Bounds on the Error Probability over Finite-State Markov Erasure Channels

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***Abstract*—We analyze the block error probability of linear block codes when used to transmit over a finite-state Markov erasure channel (FSMEC). We introduce a density evolution analysis that allows to derive the distribution of the number of erasures over a finite number of uses of the FSMEC. The DE result is used to derive upper and lower bounds on the block error probability achievable by the best (n, k) linear block code. We show that the upper bound can be generalized to specific codes (and code ensembles) for which the distance spectrum is known. An example of application to a three-state Markov erasure channel (MEC) is presented, for which the upper and the lower bounds show to be very close, hence proving an accurate estimate of the performance achievable by optimum erasure codes.**

I. Introduction

Due to their simplicity and accuracy, finite-state Markov channel (FSMC) models gained a lot of popularity to model data losses in digital (wired/wireless/satellite) networks [1]–[8]. FSMC models are particularly suitable to capture salient features of mobile communication systems working at high frequency bands, where objects obstructing the line-of-sight between the base station and the mobile user terminal may cause long signal outages. The duration of the outages can be in order of several tens of milliseconds. This fact, coupled with the high data rates that are typically targeted at high frequency bands, yields, from the physical layer channel coding viewpoint, to a nonergodic behavior: Bursts of physical layer packets may fit in the duration of a blockage event, calling for the introduction of some form of time diversity. Besides the use of long channel interleavers, an option that has been gaining popularity during the past decade is the use of a *packet-level erasure correcting code* to be applied on top of the physical layer error correcting code protecting individual packets [9]–[12]. In this case, the physical layer error correcting code is responsible of counteracting channel noise and fast-fading effects, eventually delivering an error flag when a packet cannot be decoded, whereas the erasure code task is to recover packets that are lost after physical layer decoding. Note that the symbols over which the erasure code operates are the physical layer packets. Assuming for simplicity packets of constant size equal to L bits, an (n, k) packet erasure code encodes k information packets of L bits each into n codeword packets of L bits each. Powerful classes of erasure codes that have been adopted in concatenation with physical layer codes to recover lost packets are Reed-Solomon, low-density parity-check (LDPC), and Raptor codes [13]–[15]. In particular, LDPC and Raptor codes are capable of approaching the theoretical limits on the maximum erasure correction capability over a wide range of blocklengths n [16]–[18] under a low-complexity maximum likelihood (ML) erasure decoding algorithm known as *inactivation decoding* [19]–[22].

When the underlying channel model is a FSMC model, the process describing packet erasures at the input of the packet-level erasure correcting code decoder can be modelled through a finite-state Markov erasure channel (FSMEC) [23]. Here, a specific packet erasure probability is associated with each state in the FSMC. The simplest example of FSMEC is the Gilbert erasure channel [1], [24], comprising two states: a good state, where the erasure probability is zero, and a bad state where the erasure probability is one. Lower bounds on the error probability achievable by (n, k) erasure code over Gilbert erasure channels where derived in [24], where it was shown how LDPC codes under inactivation decoding can tightly approach the bounds.

In this paper, we extend the analysis of [24] to general FSMECs. We introduce a density evolution (DE) analysis that allows to efficiently track the distribution of the number of erasures over n uses of the FSMEC. We then derive two bounds on the block error probability achievable by the best (n, k) linear block code. The first bound is a generalization of the lower bound introduced in [24]. The second bound yields an upper bound on the block error probability of random linear block codes, and it can be generalized to specific codes for which the distance spectrum is known. We show an example of application to a three-state MEC, where besides a good and a bad sate, a mixed state with a moderate erasure probability is present. The upper and the lower bounds show to be very close, hence proving an accurate estimate of the performance achievable by optimum erasure codes. Upper bounds on the block error probability of binary linear block codes and and binary linear block ensembles are derived and compared to the bounds on the performance of the best (n, k) code.

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The rest of paper is organized as follows. Section II contains preliminary definitions. The DE analysis and the derivation of the bounds is provided in Section III. The application example is given in Section IV. Conclusions follow in Section V.

II. Preliminaries

We consider transmission over a FSMEC with a (n,k) binary linear block code $\mathcal{C}$. The code rate is denoted as $R = k/n$. Each step of the Markov chain underlying the channel corresponds to the transmission of a codeword bit[1] over a binary erasure channel (BEC), where the erasure probability depends on the channel state, and within each state erasures are independent. The attention is on FSMCs defined by M states labelled from 1 to M, i.e., the state space is $\mathcal{S} = \{1,\ldots,M\}$. We denote the state at step ℓ as s_ℓ, and the corresponding random variable (r.v.) as S_ℓ. The state transition probabilities are summarized by the $M \times M$ right stochastic matrix P. We assume the Markov chain to be ergodic with stationary distribution $\boldsymbol{\pi}$. We associate to the ith state an erasure probability ϵ_i.

III. Finite-Length Analysis

We are interested in deriving upper and lower bounds on the performance achievable by an (n,k) binary linear block code $\mathcal{C}$ over FSMECs. To do so, we first derive the distribution of the number of erasures after observing the FSMEC output for n consecutive channel uses. The analysis is based on a DE technique [27, Chapter 4], and it is developed in Section III-A. Note that a derivation of the distribution of the number of erasures was already provided in [23]. The derivation of [23] is based on matrix generating functions, whereas here we provided direct recursions – the two approaches can be shown to be equivalent. We then employ the derived erasure distribution to obtain the bounds on the error probability in Section III-B

A. Density Evolution

Denote by e the number of erasures observed at the channel output after n channel uses, and by E the corresponding r.v.. We are interested in deriving the distribution P_E of the number of erasures after observing the output of the FSMEC for n consecutive channel uses. Let us denote by E_ℓ the r.v. associated to the number of erasures observed up to time ℓ, and by

$$q_\ell(e,i) := P(E_\ell = e, S_\ell = i) \tag{1}$$

the joint probability distribution of the number of erasures and of the channel state at the ℓth channel use. We can obtain P_E as

$$P_E(e) = \sum_{i=1}^{M} q_n(e,i).$$

[1]The model and the results derived in this paper trivially generalizes to the case where a symbol of the (n,k) linear block code $\mathcal{C}$ is a vector of L bits (i.e., a packet), and where at each step an L-bit symbol is transmitted over an erasure channel [25], [26]. Each L-bit vector is either correctly received, or it is completely erased with a probability that depends on the channel state.

It follows that, to compute P_E, we need to compute (1) for $\ell = n$. The result can be achieved recursively by noting that

$$q_\ell(e,i) = \epsilon_i \sum_{j=1}^{M} \mathsf{P}_{ji} q_{\ell-1}(e-1,j) + (1-\epsilon_i) \sum_{j=1}^{M} \mathsf{P}_{ji} q_{\ell-1}(e,j)$$

where the calculation is initialized as

$$q_1(e,i) = \begin{cases} (1-\epsilon_i)\pi_i & \text{if } e=0 \\ \epsilon_i \pi_i & \text{if } e=1 \\ 0 & \text{otherwise.} \end{cases}$$

B. Bounds on the Block Error Probability

We analyze next the block error probability that an (n,k) code $\mathcal{C}$ can achieve over a FSMEC. In particular, we are interested in deriving the performance achievable by the *best binary linear block code* $\mathcal{C}^\star$. We assume a decoder that outputs all the codewords that are *compatible* with the channel output, i.e., all the codewords that coincide with the channel output in all coordinates where the channel output is not erased. If the decoder outputs a single codeword, we declare a decoding success. If the decoder outputs multiple solutions, we declare a decoding error. With a slight abuse of notation, we refer to this rule as the ML decoding rule. Note that over erasure channel, for the special class of binary linear block codes, ML decoding entails a complexity that grows polynomially in n. For example, Gaussian elimination can provide the solution with a complexity order of n^3.

Let us denote by e the number of erasures affecting a codeword at the channel output. We make use of the following two observations:

i. Decoding fails whenever the number of erasures exceeds the number of equations, i.e., whenever $e > n-k$;
ii. If the code parity-check matrix is drawn randomly with entries that are independent and uniformly-distributed in $\{0,1\}$, then the probability of decoding failure is tightly upper bounded by $2^{-(n-k-e)}$ [28]–[30].

The first observation allows to derive a lower bound on the block error probability achievable by $\mathcal{C}^\star$ as $\mathsf{BEP}(\mathcal{C}^\star) \geq P_{\mathsf{S}}$ where

$$P_{\mathsf{S}} := \sum_{e=n-k+1}^{n} P_E(e). \tag{2}$$

The bound in (2) has a form that is reminiscent of the Singleton bound for the memoryless BEC [30], [31]. We hence refer to it as Singleton lower bound. Observation ii. proves to be useful to derive an upper bound on the block error probability achievable by $\mathcal{C}^\star$. In fact, by a standard random coding argument we can state that $\mathsf{BEP}(\mathcal{C}^\star)$ is upper bounded by the average error probability of binary random linear block codes. We have that $\mathcal{C}^\star$ as $\mathsf{BEP}(\mathcal{C}^\star) \leq \mathsf{E}[\mathsf{BEP}(\mathcal{C})]$ where $\mathsf{E}[\mathsf{BEP}(\mathcal{C})]$ is tightly upper bounded by

$$P_{\mathsf{B}} := \sum_{e=1}^{n} P_E(e) \min\left(1, 2^{-(n-k-e)}\right). \tag{3}$$

As (3) follows closely the principles underlying the derivation of the Berlekamp random coding bound [28], [32], we will refer to it as Berlekamp upper bound. Note that a similar bound was derived in [23], where the actual probability of decoding failure was used in place of the tight bound given by $2^{-(n-k-e)}$.

Thanks to (2) and (3), we have that

$$P_{\mathsf{S}} \leq \mathsf{BEP}\left(\mathcal{C}^{\star}\right) < P_{\mathsf{B}}.$$

As we will see, the two bounds are typically sufficiently close, yielding precise estimates on the block error probability achievable by the best (n,k) binary linear block code $\mathcal{C}^{\star}$. Before moving to an illustrative example, we should observe that an upper bound on the block error probability achievable under ML decoding by a *specific* binary linear block code $\mathcal{C}$ could be obtained if the distance spectrum $A_w(\mathcal{C})$ of the code is known. In that case, we have

$$\mathsf{BEP}\left(\mathcal{C}\right) \leq \sum_{e=1}^{n} P_E(e) \min\left(1, \sum_{w=1}^{e} \binom{e}{w} \frac{A_w(\mathcal{C})}{\binom{n}{w}}.\right) \tag{4}$$

The proof of (4) follows closely the proof of Theorem 4 in [30], with the additional care that one should consider the ensemble of codes obtained by all possible permutations of the coordinates of $\mathcal{C}$ (i.e., all the equivalent codes of $\mathcal{C}$). In this sense, the bound states the existence of an equivalent code of $\mathcal{C}$ whose error probability is upper bounded by the right-hand side of (4).

The bound in (4) can be extended to general binary linear block code ensembles. We focus next on the case of LDPC codes, due to their excellent performance when employed as erasure correcting codes. The analysis of ML decoding requires the knowledge of the ensemble average weight enumerator $\bar{A}_w$ [33], [34], while to analyze the less complex belief propagation (BP) decoder one needs the ensemble average stopping set enumerator $\bar{S}_w$ [35]. The resulting bounds are

$$\mathsf{E}\left[\mathsf{BEP}_{\mathsf{ML}}\left(\mathcal{C}\right)\right] \leq \sum_{e=1}^{n} P_E(e) \min\left(1, \sum_{w=1}^{e} \binom{e}{w} \frac{\bar{A}_w(\mathcal{C})}{\binom{n}{w}}\right) \tag{5}$$

and

$$\mathsf{E}\left[\mathsf{BEP}_{\mathsf{BP}}\left(\mathcal{C}\right)\right] \leq \sum_{e=1}^{n} P_E(e) \min\left(1, \sum_{w=1}^{e} \binom{e}{w} \frac{\bar{S}_w(\mathcal{C})}{\binom{n}{w}}\right) \tag{6}$$

under ML and BP decoding, respectively. We provide next sketch of the derivation of (5) and (6). Consider an erasure pattern with e erasures. We are interested in finding the probability that the all-zero codeword is indistinguishable from a weight-w codeword, at the channel output. The probability is given by $\binom{e}{w}/\binom{n}{w}$ for $w \leq e$, while it is zero for $w > e$, i.e., the probability that the e erasures "cover" the w coordinates where the two codewords differ. By applying the union bound, one gets the inner sum in (5) where all codewords with Hamming weight w yield the same contribution. The derivation is completed by observing that the probability of error conditioned on the number of erasures is always upper bounded by 1, and by averaging over the distribution of the number of erasures. In case of BP decoding, the derivation is similar – one needs only to compute the probability that the erasure pattern covers a size-w stopping set, in place of the probability that the erasure pattern covers a weight-w codeword.

IV. Example of Application

Three-state MECs constitute the first, non-trivial generalization of the Gilbert erasure channel. Three-state Markov chains have been used to model, among others, land-mobile satellite channels in S-band [36], clustering the received signal states as good (line-of-sight), bad (blockage) and mixed (shadowing). The model of [36] has been used to design erasure codes as fading/blockage mitigation technique [10]. As an example, we analyze a three-state MEC with state transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.85 & 0.05 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{pmatrix}$$

The first state identifies the good state (with $\epsilon_1 = 0$), the second state is the bad state (with $\epsilon_2 = 1$), whereas the third state identified a state of shadowing (mixed state), where the erasure probability ϵ_3 is moderate (see Figure 1). We computed the bounds (2), (3) for the blocklengths $n = 100$ and $n = 1000$, assuming a rate-$1/2$ code. The bounds were calculated for various values of ϵ_3, reflecting different levels of robustness of the physical layer receiver in shadowing conditions. The results are provided in Figure 2 for $n = 100$, and in Figure 3 for $n = 1000$. In both cases, the upper and the lower bounds are very close to each other, implying that they both provide accurate estimates of the performance achievable by the best (n,k) code, which should attain an error probability within the shaded area between the curves. Note that in the case of $n = 1000$, the gap between the bounds is barely visible. We run some preliminary analysis on more general FSMECs characterized by a larger number of states and various erasure probabilities, and in all cases the results confirm the tightness achieved in the three-state MECs from the example. On the same charts, we provide the upper bound on the average performance of expurgated LDPC code ensembles. In particular, we consider the regular $(3,6)$ ensemble under both BP and ML decoding, where expurgation is obtained as illustrated in [14, Chapter 2.2].

V. Conclusions

The block error probability of linear block codes over a finite-state Markov erasure channel (FSMEC) has been analyzed. The analysis relies on density evolution to obtain the distribution of the number of erasures over a finite number of uses of the FSMEC. Given the distribution of the number of erasures, upper and lower bounds on the block error probability achievable by the best (n,k) linear block code have been derived. The upper and lower bounds showed to be very close in the case of a three-state Markov erasure channel, providing an accurate estimate of the performance achievable by optimum erasure codes. We observed a similar behavior

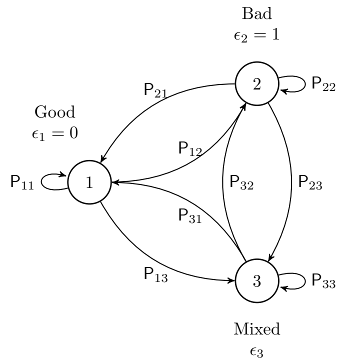

Fig. 1. Three-state MEC.

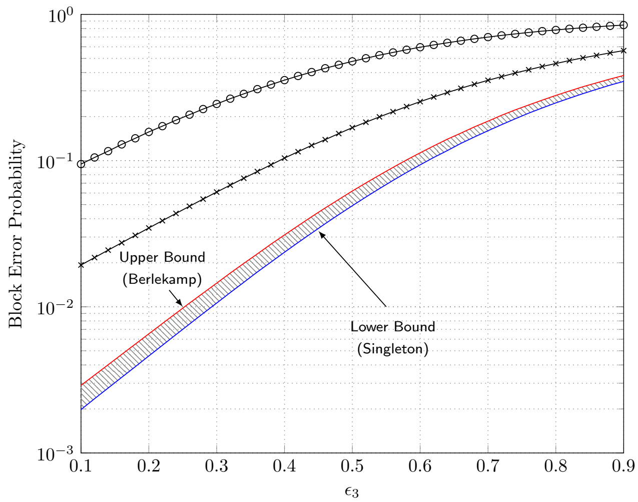

Fig. 2. Block error probability vs. erasure probability ϵ_3 of the mixed state in the three-state MEC of the example. The average block error probability for the expurgate $(3, 6)$ LDPC code ensemble is provided under ML (—×—) and BP (—○—) decoding. The block length is $n = 100$ and the code rate is $R = 1/2$.

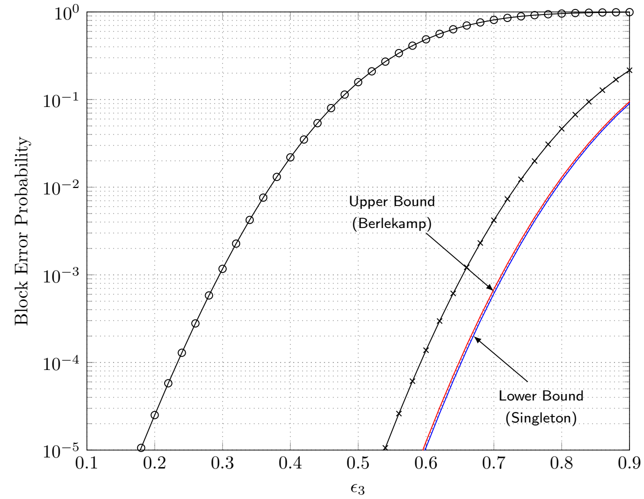

Fig. 3. Block error probability vs. erasure probability ϵ_3 of the mixed state in the three-state MEC of the example. The average block error probability for the expurgate $(3, 6)$ LDPC code ensemble is provided under ML (—×—) and BP (—○—) decoding. The block length is $n = 1000$ and the code rate is $R = 1/2$.

in a number of other examples, and we conjecture that the bounds are tight in general. Future works will target the design of codes with low-complexity erasure decoders capable of approaching the bounds.

References

[1] E. Gilbert, "Capacity of a burst-noise channel," *The Bell System Technical Journal*, vol. 39, no. 5, pp. 1253–1265, 1960.

[2] E. O. Elliott, "Estimates of error rates for codes on burst-noise channels," *Bell Syst. Tech. J.*, vol. 42, no. 5, pp. 1977–1997, Sep. 1963.

[3] E. Lutz, D. Cygan, M. Dippold, F. Dolainsky, and W. Papke, "The land mobile satellite communication channel-recording, statistics, and channel model," *IEEE Trans. Veh. Technol.*, vol. 40, no. 2, pp. 375–386, May 1991.

[4] H. S. Wang and N. Moayeri, "Finite-state markov channel-a useful model for radio communication channels," *IEEE Trans. Veh. Technol.*, vol. 44, no. 1, pp. 163–171, Feb. 1995.

[5] A. Goldsmith and P. Varaiya, "Capacity, mutual information, and coding for finite-state Markov channels," *IEEE Trans. Inf. Theory*, vol. 42, no. 3, pp. 868–886, 1996.

[6] M. Zorzi, R. R. Rao, and L. B. Milstein, "Error statistics in data transmission over fading channels," *IEEE Trans. Commun.*, vol. 46, no. 11, pp. 1468–1477, Nov. 1998.

[7] W. Turin, *Performance Analysis and modeling of digital transmission systems*. Springer Science & Business Media, 2004.

[8] I. Moreira, C. Pimentel, F. P. Barros, and D. P. B. Chaves, "Modeling fading channels with binary erasure finite-state markov channels," *IEEE Trans. Veh. Technol.*, vol. 66, no. 5, pp. 4429–4434, May 2017.

[9] ETSI TR 102 993 V1.1.1, "Digital Video Broadcasting (DVB); Upper Layer FEC for DVB Systems," Feb. 2011.

[10] A. B. Alamanac, P. Burzigotti, M. Cohen, R. D. Gaudenzi, G. Liva, S. Lipp, O. Pulvirenti, L. Roullet, and H. Stadali, "Performance Validation of the DVB-SH Standard for Satellite/Terrestrial Hybrid Mobile Broadcasting Networks," *IEEE Trans. on Broadcasting*, vol. 57, no. 4, pp. 802–825, Dec. 2011.

[11] T. A. Courtade and R. D. Wesel, "Optimal allocation of redundancy between packet-level erasure coding and physical-layer channel coding in fading channels," *IEEE Trans. Commun.*, vol. 59, no. 8, pp. 2101–2109, Aug. 2011.

[12] S. V. S. Ranganathan, T. Mu, and R. D. Wesel, "Allocating redundancy between erasure coding and channel coding when fading channel diversity grows with codeword length," *IEEE Trans. Commun.*, vol. 65, no. 8, pp. 3226–3237, Aug. 2017.

[13] I. Reed and G. Solomon, "Polynomial codes over certain finite fields," *J. Society for Industrial and Applied Mathematics*, vol. 8, no. 2, pp. 300–304, Jun. 1960.

[14] R. Gallager, "Low-density parity-check codes," Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA, USA, 1963.

[15] A. Shokrollahi, "Raptor codes," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2551–2567, Jun. 2006.

[16] M. Luby, T. Gasiba, T. Stockhammer, and M. Watson, "Reliable multimedia download delivery in cellular broadcast networks," *IEEE Trans. Broadcast.*, vol. 53, no. 1, pp. 235–246, Mar. 2007.

[17] E. Paolini, G. Liva, B. Matuz, and M. Chiani, "Maximum likelihood erasure decoding of ldpc codes: Pivoting algorithms and code design," *IEEE Trans. Commun.*, vol. 60, no. 11, pp. 3209–3220, Nov. 2012.

[18] F. L. Blasco, G. Liva, G. Bauch, and E. Paolini, "Bounds on the Error Probability of Raptor Codes under Maximum Likelihood Decoding," *IEEE Trans. Inf. Theory*, vol. 67, no. 3, pp. 1537–1558, Feb. 2021.

[19] E. Berlekamp, *Algebraic coding theory*. New York, NY, USA: McGraw-Hill, 1968.

[20] T. Richardson and R. Urbanke, “Efficient encoding of low-density parity-check codes,” *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 638–656, Feb. 2001.
[21] M. Shokrollahi, S. Lassen, and R. Karp, “Systems and processes for decoding chain reaction codes through inactivation,” Feb. 2005, US Patent 6,856,263.
[22] F. L. Blasco, G. Liva, and G. Bauch, “Inactivation Decoding of LT and Raptor Codes: Analysis and Code Design,” *IEEE Trans. Commun.*, vol. 65, no. 10, pp. 4114 – 4127, Oct. 2017.
[23] S. Kumar, J.-F. Chamberland, and H. D. Pfister, “First-passage time and large-deviation analysis for erasure channels with memory,” *IEEE Trans. Inf. Theory*, vol. 59, no. 9, pp. 5547–5565, Sep. 2013.
[24] G. Liva, B. Matuz, Z. Katona, E. Paolini, and M. Chiani, “On construction of moderate-length ldpc codes over correlated erasure channels,” in *Proc. IEEE Int. Conf. Commun.*, Dresden, Germany, Jun. 2009, pp. 1–5.
[25] J. Byers, M. Luby, M. Mitzenmacher, and A. Rege, “A Digital Fountain Approach to Reliable Distribution of Bulk Data,” in *Proc. of ACM SIGCOMM*, Vancouver, BC, Canada, Sep. 1998.
[26] M. Luby, M. Mitzenmacher, A. Shokrollahi, and D. Spielman, “Efficient erasure correcting codes,” *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 569–584, Feb. 2001.
[27] H. D. Pfister, “On the capacity of finite state channels and the analysis of convolutional accumulate-m codes,” Ph.D. dissertation, University of California, San Diego, 2003.
[28] E. Berlekamp, “The technology of error-correcting codes,” *Proc. IEEE*, vol. 68, no. 5, pp. 564–593, May 1980.
[29] G. Liva, E. Paolini, and M. Chiani, “Performance versus overhead for fountain codes over f_q,” *IEEE Commun. Lett.*, vol. 14, no. 2, pp. 178–180, Feb. 2010.
[30] ——, “Bounds on the error probability of block codes over the q-ary erasure channel,” *IEEE Trans. Commun.*, vol. 61, no. 6, pp. 2156–2165, Jun. 2013.
[31] R. Singleton, “Maximum distance q-nary codes,” *IEEE Trans. Inf. Theory*, vol. 10, no. 2, pp. 116–118, Apr. 1964.
[32] Y. Polyanskiy, H. V. Poor, and S. Verdú, “Channel coding rate in the finite blocklength regime,” *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2307–2359, May 2010.
[33] R. Gallager, “Low-density parity-check codes,” *IRE Trans. Inf. Theory*, vol. 8, no. 1, pp. 21–28, Jan. 1962.
[34] C. Di, R. Urbanke, and T. Richardson, “Weight distribution of low-density parity-check codes,” *IEEE Trans. Inf. Theory*, vol. 52, no. 11, pp. 4839–4855, Nov. 2006.
[35] A. Orlitsky, K. Viswanathan, and J. Zhang, “Stopping set distribution of LDPC code ensembles,” *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 929–953, Mar. 2005.
[36] F. Perez-Fontan, M. Vazquez-Castro, S. Buonomo, J. Poiares-Baptista, and B. Arbesser-Rastburg, “S-band LMS propagation channel behaviour for different environments, degrees of shadowing and elevation angles,” *IEEE Trans. Broadcast.*, vol. 44, no. 1, pp. 40–76, Mar. 1998.