1)$ of the normal inverse Gaussian Lévy process with the same parameters as in Figure 4.7, density of the standard normal distribution (dashed line) with the same variance 0.04 and the corresponding 1%-quantiles (left vertical lines) and 5%-quantiles (right vertical lines) .

Example 4.4.2 *(Exponential variance gamma (VG) Lévy process)* (a) As mentioned in Asmussen and Rosinski (2000), for the gamma process with $\nu(dx) = \xi x^{-1}e^{-x/\delta}dx, \delta, \xi > 0, x > 0$ the normal approximation for small jumps fails. This is a consequence of Proposition 4.3.2, since

$$\lim_{\varepsilon\to 0}\frac{\sigma^2(\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon\to 0}\frac{\xi}{\varepsilon^2}\int_0^\varepsilon xe^{-x/\delta}dx = \frac{\xi}{2}\,, \tag{4.4.2}$$

using for instance l'Hospital's rule. The limit relations of Theorem 4.3.4 hold, however, with Lévy process V having characteristic triplet $(a_V, 0, \nu_V)$ where

$$a_V = \xi(1-\sqrt{2/\xi})\wedge 0 \;\text{ and }\; \nu_V(dy) = \frac{\xi}{y}1_{(0,\sqrt{2/\xi})}(y)dy.$$

Proposition 4.3.7 gives then the approximation for the small jumps.
We show that (4.3.9) holds, which corresponds to (4.3.8).
Set

$$D_\varepsilon(t) := \sigma(\varepsilon)^{-1}(L(t) - L_\varepsilon(t))\,, \quad t \ge 0\,,$$

By Pollard (1984), Theorem V.19, (4.3.9) is equivalent to $D_\varepsilon(1) \xrightarrow{d} V(1)$, since D_ε are Lévy processes. By Kallenberg (1997), Theorem 13.14 we need to show for the characteristic

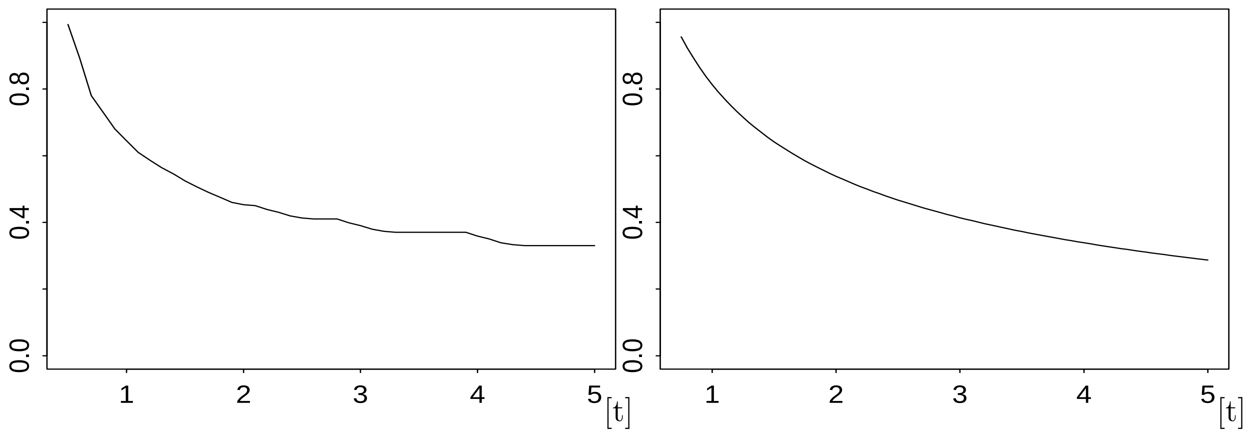


Figure 4.9: Optimal portfolio in the exponential normal inverse Gaussian Lévy model for the same parameters as in Figure 4.7 under the constraint $\mathrm{CaR}(x,\pi,T) \le \mathrm{CaR}(1000,1,0.5)$ (left), optimal portfolio in the exponential NIG-Lévy model under the constraint $\mathrm{var}(X^\pi(t)) \le 100\,000$ for the same parameters (right).

triplet $(a_D, 0, \nu_D)$ of the Lévy process D_ε

$$\lim_{\varepsilon\to 0} \nu_D([x,z]) = \nu_V([x,z]) \text{ for any } 0 < x < z \tag{4.4.3}$$

$$\lim_{\varepsilon\to 0} \int_{|y|<K} y^2 \nu_D(dy) = \int_{|y|<K} y^2 \nu_V(dy) \text{ for each } K > 0 \tag{4.4.4}$$

$$\lim_{\varepsilon\to 0} a_D = a_V \tag{4.4.5}$$

First we prove (4.4.3). By the proof of Theorem 2.1 of Asmussen and Rosinski (2000) for the process D_ε we have

$$a_D = -\frac{1}{\sigma(\varepsilon)} \int_{\sigma(\varepsilon)\wedge\varepsilon < y < \varepsilon} \xi e^{-y/\delta} dy$$

and Lévy measure $\nu_D = \nu(\sigma(\varepsilon)B \cap (0,\varepsilon))$ for any Borel set $B \subset \mathbb{R}^*$.
Hence V has Lévy measure $\nu_V(B) = \lim_{\varepsilon\to 0} \nu(\sigma(\varepsilon)B \cap (0,\varepsilon))$. For any interval $[x,z]$, $0 < x < z$, we calculate

$$\lim_{\varepsilon\to 0} \nu_D([x,z]) = \lim_{\varepsilon\to 0} \int_{\varepsilon\wedge\sigma(\varepsilon)x}^{\varepsilon\wedge\sigma(\varepsilon)z} \xi y^{-1} e^{-y/\delta} dy = \xi \ln\left(\frac{z \wedge \sqrt{2/\xi}}{x \wedge \sqrt{2/\xi}}\right) = \nu_V([x,z])\,,$$

where we have used that $e^{-y/\xi} \to 1$ as $y \to 0$.
Next we prove (4.4.4). For each $K > 0$ we calculate $\int_{|y|<K} y^2 \nu_V(dy) = \frac{\xi K^2}{2} \wedge 1$ giving with (4.4.2)

$$\int_{|y|<K} y^2 \nu_D(dy) = \frac{\sigma^2(K\sigma(\varepsilon)\wedge\varepsilon)}{\sigma^2(\varepsilon)} \to \frac{\xi K^2}{2} \wedge 1\,, \quad \varepsilon \to 0\,.$$

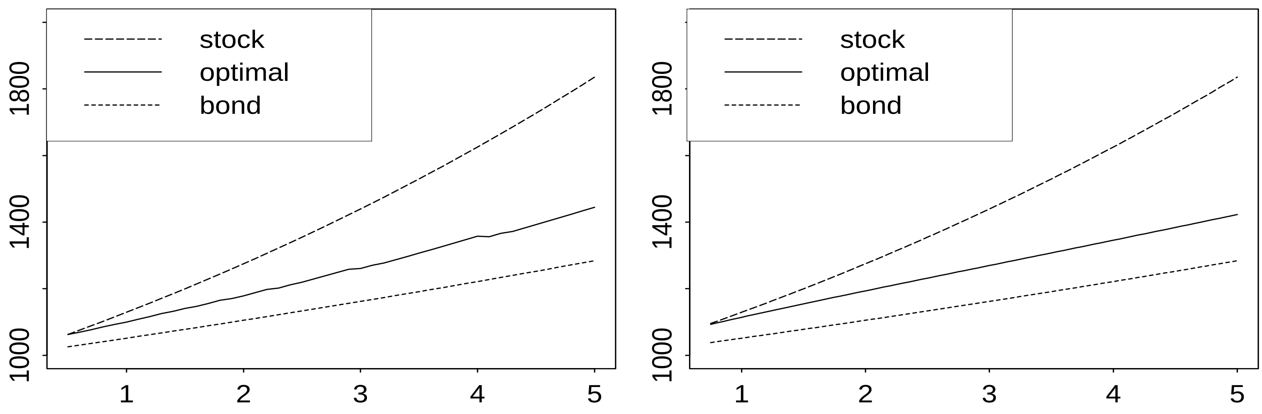


Figure 4.10: Expected terminal wealth for different investment strategies in the exponential normal inverse Gaussian Lévy model for the same parameters as in Figure 4.7 under a constraint on the CaR(left) and under a constraint on the variance (right). As the upper bound C of the CaR we used CaR(1000,1,0.5), the CaR of a pure stock strategy with time horizon $T = 0.5$, as the upper bound C of the variance we used 100 000.

Similarly we calculate

$$a_V = \lim_{\varepsilon\to 0} a_D = \lim_{\varepsilon\to 0} -\frac{1}{\sigma(\varepsilon)}\int_{\sigma(\varepsilon)\wedge\varepsilon<y<\varepsilon} \xi e^{-y/\delta}dy = \xi(1-\sqrt{2/\xi})1_{\{1-\sqrt{2/\xi}<0\}}$$

which proves (4.4.5).

(b) For the exponential variance gamma Lévy process the normal approximation for small jumps is not possible either, since by Example 4.2.7 and e.g. l'Hospital's rule

$$\begin{aligned}\frac{\sigma^2(\varepsilon)}{\varepsilon^2} &= \frac{1}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\xi\frac{x^2}{|x|}\exp(-\sqrt{\frac{2}{\theta}+\delta^2}|x|-\delta x)dx\\ &= \frac{\xi}{\varepsilon^2}\int_0^{\varepsilon}x(\exp(c_1x)+\exp(c_2x))dx \quad\to\quad \xi\,,\quad \varepsilon\to 0\,,\end{aligned}$$

where $c_1 = -\left(\sqrt{\frac{2}{\theta}+\delta^2}+\delta\right) < 0$ and $c_2 = -\left(\sqrt{\frac{2}{\theta}+\delta^2}-\delta\right) < 0$.
As in part (a) we show (4.4.3)-(4.4.5) and obtain a limit process V with characteristic triplet $(0,0,\nu_V)$, where

$$\nu_V(dy) = \frac{\xi}{y}1_{(-1/\sqrt{\xi},\,1/\sqrt{\xi})}(y)dy.$$

4.5 Proof of Theorem 4.3.4

We first derive some auxiliary results. As usual we write

$$a\Lambda := \{ax|\, x \in \Lambda\},\ e^{\Lambda} := \{e^x|\, x \in \Lambda\},\ \text{and } \Lambda - 1 := \{x-1|\, x \in \Lambda\}.$$

Lemma 4.5.1 *Let Z_ε and Y_ε be Lévy processes with characteristic triplets as in Theorem 4.3.4. Set*

$$E_\varepsilon := \frac{Z_\varepsilon}{g(\varepsilon)} \quad \text{and} \quad D_\varepsilon := \frac{Y_\varepsilon}{g(\varepsilon)}$$

Then E_ε is a Lévy process with characteristic triplet $(a_E, 0, \nu_E)$ and D_ε is a Lévy process with characteristic triplet $(a_D, 0, \nu_D)$, which both depend on ε. They satisfy the following relations:

$$\begin{aligned}
a_E &= \frac{1}{g(\varepsilon)}\left(a_Z - \int_{g(\varepsilon)<|x|\le 1} x\nu_Z(dx)\right),\\
\nu_E(\Lambda) &= \nu_Z(g(\varepsilon)\Lambda) = \nu_Y(\{x|(e^x-1)/g(\varepsilon) \in \Lambda\}) \ \text{for any Borel set } \Lambda \subset \mathbb{R}^*,\\
a_D &= \frac{1}{g(\varepsilon)}\left(a_Y - \int_{g(\varepsilon)<|x|\le 1} x\nu_Y(dx)\right),\\
\nu_D(\Lambda) &= \nu_Y(g(\varepsilon)\Lambda) = \nu_Z(e^{g(\varepsilon)\Lambda}-1) \ \text{for any Borel set } \Lambda \subset \mathbb{R}^*,\\
a_D - a_E &= \frac{1}{g(\varepsilon)}\int (\ln(x+1)1_{\{|\ln(x+1)|\le g(\varepsilon)\}} - x1_{\{|x|\le g(\varepsilon)\}})\nu_Z(dx).\\
&= \frac{1}{g(\varepsilon)}\int (x1_{\{|x|\le g(\varepsilon)\}} - (e^x-1)1_{\{|e^x-1|\le g(\varepsilon)\}})\nu_Y(dx).
\end{aligned}$$

Proof Since E_ε and D_ε have no Gaussian component, $\beta_E = \beta_D = 0$. For any Borel set $\Lambda \subset \mathbb{R}^*$, using Lemma 4.1.2 and Lemma 4.1.3 for $d = 1$ and setting $\pi = 1/g(\varepsilon)$ we obtain

$$\nu_E(\Lambda) \;=\; \nu_Z(g(\varepsilon)\Lambda) = \nu_Y(x|(e^x-1)/g(\varepsilon) \in \Lambda)$$

and analogously,

$$\nu_D(\Lambda) = \nu_Y(g(\varepsilon)\Lambda) = \nu_Z(x|\ln(x+1)/g(\varepsilon) \in \Lambda).$$

Again by Lemma 4.1.2 and Lemma 4.1.3 for $d = 1$ and setting $\pi = 1/g(\varepsilon)$ we obtain

$$a_E = \frac{1}{g(\varepsilon)}a_Z + \int \frac{x}{g(\varepsilon)}(1_{\{|x|\le g(\varepsilon)\}} - 1_{\{|x|\le 1\}})\nu_Z(dx) = \frac{1}{g(\varepsilon)}\left(a_Z - \int\limits_{g(\varepsilon)<|x|\le 1} x\nu_Z(dx)\right).$$

In a similar way we prove

$$a_D = \frac{1}{g(\varepsilon)} \left(a_Y - \int\limits_{g(\varepsilon)<|x|\leq 1} x\nu_Y(dx) \right).$$

Using Lemma 4.1.2 we obtain

$$\begin{aligned} a_D - a_E &= \frac{1}{g(\varepsilon)} \left(a_Y - a_Z + \int\limits_{g(\varepsilon)<|x|\leq 1} x(\nu_Z - \nu_Y)(dx) \right) \\ &= \frac{1}{g(\varepsilon)} \int \left(\ln(x+1) 1_{\{|\ln(x+1)|<g(\varepsilon)\}} - x 1_{\{|x|<g(\varepsilon)\}} \right) \nu_Z(dx). \end{aligned}$$

□

Lemma 4.5.2 *Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then*

$$\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{(-hg(\varepsilon), hg(\varepsilon))} x^2 \nu_Z(dx) = K(h) \quad \forall h > 0 \tag{4.5.1}$$

if and only if

$$\lim_{\varepsilon \to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu_Z(dx) = K(h) \quad \forall h > 0\,, \tag{4.5.2}$$

where $A_{\varepsilon,h} := (\exp(-hg(\varepsilon)) - 1, \exp(hg(\varepsilon)) - 1)$ for each $\varepsilon, h > 0$.

Proof Set $\nu = \nu_Z$. Let $h > 0$. Since $g(\varepsilon) \to 0$ as $\varepsilon \to 0$, there exists some $\tilde{\varepsilon} > 0$ such that $ehg(\varepsilon) < 1$ for all $0 < \varepsilon < \tilde{\varepsilon}$. By a Taylor expansion we have $e^{hg(\varepsilon)} - 1 = hg(\varepsilon)e^{\theta hg(\varepsilon)}$ for some $\theta \in (0,1)$ and hence

$$e^{-1}hg(\varepsilon) < hg(\varepsilon) < e^{hg(\varepsilon)} - 1 < ehg(\varepsilon) \tag{4.5.3}$$

and, analogously,

$$-ehg(\varepsilon) < -hg(\varepsilon) < e^{-hg(\varepsilon)} - 1 < -e^{-1}hg(\varepsilon)\,. \tag{4.5.4}$$

This leads to

$$(-K_1 g(\varepsilon), K_1 g(\varepsilon)) \subseteq A_{\varepsilon,h} \subseteq (-K_2 g(\varepsilon), K_2 g(\varepsilon)) \tag{4.5.5}$$

for $K_1 = e^{-1}h$ and $K_2 = eh$.

Assume that (4.5.1) holds. Then by a Taylor expansion around 0 we have for some $\theta = theta(x) \in (0,1)$

$$\ln(x+1) = x - \frac{x^2}{2(\theta x+1)^2}$$

giving

$$\begin{aligned}
&\frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu(dx) \\
&= \frac{1}{g^2(\varepsilon)} \int\limits_{A_{\varepsilon,h}} x^2 \nu(dx) - \frac{1}{g^2(\varepsilon)} \int_{A_{\varepsilon,h}} \frac{x^3}{(\theta x+1)^2} \nu(dx) + \frac{1}{g^2(\varepsilon)} \int_{A_{\varepsilon,h}} \frac{x^4}{4(\theta x+1)^4} \nu(dx) \\
&= I_1(\varepsilon) - I_2(\varepsilon) + I_3(\varepsilon) \,. \qquad (4.5.6)
\end{aligned}$$

First note that with (4.5.5) and (4.5.1),

$$\begin{aligned}
&|I_2(\varepsilon) - I_3(\varepsilon)| \\
&\le \frac{1}{g^2(\varepsilon)} \left(\left| \int_{A_{\varepsilon,h}} \frac{x^3}{(\theta x+1)^2} \nu(dx) \right| + \int_{A_{\varepsilon,h}} \frac{x^4}{4(\theta x+1)^4} \nu(dx) \right) \\
&\le \frac{1}{g^2(\varepsilon)} \left(\int_{(-K_2 g(\varepsilon), K_2 g(\varepsilon))} \left| \frac{x^3}{(\theta x+1)^2} \right| \nu(dx) + \int_{(-K_2 g(\varepsilon), K_2 g(\varepsilon))} \frac{x^4}{4(\theta x+1)^4} \nu(dx) \right) \\
&\le \left(\sup_{x \in (-K_2 g(\varepsilon), K_2 g(\varepsilon))} \left(\frac{|x|}{(\theta x+1)^2} + \frac{x^2}{4(\theta x+1)^4} \right) \right) \frac{1}{g^2(\varepsilon)} \int_{(-K_2 g(\varepsilon), K_2 g(\varepsilon))} x^2 \nu(dx) \\
&\le \left(\frac{K_2 g(\varepsilon)}{(1-K_2 g(\varepsilon))^2} + \frac{(K_2 g(\varepsilon))^2}{4(1-K_2 g(\varepsilon))^4} \right) \frac{1}{g^2(\varepsilon)} \int_{(-K_2 g(\varepsilon), K_2 g(\varepsilon))} x^2 \nu(dx) \\
&\to 0\,, \quad \varepsilon \to 0. \qquad (4.5.7)
\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int_{A_{\varepsilon,h}} (\ln(x+1))^2 \nu(dx) = \lim_{\varepsilon\to 0} I_1(\varepsilon) = \lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int_{(-hg(\varepsilon)\exp(-\theta_1 hg(\varepsilon)), hg(\varepsilon)\exp(\theta_2 hg(\varepsilon)))} x^2 \nu(dx)$$

for some θ_1, $\theta_2 \in (0,1)$ using a Taylor expansion. Thus, since θ_1, $\theta_2 \in (0,1)$,

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int_{|x|<hg(\varepsilon)\exp(-hg(\varepsilon))} x^2 \nu(dx) \le \lim_{\varepsilon\to 0} I_1(\varepsilon) \le \lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int_{|x|<hg(\varepsilon)\exp(hg(\varepsilon))} x^2 \nu(dx).$$

Since $g(\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain for all $\varepsilon_0 > 0$ and $\varepsilon < \varepsilon_0$ an upper bound for the right-hand side

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int_{|x|<hg(\varepsilon)\exp(hg(\varepsilon_0))} x^2 \nu(dx) = K(h\exp(hg(\varepsilon_0)))\,.$$

Since ε_0 can be chosen arbitrarily small, we obtain under condition (4.5.1)

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int_{|x|<hg(\varepsilon)\exp(hg(\varepsilon))} x^2\nu(dx) = K(h). \tag{4.5.8}$$

Similarly, we get a lower bound and hence

$$\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int_{|x|<hg(\varepsilon)\exp(-hg(\varepsilon))} x^2\nu(dx) = K(h)$$

and thus

$$\lim_{\varepsilon\to 0} I_1(\varepsilon) = K(h).$$

For the converse first note that by (4.5.7)

$$\begin{aligned}
&|I_2(\varepsilon) - I_3(\varepsilon)| \\
&\le \frac{1}{g^2(\varepsilon)} \left(\left| \int_{A_{\varepsilon,h}} \frac{x^3}{(\theta x+1)^2}\nu(dx) \right| + \int_{A_{\varepsilon,h}} \frac{x^4}{4(\theta x+1)^4}\nu(dx) \right) \\
&\le \frac{1}{g^2(\varepsilon)} \left(\int_{A_{\varepsilon,h}} \left| \frac{x^3}{(\theta x+1)^2} \right| \nu(dx) + \int_{A_{\varepsilon,h}} \frac{x^4}{4(\theta x+1)^4}\nu(dx) \right) \\
&\le \left(\sup_{x\in A_{\varepsilon,h}} \left(\frac{|x|}{(\theta x+1)^2} + \frac{x^2}{4(\theta x+1)^4} \right) \right) \frac{1}{g^2(\varepsilon)} \int_{A_{\varepsilon,h}} x^2\nu(dx) \\
&\le \left(\frac{\exp(g(\varepsilon)h)-1}{\exp(-2g(\varepsilon)h)} + \frac{(\exp(g(\varepsilon)h)-1)^2}{4\exp(-4g(\varepsilon)h)} \right) I_1(\varepsilon)
\end{aligned} \tag{4.5.9}$$

and hence $|I_2(\varepsilon) - I_3(\varepsilon)| \le T(\varepsilon)I_1(\varepsilon)$ for some positive $T(\varepsilon) \to 0$ as $\varepsilon \to 0$. So by (4.5.6)

$$I_1(\varepsilon) \le \frac{1}{g^2(\varepsilon)} \int_{A_{\varepsilon,h}} (\ln(x+1))^2\nu(dx) + T(\varepsilon)I_1(\varepsilon)$$

and hence

$$I_1(\varepsilon)(1 - T(\varepsilon)) \le \frac{1}{g^2(\varepsilon)} \int_{A_{\varepsilon,h}} (\ln(x+1))^2\nu(dx).$$

Taking limsup results in $\limsup_{\varepsilon\to 0} I_1(\varepsilon) \le K(h)$. Then by (4.5.9) $|I_2(\varepsilon) - I_3(\varepsilon)| \to 0$ and by (4.5.6) we obtain $\lim_{\varepsilon\to 0} I_1(\varepsilon) = K(h)$ for each $h > 0$. Using the same argument as for (4.5.8),

$$\begin{aligned}
K(h) &= \lim_{\varepsilon\to 0} I_1(\varepsilon) \\
&\le \lim \frac{1}{g^2(\varepsilon)} \int_{|x|<hg(\varepsilon)\exp(hg(\varepsilon))} x^2\nu(dx)
\end{aligned}$$

$$
\begin{aligned}
&= \lim \frac{1}{g^2(\varepsilon)} \int_{|x|<hg(\varepsilon)} x^2 \nu(dx) \\
&= \lim \frac{1}{g^2(\varepsilon)} \int_{|x|<hg(\varepsilon)\exp(-hg(\varepsilon))} x^2 \nu(dx) \\
&\leq \lim_{\varepsilon\to 0} I_1(\varepsilon) \\
&= K(h)
\end{aligned}
$$

we obtain (4.5.1).

Lemma 4.5.3 *Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then*

$$
\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{(-hg(\varepsilon),hg(\varepsilon))} x^2 \nu_Y(dx) = K(h), \quad \varepsilon \to 0 \tag{4.5.10}
$$

holds for each $h > 0$ if and only if

$$
\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \int\limits_{B_{\varepsilon,h}} (e^x - 1)^2 \nu_Y(dx) = K(h)\,, \quad \varepsilon \to 0\,, \tag{4.5.11}
$$

for each $h > 0$, where $B_{\varepsilon,h} := (\ln(1 - hg(\varepsilon)), \ln(1 + hg(\varepsilon)))$ for each $\varepsilon, h > 0$.

Proof Set $\nu = \nu_Y$. By (4.5.3) we obtain for $\varepsilon, h > 0$

$$
e^{-1}hg(\varepsilon) < \ln(1 + hg(\varepsilon)) < hg(\varepsilon) < ehg(\varepsilon) \tag{4.5.12}
$$

and, analogously, by (4.5.4)

$$
-ehg(\varepsilon) < \ln(1 - hg(\varepsilon)) < -hg(\varepsilon) < -e^{-1}hg(\varepsilon)\,. \tag{4.5.13}
$$

Then we obtain for $K_1 = e^{-1}h$ and $K_2 = eh$

$$
(-K_1 g(\varepsilon),\, K_1 g(\varepsilon)) \subseteq B_{\varepsilon,h} \subseteq (-K_2 g(\varepsilon),\, K_2 g(\varepsilon))\,. \tag{4.5.14}
$$

Assume that (4.5.10) holds. Then by a Taylor expansion around 0 we have $e^x - 1 = xe^{\theta x}$ for some $\theta \in (0, 1)$ and by (4.5.14) for $x \in B_{\varepsilon,h}$,

$$
xe^{-K_2 g(\varepsilon)} \leq e^x - 1 \leq xe^{K_2 g(\varepsilon)}\,,
$$

giving

$$\frac{1}{g^2(\varepsilon)}\int_{B_{\varepsilon,h}}(e^x-1)^2\nu(dx)=\frac{1}{g^2(\varepsilon)}\int_{B_{\varepsilon,h}}x^2\nu(dx)(1+o(1))\,,\quad \varepsilon\to 0\,. \tag{4.5.15}$$

By some Taylor expansion around 0 we have for some θ_1, $\theta_2 \in (0,1)$

$$B_{\varepsilon,h}=\left(-hg(\varepsilon)-\frac{(hg(\varepsilon))^2}{2(-\theta_2 hg(\varepsilon)+1)^2},hg(\varepsilon)-\frac{(hg(\varepsilon))^2}{2(\theta_1 hg(\varepsilon)+1)^2}\right)$$

and hence

$$\{|x|<hg(\varepsilon)(1-\frac{hg(\varepsilon)}{2})\}\subset B_{\varepsilon,h}\subset\{|x|<hg(\varepsilon)(1+\frac{hg(\varepsilon)}{2(1-hg(\varepsilon))^2})\}$$

Analogously to the proof of Lemma 4.5.2 we obtain $\lim\limits_{\varepsilon\to 0}\frac{1}{(g(\varepsilon))^2}\int_{B_{\varepsilon,h}}x^2\nu(dx)=K(h)$.

For the converse notice that (4.5.15) implies $\frac{1}{g^2(\varepsilon)}\int_{B_{\varepsilon,h}}x^2\nu(dx)\to K(h)$, $\varepsilon\to 0$.

Thus by the proof of the other direction

$$\begin{aligned}
K(h) &= \lim_{\varepsilon\to 0}\frac{1}{g^2(\varepsilon)}\int_{B_{\varepsilon,h}}x^2\nu(dx)\\
&\le \lim_{\varepsilon\to 0}\frac{1}{g^2(\varepsilon)}\int_{|x|<hg(\varepsilon)(1+\frac{hg(\varepsilon)}{2(1-hg(\varepsilon))^2})}x^2\nu(dx)\\
&= \lim_{\varepsilon\to 0}\frac{1}{g^2(\varepsilon)}\int_{|x|<hg(\varepsilon)}x^2\nu(dx)\\
&= \lim_{\varepsilon\to 0}\frac{1}{g^2(\varepsilon)}\int_{|x|<hg(\varepsilon)(1-\frac{hg(\varepsilon)}{2})}x^2\nu(dx)\\
&\le \lim_{\varepsilon\to 0}\frac{1}{g^2(\varepsilon)}\int_{B_{\varepsilon,h}}x^2\nu(dx)\\
&= K(h)
\end{aligned}$$

and hence we obtain (4.5.10).

Now we can prove Theorem 4.3.4.

Proof of Theorem 4.3.4. Assume that (4.3.7) holds, i.e. $E_\varepsilon(t)=Z^\varepsilon(t)/g(\varepsilon)\xrightarrow{d}V(t)$, $t\ge 0$, as $\varepsilon\to 0$. Since E_ε are Lévy processes weak convergence of the processes is equivalent to $E_\varepsilon(1)\xrightarrow{d}V(1)$ (see e.g. Pollard (1984), Theorem V.19). Let now $(a_E,0,\nu_E)$ be the characteristic triplets of the Lévy processes E_ε as derived in Lemma 4.5.1 (recall that they depend on ε). Since $\beta_E=0$, according to Kallenberg (1997), Theorem 13.14, $E_\varepsilon(1)\xrightarrow{d}V(1)$

if and only if

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\{|x|<h\}} x^2\nu_E(dx) &= \beta_V^2 + \int_{|x|<h} x^2\nu_V(dx) \quad \forall h>0\,, && (4.5.16)\\
\lim_{\varepsilon\to 0}\nu_E(\{|x|\geq c\}) &= \nu_V(\{|x|\geq c\}) \quad \forall c>0\,, && (4.5.17)\\
\lim_{\varepsilon\to 0} a_E &= a_V\,. && (4.5.18)
\end{aligned}$$

So we assume that (4.5.16)-(4.5.18) hold.

Moreover, setting $D_\varepsilon = Y^\varepsilon/g(\varepsilon)$ with characteristic triplets $(a_D, 0, \nu_D)$ (which depend on ε), we have to show

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\{|x|<h\}} x^2\nu_D(dx) &= \beta_V^2 + \int_{|x|<h} x^2\nu_V(dx) \quad \forall h>0\,, && (4.5.19)\\
\lim_{\varepsilon\to 0}\nu_D(\{|x|\geq c\}) &= \nu_V(\{|x|\geq c\}) \quad \forall c>0\,, && (4.5.20)\\
\lim_{\varepsilon\to 0} a_D &= a_V\,. && (4.5.21)
\end{aligned}$$

To prove (4.5.19) we consider

$$\begin{aligned}
\int_{|x|<h} x^2\nu_D(dx) &= E\left[\sum_{s\leq 1}(\Delta D_\varepsilon(s))^2 1_{\{|\Delta D_\varepsilon(s)|<h\}}\right]\\
&= \frac{1}{g^2(\varepsilon)}E\left[\sum_{s\leq 1}(\ln(1+\Delta Z_\varepsilon(s)))^2 1_{\{\Delta Z_\varepsilon(s)\in A_{\varepsilon,h}\}}\right]\\
&= \frac{1}{g^2(\varepsilon)}\int_{A_{\varepsilon,h}}(\ln(x+1))^2\nu_Z(dx)\,, && (4.5.22)
\end{aligned}$$

where $A_{\varepsilon,h} = (e^{-g(\varepsilon)h}-1, e^{g(\varepsilon)h}-1)$. By (4.5.16) and Lemma 4.5.2, setting $K(h) = \beta_V^2 + \int_{|x|<h} x^2\nu_V(dx)$ the right-hand side of (4.5.22) converges to $\beta_V^2 + \int_{|x|<h} x^2\nu_V(dx)$ for each $h>0$.

Now we prove (4.5.20). By Lemma 4.5.1 we have

$$\begin{aligned}
\nu_D(\{|x|\geq c\}) &= \nu_Z(e^{g(\varepsilon)\{|x|\geq c\}}-1)\\
&= \nu_Z(e^{g(\varepsilon)\{|x|\geq c\}}-1\cap\{|x|\geq cg(\varepsilon)\}) + \nu_Z(e^{g(\varepsilon)\{|x|\geq c\}}-1\cap\{|x|<cg(\varepsilon)\})
\end{aligned}$$

The first term converges to $\nu_V(\{|x|\geq c\})$, since by (4.5.17)

$$\nu_Z(\{|x|\geq cg(\varepsilon)\}) = \nu_E(\{|x|\geq c\}) \to \nu_V(\{|x|\geq c\}).$$

Since for any Borel set $\Lambda \subset \mathbb{R}^*$

$$\nu_Z(\Lambda) \inf_{x\in\Lambda} (\ln(x+1))^2 \leq \int_\Lambda (\ln(x+1))^2 \nu_Z(dx)$$

holds, we get

$$\begin{aligned}
&\nu_Z(e^{g(\varepsilon)\{|x|\geq c\}} - 1 \cap \{|x| < cg(\varepsilon)\}) \\
&= \nu_Z(\{|x| < cg(\varepsilon)\} \backslash (e^{g(\varepsilon)\{|x|<c\}} - 1)) \\
&\leq \frac{1}{(cg(\varepsilon))^2} \int_{\{|x|<cg(\varepsilon)\}\backslash(e^{g(\varepsilon)\{|x|<c\}}-1)} (\ln(1+x))^2 \nu_Z(dx) \\
&= \frac{1}{(cg(\varepsilon))^2} \int_{\{|x|<cg(\varepsilon)\}} (\ln(1+x))^2 \nu_Z(dx) \\
&\quad - \frac{1}{(cg(\varepsilon))^2} \int_{\{|x|<cg(\varepsilon)\}\cap(e^{g(\varepsilon)\{|x|<c\}}-1)} (\ln(1+x))^2 \nu_Z(dx) \\
&\to 0, \quad \varepsilon \to 0,
\end{aligned}$$

since both terms in the second last line tend to $K(c)/c^2$, where $K(h) = \beta_V^2 + \int_{|x|<h} x^2 \nu_V(dx)$. This can be seen as follows. For the first term we use Taylor's theorem in the same way as in the proof of Lemma 4.5.2 replacing $A_{\varepsilon,h}$ by $(-cg(\varepsilon), cg(\varepsilon))$. The second term tends to $K(c)/c^2$ using the same Taylor expansion and since by a Taylor expansion for $e^x - 1$ around 0

$$\begin{aligned}
\{|x| < cg(\varepsilon)\} \cap (e^{g(\varepsilon)\{|x|<c\}} - 1) &= \{|x| < cg(\varepsilon)\} \cap (-cg(\varepsilon)e^{-\theta_1 cg(\varepsilon)}, cg(\varepsilon)e^{\theta_2 cg(\varepsilon)}) \\
&= (-cg(\varepsilon)e^{-\theta_1 cg(\varepsilon)}, cg(\varepsilon))
\end{aligned}$$

for some θ_1, $\theta_2 \in (0,1)$.

Now we prove (4.5.21). By (4.5.18) we know that $a_E \to a_V$, hence we only need to show

$$|a_D - a_E| \to 0.$$

By Lemma 4.5.1 and the Taylor expansion we used in (4.5.6) we obtain for some $\theta \in (0,1)$

$$\begin{aligned}
|a_D - a_E| &= \frac{1}{g(\varepsilon)} \left| \int \left(\ln(x+1) 1_{\{|\ln(x+1)|<g(\varepsilon)\}} - x 1_{\{|x|<g(\varepsilon)\}} \right) \nu_Z(dx) \right| \\
&= \frac{1}{g(\varepsilon)} \left| \int \left(\left(x - \frac{x^2}{2(\theta x+1)^2} \right) 1_{\{e^{-g(\varepsilon)}-1<x<e^{g(\varepsilon)}-1\}} - x 1_{\{|x|<g(\varepsilon)\}} \right) \nu_Z(dx) \right|
\end{aligned}$$

$$= \frac{1}{g(\varepsilon)} \left| \int (x1_{\{e^{-g(\varepsilon)}-1<x<e^{g(\varepsilon)}-1\}} - x1_{\{|x|<g(\varepsilon)\}})\nu_Z(dx) \right.$$
$$\left. - \int_{(e^{-g(\varepsilon)}-1, e^{g(\varepsilon)}-1)} \frac{x^2}{2(\theta x+1)^2}\nu_Z(dx) \right|$$

Since

$$\frac{1}{g(\varepsilon)} \int_{(e^{-g(\varepsilon)}-1, e^{g(\varepsilon)}-1)} \frac{x^2}{2(\theta x+1)^2}\nu_Z(dx) \to 0$$

because of

$$\frac{1}{2(\theta x+1)^2} < \frac{1}{2e^{-2g(\varepsilon)}} \text{ for } x \in (e^{-g(\varepsilon)}-1, e^{g(\varepsilon)}-1),$$

we obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} |a_D - a_E|^2 \\
&= \limsup_{\varepsilon\to 0} \frac{1}{(g(\varepsilon))^2} \left| \int x(1_{\{e^{-g(\varepsilon)}-1<x<e^{g(\varepsilon)}-1\}} - 1_{\{|x|<g(\varepsilon)\}})\nu_Z(dx) \right|^2 \\
&= \limsup_{\varepsilon\to 0} \frac{1}{(g(\varepsilon))^2} \left| \int\limits_{e^{g(\varepsilon)\{|x|<1\}}-1\setminus\{|x|<g(\varepsilon)\}} x\nu_Z(dx) - \int\limits_{\{|x|<g(\varepsilon)\}\setminus e^{g(\varepsilon)\{|x|<1\}}-1} x\nu_Z(dx) \right|^2 \\
&\leq \limsup_{\varepsilon\to 0} \frac{1}{(g(\varepsilon))^2} \int\limits_{e^{g(\varepsilon)\{|x|<1\}}-1\setminus\{|x|<g(\varepsilon)\}} x^2\nu_Z(dx) \\
&\quad + \limsup_{\varepsilon\to 0} \frac{1}{(g(\varepsilon))^2} \int\limits_{\{|x|<g(\varepsilon)\}\setminus e^{g(\varepsilon)\{|x|<1\}}-1} x^2\nu_Z(dx).
\end{aligned}$$

Both terms converge to 0 as follows.

$$\frac{1}{(g(\varepsilon))^2} \int\limits_{e^{g(\varepsilon)\{|x|<1\}}-1} x^2\nu_Z(dx) \to K(1)$$

by the proof of Lemma (4.5.2),

$$\frac{1}{(g(\varepsilon))^2} \int\limits_{\{|x|<g(\varepsilon)\}} x^2\nu_Z(dx) \to K(1)$$

by (4.5.16) for $h = 1$, and

$$\frac{1}{(g(\varepsilon))^2} \int\limits_{\{|x|<g(\varepsilon)\}\cap e^{g(\varepsilon)\{|x|<1\}}-1} x^2\nu_Z(dx) \to K(1),$$

where $K(h) = \beta_V^2 + \int_{|x|<h} x^2 \nu_V(dx)$, since by a Taylor expansion for $e^x - 1$ around 0

$$\begin{aligned} \{|x| < g(\varepsilon)\} \cap e^{g(\varepsilon)\{|x|<1\}} - 1) &= \{|x| < g(\varepsilon)\} \cap (-g(\varepsilon)e^{\theta_1 g(\varepsilon)}, g(\varepsilon)e^{\theta_2 g(\varepsilon)}) \\ &= (-g(\varepsilon)e^{\theta_1 g(\varepsilon)}, g(\varepsilon)) \end{aligned}$$

for some θ_1, $\theta_2 \in (0, 1)$ and using the same argumentation as in the proof of Lemma 4.5.2. The other direction can be proved analogously.

Chapter 5

Optimal portfolios with possible bankruptcy and market crash

Asset prices in the Black-Scholes model can also written as stochastic differential equations which are equivalent to the corresponding geometric Brownian motions. E.g. by Eberlein and Keller (1995) we know that the normal distribution for asset price modelling is not very realistic, since the distribution of asset prices often has semiheavy tails, i.e. its curtosis is higher than that of the normal distribution. Thus is seems to be a natural approach to replace the Brownian motian in the stochastic differential equation by general stochastic processes with independent stationary increments, i.e. Lévy processes. But Lévy processes can have negative jumps with absolute size greater than one. Solving the stochastic differential equation by Itô's formula this leads to a positive probability for negative asset prices, which do not accure in reality. By Eberlein and Keller (1995) such a jump can be interpreted as a market crash, after which the asset price equals zero. Because of the independence of increments of the Lévy process, such a crash has always the same probability, independent of the actual asset price. For example this can be used as a realistic approach to model new economy asset prices. We optimize portfolios containing such assets with a positive crash probability.

This chapter is organized as follows. In Section 2 we consider some portfolios consisting of one riskless bond and several risky stocks. There prices follow an SDE driven by some Lévy process. To avoid negative stock prices we investigate only Lévy processes with jump heights > -1. The solution of the mean variance problem has the same structure for any price process which follows an SDE driven by a Lévy process. In Section 3 we drop the restriction on the jump heights and use an idea of Eberlein and Keller (1995). They interpret a jump leading to a negative stock price as a market crash and after this market crash the stock price equals zero. This crash possibility leads to a much more complicated structure of the mean variance problem.

5.1 The general market model

We consider a standard Black-Scholes type market consisting of a *riskless bond* and several *risky stocks*, which follow SDEs driven by Lévy processes. Their respective prices $(P_0(t))_{t\geq 0}$ and $(P_i(t))_{t\geq 0}$, $i = 1, \dots, d$, evolve according to the equations

$$\begin{aligned} dP_0(t) &= P_0(t) r dt, & P_0(0) &= 1\,, \\ dP_i(t) &= P_i(t-)\left(b_i dt + \sum_{j=1}^{d} \sigma_{ij}\left(\sum_{l=1}^{d} \beta_{lj} dW_l(t) + \sum_{l=1}^{d} \delta_{lj} d\widetilde{L}_l(t)\right)\right), & P_i(0) &= p_i\,. \end{aligned} \tag{5.1.1}$$

Here $\widetilde{L} = (\widetilde{L}(t))_{t\geq 0} = (\widehat{L}_1(t), \dots, \widehat{L}_d(t))$ is a d-dimensional Lévy process (stationary independent increments with cadlag sample paths) with independent components and without Gaussian part. Assets on the same market show some correlation structure, which we model by the linear combination of independent Lévy processes. The arbitrary matrices β, and σ and the diagonal matrix δ, $\delta_{jj} \in \{0, 1\}$, give us the opportunity to choose different scaling factors for the Wiener processes and the pure jump processes and even to have to correlated assets, where only one contains jumps. For different sorts of portfolios with restrict this general model in different ways in the following Sections. Detailed explanations on the rectricted models are given in these Sections.

5.2 Optimal portfolios in the Lévy-Black-Scholes SDE setting

If we assume in (5.1.1) $\delta_{jl} = 1_{\{j=l\}}$ and β to be diagonal we obtain

$$\begin{array}{rclrcl} dP_0(t) & = & P_0(t)r dt\,, & P_0(0) & = & 1\,, \\ dP_i(t) & = & P_i(t-)\left(b_i dt + \sum_{j=1}^d \sigma_{ij} dL_j(t)\right)\,, & P_i(0) & = & p_i\,. \end{array} \tag{5.2.1}$$

Here $L = (L(t))_{t\geq 0} = (L_1(t), \ldots, L_d(t))$ is a d-dimensional Lévy process (stationary independent increments with cadlag sample paths) with independent components ($L_j(t) = \beta_{jj} W_j(t) + \widetilde{L}_j(t)$ in (5.1.1) with the restrictions above). Since the assets are on the same market, they show some correlation structure which we model by a linear combination of the same Lévy processes $L_1, \ldots, L_d$ for each asset price. We define this model analogously to the Black-Scholes model in Emmer, Klüppelberg and Korn (2001), but replace the Brownian motion W by the Lévy process L. Thus this model can be seen as a generalization of the Black-Scholes model since the d-dimensional standard Brownian motion is a Lévy process with triplet $(0, E_d, 0)$, where E_d is the d-dimensional unit matrix. The Lévy process L has characteristic triplet (a, β, ν), where $a \in \mathbb{R}^d$, β is an arbitrary d-dimensional diagonal matrix. Because of the independence of the components β has to be diagonal. We did not define β as the unit matrix since then the model would not include any pure jump process. So the independent Wiener processes $(\beta W)_i$, $i = 1, \ldots, d$ can have different variances and we can choose different scaling factors for the Wiener process and the non Gaussian components. Since the components of L are independent we obtain for the Lévy measure ν and a d-dimensional rectangle $A = \times_{i=1}^d (a_i, b_i] \subset \mathbb{R}^d$ that $\nu(A) = \sum_{i=1}^d \nu_i(a_i, b_i]$, where ν_i is the Lévy measure of L_i for $i = 1, \ldots, d$, i.e. the Lévy measure is supported on the union of the coordinate axes (see Sato (1999), E12.10, p.67). Hence, because of independence, the jumps of the different components occur at different times. We restrict the jump size ΔL by $\Delta L > -1$ to avoid negative stock prizes, $r \in \mathbb{R}$ is the *riskless interest rate* and $(\sigma_{ij})_{i,j\in\{1,\ldots,d\}}$ is an invertible matrix with $0 \leq \sigma_{ij} \leq 1$ by for $1 \leq i, j \leq d$ again in order to avoid negative stock prices. The vector $b \in \mathbb{R}^d$ can be chosen such that each stock has a certain *stock-appreciation rate.*

Let $\pi(t) = (\pi_1(t), \ldots, \pi_d(t)) \in \mathbb{R}^d$ be an admissible portfolio process, i.e. $\pi(t)$ is the fraction of the wealth $X^\pi(t)$, which is invested in the risky asset i (see Korn (1997), Section 2.1 for relevant definitions). The fraction of the investment in the bond is $\pi_0(t) = 1 - \pi(t)\underline{1}$, where $\underline{1} = (1, \ldots, 1)'$ denotes the vector (of appropriate dimension) having unit components. Throughout the chapter, we restrict ourselves to constant portfolios; i.e. $\pi(t) = \pi, t \in [0, T]$, for some fixed planning horizon T. This means that the fractions of wealth in the different stocks and the bond remain constant on $[0, T]$. Thus one has to trade at every time instant if $\pi_i \notin \{0, 1\}$, for all $1 \leq i \leq d$ since stock prices evolve randomly. In order to avoid negative wealth we require that $\pi \in [0, 1]^d$ and $\pi'\underline{1} \leq 1$. Denoting by $(X^\pi(t))_{t\geq 0}$ the *wealth process*, it follows the dynamic

$$dX^\pi(t) = X^\pi(t-)\left\{((1 - \pi'\underline{1})r + \pi'b)dt + \pi'\sigma dL(t)\right\} , \quad X^\pi(0) = x\,, \tag{5.2.2}$$

where $x \in \mathbb{R}$ denotes the initial capital of the investor.

As L is a semimartingale, general Itô calculus leads to the following explicit formula for the wealth process:

$$\begin{aligned} X^\pi(t) &= x\exp((r + \pi'(b - r1))t)\mathcal{E}(\pi\sigma' L(t)) \\ &= x\exp(a_X t + \pi'\sigma\beta W(t))\widetilde{X}^\pi(t),\ t \geq 0, \end{aligned} \tag{5.2.3}$$

where $\mathcal{E}$ defines the stochastic exponential of a process and a_X is as defined in Lemma 5.2.3 and

$$\begin{aligned} \ln \widetilde{X}^\pi(t) &= \int_0^t \int_{\mathbb{R}^d} \ln(1 + \pi\sigma x) 1_{\{|\ln(1+\pi\sigma x)| > 1\}} M_L(dx, ds) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \ln(1 + \pi\sigma x) 1_{\{|\ln(1+\pi\sigma x)| \leq 1\}} (M_L(dx, ds) - \nu_L(dx)ds),\ t \geq 0. \end{aligned}$$

Remark 5.2.1 Note that a jump of $\Delta L(t)$ of L leads to a jump $\Delta \ln X^\pi(t)$ of $\ln X^\pi$ of height $\ln(1 + \pi'\sigma\Delta L(t))$ and hence $\Delta X^\pi(t) > \ln(1 - \pi'\underline{1})$ by the restrictions on π, σ_{ij} and $\Delta L(t)$; recall also that jumps of the independent components of L do not occur at the same time.

Remark 5.2.2 The wealth process X^π is an exponential Lévy process. We calculate the characteristic triplet of its logarithm in the following Lemma.

Lemma 5.2.3 *Consider model (5.2.1) with Lévy process L and characteristic triplet $(a, \beta'\beta, \nu)$. The process $\ln X^\pi$ is a Lévy process with characteristic triplet (a_X, β_X, ν_X) given by*

$$
\begin{aligned}
a_X &= r + \pi(b - r + \sigma a) - \|\pi'\sigma\beta\|^2/2 \\
&\quad + \int (\ln(1+\pi'\sigma x) 1_{\{|\ln(1+\pi'\sigma x)| \le 1\}} - \pi'\sigma x 1_{\{|x|\le 1\}}) \nu_L(dx) \\
\beta_X^2 &= \|\pi'\sigma\beta_L\|^2 \\
\nu_X(A) &= \nu_L(x | \ln(1+\pi'\sigma x) \in A) \text{ for any Borel set } A \subset \mathbb{R}^*.
\end{aligned}
$$

Proof We have calculated the characteristic triplet of $\ln(\mathcal{E}(\pi L(t)))$ in Lemmas 4.1.2 and 4.1.3. By equation (5.2.3) and the uniqueness of the characteristic triplet we obtain (a_X, β_X, ν_X). □

Extending the characteristic function of $\ln(X^\pi(t))$ on $\mathbb{C}$ as in Lemma 4.1.6 we obtain for all $k \in \mathbb{N}$, such that the moment exists,

$$
E\left[(X^\pi(t))^k\right] = x^k \exp(kt(a_X + k\beta_X^2/2)) E\left[(\widetilde{X}^\pi(t))^k\right], \quad t \ge 0. \tag{5.2.4}
$$

and

$$
E\left[(\widetilde{X}^\pi(t))^k\right] = \exp(\widetilde{\mu}_k t), \quad t \ge 0
$$

where

$$
\widetilde{\mu}_k = \int_{\mathbb{R}^d} \left((1+\pi'\sigma x)^k - 1 - k\ln(1+\pi'\sigma x) 1_{\{|\ln(1+\pi'\sigma x)|\le 1\}}\right) \nu(dx)
$$

and ν is the Lévy measure of L.

In particular,

$$
E\left[\widetilde{X}^\pi(t)\right] = \exp\left(t \int_{\mathbb{R}^d} (\pi'\sigma x - \ln(1+\pi'\sigma x) 1_{\{|\ln(1+\pi'\sigma x)|\le 1\}}) \nu(dx)\right).
$$

Proposition 5.2.4 *Let L be a d-dimensional Lévy process and assume that $E(L_i(1)) < \infty$ and $\operatorname{var}(L_i(1)) < \infty$ for all $i \in \{1, \dots, d\}$. Let $X^\pi(t)$ be as in equation 5.2.3. Then*

$$
\begin{aligned}
E[X^\pi(t)] &= x \exp((r + \pi'(b - r\underline{1} + \sigma E[L(1)]))t) \\
\operatorname{var}(X^\pi(t)) &= x^2 \exp(2(r + \pi'(b - r\underline{1} + \sigma E[L(1)]))t) \left((\operatorname{var}(\pi'\sigma L(1)))t) - 1\right)
\end{aligned}
$$

Proof We obtain

$$E[X^\pi(t)] = x\exp\left(t\left(r+\pi'(b-r\underline{1}+\sigma a+\int_{\mathbb{R}^d}\sigma x 1_{\{|x|\geq 1\}}\nu(dx))\right)\right) \tag{5.2.5}$$

$$\begin{aligned}\operatorname{var}(X^\pi(t)) &= x^2\exp\left(2t\left(r+\pi'(b-r\underline{1}+\sigma a+\int_{\mathbb{R}^d}\sigma x 1_{\{|x|\geq 1\}}\nu(dx))\right)\right)\\ &\quad\times\left(\exp\left(t(\|\pi'\sigma\beta\|^2+\int(\pi'\sigma x)^2\nu(dx))\right)-1\right)\end{aligned} \tag{5.2.6}$$

By Sato (1999), Example 25.12, we have

$$E[L(1)] = a_L + \int_{|x|>1} x\nu_L(dx)$$

and

$$\operatorname{var}(\pi'\sigma L(t)) = \|\pi'\sigma\beta\|^2 + \int_{\mathbb{R}^d}(\pi'\sigma x)^2\nu(dx)$$

Plugging these expressions into (5.2.5) and (5.2.6) leads to the assertion. □

Remark 5.2.5 If the jump part of L has finite variation (5.2.5) and (5.2.6), can be written as:

$$E[X^\pi(t)] = x\exp\left((\pi(b+\gamma+\mu-r)+r)t\right), \tag{5.2.7}$$

$$\operatorname{var}(X^\pi(t)) = x^2\exp\left(2(\pi(b+\gamma+\mu-r)+r)t\right)\left(\exp(\pi^2(\beta^2+\mu_2)t)-1\right), \tag{5.2.8}$$

where

$$\begin{aligned}\mu=\mu_1 &= \int x\nu(dx) = E\left[\textstyle\sum_{0<s\leq 1}\Delta L(s)\right]\\ \mu_2 &= \int x^2\nu(dx) = E\left[\textstyle\sum_{0<s\leq 1}(\Delta L(s))^2\right].\end{aligned} \tag{5.2.9}$$

For a pure jump process we have $\mu = E[L(1)]$ and $\mu_2 = \operatorname{var}(L(1))$ by Protter (1990), Theorem I.38.

Next we consider the following classical optimization problem using the variance as risk measure.

$$\max_{\{\pi\in[0,1]^d|\pi'\underline{1}\leq 1\}} E[X^\pi(T)] \quad \text{subject to} \quad \mathrm{var}(x,\pi,T)\leq C\,, \tag{5.2.10}$$

where T is some given planning horizon and C is a given bound for the risk.

The solution of this problem for the classical Black-Scholes model (geometric Brownian motion with $\mu = 0$) can be found in Emmer, Klüppelberg and Korn (2001).

Theorem 5.2.6 *Let L be a Lévy process with Lévy-Khinchine representation (4.0.2). Then the optimal solution of problem (5.2.10) is given by*

$$\pi^* = \varepsilon^* \frac{((\sigma\beta)(\sigma\beta)')^{-1}(b-r\underline{1}+\sigma E(L(1))}{\|(\sigma\beta)^{-1}(b-r\underline{1}+\sigma E(L(1))\|} \tag{5.2.11}$$

where ε^* is the unique positive solution of

$$\begin{aligned}
&\varepsilon\|(\sigma\beta)^{-1}(b-r\underline{1}+\sigma E(L(1))\|T\\
&+\ \frac{1}{2}\ln\left(\frac{x^2}{C}\left(\exp\left(\varepsilon^2\mathrm{var}\left(\frac{((\sigma\beta)(\sigma\beta)')^{-1}(b-r\underline{1}+\sigma E(L(1))}{\|(\sigma\beta)^{-1}(b-r\underline{1}+\sigma E(L(1)))\|}\sigma L(1)\right)T\right)-1\right)\right)\\
&+\ rT=0\,.
\end{aligned} \tag{5.2.12}$$

subject to $\pi^*\in[0,1]^d$ and $\pi^*\underline{1}\leq 1$.

Remark 5.2.7 If the solution π^* to (5.2.12) satisfies $\pi^*\in[0,1]^d$ and $\pi^*\underline{1}\leq 1$, then π^* is the solution of the constraint optimization problem. If the solution to (5.2.12) does not satisfy the constraints, then the problem can be solved by the Lagrange method using some numerical algorithm, for example the SQP method (sequential quadratic programming) (see e.g. Nocedal and Wright (1999) and Boggs and Tolle (1995).
If for $d = 1$ the solution of (5.2.12) leads to $\pi^* > 1$, the optimal solution is $\pi^* = 1$.

Proof Mean and variance of the portfolio's wealth have for any exponential Lévy process the same form as for geometric Brownian motion. In that case the optimization problem has been solved in Proposition 2.9. in Emmer, Klüppelberg and Korn (2001). The general result follows then just from comparison of constants. The idea behind this solution is to find the portfolio with the highest terminal wealth over all portfolios satisfying $\|\pi'\sigma\beta\| = \varepsilon$, which is given by (5.2.11). Plugging this into the explicit form of the variance given in Proposition 5.2.4 we obtain constraint (5.2.12). The only difference to the optimization in Emmer, Klüppelberg and Korn (2001) is the constraint $\pi^* \in [0,1]^d$ and $\pi^*\underline{1} \leq 1$. In the following we consider some examples in order to understand the influence of the jumps on the choice of the optimal portfolio. All examples are for the case $d = 1$. Hence w.l.o.g. we choose $\sigma = 1$.

Example 5.2.8 *(Brownian motion with jumps)*
Let $Y, Y_1, Y_2, \ldots$ be iid random variables with distribution function p on $\mathbb{R}^*$ and $(N(t))_{t\geq 0}$ a Poisson process with parameter $c > 0$, independent of the Y_i. Then $\overline{L(t)} := \sum_{i=1}^{N(t)} Y_i$, $t \geq 0$, defines a compound Poisson process, a Lévy process with Lévy measure $\nu(dx) = cp(dx)$. For μ and μ_2 as defined in (5.2.9) we obtain

$$\mu = c\int_{-\infty}^{\infty} xp(dx) = cEY \quad \text{and} \quad \mu_2 = c\int_{-\infty}^{\infty} x^2 p(dx) = cE[Y^2]\,.$$

Here the Lévy process is the sum of a Brownian motion with drift $\beta W + \gamma t$ and the compound Poisson process $\overline{L}$ with intensity c and distribution function p as distribution for the jump heights. For illustrative purpose we restrict this example to one coumpound Poisson process, we could as well take different ones. The drift $\gamma = -\mu = -c\int xp(dx)$ is chosen such that it compensates the jumps. The Lévy measure is $\nu(dx) = cp(dx)$ and hence also μ and μ_2 are as above for the compound Poisson process $\overline{L}$. Since $\gamma = -\mu$, and by (5.2.3), (5.2.7) and (5.2.8) we obtain for $t \geq 0$,

$$\begin{aligned}
X^\pi(t) &= x\exp\left(t(\pi(b-r+\gamma)+r)\right)\exp\left(\pi(\beta W(t)) - \frac{1}{2}\pi^2\beta^2 t\right)\prod_{i=1}^{N(t)}(1+\pi Y_i)\,,\\
E[X^\pi(t)] &= x\exp\left(t(\pi(b-r)+r)\right),\\
\operatorname{var}(X^\pi(t)) &= x^2\exp\left(2t(\pi(b-r)+r)\right)\left(\exp\left((\pi^2\beta^2+\pi^2 c\int x^2 p(dx))t\right)-1\right).
\end{aligned}$$

The compound Poisson process ($\beta = 0$) and the Brownian motion ($c = 0$) are special cases of this example. Figures 5.1 and 5.2 show sample paths and the optimal portfolio for a jump scenario, namely possible jumps of height -0.1 with intensity 2, i.e. we expect 2 jumps per year.

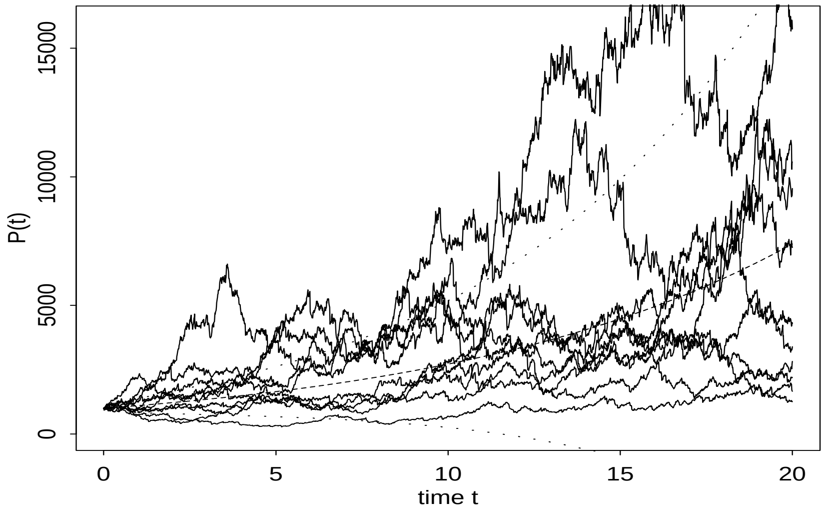


Figure 5.1: Ten paths of an asset in the Black-Scholes model with compensated jumps of height -0.1 and intensity 2, its expectation and standard deviation.The parameters are $x = 1\,000$, $b = 0.1$ and $r = 0.05$.

5.3 Optimal portfolios in the Lévy-Black-Scholes SDE setting with a possible crash

In this Section, we consider the same model as in Section 5.1, but include the possibility of bankruptcy. We want to study the effect of such a bankruptcy. Consequently we restrict ourselves to two basic portfolios: portfolio 1 consists of one riskless bond and one asset with crash possibility. Portfolio 2 consists of two assets, one Black-Scholes asset and one with crash possibility, but not containing a riskless bond. In (5.1.1) a jump of height ≤ -1 of L leads to a negative stock price. As suggested in Eberlein and Keller (1995) we interpret such a jump of L as a market crash and after this crash the stock price equals zero, what remains is the bond investment (in portfolio 1) or the Black-Scholes investment (in portfolio 2), respectively. Thus up to this bankruptcy the wealth process is as in Section 5.2, afterwards the fraction which was in the crash stock just before crash time is lost and the remaining wealth stays in the bond or the Black-Scholes stock until the end of the planning horizon.

Remark 5.3.1 Because of the independent increments of a Lévy process the crash probability does not depend on the actual stock price, i.e. if the stock price is very high a crash is as likely as if the price is very low.

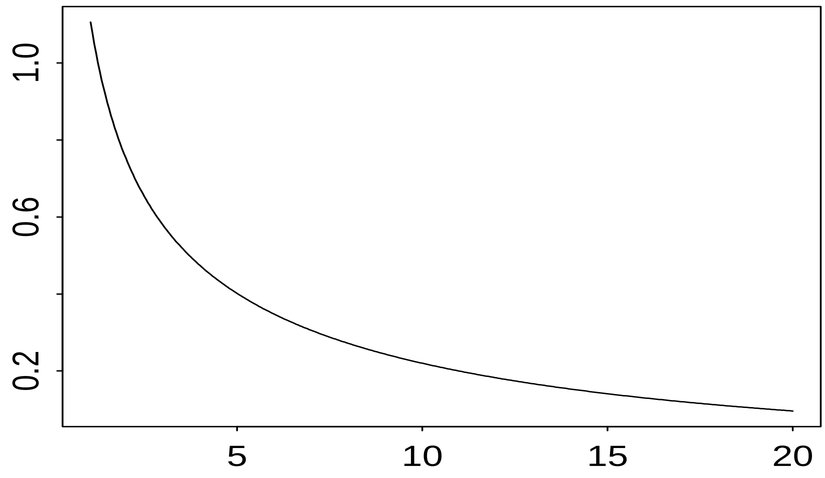


Figure 5.2: Optimal portfolio in the Black-Scholes model with jumps of height -0.1 and intensity 2 under the constraint $\mathrm{var}(X^\pi(t) \le 100\,000$. The parameters are the same as in Figure 5.1.

We first analyse portfolio 1 (consisting of one riskless bond and one stock with crash possibility), where we assume w.l.o.g. $\sigma = 1$. In this case the general model (5.1.1) and the model in Section 5.2 coincide.

Let

$$\tau = \inf\{t > 0 : \Delta L(t) \le -1\}$$

be the *crash time*. The wealth process is given by

$$X_C^\pi(t) = X^\pi(t) 1_{\{\tau > t\}} + (1-\pi) X^\pi(\tau-) \exp(r(t-\tau)) 1_{\{t \ge \tau\}}\,, \quad t \ge 0\,. \tag{5.3.1}$$

where X^π is the wealth process without crash possibility as in (5.2.3). For the portfolio optimization we have to calculate the moments ($k \in \mathbb{N}$) of the wealth process.

$$E[(X_C^\pi(t))^k] = E[(X^\pi(t))^k 1_{\{\tau > t\}} + ((1-\pi) X^\pi(\tau-) \exp(r(t-\tau)))^k 1_{\{t \ge \tau\}}]\,, \quad t \ge 0 \tag{5.3.2}$$

Theorem 5.3.2 *Let L be the Lévy process with characteristic triplet (a_L, β_L, ν_L). Define*

$$\widehat{L}(t) := L(t) - \sum_{s \le t} \Delta L(s) 1_{\{\Delta L(s) \le -1\}}\,, \quad t \ge 0\,,$$

and $\widehat{X}^\pi$ as X^π in (5.2.3) with L replaced by $\widehat{L}$. Then for $k \in \mathbb{N}$, provided the moment is finite and $E[(\widehat{X}^\pi(t))^{k(1+\delta)}] < \infty$ for some $\delta > 0$,

$$E[(X_C^\pi(t))^k] = x^k \exp(krt) \left((1-\pi)^k \nu((-\infty,-1)) \frac{1}{a_k} (\exp(a_k t) - 1) + \exp(a_k t) \right)\,, \quad t \ge 0\,,$$

where

$$a_k = k\pi(b-r+a_L)+\frac{1}{2}\pi^2\beta^2 k(k-1)+\int_{(-1,\infty)}((1+\pi x)^k-1-k\pi x 1_{\{|x|<1\}})\nu(dx)-\nu((-\infty,-1]) .$$

Proof By definition of $\widehat{X}^\pi(t)$ we have

$$\begin{aligned}\widehat{X}^\pi(t) &= x\exp\Bigg(t\Bigg(r+(b-r+a_L)\pi-\frac{1}{2}\beta^2\pi^2 \\ &\quad +\int\left(\ln(1+\pi x 1_{\{x>-1\}})\right)1_{\{|\ln(1+\pi x)|\le 1\}}-\pi x 1_{\{|x|<1\}}\big)\,\nu_L(dx)\Bigg)+\pi\beta W(t)\Bigg) \\ &\quad \times\overline{X^\pi(t)}\end{aligned}$$

where

$$\begin{aligned}\ln\overline{X^\pi(t)} &= \int_0^t\int_{-1}^\infty \ln(1+\pi x)1_{\{|\ln(1+\pi x)|>1\}}M_L(ds,dx) \\ &\quad +\int_0^t\int_{-1}^\infty \ln(1+\pi x)1_{\{|\ln(1+\pi x)|\le 1\}}(M_L(ds,dx)-ds\nu(dx)),\end{aligned}$$

By Protter (1990), Theorem I.39, the Lévy processes $\sum_{s\le t}\Delta L(s)1_{\{\Delta L(s)\le -1\}}$ and $\sum_{s\le t}\Delta L(s)1_{\{\Delta L(s)>-1\}}$ are independent and by Protter (1990), Theorem II.36, $\mathcal{E}(\pi L)$ can be written as $\mathcal{E}(\pi\widehat{L})\mathcal{E}(\pi\check{L})$, where $\check{L}:=\sum_{s\le t}\Delta L(s)1_{\{\Delta L(s)\le -1\}}$ and hence for $s>t$,

$$\begin{aligned}E[(X^\pi(t))^k|\tau=s] &= E[(\widehat{X}^\pi(t))^k(\mathcal{E}(\pi\check{L}(t)))^k|\sum_{0<u\le s}1_{\{\Delta L(u)\le -1\}}=0] \\ &= E[(\widehat{X}^\pi(t))^k]\end{aligned} \tag{5.3.3}$$

By (5.2.3) and (5.2.4) we get

$$\begin{aligned}E[(\widehat{X}^\pi(t))^k] &= x^k\exp\Bigg(t\Bigg(k(r+\pi(b-r+a_L)+\frac{1}{2}\pi^2\beta^2(k-1)) \\ &\quad +\int_{(-1,\infty)}((1+\pi x)^k-1-k\pi x 1_{\{|x|\le 1\}})\nu(dx)\Bigg)\Bigg).\end{aligned}$$

By Proposition 5.2.4 $E\widehat{X}^\pi(t)$ increases in t, hence

$$\sup_{r<t}E[(\widehat{X}^\pi(r))^{k(1+\delta)}]=E[(\widehat{X}^\pi(t))^{k(1+\delta)}]<\infty,$$

for all $t>0$. Thus $(\widehat{X}^\pi(r))_{0\le r\le t}$ is uniformly integrable and hence

$$\lim_{r\uparrow t}E[(\widehat{X}^\pi(r))^k]=E[\lim_{r\uparrow t}(\widehat{X}^\pi(r))^k],\ t>0 \tag{5.3.4}$$

Furthermore

$$P(\tau > t) = P\left(\sum_{s\le t} 1_{\{\Delta L(s)\in(-\infty,-1]\}} = 0\right) = \exp(-\nu((-\infty,-1])t),\ t > 0.$$

Setting $F_\tau(t) = P(\tau \le t)$, $t > 0$, we obtain by (5.3.1) and (5.3.3)

$$\begin{aligned}
E[(X_C^\pi(t))^k] &= \int_0^\infty E[(X_C^\pi(t))^k|\tau = s]F_\tau(ds) \\
&= \int_0^t E[(X_C^\pi(t))^k|\tau = s]F_\tau(ds) + E[(\widehat{X}^\pi(t))^k]P(\tau > t) \\
&= \int_0^t (1-\pi)^k E[(X^\pi(s-))^k \exp(kr(t-s))|\tau = s]F_\tau(ds) + E[(\widehat{X}^\pi(t))^k]P(\tau > t).
\end{aligned}$$

(5.3.3) and (5.3.4) lead to

$$\begin{aligned}
E[(X_C^\pi(t))^k] &= \int_0^t (1-\pi)^k E[(\widehat{X}^\pi(s-))^k \exp(kr(t-s))]F_\tau(ds) + E[(\widehat{X}^\pi(t))^k]P(\tau > t) \\
&= (1-\pi)^k e^{krt}\int_0^t E[(\widehat{X}^\pi(s-))^k]e^{-krs}\nu((-\infty,-1])e^{-\nu((-\infty,-1])s}ds + E[(\widehat{X}^\pi(t))^k]P(\tau > t) \\
&= (1-\pi)^k e^{krt}\nu((-\infty,-1])x^k \int_0^t \exp(a_k s)ds + x^k e^{krt}\exp(a_k t) \\
&= x^k e^{krt}\left((1-\pi)^k\nu((-\infty,-1])\frac{1}{a_k}\left(\exp(a_k t) - 1\right) + \exp(a_k t)\right)
\end{aligned}$$

where

$$a_k = k\pi(b-r+a_L)+\frac{1}{2}\pi^2\beta^2k(k-1)+\int_{(-1,\infty)}((1+\pi x)^k-1-k\pi x1_{\{|x|<1\}})\nu(dx)-\nu((-\infty,-1]).$$

We need the following results explicitly:

$$E[(X_C^\pi(t))] = xe^{rt}\left(\exp(f(\pi)t)\left(\frac{(1-\pi)\nu((-\infty,-1])}{f(\pi)}+1\right) - \frac{(1-\pi)\nu((-\infty,-1))}{f(\pi)}\right)$$

$$\begin{aligned}
\mathrm{var}[(X_C^\pi(t))] &= (xe^{rt})^2\left[e^{g(\pi)t}\left(\frac{(1-\pi)^2\nu((-\infty,-1])}{g(\pi)}+1\right) - \frac{(1-\pi)^2\nu((-\infty,-1])}{g(\pi)}\right. \\
&\quad \left. - \left(e^{f(\pi)t}\left(\frac{(1-\pi)\nu((-\infty,-1])}{f(\pi)}+1\right) - \frac{(1-\pi)\nu((-\infty,-1])}{f(\pi)}\right)^2\right]
\end{aligned}$$

where

$$f(\pi) = -\nu((-\infty,-1]) + \pi\left(b - r + a_L + \int_{[1,\infty)} x\nu(dx)\right)$$

and

$$g(\pi) = -\nu((-\infty,-1]) + 2\pi \left(b - r + a_L + \int\limits_{[1,\infty)} x\nu(dx) \right) + \pi^2 \left(\beta^2 + \int\limits_{(-1,\infty)} x^2 \nu(dx) \right)$$

Now we consider the following optimization problem

$$\boxed{\max_{\pi\in[0,1]} E(X^\pi(T)) \quad \text{subject to} \quad \operatorname{var}(x,\pi,T) \le C\,, \qquad (5.3.5)}$$

As long as the risky asset has a higher expectation than the bond, the solution is the largest $\pi \in [0,1]$ such that the variance constraint is fulfilled.

Example 5.3.3 *(Brownian motion with jumps and crash possibility)*
Let $Y_1, Y_2, \ldots$ be iid random variables with distribution p on $\mathbb{R}\backslash\{0\}$ and $(N(t))_{t\ge 0}$ a Poisson process with parameter $c > 0$, independent of the $Y_i's$. Then $L(t) := \sum_{i=1}^{N(t)} Y_i$, $t \ge 0$, defines a compound Poisson process which is a Lévy process with Lévy measure cp. Here the Lévy process is the sum of a Brownian motion with drift $\beta W(t) + \gamma t$ and a compound Poisson process with intensity c and p as distribution of the jump heights.
We optimize portfolios for an intensity $c = 0.5$, i.e. we expect one jump within two years, and $P(Y_i = -0.5) = P(Y_i = 0.5) = 0.4$, $P(Y_i = -1) = P(Y_i = 1) = 0.1$ and $\beta = 0.2$ under the constraint $\operatorname{var}(X_C^\pi(t)) \le 100\,000$.

Now we consider a portfolio consisting of one Lévy stock with crash possibility and one Black-Scholes stock. Therefore we restrict the model introduced in Section 5.1 by $\delta_{11} = 1$, $\delta_{22} = 0$, and $\sigma = E_d$. With the restrictions of Section 5.2 it is not possible to have two correlated assets, where only one has jumps. The quantity $\pi \in [0,1]$ is defined as the fraction of wealth invested in the stock with crash possibility and $1-\pi$ the fraction of investment in the Black-Scholes asset; there is no bond investment. Since the Black-Scholes asset has no jump part the correlation structure of the two assets is only given by the Brownian motion. So we model the correlation structure by the matrix β. Then the price of the Black-Scholes asset is given by

$$dP_2(t) = P_2(t) \left(b_2 dt + \sum_1^2 \beta_{2j} dW_j(t) \right),\ P_2(0) = p_2.$$

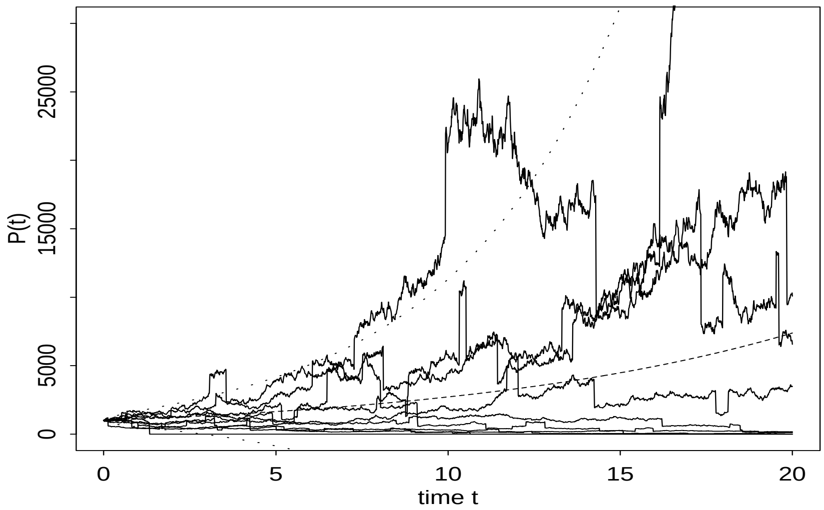


Figure 5.3: Ten paths of an asset in the Black-Scholes model with jumps and a possible crash, its expectation and standard deviation. The parameters are $x = 1\,000, b = 0.1, r = 0.05, c = 0.5, P(Y_i = -0.5) = P(Y_i = 0.5) = 0.4$ and $P(Y_i = -1) = P(Y_i = 1) = 0.1$.

As explained before the Lévy process in the crash asset has also a jump part. So let the price of the crash asset before crash time be given by

$$dP_1(t) = P_1(t-)\left(b_1 dt + dL(t)\right),\ P_1(0) = p_1,$$

where (a_L, β_L^2, ν_L) with $\beta_L^2 = \beta_{11}^2 + \beta_{12}^2$ is the characteristic triplet of L, i.e. $\sum_1^2 \beta_{1j} W_j(t),\ t \geq 0$ is its Brownian component. In terms of (5.1.1) $L(t) = \beta_{11} W_1(t) + \beta_{12} W_2(t) + d\widetilde{L}_1$. Analogously to the wealth process 5.2.2 in Section 2 the wealth process before crash time evolves according to

$$X^\pi(t) = x \exp(a_{CX} t + \begin{pmatrix} \pi \\ 1-\pi \end{pmatrix}' \beta W(t)) \widetilde{X}^\pi(t),\ t > 0 \tag{5.3.6}$$

where

$$a_{CX} = b_2 + \pi(b_1 + a_L - b_2) - \| \begin{pmatrix} \pi \\ 1-\pi \end{pmatrix}' \beta \|^2 + \int (\ln(1+\pi x) 1_{\{|\ln(1+\pi x)| \leq 1\}} - \pi x 1_{\{|x| \leq 1\}}) \nu_L(dx)$$

and

$$\begin{aligned} \ln \widetilde{X}^\pi(t) \quad &= \quad \int_0^t \int \ln(1+\pi x) 1_{\{|\ln(1+\pi x)| > 1\}} M_L(dx, ds) \\ &\quad + \int_0^t \int \ln(1+\pi x) 1_{\{|\ln(1+\pi x)| \leq 1\}} (M_L(dx, ds) - \nu_L(dx) ds), \quad t \geq 0. \end{aligned}$$

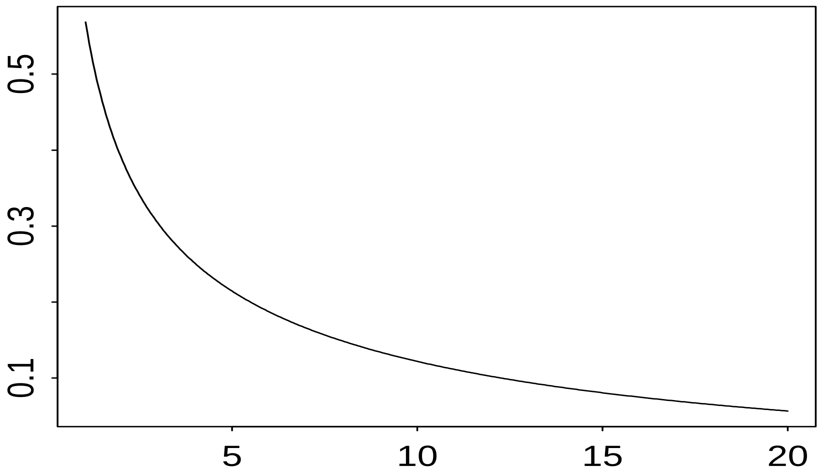


Figure 5.4: Optimal portfolio an asset in the Black-Scholes model with jumps and a possible crash and a bond for the same parameters as in Figure 5.3 under the constraint $\mathrm{var}(X_C^\pi(t)) \leq 100\,000$.

Denoting by X_C^π the wealth process of the portfolio with crash possibility, it follows

$$\begin{aligned} X_C^\pi(t) &= 1_{\{\tau>t\}} X^\pi(t) + 1_{\{t\geq\tau\}}(1-\pi)X^\pi(\tau-) \times \\ &\quad \exp(b_2(t-\tau) - \frac{1}{2}\sum_1^2 \beta_{2j}^2(t-\tau) + \sum_1^2 \beta_{2j} W_j(t-\tau)),\ t \geq 0. \end{aligned} \tag{5.3.7}$$

For the portfolio optimization we have to calculate the moments of the wealth process

$$\begin{aligned} E[(X_C^\pi(t))^k] &= E[(X^\pi(t))^k 1_{\{\tau>t\}} + 1_{\{t\geq\tau\}}\left((1-\pi)X^\pi(\tau-)\right)^k \qquad (5.3.8) \\ &\quad \times \exp\left(k(b_2(t-\tau) - \frac{1}{2}\sum_1^2 \beta_{2j}^2(t-\tau) + \sum_1^2 \beta_{2j} W_j(t-\tau))\right)],\ t \geq 0. \end{aligned}$$

Theorem 5.3.4 *Let (a_L, β_L, ν_L) be the characteristic triplet of the Lévy process L. Define*

$$\widehat{L}(t) := L(t) - \sum_{s\leq t} \Delta L(s) 1_{\{\Delta L(s)\leq -1\}}\,, \quad t \geq 0\,,$$

and define $\widehat{X}^\pi$ as X^π with L replaced by $\widehat{L}$. Then for $k \in \mathbb{N}$, provided the moment is finite and $E[(\widehat{X}^\pi(t))^{k(1+\delta)}] < \infty$ for some $\delta > 0$,

$$\begin{aligned} E[(X_C^\pi(t))^k] &= x^k \exp((kb_2 + \frac{1}{2}(k^2-k)\sum_1^2 \beta_{2j}^2)t) \\ &\quad \times \left((1-\pi)^k \frac{\nu((-\infty,-1])}{\widetilde{a}_k}(\exp(\widetilde{a}_k t) - 1) + \exp(\widetilde{a}_k t)\right),\ t \geq 0, \end{aligned} \tag{5.3.9}$$

where

$$\widetilde{a}_k = k\pi(b_1 + a_L - b_2) + \frac{1}{2}(k^2 - k)\left(\|\begin{pmatrix}\pi\\ 1-\pi\end{pmatrix}'\beta\|^2 - \sum_1^2 \beta_{2j}^2\right) + \int_{(-1,\infty)}((1+\pi x)^k - 1 - k\pi x 1_{\{|x|<1\}})\nu(dx) - \nu((-\infty,-1])$$

Proof With the same argumentation as in the proof of Theorem 5.3.4 we obtain

$$E[(\widehat{X}^\pi(t))^k] = x^k \exp\left(t(k(b_2 + \pi(b_1 + a_L - b_2) + \frac{1}{2}(k-1)\left(\|\begin{pmatrix}\pi\\ 1-\pi\end{pmatrix}'\beta\|^2\right)) + \int_{(-1,\infty)}((1+\pi x)^k - 1 - k\pi x 1_{\{|x|<1\}})\nu(dx))\right).$$

and

$$E[(X^\pi(t))^k | \sum_{s\le t} 1_{\{\Delta L \le -1\}} = 0] = E[(\widehat{X}^\pi(t))^k] \tag{5.3.10}$$

Furthermore $P(\tau > t) = \exp(-\nu((-\infty,-1])t)$ and

$$\lim_{r\uparrow t} E((\widehat{X}^\pi(r))^k) = E(\lim_{r\uparrow t}(\widehat{X}^\pi(r))^k),\ t > 0 \tag{5.3.11}$$

for the same reasons as in Theorem 5.3.2.

Setting $P(\tau \le t) = F_\tau(t)$, $t > 0$ we obtain by (5.3.6) and (5.3.10)

$$\begin{aligned}
E[(X_C^\pi(t))^k] &= \int_0^\infty E[(X_C^\pi(t))^k | \tau = s] F_\tau(ds) \\
&= \int_0^t E[(X_C^\pi(t))^k | \tau = s] F_\tau(ds) + E[(\widehat{X}^\pi(t))^k] P(\tau > t) \\
&= \int_0^t (1-\pi)^k E[(X^\pi(s-))^k \exp(k((b_2 - \frac{1}{2}\sum_1^2 \beta_{2j}^2)(t-s) + \sum_1^2 \beta_{2j} W_{2j}(t-s))) | \tau = s] F_\tau(ds) \\
&\quad + E[(\widehat{X}^\pi(t))^k] P(\tau > t)
\end{aligned}$$

(5.3.10) and (5.3.11) lead to

$$\begin{aligned}
&E[(X_C^\pi(t))^k] \\
&= \int_0^t (1-\pi)^k E[(\widehat{X}^\pi(s-))^k] E[\exp(k((b_2 - \frac{1}{2}\sum_1^2 \beta_{2j}^2)(t-s) + \sum_1^2 \beta_{2j} W_{2j}(t-s)))] F_\tau(ds) \\
&\quad + E[(\widehat{X}^\pi(t))^k] P(\tau > t) \\
&= (1-\pi)^k \exp((kb_2 + \frac{1}{2}(k^2 - k)\sum_1^2 \beta_{2j}^2)t) \times
\end{aligned}$$

$$
\begin{aligned}
& \int_0^t E[(\widehat{X}^\pi(s-))^k]\exp(-(kb_2+\frac{1}{2}(k^2-k)\sum_1^2\beta_{2j}^2)s)\nu((-\infty,-1])e^{-\nu((-\infty,-1])s}ds \\
& +E[(\widehat{X}^\pi(t))^k]P(\tau>t) \\
= \;& x^k(1-\pi)^k\exp((kb_2+\frac{1}{2}(k^2-k)\sum_1^2\beta_{2j}^2)t)\nu((-\infty,-1])\int_0^t\exp(\widetilde{a}_k s)ds+ \\
& x^k\exp((kb_2+\frac{1}{2}(k^2-k)\sum_1^2\beta_{2j}^2)t)\exp(\widetilde{a}_k t) \\
= \;& x^k\exp((kb_2+\frac{1}{2}(k^2-k)\sum_1^2\beta_{2j}^2)t)\left((1-\pi)^k\frac{\nu((-\infty,-1])}{\widetilde{a}_k}(\exp(\widetilde{a}_k t)-1)+\exp(\widetilde{a}_k t)\right)
\end{aligned}
$$

We shall need the following results explicitly:

$$
\begin{aligned}
E[(X_C^\pi(t))] \;=\;& xe^{b_2t}\left(\exp(\widetilde{f}(\pi)t)\left(\frac{(1-\pi)\nu((-\infty,-1])}{\widetilde{f}(\pi)}+1\right)\right. \\
& \left.-\frac{(1-\pi)\nu((-\infty,-1))}{\widetilde{f}(\pi)}\right)
\end{aligned}
\tag{5.3.12}
$$

$$
\begin{aligned}
& \operatorname{var}[(X_C^\pi(t))] \\
= \;& (xe^{b_2t})^2\left[\exp(\sum_1^2\beta_{2j}^2t)\left(e^{\widetilde{g}(\pi)t}\left(\frac{(1-\pi)^2\nu((-\infty,-1])}{\widetilde{g}(\pi)}+1\right)-\frac{(1-\pi)^2\nu((-\infty,-1])}{\widetilde{g}(\pi)}\right)\right. \\
& \left.-\left(e^{\widetilde{f}(\pi)t}\left(\frac{(1-\pi)\nu((-\infty,-1])}{\widetilde{f}(\pi)}+1\right)-\frac{(1-\pi)\nu((-\infty,-1])}{\widetilde{f}(\pi)}\right)^2\right]
\end{aligned}
\tag{5.3.13}
$$

where

$$
\widetilde{f}(\pi)=-\nu((-\infty,-1])+\pi\left(b_1-b_2+a_L+\int_{[1,\infty)}x\nu(dx)\right)
$$

and

$$
\begin{aligned}
\widetilde{g}(\pi) \;=\;& -\nu((-\infty,-1])+2\pi\left(b_1-b_2+a_L+\int_{[1,\infty)}x\nu(dx)\right)+\pi^2\int_{(-1,\infty)}x^2\nu(dx) \\
& +\|\begin{pmatrix}\pi\\1-\pi\end{pmatrix}'\beta\|^2-\sum_1^2\beta_{2j}^2
\end{aligned}
$$

Now we consider the optimization problem (5.3.5) also for the case of a portfolio of a Black-Scholes asset and an asset with crash possibility. In this case C has to be chosen larger than $\min(\mathrm{var}(x,\pi=0,T),\mathrm{var}(x,\pi=1,T))$ to ensure that the variance constraint can be satisfied.

If the crash asset to has a larger expectation and also a larger variance, the solution is the largest $\pi\in[0,1]$ such that the variance constraint is satisfied. If the crash asset has a larger expectation, but lower variance, the optimal portfolio is a pure crash portfolio.

Remark 5.3.5 Note that a lower variance and larger expectation of the crash stock always leads to a pure crash investment for any planning horizon, although we know that for large planning horizons the probability $P(\tau\le T)=1-P(\tau>T)=1-\exp(-\nu((-\infty,-1])T)$ of a crash within the planning horizon and hence to loose the whole wealth becomes very large and tends to 1 for $T\to\infty$. This is a deficiency of the variance as a risk measure.

Example 5.3.6 Here the Lévy process is the sum of a Brownian motion $\sum_1^2\beta_{1j}W_j(t)$ and a compound Poisson process with intensity c and p as distribution of the jump heights as explained in Example (5.3.3). We optimized portfolios for intensity $c=0.5$, i.e. we expect one jump within two years, $P(Y_i=-0.5)=P(Y_i=0.5)=0.4$, $P(Y_i=-1)=P(Y_i=1)=0.1$ and $\beta_{11}=\beta_{22}=0.1$, $\beta_{12}=\beta_{21}=\sqrt{0.03}$, $a_L=0$, $b_1=0.15$, $b_2=0.1$ under the constraint $\mathrm{var}(X_C^\pi(t))\le 600\,000$. Since the crash asset has larger expectation and variance than the Black-Scholes asset in this case, the optimal π is the largest one $\in[0,1]$, such that the variance constraint is fulfilled. Here we have to choose another bound C for the variance than in the case of a portfolio including a riskless bond investment, since we have only the choice between two risky assets here. For comparison we optimized portfolios consisting of two Black-Scholes assets without crash possibility with the same appreciation rates. If we only drop the jumps in asset 1 with the same β_{11} and β_{12} as before, asset 1 and asset 2 have the same variance, but asset 1 has a larger expectation. Hence the optimal portfolio is $\pi=1$ for all panning horizons T. Thus, now we choose $\beta_{11}=0.3$ and $\beta_{12}=\sqrt{0.15}$ such that the variance of asset 1 without crash possibility is the same as the variance of asset 1 before in the crash portfolio. As expected we see that the optimal portfolios behave similar.

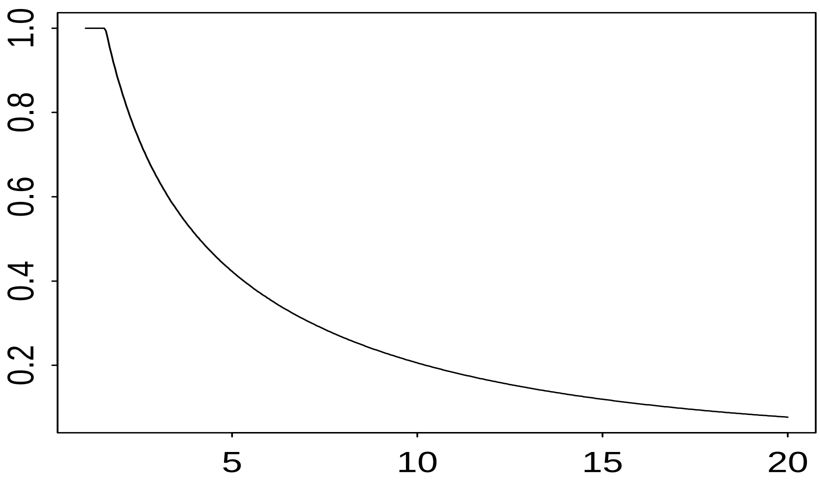


Figure 5.5: Optimal portfolio for one asset in the Black-Scholes model with jumps and a possible crash and one asset in the Black-Scholes model for the parameters $a_L = 0$, $c = 0.5$, $P(Y_i = -0.5) = P(Y_i = 0.5) = 0.4$, $P(Y_i = -1) = P(Y_i = 1) = 0.1$, $\beta_{11} = \beta_{22} = 0.1$, $\beta_{12} = \beta_{21} = \sqrt{0.03}$, $b_1 = 0.15$, $b_2 = 0.1$ under the constraint $\mathrm{var}(X_C^\pi(t)) \leq 600\,000$.

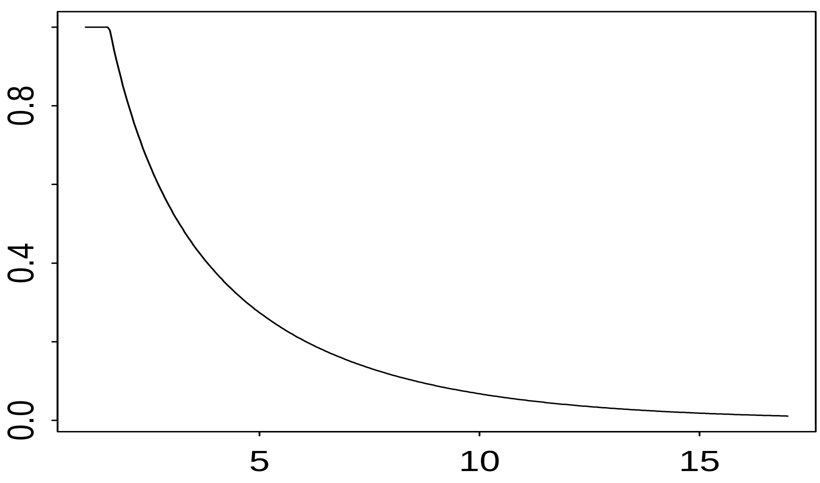


Figure 5.6: Optimal portfolio for two assets in the Black-Scholes model for the parameters $a_L = 0$, $\beta_{11} = 0.3$, $\beta_{12} = \sqrt{0.15}$, $\beta_{21} = \sqrt{0.03}$, $\beta_{22} = 0.1$, $b_1 = 0.15XS$, $b_2 = 0.1$ under the constraint $\mathrm{var}(X_C^\pi(t)) \leq 600\,000$.

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Fields of research: Portfolio optimization, Mathematical Finance, Lévy processes.

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1990 – 1996	Study of mathematics at Ludwig-Maximilians-Universität Munich, specializing in probability theory. Subject of Diploma thesis: "Über die S-Stetigkeit von Martingalen".
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01/1995 – 04/1996	The Boston Consulting Group
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- Informal Workshop on Lévy Processes, Aarhus, Denmark, September 2000
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- Workshop ”Extreme Value Theory and Financial Risk”, Munich, Germany, December 1999
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- 1st International Conference on Risk Management in Finance, Munich, Germany, October 1998

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Research Visits

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Publications

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