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Abstract Fractional Cauchy Problem: Existence of Propagators and Inhomogeneous Solution Representation

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Abstract: We consider a Cauchy problem for the inhomogeneous differential equation given in terms of an unbounded linear operator A and the Caputo fractional derivative of order $\alpha \in (0, 2)$ in time. The previously known representation of the mild solution to such a problem does not have a conventional variation-of-constants like form, with the propagator derived from the associated homogeneous problem. Instead, it relies on the existence of two propagators with different analytical properties. This fact limits theoretical and especially numerical applicability of the existing solution representation. Here, we propose an alternative representation of the mild solution to the given problem that consolidates the solution formulas for sub-parabolic, parabolic and sub-hyperbolic equations with a positive sectorial operator A and non-zero initial data. The new representation is solely based on the propagator of the homogeneous problem and, therefore, can be considered as a more natural fractional extension of the solution to the classical parabolic Cauchy problem. By exploiting a trade-off between the regularity assumptions on the initial data in terms of the fractional powers of A and the regularity assumptions on the right-hand side in time, we show that the proposed solution formula is strongly convergent for $t \geq 0$ under considerably weaker assumptions compared to the standard results from the literature. Crucially, the achieved relaxation of space regularity assumptions ensures that the new solution representation is practically feasible for any $\alpha \in (0, 2)$ and is amenable to the numerical evaluation using uniformly accurate quadrature-based algorithms.

Keywords: inhomogeneous Cauchy problem; Caputo fractional derivative; sub-parabolic problem; sub-hyperbolic problem; mild solution; propagator; contour representation

MSC: 34A08; 35R11; 34G10; 35R20; 65L05; 65J08; 65J10

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1. Problem Setting

We consider a Cauchy problem for the following fractional order differential equation:

$$\partial_t^\alpha u(t) + Au(t) = f(t), \quad \alpha \in (0, 2),\ t \in (0, T]. \tag{1}$$

Here, ∂_t^α denotes the Caputo fractional derivative of order α with respect to t (see, e.g., ref. [1] (p. 91))

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds, \tag{2}$$

with $u^{(n)}(s)$ being the standard integer-order derivative of $u(s)$; $\Gamma(\cdot)$ is Euler's Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, and $n = \lceil \alpha \rceil$ is the smallest integer greater or equal to α. In this work, we focus on the linear case of (1), so the function $f \in C([0, T], X)$ only depends on time and space. The choice of range $\alpha \in (0, 2)$ for the fractional derivative parameter is motivated by the applications of (1) to the modeling of physical phenomena [2–5]. If $n = 1$, the given differential equation is called sub-parabolic. Similarly, when $n = 2$, this equation is called sub-hyperbolic.

The time-independent operator A in (1) is assumed to be closed and linear with the domain $D(A)$ dense in a Banach space $X = X(\|\cdot\|, \Omega)$ and the spectrum $\mathrm{Sp}(A)$ contained in the sectorial region $\Sigma(\rho_s, \varphi_s)$ of the complex plane:

$$\Sigma(\rho_s, \varphi_s) = \left\{ z = \rho_s + \rho e^{i\theta} : \quad \rho \in [0, \infty),\ |\theta| < \varphi_s \right\}, \quad \rho_s > 0,\ \varphi_s < \frac{\pi}{2}. \tag{3}$$

The numbers ρ_s and φ_s are called the spectral parameters of A. In addition to the assumptions on the spectrum, we suppose that the resolvent of A: $R(z, A) \equiv (zI - A)^{-1}$ satisfies the bound

$$\left\| (zI - A)^{-1} x \right\| \le M \frac{\|x\|}{1 + |z|}, \quad M > 0, \tag{4}$$

for all $x \in X$ and any fixed z outside the sector $\Sigma(\rho_s, \varphi_s)$ and on its boundary. Following the established convention [6], we will call such operators strongly positive. The class of strongly positive operators includes second-order elliptic partial differential operators [7], as well as more general strongly elliptic pseudo-differential operators defined over the bounded domain [8]. The spectral parameters ρ_s and φ_s can be estimated from the coefficients of the differential expression for such A; or using the associated sesquilinear form, if X has an additional Hilbert space structure and admits the embedding into a Gelfand triple (V, X, V^*). The precise results on that behalf are stated in Sections 2.1–2.2 of [9].

To get an intuitive reasoning regarding the solution to the given equation, let us proceed informally at first. We apply the Laplace transform $\mathcal{L}$ to both parts of (1) and evaluate $\mathcal{L}\{\partial_t^\alpha u\}$ using the formula (2.140) from [10]:

$$\mathcal{L}\{\partial_t^\alpha u\}(z) = z^\alpha \mathcal{L}\{u\}(z) - \sum_{k=0}^{n-1} z^{\alpha-1-k} u^{(k)}(0).$$

This allows us to rewrite the transformed equation as

$$z^\alpha \widehat{u}(z) - \sum_{k=0}^{n-1} z^{\alpha-1-k} u^{(k)}(0) + A\widehat{u}(z) = \widehat{f}(z).$$

Henceforth, the functions $\widehat{u}(z)$, $\widehat{f}(z)$ will be used to denote $\mathcal{L}\{u\}(z)$, $\mathcal{L}\{f\}(z)$, correspondingly. If the operator $(z^\alpha I + A)$ is invertible, this linear equation admits the solution

$$\widehat{u}(z) = (z^\alpha I + A)^{-1} \left(\sum_{k=0}^{n-1} z^{\alpha-1-k} u^{(k)}(0) + \widehat{f}(z) \right), \tag{5}$$

which leads us to the representation of $u(t)$:

$$u(t) = \sum_{k=0}^{n-1} \mathcal{L}^{-1}\left\{ z^{\alpha-1-k} (Iz^\alpha + A)^{-1} u^{(k)}(0) \right\} + \mathcal{L}^{-1}\left\{ (Iz^\alpha + A)^{-1} \widehat{f}(z) \right\},$$

valid as long as the inverse Laplace transform $\mathcal{L}^{-1}$ of the right-hand side from (5) exists. One can employ a well-known property $\widehat{f}\,\widehat{g} = \mathcal{L}\{f * g\}$ of the convolution to deduce that

$$\begin{aligned} \mathcal{L}^{-1}\left\{ (Iz^\alpha + A)^{-1} \widehat{f}(z) \right\} &= \mathcal{L}^{-1}\left\{ (Iz^\alpha + A)^{-1} \right\} * \mathcal{L}^{-1}\{\widehat{f}(z)\} \\ &= \frac{1}{2\pi i} \int_0^t \int_{\Gamma_I} e^{z(t-s)} (z^\alpha I + A)^{-1} \, dz f(s) \, ds. \end{aligned}$$

By substituting the last expression into the previously obtained formula for $u(t)$, we obtain

$$\begin{aligned} u(t) =& \frac{1}{2\pi i}\sum_{k=0}^{n-1}\int_{\Gamma_I} \mathrm{e}^{zt} z^{\alpha-1-k}(z^\alpha I + A)^{-1} u^{(k)}(0)\,dz \\ &+ \frac{1}{2\pi i}\int_0^t \int_{\Gamma_I} \mathrm{e}^{z(t-s)}(z^\alpha I + A)^{-1} f(s)\,dz\,ds. \end{aligned} \tag{6}$$

The integration contour Γ_I in the Bromwich integrals above should be oriented counter-clockwise with respect to the singularities of the integrands. For the time being, let us assume the existence of such Γ_I and that the integrals in (6) are well-defined for the given A, $f(t)$, and α. Formula (6) determines a unique solution to (1) when the initial values of $u(t)$ are prescribed by the conditions

$$\begin{cases} u(0) = u_0, & \text{if } 0 < \alpha \le 1, \\ u(0) = u_0,\ u'(0) = u_1, & \text{if } 1 < \alpha < 2. \end{cases} \tag{7}$$

The fractional Cauchy problem (1), (7) is the main subject of the current work. The theory of such problems for differential operators was developed in [2,11–13]. The abstract setting, considered here, has been theoretically studied in [14,15] for $\alpha \in (0,1)$, and in [16] for $\alpha \in [1,2)$. Later, various particular cases of problem (1), (7) were analyzed in [17–24].

The inhomogeneous solution formula for (1), (7) also has a rich history. The authors of the pioneering works [14–16] established the solution existence using only the homogeneous version of (6). Shortly after, in [17], El-Borai derived the inhomogeneous solution representation for the abstract analog of (1), (7), with the Riemann–Liouville time-fractional derivative in place of ∂_t^α. He used the fractional propagator $P_\alpha(t) = \int\limits_0^\infty t^{-\alpha} \sum\limits_{n=0}^{\infty} \frac{(-st^{-\alpha})^n}{n!\Gamma(1-\alpha(n+1))} \exp(-As)\,ds$, which stems from the so-called subordination principle [16,25,26]. In the latter works [24,27], the proposed formula was extended to problem (1), (7) and its generalizations. Due to the properties of $P_\alpha(t)$ [28,29], its applicability is limited to $\alpha \in (0,1)$. Therefore, in order to generalize beyond the sub-parabolic case and justify (6), one has to adopt an alternative notion of the propagator $S_\alpha(t) = \frac{1}{2\pi i}\int\limits_{\Gamma_I} e^{zt} z^{\alpha-1}(z^\alpha I + A)^{-1} dz$, which was also proposed in [16]. For $\alpha \in (0,1)$, the formal justification of (6) with the help of $S_\alpha(t)$ was carried out in [30]. This work is notable for containing both the global and local existence results. In [21], the solution formula was extended using the Laplace transform to the case $\alpha \in (1,2)$. Actually, the formula provided in [21] (Definition 3.2) expresses the solution of (1), (7) solely in terms of $S_\alpha(t)$ and the convolution operations: $u(t) = S_\alpha(t)u_0 + 1 * S_\alpha(t)u_1 + (t^{\alpha-2}/\Gamma(\alpha-1) * S_\alpha * f)(t)$; but, due to the singularity of $t^{\alpha-2}$, this representation is of practical reference for $\alpha \in (0,2)$ only when reduced to (6).

Meanwhile, the authors of [18] adopted an alternative solution derivation procedure. Instead of working with the Laplace transformed formula from (5), they considered the integral analog of (1) and used the respective theory of the abstract integral equations [25], which can be applied without specifying the precise notion of a propagator. This resulted in three distinct mild solution representations [18] (Lemma 1.1), which are progressively attuned to the local time regularity of the propagator and $f(t)$. However, these formulas involve the derivative of the propagator; hence, they are also practically unsuitable for the reasons to be explained below.

If the operator A is bounded, all reviewed representations are functionally equivalent to a well-known solution formula, viz. (12), given in terms of the two-parameter Mittag–Leffler functions [1,31].

It is worth noting that (6) might be viewed as an extension of the standard mild solution formula for the abstract first-order Cauchy problem [32–34]. Indeed, when $\alpha = 1$, the contour integrals from (6) are reduced to the Dunford–Cauchy representation of $\exp(-At)u(0)$ and $\exp(-A(t-s))f(s)$:

$$\exp(-At) = \frac{1}{2\pi i}\int\limits_{\Gamma_I} \mathrm{e}^{zt}(zI+A)^{-1}\,dz,$$

defined in terms of the holomorphic function calculus for sectorial operators [7]. If $\alpha \neq 1$, such an interpretation of the contour integrals as operator functions can no longer be applied because the contour Γ_I encircles $\mathrm{Sp}(-A)$ and the scalar-part singularity of the first term in (6), located at $z = 0$.

The current study is motivated by an important observation regarding the aforementioned representation of the solution to problem (1), (7). For a sufficiently large $z \in \Gamma_I$, the norms of the integrands pertaining to the homogeneous part of (6) are asymptotically equal to $|e^{zt}||z|^{-1-k}$, $k = 0,1$. Consequently, they decay at least linearly in z, even if $t = 0$. Under the same conditions, the norm of the remaining integrand from (6) is asymptotically equal to $\left|e^{z(t-s)}\right||z|^{-\alpha}$, which leads to a slower-than-linear decay with respect to z for $t = s$ and $\alpha < 1$. This fact is obviously detrimental to the practical applications of (6) relying on the quadrature-based numerical evaluation of the solution [35–38]. The same observation applies to the solution representations from [18]. The goal of this paper is to present an alternative and more algorithmically relevant form of the mild solution to problem (1), (7), valid for any $\alpha \in (0,2)$.

In Section 2, we show that the uniform strong convergence of the integrals in (6) for $\alpha \in (0,2)$ and $t \in [0,T]$ necessitate the additional space regularity assumptions on the constituents of (1) and (7): $u_0 \in D(A^\gamma)$, $f(t) \in D(A^\kappa)$ with some $\gamma > 0$ and $\kappa > \frac{1-\alpha}{\alpha}$ (see Lemma 2). For $\alpha < 1$, this leads to the considerable reduction of the class of admissible right-hand sides. In this section, we also further discuss the connection between Formula (6) and other existing solution representations, the corresponding propagators as well as their validity and properties.

Guided by the gathered knowledge, in Section 3, we derive an alternative formula for the mild solution to the general inhomogeneous version of problem (1), (7) with $\alpha \in (0,2)$. It is the main theoretical result of the work. The derived formula exhibits strong convergence for $t \geq 0$ under the minimal space regularity assumptions for u_0 and $f(t)$, akin to those of the standard parabolic Cauchy problem. This property is achieved at the expense of imposing stricter regularity conditions on the right-hand side in time $f \in W^{1,1}([0,T],X)$ for $\alpha < 1$. To justify the new solution representation, we rely upon the results from the theory of abstract integral equations [25] instead of the usual toolkit from the holomorphic function calculus. The new representation is thoroughly validated in [38], where it is used as a base for the exponentially convergent numerical method.

2. Propagators and Existing Solution Representation Formulas

We start by stating several basic facts regarding problem (1), (7).

Definition 1 ([16])**.** *Let $u_0, u_1 \in X$ and $f \in C([0,T],X)$. A function $u \in C([0,T],D(A))$ is called a strong solution of* (1), (7) *if*

$$u \in C^{n-1}([0,T],X), \qquad \int\limits_0^t (t-s)^{n-\alpha-1}\left(u(s) - \sum_{k=0}^{n-1}\frac{s^k}{k!}u^{(k)}(0)\right)ds \in C^n([0,T],X), \tag{8}$$

and (1), (7) *hold.*

Definition 2 ([16])**.** *Problem* (1), (7) *is called well-posed if for any given $x_0, x_1 \in D(A)$, there exists a unique strong solution $u(t,x_0,x_1)$ of* (1), (7) *and for arbitrary sequences $x_{kp} \in D(A)$, the fact that $x_{kp} \to 0$ as $p \to \infty$, $k = 0,1$ implies $u(t,x_{0p},x_{1p}) \to 0$ uniformly on compact intervals.*

Next, we introduce the concept of the propagator for the given problem. It coincides with the solution operator of (1), (7) when $u_1 = 0$ and $f(t) \equiv 0$, $t \geq 0$.

Definition 3 ([25])**.** *Let $\alpha \in (0,2)$. A bounded linear operator $S_\alpha(t) : X \to X$ is called a propagator of* (1)*,* (7) *if the following three conditions are satisfied:*

P1. *$S_\alpha(t)$ is strongly continuous on X for $t \geq 0$ and $S_\alpha(0) = I$.*

P2. *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$.*

P3. *$u(t) = S_\alpha(t)u_0$ is the solution of*

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)ds, \tag{9}$$

for all $u_0 \in D(A)$, $t \geq 0$.

In this work, we focus on the mild solution to the given problem. Let us consider a Volterra integral equation

$$u(t) = \sum_{k=0}^{n-1} u_k t^k - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (Au(s) - f(s))ds. \tag{10}$$

Similar to (9), this equation serves as an integral analog of problem (1), (7) when u_1 and $f(t)$ are not identically equal to zero. To show this, one needs to apply the Riemann–Liouville integral operator

$$J_\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds, \quad v \in L^1([0,T], X) \tag{11}$$

to both sides of (1) and evaluate $J_\alpha \partial_t^\alpha$ with the help of the formula $J_\alpha \partial_t^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{\Gamma(k+1)} t^k$, that is valid if $u(t)$ satisfies the regularity assumptions from Definition 1 (see [16] (Section 1.2)). The highlighted analogy suggests us to adopt the following definition of the mild solution from [30].

Definition 4. *For any given $\alpha \in (0,2)$, $u_0, u_1 \in X$, and $f \in C([0,T], X)$, a function $u \in C([0,T], D(A))$ is called a mild solution of* (1)*,* (7) *if it satisfies the integral equation* (10)*.*

Every strong solution to the given problem is also its mild solution. The backward conjecture is false since the solution $u(t)$ to (10) does not, in general, satisfy conditions (8), so the derivative $\partial_t^\alpha u(t)$ might not exist. The notions of strong and mild solutions coincide when $f(t) \equiv 0$ and the second initial condition exhibits additional spatial regularity $u_1 \in D(A)$. For this reason, the well-posedness of the homogeneous version of problem (1), (7) in the sense of Definition 2 is equivalent to the well-posedness of the integral equation (10) in the sense of Definition 1.2 from [25]. This observation is based on Formula (12), the analyticity of the involving solution operators, and the bound (2.27) from [16]. Furthermore according to Proposition 1.1 of [25], Equation (10) is well-posed if and only if the given problem admits the propagator satisfying Definition 3.

To demonstrate a concrete example of the propagator of (1), (7), let us assume for the moment that A is bounded. Then, the inverse Laplace transform of the terms in (6) can be evaluated explicitly. This gives rise to an alternative formula for the inhomogeneous solution of problem (1), (7):

$$u(t) = \sum_{k=1}^{n} E_{\alpha,k}(-At^{\alpha})u_{k-1} + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-A(t-s)^{\alpha}) f(s)\, ds, \tag{12}$$

valid for the bounded linear operator A and any $u_0, u_1 \in X$. This representation of $u(t)$ in terms of a linear combination of the two-parameter Mittag–Leffler functions $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$ is well-known in the field of ordinary fractional differential equations [1,10,31], where it has far-reaching theoretical and practical applications. Through the direct transformations of the series expansion for $E_{\alpha,1}(-At^{\alpha})$, it is easy to conclude that this operator from (12) satisfies the conditions P1 and P2 of Definition 3. The validity of P3 follows from the equivalence between (9) and the homogeneous version of the given problem, which can be demonstrated by applying J_α to each part of (1). Therefore, for any finite $t \geq 0$ and bounded A, the function $E_{\alpha,1}(-At^{\alpha})$ is the propagator of (1), (7) in the sense of Definition 3. Note that its time derivative of the n-th order has a singularity at $t = 0$, which is caused by the structure of the kernel in (2). This is a general property of the fractional-order solution to problem (1), (7).

Unfortunately, the described notion of the propagator has limited practical utility for the evaluation of (12) because it is ultimately confined to the case when all operator powers $A^k u_0$, $k \in \mathbb{N}$ are bounded. Additionally, the convergence of the series for $E_{\alpha,\beta}(z)$ might be slow for certain combinations of α, β, even if z is a scalar [39].

Next, we would like to extend our analysis to the targeted class of strongly positive operators A. At this point, it is instructive to provide some background on the connection between the chosen range $(0,2)$ for the fractional-order parameter α and the aforementioned class of A. As we have already mentioned, the bulk of existing research is devoted to the particular cases of (1) when A is a strongly elliptic linear partial differential or, more generally, a pseudo-differential operator with the domain $D(A)$ that is dense in X. These cases are encompassed by the class of strongly positive operators [8,40], considered in the present work. Another common property of the solutions to (1), (7), with $\alpha \in (0,2)$, is their decaying behavior as $t \to \infty$, typical for the classical first-order Cauchy problem ($\alpha = 1$). The case $\alpha = 2$ is essentially different in nature, as it gives rise to the non-decaying-in-time propagator $\cos \sqrt{A}t$ (see [34]), which is well-defined only for the class of operators with the spectrum in parabola [41,42]. The class of admissible A becomes even more restrictive when the fractional-order parameter α is greater than 2. Namely, for such α, the operator A should necessarily be bounded; otherwise, problem (1), (7) is no longer properly defined [16] (Thm. 2.6) (see also [43] (p. 99)). Therefore, the range $\alpha \in (0,2)$ is both natural and the maximum possible for the targeted class of A.

2.1. Contour-Based Definition of the Propagator

The extension of solution formula (12) to the class of sectorial operators requires a more general notion of the propagator $S_\alpha(t)$, formulated in terms of the contour representation, which is directly compatible with (6). The following lemma is based on the relevant results of [16] (Section 2.2). Parts of its proof are reused in the sequel; hence, for the convenience of the reader, we provide it in full detail.

Lemma 1. *Let $\alpha \in (0,2)$. Assume that A is a strongly sectorial operator, with the spectral parameters ρ_s and φ_s. If $\beta \geq 1$, the operator function $S_{\alpha,\beta}(t)$:*

$$S_{\alpha,\beta}(t)x = \frac{1}{2\pi i} \int_{\Gamma_I} e^{zt} z^{\alpha-\beta} (z^{\alpha} I + A)^{-1} x dz \tag{13}$$

is well-defined for any $x \in X$. Moreover, under these conditions, $S_\alpha(t) \equiv S_{\alpha,1}(t)$ is the propagator of (1), (7). The contour Γ_I is chosen in such a way that the integral in (13) is convergent and the curve z^{α}, $z \in \Gamma_I$ is positively oriented with respect to $\mathrm{Sp}(-A) \cup \{0\}$.

Proof. First, let us check that the operator function $S_{\alpha,\beta}$ is well-defined for any $\beta > 1$ and $t \geq 0$. To show this, it is enough to demonstrate that the integral in (13) is uniformly convergent with respect to t. From definition (13), one can deduce that

$$\|S_{\alpha,\beta}(t)x\| = \frac{1}{2\pi}\left\|\int\limits_{\Gamma_I} e^{zt}z^{\alpha-\beta}(z^\alpha I + A)^{-1}x\,dz\right\| \leq C_1 \int\limits_{\Gamma_I} \left|e^{zt}z^{\alpha-\beta}\right| \|R(z^\alpha, -A)x\|\,dz.$$

We apply the resolvent bound from (4) to the last integral and obtain the estimate

$$\|S_{\alpha,\beta}(t)x\| \leq C\|x\| \int\limits_{\Gamma_I} \frac{|z|^{\alpha-\beta}e^{t\Re z}}{1+|z|^\alpha}\,dz \leq C\|x\| \sup_{z\in\Gamma_I} e^{t\Re z} \int\limits_{\Gamma_I} \frac{|z|^{\alpha-\beta}}{1+|z|^\alpha}\,dz,\ \forall x \in X. \tag{14}$$

Here and below, $C > 0$ denotes a generic finite constant. The integral in (14) is absolutely convergent for an arbitrary $\beta > 1$ since the integrand is bounded and decays as $|z|^{-\beta}$ when z tends to infinity on the contour Γ_I. In order to convert (14) into the proper bound, we choose Γ_I to be a composition of two rays $-\rho e^{\pm i\varphi_s}$, $\rho \in [r, +\infty]$, which are parallel to the boundary of $-\Sigma(\rho_s, \varphi_s)$ and a longer arc of the circle re^{is}, $s \in [-\pi+\varphi_s, \pi-\varphi_s]$. Recall that the function z^α maps the sector $\Sigma(0, \varphi_s)$ into the sector $\Sigma(0, \alpha\varphi_s)$; thus, the condition $\varphi_s < \pi/2$ implies that the curve z^α, $z \in \Gamma_I$ is free of self-intersections for all $\alpha \in (0,2)$. Then, for any given $1 < r < \infty$, the last integral in (14) can be estimated as

$$\begin{aligned}
\int\limits_{\Gamma_I} \frac{|z|^{\alpha-\beta}}{1+|z|^\alpha}\,dz &= 2\int\limits_r^\infty \frac{|\rho e^{i\varphi_s}|^{\alpha-\beta}}{1+|\rho e^{i\varphi_s}|^\alpha}\,d\rho + \int\limits_{-\pi+\varphi_s}^{\pi-\varphi_s} \frac{|re^{is}|^{\alpha-\beta}}{1+|re^{is}|^\alpha}\,ds \\
&= 2\int\limits_r^\infty \frac{\rho^{\alpha-\beta}}{1+\rho^\alpha}\,d\rho + \int\limits_{-\pi+\varphi_s}^{\pi-\varphi_s} \frac{r^{\alpha-\beta}}{1+r^\alpha}\,ds \leq 2\int\limits_r^\infty \rho^{-\beta}d\rho + \frac{s}{r^\beta}\bigg|_{-\pi+\varphi_s}^{\pi-\varphi_s} \\
&= \frac{2(\pi-\varphi_s)}{r^\beta} - 2\frac{\rho^{1-\beta}}{\beta-1}\bigg|_r^{+\infty} = \frac{2(\pi-\varphi_s)}{r^\beta} + \frac{2r^{1-\beta}}{\beta-1}.
\end{aligned}$$

Additionally, the value of $\sup\limits_{z\in\Gamma_I} e^{t\Re z}$ from (14) is equal to e^{rt}, with r being the arc radius of the chosen Γ_I. This observation leads us to the estimate

$$\|S_{\alpha,\beta}(t)x\| \leq C\frac{e^{rt}}{r^{\beta-1}}\left(\frac{1}{\beta-1} + \frac{(\pi-\varphi_s)}{r}\right)\|x\|, \quad C > 0,\ r > 1, \tag{15}$$

which clearly shows that $\|S_{\alpha,\beta}x\|$ is bounded for $\beta > 1$. Furthermore, due to the fact that the dependence on t in (14) and (15) is expressed only by the scalar factor e^{rt}, the integral in Formula (13) convergences uniformly in $t \in [0, T]$.

In order to treat $S_\alpha(t)$, we recall that for any $x \in D(A^{m+1})$, the resolvent $R(z, A)x$ satisfies the identity (2.25) from [6]:

$$R(z, A)x = \sum_{k=1}^{m+1} \frac{A^{k-1}}{z^k}x + \frac{1}{z^{m+1}}R(z, A)A^{m+1}x, \quad m \in \mathbb{Z}_+. \tag{16}$$

Now, assume that t is strictly positive. Together with (4), such an assumption guarantees the strong convergence of the integral in (13) on the contour Γ_I described above. Then, by applying identity (16) with $m = 0$ to this convergent representation of $S_\alpha(t)$, we obtain

$$S_\alpha(t)x = \frac{1}{2\pi i}\int\limits_{\Gamma_I} \frac{e^{zt}}{z}x\,dz - \frac{1}{2\pi i}\int\limits_{\Gamma_I} \frac{e^{zt}}{z}(z^\alpha I + A)^{-1}Ax\,dz = x - S_{\alpha,1+\alpha}(t)Ax,$$

for any $x \in D(A)$. The term $S_{\alpha,1+\alpha}(t)Ax$ satisfies estimate (15) with $\beta = 1+\alpha$ and any $r > 1$. Consequently, the last formula can be used to extend the definition of the propagator $S_\alpha(t)$ to the closed interval $t \in [0,T]$, where it remains bounded and strongly continuous. The generalization of this result to all $x \in X$ follows from the dense embedding $D(A) \subseteq X$ and the closedness of A [7]. Additionally, bound (15) stipulates that the norm of $S_{\alpha,1+\alpha}(0)Ax$ can be made arbitrarily small by increasing r. It implies that $S_\alpha(0)x = x$ for any $x \in D(A)$. The equality $S_\alpha(0) = I$ on X follows from the strong continuity of $S_\alpha(t)$ established earlier. This concludes the proof of P1 from Definition 3.

To show the validity of P2 and P3, we proceed by establishing the correspondence between $S_\alpha(t)$ and the solution operator

$$H(a,B,t) = \frac{1}{2\pi i}\int_{\Gamma_I} \frac{e^{zt}}{z}(I - \widehat{g_\alpha}(z)B)^{-1}dz,$$

studied by Prüss [25] in the context of a more general integral equation

$$v(t) = v_0 + \int_0^t g_\alpha(t-s)Bv(s)ds.$$

In the case of (9), we have $B = -A$, $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $v_0 = u(0)$, so

$$\begin{aligned} u(t) &= H\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, -A, t\right)u(0) = \frac{1}{2\pi i}\int_{\Gamma_I} \frac{e^{zt}}{z}\left(I + z^{-\alpha}A\right)^{-1}u(0)dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_I} e^{zt}z^{\alpha-1}(Iz^\alpha + A)^{-1}u(0)dz, \end{aligned}$$

according to Theorem 1.3 from [25]. Now, the validity of P2 and P3 follows from the results of Theorem 2.1, Example 2.1 in [25], which are applicable because of the bound on $(Iz^\alpha + A)^{-1}$ imposed by (4) and the inequality $\alpha\varphi_s < \pi$. □

We point out that the propagator $S_{\alpha,\beta}(t)$ from Lemma 1 is defined only for $\beta \geq 1$, although its representation (13) is formally equivalent to the integral formula for $E_{\alpha,\beta}(-At^\alpha)$ with other admissible β [16]. Our motivation for doing so is to distinguish the precise meaning of a propagator from a more general object $E_{\alpha,\beta}(-At^\alpha)$ while still being able to cover all cases of α, β relevant to the given problem in its full generality.

The part of the above proof connected with bound (15) will become instrumental in the forthcoming analysis. We state it as a corollary.

Corollary 1. *Let the operator A and the fractional-order parameter α fulfill the assumptions of Lemma 1. If $\beta > 1$, then the integral from definition (13) of the operator function $S_{\alpha,\beta}(t)x$ is strongly convergent for any $t \in [0,T]$ and $x \in X$. Moreover, $S_{\alpha,\beta}(0)x = 0$.*

By slightly abusing the notation of Lemma 1, we can rewrite (6) in a more compact form:

$$u(t) = \sum_{k=1}^{n} S_{\alpha,k}(t)u_{k-1} + \int_0^t (t-s)^{\alpha-1}S_{\alpha,\alpha}(t-s)f(s)\,ds, \tag{17}$$

The rigorous proof of this solution formula was provided in [21,30]. Our aim here is to investigate the conditions on u_0, u_1, and $f(t)$ needed to maintain the strong convergence of the integral representations for $S_\alpha(t)u_0$, $S_{\alpha,2}(t)u_1$, and $S_{\alpha,\alpha}(t-s)f(s)$ for any $t \in [0,T]$. To understand why this investigation is important, let us recall that the n-th derivative of the solution to the given fractional Cauchy problem is unbounded at $t = 0$. Such behavior of $u(t)$ has a profound impact on the properties of the finite-difference methods for (1), (7). Their convergence order is typically limited by n, even for the multi-step methods (see [44,45] and the references therein). Besides that, at each time-step, these methods need to query the solution history in order to evaluate $\partial_t^\alpha u(t)$ numerically, which makes them computationally costly and memory constrained [46].

The mentioned drawbacks of finite-difference methods have put an additional spotlight on the alternative numerical solution evaluation strategies [47–49] and, in particular, on the quadrature-based numerical methods [35–38]. Unlike finite differences, the methods from the latter group perform the direct numerical evaluation of (17) via the quadrature of the integral in Formula (13), defining the operator function $S_{\alpha,\beta}(t)$. Such a strategy offers a unique combination of advantages, including multi-level parallelism, the concurrent evaluation of solution for different values of t, and, most notably, the exponential convergence of the approximation for any given t, including $t = 0$. The last property is contingent upon the strong convergence of the contour integral in (13).

We observe that the strong convergence of the integral definition for $S_{\alpha,2}(t)u_1$ with any $u_1 \in X$ and $t \in [0, T]$, established in Corollary 1, cannot be proven in general for two other components $S_\alpha(t)u_0$ and $S_{\alpha,\alpha}(t-s)f(s)$ of (17). The convergence of the respective integrals degrades when the scalar argument of $S_\alpha(t)$ or $S_{\alpha,\alpha}(t-s)$ approaches 0, as shown by (14). In the proof of Lemma 1, we demonstrated that the strong convergence of $S_\alpha(0)u_0$ can be reestablished by redefining the formula for the propagator via the identity $S_\alpha(t)u_0 = u_0 - S_{\alpha,\alpha+1}(t)Au_0$, valid for $u_0 \in D(A)$. The next proposition supplies a more refined version of this technique.

Proposition 1 ([38])**.** *Let A be the sectorial operator satisfying the assumptions of Lemma 1. If $x \in D(A^{m+\gamma})$ and $z^\alpha \notin \mathrm{Sp}(A) \cup \{0\}$, then for any $\gamma \geq 0$,*

$$\left\| z^{\alpha-\beta}(z^\alpha I - A)^{-1}x - \frac{1}{z^\beta}\sum_{k=0}^{m}\frac{A^k x}{z^{\alpha k}} \right\| \leq \frac{K(1+M)\|A^{m+\gamma}x\|}{|z|^{m\alpha+\beta}(1+|z|^\alpha)^\gamma}, \tag{18}$$

with some constant $K > 0$ and M defined by (4).

This result allows us to relax the space regularity assumptions for $S_{\alpha,1}(t)u_0$ to $u_0 \in D(A^\gamma)$, with some $\gamma > 0$, and still maintain the strong convergence of its definition (13) for $t \in [0, T]$. In the case of $S_{\alpha,\alpha}(t-s)f(s)$, we have a more ambiguous situation since the integrand in (13) for $\beta = \alpha$ does not have a singularity at $z = 0$. Thus, if we assume that $\mathrm{Sp}(A)$ is separated from the origin, the integration contour Γ_I can be chosen to reside entirely in the same half-plane as $\mathrm{Sp}(-A)$. The strong convergence of $S_{\alpha,\alpha}(t-s)f(s)$ for the targeted range of t, s can be achieved via (16) only for such a subclass of sectorial operators.

Lemma 2. *Assume that operator A satisfies the assumptions of Lemma 1 and $f \in L^1([0, T], X)$. The operator function $S_{\alpha,\alpha}(t)$ is well-defined with any $\alpha \in (0, 2)$, and the corresponding integral for $S_{\alpha,\alpha}(t-s)f(s)$ is strongly convergent for any $t \in [0, T]$ and all $s \in [0, t]$ if there exists $\delta > \frac{1-\alpha}{\alpha}$, such that $f(t) \in D(A^\delta)$ and the spectrum $\mathrm{Sp}(A)$ is separated from the origin, i.e., $\rho_s > \rho_0$ for some $\rho_0 > 0$. If $\mathrm{Sp}(A)$ is arbitrarily close to 0, then the restriction $f(s) \equiv 0$ is necessary to obtain the strong convergence of $S_{\alpha,\alpha}(t-s)f(s)$ via (16).*

Proof. Let us first deal with the situation where $\rho_s \geq \rho_0 > 0$. If $\alpha > 1$, the sought convergence readily follows from (14). Thus, we focus on $\alpha \leq 1$. If $f(s) \in D(A^\delta)$,

$$
\begin{aligned}
\|S_{\alpha,\alpha}(t-s)f(s)\| =& \left\| \frac{1}{2\pi i} \int\limits_{\Gamma_I} e^{z(t-s)} (z^\alpha I + A)^{-1} f(s)\, dz \right\| \\
\le& \frac{1}{2\pi} \left\| \int\limits_{\Gamma_I} e^{z(t-s)} \left((z^\alpha I + A)^{-1} - \sum_{k=0}^{m} \frac{(-A)^k}{z^{\alpha(k+1)}} \right) f(s)\, dz \right\| \\
&+ \left\| \frac{1}{2\pi i} \int\limits_{\Gamma_I} e^{z(t-s)} \sum_{k=0}^{m} \frac{(-A)^k f(s)}{z^{\alpha(k+1)}}\, dz \right\|, \quad m = \lfloor \delta \rfloor.
\end{aligned}
\tag{19}
$$

The integrands from the last term are analytic inside the region encircled by Γ_I and remain bounded because the norms

$$
\|A^k x\| = \|A^{k-m} A^m x\| \le \frac{M}{2\pi} \int\limits_{\Gamma_I} \frac{|z|^{k-m}}{1+|z|} \|A^m x\|\, dz \tag{20}
$$

are bounded for $k < m$. Consequently, the second term is equal to zero by the Cauchy integral theorem. The first norm from the above bound for $\|S_{\alpha,\alpha}(t-s)f(s)\|$ is estimated via Proposition 1, yielding

$$
\|S_{\alpha,\alpha}(t-s)f(s)\| \le C \int\limits_{\Gamma_I} \frac{\|A^\delta f(s)\|}{|z|^{\alpha(m+1)}(1+|z|^\alpha)^{\delta-m}}\, dz, \quad s \in [0,t],\ t \in [0,T]. \tag{21}
$$

The strong convergence of the integral in (21) is assured by the condition $\alpha(m+1) + \alpha(\delta - m) > 1$, which gives us the target inequality for δ.

Next, we consider the case when the gap between $\mathrm{Sp}(A)$ and 0 is arbitrarily small. Then, similarly to $S_\alpha(t)$, we are forced to use the contour Γ_I that encircles $-\Sigma(\rho_s, \varphi_s) \cup \{0\}$. In this case, the scalar integrals from the second term in (19) can be regarded as the inverse Laplace transform of $z^{-\alpha(k+1)}$, so

$$
\frac{1}{2\pi i} \int\limits_{\Gamma_I} e^{z(t-s)} \sum_{k=0}^{m} \frac{(-A)^k f(s)}{z^{\alpha(k+1)}}\, dz = \sum_{k=0}^{m} \frac{(t-s)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} A^k f(s).
$$

By inspecting the leading term of the last sum, we deduce that it is bounded for any $t \in [0,T]$ and $s \in [0,t]$, if and only if $f(s) \equiv 0$. This completes the proof. $\square$

This lemma highlights two key deficiencies of the solution representation (17) or, equivalently, (6). Firstly, the stable and accurate numerical approximation of the inhomogeneous part of (17) is achievable only under the additional space regularity assumption on the right-hand side $f(t)$ of (1). For $\alpha < 1$, it is generally more restrictive than the corresponding assumption $u_0 \in D(A^\gamma)$, $\gamma > 0$, needed for the homogeneous part. If $\alpha < 1/2$, the required space regularity might simply become incompatible with certain combinations of $f(t)$ and A since the domain of $A^{\frac{1-\alpha}{\alpha}}$ could no longer be dense in X. Furthermore, the evaluation of the resolvent arguments $A^k f(s)$, $k > 1$ from (16), needed to enforce the convergence of $S_{\alpha,\alpha}(t-s)f(s)$ with $\alpha < 1/2$, is a challenging numerical problem in itself. It cannot be solved accurately without special precautions when A is a finite-difference or finite-element discretization of some elliptic partial differential operator. Recall that in the integer order case of problem (1), (7), the space regularity assumptions for the homogeneous and inhomogeneous parts of the solution are equivalent and quite mild: $u_0, f(t) \in D(A^\gamma)$, $\gamma > 0$ since the same propagator $\exp(-At) = S_1(t)$ is used for both parts.

Secondly, the lemma's requirement about the spectrum separation renders the evaluation of $S_{\alpha,\alpha}(t)$ practically unfeasible for an important family of nonlinear problems with singularly perturbed A [50]. Furthermore, the existing accuracy theory [6,35,36] suggests that a slow convergence of the quadrature approximation of $S_{\alpha,\alpha}(t)$ is to be expected when the spectral parameter ρ_s is positive but small (e.g., [51,52]).

The convergence versus regularity behavior of the formulas that involve $S'_{\alpha,\beta}(t)$ can be analyzed in a similar fashion, albeit the use of Proposition 1 should, this time, be prepended by the transformation $S'_{\alpha,\beta}(t) = S_{\alpha,\beta-1}(t)$ stemming from (13). In this way, we get the condition $f(t) \in D(A^{\delta})$, $\delta > 1/\alpha$ for the solution representations from Lemma 1.1 of [18], which is even more restrictive than the corresponding condition $\delta > \frac{1-\alpha}{\alpha}$ for (17) or (6).

2.2. Representation of Solution via Subordination Principle

In the sub-parabolic case, the space regularity assumptions imposed by Lemma 2 can be relaxed by introducing yet another notion of the fractional propagator [16,25,26]:

$$P_{\alpha}(t) = \int_0^{\infty} \Phi_{\alpha}(s) \exp\left(-Ast^{\alpha}\right) ds, \quad \Phi_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\alpha(n+1))}. \tag{22}$$

The so-called M-Wright function $\Phi_{\alpha}(z)$ from (22) obeys the law [53]

$$\int_0^{\infty} z^n \Phi_{\alpha}(z)\, dz = \frac{\Gamma(n+1)}{\Gamma(\alpha n+1)}, \quad n > 1,\ \alpha \in [0,1). \tag{23}$$

Then, the mild solution of problem (1), (7) with $\alpha \in (0,1)$ can be represented as follows [17,24]

$$u(t) = P_{\alpha}(t)u_0 + \int_0^{t} (t-s)^{\alpha-1} P_{\alpha,\alpha}(t-s) f(s)\, ds, \tag{24}$$

where the operator function $P_{\alpha,\alpha}(t) = \alpha \int_0^{\infty} s\Phi_{\alpha}(s) \exp\left(-Ast^{\alpha}\right) ds$ is well-defined for all $t \in [0,T]$ owing to (23). The strong convergence of (24) is guaranteed under the minimal space regularity assumptions on both u_0 and $f(t)$: $u_0 \in D(A^{\gamma})$, $f(t) \in D(A^{\kappa})$, $\gamma, \kappa > 0$. From the application point of view, however, the representation of the solution to (1), (7) by means of (24) introduces an additional level of complexity associated with the numerical evaluation of the time integrals in $P_{\alpha}(t)$ and $P_{\alpha,\alpha}(t)$. For that reason, the numerical potential of (22) has been largely left unexplored, with the exception of [54]. The direct approximation of (24) might become a worthwhile complement to the numerical methods from [36–38], provided that the underlying integrals can be numerically evaluated in an accurate and efficient manner for small values of $\alpha \in (0, 0.1)$ or when $\alpha\varphi_s > \pi/2$.

The evidence supplied in [38] indicates that all the mentioned complications with (17) can be avoided if the solution formula involves only the propagators $S_{\alpha}(t)$, $S_{\alpha,2}(t)$. The next section is devoted to deriving the alternative representation of the solution to the fractional Cauchy problem (1), (7) that fulfills this property.

3. New form of Mild Solution Representation

According to Proposition 1.2 from [25], the mild solution $u(t)$ of (1), (7) can be formally expressed through the variation of parameters formula

$$u(t) = \frac{d}{dt} \int_0^{t} S_{\alpha}(t-s) v(s)\, ds, \tag{25}$$

where the function $v(t) = \sum_{k=0}^{n-1} u_k t^k + J_{\alpha} f(t)$ is determined from (10). The next theorem presents a more convenient version of (25) and formalizes the conditions for its existence.

Theorem 1. *Let $\alpha \in (0,2)$. Assume that A is a strongly positive operator with the domain $D(A)$ and the spectral parameters ρ_s, φ_s. If $f \in W^{1,1}([0,T],X)$, $u_0, u_1 \in D(A)$, then there exists a unique mild solution $u(t)$ to problem (1), (7) that can be represented as follows*

$$u(t) \ = \ S_\alpha(t)u_0 + S_{\alpha,2}(t)u_1 + J_\alpha S_\alpha(t)f(0) + \int_0^t S_\alpha(t-s)J_\alpha f'(s)ds. \tag{26}$$

Here, the initial value $u_1 \equiv 0$ for $\alpha \in (0,1]$ and $S_{\alpha,\beta}(t)$ is defined by (13).

Proof. The assumptions imposed on A in the formulation of the theorem, together with the results of Lemma 1, guarantee the existence and well-definedness of the propagator $S_\alpha(t)$. Equation (10) can be regarded as an abstract integral equation

$$u(t) \ = \ v(t) - (g_\alpha * Au)(t), \tag{27}$$

with $g_\alpha(t) \ = \ t^{\alpha-1}/\Gamma(\alpha)$, $v(t) \ = \ u_0 + tu_1 + J_\alpha f(t)$. Here, the symbol $*$ is again used to denote the convolution $(f * g)(t) = \int_0^t f(t-s)g(s)\, ds$.

Let $\alpha > 1$. Then, the function $v(t)$ is differentiable and

$$\begin{aligned} v'(t) \ &= \ u_1 + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} f(s)\, ds \\ &= u_1 + g_\alpha(t-s)f(s)\Big|_0^t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f'(s)\, ds \\ &= \ u_1 + g_\alpha(t)f(0) + J_\alpha f'(t). \end{aligned}$$

This formula, together with the theorem's assumptions regarding the regularity of $f(t)$, ensures that $v \in W^{1,1}([0,T],X)$. Hence, we are allowed to apply Proposition 1.2 from [25], which states that the function

$$u(t) \ = \ S_\alpha(t)v(0) + \big(S_\alpha * v'\big)(t) \tag{28}$$

is the unique solution to the Volterra integral equation $u(t) \ = \ f(t) - AJ_\alpha u(t)$. We substitute $v(0)$, $v'(t)$ into (28) and obtain

$$u(t) \ = \ S_\alpha(t)u_0 + (S_\alpha * u_1)(t) + J_\alpha S_\alpha(t)f(0) + \big(S_\alpha * J_\alpha f'\big)(t). \tag{29}$$

The terms involving $f(0)$ and $f'(t)$ belong to $D(A)$ for any fixed $t \in [0,T]$ according to Proposition 1.1 from [25]. Property P2 in Definition 3 declares that the propagator $S_\alpha(t)$ commutes with A since $u_0, u_1 \in D(A)$. Hence, the function $u(t)$ defined by (29) is the unique solution to (27) and thus to (10). After having demonstrated this, we further simplify the operator acting on the second initial condition u_1:

$$\begin{aligned} (S_\alpha * u_1)(t) &= \frac{1}{2\pi i} \int_{\Gamma_I} \int_0^t e^{z(t-s)} z^{\alpha-1} (z^\alpha I + A)^{-1} u_1\, ds\, dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_I} \left(-\frac{1}{z} e^{z(t-s)} \right)\Big|_0^t z^{\alpha-1} (z^\alpha I + A)^{-1} u_1\, dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_I} (e^{zt} - 1) z^{\alpha-2} (z^\alpha I + A)^{-1} u_1\, dz \\ &= S_{\alpha,2}(t)u_1 - S_{\alpha,2}(0)u_1. \end{aligned}$$

The term $S_{\alpha,2}(0)u_1$ is equal to zero, as stated in Corollary 1. Therefore, the combination of the last formula and (29) gives us (26).

Now, we assume that $\alpha \le 1$. In this case, the operator-independent part $v(t)$ of Equation (27) takes the form $v(t) = u_0 + J_\alpha f(t)$. For such α and $v(t)$, the unique solution of this equation is provided by formula (1.11) from [25]:

$$u(t) = S_\alpha(t)u_0 + J_\alpha S_\alpha(t)f(0) + \left(g_\alpha * S_\alpha * f'\right)(t).$$

As a final step needed to transform this representation into (26), we utilize the associativity of the convolution:

$$\left(g_\alpha * S_\alpha * f'\right)(t) = \int_0^t S_\alpha(t-s)J_\alpha f'(s)\,ds,$$

which holds true in the consequence of Fubini's theorem [55] (Thm. 8.7). The constructed solution representation involves only the linear bounded and continuous in t operators; hence, $u \in C([0,T], D(A))$ for any given combination of u_0, u_1, and $f(t)$. □

As we can see from Theorem 1, the inhomogeneous part of the newly derived solution representation relies only on the original problem's propagator $S_\alpha(t)$. By Proposition 1, this implies that the following space regularity assumptions are needed for the strong convergence of the integrals involved in the solution representation (26) with $t \ge 0$:

$$\exists \gamma, \kappa > \min\{0, 1-\alpha\}: \ u_0 \in D(A^\gamma),\ f(0), f'(s) \in D(A^\kappa), \quad s \in (0,T]. \tag{30}$$

For the affected range $\alpha \in (0,1]$, formula (30) is more permissive with respect to the space regularity of the function f than the constraints imposed by Lemma 2. In particular, inequality (20) reveals that the domain of A^κ with any fixed $\kappa \in (0,1]$ is dense in X. Hence, the strong convergence of the operator function definitions in (26) for $\alpha < 1/2$ is practically realizable for any positive sectorial operator A. This is not always the case for the solution formula (17), as discussed in Section 2.

When $\alpha \to 1$, Formula (26) recovers the mild solution to the integer-order Cauchy problem, which requires precisely the same conditions for the strong convergence as (30). On that account, the regularity imposed by (30) is as weak as possible, whereas formula (26) can be regarded as a more natural fractional extension of the integer-order solution compared to the previously known representation (17) or (6). Moreover, if the conditions of Theorem 1 are met and $u_1 \equiv 0$, then the constructed solution is continuous with respect to α for the whole range $\alpha \in (0,2)$. The mentioned properties of (26) should prove to be useful for certain parameter identification problems [56,57], as well as for the final value problems [58,59] associated with (1), (7).

It is noteworthy to highlight that the conditions from (30) fit well into the existing solution theory of fractional Cauchy problems. For instance, when $A = -\Delta$, the condition $u_0 \in D(A^\gamma)$ is equivalent to the regularity assumption $u_0 \in \dot{H}^{2\gamma}(\Omega)$, $\gamma \in (0,1]$, encompassing most of the realistic application scenarios. Then, the solution $u(t)$ to the homogeneous version of (1), (7) with such A, u_0, and $\alpha < 1$ satisfies the energy estimate from [60] (Theorem 2.1)

$$\|\partial_t^m u(t)\|_{\dot{H}^p(\Omega)} \le c t^{\alpha \frac{2\gamma - p}{2} - m} \|u_0\|_{\dot{H}^{2\gamma}(\Omega)}, \quad 0 \le p - 2\gamma \le 2,$$

indicating that the propagator $S_\alpha(t)$ exhibits a characteristic weak singularity at $t = 0$. For any other fixed $t > 0$, it has only a limited in-space smoothing effect, unlike the integer-order propagator $\exp(-At)$. A more in-depth review of the results on the connections between the regularity of the initial data and the solution to (1), (7) with such A is given in [61]. For the abstract solution regularity estimates and their discretized analogs, we refer the reader to [16,62–64].

Unsurprisingly, all the mentioned benefits of the new solution representation come at a price. Formula (26) requires more operations to evaluate the solution compared to (17), and it is dependent on $f'(t)$. In [38], we argue that such a trade-off is still worth to consider from a numerical standpoint because the action of the integer-order derivative is local, and the values of $J_\alpha f'(t)$, $t \in [0,T]$ can be pre-computed efficiently, even if $f'(t)$ has a singularity at one or both interval endpoints. Other techniques to evaluate the Riemann–Liouville integral operator J_α with appealing numerical properties are proposed in [49,65].

The computational efficiency of (26) is also affected by the order in which the operators J_α and $S_\alpha(t)$ act on $f(0)$ and $f'(t)$. The calculated values of $(z^\alpha I + A)^{-1} f(0)$ from (13) can be reused during the quadrature evaluation of $S_\alpha(t)f(0)$ for the different values of t. This explains the particular arrangement of terms in (26). A more detailed discussion on that matter is provided in the numerical part of the current study [38].

If $\alpha \in (1,2)$, we can get rid of $f'(t)$ in (26) using the formula $J_\alpha f'(t) = J_{\alpha-1} f(t) - J_\alpha f(0)$ and exchange $J_{\alpha-1}$ with $S_\alpha(t)$ by the associativity of the convolution. In this way, we arrive at the solution representation from [21,22]:

$$u(t) \;=\; S_\alpha(t)u_0 + \int\limits_0^t S_\alpha(s)u_1\, ds + J_{\alpha-1}(S_\alpha * f)\,(t), \quad f \in L^1([0,T],X). \tag{31}$$

The results of [18] use a similar derivation procedure as the one employed in Theorem 1; thus, the correspondence between the respective solution formulas is obvious. The equivalence between (26) and (17) is established via (31) and the identity $J_{\alpha-1}S_{\alpha,1}(t) = S_{\alpha,\alpha}(t)$, which was proven in [21].

4. Conclusions

This work is devoted to the aspects of the solution theory for the abstract fractional Cauchy problem (1), (7) that shed new light on the connections between the practical versatility of the mild solution representation, the strong convergence of the involved integral operators, and the space regularity of the problem's initial data $u_0, u_1, f(t)$. The uniform strong convergence of the integrals from the previously known representation (17) is guaranteed under the assumptions $u_0 \in D(A^\gamma)$, $f(t) \in D(A^\kappa)$ with some $\gamma > 0$ and $\kappa > (1-\alpha)/\alpha$. For sub-parabolic problems, the fulfillment of such assumptions leads to the disproportionately limiting constraints on the space regularity of $f(t)$ that become severe if $\alpha < 1/2$. This fact makes formula (17) unsuitable for the quadrature-based numerical evaluation of the mild solution. The issue with requiring the additional regularity of $f(t)$ is not related to the nature of (1), (7) because the homogeneous part of the solution, constructed from the native problem's propagator $S_\alpha(t)$ and its integral $\int\limits_0^t S_\alpha(t-s)u_1\, ds = S_{\alpha,2}(t)u_1$, remains unaffected. It is, rather, related to the particular structure of representation (17) and its use of the propagator $S_{\alpha,\alpha}(t)$, originally associated with a different fractional Cauchy problem.

To circumvent this issue, we derived the alternative representation (26) of the mild solution to problem (1), (7), that is free of $S_{\alpha,\alpha}(t)$ and remains valid for the same range of fractional-order parameters $\alpha \in (0,2)$ as (17). The strong convergence of the integrals in (26) for $t \in [0,T]$ is guaranteed under much more permissive space regularity assumptions (30), which are actually equivalent to the corresponding assumptions for the classical parabolic Cauchy problem [6]. In that respect, the space regularity induced by (30) can be regarded as minimal possible. Moreover, when $A = -\Delta$, the conditions in (30) are fully compatible with the existing energy estimates (e.g. [61]), describing the regularity of the solution in the scale of spaces $\dot{H}^s(\Omega)$. This observation provides an additional incentive to use the numerical method from the companion work [38] for applied problems with limited spatial regularity.

Even though the presented results were mostly driven by numerical applications, the proposed solution representation could also help to get some new theoretical insights. Developing more refined estimates for the inhomogeneous version of the given problem, that are based on the asymptotics of $S_\alpha(t)$, constitutes one of these directions. An application of the new solution formula to nonlinear extensions of the considered fractional Cauchy problem is another promising direction of research. Recall that according to the general theory, the solution of (1), (7) with $f(t) = f(t, u(t))$ has to satisfy Equation (26). In this regard, the results of Theorem 1, establishing the validity of (26) with no additional space regularity assumptions on f, should be useful. Such semi-linear extensions of the given problem will be considered in the future works.

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