Distributed Optimal Control with Recovered Robustness for Uncertain Network Systems: A Complementary Design Approach

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***Abstract*—This paper considers the distributed robust suboptimal consensus control problem of uncertain linear multi-agent systems, with both H_2 and H_∞ performance requirements. A novel two-step complementary design approach is proposed. In the first step, a distributed control law is designed for the nominal multi-agent system to achieve consensus with a prescribed H_2 performance. In the second step, an extra control input, depending on some carefully chosen residual signals indicating the modeling mismatch, is designed to complement the H_2 performance by providing robustness guarantee in terms of H_∞ requirement with respect to disturbances or uncertainties. The proposed complementary design approach provides an additional degree of freedom for design, having two separate controls to deal with the H_2 performance and the robustness of consensus, respectively. Thereby, it does not need to make much trade-off, and can be expected to be much less conservative than the trade-off design such as the mixed H_2/H_∞ control method. Besides, this complementary approach will recover the achievable H_2 performance when external disturbances or uncertainties do not exist. The effectiveness of the theoretical results and the advantages of the complementary approach are validated via numerical simulations.**

***Index Terms*—Robust control, cooperative control, consensus, distributed control, optimal control, H_∞ control, H_2 control.**

I. Introduction

Optimality and robustness are two main issues and missions in the feedback control theory [1]. Optimality requires an optimal or suboptimal controller to ensure that a closed-loop system satisfies certain predefined performance criteria, with the linear quadratic regulator (LQR) and the linear quadratic Gaussian (LQG)/H_2 problems as typical examples. Robustness, on the other hand, characterizes the property that a system still works well in the presence of external disturbances or model uncertainties, which can be addressed in the framework of H_∞ control and μ synthesis. For networked multi-agent systems, the optimality and robustness problems encounter new inherent challenges, since the control laws need to be distributed in the sense that only local information between neighboring agents can be utilized and meanwhile the control laws are subject to structural constraints imposed by the network topology [2]. The potential applications of distributed robust optimal control are broad, ranging from UAV formation to wireless sensor networks and intelligent transportation systems.

In the last two decades, many advances have been reported on optimality issues of networked multi-agent systems. In the distributed LQR framework, several approaches (e.g. suboptimal design [3], [4], hierarchical control [5], inverse optimal method [6] and decentralized computation [7]) have been proposed. The H_2 consensus problems of multi-agent systems were also investigated using distributed protocols in [14] by static state feedback and in [15] by dynamic output feedback. Moreover, through studying the H_2 norms of multi-agent networks from white noises to the performance variables, the *coherence* and *centrality* of networks were formulated and discussed in [8]–[13]. Meanwhile, robustness issues of networked multi-agent systems have also attracted much attention. In [14], [16]–[19], disturbance attenuation problems of linear consensus networks were studied from the H_∞ control perspective. In [20]–[22], robust synchronization and consensus problems of multi-agent systems were investigated, where the agent dynamics are subject to multiplicative or coprime factor uncertainties. In fact, not only the agent dynamics could be perturbed by uncertainties, the interactions among neighboring agents could also be subject to uncertainties. In the cases that the communication channels among agents are subject to multiplicative stochastic uncertainties, robust consensus problems of multi-agent systems were considered in [23]–[26], in the sense of mean square and almost sure stability. Moreover, in [27], [28] robust consensus over deterministic uncertain network graphs was also studied. While distributed H_2 control, distributed H_∞ control and robust distributed control problems, respectively, have been extensively studied, there are few works that deal with the problem of distributed robust optimal control [35], [41].

It is well-known that there is an intrinsic conflict between optimality and robustness in the standard feedback framework [1], [29]. Therefore, in the case of multi-objective design, e.g., the mixed H_2/H_∞ control [30] and the H_∞ Guassian control [31], a trade-off has to be made between the achievable optimal performance and robustness [29]. These trade-off design approaches suffer from a fundamental drawback of severe conservativeness, since a single controller is developed to address the conflicting requirements simultaneously. For example, the mixed H_2/H_∞ control is generally worse than the H_∞ control in terms of the robustness and worse than the LQG control in terms of the optimality [31]. Inspired by the structure in [29], a new design paradigm is proposed in [32], consisting of an LQG control designed for the nominal plant and an operator Q as a separate degree of freedom. The operator Q provides an extra control action to recover the robustness performance for the closed-loop system. This new paradigm is shown to be able to avoid trade-off and to reduce the conflict between the robustness and achievable suboptimality/optimality. Note that the design paradigms in [29], [32] are applicable only to single-agent systems. So far, novel non-trade-off design schemes for multi-agent systems have not witnessed significant progress, due to the severe difficulties caused by the requirement of distributed control and structural uncertainties and by constraints imposed by the network graphs.

Motivated by the above, in the present paper we consider the distributed robust optimal consensus problems for linear multi-agent systems, taking into account both the optimality and robustness at

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the same time. Specifically, the objective of this paper is to present a novel *non-trade-off* complementary design approach to the robust optimal consensus problems for linear multi-agent systems. This complementary approach consists of two steps. In the first step, we design a distributed control law for the nominal multi-agent system, without considering the disturbances or uncertainties, to ensure that consensus is achieved with a prescribed H_2 performance. In the second step, a separate control input, activated by some carefully chosen residual signals indicating the modeling mismatch, will be designed to ensure robustness in terms of the H_∞ requirement. Two cases are considered, namely, the case that *relative outputs* or *absolute outputs* of neighboring agents are available. A distinct feature of this complementary approach is that the design of H_2 consensus control in this first step is independent of the second step and the extra control action in the second step will complement the H_2 performance by providing a robustness guarantee with respect to disturbances or uncertainties.

Compared to the trade-off approach, e.g., the mixed H_2/H_∞ control design, the proposed complementary design has at least two main advantages. Firstly, since the extra control provides an additional degree of freedom for design, the complementary approach has two separate controls to deal with the H_2 performance and the robustness of consensus, respectively. Thereby, this approach does not need to make much trade-off, and can be expected to be much less conservative than the trade-off approach where one control tackles two conflicting performances. Secondly, the control action of the second step is proportional to the residual signal which quantifies the modeling mismatch level, thereby having some online "adaptivity" with respect to modeling mismatches. This complementary approach will yield the same achievable H_2 performance when modeling mismatches do not exist. By contrast, the trade-off approach always considers the *a priori* worst case (for both H_2 performance and H_∞ robustness) and still yields the same conservative performance even when disturbances or uncertainties do not exist.

The remainder of this paper is organized as follows. Some mathematical preliminaries including graph theory and results on H_2 and H_∞ performances are summarized in Section II. The H_2 and H_∞ consensus problem is formulated in Section III. A two-step complementary design approach is proposed for the H_2 and H_∞ consensus problem in Section IV. A simulation example that illustrates the proposed theoretical results are presented in Section V. Finally, conclusions are given in Section VI.

II. Mathematical Preliminaries

A. Graph Theory

The information flow among the agents can be conveniently modeled by a graph. An undirected graph is defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \cdots, N\}$ is the set of *nodes* (each node represents an agent) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of unordered pairs of nodes, called *edges*. An undirected graph is connected, if there exists a path between every two distinct nodes. For an undirected graph $\mathcal{G}$, its adjacency matrix, denoted by $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$, is defined such that $a_{ii} = 0$, $a_{ij} = a_{ji} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}] \in \mathbf{R}^{N\times N}$ associated with $\mathcal{G}$ is defined as $\mathcal{L}_{ii} = \sum_{j=1}^{N} a_{ij}$ and $\mathcal{L}_{ij} = -a_{ij}$, $i \neq j$.

Lemma 1 ([2])**.** For an undirected graph $\mathcal{G}$, zero is an eigenvalue of $\mathcal{L}$ with $\mathbf{1}$ as an eigenvector and all nonzero eigenvalues are positive. Moreover, zero is a simple eigenvalue of $\mathcal{L}$ if and only if $\mathcal{G}$ is connected.

B. Results on H_2 and H_∞ Performances

In this subsection, we summarize results on H_2 and H_∞ performances of linear systems.

Consider the linear system

$$\dot{x} = Ax + Bw, \quad y = Cx, \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $y \in \mathbf{R}^m$ is the measured output, and $w \in \mathbf{R}^q$ is the external disturbance.

Let $G(s) = C(sI - A)^{-1}B$ be the transfer function matrix of (1). The H_2 norm of G is defined to be $\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G^*(j\omega)G(j\omega))d\omega$. We then review the following well-known result on the H_2 performance [1].

Lemma 2. Let $\gamma_2 > 0$. The following statements are equivalent:

i) A is stable and $\|G\|_2 < \gamma_2$.
ii) There exists $X > 0$ such that

$$AX + XA^T + BB^T < 0, \ \mathrm{tr}(CXC^T) < \gamma_2^2.$$

iii) There exist $P > 0$ and $Q > 0$ such that

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & -I \end{bmatrix} < 0, \ \begin{bmatrix} P & C^T \\ C & Q \end{bmatrix} > 0, \ \mathrm{tr}(Q) < \gamma_2^2.$$

Next, we review the H_∞ performance of (1). If A is stable, the H_∞ norm of (1) is defined to be $\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma(G(j\omega))$, where $\sigma(G(j\omega))$ is the maximum singular value of $G(j\omega)$. The following lemma presents a well-known result on the H_∞ performance [1], [33].

Lemma 3. Let $\gamma_\infty > 0$. The following statements are equivalent:

i) A is stable and $\|G\|_\infty < \gamma_\infty$.
ii) There exists $X > 0$ such that

$$A^TX + XA + \frac{1}{\gamma_\infty^2}XBB^TX + C^TC < 0.$$

III. Formulation of H_2 and H_∞ Consensus Problem

Consider a multi-agent system, consisting of N identical uncertain linear agents subject to different noises and external disturbances, the dynamics of the i-th agent are described by

$$\begin{aligned} \dot{x}_i &= Ax_i + B_0w_{0i} + B_1w_i + B_2u_i, \\ y_i &= C_2x_i + D_0w_{0i} + D_1w_i, \quad i = 1, \cdots, N, \end{aligned} \tag{2}$$

where $x_i \in \mathbf{R}^n$ is the state, $u_i \in \mathbf{R}^p$ is the control input, and $y_i \in \mathbf{R}^{m_1}$ is the measurement output, respectively. In (2), $w_i \in \mathbf{R}^{q_1}$ denotes the external disturbance signal, representing the modeling uncertainty and/or unmodeled dynamics of the i-th agent, $w_{0i} \in \mathbf{R}^{q_2}$ is a white noise signal with $\mathbf{E}\{w_{0i}(t)\} = 0$ and $\mathbf{E}\{w_{0i}(t)w_{0i}(\tau)^T\} = \delta(t - \tau)I$, where $\mathbf{E}\{\cdot\}$ and $\delta(t)$ denote the expectation operator and the Dirac function. The matrices A, B_0, B_1, B_2, C_2, D_0 and D_1 are of suitable dimensions. The pair (A, B_2) is assumed to be stabilizable and the pair (C_2, A) is assumed to be detectable. The communication graph among the N agents is represented by an undirected graph $\mathcal{G}$.

The agents in (2) are said to achieve consensus if there exist control laws u_i such that, given $w_{0i} = 0$ and $w_i = 0$, $x_i - x_j \to 0$ as $t \to \infty$ for all $i, j = 1, \ldots, N$. In this paper, output variables z_i, $i = 1, \cdots, N$, as defined in (3) (see also [2], [14], [16]), are adopted to quantify the consensus performance,

$$z_i = \frac{1}{N}\sum_{j=1}^{N} C_1(x_i - x_j), \quad i = 1, \cdots, N, \tag{3}$$

where $z_i \in \mathbf{R}^{m_2}$, and $C_1 \in \mathbf{R}^{m_2\times n}$ is a given constant weighting matrix. Note that other candidate performance variables could also be applied, for example, those depending on the specific network topology $\mathcal{G}$ as in [15], [34].

Let $\mathbf{x} = [x_1^T, \cdots, x_N^T]^T$, $\mathbf{u} = [u_1^T, \cdots, u_N^T]^T$, $\mathbf{y} = [y_1^T, \cdots, y_N^T]^T$, $\mathbf{z} = [z_1^T, \cdots, z_N^T]^T$, $\mathbf{w}_0 = [w_{01}^T, \cdots, w_{0N}^T]^T$, and $\mathbf{w} = [w_1^T, \cdots, w_N^T]^T$. The agents in (2) can be written in compact form as

$$\begin{aligned}\dot{\mathbf{x}} &= (I \otimes A)\mathbf{x} + (I \otimes B_0)\mathbf{w}_0 + (I \otimes B_1)\mathbf{w} + (I \otimes B_2)\mathbf{u}, \\ \mathbf{y} &= (I \otimes C_2)\mathbf{x} + (I \otimes D_0)\mathbf{w}_0 + (I \otimes D_1)\mathbf{w}, \\ \mathbf{z} &= (\mathcal{M} \otimes C_1)\mathbf{x},\end{aligned} \tag{4}$$

where $\mathcal{M} \triangleq I - \frac{1}{N}\mathbf{1}\mathbf{1}^T$. Let $T_{\mathbf{wz}}(s)$ and $T_{\mathbf{w}_0\mathbf{z}}(s)$ denote the closed-loop transfer function matrices of (4) from $\mathbf{w}$ to $\mathbf{z}$ and from $\mathbf{w}_0$ to $\mathbf{z}$, respectively, under a distributed feedback control law $\mathbf{u}$. The following main problem to be addressed in this paper can then be formulated:

> **Main Problem:** For the multi-agent system in (2) and (3), given constants $\gamma_2 > 0$ and $\gamma_\infty > 0$, find a distributed feedback control law $\mathbf{u}$ such that $\|T_{\mathbf{w}_0\mathbf{z}}(s)\|_2 < \gamma_2$ and $\|T_{\mathbf{wz}}(s)\|_\infty < \gamma_\infty$ and the agents in (2) achieve consensus, that is, $x_i - x_j \to 0$ as $t \to \infty$ for all $i, j = 1, \ldots, N$ if both $w_{0i} = 0$ and $w_i = 0$.

We shall call this distributed robust optimal control formulation 'H_2 and H_∞ Consensus Problem', referring to $\|T_{\mathbf{w}_0\mathbf{z}}\|_2 < \gamma_2$ as H_2 consensus and $\|T_{\mathbf{wz}}\|_\infty < \gamma_\infty$ as H_∞ consensus, respectively.

One method to solve the H_2 and H_∞ consensus problem is the standard trade-off mixed H_2/H_∞ design [35], [36], i.e., using one single control law such that both performance criteria are satisfied. However, it is well understood that there is an intrinsic conflict between the H_2 performance and H_∞ robustness in the mixed H_2/H_∞ design [1], [29]. To the best of our knowledge so far, there has been no effective solution to the consensus problem for multi-agent systems that could guarantee non-compromised H_2 and H_∞ performance.

In the present paper, we will propose a novel *non-trade-off complementary design* for obtaining distributed control laws that address the said H_2 and H_∞ consensus problem. The proposed design contains two steps. In the first step, a distributed control law is proposed, which achieves H_2 consensus for the controlled multi-agent system. In the second step, an extra distributed control law is designed which achieves H_∞ consensus for the overall network. In particular, we will provide two design methods for obtaining such distributed control laws that solve the H_2 and H_∞ consensus problem, based on relative output feedback and absolute output feedback, respectively.

Note that the complementary design can be expected to be much less conservative than the trade-off approach, as it has two separate controls to deal with the H_2 performance and the H_∞ robustness of consensus, respectively.

IV. A Two-step Complementary Approach to the Distributed H_2 and H_∞ Control Problem

In this section, we will provide two design methods for obtaining distributed control laws that solve the H_2 and H_∞ consensus problem with the proposed non-trade-off complementary approach, based on *relative output feedback* and *absolute output feedback*, respectively.

A. Relative Output Feedback Case

In this subsection, we consider the case where only the *relative output* information of the neighboring agents is accessible to each agent. In this case, the structure of the proposed complementary design is depicted in Fig. 1.

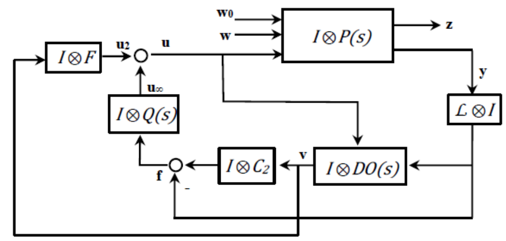

Fig. 1. The controller structure of the complementary design based on relative outputs. In this structure, $\mathbf{v} = [v_1^T, \cdots, v_N^T]^T$, $\mathbf{u}_2 = [u_{21}^T, \cdots, u_{2N}^T]^T$, $\mathbf{u}_\infty = [u_{\infty 1}^T, \cdots, u_{\infty N}^T]^T$, $P(s)$ denotes the agent dynamics in (2), $DO(s)$ represents the distributed observer for each agent, with v_i as the protocol state, $\mathbf{f} = [f_1^T, \cdots, f_N^T]^T$ is the residual signal, and $Q(s)$ is the extra controller to compensate for $\mathbf{w}$. The rest variables are defined as in Section III.

*1) **Step One:*** In the first step, we consider the H_2 consensus problem for the case with nominal agent dynamics, i.e., we consider only the noise w_{0i} (without considering external disturbances or uncertainties w_i, i.e. we let $w_i = 0$.). Relying on the relative output information of neighboring agents, we employ the following distributed observer-based protocol [2], [20]:

$$\begin{aligned}\dot{v}_i &= (A - GC_2)v_i + \sum_{j=1}^{N} a_{ij}\Big(B_2F(v_i - v_j) + G(y_i - y_j)\Big), \\ u_{2i} &= Fv_i, \quad i = 1, \cdots, N,\end{aligned} \tag{5}$$

where $v_i \in \mathbf{R}^n$ is the protocol state, u_{2i} is the input of the i-th agent in this step, the matrices F and G are feedback gains to be designed. The coefficient a_{ij} is the ij-th entry of the adjacency matrix of the communication graph among the agents. Since in the this step, we only take care of the influence of the noise w_{0i} on the performance outputs z_i, we consider only the *outer loop* in Fig. 1. The control input u_i of agent i in this case is equal to u_{2i}, with $u_{\infty i} = 0$. Define the error variables

$$e_i \triangleq v_i - \sum_{j=1}^{N} a_{ij}(x_i - x_j), \quad i = 1, \cdots, N. \tag{6}$$

We then have

$$\dot{e}_i = (A - GC_2)e_i + (GD_0 - B_0)\sum_{j=1}^{N} a_{ij}(w_{0i} - w_{0j}). \tag{7}$$

Therefore, if G is chosen such that $A - GC_2$ is Hurwitz, v_i in (5) is actually an estimate of $\sum_{j=1}^{N} a_{ij}(x_i - x_j)$ for agent i. That is, $DO(s)$ in Fig. 1 is in fact represented by the distributed observer in (5).

Denote $\mathbf{u}_2 = (u_{21}^T, u_{22}^T, \ldots, u_{2N}^T)^T$. The distributed observer-based protocol can be written in compact form as

$$\begin{aligned}\dot{\mathbf{v}} &= (I \otimes (A - GC_2))\mathbf{v} + (L \otimes B_2F)\mathbf{v} + (L \otimes G)\mathbf{y}, \\ \mathbf{u}_2 &= (I \otimes F)\mathbf{v}.\end{aligned} \tag{8}$$

Denote $\mathbf{v} = [v_1^T, \cdots, v_N^T]^T$ and $\boldsymbol{\xi} = \begin{bmatrix}\mathbf{x}^T & \mathbf{v}^T\end{bmatrix}^T$. By substituting (5) into (2), the closed-loop network dynamics can then be written in compact form as

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \mathscr{A}\boldsymbol{\xi} + \mathscr{B}_0\mathbf{w}_0, \\ \mathbf{z} &= \mathscr{C}_1\boldsymbol{\xi},\end{aligned} \tag{9}$$

where

$$\begin{aligned}\mathscr{A} &= \begin{bmatrix} I \otimes A & I \otimes B_2F \\ \mathcal{L} \otimes GC_2 & I \otimes (A - GC_2) + \mathcal{L} \otimes B_2F \end{bmatrix}, \\ \mathscr{B}_0 &= \begin{bmatrix} I \otimes B_0 \\ \mathcal{L} \otimes GD_0 \end{bmatrix}, \ \mathscr{C}_1 = \mathcal{M} \otimes \begin{bmatrix} C_1 & 0 \end{bmatrix}.\end{aligned} \tag{10}$$

The following theorem provides a necessary and sufficient condition for the H_2 suboptimal consensus problem.

Theorem 1. Assume that the graph $\mathcal{G}$ is a connected undirected graph with the Laplacian matrix $\mathcal{L}$. Let $\gamma_2 > 0$. Then, the distributed protocol (5) achieves H_2 consensus for the network (9) if and only if the following $N-1$ subsystems

$$\begin{aligned}\dot{\tilde{\xi}}_i &= \begin{bmatrix} A & \lambda_i B_2 F \\ GC_2 & A - GC_2 + \lambda_i B_2 F \end{bmatrix} \tilde{\xi}_i + \begin{bmatrix} B_0 \\ GD_0 \end{bmatrix} \tilde{w}_{0i}, \\ \tilde{z}_i &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \tilde{\xi}_i, \quad i = 2, \cdots, N,\end{aligned} \tag{11}$$

are internally stable and $\sum_{j=2}^N \|\tilde{T}_{\tilde{w}_{0i}\tilde{z}_i}\|_2^2 < \gamma_2^2$, where $\lambda_2, \ldots, \lambda_N$ are the nonzero eigenvalues of $\mathcal{L}$, and $\tilde{T}_{\tilde{w}_{0i}\tilde{z}_i}$ denotes the transfer function matrix of (11) from $\tilde{w}_{0i}$ to $\tilde{z}_i$.

Proof. The result can be proved by following similar lines in [2], [15]. The key steps are sketched here for clarity. First, we apply the unitary transformation $U \otimes I$ onto the dynamics of the consensus error $(\mathcal{M} \otimes I)\xi$, where U is a unitary matrix such that $U^T \mathcal{L} U = \mathrm{diag}(0, \lambda_2, \cdots, \lambda_N)$. Note that $U^T \mathcal{M} U = \mathrm{diag}(0, 1, \cdots, 1)$, see e.g., [2]. Next, by observing that the H_2 norm is invariant under unitary transformations, we can get that the H_2 suboptimal consensus problem is solved if and only if the following $N-1$ subsystems

$$\begin{aligned}\dot{\bar{\xi}}_i &= \begin{bmatrix} A & B_2 F \\ \lambda_i GC_2 & A - GC_2 + \lambda_i B_2 F \end{bmatrix} \bar{\xi}_i + \begin{bmatrix} B_0 \\ \lambda_i GD_0 \end{bmatrix} \tilde{w}_{0i}, \\ \tilde{z}_i &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \bar{\xi}_i, \quad i = 2, \cdots, N,\end{aligned} \tag{12}$$

are internally stable and $\sum_{j=2}^N \|\tilde{T}_{\tilde{w}_{0i}\tilde{z}_i}\|_2^2 < \gamma_2^2$. Now, by letting $\tilde{\xi}_i = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\lambda_i} I \end{bmatrix} \bar{\xi}_i$, evidently the subsystems in (12) are equivalent to those in (11). □

Before moving forwards, we need to make the following assumption and introduce a lemma.

Assumption 1. The system matrices in (2) satisfy that $D_0 B_0^T = 0$ and $D_0 D_0^T = I$.

Note that Assumption 1 is made to simplify the notation and can be easily removed, see e.g. [15], [37].

Lemma 4 ([15], [37])**.** Suppose Assumption 1 holds. Consider the i-th subsystem in (11) with $\lambda_i = 1$. Let $P > 0$ and $Q > 0$, respectively, satisfy the following inequalities:

$$(A + B_2 F)^T P + P(A + B_2 F) + C_1^T C_1 < 0, \tag{13}$$

$$AQ + QA^T - QC_2^T C_2 Q + B_0 B_0^T < 0. \tag{14}$$

If the inequality

$$\mathrm{tr}(C_2 QPQC_2^T) + \mathrm{tr}(C_1 QC_1^T) < \gamma^2$$

holds, then $\tilde{T}_{\tilde{w}_{0i}\tilde{z}_i}$, with $G = QC_2^T$ and $\lambda_i = 1$, satisfies that $\|\tilde{T}_{\tilde{w}_{0i}\tilde{z}_i}\|_2 < \gamma$.

The following theorem provides a design method for obtaining distributed protocols (5) that achieves H_2 suboptimal consensus.

Theorem 2. Assume that Assumption 1 holds and the graph $\mathcal{G}$ is a connected undirected graph with the Laplacian matrix $\mathcal{L}$. Let $\gamma_2 > 0$. Let $Q > 0$ be a solution to (14). Let $\bar{P} > 0, W > 0, \tau > 0$ be solutions to the following LMIs:

$$\begin{bmatrix} \bar{P}A^T + A\bar{P} - \tau B_2 B_2^T & \bar{P}C_1^T \\ C_1 \bar{P} & -I \end{bmatrix} < 0, \tag{15}$$

$$\begin{bmatrix} \bar{P} & C_2 Q \\ QC_2^T & W \end{bmatrix} > 0, \tag{16}$$

$$\mathrm{tr}(W) + \mathrm{tr}(C_1 QC_1^T) < \frac{\gamma_2^2}{N-1}. \tag{17}$$

Then, the protocol (5) with $G = QC_2^T$, $F = -cB_2^T \bar{P}^{-1}$ and $c \geq \frac{\tau}{2\lambda_2}$ achieves H_2 consensus.

Proof. In light of Theorem 1, the network (9) achieves H_2 consensus if the $N-1$ subsystems in (11) are internally stable and $\|\tilde{T}_{\tilde{w}_{0i}\tilde{z}_i}\|_2^2 < \frac{\gamma_2^2}{N-1}$. According to Lemma 4, the i-th subsystem in (11) is internally stable and $\|\tilde{T}_{\tilde{w}_{0i}\tilde{z}_i}\|_2^2 < \frac{\gamma_2^2}{N-1}$, if there exist $Q > 0$ satisfying (14) and $P > 0$ such that

$$(A + \lambda_i B_2 F)^T P + P(A + \lambda_i B_2 F) + C_1^T C_1 < 0, \tag{18}$$

and

$$\mathrm{tr}(C_2 QPQC_2^T) + \mathrm{tr}(C_1 QC_1^T) < \frac{\gamma_2^2}{N-1}. \tag{19}$$

Let $\bar{P} = P^{-1}$. Multiplying on both sides of (18) by $\bar{P}$ and in light of Schur Complement Lemma [38], we obtain that (18) and (19) hold if and only if

$$\begin{bmatrix} \bar{P}(A + \lambda_i B_2 F)^T + (A + \lambda_i B_2 F)\bar{P} & \bar{P}C_1^T \\ C_1 \bar{P} & -I \end{bmatrix} < 0, \tag{20}$$

and the inequalities (16) and (17) hold at the same time. Evidently, if we choose $F = -cB_2^T \bar{P}^{-1}$ and $c \geq \frac{\tau}{2\lambda_2}$, then (15) implies (20) and thereby (18). □

Remark 1. The separation property of observed-based controllers shown in [15], [37] is employed in this theorem. The observer gain matrix G and the feedback gain F are designed in a decoupled way. Moreover, the feasibility of (15) is equivalent to that of (20). Note that by letting $F\bar{P} = V$ and $\lambda_i = 1$, we know that (20) holds, then

$$\bar{P}A + A^T \bar{P} + B_2 V + V^T B_2^T + \bar{P}C_1^T C_1 \bar{P} < 0,$$

which, in light of Finsler's Lemma [2], [39], is equivalent to that there exist $\bar{P} > 0$ and $\tau > 0$ such that (15) holds. Therefore, (20) implies (15). The converse was shown in the proof.

*2) **Step Two:*** In the second step, we design an additional regulating control input $u_{\infty i}$ to deal with the external disturbances w_i and to guarantee the H_∞ robustness while not significantly compromising the H_2 performance. Note that, in the second step, we only consider the effect of w_i.

Under the H_2 consensus protocol (5) in the first step, the augmented agent dynamics are described by

$$\begin{aligned}\dot{x}_i &= Ax_i + B_2 u_i + B_1 w_i, \\ \dot{v}_i &= (A - GC_2)v_i + \sum_{j=1}^N a_{ij}\Big(B_2 F(v_i - v_j) \\ &\quad + GC_2(x_i - x_j) + GD_1(w_i - w_j)\Big), \\ u_i &= u_{2i} + u_{\infty i}, \\ u_{2i} &= Fv_i, \quad i = 1, \cdots, N,\end{aligned} \tag{21}$$

where the gain matrices F and G are designed in the first step.

The residual signal $f = [f_1^T, \cdots, f_N^T]^T$ in Fig. 1 is used in the second step to activate the *inner loop*, which is defined as the stacked error between the actual relative outputs of neighboring agents and their observed ones given by a distributed observer. Therefore, the residual signal f builds on the protocol (5) and is given by

$$\begin{aligned}f_i &\triangleq C_2 v_i - \sum_{j=1}^N a_{ij}(y_i - y_j) \\ &= C_2 e_i - D_1 \sum_{j=1}^N a_{ij}(w_i - w_j), \ i = 1, \cdots, N,\end{aligned} \tag{22}$$

where e_i is defined as in (6).

In this step, we consider a distributed protocol of the form

$$\begin{aligned} \dot{n}_i &= A_c n_i + B_c f_i, \\ u_{\infty i} &= C_c n_i + D_c f_i, \end{aligned} \tag{23}$$

where $n_i \in \mathbf{R}^n$ is the state of the protocol, and A_c, B_c, C_c, D_c are protocol matrices to be designed. In this case, $Q(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$ in Fig. 1. It should be mentioned that here we assume that $u_{\infty i}$ is a general dynamic controller with f_i as its input. Special forms such as observer-based ones in [20], [21] can be also considered.

Note that the error e_i in the current case satisfies

$$\dot{e}_i = (A - GC_2)e_i + (GD_1 - B_1)\sum_{j=1}^{N} a_{ij}(w_i - w_j). \tag{24}$$

Evidently, if the external disturbances w_i are absent, then it follows from (24) and (22) that e_i and subsequently f_i will asymptotically converge to zero. If the external disturbances w_i are not equal to zero, the inner loop will be activated as f_i are bounded non-zero signals and subsequently $u_{\infty i}$ will be implemented to recover robustness.

Denote $\mathbf{e} = [e_1^T, \cdots, e_N^T]^T$, $\mathbf{n} = [n_1^T, \cdots, n_N^T]^T$ and $\boldsymbol{\zeta} = \begin{bmatrix}\mathbf{x}^T & \mathbf{e}^T\end{bmatrix}^T$. Using (21) and (23), we obtain the closed-loop network dynamics in compact form as

$$\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{\zeta}} \\ \dot{\mathbf{n}} \end{bmatrix} &= \begin{bmatrix} \bar{\mathscr{A}} + \mathscr{B}_2\mathscr{D}_c\mathscr{C}_2 & \mathscr{B}_2\mathscr{C}_c \\ \mathscr{B}_c\mathscr{C}_2 & \mathscr{A}_c \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta} \\ \mathbf{n} \end{bmatrix} + \begin{bmatrix} \mathscr{B}_1 - \mathscr{B}_2\mathscr{D}_c\mathscr{D}_1 \\ -\mathscr{B}_c\mathscr{D}_1 \end{bmatrix} \mathbf{w}, \\ \mathbf{z} &= \begin{bmatrix} \mathscr{C}_1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta} \\ \mathbf{n} \end{bmatrix}, \end{aligned} \tag{25}$$

where $\mathscr{C}_1$ is defined in (10), and

$$\begin{aligned} \bar{\mathscr{A}} &= \begin{bmatrix} I \otimes A + \mathcal{L} \otimes B_2F & I \otimes B_2F \\ 0 & I \otimes (A - GC_2) \end{bmatrix}, \\ \mathscr{B}_2 &= \begin{bmatrix} I \otimes B_2 \\ 0 \end{bmatrix},\ \mathscr{B}_1 = \begin{bmatrix} I \otimes B_1 \\ \mathcal{L} \otimes (GD_1 - B_1) \end{bmatrix}, \\ \mathscr{C}_2 &= I \otimes \begin{bmatrix} 0 & C_2 \end{bmatrix},\ \mathscr{D}_1 = \mathcal{L} \otimes D_1,\ \mathscr{A}_c = I \otimes A_c, \\ \mathscr{B}_c &= I \otimes B_c,\ \mathscr{C}_c = I \otimes C_c,\ \mathscr{D}_c = I \otimes D_c. \end{aligned} \tag{26}$$

By following similar steps in deriving Theorem 1, it is not difficult to obtain the following result.

Theorem 3. Assume that graph $\mathcal{G}$ is a connected undirected graph with the Laplacian matrix $\mathcal{L}$. Let $\gamma_\infty > 0$. Then, the network (25) achieves H_∞ consensus if and only if the following $N-1$ subsystems:

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{\zeta}}_i \\ \dot{\tilde{n}}_i \end{bmatrix} &= (\mathfrak{A}_i + \mathfrak{B}_2\mathfrak{K}\mathfrak{C}_2) \begin{bmatrix} \tilde{\zeta}_i \\ \tilde{n}_i \end{bmatrix} + (\mathfrak{B}_{1i} + \mathfrak{B}_2\mathfrak{K}\mathfrak{D}_{1i})\tilde{w}_i, \\ \tilde{z}_i &= \mathfrak{C}_1 \begin{bmatrix} \tilde{\xi}_i \\ \tilde{n}_i \end{bmatrix},\ i = 2, \cdots, N, \end{aligned} \tag{27}$$

with

$$\begin{aligned} \mathfrak{A}_i &= \begin{bmatrix} A + \lambda_i B_2F & B_2F & 0 \\ 0 & A - GC_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathfrak{B}_2 = \begin{bmatrix} 0 & B_2 \\ 0 & 0 \\ I & 0 \end{bmatrix}, \\ \mathfrak{K} &= \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix},\ \mathfrak{C}_2 = \begin{bmatrix} 0 & 0 & I \\ 0 & C_2 & 0 \end{bmatrix},\ \mathfrak{C}_1^T = \begin{bmatrix} C_1^T \\ 0 \\ 0 \end{bmatrix}, \\ \mathfrak{B}_{1i} &= \begin{bmatrix} B_1 \\ \lambda_i(GD_1 - B_1) \\ 0 \end{bmatrix},\ \mathfrak{D}_{1i} = \begin{bmatrix} 0 \\ -\lambda_i D_1 \end{bmatrix}. \end{aligned}$$

are internally stable and the associated transfer function matrices $\tilde{T}_{\tilde{w}_i\tilde{z}_i}$ from $\tilde{w}_i$ to $\tilde{z}_i$ satisfy $\|\tilde{T}_{\tilde{w}_i\tilde{z}_i}\|_\infty < \gamma_\infty$, where $\lambda_2, \ldots, \lambda_N$ are the nonzero eigenvalues of the Laplacian matrix $\mathcal{L}$.

The following theorem provides a design method for obtaining the distributed control law (23).

Theorem 4. Assume that $\mathcal{G}$ is a connected undirected graph with the Laplacian matrix $\mathcal{L}$ and that B_2 is of full column rank. Let $\gamma_\infty > 0$. Then the network (25) achieves H_∞ consensus if there exist positive definite matrices S_{11} and S_{22}, and a matrix V_1 such that

$$S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}, \qquad V = \begin{bmatrix} V_1 \\ 0 \end{bmatrix} \tag{28}$$

satisfying the following LMI's

$$\begin{bmatrix} \Upsilon_{1i} & S\bar{\mathfrak{B}}_{1i} + V\mathfrak{D}_{1i} \\ (S\bar{\mathfrak{B}}_{1i} + V\mathfrak{D}_{1i})^T & -\gamma_\infty^2 I \end{bmatrix} < 0, \tag{29}$$

for $i = 2, N$, where

$$\begin{aligned} &\Upsilon_{1i} = \bar{\mathfrak{A}}_i^T S + S\bar{\mathfrak{A}}_i + \bar{\mathfrak{C}}_2^T V^T + V\bar{\mathfrak{C}}_2 + \bar{\mathfrak{C}}_1^T\bar{\mathfrak{C}}_1, \\ &\bar{\mathfrak{A}}_i = T\mathfrak{A}_iT^{-1},\ \bar{\mathfrak{B}}_{1i} = T\mathfrak{B}_{1i},\ \bar{\mathfrak{C}}_2 = \mathfrak{C}_2T^{-1},\ \bar{\mathfrak{C}}_1 = \mathfrak{C}_1T^{-1}, \end{aligned}$$

and T is a nonsingular matrix such that $T\mathfrak{B}_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}$. Then, the system matrix $\mathfrak{K}$ of (23) is given by

$$\mathfrak{K} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = S_{11}^{-1}V_1. \tag{30}$$

Proof. In virtue of Theorem 3 and Lemma 3, it follows that the $N-1$ subsystems in (27) are internally stable and $\|\tilde{T}_{\tilde{w}_i\tilde{z}_i}\|_\infty < \gamma_\infty$ if and only if there exist matrices $S_i > 0$ such that

$$\begin{aligned} &(\mathfrak{A}_i + \mathfrak{B}_2\mathfrak{K}\mathfrak{C}_2)^T S_i + S_i(\mathfrak{A}_i + \mathfrak{B}_2\mathfrak{K}\mathfrak{C}_2) \\ &+ \frac{1}{\gamma_\infty^2} S_i(\mathfrak{B}_{1i} + \mathfrak{B}_2\mathfrak{K}\mathfrak{D}_{1i})(\mathfrak{B}_{1i} + \mathfrak{B}_2\mathfrak{K}\mathfrak{D}_{1i})^T S_i \\ &+ \mathfrak{C}_1^T\mathfrak{C}_1 < 0, \quad i = 2, \cdots, N. \end{aligned} \tag{31}$$

Following the steps in [40, Theorem 2], by using a Schur complement, the above inequalities (31) are equivalent to

$$\begin{bmatrix} \Phi_{1i} & \Phi_{2i} \\ \Phi_{2i}^T & -\gamma_\infty^2 I \end{bmatrix} < 0, \quad i = 2, \cdots, N \tag{32}$$

with

$$\begin{aligned} &\Phi_{1i} = (\mathfrak{A}_i + \mathfrak{B}_2\mathfrak{K}\mathfrak{C}_2)^T S_i + S_i(\mathfrak{A}_i + \mathfrak{B}_2\mathfrak{K}\mathfrak{C}_2) + \mathfrak{C}_1^T\mathfrak{C}_1, \\ &\Phi_{2i} = S_i(\mathfrak{B}_{1i} + \mathfrak{B}_2\mathfrak{K}\mathfrak{D}_{1i}). \end{aligned}$$

Since the matrix B_2 is of full column rank, there exists a matrix T such that

$$T\mathfrak{B}_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

By pre-multiplying $\bar{T} = \begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix}$ and post-multiplying $\bar{T}^T$ on (32), it follows that (32) holds if and only if

$$\begin{bmatrix} \bar{\Phi}_{1i} & \bar{\Phi}_{2i} \\ \bar{\Phi}_{2i}^T & -\gamma_\infty^2 I \end{bmatrix} < 0, \quad i = 2, \ldots, N, \tag{33}$$

where

$$\begin{aligned} &\bar{\Phi}_{1i} = \bar{\mathfrak{A}}_i^T\bar{S}_i + \bar{S}_i\bar{\mathfrak{A}}_i + \bar{S}_i\bar{\mathfrak{B}}_2\mathfrak{K}\bar{\mathfrak{C}}_2 + (\bar{S}_i\bar{\mathfrak{B}}_2\mathfrak{K}\bar{\mathfrak{C}}_2)^T + \bar{\mathfrak{C}}_1^T\bar{\mathfrak{C}}_1, \\ &\bar{\Phi}_{2i} = \bar{S}_i\bar{\mathfrak{B}}_{1i} + \bar{S}_i\bar{\mathfrak{B}}_2\mathfrak{K}\mathfrak{D}_{1i},\ \bar{S}_i = T^{-T}S_iT^{-1}, \\ &\bar{\mathfrak{B}}_2 = T\mathfrak{B}_2 = \begin{bmatrix} I & 0 \end{bmatrix}^T. \end{aligned}$$

Now, let $S = \bar{S}_i$ and $\bar{S}_i\bar{\mathfrak{B}}_2\mathfrak{K} = V$. Recall that (28), then (33) holds if (29) holds for $i = 2, \ldots, N$. In this case, due to

$$\begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \mathfrak{K} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix},$$

the system matrix is given by (30).

Finally, note that the inequalities in (29) are linear matrix inequalities with respect to the unknown variables, we need to check only the two LMIs in (29) for $i = 2$ and N, and the other $N-3$ LMIs in (29) corresponding to $i = 3, \cdots, N-1$, also hold, with their variables chosen to be some convex combinations of those satisfying the two LMIs in (29) for $i = 2, N$. This completes the proof. $\square$

Remark 2. In the novel control structure in Fig. 1, the design of H_2 consensus control of the outer loop is independent of the control $Q(s)$ in (23) of the inner loop. The extra control $Q(s)$ relies on the residual signal $\mathbf{f}$, which is the stacked error between the actual relative outputs of neighboring agents $\sum_{j=1}^N a_{ij}(y_i - y_j)$ and their observed ones $C_2 v_i$ given by the distributed observer (5). The extra control action of the inner loop, activated by residual signal $\mathbf{f}$ in the presence of external disturbances or uncertainties w_i, will complement the H_2 performance by providing H_∞ robustness guarantee with respect to w_i. This is the reason why this approach is called a complementary design approach.

Remark 3. The two-step complementary approach proposed in this section, compared to the trade-off approach, has at least two main advantages:

i) The extra control $Q(s)$ in (23) provides an additional degree of freedom for design. Therefore, the current complementary approach has two separate control inputs to deal with the H_2 performance and the H_∞ robustness of consensus, respectively, thereby does not need to make much trade-off, and can be expected to be much less conservative,while the trade-off design has only one control to tackle two conflicting performances simultaneously. Although Theorems 2 and 4 in this section are conservative, however, it should be pointed out that the conservatism of these results is not caused by the complementary approach. On the contrary, it in fact highlights the difficulty of distributed control.

ii) The control action of the inner loop is proportional to the residual signal which quantifies the modeling mismatch level, thereby having some online "adaptivity" with respect to modeling errors. This complementary approach will yield the same achievable H_2 performance when modeling mismatches do not exist, because in this case the inner loop will be de-activated. By contrast, the trade-off approach always considers the *a priori* worst case and produce the same conservative performance even when disturbances or uncertainties disappear.

Remark 4. It should be mentioned that the order of the overall control law designed by the complementary approach is higher than the trade-off approach, since both distributed protocols (5) and (23) are required in the former approach while only one dynamic controller is needed in the latter. This is the price to provide more degree of design freedom and it is actually not a big issue considering the abundance of cheep storage and computing resources. Besides, the extra control action of the inner loop, providing robustness guarantee, will inject some white noises into the closed-loop network dynamics, and thereby will make certain compromise of the H_2 performance.

B. Absolute Output Feedback Case

In this subsection, we consider the case where *absolute output* information of each agent is available. In this case, we adopt local observers for the agents, instead of the distributed observers as in the previous subsection. The structure of the complementary design approach in this case is depicted in Fig. 2.

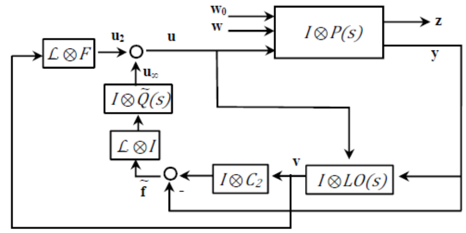

Fig. 2. The controller structure of the complementary approach based on absolute outputs, where $LO(s)$ represents the local observer for each agent, $\tilde{\mathbf{f}}$ is the residual signal, and $\tilde{Q}(s)$ is the additional controller based on $\tilde{\mathbf{f}}$, and the rest of variables are defined as in Fig. 1.

*1) **Step One:*** Based on the absolute output y_i, we propose for each agent the following Luenberger observer:

$$\dot{\breve{v}}_i = A\breve{v}_i + Bu_i + \breve{G}(y_i - C_2\breve{v}_i), \tag{34}$$

where $\breve{G}$ is the observer gain to be designed. In the first step, we consider only the outer loop, therefore, $u_i = u_{2i}$. For the H_2 consensus problem, we design the following protocol:

$$u_{2i} = \breve{F}\sum_{j=1}^N a_{ij}(\breve{v}_i - \breve{v}_j), \tag{35}$$

where $\breve{F}$ is the feedback gain to be designed. Denote $\breve{\mathbf{v}} = [\breve{v}_1^T, \cdots, \breve{v}_N^T]^T$ and $\breve{\boldsymbol{\xi}} = \begin{bmatrix}\mathbf{x}^T & \breve{\mathbf{v}}^T\end{bmatrix}^T$. The closed-loop network dynamics in this case can be written in compact form as

$$\begin{aligned}\dot{\breve{\boldsymbol{\xi}}} &= \begin{bmatrix} I\otimes A & \mathcal{L}\otimes B_2\breve{F} \\ I\otimes \breve{G}C_2 & I\otimes(A-\breve{G}C_2)+\mathcal{L}\otimes B_2\breve{F}\end{bmatrix}\breve{\boldsymbol{\xi}} + \begin{bmatrix} I\otimes B_0 \\ I\otimes\breve{G}D_0\end{bmatrix}\mathbf{w}_0,\\ \mathbf{z} &= \mathcal{M}\otimes\begin{bmatrix}C_1 & 0\end{bmatrix}\breve{\boldsymbol{\xi}}.\end{aligned} \tag{36}$$

It is easy to verify that the H_2 consensus problem of (36) can be reduced to the same condition as in Theorem 1. Then, Theorem 2 can also be used to design the protocol (35).

*2) **Step Two:*** In the second step, we define the residual signals $\tilde{f}_i$ as follows:

$$\tilde{f}_i \triangleq C_2\breve{v}_i - y_i,\ \ i = 1, \cdots, N, \tag{37}$$

which is actually the local estimated output error. Therefore, $\tilde{f}_i$ quantifies the difference between the actual plant and the ideal plant, and we can see that $\tilde{f}_i \to 0$ as $t \to \infty$ if there exist no disturbances or uncertainties.

Since $\sum_{j=1}^N a_{ij}(x_i - x_j)$ is the consensus error and $\breve{v}_i$ is the estimate of x_i for agent i, we can see that $\sum_{j=1}^N a_{ij}(\tilde{f}_i - \tilde{f}_j)$ denotes the estimated output error of the consensus error. Therefore, in this case we design the control input $u_{\infty i}$ for the inner loop based on $\sum_{j=1}^N a_{ij}(\tilde{f}_i - \tilde{f}_j)$, instead of $\tilde{f}_i$ as in the previous subsection. Specifically, we consider a distributed protocol of the form

$$\begin{aligned}\dot{n}_i &= \breve{A}_c n_i + \breve{B}_c\sum_{j=1}^N a_{ij}(\tilde{f}_i - \tilde{f}_j),\\ u_{\infty i} &= \breve{C}_c n_i + \breve{D}_c\sum_{j=1}^N a_{ij}(\tilde{f}_i - \tilde{f}_j),\end{aligned} \tag{38}$$

where $n_i \in \mathbf{R}^n$ is the state of the protocol, and $\breve{A}_c$, $\breve{B}_c$, $\breve{C}_c$, $\breve{D}_c$ are protocol matrices to be designed. In this case, $Q(s) = \left[\begin{array}{c|c}\breve{A}_c & \breve{B}_c \\ \hline \breve{C}_c & \breve{D}_c\end{array}\right]$ in Fig. 2.

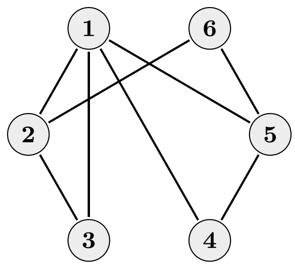

Fig. 3. The communication graph among the six agents.

Let $\breve{e}_i = \breve{v}_i - x_i$. Denote $\breve{\mathbf{e}} = [\breve{e}_1^T, \cdots, \breve{e}_N^T]^T$, $\breve{\boldsymbol{\zeta}} = \begin{bmatrix}\mathbf{x}^T & \breve{\mathbf{e}}^T\end{bmatrix}^T$ and $\breve{\mathbf{n}} = [\breve{n}_1^T, \cdots, \breve{n}_N^T]^T$. Then, it follows from (2), (35) and (38) that the closed-loop network dynamics are given by

$$\begin{aligned}
\begin{bmatrix}\dot{\breve{\boldsymbol{\zeta}}}\\ \dot{\breve{\mathbf{n}}}\end{bmatrix} &= \begin{bmatrix}\breve{\mathscr{A}} + \mathscr{B}_2\breve{\mathscr{D}}_c\mathscr{C}_2 & \mathscr{B}_2\breve{\mathscr{C}}_c\\ \breve{\mathscr{B}}_c\mathscr{C}_2 & \breve{\mathscr{A}}_c\end{bmatrix}\begin{bmatrix}\breve{\boldsymbol{\zeta}}\\ \breve{\mathbf{n}}\end{bmatrix} + \begin{bmatrix}\breve{\mathscr{B}}_1 - \mathscr{B}_2\breve{\mathscr{D}}_c\breve{\mathscr{D}}_1\\ -\breve{\mathscr{B}}_c\breve{\mathscr{D}}_1\end{bmatrix}\mathbf{w},\\
\mathbf{z} &= \begin{bmatrix}\mathscr{C}_1 & 0\end{bmatrix}\begin{bmatrix}\breve{\boldsymbol{\zeta}}\\ \breve{\mathbf{n}}\end{bmatrix},
\end{aligned} \tag{39}$$

where $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{B}_2$ are defined in (10), and

$$\begin{aligned}
\breve{\mathscr{A}} &= \begin{bmatrix}I\otimes A + \mathcal{L}\otimes B_2\breve{F} & \mathcal{L}\otimes B_2\breve{F}\\ 0 & I\otimes(A - \breve{G}C_2)\end{bmatrix},\\
\breve{\mathscr{B}}_1 &= \begin{bmatrix}I\otimes B_1\\ \mathcal{L}\otimes(\breve{G}D_1 - B_1)\end{bmatrix},\ \breve{\mathscr{D}}_1 = I\otimes D_1,\ \breve{\mathscr{A}}_c = I\otimes\breve{A}_c,\\
\breve{\mathscr{B}}_c &= I\otimes\breve{B}_c,\ \breve{\mathscr{C}}_c = I\otimes\breve{C}_c,\ \breve{\mathscr{D}}_c = I\otimes\breve{D}_c.
\end{aligned}$$

Similarly as in Theorems 3 and 4, the control law (38) can be constructed to achieve H_∞ consensus with prescribed index. The details are omitted here for conciseness.

V. Simulation Example

In this section we will use a simulation example to illustrate the proposed complementary H_2 and H_∞ design by dynamic output feedback, as in Theorems 2 and 4 in Subsection IV-A, for obtaining distributed protocols.

Consider a multi-agent system that consists of six agents. The dynamics of each agent is given by (2), where

$$\begin{aligned}
&A = \begin{bmatrix}-2 & 2\\ -1 & 1\end{bmatrix},\ B_0 = \begin{bmatrix}0 & 0\\ 0.5 & 0\end{bmatrix},\ B_1 = \begin{bmatrix}1\\ 0.6\end{bmatrix},\ B_2 = \begin{bmatrix}1\\ -2\end{bmatrix},\\
&C_1 = \begin{bmatrix}1 & 0\end{bmatrix},\ C_2 = \begin{bmatrix}1 & 0.8\end{bmatrix},\ D_1 = 0.1,\ D_0 = \begin{bmatrix}0 & 1\end{bmatrix}.
\end{aligned}$$

The communication graph among the agents is shown in Fig. 3, which is a connected undirected graph with the Laplacian matrix $\mathcal{L}$. The smallest nonzero and the largest eigenvalues of $\mathcal{L}$ are computed to be $\lambda_2 = 1.3820$ and $\lambda_N = 5.3028$.

In the first step of the complementary design, we obtain via Theorem 2 a distributed control law that takes care of the H_2 performance. We choose $\gamma_2 = 2$ and compute the control gains $F = \begin{bmatrix}0.0898 & 2.0360\end{bmatrix}$ and $G = \begin{bmatrix}0.5584 & 0.7792\end{bmatrix}^T$. Next in the second step, we obtain via Theorem 4 a distributed control law that deals with the H_∞ consensus. We compute the protocol gains to be $A_c = \begin{bmatrix}-0.5468 & 0\\ 0 & -0.5468\end{bmatrix}$, $B_c = \begin{bmatrix}0\\ 0\end{bmatrix}$, $C_c = \begin{bmatrix}0 & 0\end{bmatrix}$ and $D_c = -1.1570$. The associated computed upper bound for the H_∞ robustness is $\gamma_{\infty,\min} = 1.7982$. In Fig. 4, we have plotted the state trajectories of the agents. It can be seen that indeed the proposed distributed protocols together achieve consensus.

As a comparison, we will next compare the output performance of the proposed complementary approach with that of the mixed H_2 and H_∞ control design in [41], in presence of the external noise and disturbance. In particular, we choose the external noise

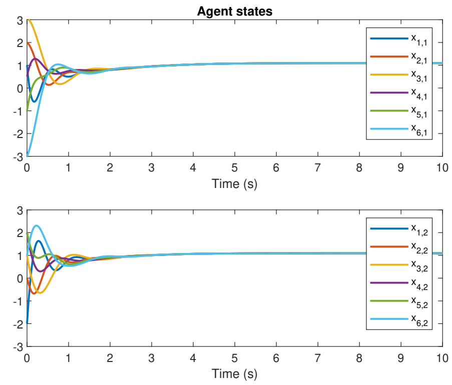

Fig. 4. Complementary H_2 and H_∞ design by output feedback: plots of the agent state vector $\mathbf{x}^1 = (x_{1,1}, \ldots, x_{6,1})^T$ (upper plot) and $\mathbf{x}^2 = (x_{1,2}, \ldots, x_{6,2})^T$ (lower plot).

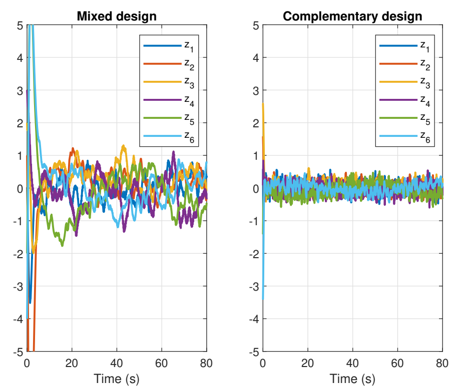

Fig. 5. Plots of the agent output vector $\mathbf{z} = (z_1, \ldots, z_6)^T$: mixed H_2 and H_∞ design (left-hand side plot) and complementary H_2 and H_∞ design (right-hand side plot)

w_{0i} to be a uniformly distributed signal, generated by Matlab command `30*rand()`. We choose the disturbance w_i to be $w_1 = 3\sin(110t) + 0.2,\ w_2 = \sin(30t) + 0.4, w_3 = -2\sin(110t) + 0.6, w_4 = 3\sin(30t) - 0.6, w_5 = -2\sin(0.1t) - 0.4, w_6 = \sin(0.1t) - 0.2$. The plots of the trajectories of the performance outputs z_i are given in Fig. 5. It can be seen that, in the presence of noises and disturbances, indeed the proposed complementary approach guarantees a better performance than the mixed design.

VI. Conclusions

In this paper, we have presented a novel complementary approach to the distributed H_2 and H_∞ consensus problem of multi-agent systems. Through introducing an extra control input that depends on some carefully chosen residual signals, which indicates the modeling mismatch, the complementary approach provides an additional degree of freedom for control design and complements the H_2 performance of consensus by providing the H_∞ robustness guarantee. This complementary approach does not involves much trade-off, and can be expected to be much less conservative than the trade-off design. Extending the proposed complementary approach to the case of directed graphs or switching topology is an interesting direction for future research [42], [43].

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