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The antiferromagnetic *XY* model on the triangular lattice: chirality transitions at the surface scaling

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Abstract

We study the discrete-to-continuum variational limit of the antiferromagnetic *XY* model on the two-dimensional triangular lattice. The system is fully frustrated and displays two families of ground states distinguished by the chirality of the spin field. We compute the Γ-limit of the energy in a regime which detects chirality transitions on one-dimensional interfaces between the two admissible chirality phases.

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1 Introduction

Ordering problems in magnetism have been extensively studied by both the physics and the mathematics communities. Researchers have been attracted by the rich phase diagrams and critical behaviors of magnetic models which are often the result of difficult-to-detect optimization effects taking place at several energy and length scales. The reason for such a complex behavior can be traced back to the presence of many competing mechanisms which

give rise to frustration. *Frustration* in the context of spin systems (here, as it is customary in the statistical mechanics literature, we will often refer to magnets as to spins) refers to the situation where spins cannot find an orientation that simultaneously minimizes all the pairwise exchange interactions. Such interactions are said to be ferromagnetic or antiferromagnetic if they favour alignment or antialignment, respectively. Often frustration occurs in those systems where spins are subject to conflicting short range ferromagnetic and long range antiferromagnetic interactions, as when modulated phases appear (see, e.g. the expository paper [37]). For antiferromagnetic lattice systems, that is systems of lattice spins subject to only antiferromagnetic interactions, frustration can also stem from the relative spatial arrangement of spins induced by the geometry of the lattice. In this case frustration is often referred to as geometric frustration. As a consequence of geometric frustration magnetic compounds show complex geometric patterns that induce often unexpected effects whose understanding is one of the primary subjects in statistical and condensed matter physics as it can help to better explain the nature of phase transitions in magnetic materials [30,32, 33]. From a mathematical perspective, several interesting questions can be addressed. In this paper we are interested in the variational coarse graining of the system, in the line of what is by now addressed to as the "discrete-to-continuum variational analysis of discrete systems". Within this line of investigation the analysis of spin systems turns out to be a difficult nonlinear optimization problem requiring the combination of several methods ranging from simple discrete optimization procedures to sophisticated techniques in geometric measure theory and the calculus of variations. While models where frustration is induced by the competition of ferromagnetic/antiferromagnetic interactions have been already studied from a variational perspective (see, e.g. [1,12,20,24,28,36]), what we present here is the first discrete-to-continuum result for a geometrically frustrated system.

We carry out the discrete-to-continuum variational analysis (at zero temperature) of a geometrically frustrated spin model in a specific energetic regime and we characterize the effective behavior of its low-energy states, that is states that can deviate from the global minimizers (ground states) by a certain small amount of energy. More precisely we consider a 2-dimensional nearest-neighbors antiferromagnetic planar spin model on the triangular lattice, *cf.* [30, Chapter 1]. Despite being considered one of the most elementary geometrically frustrated spin models, its variational analysis turns out to be quite a delicate task. More in detail, we let $\varepsilon > 0$ be a small parameter and we consider the triangular lattice $\mathcal{L}_\varepsilon$ with spacing ε (see Subsection 2.2 for the precise definition). To every spin field $u \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ we associate the energy

$$\sum_{\substack{\varepsilon\sigma, \varepsilon\sigma' \in \mathcal{L}_\varepsilon \\ |\sigma - \sigma'| = 1}} \langle u(\varepsilon\sigma), u(\varepsilon\sigma') \rangle, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. (Below, the energy will be restricted to bounded regions in the plane.) This model is antiferromagnetic since the interaction energy between two neighboring spins is minimized by two opposite vectors. Such an order in the magnetic alignment, also known as antiferromagnetic order, is frustrated by the geometry of the triangular lattice, which inhibits a configuration where each pair of neighboring spins are opposite or, equivalently, where each interaction is minimized. This suggests that the antiferromagnetic *XY* model depends substantially on the geometry of the lattice, which affects the structure of the ground states, the choice of the relevant variables and of the energy scalings. Notice, for example, that on a square lattice the system would not be frustrated, as opposite vectors distributed in a checkerboard structure minimize each interaction. In fact, on the square lattice a straightforward change of variable allows one to recast the antiferromagnetic *XY* model

into the ferromagnetic *XY* model [2, Remark 3], which is driven by an energy with neighboring interactions $-\langle u(\varepsilon\sigma), u(\varepsilon\sigma') \rangle$. The latter model has been thoroughly investigated in the last decade both on the square lattice [2,3,5,21,22] and on the triangular lattice [18,29]. Independently of the geometry of the lattice, it has been proved that spin fields that deviate from the ground states by an amount of energy which diverges logarithmically as ε vanishes form of topological charges (vortex-like singularities of the spin field as those arising in the Ginzburg-Landau model [10,35]), when subject to boundary conditions or external magnetic fields. In [9] we show how such a phenomenon also occurs in the antiferromagnetic *XY* model on the triangular lattice.

We now come back to our model (1.1). In order to identify the relevant variable of the system, we first need to characterize the ground states of the antiferromagnetic *XY* system in (1.1). To this end it is convenient to rearrange the indices of the sum in (1.1) and to recast the energy as a sum over all triangular plaquettes T with vertices $\varepsilon i, \varepsilon j, \varepsilon k \in \mathcal{L}_\varepsilon$

$$\sum_T \left(\langle u(\varepsilon i), u(\varepsilon j) \rangle + \langle u(\varepsilon j), u(\varepsilon k) \rangle + \langle u(\varepsilon k), u(\varepsilon i) \rangle \right)$$
$$= \frac{1}{2} \sum_T \left(|u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k)|^2 - 3 \right). \tag{1.2}$$

In each triangle T the energy is minimized (and is equal to $-\frac{3}{2}$) if and only if $u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k) = 0$, namely, when the vectors of a triple $(u(\varepsilon i), u(\varepsilon j), u(\varepsilon k))$ point at the vertices of an equilateral triangle. By the $\mathbb{S}^1$-symmetry, every rotation of a minimizing triple $(u(\varepsilon i), u(\varepsilon j), u(\varepsilon k))$ is minimizing, too. The ground states in this model feature an additional symmetry, usually referred to as $\mathbb{Z}_2$-symmetry: triple obtained by from a minimizing triple via a permutation of negative sign as $(u(\varepsilon i), u(\varepsilon k), u(\varepsilon j))$ is also minimizing. This determines two families of ground states, *i.e.*, spin fields for which the energy is minimized in each plaquette, see Fig. 1. These two families can be distinguished through the *chirality*, a scalar which quantifies the handedness of a certain spin structure. To define the chirality of a spin field u in a triangle T, we need a consistent ordering of its vertices $\varepsilon i, \varepsilon j, \varepsilon k$. We assume that $\varepsilon i \in \mathcal{L}_\varepsilon^1, \varepsilon j \in \mathcal{L}_\varepsilon^2, \varepsilon k \in \mathcal{L}_\varepsilon^3$, where $\mathcal{L}_\varepsilon^1, \mathcal{L}_\varepsilon^2, \mathcal{L}_\varepsilon^3$ are the sublattices as in Fig. 1, and we set (see (2.1) for the precise definition)

$$\chi(u, T) = \frac{2}{3\sqrt{3}} \left(u(\varepsilon i) \times u(\varepsilon j) + u(\varepsilon j) \times u(\varepsilon k) + u(\varepsilon k) \times u(\varepsilon i) \right) \in [-1, 1],$$

where the symbol $\times$ stands for the cross product. We denote by $\chi(u) \in L^\infty(\mathbb{R}^2)$ the function equal to $\chi(u, T)$ on the interior of each plaquette T. The ground states are exactly those configurations u that satisfy either $\chi(u) \equiv 1$ or $\chi(u) \equiv -1$, *cf.* Remark 2.2.

In this paper we analyze the energy regime at which the two families of ground states coexist and at the same time the energy of the system concentrates at the interface between the two chiral phases $\{\chi = 1\}$ and $\{\chi = -1\}$. We fix $\Omega \subset \mathbb{R}^2$ open, bounded, and with Lipschitz boundary and we consider the energy (1.2) restricted to Ω, *i.e.*, computed only on plaquettes of $\mathcal{L}_\varepsilon$ contained in Ω. We refer the energy to its minimum by removing the energy of the ground states ($-\frac{3}{2}$ for each plaquette) and we divide it by the number of lattice points in Ω (of order $1/\varepsilon^2$). We obtain (up to a multiplicative constant) the energy per particle given by

$$E_\varepsilon(u) = \sum_{T \subset \Omega} \varepsilon^2 |u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k)|^2.$$
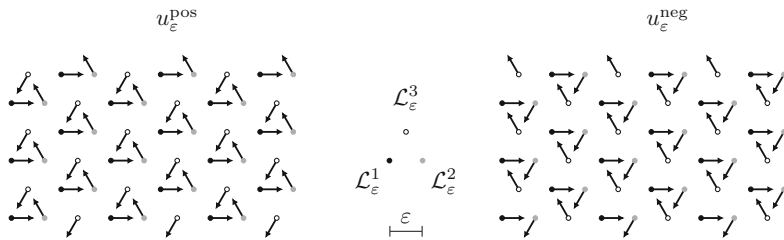
Fig. 1 A ground state $u_\varepsilon^{\text{pos}}$ with positive chirality and a ground state $u_\varepsilon^{\text{neg}}$ with negative chirality. Any other ground state of the system is obtained by composing one of these two configurations with a constant rotation. In the center: three points of the sublattices $\mathcal{L}_\varepsilon^1$, $\mathcal{L}_\varepsilon^2$, and $\mathcal{L}_\varepsilon^3$ in black, gray, and white, respectively

We are interested to the asymptotic behavior of the energy above as $\varepsilon \to 0$ on sequences of spin fields $u_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ that can deviate from ground states yet satisfying a bound $E_\varepsilon(u_\varepsilon) \leq C\varepsilon$. To this end we define the energy $F_\varepsilon(u) := \frac{1}{\varepsilon} E_\varepsilon(u)$ and study sequences of spin fields with equibounded F_ε energy. Due to the $\mathbb{S}^1$-symmetry, the energy at this regime cannot distinguish ground states with the same chirality, so that the relevant order parameter of the model is, in fact, not the spin field but its chirality: in Proposition 3.1 we prove that a sequence (u_ε) satisfying $F_\varepsilon(u_\varepsilon) \leq C$ admits a subsequence (not relabeled) such that $\chi(u_\varepsilon) \to \chi$ strongly in $L^1(\Omega)$ for some $\chi \in BV(\Omega; \{-1, 1\})$, *i.e.*, the admissible chiralities in the continuum limit are -1 and 1 and the chirality phases $\{\chi = -1\}$ and $\{\chi = 1\}$ have finite perimeter in Ω. This suggests that the model shares similarities with systems having finitely many phases, such as Ising models [1,17,34] or Potts models [23]. However, a crucial difference consists in the fact that in our case the variable that shows a phase transition is not the spin variable itself, but the chirality, which depends on the spin field in a nonlinear way. This is a source of difficulties that will be explained below.

To describe the asymptotic behavior of the system it is convenient to introduce the functionals depending only on functions $\chi \in L^1(\Omega)$ defined by $\mathscr{F}_\varepsilon(\chi) := \inf\{F_\varepsilon(u) \colon u \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1 \text{ such that } \chi = \chi(u, T) \text{ on every } T \subset \Omega\}$ (equal to $+\infty$ if χ is not the chirality of a spin field). The main result in this paper is Theorem 2.5, where we prove that the Γ-limit of $\mathscr{F}_\varepsilon$ with respect to the L^1-convergence is an anisotropic surface energy given by

$$\mathscr{F}(\chi) = \int_{J_\chi} \varphi(\nu_\chi) \, \mathrm{d}\mathcal{H}^1 \quad \text{for } \chi \in BV(\Omega; \{-1, 1\}),$$

extended to $+\infty$ otherwise in $L^1(\Omega)$, where J_χ is the interface between $\{\chi = -1\}$ and $\{\chi = 1\}$ and ν_χ is the normal to J_χ. The density φ is given by the following asymptotic formula

$$\varphi(\nu) = \lim_{\varepsilon \to 0} \min \left\{ F_\varepsilon(u, Q^\nu) \colon u = u_\varepsilon^{\text{pos}} \text{ on } \partial_\varepsilon^+ Q^\nu, \ u = u_\varepsilon^{\text{neg}} \text{ on } \partial_\varepsilon^- Q^\nu \right\}, \tag{1.3}$$

where Q^ν is the square with one side orthogonal to ν, $u_\varepsilon^{\text{pos}}$ and $u_\varepsilon^{\text{neg}}$ are the ground states depicted in Fig. 1, and $\partial_\varepsilon^\pm Q^\nu$ are a discrete version of the top/bottom parts of ∂Q^ν. Asymptotic formulas like (1.3) are common in discrete-to-continuum variational analyses and are often used to represent Γ-limits of discrete energies [6,8,12,14,15,31]. However, proving an asymptotic lower bound with the density (1.3) for this model requires additional care and is the technically most demanding contribution of this paper. We conclude this introduction by describing the main difficulties that arise in the proof.

Via a classical blow-up argument (see Proposition 4.1) we obtain an asymptotic lower bound with the surface density

$$\psi(v) = \inf \left\{ \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, Q^v) \colon \chi(u_\varepsilon) \to \chi_v \text{ in } L^1(Q^v) \right\}, \tag{1.4}$$

where χ_v is the pure-jump function which takes the values $\chi_v(x) = \pm 1$ for $\pm \langle x, v \rangle > 0$. Hence, the proof of the asymptotic lower bound boils down to the proof of the inequality $\psi(v) \geq \varphi(v)$. To obtain the latter inequality, we need to modify sequences (u_ε) with $\chi(u_\varepsilon) \to \chi_v$ in $L^1(Q^v)$ without increasing their energy in such a way that they attain the boundary conditions required in (1.3). A common approach to deal with this modification consists in selecting (via a well-known slicing/averaging argument due to De Giorgi) a low-energy frame contained in Q^v and close to ∂Q^v where the sequence can be modified using a cut-off function that interpolates to the boundary values. In our problem, instead, a cut-off modification of $\chi(u_\varepsilon)$ may generate a sequence of functions that are not chiralities of spin fields (and thus have infinite energy $\mathscr{F}_\varepsilon$). Consequently, we have to operate directly on the sequence (u_ε), on whose convergence we have no information due to the invariance of the system under rotation of the spin field (the $\mathbb{S}^1$-symmetry discussed above). We turn however the $\mathbb{S}^1$-symmetry to our advantage to define the needed modification. Inside a one-dimensional slice of $\mathcal{L}_\varepsilon$, a spin field close to a ground state in one triangle can be slowly rotated to reach any other ground state with the same chirality by paying an amount of energy proportional to the energy in the starting triangle (see Lemma 4.5). This one-dimensional construction can then be reproduced in the whole Q^v starting from triangles in a low-energy frame close to ∂Q^v in such a way that the modified spin field attains the fixed ground states $u_\varepsilon^{\text{pos}}$ and $u_\varepsilon^{\text{neg}}$ at the (discrete) boundary. However, for this procedure to be successful, the usual slicing/averaging method to find a low-energy frame close to ∂Q^v is not enough. We need to improve it and to find a frame with a better (smaller) energy bound. To this end, we proceed as follows. In Lemma 4.3 we show that $\psi(v)$ can be equivalently defined using in place of Q^v any rectangle coinciding with Q^v along the interface, but with arbitrarily small height (similar results appeared in different contexts, e.g. [16,17,19,25–27,31]). Hence the energy of any sequence (u_ε) admissible for (1.4) concentrates arbitrarily close to the jump set of χ_v, *i.e.*, the interface $\{\langle x, v \rangle = 0\}$. (This is, in general, not true for discrete systems, as observed in [11].) With this result at hand, in Lemma 4.4 we can apply the averaging method with the advantage of knowing that in most of the space the total energy is going to vanish, thus finally deducing the existence of a frame close to ∂Q^v with the wished (small enough) energy bound. Even at this point, to reproduce the one-dimensional interpolation along this frame requires additional care. In fact, to conclude the argument one still needs to prove that the winding number of the spin field in the low-energy frame can be properly controlled (Step 3 of Proposition 4.2).

2 Setting of the problem and statement of the main result

2.1 General notation

Throughout this paper $\Omega \subset \mathbb{R}^2$ is an open, bounded set with Lipschitz boundary. For every $A \subset \mathbb{R}^2$ measurable we denote by $|A|$ its 2-dimensional Lebesgue measure. With $\mathcal{H}^1$ we indicate the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. Given two points $x, y \in \mathbb{R}^2$ we use the notation $[x; y] := \{\lambda x + (1 - \lambda)y \colon \lambda \in [0, 1]\}$ for the segment joining x and y. The set $\mathbb{S}^1 := \{v \in \mathbb{R}^2 \colon |v| = 1\}$ is the set of all 2-dimensional unit vectors. For every such

vector $v = (v_1, v_2) \in \mathbb{S}^1$ we denote by $v^{\perp} := (-v_2, v_1) \in \mathbb{S}^1$ the unit vector orthogonal to v obtained by rotating v counterclockwise by $\pi/2$. Given $v, w \in \mathbb{S}^1$ we denote by $\langle v, w \rangle$ their scalar product and by $v \times w = -\langle v, w^{\perp} \rangle$ their cross product. We denote by ι the imaginary unit in the complex plane. It will be often convenient to write vectors in $\mathbb{S}^1$ as $\exp(\iota\theta)$, $\theta \in \mathbb{R}$. We denote by $R_{\ell,h}^{v}$ the rectangle of length $\ell > 0$ and height $h > 0$ with two sides orthogonal to $v \in \mathbb{S}^1$ given by

$$R_{\ell,h}^{v} := \{x \in \mathbb{R}^2 \colon |\langle x, v^{\perp} \rangle| < \ell/2, \ |\langle x, v \rangle| < h/2\},$$

extending the definition to the case $\ell = +\infty$ by setting $R_{\infty,h}^{v} := \{x \in \mathbb{R}^2 \colon |\langle x, v \rangle| < h/2\}$. Given $\rho > 0$ we define the cube centered at the origin with side length ρ and one face orthogonal to v by $Q_{\rho}^{v} := R_{\rho,\rho}^{v}$. For $\rho = 1$ we simply write Q^{v} instead of Q_1^{v}. By $L^{v} := \{x \in \mathbb{R}^2 \colon \langle x, v \rangle = 0\}$ we denote the line orthogonal to v passing through the origin, while $H_+^{v} := \{x \in \mathbb{R}^2 \colon \langle x, v \rangle \geq 0\}$ and $H_-^{v} := \mathbb{R}^2 \setminus H_+^{v}$ stand for the two half spaces separated by L^{v}. Given $x_0 \in \mathbb{R}^2$ we set $Q_{\rho}^{v}(x_0) := x_0 + Q_{\rho}^{v}$, $R_{\ell,h}^{v}(x_0) := x_0 + R_{\ell,h}^{v}$, $L^{v}(x_0) := x_0 + L^{v}$ and $H_{\pm}^{v}(x_0) := x_0 + H_{\pm}^{v}$.

2.2 Triangular lattices and discrete energies

In this paragraph we define the discrete energy functionals we consider in this paper. To this end we first define the triangular lattice $\mathcal{L}$. It is given by

$$\mathcal{L} := \{z_1 \hat{e}_1 + z_2 \hat{e}_2 \colon z_1, z_2 \in \mathbb{Z}\},$$

with $\hat{e}_1 = (1, 0)$, and $\hat{e}_2 = \frac{1}{2}(1, \sqrt{3})$. For later use, we find it convenient here to introduce $\hat{e}_3 := \frac{1}{2}(-1, \sqrt{3})$ as a further unit vector connecting points of $\mathcal{L}$ and to define three pairwise disjoint sublattices of $\mathcal{L}$, denoted by $\mathcal{L}^1$, $\mathcal{L}^2$, and $\mathcal{L}^3$ (see Fig. 1), by

$$\mathcal{L}^1 := \{z_1(\hat{e}_1 + \hat{e}_2) + z_2(\hat{e}_2 + \hat{e}_3) \colon z_1, z_2 \in \mathbb{Z}\}, \quad \mathcal{L}^2 := \mathcal{L}^1 + \hat{e}_1, \quad \mathcal{L}^3 := \mathcal{L}^1 + \hat{e}_2.$$

Eventually, we define the family of triangles subordinated to the lattice $\mathcal{L}$ by setting

$$\mathcal{T}(\mathbb{R}^2) := \{T = \mathrm{conv}\{i, j, k\} \colon i, j, k \in \mathcal{L}, \ |i - j| = |j - k| = |k - i| = 1\},$$

where $\mathrm{conv}\{i, j, k\}$ denotes the closed convex hull of i, j, k. It is also convenient to introduce the families of upward/downward facing triangles

$$\mathcal{T}^{\pm}(\mathbb{R}^2) := \left\{T = \mathrm{conv}\{i, j, k\} \in \mathcal{T}(\mathbb{R}^2) \colon i \in \mathcal{L}^1, j \in \mathcal{L}^2, k \in \mathcal{L}^3, \pm(j - i) \times (k - i) > 0\right\}.$$

For $\varepsilon > 0$, we consider rescaled versions of $\mathcal{L}$ and $\mathcal{T}(\mathbb{R}^2)$ given by $\mathcal{L}_{\varepsilon} := \varepsilon\mathcal{L}$ and $\mathcal{T}_{\varepsilon}(\mathbb{R}^2) := \varepsilon\mathcal{T}(\mathbb{R}^2)$, $\mathcal{T}_{\varepsilon}^{\pm}(\mathbb{R}^2) := \varepsilon\mathcal{T}^{\pm}(\mathbb{R}^2)$. With this notation every $T \in \mathcal{T}_{\varepsilon}(\mathbb{R}^2)$ has vertices $\varepsilon i, \varepsilon j, \varepsilon k \in \mathcal{L}_{\varepsilon}$. The same notation applies to the sublattices, namely $\mathcal{L}_{\varepsilon}^{\alpha} := \varepsilon\mathcal{L}^{\alpha}$ for $\alpha \in \{1, 2, 3\}$. Given a Borel set $A \subset \mathbb{R}^2$ we denote by $\mathcal{T}_{\varepsilon}(A) := \{T \in \mathcal{T}_{\varepsilon}(\mathbb{R}^2) \colon T \subset A\}$ the subfamily of triangles contained in A. Eventually, we introduce the set of admissible configurations as the set of all *spin fields*

$$\mathcal{SF}_{\varepsilon} := \{u \colon \mathcal{L}_{\varepsilon} \to \mathbb{S}^1\}.$$

In the case $\varepsilon = 1$ we set $\mathcal{SF} := \mathcal{SF}_1$. For $u \in \mathcal{SF}_{\varepsilon}$ we now define the discrete energies $F_{\varepsilon}(u)$ as follows: for every $T \in \mathcal{T}_{\varepsilon}(\mathbb{R}^2)$ we set

$$F_{\varepsilon}(u, T) := \varepsilon|u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k)|^2,$$

and we extend the energy to any Borel set $A \subset \mathbb{R}^2$ by setting

$$F_\varepsilon(u, A) := \sum_{T \in \mathcal{T}_\varepsilon(A)} F_\varepsilon(u, T).$$

If $A = \Omega$ we omit the dependence on the set and write $F_\varepsilon(u) := F_\varepsilon(u, \Omega)$.

2.3 Chirality

In this section we introduce the relevant order parameter to analyze the asymptotic behavior of F_ε, namely the chirality χ. More in detail, given $u \in \mathcal{SF}_\varepsilon$ and $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ with $i \in \mathcal{L}^1$, $j \in \mathcal{L}^2$ and $k \in \mathcal{L}^3$ we set

$$\chi(u, T) := \frac{2}{3\sqrt{3}} \left(u(\varepsilon i) \times u(\varepsilon j) + u(\varepsilon j) \times u(\varepsilon k) + u(\varepsilon k) \times u(\varepsilon i) \right). \tag{2.1}$$

Moreover, we define $\chi(u) \colon \Omega \to \mathbb{R}$ by setting $\chi(u)(x) := \chi_\varepsilon(u, T)$ if $x \in \mathrm{int}\, T$. Given $u \in \mathcal{SF}_\varepsilon$ and $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ it is sometimes convenient to rewrite $\chi_\varepsilon(u, T)$ and $F_\varepsilon(u, T)$ in terms of the angular lift of u. More precisely, let $\theta(\varepsilon i), \theta(\varepsilon j), \theta(\varepsilon k) \in \mathbb{R}$ be such that $u(\varepsilon \alpha) = \exp(\iota \theta(\varepsilon \alpha))$, $\alpha \in \{i, j, k\}$. Then

$$\chi(u, T) = \frac{2}{3\sqrt{3}} \left(\sin\left(\theta(\varepsilon j) - \theta(\varepsilon i)\right) + \sin\left(\theta(\varepsilon k) - \theta(\varepsilon j)\right) + \sin\left(\theta(\varepsilon i) - \theta(\varepsilon k)\right) \right),$$
$$\tag{2.2}$$

$$F_\varepsilon(u, T) = 3\varepsilon + 2\varepsilon \left(\cos\left(\theta(\varepsilon j) - \theta(\varepsilon i)\right) + \cos\left(\theta(\varepsilon k) - \theta(\varepsilon j)\right) + \cos\left(\theta(\varepsilon i) - \theta(\varepsilon k)\right) \right).$$
$$\tag{2.3}$$

The next lemma is useful to relate the chirality and the energy in a triangle.

Lemma 2.1 *Let $f, g : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}$ be given by*

$$f(\theta_1, \theta_2) := \sin(\theta_1) + \sin(\theta_2 - \theta_1) - \sin(\theta_2),$$
$$g(\theta_1, \theta_2) := \cos(\theta_1) + \cos(\theta_2 - \theta_1) + \cos(\theta_2).$$

Then f and g have the following properties:

(i) *$f(\theta_1, \theta_2) \in [-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}]$ for every $\theta_1, \theta_2 \in [0, 2\pi)$. Moreover, $f(\theta_1, \theta_2) \in \{-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\}$ if and only if $g(\theta_1, \theta_2) = -\frac{3}{2}$.*
(ii) *$f(\theta_1, \theta_1) = f(\theta_1, 0) = f(0, \theta_2) = 0$ for every $\theta_1, \theta_2 \in [0, 2\pi)$. In addition, for every $\theta_2 \in (0, 2\pi)$ there holds $f(\cdot, \theta_2) > 0$ on $(0, \theta_2)$ and $f(\cdot, \theta_2) < 0$ on $(\theta_2, 2\pi)$.*

Proof Since there obviously holds $f(\theta_1, \theta_1) = f(\theta_1, 0) = f(0, \theta_2) = 0$, we only need to prove (i) and the second part of (ii). To prove (i) we show that $\min f = -\frac{3\sqrt{3}}{2}$ and $\max f = \frac{3\sqrt{3}}{2}$ and we relate minimizers and maximizers of f to minimizers of g. To this end we start computing

$$\nabla f(\theta_1, \theta_2) = \begin{pmatrix} \cos(\theta_1) - \cos(\theta_2 - \theta_1) \\ \cos(\theta_2 - \theta_1) - \cos(\theta_2) \end{pmatrix} \quad \text{and} \quad \nabla g(\theta_1, \theta_2) = \begin{pmatrix} -\sin(\theta_1) + \sin(\theta_2 - \theta_1) \\ -\sin(\theta_2 - \theta_1) - \sin(\theta_2) \end{pmatrix}.$$

A direct calculation shows that $\nabla f(\theta_1, \theta_2) = 0$ for some $(\theta_1, \theta_2) \in (0, 2\pi) \times (0, 2\pi)$ if and only if

$$\theta_1 = \frac{\theta_2}{2} + z_1 \pi \quad \text{and} \quad \theta_2 = \frac{\theta_1}{2} + z_2 \pi, \quad \text{for some } z_1, z_2 \in \{0, 1\}. \tag{2.4}$$

For $(\theta_1, \theta_2) \in (0, 2\pi) \times (0, 2\pi)$ this can only be satisfied if

$$(\theta_1, \theta_2) = (\tfrac{2\pi}{3}, \tfrac{4\pi}{3}) \quad \text{or} \quad (\theta_1, \theta_2) = (\tfrac{4\pi}{3}, \tfrac{2\pi}{3}). \tag{2.5}$$

Then, since $f = 0$ on the boundary of $[0, 2\pi] \times [0, 2\pi]$, we deduce that

$$\min_{[0, 2\pi] \times [0, 2\pi]} f = f\left(\left(\frac{4\pi}{3}, \frac{2\pi}{3}\right)\right) = -\frac{3\sqrt{3}}{2} \quad \text{and}$$

$$\max_{[0, 2\pi] \times [0, 2\pi]} f = f\left(\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)\right) = \frac{3\sqrt{3}}{2}.$$

Moreover, $g((\tfrac{2\pi}{3}, \tfrac{4\pi}{3})) = g((\tfrac{4\pi}{3}, \tfrac{2\pi}{3})) = -\tfrac{3}{2}$, which shows one direction of the second part of (i). To prove the opposite direction, let us assume that $(\bar{\theta}_1, \bar{\theta}_2) \in (0, 2\pi) \times (0, 2\pi)$ is such that $g((\bar{\theta}_1, \bar{\theta}_2)) = \min g$. Then necessarily $\nabla g(\bar{\theta}_1, \bar{\theta}_2) = 0$, from which we deduce that $(\bar{\theta}_1, \bar{\theta}_2)$ must satisfy (2.4) (the possibility that $\bar{\theta}_1 = \pi$ or $\bar{\theta}_2 = \pi$ are ruled out by the fact that $g(\cdot, \pi) = g(\pi, \cdot) = -1$). The pairs $(\bar{\theta}_1, \bar{\theta}_2)$ satisfying (2.4) are either $(\bar{\theta}_1, \bar{\theta}_2) = (\tfrac{2\pi}{3}, \tfrac{4\pi}{3})$ or $(\bar{\theta}_1, \bar{\theta}_2) = (\tfrac{4\pi}{3}, \tfrac{2\pi}{3})$ and in both cases it holds $g(\bar{\theta}_1, \bar{\theta}_2) = -\tfrac{3}{2}$. This yields that $\min g = -\tfrac{3}{2}$ and that the opposite direction of (i) holds, upon noticing that $g \geq -1$ on the boundary of $[0, 2\pi] \times [0, 2\pi]$. To complete the proof of (ii) let us fix $\theta_2 \in (0, 2\pi)$ and consider $f(\cdot, \theta_2)$ as a function of θ_1. Then (2.4) shows that $\frac{\partial f}{\partial \theta_1}(\theta_1, \theta_2) = 0$ if and only if $\theta_1 \in \{\theta_2^{\text{pos}}, \theta_2^{\text{neg}}\}$, where

$$\theta_2^{\text{pos}} := \frac{\theta_2}{2} \in (0, \theta_2), \quad \theta_2^{\text{neg}} := \frac{\theta_2}{2} + \pi \in (\theta_2, 2\pi). \tag{2.6}$$

Moreover, upon extending $f(\cdot, \theta_2)$ to an open interval containing $(0, 2\pi)$, we get

$$\frac{\partial f}{\partial \theta_1}(0, \theta_2) = \frac{\partial f}{\partial \theta_1}(2\pi, \theta_2) = 1 - \cos(\theta_2) > 0 \quad \text{and} \quad \frac{\partial f}{\partial \theta_1}(\theta_2, \theta_2) = \cos(\theta_2) - 1 < 0.$$

In particular, from the intermediate value theorem we deduce that $f(\cdot, \theta_2)$ is strictly increasing on $(0, \theta_2^{\text{pos}})$ and strictly decreasing on $(\theta_2^{\text{pos}}, \theta_2)$. Since in addition $f(0, \theta_2) = f(\theta_2, \theta_2) = 0$ this implies that $f(\cdot, \theta_2) > 0$ on $(0, \theta_2)$. Arguing similarly on the intervals $(\theta_2, \theta_2^{\text{neg}})$ and $(\theta_2^{\text{neg}}, 2\pi)$ we obtain $f(\cdot, \theta_2) < 0$ on $(\theta_2, 2\pi)$, which proves (ii). □

Remark 2.2 Using the expressions of $\chi(u, T)$ and $F_\varepsilon(u, T)$ in (2.2)–(2.3) one can show that $\chi(u, T) \in [-1, 1]$ and $\chi(u, T) \in \{-1, 1\}$ if and only if $F_\varepsilon(u, T) = 0$, *i.e.*, configurations that maximize or minimize $\chi(\cdot, T)$ are at the same time minimizers for $F_\varepsilon(\cdot, T)$. This follows from Lemma 2.1 (i) upon noticing that in (2.2)–(2.3) it is not restrictive to assume that $\theta(\varepsilon i) = 0$, since both χ_ε and F_ε are invariant under rotations in u. We observe that also a quantitative version of this property holds. Namely, a continuity argument shows that for every $\delta > 0$ there exists $C_\delta > 0$ such that for every $u \in \mathcal{SF}_\varepsilon$ and every $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ the following implication holds:

$$\chi(u, T) \in (-1 + \delta, 1 - \delta) \implies F_\varepsilon(u, T) \geq C_\delta \varepsilon. \tag{2.7}$$

Remark 2.3 As a consequence of Lemma 2.1 (ii) one obtains the following characterization of the sign of the chirality. Let $\theta(\varepsilon j) \in [0, 2\pi)$ be the angle between $u(\varepsilon i)$ and $u(\varepsilon j)$ and let $\theta(\varepsilon k) \in [0, 2\pi)$ the angle between $u(\varepsilon i)$ and $u(\varepsilon k)$. Then $\chi(u, T) > 0$ if and only if $\theta(\varepsilon j) < \theta(\varepsilon k)$ and $\chi(u, T) < 0$ if and only if $\theta(\varepsilon j) > \theta(\varepsilon k)$. In other words, a positive chirality on $T = \text{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ corresponds to a counterclockwise ordering of $(u(\varepsilon i), u(\varepsilon j), u(\varepsilon k))$ on $\mathbb{S}^1$, while a negative chirality corresponds to a clockwise ordering on $\mathbb{S}^1$ (Fig. 2).
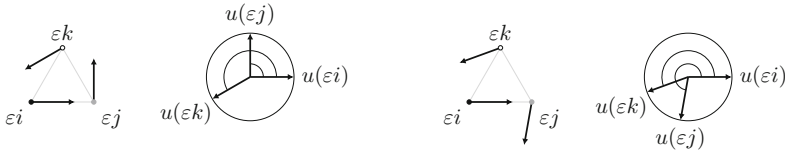
Fig. 2 On the left: a configuration of vectors with positive chirality which shows the criterion explained in Remark 2.3. On the right: negative chirality
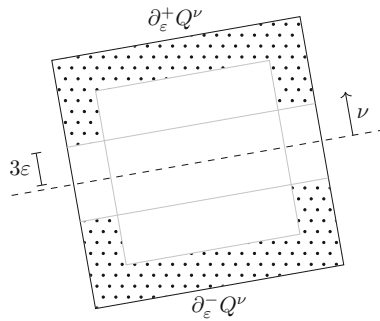
Fig. 3 Discrete boundary of the square Q^ν

2.4 Statement of the main result

Notice that $\chi(u) \in L^1(\Omega)$. We then extend F_ε to $L^1(\Omega)$ by setting

$$\mathscr{F}_\varepsilon(\chi) = \inf\{F_\varepsilon(u) \colon u \in \mathcal{SF}_\varepsilon, \ \chi(u) = \chi\}, \tag{2.8}$$

with the convention $\inf \emptyset = +\infty$.

Remark 2.4 If $\chi \in L^1(\Omega)$ is such that $\chi = \chi(u)$ for some $u \in \mathcal{SF}_\varepsilon$, then the infimum in (2.8) is actually a minimum.

To state the main theorem we need to introduce two ground states, that we name $u_\varepsilon^{\mathrm{pos}}, u_\varepsilon^{\mathrm{neg}} \in \mathcal{SF}_\varepsilon$ which have a uniform chirality equal to $+1$ and -1, respectively. They are given by

$$u_\varepsilon^{\mathrm{pos}}(x) := \begin{cases} \exp(\iota 0) & \text{if } x \in \mathcal{L}_\varepsilon^1, \\ \exp(\iota 2\pi/3) & \text{if } x \in \mathcal{L}_\varepsilon^2, \\ \exp(\iota 4\pi/3) & \text{if } x \in \mathcal{L}_\varepsilon^3, \end{cases} \qquad u_\varepsilon^{\mathrm{neg}}(x) := \begin{cases} \exp(\iota 0) & \text{if } x \in \mathcal{L}_\varepsilon^1, \\ \exp(\iota 4\pi/3) & \text{if } x \in \mathcal{L}_\varepsilon^2, \\ \exp(\iota 2\pi/3) & \text{if } x \in \mathcal{L}_\varepsilon^3, \end{cases}$$

for every $x \in \mathcal{L}_\varepsilon$. We also set $u^{\mathrm{pos}} := u_1^{\mathrm{pos}}$, $u^{\mathrm{neg}} := u_1^{\mathrm{neg}}$. The ground states u^{pos} and u^{neg} will be used as boundary conditions on the discrete boundary of the square Q^ν given by (see Fig. 3)

$$\partial_\varepsilon^\pm Q^\nu = \{x \in \mathcal{L}_\varepsilon \colon \pm \langle \nu, x \rangle \geq 3\varepsilon, \ \mathrm{dist}(x, \partial Q^\nu) \leq 3\varepsilon\}. \tag{2.9}$$

Theorem 2.5 *The energies $\mathscr{F}_\varepsilon \colon L^1(\Omega) \to [0, +\infty]$ defined by (2.8) Γ-converge in the strong $L^1(\Omega)$-topology to the functional $\mathscr{F} \colon L^1(\Omega) \to [0, +\infty]$ given by*

$$\mathscr{F}(\chi) := \begin{cases} \displaystyle\int_{J_\chi} \varphi(\nu_\chi) \, d\mathcal{H}^1 & \text{if } \chi \in BV(\Omega; \{-1, 1\}), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{2.10}$$

where $\varphi \colon \mathbb{S}^1 \to [0, +\infty)$ is defined by

$$\varphi(\nu) := \lim_{\varepsilon \to 0} \min \left\{ F_\varepsilon(u, Q^\nu) \colon u = u_\varepsilon^{\mathrm{pos}} \text{ on } \partial_\varepsilon^+ Q^\nu, \ u = u_\varepsilon^{\mathrm{neg}} \text{ on } \partial_\varepsilon^- Q^\nu \right\}. \tag{2.11}$$

The proof of Theorem 2.5 will be carried out in Sections 4 and 5 , where we prove separately the asymptotic lower bound (Proposition 4.1) and the asymptotic upper bound (Proposition 5.1), respectively.

Remark 2.6 By standard arguments in the analysis of asymptotic cell formulas (see e.g. [4, Proposition 4.6]) one can show that the limit in (2.11) actually exists, so that φ is well defined. Note that, by the symmetries of the interaction energies, there holds $\varphi(-\nu) = \varphi(\nu)$. Moreover, one can show (cf. [4, Proposition 4.7]) that the one-homogeneous extension of φ is convex, hence continuous.

Remark 2.7 By a scaling argument we note that for all $\rho > 0$ there holds

$$\varphi(\nu) = \lim_{\varepsilon \to 0} \frac{1}{\rho} \min \left\{ F_\varepsilon(u, Q_\rho^\nu) \colon u = u_\varepsilon^{\mathrm{pos}} \text{ on } \partial_\varepsilon^+ Q_\rho^\nu, \ u = u_\varepsilon^{\mathrm{neg}} \text{ on } \partial_\varepsilon^- Q_\rho^\nu \right\}, \tag{2.12}$$

where $\partial_\varepsilon^\pm Q_\rho^\nu$ are defined according to (2.9) with Q_ρ^ν in place of Q^ν.

3 Compactness

Proposition 3.1 *Let (u_ε) be a sequence of spin fields $u_\varepsilon \in \mathcal{SF}_\varepsilon$ satisfying*

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty. \tag{3.1}$$

Then there exists $\chi \in BV(\Omega; \{-1, 1\})$ such that up to subsequences $\chi(u_\varepsilon) \to \chi$ in $L^1(\Omega)$.

To prove Proposition 3.1 we first estimate from below the energy of a spin field u on two neighboring triangles where $\chi(u)$ changes sign. Given a triangle $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ we introduce the class $\mathcal{N}_\varepsilon(T)$ of its neighboring triangles, namely those triangles in $\mathcal{T}_\varepsilon(\mathbb{R}^2)$ that share a side with T. More precisely, we define

$$\mathcal{N}_\varepsilon(T) := \{T' \in \mathcal{T}_\varepsilon(\mathbb{R}^2) \colon \mathcal{H}^1(T \cap T') = \varepsilon\}. \tag{3.2}$$

Lemma 3.2 *Let $u \in \mathcal{SF}_\varepsilon$ and suppose that $T^{\mathrm{pos}}, T^{\mathrm{neg}} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ with $T^{\mathrm{neg}} \in \mathcal{N}_\varepsilon(T^{\mathrm{pos}})$ are such that $\chi(u, T^{\mathrm{pos}}) \geq 0$ and $\chi(u, T^{\mathrm{neg}}) \leq 0$. Then $F_\varepsilon(u, T^{\mathrm{pos}} \cup T^{\mathrm{neg}}) \geq \frac{5}{3}\varepsilon$.*

Proof It is not restrictive to assume that $T^{\mathrm{pos}} = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ and $T^{\mathrm{neg}} = \mathrm{conv}\{\varepsilon i, \varepsilon j', \varepsilon k\}$ with $i \in \mathcal{L}^1$, $j, j' \in \mathcal{L}^2$ and $k \in \mathcal{L}^3$. Moreover, we can assume that $u(\varepsilon i) = \hat{e}_1$, that is, $\theta(\varepsilon i) = 0$ according to the notation in (2.2)–(2.3). Then, using the function $g \colon [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}$ defined in Lemma 2.1, we can rewrite $F_\varepsilon(u, T^{\mathrm{pos}} \cup T^{\mathrm{neg}})$ as

$$F_\varepsilon(u, T^{\mathrm{pos}} \cup T^{\mathrm{neg}}) = 6\varepsilon + 2\varepsilon \left(g \left(\theta(\varepsilon j), \theta(\varepsilon k) \right) + g \left(\theta(\varepsilon j'), \theta(\varepsilon k) \right) \right).$$

Moreover, thanks to Lemma 2.1 (ii) the chirality constraint reads $0 \leq \theta(\varepsilon j) \leq \theta(\varepsilon k) \leq \theta(\varepsilon j')$. Thus, the statement is proved if we show that for all $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$ with $0 \leq \theta_1 \leq \theta_2 \leq \theta_3$ there holds

$$6 + 2\left(g(\theta_1, \theta_2) + g(\theta_3, \theta_2)\right) \geq \frac{5}{3}. \tag{3.3}$$

We first observe that (3.3) trivially holds if $\theta_2 = 0$ or $\theta_2 = \pi$. Indeed, if $\theta_2 = 0$, then also $\theta_1 = 0$, hence $g(\theta_1, \theta_2) + g(\theta_3, \theta_2) = 4 + 2\cos(\theta_3) \geq 2$, thus (3.3) is satisfied. If, instead, $\theta_2 = \pi$, then a direct computation shows that $g(\theta_1, \theta_2) + g(\theta_3, \theta_2) = -2$ for every $\theta_1, \theta_3 \in [0, 2\pi)$, which directly gives (3.3).

Suppose now that $\theta_2 \in (0, 2\pi) \setminus \{\pi\}$ and let us minimize $g(\cdot, \theta_2)$ on the two intervals $[0, \theta_2]$ and $[\theta_2, 2\pi)$. As in the proof of Lemma 2.1 we obtain that $\frac{\partial g}{\partial \theta_1}(\theta_1, \theta_2) = 0$ if and only if $\theta_1 \in \{\theta_2^{\mathrm{pos}}, \theta_2^{\mathrm{neg}}\}$ with $\theta_2^{\mathrm{pos}}, \theta_2^{\mathrm{neg}}$ as in (2.6). Moreover, we have

$$\frac{\partial^2 g}{\partial \theta_1^2}(\theta_2^{\mathrm{pos}}, \theta_2) = -2\cos\left(\frac{\theta_2}{2}\right) \quad \text{and} \quad \frac{\partial^2 g}{\partial \theta_1^2}(\theta_2^{\mathrm{neg}}, \theta_2) = 2\cos\left(\frac{\theta_2}{2}\right). \tag{3.4}$$

Thus, either $\theta_2^{\mathrm{pos}} \in (0, \theta_2)$ or $\theta_2^{\mathrm{neg}} \in (\theta_2, 2\pi)$ is a minimizer for $g(\cdot, \theta_2)$, depending on wether $\theta_2 \in (0, \pi)$ or $\theta_2 \in (\pi, 2\pi)$. Suppose first that $\theta_2 \in (\pi, 2\pi)$. Then (3.4) implies that $g(\cdot, \theta_2)$ is minimized in $[0, \theta_2)$ by θ_2^{pos}, while in $[\theta_2, 2\pi)$ it attains its minimum on the boundary, that is at θ_2. This yields

$$g(\theta_1, \theta_2) + g(\theta_3, \theta_2) \geq g(\theta_2^{\mathrm{pos}}, \theta_2) + g(\theta_2, \theta_2) = 2\cos\left(\frac{\theta_2}{2}\right) + 3\cos(\theta_2) + 1, \tag{3.5}$$

for every $\theta_1 \in [0, \theta_2]$ and $\theta_3 \in [\theta_2, 2\pi)$. Using the equality $\cos(\theta_2) = 2\cos^2(\frac{\theta_2}{2}) - 1$, the estimate in (3.5) can be continued via

$$g(\theta_1, \theta_2) + g(\theta_3, \theta_2) \geq 6\cos^2\left(\frac{\theta_2}{2}\right) + 2\cos\left(\frac{\theta_2}{2}\right) - 2. \tag{3.6}$$

Since the mapping $t \mapsto 6t^2 + 2t - 2$ admits its minimum at $t = -1/6$, from (3.6) we finally deduce that

$$g(\theta_1, \theta_2) + g(\theta_3, \theta_2) \geq -\frac{13}{6},$$

which is equivalent to (3.3). Eventually, the case $\theta_2 \in (0, \pi)$ follows similarly by exchanging the roles of θ_1 and θ_3 and replacing θ_2^{pos} by θ_2^{neg}. □

Based on Lemma 3.2 we now prove Proposition 3.1.

Proof of Proposition 3.1 We divide the proof in two steps. First, we construct a sequence $(\hat{\chi}_\varepsilon)$ of auxiliary functions $\hat{\chi}_\varepsilon : \Omega \to \{-1, 1\}$ whose level sets $\{\hat{\chi}_\varepsilon = 1\}$ have uniformly bounded perimeter. Second, we show that the constructed auxiliary functions are close in $L^1(\Omega)$ to the original chirality functions $\chi(u_\varepsilon)$ defined according to (2.1).
Step 1. (Compactness of the auxiliary functions) Let $\varepsilon > 0$ and define $\hat{\chi}_\varepsilon : \Omega \to \{-1, 1\}$ by

$$\hat{\chi}_\varepsilon := \begin{cases} 1 & \text{if } \chi(u_\varepsilon) > 0, \\ -1 & \text{otherwise.} \end{cases}$$

We claim that for every $\Omega' \subset\subset \Omega$ we have

$$\mathcal{H}^1(\partial\{\hat{\chi}_\varepsilon = 1\} \cap \Omega') \leq C F_\varepsilon(u_\varepsilon). \tag{3.7}$$

Then the uniform bound (3.1) together with [7, Theorem 3.39 and Remark 3.37] yields the existence of a function $\chi \in BV(\Omega; \{-1, 1\})$ and a subsequence (not relabelled) such that $\hat{\chi}_\varepsilon \to \chi$ in $L^1(\Omega)$. To prove (3.7) it is convenient to consider the class of triangles

$$\mathcal{T}_\varepsilon^{\text{pos}} := \{T \in \mathcal{T}_\varepsilon(\Omega) \colon \chi(u_\varepsilon, T) > 0 \text{ and } \chi(u_\varepsilon, T') \leq 0 \text{ for some } T' \in \mathcal{N}_\varepsilon(T) \cap \mathcal{T}_\varepsilon(\Omega)\},$$

where $\mathcal{N}_\varepsilon(T)$ is as in (3.2). Let $\Omega' \subset\subset \Omega$. By the very definition of $\hat{\chi}_\varepsilon$ and of $\chi(u_\varepsilon)$ we have

$$\partial\{\hat{\chi}_\varepsilon = 1\} \cap \Omega' \subset \partial\left(\bigcup_{T \in \mathcal{T}_\varepsilon^{\text{pos}}} T\right),$$

provided $\sqrt{3}\varepsilon < \text{dist}(\Omega', \partial\Omega)$. Estimating the $\mathcal{H}^1$-measure of the latter set in terms of the cardinality of $\mathcal{T}_\varepsilon^{\text{pos}}$ we thus infer

$$\mathcal{H}^1(\partial\{\hat{\chi}_\varepsilon = 1\} \cap \Omega') \leq 3\varepsilon \#\mathcal{T}_\varepsilon^{\text{pos}}. \tag{3.8}$$

The last term in (3.8) can be bounded using Lemma 3.2. Indeed, from Lemma 3.2 we deduce that

$$\frac{5}{3}\varepsilon \#\mathcal{T}_\varepsilon^{\text{pos}} \leq \sum_{T \in \mathcal{T}_\varepsilon(\Omega)} \sum_{T' \in \mathcal{N}_\varepsilon(T) \cap \mathcal{T}_\varepsilon(\Omega)} F_\varepsilon(u_\varepsilon, T \cup T') \leq 3 F_\varepsilon(u_\varepsilon), \tag{3.9}$$

where the additional factor 3 comes from the fact that each triangle is counted 3 times. Thus, (3.7) follows from (3.8) and (3.9).

Step 2. (Closeness to $\chi(u_\varepsilon)$) We claim that for every $\delta > 0$ and every $\Omega' \subset\subset \Omega$ there holds

$$\lim_{\varepsilon \to 0} \left|\{|\hat{\chi}_\varepsilon - \chi(u_\varepsilon)| > \delta\} \cap \Omega'\right| = 0, \tag{3.10}$$

i.e., the functions $\hat{\chi}_\varepsilon - \chi(u_\varepsilon)$ converge to 0 locally in measure. Since $\|\hat{\chi}_\varepsilon - \chi(u_\varepsilon)\|_\infty \leq 2$, this implies that $(\hat{\chi}_\varepsilon - \chi(u_\varepsilon)) \to 0$ in $L^1(\Omega)$, which concludes the proof of the Proposition 3.1 thanks to Step 1. It remains to prove the claim (3.10). Let $\Omega' \subset\subset \Omega$ and $\delta > 0$ and let C_δ be given by (2.7). Setting

$$\mathcal{T}_\varepsilon^\delta := \{T \in \mathcal{T}_\varepsilon(\Omega) \colon \chi(u_\varepsilon, T) \in (-1 + \delta, 1 - \delta)\},$$

for ε sufficiently small we deduce that

$$|\{|\hat{\chi}_\varepsilon - \chi(u_\varepsilon)| > \delta\} \cap \Omega'| \leq \frac{\sqrt{3}}{4}\varepsilon^2 \#\mathcal{T}_\varepsilon^\delta \leq \frac{\sqrt{3}}{4}\varepsilon C_\delta^{-1} \sum_{T \in \mathcal{T}_\varepsilon^\delta} F_\varepsilon(u_\varepsilon, T) \leq \frac{\sqrt{3}}{4}\varepsilon C_\delta^{-1} F_\varepsilon(u_\varepsilon).$$

Hence, (3.10) follows from the uniform bound (3.1). $\qquad\square$

4 Lower bound

In this section we start proving the main result of our paper, namely Theorem 2.5 by presenting the optimal lower bound estimate on the energy $\mathscr{F}_\varepsilon$, the technically most demanding part of our contribution. We begin with a blow-up argument that gives us a first asymptotic lower bound.

Proposition 4.1 *Let $\mathscr{F}_\varepsilon$ be as in (2.8). Then for every $\chi \in L^1(\Omega)$ we have*

$$\Gamma\text{-}\liminf_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\chi) \geq \mathscr{F}(\chi),$$

where $\mathscr{F}$ is given by (2.10) and the Γ- lim inf is with respect to the strong topology in $L^1(\Omega)$.

Proof Let $\chi_\varepsilon \to \chi$ in $L^1(\Omega)$. We assume that $\liminf_\varepsilon \mathscr{F}_\varepsilon(\chi_\varepsilon) < +\infty$, otherwise we have nothing to prove. Moreover, upon extracting a (not relabeled) subsequence we can assume the liminf to be a limit and hence $\sup_\varepsilon \mathscr{F}_\varepsilon(\chi_\varepsilon) < +\infty$. In view of Remark 2.4 we can find a sequence of spin fields $u_\varepsilon \in \mathcal{SF}_\varepsilon$ with $\chi(u_\varepsilon) = \chi_\varepsilon$ and $\mathscr{F}_\varepsilon(\chi_\varepsilon) = F_\varepsilon(u_\varepsilon)$. In particular, $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Thus, from Proposition 3.1 we deduce that $\chi \in BV(\Omega; \{-1, 1\})$. As a consequence, to prove the statement of the proposition it suffices to show that

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \int_{J_\chi} \varphi(\nu_\chi) \, \mathrm{d}\mathcal{H}^1, \tag{4.1}$$

where φ is as in (2.11). To prove (4.1) we consider the sequence of non-negative finite Radon measures μ_ε given by

$$\mu_\varepsilon := \sum_{T \in \mathcal{T}_\varepsilon(\Omega)} \varepsilon |u_\varepsilon(\varepsilon i) + u_\varepsilon(\varepsilon j) + u_\varepsilon(\varepsilon k)|^2 \delta_{\varepsilon i} \,,$$

where $\delta_{\varepsilon i}$ denotes the Dirac delta in εi. From the condition $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ it follows that $\sup_\varepsilon \mu_\varepsilon(\Omega) < +\infty$, hence there exists a non-negative finite Radon measure μ such that up to subsequences (not relabeled) $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$. By the Radon-Nikodým Theorem the measure μ can be decomposed in the sum of two mutually singular non-negative measures as

$$\mu = \mu_j \mathcal{H}^1 \llcorner J_\chi + \mu_s \,.$$

Then, to establish (4.1) it is sufficient to show that

$$\mu_j(x_0) \geq \varphi(\nu_\chi(x_0)) \quad \text{for } \mathcal{H}^1\text{-a.e. } x_0 \in J_\chi \,, \tag{4.2}$$

where $\nu_\chi(x_0)$ denotes the measure theoretic normal to J_χ at x_0. To verify (4.2) we choose $x_0 \in J_\chi$ satisfying

(i) $\mu_j(x_0) = \dfrac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^1 \llcorner J_\chi}(x_0) = \lim\limits_{\rho \to 0} \dfrac{\mu(Q_\rho^\nu(x_0))}{\rho}$, where we have set $\nu := \nu_\chi(x_0)$,

(ii) $\lim\limits_{\rho \to 0} \dfrac{1}{\rho^2} \displaystyle\int_{Q_\rho^\nu(x_0) \cap H_+^\nu(x_0)} |\chi_\varepsilon(x) - 1| \, \mathrm{d}x = 0 = \lim\limits_{\rho \to 0} \dfrac{1}{\rho^2} \displaystyle\int_{Q_\rho^\nu(x_0) \cap H_-^\nu(x_0)} |\chi_\varepsilon(x) + 1| \, \mathrm{d}x,$

and we notice that (i) and (ii) are satisfied for $\mathcal{H}^1$-a.e.$x_0 \in J_\chi$ thanks to the Besicovitch derivation Theorem and the definition of approximate jump point, respectively. Moreover, since μ is a finite Radon measure, we can choose a sequence $\rho_n \to 0$ along which $\mu(\partial Q_{\rho_n}^\nu(x_0)) = 0$. Thanks to [7, Proposition 1.62 (a)], the convergence $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ together with (i) implies that

$$\mu_j(x_0) = \lim_{n \to +\infty} \frac{\mu(Q_{\rho_n}^\nu(x_0))}{\rho_n} = \lim_{n \to +\infty} \lim_{\varepsilon \to 0} \frac{\mu_\varepsilon(Q_{\rho_n}^\nu(x_0))}{\rho_n}$$

$$\geq \lim_{n \to +\infty} \limsup_{\varepsilon \to 0} \frac{1}{\rho_n} F_\varepsilon(u_\varepsilon, Q_{\rho_n}^\nu(x_0)) \,, \tag{4.3}$$

where the last inequality follows from the positivity of the energy. Notice that for every $n \in \mathbb{N}$ there exist sequences (ρ_n^ε) and (x_0^ε) with $\lim_\varepsilon \rho_n^\varepsilon = \rho_n$, $\lim_\varepsilon x_0^\varepsilon = x_0$, $x_0^\varepsilon \in \mathcal{L}_\varepsilon$, and

$$\mathcal{T}_\varepsilon(Q_{\rho_n^\varepsilon}^\nu(x_0^\varepsilon)) \subset \mathcal{T}_\varepsilon(Q_{\rho_n}^\nu(x_0)) \,. \tag{4.4}$$

In fact, if we write x_0 in terms of the basis $\hat{e}_1$, $\hat{e}_2$ as $x_0 = a_1\hat{e}_1 + a_2\hat{e}_2$ for some $a_1, a_2 \in \mathbb{R}$, we obtain the required sequence (x_0^ε) by setting

$$x_0^\varepsilon := \varepsilon \left\lfloor \frac{a_1}{\varepsilon} \right\rfloor \hat{e}_1 + \varepsilon \left\lfloor \frac{a_2}{\varepsilon} \right\rfloor \hat{e}_2 \in \mathcal{L}_\varepsilon \,.$$

Then, upon noticing that $|x_0^\varepsilon - x_0| \le 2\varepsilon$, it suffices to set $\rho_n^\varepsilon := \rho_n - 4\varepsilon$. Indeed, if $T \in \mathcal{T}_\varepsilon(Q_{\rho_n^\varepsilon}^\nu(x_0^\varepsilon))$, by definition we have that for every $x \in T$

$$|\langle x - x_0^\varepsilon, \nu \rangle| < \frac{\rho_n^\varepsilon}{2} \quad \text{and} \quad |\langle x - x_0^\varepsilon, \nu^\perp \rangle| < \frac{\rho_n^\varepsilon}{2} \,,$$

so that for any $x \in T$ there also holds

$$|\langle x - x_0, \nu \rangle| \le |\langle x - x_0^\varepsilon, \nu \rangle| + |x_0^\varepsilon - x_0| < \frac{\rho_n}{2} \,,$$

and similarly $|\langle x - x_0, \nu^\perp \rangle| < \rho_n/2$, hence $T \in \mathcal{T}_\varepsilon(Q_{\rho_n}^\nu(x_0))$. As a consequence, (4.4) holds true. In particular, we obtain the following estimate

$$\begin{aligned} \frac{1}{\rho_n} F_\varepsilon(u_\varepsilon, Q_{\rho_n}^\nu(x_0)) &\ge \frac{\rho_n^\varepsilon}{\rho_n} \frac{\varepsilon}{\rho_n^\varepsilon} \sum_{T \in \mathcal{T}_\varepsilon(Q_{\rho_n}^\nu(x_0^\varepsilon))} |u_\varepsilon(\varepsilon i) + u_\varepsilon(\varepsilon j) + u_\varepsilon(\varepsilon k)|^2 \\ &= \frac{\rho_n^\varepsilon}{\rho_n} \sum_{T \in \mathcal{T}_{\sigma_n^\varepsilon}(Q^\nu)} \sigma_n^\varepsilon |v_{\varepsilon,n}(\sigma_n^\varepsilon i) + v_{\varepsilon,n}(\sigma_n^\varepsilon j) + v_{\varepsilon,n}(\sigma_n^\varepsilon k)|^2, \end{aligned} \tag{4.5}$$

where we have set $\sigma_n^\varepsilon := \varepsilon/\rho_n^\varepsilon$ and $v_{\varepsilon,n}(z) := u_\varepsilon(x_0^\varepsilon + \rho_n^\varepsilon z)$ for every $z \in \mathcal{L}_{\sigma_n^\varepsilon}$. Let $\chi_\nu \colon \mathbb{R}^2 \to \{-1, 1\}$ be given by

$$\chi_\nu(x) := \begin{cases} 1 & \text{if } \langle x, \nu \rangle \ge 0\,, \\ -1 & \text{if } \langle x, \nu \rangle < 0\,. \end{cases}$$

Then (ii) ensures that $\chi(v_{\varepsilon,n}) \to \chi_\nu$ in $L^1(Q^\nu)$ as first $\varepsilon \to 0$ and then $n \to +\infty$. Thus, gathering (4.3)–(4.5) and applying a diagonal argument we find a sequence $\sigma_m := \varepsilon_m/\rho_{n_m}$ converging to 0 as $m \to +\infty$ such that for $v_m := v_{\varepsilon_m, n_m}$ there holds $\chi(v_m) \to \chi_\nu$ in $L^1(Q^\nu)$ and

$$\mu_j(x_0) \ge \liminf_{m \to +\infty} F_{\sigma_m}(v_m, Q^\nu)\,.$$

For $\ell, h > 0$ let us finally introduce the minimization problem

$$\psi(\ell, h, \nu) := \frac{1}{\ell} \inf \left\{ \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, R_{\ell,h}^\nu) \colon \chi(u_\varepsilon) \to \chi_\nu \text{ in } L^1(R_{\ell,h}^\nu) \right\}, \tag{4.6}$$

so that the sequence (v_m) is admissible for $\psi(1, 1, \nu)$. Then (4.2) follows from Proposition 4.2 below, concluding the proof of Proposition 4.1. $\qquad\square$

Proposition 4.2 *Let ψ be the function defined in (4.6). Then $\psi(1, 1, \nu) \ge \varphi(\nu)$ for every $\nu \in \mathbb{S}^1$.*

To prove Proposition 4.2 it is necessary to modify admissible sequences for the infimum problem defining $\psi(1, 1, \nu)$ in such a way that they satisfy the boundary conditions required in the minimum problem defining $\varphi(\nu)$, without essentially increasing the energy. This will be done by a careful interpolation procedure based on several auxiliary results and estimates that we prefer to state in separate lemmas below. As a first step towards the proof of Proposition 4.2 we show that $\psi(\ell, h, \nu)$ is independent of ℓ and h, which in turn will allow us to conclude

that the energy of admissible functions for $\psi(1, 1, \nu)$ concentrates close to the line segment L^ν (see Lemma 4.4 below).

Lemma 4.3 *Let* $\psi: (0, +\infty) \times (0, +\infty) \times \mathbb{S}^1 \to [0, +\infty]$ *be given by* (4.6); *then* $\psi(\cdot, \cdot, \nu)$ *is independent of* ℓ, h *for every* $\nu \in \mathbb{S}^1$.

Proof Let $\nu \in \mathbb{S}^1$ be fixed. To show that $\psi(\cdot, \cdot, \nu)$ does not depend on ℓ, h it suffices to show that for every $\ell, h, \lambda > 0$ the following identities hold

$$\psi(\lambda\ell, h, \nu) = \psi(\ell, h, \nu) \quad \text{and} \quad \psi(\ell, \lambda h, \nu) = \psi(\ell, h, \nu). \tag{4.7}$$

Let us fix $\ell, h > 0$. We first observe that

$$\psi(\ell, \lambda h, \nu) \geq \psi(\ell, h, \nu) \quad \text{for every } \lambda \in [1, +\infty), \tag{4.8}$$

$$\psi(\ell, h, \nu) \geq \psi(\ell, \lambda h, \nu) \quad \text{for every } \lambda \in (0, 1), \tag{4.9}$$

since F_ε is increasing as a set function. The proof of (4.7) is now divided into three steps.
Step 1. ψ is invariant under dilations, *i.e.*,

$$\psi(\lambda\ell, \lambda h, \nu) = \psi(\ell, h, \nu) \quad \text{for every } \lambda > 0. \tag{4.10}$$

Let (u_ε) be any sequence of spin fields $u_\varepsilon: \mathcal{L}_\varepsilon \to \mathbb{S}^1$ with $\chi(u_\varepsilon) \to \chi_\nu$ in $L^1(R^\nu_{\lambda\ell, \lambda h})$. We define the rescaled functions $v_\varepsilon: \mathcal{L}_{\varepsilon/\lambda} \to \mathbb{S}^1$ by setting $v_\varepsilon(z) := u_\varepsilon(\lambda z)$ for every $z \in \mathcal{L}_{\varepsilon/\lambda}$. Then $\chi(v_\varepsilon) \to \chi_\nu$ in $L^1(R^\nu_{\ell, h})$ and

$$
\begin{aligned}
F_{\frac{\varepsilon}{\lambda}}(v_\varepsilon, R^\nu_{\ell, h}) &= \sum_{T \in \mathcal{T}_{\frac{\varepsilon}{\lambda}}(R^\nu_{\ell, h})} \frac{\varepsilon}{\lambda} |v_\varepsilon(\tfrac{\varepsilon}{\lambda}i) + v_\varepsilon(\tfrac{\varepsilon}{\lambda}j) + v_\varepsilon(\tfrac{\varepsilon}{\lambda}k)|^2 \\
&= \frac{1}{\lambda} \sum_{T \in \mathcal{T}_\varepsilon(R^\nu_{\lambda\ell, \lambda h})} \varepsilon |u_\varepsilon(\varepsilon i) + u_\varepsilon(\varepsilon j) + u_\varepsilon(\varepsilon k)|^2 = \frac{1}{\lambda} F_\varepsilon(u_\varepsilon, R^\nu_{\lambda\ell, \lambda h}).
\end{aligned}
$$

Setting $\eta := \varepsilon/\lambda \to 0$ as $\varepsilon \to 0$ and passing to the infimum over all admissible sequences (u_ε) we deduce that

$$\psi(\lambda\ell, \lambda h, \nu) \geq \frac{1}{\ell} \inf \left\{ \liminf_{\eta \to 0} F_\eta(v_\eta, R^\nu_{\ell, h}): \chi(v_\eta) \to \chi_\nu \text{ in } L^1(R^\nu_{\ell, h}) \right\} = \psi(\ell, h, \nu).$$

The opposite inequality and hence (4.10) follow by observing that

$$\psi(\ell, h, \nu) = \psi(\lambda^{-1}(\lambda\ell), \lambda^{-1}(\lambda h), \nu) \geq \psi(\lambda\ell, \lambda h, \nu).$$

Note that thanks to (4.10) it suffices to show the first equality in (4.7). In fact, if the first equality in (4.7) is true, from (4.10) we directly deduce that

$$\psi(\ell, \lambda h, \nu) = \psi(\lambda^{-1}\ell, h, \nu) = \psi(\ell, h, \nu) \quad \text{for every } \lambda > 0.$$

Step 2. We continue establishing the first equality in (4.7) by showing that

$$\psi(N\ell, h, \nu) = \psi(\ell, h, \nu) \quad \text{for every } N \in \mathbb{N}. \tag{4.11}$$

For $N \in \mathbb{N}$ fixed let (u_ε) be a sequence of spin fields satisfying $\chi(u_\varepsilon) \to \chi_\nu$ in $L^1(R^\nu_{N\ell, h})$. We subdivide the rectangle $R^\nu_{N\ell, h}$ in N open rectangles of the form

$$R^\nu_{\ell, h}(x_m) \quad \text{with } x_m := \left(m - \frac{N-1}{2} \right) \ell \nu^\perp \text{ for } m \in \{0, \ldots, N-1\}.$$

Notice that $x \in R_{\ell,h}^\nu(x_m)$ if and only if

$$\left| \langle x, \nu^\perp \rangle - \left(m - \frac{N-1}{2} \right) \ell \right| < \frac{\ell}{2} \quad \text{and} \quad |\langle x, \nu \rangle| < \frac{h}{2} \,,$$

and therefore $R_{\ell,h}^\nu(x_m) \subset R_{N\ell,h}^\nu$ for all $m \in \{0, \dots, N-1\}$. By choosing $m_0 \in \{0, \dots, N-1\}$ such that $F_\varepsilon(u_\varepsilon, R_{\ell,h}^\nu(x_{m_0})) \leq F_\varepsilon(u_\varepsilon, R_{\ell,h}^\nu(x_m))$ for every $m \in \{0, \dots, N-1\}$ we obtain the estimate

$$\frac{1}{N\ell} F_\varepsilon(u_\varepsilon, R_{N\ell,h}^\nu) \geq \frac{1}{N\ell} \sum_{m=0}^{N-1} F_\varepsilon(u_\varepsilon, R_{\ell,h}^\nu(x_m)) \geq \frac{1}{\ell} F_\varepsilon(u_\varepsilon, R_{\ell,h}^\nu(x_{m_0})) \,. \tag{4.12}$$

We now define a suitable shifted version of u_ε whose energy is concentrated in a rectangle centered at zero. To this end, as in the proof of Proposition 4.1 it is convenient to write the vector $\nu^\perp$ in terms of the basis $\{\hat{e}_1, \hat{e}_2\}$ as $\nu^\perp = a_1 \hat{e}_1 + a_2 \hat{e}_2$ for some $a_1, a_2 \in \mathbb{R}$ and to introduce the vector $x_{m_0}^\varepsilon \in \mathcal{L}_\varepsilon$ given by

$$x_{m_0}^\varepsilon := \varepsilon \left\lfloor \frac{\left(m_0 - \frac{N-1}{2} \right) \ell a_1}{\varepsilon} \right\rfloor \hat{e}_1 + \varepsilon \left\lfloor \frac{\left(m_0 - \frac{N-1}{2} \right) \ell a_2}{\varepsilon} \right\rfloor \hat{e}_2 \,.$$

We then define spin fields $v_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ by setting $v_\varepsilon(z) := u_\varepsilon(z + x_{m_0}^\varepsilon)$. As in the proof of Proposition 4.1 we notice that $|x_{m_0}^\varepsilon - x_{m_0}| \leq 2\varepsilon$, $\chi(v_\varepsilon) \to \chi_\nu$ in $L^1(R_{\ell,h}^\nu)$ and $R_{\ell-4\varepsilon,h-4\varepsilon}^\nu(x_{m_0}^\varepsilon) \subset R_{\ell,h}^\nu(x_{m_0})$. Let us fix $\lambda \in (0,1)$ and $\varepsilon_\lambda > 0$ sufficiently small such that $\ell - 4\varepsilon_\lambda > \lambda\ell$, $h - 4\varepsilon_\lambda > \lambda h$. Then for every $\varepsilon \in (0, \varepsilon_\lambda)$ there holds $\mathcal{T}_\varepsilon(R_{\lambda\ell,\lambda h}^\nu(x_{m_0}^\varepsilon)) \subset \mathcal{T}_\varepsilon(R_{\ell,h}^\nu(x_{m_0}))$, hence

$$\frac{1}{\ell} F_\varepsilon(v_\varepsilon, R_{\lambda\ell,\lambda h}^\nu) \leq \frac{1}{\ell} F_\varepsilon(u_\varepsilon, R_{\ell,h}^\nu(x_{m_0})) \,.$$

Moreover, since v_ε is admissible for $\psi(\lambda\ell, \lambda h, \nu)$, we have

$$\lambda \psi(\lambda\ell, \lambda h, \nu) \leq \frac{1}{\ell} \liminf_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon, R_{\lambda\ell,\lambda h}^\nu) \,. \tag{4.13}$$

Combining (4.10) in Step 1 with (4.12)–(4.13), in view of the arbitrariness of u_ε we finally obtain

$$\lambda \psi(\ell, h, \nu) = \lambda \psi(\lambda\ell, \lambda h, \nu) \leq \psi(N\ell, h, \nu) \,.$$

Thus, by letting $\lambda \to 1$ we deduce that $\psi(\ell, h, \nu) \leq \psi(N\ell, h, \nu)$. Finally, (4.11) follows from (4.10) and (4.9) by observing that

$$\psi(\ell, h, \nu) \leq \psi(N\ell, h, \nu) = \psi(\ell, \tfrac{h}{N}, \nu) \leq \psi(\ell, h, \nu) \,.$$

Step 3. We prove the first equality in (4.7). Suppose first that $\lambda \in (0, +\infty) \cap \mathbb{Q}$. Then $\lambda = N/M$ for some $N, M \in \mathbb{N}$, hence applying twice (4.11) yields
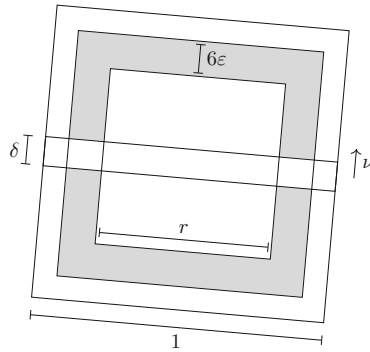
$$\psi(\lambda\ell, h, \nu) = \psi(\tfrac{N}{M}\ell, h, \nu) = \psi(\tfrac{1}{M}\ell, h, \nu) = \psi(M(\tfrac{1}{M}\ell), h, \nu) = \psi(\ell, h, \nu) \,. \tag{4.14}$$

Suppose now that $\lambda \in (0, +\infty)$ and let $(\lambda_n) \subset (0, +\infty) \cap \mathbb{Q}$ with $\lambda_n \to \lambda$ as $n \to +\infty$, $\lambda_n > \lambda$ for every $n \in \mathbb{N}$. Thanks to (4.10) and (4.14) we deduce that

$$\psi(\lambda\ell, h, \nu) = \psi(\lambda_n\ell, \tfrac{\lambda_n}{\lambda} h, \nu) = \psi(\ell, \tfrac{\lambda_n}{\lambda} h, \nu) \geq \psi(\ell, h, \nu) \,,$$

where the last inequality follows from (4.8), since $\lambda_n/\lambda > 1$. To prove the opposite inequality it suffices to take a sequence $(\lambda_n) \subset (0, +\infty) \cap \mathbb{Q}$ converging to λ with $\lambda_n < \lambda$. Then, arguing

Fig. 4 The grey region depicts a possible element $S_{\varepsilon,r} \in \mathscr{S}^v_{\varepsilon,\delta}$

as before and now applying (4.8) we obtain

$$\psi(\lambda\ell, h, v) = \psi(\ell, \tfrac{\lambda_n}{\lambda}h, v) \leq \psi(\ell, h, v),$$

hence equality follows.

□

On account of Lemma 4.3 we show that for a sequence (u_ε) realizing the infimum in the definition of $\psi(1, 1, v)$ the energy concentrates close to the line L^v. As a consequence, we obtain that outside a small neighborhood of L^v there exists a suitable strip on which the energy is of order $o(\varepsilon)$. To be more precise, for fixed $v \in \mathbb{S}^1$, $\delta > 0$, and every $\varepsilon > 0$ we introduce the class $\mathscr{S}^v_{\varepsilon,\delta}$ of strips

$$\mathscr{S}^v_{\varepsilon,\delta} := \left\{ Q^v_{r+12\varepsilon} \setminus \left(\overline{Q}^v_r \cup \overline{R}^v_{1,\delta} \right) : r \in (1 - 3\delta, 1 - 2\delta) \right\}. \tag{4.15}$$

We denote the elements of $\mathscr{S}^v_{\varepsilon,\delta}$ by $S_{\varepsilon,r}$ (see Fig. 4). Then the following result holds true.

Lemma 4.4 *Let $v \in \mathbb{S}^1$ and let (u_ε) be a sequence such that $\chi(u_\varepsilon) \to \chi_v$ in $L^1(Q^v)$ and $F_\varepsilon(u_\varepsilon, Q^v) \to \psi(1, 1, v)$. Then for every $\delta > 0$ there exists a sequence $\sigma_\varepsilon \to 0$ (depending on δ) and a strip $S_\varepsilon = S_{\varepsilon,r_\varepsilon} \in \mathscr{S}^v_{\varepsilon,\delta}$ such that*

$$F_\varepsilon(u_\varepsilon, S_\varepsilon) + \|\chi(u_\varepsilon) - \chi_v\|_{L^1(S_\varepsilon)} \leq \varepsilon\sigma_\varepsilon. \tag{4.16}$$

Proof Let $v \in \mathbb{S}^1$ and (u_ε) be as in the statement and let $\delta > 0$ be fixed. For every Borel set $A \subset Q^v$ set

$$G_\varepsilon(u_\varepsilon, A) := F_\varepsilon(u_\varepsilon, A) + \int_A |\chi(u_\varepsilon) - \chi_v| \, dx.$$

We consider for ε small enough the family of pairwise disjoint strips $S_{\varepsilon,r^m_\varepsilon} \in \mathscr{S}^v_{\varepsilon,\delta}$ with $r^m_\varepsilon = 1 - 3\delta + 12m\varepsilon$ and $m \in \{0, \ldots, \lfloor\frac{\delta}{12\varepsilon}\rfloor - 1\}$ and we notice that

$$\bigcup_{m=0}^{\lfloor\frac{\delta}{12\varepsilon}\rfloor-1} S_{\varepsilon,r^m_\varepsilon} \subset Q^v_{1-2\delta} \setminus (\overline{Q}^v_{1-3\delta} \cup \overline{R}^v_{1,\delta}) \subset Q^v \setminus \overline{R}^v_{1,\delta}.$$

This implies in particular that

$$\sum_{m=0}^{\lfloor \frac{\delta}{12\varepsilon} \rfloor - 1} G_\varepsilon(u_\varepsilon, S_{\varepsilon, r_\varepsilon^m}) \leq G_\varepsilon \left(u_\varepsilon, \bigcup_{m=0}^{\lfloor \frac{\delta}{12\varepsilon} \rfloor - 1} S_{\varepsilon, r_\varepsilon^m} \right) \leq F_\varepsilon(u_\varepsilon, Q^\nu \backslash \overline{R}_{1,\delta}^\nu) + \int_{Q^\nu \backslash \overline{R}_{1,\delta}^\nu} |\chi(u_\varepsilon) - \chi_\nu| \, dx.$$

Averaging over $m \in \{0, \ldots, \lfloor \frac{\delta}{12\varepsilon} \rfloor - 1\}$ we thus find $m(\varepsilon)$ such that the strip $S_{\varepsilon, r_\varepsilon^{m(\varepsilon)}}$ satisfies

$$G_\varepsilon(u_\varepsilon, S_{\varepsilon, r_\varepsilon^{m(\varepsilon)}}) \leq \left\lfloor \frac{\delta}{12\varepsilon} \right\rfloor^{-1} \left(F_\varepsilon(u_\varepsilon, Q^\nu \backslash \overline{R}_{1,\delta}^\nu) + \|\chi(u_\varepsilon) - \chi_\nu\|_{L^1(Q^\nu)} \right). \tag{4.17}$$

Notice that $F_\varepsilon(u_\varepsilon, Q^\nu \backslash \overline{R}_{1,\delta}^\nu) \to 0$ as $\varepsilon \to 0$. In fact, Lemma 4.3 together with the choice of (u_ε) yields

$$\psi(1, 1, \nu) = \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, Q^\nu) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, R_{1,\delta}^\nu) \geq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, R_{1,\delta}^\nu)$$
$$\geq \psi(1, \delta, \nu) = \psi(1, 1, \nu),$$

from which we readily deduce that $F_\varepsilon(u_\varepsilon, R_{1,\delta}^\nu) \to \psi(1, 1, \nu)$ as $\varepsilon \to 0$, hence

$$F_\varepsilon(u_\varepsilon, Q^\nu \backslash \overline{R}_{1,\delta}^\nu) \leq F_\varepsilon(u_\varepsilon, Q^\nu) - F_\varepsilon(u_\varepsilon, R_{1,\delta}^\nu) \to 0 \text{ as } \varepsilon \to 0.$$

Thus, in view of (4.17), it suffices to set $\sigma_\varepsilon := \frac{13}{\delta}(F_\varepsilon(u_\varepsilon, Q^\nu \backslash \overline{R}_{1,\delta}^\nu) + \|\chi(u_\varepsilon) - \chi_\nu\|_{L^1(Q^\nu)})$ and $r_\varepsilon := r_\varepsilon^{m(\varepsilon)}$ to find the required strip $S_{\varepsilon, r_\varepsilon} \in \mathscr{S}_{\varepsilon,\delta}^\nu$ satisfying (4.16). $\qquad\square$

We are now in a position to start with the interpolation procedure mentioned before. The final interpolation procedure will be based on a one-dimensional construction that we introduce below.

4.1 One-dimensional interpolation

To define the one-dimensional interpolation we consider slices in the triangular lattice. To this end, let $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ be as in Section 2.2. Given $\alpha \in \{1, 2, 3\}$ we consider the orthogonal vector $\hat{e}_\alpha^\perp$ to $\hat{e}_\alpha$ and we define the slice in the direction $\hat{e}_\alpha$ by

$$\Sigma^\alpha := \left\{ s\hat{e}_\alpha + t\hat{e}_\alpha^\perp : s \in \mathbb{R}, \ t \in [0, \tfrac{\sqrt{3}}{2}] \right\}.$$
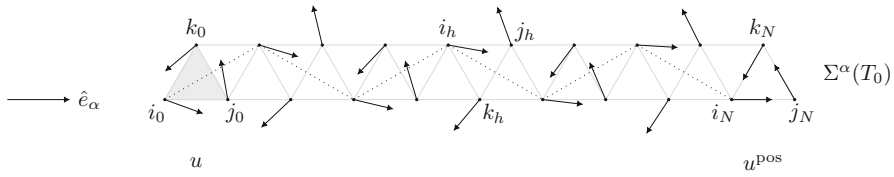
Given $z \in \mathbb{Z}$, we define

$$\Sigma^{\alpha,z} := \Sigma^\alpha + \tfrac{\sqrt{3}}{2} z \, \hat{e}_\alpha^\perp = \left\{ s\hat{e}_\alpha + t\hat{e}_\alpha^\perp : s \in \mathbb{R}, \ t \in [\tfrac{\sqrt{3}}{2}z, \tfrac{\sqrt{3}}{2}(z + 1)] \right\}.$$

Finally, for every ε we set

$$\Sigma_\varepsilon^{\alpha,z} := \varepsilon \Sigma^{\alpha,z}. \tag{4.18}$$

We shall define the one-dimensional interpolation in a slice Σ^α starting from a triangle $T_0 \in \mathcal{T}(\mathbb{R}^2)$ such that $T_0 \subset \Sigma^\alpha$. Let us denote by $i_0 \in \mathcal{L}^1$, $j_0 \in \mathcal{L}^2$, $k_0 \in \mathcal{L}^3$ the vertices of T_0. Note that $\langle i_0, \hat{e}_\alpha^\perp \rangle, \langle j_0, \hat{e}_\alpha^\perp \rangle, \langle k_0, \hat{e}_\alpha^\perp \rangle \in \{0, \tfrac{\sqrt{3}}{2}\}$. We define the lattice points $i_h \in \mathcal{L}^1$, $j_h \in \mathcal{L}^2$, $k_h \in \mathcal{L}^3$ and the triangle T_h with the following recursive formula: we set $\tau(0) := 1$,

Fig. 5 Example of interpolation from u to u^{pos} in the slice $\Sigma^\alpha(T_0)$ starting from the triangle T_0 (in grey)

$\tau(\frac{\sqrt{3}}{2}) := -1$ and for $h \in \mathbb{N}$

$$
\begin{aligned}
i_{h+1} &:= i_h + \hat{e}_\alpha + \tfrac{1}{2}\hat{e}_\alpha + \tau(\langle i_h, \hat{e}_\alpha^\perp \rangle)\tfrac{\sqrt{3}}{2}\hat{e}_\alpha^\perp \,, \\
j_{h+1} &:= j_h + \hat{e}_\alpha + \tfrac{1}{2}\hat{e}_\alpha + \tau(\langle j_h, \hat{e}_\alpha^\perp \rangle)\tfrac{\sqrt{3}}{2}\hat{e}_\alpha^\perp \,, \\
k_{h+1} &:= k_h + \hat{e}_\alpha + \tfrac{1}{2}\hat{e}_\alpha + \tau(\langle k_h, \hat{e}_\alpha^\perp \rangle)\tfrac{\sqrt{3}}{2}\hat{e}_\alpha^\perp \,, \\
T_{h+1} &:= \operatorname{conv}\{i_{h+1}, j_{h+1}, k_{h+1}\} \subset \Sigma^\alpha \,,
\end{aligned}
\tag{4.19}
$$

(see Fig. 5). Observe that $\tau(\langle i_{h+1}, \hat{e}_\alpha^\perp \rangle) = -\tau(\langle i_h, \hat{e}_\alpha^\perp \rangle)$, the analogous equality being true also for j_h and k_h. Moreover, $T_{2h} = T_0 + 3h\hat{e}_\alpha$.

We define the half-slice $\Sigma^\alpha(T_0)$ of the lattice $\mathcal{L}$ starting from T_0 by

$$
\Sigma^\alpha(T_0) := \operatorname{conv}\{T_h : h \in \mathbb{N}\}\,.
\tag{4.20}
$$

Given $u \colon \mathcal{L} \to \mathbb{S}^1$ and $N, m \in \mathbb{N}$, we now define in the half-slice $\Sigma^\alpha(T_0)$ a one-parameter family (parametrized by m) of spin fields which coincides with u on T_0 and with the fixed ground state u^{pos} on T_h for $h \geq N$. We construct the interpolation in such a way that the configuration of spins rotates a fixed amount of times by 2π. To make the construction precise, we first say that the three angles $\theta(i_0) \in \mathbb{R}$ (not necessarily in $[0, 2\pi)$), $\theta(j_0) \in [\theta(i_0) - \pi, \theta(i_0) + \pi)$ and $\theta(k_0) \in [\theta(j_0) - \pi, \theta(j_0) + \pi)$ represent a lifting of u in T_0 if $u(i_0) = \exp(\iota\theta(i_0))$, $u(j_0) = \exp(\iota\theta(j_0))$ and $u(k_0) = \exp(\iota\theta(k_0))$. We then define the interpolated angles $\theta(i_h), \theta(j_h), \theta(k_h)$ for $h = 0, \ldots, N$ by

$$
\begin{aligned}
\theta(i_h) &:= \theta(i_0) + h\frac{2\pi m - \theta(i_0)}{N} = \left(1 - \frac{h}{N}\right)\theta(i_0) + \frac{h}{N}2\pi m \,, \\
\theta(j_h) &:= \theta(j_0) + h\frac{2\pi m + \frac{2\pi}{3} - \theta(j_0)}{N} = \left(1 - \frac{h}{N}\right)\theta(j_0) + \frac{h}{N}2\pi m + \frac{h}{N}\frac{2\pi}{3} \,, \\
\theta(k_h) &:= \theta(k_0) + h\frac{2\pi m + \frac{4\pi}{3} - \theta(k_0)}{N} = \left(1 - \frac{h}{N}\right)\theta(k_0) + \frac{h}{N}2\pi m + \frac{h}{N}\frac{4\pi}{3} \,,
\end{aligned}
\tag{4.21}
$$

and $\theta(i_h) := 2\pi m$, $\theta(j_h) := 2\pi m + \frac{2\pi}{3}$, $\theta(k_h) := 2\pi m + \frac{4\pi}{3}$ for $h \geq N + 1$ (see Fig. 5). Eventually, we define $u^{N,m} \colon \mathcal{L} \cap \Sigma^\alpha(T_0) \to \mathbb{S}^1$ by setting

$$
u^{N,m}(i_h) := \exp(\iota\theta(i_h))\,, \quad u^{N,m}(j_h) := \exp(\iota\theta(j_h))\,, \quad u^{N,m}(k_h) := \exp(\iota\theta(k_h))\,.
\tag{4.22}
$$

Note that $u^{N,m} = u^{\text{pos}}$ on T_h for $h \geq N$.

In the next lemma we estimate the energy of the interpolation on $\Sigma^\alpha(T_0)$ in terms of the energy on the initial triangle T_0 plus an error depending on the number of steps N and on m. We assume that the configuration of spins in the initial triangle is sufficiently close to a ground state with chirality 1 (not necessarily u^{pos}).

Lemma 4.5 *Let $T_0 \in \mathcal{T}(\mathbb{R}^2)$ be a triangle of vertices $i_0 \in \mathcal{L}^1$, $j_0 \in \mathcal{L}^2$, and $k_0 \in \mathcal{L}^3$. Let $u \colon \mathcal{L} \to \mathbb{S}^1$ and let $\theta(i_0) \in \mathbb{R}$, $\theta(j_0) \in [\theta(i_0) - \pi, \theta(i_0) + \pi)$ and $\theta(k_0) \in [\theta(j_0) - \pi, \theta(j_0) + \pi)$ be three angles representing a lifting of u in T_0 satisfying*

$$\left| \theta(j_0) - \theta(i_0) - \frac{2\pi}{3} \right| \leq \frac{1}{4}, \quad \left| \theta(k_0) - \theta(j_0) - \frac{2\pi}{3} \right| \leq \frac{1}{4}. \tag{4.23}$$

Let $N, m \in \mathbb{N}$ and assume that

$$2\pi m \geq |\theta(i_0)| + 2\pi. \tag{4.24}$$

Let $u^{N,m}$ be the interpolation on $\Sigma^\alpha(T_0)$ defined according to (4.22). Then there exists a constant $C > 0$ independent of N and m such that

$$F_1(u^{N,m}, \Sigma^\alpha(T_0)) \leq C \left(N F_1(u, T_0) + \frac{m^2}{N} \right).$$

Proof It is not restrictive to assume that $j_0 - i_0 = \hat{e}_\alpha$ as in Fig. 5. We shall estimate each of the terms in the sum

$$
\begin{aligned}
F_1(u^{N,m}, \Sigma^\alpha(T_0)) &= \sum_{h=0}^{N-1} |u^{N,m}(i_h) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2 \\
&+ \sum_{h=0}^{N-1} |u^{N,m}(i_{h+1}) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2 \\
&+ \sum_{h=0}^{N-1} |u^{N,m}(i_{h+1}) + u^{N,m}(j_h) + u^{N,m}(k_{h+1})|^2,
\end{aligned}
\tag{4.25}
$$

where we used that for $h \geq N$ we have that

$$|u^{N,m}(i_h) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2 = |u^{\text{pos}}(i_h) + u^{\text{pos}}(j_h) + u^{\text{pos}}(k_h)|^2 = 0,$$

being u^{pos} a ground state. Adopting the notation for the angles used in the construction in (4.22), we recast the energy in the first term of the sum as

$$
\begin{aligned}
&|u^{N,m}(i_h) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2 \\
&= 3 + 2\cos(\theta(j_h) - \theta(i_h)) + 2\cos(\theta(k_h) - \theta(j_h)) + 2\cos(\theta(i_h) - \theta(k_h)).
\end{aligned}
\tag{4.26}
$$

Note that, by (4.21) and (4.23),

$$
\begin{aligned}
\left| \theta(j_h) - \theta(i_h) - \frac{2\pi}{3} \right| &\leq \left| \theta(j_0) - \theta(i_0) - \frac{2\pi}{3} \right| \leq \frac{1}{4}, \\
\left| \theta(k_h) - \theta(j_h) - \frac{2\pi}{3} \right| &\leq \left| \theta(k_0) - \theta(j_0) - \frac{2\pi}{3} \right| \leq \frac{1}{4}, \\
\left| \theta(k_h) - \theta(i_h) - \frac{4\pi}{3} \right| &\leq \left| \theta(k_0) - \theta(i_0) - \frac{4\pi}{3} \right| \leq \frac{1}{2}.
\end{aligned}
\tag{4.27}
$$

By Taylor's formula, there exists $\zeta \in [\phi, 2\pi/3]$ such that $1 + 2\cos(\phi) = -\sqrt{3}(\phi - \frac{2\pi}{3}) + \frac{1}{2}(\phi - \frac{2\pi}{3})^2 + \frac{1}{3}\sin(\zeta)(\phi - \frac{2\pi}{3})^3$. As a result we obtain the estimates

$$\frac{1}{3}\left(\phi - \frac{2\pi}{3} \right)^2 \leq 1 + 2\cos(\phi) + \sqrt{3}\left(\phi - \frac{2\pi}{3} \right) \leq \frac{2}{3}\left(\phi - \frac{2\pi}{3} \right)^2, \quad \text{for } \left| \phi - \frac{2\pi}{3} \right| \leq \frac{1}{2}.$$

Analogously,

$$\frac{1}{3}\left(\phi - \frac{4\pi}{3}\right)^2 \le 1 + 2\cos(\phi) - \sqrt{3}\left(\phi - \frac{4\pi}{3}\right) \le \frac{2}{3}\left(\phi - \frac{4\pi}{3}\right)^2, \quad \text{for } \left|\phi - \frac{4\pi}{3}\right| \le \frac{1}{2}.$$

Then by (4.26), (4.27), and the two previous estimates we infer that

$$|u^{N,m}(i_h) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2$$

$$\le \frac{2}{3}\left[\left(\theta(j_h) - \theta(i_h) - \frac{2\pi}{3}\right)^2 + \left(\theta(k_h) - \theta(j_h) - \frac{2\pi}{3}\right)^2 + \left(\theta(k_h) - \theta(i_h) - \frac{4\pi}{3}\right)^2\right]$$

$$\le \frac{2}{3}\left[\left(\theta(j_0) - \theta(i_0) - \frac{2\pi}{3}\right)^2 + \left(\theta(k_0) - \theta(j_0) - \frac{2\pi}{3}\right)^2 + \left(\theta(k_0) - \theta(i_0) - \frac{4\pi}{3}\right)^2\right]$$

$$\le \frac{2}{3}3|u^{N,m}(i_0) + u^{N,m}(j_0) + u^{N,m}(k_0)|^2 = 2F_1(u, T_0).$$

This proves that

$$\sum_{h=0}^{N-1} |u^{N,m}(i_h) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2 \le 2NF_1(u, T_0).$$

Let us now consider the second term in the sum in the right-hand side of (4.25). For every $h = 0, \dots, N-1$ we have

$$|u^{N,m}(i_{h+1}) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2$$

$$\le 2|u^{N,m}(i_h) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2 + 2|u^{N,m}(i_{h+1}) - u^{N,m}(i_h)|^2.$$

The first term is estimated as via $2F_1(u, T_0)$. As for $|u^{N,m}(i_{h+1}) - u^{N,m}(i_h)|^2$, by (4.21) we have that

$$|u^{N,m}(i_{h+1}) - u^{N,m}(i_h)|^2 = 2 - 2\cos(\theta(i_{h+1}) - \theta(i_h)) = 2 - 2\cos\left(\frac{2\pi m - \theta(i_0)}{N}\right).$$

Using the fact that $1 - \cos(t) \le \frac{t^2}{2}$ we deduce

$$|u^{N,m}(i_{h+1}) - u^{N,m}(i_h)|^2 \le \left(\frac{2\pi m - \theta(i_0)}{N}\right)^2 \le C\frac{m^2}{N^2},$$

since $|\theta(i_0)| \le 2\pi m$. Hence

$$\sum_{h=0}^{N-1} |u^{N,m}(i_{h+1}) + u^{N,m}(j_h) + u^{N,m}(k_h)|^2 \le CNF_1(u, T_0) + C\frac{m^2}{N}.$$

The third term in the right-hand side in (4.25) is treated analogously using the inequality

$$|u^{N,m}(k_{h+1}) - u^{N,m}(k_h)|^2 = 2 - 2\cos(\theta(k_{h+1}) - \theta(k_h)) = 2 - 2\cos\left(\frac{2\pi m + \frac{4\pi}{3} - \theta(k_0)}{N}\right)$$

$$\le \left(\frac{2\pi m + \frac{4\pi}{3} - \theta(k_0)}{N}\right)^2 \le C\frac{m^2}{N^2},$$

where we used (4.23) to get that $|\theta(k_0)| + \frac{4\pi}{3} \le |\theta(i_0)| + \frac{1}{2} + \frac{4\pi}{3} \le |\theta(i_0)| + 2\pi \le 2\pi(m+1)$. $\qquad\square$

We are now in a position to prove Proposition 4.2 and thus conclude the proof of the lower bound in Proposition 4.1.

Proof of Proposition 4.2 For the reader's convenience, we recall here the definitions of $\varphi(\nu)$ and $\psi(1, 1, \nu)$:

$$\varphi(\nu) = \lim_{\varepsilon \to 0} \min\{F_\varepsilon(u, Q^\nu) : u = u_\varepsilon^{\text{pos}} \text{ on } \partial_\varepsilon^+ Q^\nu \text{ and } u = u_\varepsilon^{\text{neg}} \text{ on } \partial_\varepsilon^- Q^\nu\},$$

$$\psi(1, 1, \nu) = \inf\left\{\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, Q^\nu) : \chi(u_\varepsilon) \to \chi_\nu \text{ in } L^1(Q^\nu)\right\}.$$

Let us fix a sequence (u_ε) such that $\chi(u_\varepsilon) \to \chi_\nu$ in $L^1(Q^\nu)$ and $F_\varepsilon(u_\varepsilon, Q^\nu) \to \psi(1, 1, \nu)$. The aim of this proof is to define a modification $\widetilde{u}_\varepsilon$ of u_ε such that

$$\widetilde{u}_\varepsilon = u_\varepsilon^{\text{pos}} \text{ on } \partial_\varepsilon^+ Q^\nu \text{ and } \widetilde{u}_\varepsilon = u_\varepsilon^{\text{neg}} \text{ on } \partial_\varepsilon^- Q^\nu, \tag{4.28}$$

$$\limsup_{\varepsilon \to 0} F_\varepsilon(\widetilde{u}_\varepsilon, Q^\nu) \leq \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, Q^\nu). \tag{4.29}$$

This allows us to conclude that $\varphi(\nu) \leq \psi(1, 1, \nu)$.

The construction of the modified sequence $(\widetilde{u}_\varepsilon)$ is divided in several steps.

Step 1. (Choosing a strip with low energy). We begin the construction by exploiting the property that the energy of (u_ε) concentrates close to the interface $Q^\nu \cap L^\nu$ in order to choose a strip with low energy. Given $\delta \in (0, \frac{1}{3})$, we consider the family of strips $\mathscr{S}_{\varepsilon,\delta}^\nu$ defined in (4.15) and we apply Lemma 4.4 to deduce the existence of a strip $S_\varepsilon = S_{\varepsilon,r_\varepsilon} = Q_{r_\varepsilon+12\varepsilon}^\nu \setminus \left(\overline{Q}_{r_\varepsilon}^\nu \cup \overline{R}_{1,\delta}^\nu\right) \in \mathscr{S}_{\varepsilon,\delta}^\nu$ such that

$$F_\varepsilon(u_\varepsilon, S_\varepsilon) + \|\chi(u_\varepsilon) - \chi_\nu\|_{L^1(S_\varepsilon)} \leq \varepsilon \sigma_\varepsilon, \tag{4.30}$$

where $\sigma_\varepsilon \to 0$. The modification $\widetilde{u}_\varepsilon$ of u_ε will coincide with $u_\varepsilon^{\text{pos}}$ and $u_\varepsilon^{\text{neg}}$ in $Q^\nu \setminus (\overline{Q}_{1-\delta}^\nu \cup \overline{R}_{1,\delta}^\nu)$ (notice that the square $Q_{1-\delta}^\nu$ contains the closure of S_ε, *cf.* (4.15)). In the triangles contained in S_ε the energy is low and thus u_ε is close to ground states, yet not necessarily $u_\varepsilon^{\text{pos}}$ or $u_\varepsilon^{\text{neg}}$. There $\widetilde{u}_\varepsilon$ will start to interpolate from the configuration u_ε until it reaches the fixed ground state $u_\varepsilon^{\text{pos}}$ or $u_\varepsilon^{\text{neg}}$ close to the boundary.

We shall describe in detail how to define $\widetilde{u}_\varepsilon$ in the top part of the cube given by $Q_+^\nu = Q^\nu \cap \{x : \langle x, \nu \rangle > 0\}$, where the chirality of u_ε converges to 1. The construction in $Q_-^\nu \cap \{x : \langle x, \nu \rangle < 0\}$ is completely analogous.

Step 2. (Choosing triangles with low energy). We show here how to choose the triangles with low energy where to start the modification of u_ε. Let us consider the line

$$L_\varepsilon := \{x \in \mathbb{R}^2 : \langle x, \nu \rangle = \tfrac{r_\varepsilon}{2} + 3\varepsilon\},$$

which cuts in two the top part of the strip given by the rectangle

$$S_\varepsilon^{\text{top}} := R_{r_\varepsilon,6\varepsilon}^\nu \left((\tfrac{r_\varepsilon}{2} + 3\varepsilon)\nu\right) = (L_\varepsilon + B_{3\varepsilon}(0)) \cap R_{r_\varepsilon,1}^\nu \subset S_\varepsilon. \tag{4.31}$$

We describe now how to start the modification in $S_\varepsilon^{\text{top}}$. The modification in the other parts

$$S_\varepsilon^{\text{left}} := R_{r_\varepsilon,6\varepsilon}^{\nu^\perp} \left((\tfrac{r_\varepsilon}{2} + 3\varepsilon)\nu^\perp\right) \setminus \overline{R}_{1,\delta}^\nu,$$

$$S_\varepsilon^{\text{right}} := R_{r_\varepsilon,6\varepsilon}^{\nu^\perp} \left(-(\tfrac{r_\varepsilon}{2} + 3\varepsilon)\nu^\perp\right) \setminus \overline{R}_{1,\delta}^\nu,$$

cf. Figure 6, will be only sketched since it is completely analogous.

We consider now the slices $(\Sigma_\varepsilon^{\alpha,z})_{z \in \mathbb{Z}}$ of the ε-triangular lattice defined in (4.18). We choose $\alpha \in \{1, 2, 3\}$ such that $|\langle \hat{e}_\alpha, \nu \rangle| \geq \frac{\sqrt{3}}{2}$, namely the best approximation of ν in the
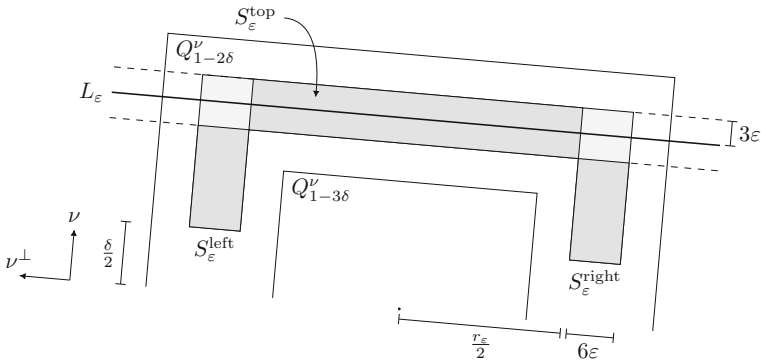
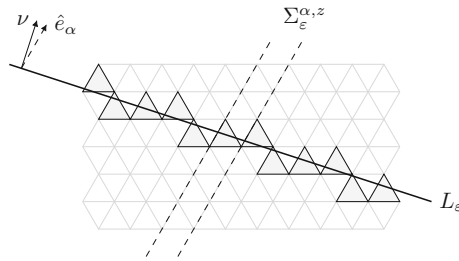**Fig. 6** Parts of the strip S_ε in Q_+^ν

Fig. 7 A chain of triangles $(T_z)_{z\in\mathbb{Z}}$

set $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. Equivalently, $|\langle\hat{e}_\alpha, \nu^\perp\rangle| \leq \frac{1}{2}$, where $\nu^\perp$ is the direction of L_ε. (For $S_\varepsilon^{\mathrm{right}}$ and $S_\varepsilon^{\mathrm{left}}$ we consider a different direction, namely $\beta \in \{1, 2, 3\}$ such that $|\langle\hat{e}_\beta, \nu^\perp\rangle| \geq \frac{\sqrt{3}}{2}$.)

We can find a chain of closed triangles which intersect L_ε such that each slice in the direction $\hat{e}_\alpha$ contains only one triangle of the chain. Specifically, there exist $(T_z)_{z\in\mathbb{Z}}$, satisfying

$$T_z \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2)\,, \quad T_z \subset \Sigma_\varepsilon^{\alpha,z}\,, \quad T_z \cap L_\varepsilon \neq \emptyset\,, \quad T_z \cap T_{z+1} \neq \emptyset\,, \tag{4.32}$$

for every $z \in \mathbb{Z}$, *cf.* Fig. 7. We prove this statement in Lemma 4.6 below, since the geometric argument is irrelevant for the present discussion.

The modification of u_ε starts in the triangles T_z of the chain contained in $S_\varepsilon^{\mathrm{top}}$. For this reason it is convenient to consider

$$\mathcal{Z}_\varepsilon^{\mathrm{top}} := \{z \in \mathbb{Z} : T_z \subset S_\varepsilon^{\mathrm{top}}\} \quad \text{and} \quad z_0 \in \operatorname{argmin}\mathcal{Z}_\varepsilon^{\mathrm{top}}\,.$$

For future purposes we observe that

$$\frac{\sqrt{3}}{4}\varepsilon^2 \#\mathcal{Z}_\varepsilon^{\mathrm{top}} = \Big|\bigcup_{z\in\mathcal{Z}_\varepsilon^{\mathrm{top}}} T_z\Big| \leq |S_\varepsilon^{\mathrm{top}}| = 6\varepsilon r_\varepsilon \leq 6\varepsilon \implies \#\mathcal{Z}_\varepsilon^{\mathrm{top}} \leq \frac{C_1}{\varepsilon}\,, \tag{4.33}$$

for some positive constant C_1 and for ε small enough.

Step 3. (Estimating the maximal winding number). The energy regime we are working in does not rule out the possibility that inside the strip S_ε the configuration of spin field displays global rotations. However, the bound of the energy in S_ε allows us to estimate the maximal number of complete turns of 2π. To present precisely the estimate, we define in the triangles

chosen in Step 2 the liftings $\theta_\varepsilon \in \mathbb{R}$ of u_ε according to the following recursive argument. Given $z \in \mathcal{Z}_\varepsilon^{\text{top}}$ we denote by $i_z \in \mathcal{L}^1$, $j_z \in \mathcal{L}^2$, $k_z \in \mathcal{L}^3$ the points in the sublattices such that εi_z, εj_z, εk_z are the vertices of the triangle T_z (some points might have multiple labels). We now define recursively angles $\theta_\varepsilon(\varepsilon i_z)$, $\theta_\varepsilon(\varepsilon j_z)$, $\theta_\varepsilon(\varepsilon k_z)$ in suitably chosen intervals of length 2π satisfying $u_\varepsilon(\varepsilon i_z) = \exp(\iota\theta_\varepsilon(\varepsilon i_z))$, $u_\varepsilon(\varepsilon j_z) = \exp(\iota\theta_\varepsilon(\varepsilon j_z))$, $u_\varepsilon(\varepsilon k_z) = \exp(\iota\theta_\varepsilon(\varepsilon k_z))$ as follows. We choose

$$
\begin{aligned}
\theta_\varepsilon(\varepsilon i_{z_0}) &\in [0, 2\pi), \\
\theta_\varepsilon(\varepsilon j_{z_0}) &\in [\theta_\varepsilon(\varepsilon i_{z_0}) - \pi, \theta_\varepsilon(\varepsilon i_{z_0}) + \pi), \\
\theta_\varepsilon(\varepsilon k_{z_0}) &\in [\theta_\varepsilon(\varepsilon j_{z_0}) - \pi, \theta_\varepsilon(\varepsilon j_{z_0}) + \pi), \\
\theta_\varepsilon(\varepsilon i_{z+1}) &\in [\theta_\varepsilon(\varepsilon i_z) - \pi, \theta_\varepsilon(\varepsilon i_z) + \pi).
\end{aligned}
$$

The choice of $\theta_\varepsilon(\varepsilon j_z)$ and $\theta_\varepsilon(\varepsilon k_z)$ is made according to the same recursive procedure above, but taking as starting point (instead of $\theta_\varepsilon(\varepsilon i_{z_0})$) the angles $\theta_\varepsilon(\varepsilon j_{z_0})$ and $\theta_\varepsilon(\varepsilon k_{z_0})$, respectively. We claim that

$$
\frac{1}{2\pi} \sup_{\substack{z \in \mathcal{Z}_\varepsilon^{\text{top}} \\ z \geq z_0}} \left\{ |\theta_\varepsilon(\varepsilon i_z) - \theta_\varepsilon(\varepsilon i_{z_0})|, \ |\theta_\varepsilon(\varepsilon j_z) - \theta_\varepsilon(\varepsilon j_{z_0})|, \ |\theta_\varepsilon(\varepsilon k_z) - \theta_\varepsilon(\varepsilon k_{z_0})| \right\} \leq C_2 \sqrt{\frac{\sigma_\varepsilon}{\varepsilon}},
$$

(4.34)

for some positive constant C_2. To prove the claim, let us fix $z_* \in \mathcal{Z}_\varepsilon^{\text{top}}$, $z_* \geq z_0$. Note that $z_* - z_0 \leq \frac{C_1}{\varepsilon}$ by (4.33). Jensen's inequality implies that

$$
\begin{aligned}
&|\theta_\varepsilon(\varepsilon i_{z_*}) - \theta_\varepsilon(\varepsilon i_{z_0})|^2 \\
&\leq \left(\sum_{z=z_0}^{z_*-1} |\theta_\varepsilon(\varepsilon i_{z+1}) - \theta_\varepsilon(\varepsilon i_z)| \right)^2 \leq (z_* - z_0) \sum_{z=z_0}^{z_*-1} |\theta_\varepsilon(\varepsilon i_{z+1}) - \theta_\varepsilon(\varepsilon i_z)|^2 \\
&\leq \frac{C_1}{\varepsilon} \sum_{z=z_0}^{z_*-1} |\theta_\varepsilon(\varepsilon i_{z+1}) - \theta_\varepsilon(\varepsilon i_z)|^2 \leq \frac{C}{\varepsilon} \sum_{z=z_0}^{z_*-1} 2 - 2\cos(\theta_\varepsilon(\varepsilon i_{z+1}) - \theta_\varepsilon(\varepsilon i_z)) \\
&= \frac{C}{\varepsilon} \sum_{z=z_0}^{z_*-1} |u_\varepsilon(\varepsilon i_{z+1}) - u_\varepsilon(\varepsilon i_z)|^2
\end{aligned}
$$

(4.35)

for some positive constants C, where we used the fact that $1 - \cos(\phi) \geq \frac{1}{12}\phi^2$ for every $|\phi| \leq \pi$. We start observing that the regular hexagon H_z containing T_z and T_{z+1} is contained in S_ε. Indeed, let $x \in H_z$ and let $y \in T_z \cap L_\varepsilon \subset S_\varepsilon^{\text{top}}$. Then $\text{dist}(x, L_\varepsilon) \leq |x - y| \leq \text{diam } H_z = 2\varepsilon < 3\varepsilon$. Hence, cf. (4.31), $x \in (L_\varepsilon + B_{3\varepsilon}(0)) \cap R_{r_\varepsilon+6\varepsilon,1}^\nu \subset S_\varepsilon$. Let us show that

$$
|u_\varepsilon(\varepsilon i_{z+1}) - u_\varepsilon(\varepsilon i_z)|^2 \leq \frac{2}{\varepsilon} F_\varepsilon(u_\varepsilon, H_z).
$$

(4.36)

Indeed, if $T_z \cap T_{z+1} = \{\varepsilon i_z\} = \{\varepsilon i_{z+1}\}$, then $|u_\varepsilon(\varepsilon i_{z+1}) - u_\varepsilon(\varepsilon i_z)|^2 = 0$; if $T_z \cap T_{z+1} = \{\varepsilon j_z\} = \{\varepsilon j_{z+1}\}$ (and analogously if $T_z \cap T_{z+1} = \{\varepsilon k_z\} = \{\varepsilon k_{z+1}\}$), then we let T' be the third triangle in $\text{conv}\{T_z, T_{z+1}\}$. The triangle T' is either $\text{conv}\{\varepsilon i_z, \varepsilon j_z, \varepsilon k_{z+1}\}$ or $\text{conv}\{\varepsilon i_{z+1}, \varepsilon j_z, \varepsilon k_z\}$ and is always contained in H_z, see Fig. 8. Letting εk be its vertex in
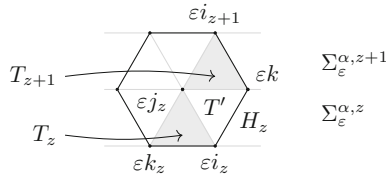
Fig. 8 Triangle T' in a possible configuration of T_z and T_{z+1}

$\mathcal{L}_\varepsilon^3$ (either εk_z or εk_{z+1}) we have that

$$|u_\varepsilon(\varepsilon i_{z+1}) - u_\varepsilon(\varepsilon i_z)|^2$$
$$\leq 2\,|u_\varepsilon(\varepsilon i_{z+1}) + u_\varepsilon(\varepsilon j_{z+1}) + u_\varepsilon(\varepsilon k)|^2 + 2\,|u_\varepsilon(\varepsilon i_z) + u_\varepsilon(\varepsilon j_z) + u_\varepsilon(\varepsilon k)|^2$$
$$\leq \frac{2}{\varepsilon} F_\varepsilon(u_\varepsilon, H_z)\,.$$

Then we estimate the last sum in (4.35) using (4.36) by

$$\sum_{z=z_0}^{z_*-1} |u_\varepsilon(\varepsilon i_{z+1}) - u_\varepsilon(\varepsilon i_z)|^2 \leq \sum_{z=z_0}^{z_*-1} \frac{2}{\varepsilon} F_\varepsilon(u_\varepsilon, H_z) \leq \frac{C}{\varepsilon} F_\varepsilon(u_\varepsilon, S_\varepsilon) \leq C\sigma_\varepsilon\,,$$

for some positive constant C. In conclusion, by (4.35) we have that

$$|\theta_\varepsilon(\varepsilon i_{z_*}) - \theta_\varepsilon(\varepsilon i_{z_0})| \leq C\sqrt{\frac{\sigma_\varepsilon}{\varepsilon}}\,.$$

Arguing in an analogous way for $|\theta_\varepsilon(\varepsilon j_{z_*}) - \theta_\varepsilon(\varepsilon j_{z_0})|$ and $|\theta_\varepsilon(\varepsilon k_{z_*}) - \theta_\varepsilon(\varepsilon k_{z_0})|$, we conclude the proof of the claim (4.34).

We consider the bound on the maximal winding number given by

$$m_\varepsilon := \left\lceil C_2\sqrt{\frac{\sigma_\varepsilon}{\varepsilon}} \right\rceil + 8\,, \tag{4.37}$$

where $\lceil C_2\sqrt{\frac{\sigma_\varepsilon}{\varepsilon}} \rceil$ is the smallest natural number grater than or equal to $C_2\sqrt{\frac{\sigma_\varepsilon}{\varepsilon}}$ and C_2 is the constant given in (4.34).

Step 4. (Modification on slices). We define the modification on the slices $\Sigma_\varepsilon^{\alpha,z}$ starting from triangles T_z with $z \in \mathcal{Z}_\varepsilon^{\text{top}}$ by reproducing the construction in Lemma 4.5. Here we make precise the choice of the parameters for this construction and the notation. Let us assume, without loss of generality, that $\langle \hat{e}_\alpha, \nu \rangle \geq \frac{\sqrt{3}}{2}$ (if, instead, $\langle \hat{e}_\alpha, \nu \rangle \leq -\frac{\sqrt{3}}{2}$ we work with $-\hat{e}_\alpha$). For $z \in \mathcal{Z}_\varepsilon^{\text{top}}$ we let $i_z^0 := i_z \in \mathcal{L}^1$, $j_z^0 := j_z \in \mathcal{L}^2$, $k_z^0 := k_z \in \mathcal{L}^3$ where $\varepsilon i_z, \varepsilon j_z, \varepsilon k_z$ are the vertices of T_z. As in (4.19), we define the lattice points $i_z^h \in \mathcal{L}^1$, $j_z^h \in \mathcal{L}^2$, $k_z^h \in \mathcal{L}^3$ and the triangle T_z^h with the following recursive formula: for $h \in \mathbb{N}$ we set

$$\begin{aligned}
i_z^{h+1} &:= i_z^h + \hat{e}_\alpha + \tfrac{1}{2}\hat{e}_\alpha + \tau(\langle i_z^h, \hat{e}_\alpha^\perp \rangle)\tfrac{\sqrt{3}}{2}\hat{e}_\alpha^\perp\,, \\
j_z^{h+1} &:= j_z^h + \hat{e}_\alpha + \tfrac{1}{2}\hat{e}_\alpha + \tau(\langle j_z^h, \hat{e}_\alpha^\perp \rangle)\tfrac{\sqrt{3}}{2}\hat{e}_\alpha^\perp\,, \\
k_z^{h+1} &:= k_z^h + \hat{e}_\alpha + \tfrac{1}{2}\hat{e}_\alpha + \tau(\langle k_z^h, \hat{e}_\alpha^\perp \rangle)\tfrac{\sqrt{3}}{2}\hat{e}_\alpha^\perp\,, \\
T_z^{h+1} &:= \text{conv}\{\varepsilon i_z^{h+1}, \varepsilon j_z^{h+1}, \varepsilon k_z^{h+1}\} \subset \Sigma_\varepsilon^{\alpha,z}\,,
\end{aligned} \tag{4.38}$$

where $\tau(0) := 1$, $\tau(\frac{\sqrt{3}}{2}) := -1$. As in (4.20), we define the half-slice $\Sigma_\varepsilon^{\alpha,z}(T_z)$ of the lattice $\mathcal{L}_\varepsilon$ starting from T_z by $\Sigma_\varepsilon^{\alpha,z}(T_z) := \text{conv}\{T_z^h : h \in \mathbb{N}\}$.

Number of interpolation steps. The number of interpolation steps will be defined by finding the first shifted triangle $T_{z_0}^{2h}$ in the half-slice $\Sigma_\varepsilon^{\alpha,z_0}(T_{z_0})$ that is well contained in $R_{\infty,1-\delta}^\nu \setminus \overline{R}_{\infty,1-2\delta}^\nu$. Specifically, we define

$$N_\varepsilon := \min\{2h : h \in \mathbb{N}, \ T_{z_0}^{2h} \subset R_{\infty,1-5\delta/4}^\nu \setminus \overline{R}_{\infty,1-7\delta/4}^\nu\}.$$

Given another $z \in \mathcal{Z}_\varepsilon^{\text{top}}$, we have that

$$T_z^{N_\varepsilon} \subset R_{\infty,1-\delta}^\nu \setminus \overline{R}_{\infty,1-2\delta}^\nu. \tag{4.39}$$

Indeed, let $y = y_0 + 3\varepsilon\frac{N_\varepsilon}{2}\hat{e}_\alpha \in T_z^{N_\varepsilon}$ with $y_0 \in T_z$. Let $x_0 \in T_{z_0} \cap L_\varepsilon$, cf. (4.32), and let $x := x_0 + 3\varepsilon\frac{N_\varepsilon}{2}\hat{e}_\alpha \in T_{z_0}^{N_\varepsilon}$. Since $y_0 \in L_\varepsilon + B_{3\varepsilon}(0)$, we have that $|\langle y_0 - x_0, \nu\rangle| < 3\varepsilon$ and thus $|\langle y - x, \nu\rangle| < 3\varepsilon$, *i.e.*, y belongs to the 3ε-neighborhood of $R_{\infty,1-5\delta/4}^\nu \setminus \overline{R}_{\infty,1-7\delta/4}^\nu$, which is contained in $R_{\infty,1-\delta}^\nu \setminus \overline{R}_{\infty,1-2\delta}^\nu$.

Observe that

$$N_\varepsilon \leq \frac{C_3}{\varepsilon} \tag{4.40}$$

for some positive constant C_3. To prove this, let $x_0 \in T_{z_0}$ and $x := x_0 + 3\varepsilon\frac{N_\varepsilon}{2}\hat{e}_\alpha \in T_{z_0}^{N_\varepsilon}$. The segment $[x_0; x]$ is fully contained in $R_{\infty,1-\delta}^\nu \setminus \overline{R}_{\infty,1-3\delta}^\nu$ and thus $\delta \geq |\langle x - x_0, \nu\rangle| = 3\varepsilon\frac{N_\varepsilon}{2}\langle \hat{e}_\alpha, \nu\rangle \geq 3\varepsilon\frac{N_\varepsilon}{2}\frac{\sqrt{3}}{2}$.

Winding number. We choose m_ε given by (4.37). We consider the angles $\theta_\varepsilon(\varepsilon i_z)$, $\theta_\varepsilon(\varepsilon j_z)$, $\theta_\varepsilon(\varepsilon k_z)$ introduced in Step 3. By (4.37) and (4.34) we infer that

$$2\pi m_\varepsilon \geq 2\pi C_2\sqrt{\frac{\sigma_\varepsilon}{\varepsilon}} + 16\pi \geq 2\pi|\theta_\varepsilon(\varepsilon i_z) - \theta_\varepsilon(\varepsilon i_{z_0})| + 16\pi \geq 2\pi|\theta_\varepsilon(\varepsilon i_z)| + 2\pi,$$

hence (4.24) is satisfied.

Checking the assumptions on the angles. We check that the assumptions (4.23) are satisfied. First, we claim that for ε small enough the configuration u_ε has positive chirality in every triangle $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ contained in $S_\varepsilon^{\text{top}}$. To prove it, let us start by showing that the sign of the chirality is constant arguing by contradiction. Assume that there exist two triangles $T', T'' \subset S_\varepsilon^{\text{top}}$ with a common side such that $\chi(u_\varepsilon) \leq 0$ in T' and $\chi(u_\varepsilon) \geq 0$ in T''. Then by (4.30) and Lemma 3.2 we would get

$$\varepsilon\sigma_\varepsilon \geq F_\varepsilon(u_\varepsilon, S_\varepsilon) \geq F_\varepsilon(u_\varepsilon, T' \cup T'') \geq \frac{5}{3}\varepsilon,$$

which contradicts the condition $\sigma_\varepsilon \to 0$. Therefore $\chi(u_\varepsilon)$ has constant sign in $S_\varepsilon^{\text{top}}$. In fact, $\chi(u_\varepsilon) > 0$ in $S_\varepsilon^{\text{top}}$. If instead $\chi(u_\varepsilon) \leq 0$ in $S_\varepsilon^{\text{top}}$, by (4.30) we would have that

$$\varepsilon\sigma_\varepsilon \geq \|\chi(u_\varepsilon) - \chi_\nu\|_{L^1(S_\varepsilon^{\text{top}})} = \int_{S_\varepsilon^{\text{top}}} (1 - \chi(u_\varepsilon))\,\mathrm{d}x \geq |S_\varepsilon^{\text{top}}| = 6\varepsilon r_\varepsilon \geq 6\varepsilon\left(\frac{1}{2} - \frac{3}{2}\delta\right),$$

which contradicts $\sigma_\varepsilon \to 0$. In conclusion, $\chi(u_\varepsilon) > 0$ in $S_\varepsilon^{\text{top}}$.

Let now $z \in \mathcal{Z}_\varepsilon^{\text{top}}$. We have

$$|u_\varepsilon(\varepsilon i_z) + u_\varepsilon(\varepsilon j_z) + u_\varepsilon(\varepsilon k_z)|^2 = \frac{1}{\varepsilon}F_\varepsilon(u_\varepsilon, T_z) \leq \frac{1}{\varepsilon}F_\varepsilon(u_\varepsilon, S_\varepsilon) \leq \sigma_\varepsilon.$$

Since $\chi(u_\varepsilon) > 0$ in T_z, for ε small enough u_ε is close to a ground state with chirality 1 and therefore, using (2.3) and Lemma 2.1 (see also (2.5)),

$$\left| \theta_\varepsilon(\varepsilon j_z) - \theta_\varepsilon(\varepsilon i_z) - \frac{2\pi}{3} \right| \leq \frac{1}{4}, \quad \left| \theta_\varepsilon(\varepsilon k_z) - \theta_\varepsilon(\varepsilon j_z) - \frac{2\pi}{3} \right| \leq \frac{1}{4}. \qquad (4.41)$$

Definition of the interpolation. We are in a position to define the interpolation. We reproduce the one-dimensional construction of Lemma 4.5 by suitably translating and scaling it, providing the precise notation as it will be useful for later estimates. We shall define the interpolation only on slices starting from every other triangle T_z, for the constructions on two slices $\Sigma_\varepsilon^{\alpha,z}$ and $\Sigma_\varepsilon^{\alpha,z+2}$ completely determine the values of the modified spin configuration in $\Sigma_\varepsilon^{\alpha,z+1}$. For this reason, let $z \in \mathcal{Z}_\varepsilon^{\text{top}}$ be such that $z \equiv z_0 \mod 2$. We then define the interpolated angles $\theta(\varepsilon i_z^h)$, $\theta(\varepsilon j_z^h)$, $\theta(\varepsilon k_z^h)$ for $h = 0, \ldots, N_\varepsilon$ as in (4.21) by (recall that $i_z^0 = i_z, j_z^0 = j_z, k_z^0 = k_z$)

$$\theta_\varepsilon(\varepsilon i_z^h) := \theta_\varepsilon(\varepsilon i_z) + h \frac{2\pi m_\varepsilon - \theta_\varepsilon(\varepsilon i_z)}{N_\varepsilon} = \left(1 - \frac{h}{N_\varepsilon}\right)\theta_\varepsilon(\varepsilon i_z) + \frac{h}{N_\varepsilon} 2\pi m_\varepsilon ,$$

$$\theta_\varepsilon(\varepsilon j_z^h) := \theta_\varepsilon(\varepsilon j_z) + h \frac{2\pi m_\varepsilon + \frac{2\pi}{3} - \theta_\varepsilon(\varepsilon j_z)}{N_\varepsilon} = \left(1 - \frac{h}{N_\varepsilon}\right)\theta_\varepsilon(\varepsilon j_z) + \frac{h}{N_\varepsilon} 2\pi m_\varepsilon + \frac{h}{N_\varepsilon}\frac{2\pi}{3} ,$$

$$\theta_\varepsilon(\varepsilon k_z^h) := \theta_\varepsilon(\varepsilon k_z) + h \frac{2\pi m_\varepsilon + \frac{4\pi}{3} - \theta_\varepsilon(\varepsilon k_z)}{N_\varepsilon} = \left(1 - \frac{h}{N_\varepsilon}\right)\theta_\varepsilon(\varepsilon k_z) + \frac{h}{N_\varepsilon} 2\pi m_\varepsilon + \frac{h}{N_\varepsilon}\frac{4\pi}{3} ,$$
$$(4.42)$$

and $\theta_\varepsilon(\varepsilon i_z^h) := 2\pi m_\varepsilon, \theta_\varepsilon(\varepsilon j_z^h) := 2\pi m_\varepsilon + \frac{2\pi}{3}, \theta_\varepsilon(\varepsilon k_z^h) := 2\pi m_\varepsilon + \frac{4\pi}{3}$ for $h \geq N_\varepsilon + 1$.

Eventually, we define $u_\varepsilon^{\text{top}} : \mathcal{L}_\varepsilon \cap \Sigma_\varepsilon^{\alpha,z}(T_z) \to \mathbb{S}^1$ by setting

$$u_\varepsilon^{\text{top}}(\varepsilon i_z^h) := \exp(\iota\theta(\varepsilon i_z^h)), \quad u_\varepsilon^{\text{top}}(\varepsilon j_z^h) := \exp(\iota\theta(\varepsilon j_z^h)), \quad u_\varepsilon^{\text{top}}(\varepsilon k_z^h) := \exp(\iota\theta(\varepsilon k_z^h)).$$

By (4.39) we have that

$$u_\varepsilon^{\text{top}}|_T = u_\varepsilon^{\text{pos}}|_T \quad \text{if } T \subset \Sigma_\varepsilon^{\alpha,z}(T_z) \setminus \overline{R}_{\infty,1-\delta}^\nu. \qquad (4.43)$$

Estimate on "even" slices. We observe that the construction of $u_\varepsilon^{\text{top}}$ is simply a translation and a scaling of the construction in Lemma 4.5. As the assumption (4.23) is satisfied, *cf.* (4.41), we can apply Lemma 4.5 to deduce that

$$F_\varepsilon(u_\varepsilon^{\text{top}}, \Sigma_\varepsilon^{\alpha,z}(T_z)) \leq C\left(N_\varepsilon F_\varepsilon(u_\varepsilon, T_z) + \varepsilon \frac{m_\varepsilon^2}{N_\varepsilon}\right). \qquad (4.44)$$

Estimate on "odd" slices. We estimate the energy on the missing half-slices. Let us fix $z, z+1, z+2 \in \mathcal{Z}_\varepsilon^{\text{top}}$ with $z \equiv z_0 \mod 2$. Let T be a triangle contained in $\Sigma_\varepsilon^{\alpha,z+1}(T_{z+1})$. Then T shares two vertices with one triangle contained in $\Sigma_\varepsilon^{\alpha,z}(T_z)$ or with one triangle contained in $\Sigma_\varepsilon^{\alpha,z+2}(T_{z+2})$. Let us assume, without loss of generality, that the two shared vertices are the vertices $\varepsilon j' \in \mathcal{L}_\varepsilon^2$ and $\varepsilon k' \in \mathcal{L}_\varepsilon^3$ of some triangle $T' \subset \Sigma_\varepsilon^{\alpha,z}(T_z)$. The third vertex of T' is of the type $\varepsilon i_z^{h'} \in \mathcal{L}_\varepsilon^1$ for some $h' \in \mathbb{N}$. Moreover, the third vertex of T is shared with a triangle T_{z+2}^h, $h \in \mathbb{N}$, and is of the type $\varepsilon i_{z+2}^h \in \mathcal{L}_\varepsilon^3$. We remark that $|h' - h| \leq 2$. Indeed, by (4.38) we have that

$$i_z^{h'} = i_z + h'\frac{3}{2}\hat{e}_\alpha \pm \frac{\sqrt{3}}{2}\hat{e}_\alpha^\perp ,$$
$$i_{z+2}^h = i_{z+2} + h\frac{3}{2}\hat{e}_\alpha \pm \frac{\sqrt{3}}{2}\hat{e}_\alpha^\perp .$$

From the assumptions on the position of the two triangles together with the definition of $i_z^{h'}, i_{z+2}^h$ it follows that

$$0 = \langle i_z^{h'} - i_{z+2}^h, \hat{e}_\alpha \rangle = \langle i_z - i_{z+2}, \hat{e}_\alpha \rangle + (h'-h)\frac{3}{2} \quad \Longrightarrow \quad |h'-h| = \frac{2}{3}|\langle i_z - i_{z+2}, \hat{e}_\alpha \rangle| \leq 2\,,$$

where in the last inequality we used the fact that $T_z \cap T_{z+1} \neq \emptyset$ and $T_{z+1} \cap T_{z+2} \neq \emptyset$.

We estimate the energy in the triangle T by

$$\begin{aligned}
F_\varepsilon(u_\varepsilon^{\text{top}}, T) &= \varepsilon |u_\varepsilon^{\text{top}}(\varepsilon i_{z+2}^h) + u_\varepsilon^{\text{top}}(\varepsilon j') + u_\varepsilon^{\text{top}}(\varepsilon k')|^2 \\
&\leq 2\varepsilon |u_\varepsilon^{\text{top}}(\varepsilon i_z^{h'}) + u_\varepsilon^{\text{top}}(\varepsilon j') + u_\varepsilon^{\text{top}}(\varepsilon k')|^2 + 2\varepsilon |u_\varepsilon^{\text{top}}(\varepsilon i_z^{h'}) - u_\varepsilon^{\text{top}}(\varepsilon i_{z+2}^h)|^2 \\
&= 2 F_\varepsilon(u_\varepsilon^{\text{top}}, T') + 2\varepsilon |u_\varepsilon^{\text{top}}(\varepsilon i_z^{h'}) - u_\varepsilon^{\text{top}}(\varepsilon i_{z+2}^h)|^2\,.
\end{aligned}$$

Note that (4.34) and (4.37) imply

$$|\theta_\varepsilon(\varepsilon i_{z+2})| \leq |\theta_\varepsilon(\varepsilon i_{z+2}) - \theta_\varepsilon(\varepsilon i_{z_0})| + |\theta_\varepsilon(\varepsilon i_{z_0})| \leq |\theta_\varepsilon(\varepsilon i_{z+2}) - \theta_\varepsilon(\varepsilon i_{z_0})| + 2\pi \leq 2\pi m_\varepsilon\,.$$

From (4.42), from the previous estimate, and since $|h'-h| \leq 2$ it follows that

$$\begin{aligned}
|u_\varepsilon^{\text{top}}(\varepsilon i_z^h) - u_\varepsilon^{\text{top}}(\varepsilon i_{z+2}^{h'})|^2 &= 2 - 2\cos(\theta_\varepsilon(\varepsilon i_z^h) - \theta_\varepsilon(\varepsilon i_{z+2}^{h'})) \leq |\theta_\varepsilon(\varepsilon i_z^h) - \theta_\varepsilon(\varepsilon i_{z+2}^{h'})|^2 \\
&= \left| \left(1 - \frac{h}{N_\varepsilon}\right)(\theta_\varepsilon(\varepsilon i_z) - \theta_\varepsilon(\varepsilon i_{z+2})) + \frac{h-h'}{N_\varepsilon}(2\pi m_\varepsilon - \theta_\varepsilon(\varepsilon i_{z+2})) \right|^2 \\
&\leq 2|\theta_\varepsilon(\varepsilon i_z) - \theta_\varepsilon(\varepsilon i_{z+2})|^2 + 2\left| \frac{h-h'}{N_\varepsilon}\right|^2 |2\pi m_\varepsilon - \theta_\varepsilon(\varepsilon i_{z+2})|^2 \\
&\leq 2|\theta_\varepsilon(\varepsilon i_z) - \theta_\varepsilon(\varepsilon i_{z+2})|^2 + C\frac{m_\varepsilon^2}{N_\varepsilon^2}
\end{aligned}$$

It remains to estimate $|\theta_\varepsilon(\varepsilon i_z) - \theta_\varepsilon(\varepsilon i_{z+2})|^2$. Using the fact that $1 - \cos(\phi) \geq \frac{1}{12}\phi^2$ for every $|\phi| \leq \pi$ and by (4.36) we obtain that

$$\begin{aligned}
|\theta_\varepsilon(\varepsilon i_z) - \theta_\varepsilon(\varepsilon i_{z+2})|^2 &\leq 2|\theta_\varepsilon(\varepsilon i_z) - \theta_\varepsilon(\varepsilon i_{z+1})|^2 + 2|\theta_\varepsilon(\varepsilon i_{z+1}) - \theta_\varepsilon(\varepsilon i_{z+2})|^2 \\
&\leq C|u_\varepsilon(\varepsilon i_z) - u_\varepsilon(\varepsilon i_{z+1})|^2 + C|u_\varepsilon(\varepsilon i_{z+1}) - u_\varepsilon(\varepsilon i_{z+2})|^2 \\
&\leq \frac{C}{\varepsilon}(F_\varepsilon(u_\varepsilon, H_z) + F_\varepsilon(u_\varepsilon, H_{z+1}))\,,
\end{aligned}$$

where H_z is an hexagon containing T_z and T_{z+1} and H_{z+1} is an hexagon containing T_{z+1} and T_{z+2}. In conclusion, we have that

$$F_\varepsilon(u_\varepsilon^{\text{top}}, T) \leq C\left(F_\varepsilon(u_\varepsilon^{\text{top}}, T') + F_\varepsilon(u_\varepsilon, H_z) + F_\varepsilon(u_\varepsilon, H_{z+1}) + \varepsilon\frac{m_\varepsilon^2}{N_\varepsilon^2} \right)\,.$$

Summing over all triangles in $\Sigma_\varepsilon^{\alpha,z}(T_{z+1})$ (their number is CN_ε) we deduce that

$$\begin{aligned}
F_\varepsilon(&u_\varepsilon^{\text{top}}, \Sigma_\varepsilon^{\alpha,z+1}(T_{z+1})) \\
&\leq C\Big(F_\varepsilon(u_\varepsilon^{\text{top}}, \Sigma_\varepsilon^{\alpha,z}(T_z)) + F_\varepsilon(u_\varepsilon^{\text{top}}, \Sigma_\varepsilon^{\alpha,z+2}(T_{z+2})) + N_\varepsilon F_\varepsilon(u_\varepsilon, H_z) \\
&\quad + N_\varepsilon F_\varepsilon(u_\varepsilon, H_{z+1}) + \varepsilon\frac{m_\varepsilon^2}{N_\varepsilon} \Big)\,.
\end{aligned} \quad (4.45)$$

Final estimate on top part. By (4.45), (4.44), summing over z and by (4.30), (4.33), (4.37), and (4.40) we conclude that[1]

$$\sum_{z \in \mathcal{Z}_\varepsilon^{\text{top}}} F_\varepsilon(u_\varepsilon^{\text{top}}, \Sigma_\varepsilon^{\alpha,z}(T_z))$$

$$\leq \sum_{\substack{z \in \mathcal{Z}_\varepsilon^{\text{top}} \\ z \equiv z_0 \bmod 2}} C \left(F_\varepsilon(u_\varepsilon^{\text{top}}, \Sigma_\varepsilon^{\alpha,z}(T_z)) + N_\varepsilon F_\varepsilon(u_\varepsilon, H_z) + N_\varepsilon F_\varepsilon(u_\varepsilon, H_{z+1}) + \varepsilon \frac{m_\varepsilon^2}{N_\varepsilon} \right)$$

$$\leq \sum_{\substack{z \in \mathcal{Z}_\varepsilon^{\text{top}} \\ z \equiv z_0 \bmod 2}} C \left(N_\varepsilon F_\varepsilon(u_\varepsilon, T_z) + N_\varepsilon F_\varepsilon(u_\varepsilon, H_z) + N_\varepsilon F_\varepsilon(u_\varepsilon, H_{z+1}) + \varepsilon \frac{m_\varepsilon^2}{N_\varepsilon} \right) \tag{4.46}$$

$$\leq C N_\varepsilon F_\varepsilon(u_\varepsilon, S_\varepsilon) + C\varepsilon \frac{m_\varepsilon^2}{N_\varepsilon} \# \mathcal{Z}_\varepsilon^{\text{top}} \leq C \frac{C_3}{\varepsilon} \varepsilon \sigma_\varepsilon + C\varepsilon \left(C_2 \sqrt{\frac{\sigma_\varepsilon}{\varepsilon}} + 4 \right)^2 \frac{\varepsilon}{C_3} \frac{C_1}{\varepsilon}$$

$$\leq C\, (\sigma_\varepsilon + \varepsilon)\,.$$

Step 5. (parts of the square). The modification starting from $S_\varepsilon^{\text{left}}$ and $S_\varepsilon^{\text{right}}$ is completely analogous. We recall that $\beta \in \{1, 2, 3\}$ is such that $|\langle \hat{e}_\beta, \nu^\perp \rangle| \geq \frac{\sqrt{3}}{2}$. We consider chains of triangles contained in $S_\varepsilon^{\text{left}}$ and $S_\varepsilon^{\text{right}}$ given by Lemma 4.6 (suitably adapted). In half-slices in the direction $\hat{e}_\beta$ starting from triangles of these chains and approaching the boundary ∂Q^ν, we define $u_\varepsilon^{\text{left}}$ and $u_\varepsilon^{\text{right}}$ as in Step 4.

We are finally in a position to define $\widetilde{u}_\varepsilon$ in Q_+^ν. We fix $\delta \in (0, \frac{1}{8})$ and we consider the two-barred cross-shaped set (the white region in Fig. 9)

$$P_\delta := R_{1-5\delta,1}^\nu \cup (R_{1,1-5\delta}^\nu \setminus \overline{R}_{1,3\delta}^\nu)\,.$$

Given $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ such that $T \subset Q^\nu$, we distinguish some cases.
*Case* $T \subset P_\delta \cap Q_{r_\varepsilon + 6\varepsilon}^\nu$: We set

$$\text{if } T \subset P_\delta \cap Q_{r_\varepsilon + 6\varepsilon}^\nu : \quad \widetilde{u}_\varepsilon|_T := u_\varepsilon|_T \tag{4.47}$$

Case $T \subset R_{1-5\delta,1}^\nu \setminus \overline{Q}_{r_\varepsilon}^\nu$ (part of the cross-shaped set P_δ aligned with ν): We give the definition in the case $T \subset Q_+^\nu$ (the case $T \subset Q_-^\nu$ being analogous). Let $y_0 \in T$. Let us consider the slice $\Sigma_\varepsilon^{\alpha,z}$ such that $T \subset \Sigma_\varepsilon^{\alpha,z}$ and let us show that $z \in \mathcal{Z}_\varepsilon^{\text{top}}$. Let $x \in T_z$ and first of all note that $x \in L_\varepsilon + B_{3\varepsilon}(0)$. Since T and T_z are contained in the same slice, by definition of $\Sigma_\varepsilon^{\alpha,z}$ we can find $s \in \mathbb{R}$ such that $x_0 := y_0 + s\hat{e}_\alpha \in T_z$. Since $y_0 \notin \overline{Q}_{r_\varepsilon}^\nu$, the segment $[x_0; y_0]$ is contained in $R_{\infty,1}^\nu \setminus \overline{R}_{\infty,r_\varepsilon}^\nu$, thus $|s|\frac{\sqrt{3}}{2} \leq |s||\langle \hat{e}_\alpha, \nu \rangle| = |\langle x_0 - y_0, \nu \rangle| \leq \frac{1}{2}(1 - r_\varepsilon)$, i.e., $|s| \leq \frac{1}{\sqrt{3}}(1 - r_\varepsilon) < \sqrt{3}\delta$. Then, using that $|\langle \hat{e}_\alpha, \nu^\perp \rangle| \leq \frac{1}{2}$,

$$|\langle x, \nu^\perp \rangle| \leq |\langle x - x_0, \nu^\perp \rangle| + |\langle x_0 - y_0, \nu^\perp \rangle| + |\langle y_0, \nu^\perp \rangle|$$

$$\leq \varepsilon + |s||\langle \hat{e}_\alpha, \nu^\perp \rangle| + \frac{1}{2} - \frac{5}{2}\delta < \varepsilon + \frac{\sqrt{3}}{2}\delta + \frac{1}{2} - \frac{5}{2}\delta < \frac{1}{2} - \frac{3}{2}\delta < \frac{r_\varepsilon}{2}\,,$$

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[1] In this estimate it becomes evident that it was crucial to prove that the energy concentrates close to the interface. A classical averaging/slicing argument would only provide a bound on the strip S_ε of the type $F_\varepsilon(u_\varepsilon, S_\varepsilon) \leq C\varepsilon$. This would not suffice to conclude that the modified sequence does not increase the energy, as the right-hand side in this estimate would end up to be a constant.
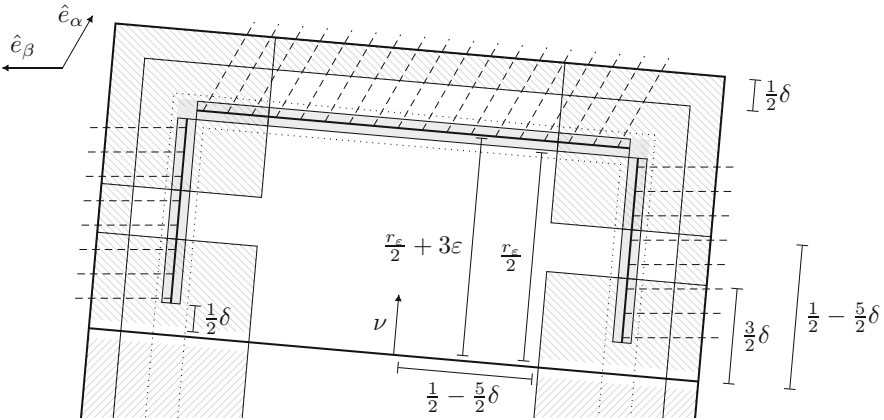
Fig. 9 Definition of $\widetilde{u}_\varepsilon$ in Q_+^ν: in the hatched regions it is equal to u^{pos}; in the white region enclosed by $Q_{r_\varepsilon+6\varepsilon}^\nu$ it is equal to u_ε; outside of $Q_{r_\varepsilon}^\nu$ it is defined through the interpolation $u_\varepsilon^{\text{top}}$, $u_\varepsilon^{\text{left}}$, and $u_\varepsilon^{\text{right}}$ constructed with the slices in the lattice directions $\hat{e}_\alpha$ and $\hat{e}_\beta$

i.e., $x \in R_{r_\varepsilon,1}^\nu$ and hence $x \in (L_\varepsilon + B_{3\varepsilon}(0)) \cap R_{r_\varepsilon,1}^\nu = S_\varepsilon^{\text{top}}$. We set

$$\text{if } T \subset R_{1-5\delta,1}^\nu \setminus \overline{Q}_{r_\varepsilon}^\nu : \quad \widetilde{u}_\varepsilon|_T := \begin{cases} u_\varepsilon^{\text{top}}|_T & \text{if } T \subset \Sigma_\varepsilon^{\alpha,z}(T_z), \\ u_\varepsilon|_T & \text{otherwise.} \end{cases} \tag{4.48}$$

The definition is consistent with the previous case: if $T \subset Q_{r_\varepsilon+6\varepsilon}^\nu \setminus \overline{Q}_{r_\varepsilon}^\nu$, then T is not contained in any half-slice $\Sigma_\varepsilon^{\alpha,z}(T_z)$ (because $T_z \cap \partial Q_{r_\varepsilon+6\varepsilon}^\nu \neq \emptyset$) and thus $\widetilde{u}_\varepsilon|_T = u_\varepsilon|_T$, in accordance with (4.47). If $T \subset R_{1-5\delta,1}^\nu \setminus \overline{Q}_{r_\varepsilon}^\nu$ but T is not contained in any half-slice $\Sigma_\varepsilon^{\alpha,z}(T_z)$, then $T \subset S_\varepsilon$. In particular, by (4.46) and (4.30) we infer that

$$F_\varepsilon(\widetilde{u}_\varepsilon, R_{1-5\delta,1}^\nu \setminus \overline{Q}_{r_\varepsilon}^\nu) \leq \sum_{z \in \mathcal{Z}_\varepsilon^{\text{top}}} F_\varepsilon(u_\varepsilon^{\text{top}}, \Sigma_\varepsilon^{\alpha,z}(T_z)) + F_\varepsilon(u_\varepsilon, S_\varepsilon) \leq C(\sigma_\varepsilon + \varepsilon). \tag{4.49}$$

Case $T \subset (R_{1,1-5\delta}^\nu \setminus \overline{R}_{1,3\delta}^\nu) \setminus \overline{Q}_{r_\varepsilon}^\nu$ (part of the cross-shaped set P_δ aligned with $\nu^\perp$): As in the previous case, assuming $T \subset Q_+^\nu$, we define $\widetilde{u}_\varepsilon|_T := u_\varepsilon^{\text{left}}|_T$ if T is contained in a half-slice starting from a triangle in $S_\varepsilon^{\text{left}}$, $\widetilde{u}_\varepsilon|_T := u_\varepsilon^{\text{right}}|_T$ if T is contained in a half-slice starting from a triangle in $S_\varepsilon^{\text{right}}$, and $\widetilde{u}_\varepsilon|_T := u_\varepsilon|_T$ otherwise. As before, the definition is compatible with (4.47). Similarly to (4.49) we obtain that

$$F_\varepsilon(\widetilde{u}_\varepsilon, (R_{1,1-5\delta}^\nu \setminus \overline{R}_{1,3\delta}^\nu) \setminus \overline{Q}_{r_\varepsilon}^\nu) \leq C(\sigma_\varepsilon + \varepsilon). \tag{4.50}$$

Case $T \cap (\mathbb{R}^2 \setminus P_\delta) \neq \emptyset$: let x be a vertex of T and assume that x is not the vertex of a triangle T' covered by the previous cases. Then we set $\widetilde{u}_\varepsilon(x) := u_\varepsilon^{\text{pos}}(x)$ if $\langle x, \nu \rangle \geq 0$ and $\widetilde{u}_\varepsilon(x) := u_\varepsilon^{\text{neg}}(x)$ if $\langle x, \nu \rangle < 0$. In particular,

$$\begin{aligned} \text{if } T \subset Q_+^\nu \setminus \overline{P}_\delta : \quad & \widetilde{u}_\varepsilon|_T = u_\varepsilon^{\text{pos}}|_T, \\ \text{if } T \subset Q_-^\nu \setminus \overline{P}_\delta : \quad & \widetilde{u}_\varepsilon|_T = u_\varepsilon^{\text{neg}}|_T. \end{aligned} \tag{4.51}$$

We remark that

$$\varepsilon^2 \#\{T \subset Q^\nu : T \cap \partial P_\delta \neq \emptyset\} \leq C\delta\varepsilon \implies \#\{T \subset Q^\nu : T \cap \partial P_\delta \neq \emptyset\} \leq C\frac{\delta}{\varepsilon} \tag{4.52}$$

and

$$\#\{T \subset Q^\nu \setminus \overline{P}_\delta : T \cap L^\nu \neq \emptyset\} \leq C\frac{\delta}{\varepsilon}. \tag{4.53}$$

Let us check that $\widetilde{u}_\varepsilon$ attains the desired boundary conditions (4.28). Let $T \subset Q_+^\nu \setminus \overline{Q}_{1-\delta}^\nu$. If $T \subset (Q_+^\nu \setminus \overline{Q}_{1-\delta}^\nu) \cap R_{1-5\delta,1}^\nu$ (and similarly if $T \subset (Q_+^\nu \setminus \overline{Q}_{1-\delta}^\nu) \cap (R_{1,1-5\delta}^\nu \setminus \overline{R}_{1,3\delta}^\nu)$), then we are in the case covered by (4.48). By (4.43) we have $\widetilde{u}_\varepsilon|_T = u_\varepsilon^{\mathrm{top}}|_T = u_\varepsilon^{\mathrm{pos}}|_T$. Otherwise, if $T \cap (\mathbb{R}^2 \setminus P_\delta) \neq \emptyset$, let x be a vertex of T and assume that x is not the vertex of a triangle T' covered by the previous cases. Then, by definition, $\widetilde{u}_\varepsilon(x) := u_\varepsilon^{\mathrm{pos}}(x)$. We argue analogously if $T \subset Q_-^\nu \setminus \overline{Q}_{1-\delta}^\nu$. Finally, if $T \cap L^\nu \neq \emptyset$, then $T \subset L^\nu + B_{2\varepsilon}(0)$ and thus it is not relevant for the boundary conditions by the definition of discrete boundary $\partial_\varepsilon^\pm Q^\nu$.

Step 6. (Energy estimate). By (4.52), (4.51), and (4.53) we have that

$$\begin{aligned}
F_\varepsilon(\widetilde{u}_\varepsilon, Q^\nu) &\leq F_\varepsilon(\widetilde{u}_\varepsilon, P_\delta) + F_\varepsilon(\widetilde{u}_\varepsilon, Q^\nu \setminus \overline{P}_\delta) + \sum_{T \cap \partial P_\delta \neq \emptyset} F_\varepsilon(\widetilde{u}_\varepsilon, T) \\
&\leq F_\varepsilon(\widetilde{u}_\varepsilon, P_\delta) + F_\varepsilon(u_\varepsilon^{\mathrm{pos}}, Q_+^\nu \setminus \overline{P}_\delta) + F_\varepsilon(u_\varepsilon^{\mathrm{neg}}, Q_-^\nu \setminus \overline{P}_\delta) \\
&\quad + \sum_{\substack{T \subset Q^\nu \setminus \overline{P}_\delta \\ T \cap L^\nu \neq \emptyset}} F_\varepsilon(\widetilde{u}_\varepsilon, T) + C\delta \\
&\leq F_\varepsilon(\widetilde{u}_\varepsilon, P_\delta) + C\delta.
\end{aligned}$$

Moreover, by (4.47), (4.49), and (4.50) we deduce that

$$\begin{aligned}
F_\varepsilon(\widetilde{u}_\varepsilon, P_\delta) &\leq F_\varepsilon(u_\varepsilon, P_\delta \cap Q_{r_\varepsilon + 6\varepsilon}^\nu) + F_\varepsilon(\widetilde{u}_\varepsilon, R_{1-5\delta,1}^\nu \setminus \overline{Q}_{r_\varepsilon}^\nu) + F_\varepsilon(\widetilde{u}_\varepsilon, (R_{1,1-5\delta}^\nu \setminus \overline{R}_{1,3\delta}^\nu) \setminus \overline{Q}_{r_\varepsilon}^\nu) \\
&\leq F_\varepsilon(u_\varepsilon, Q^\nu) + C(\sigma_\varepsilon + \varepsilon).
\end{aligned}$$

In conclusion,

$$\limsup_{\varepsilon \to 0} F_\varepsilon(\widetilde{u}_\varepsilon, Q^\nu) \leq \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, Q^\nu) + C\delta.$$

Eventually, letting $\delta \to 0$ and with a diagonal argument, we construct a sequence which satisfies (4.29).

□

In the proof of Proposition 4.2 we applied the following lemma.

Lemma 4.6 *Let $\Sigma_\varepsilon^{\alpha,z}$ be the slices of the triangular lattice defined in (4.18). Let L be a line in $\mathbb{R}^2$ orthogonal to ν and assume that $|\langle \hat{e}_\alpha, \nu^\perp \rangle| \leq \frac{1}{2}$. Then there exists a chain of triangles $(T_z)_{z \in \mathbb{Z}}$ satisfying for every $z \in \mathbb{Z}$*

$$T_z \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2), \quad T_z \subset \Sigma_\varepsilon^{\alpha,z}, \quad T_z \cap L \neq \emptyset, \quad T_z \cap T_{z+1} \neq \emptyset. \tag{4.54}$$

Proof It is enough to prove the following:
Claim: Let $z \in \mathbb{Z}$ and let $T_z \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2)$ be such that $T_z \subset \Sigma_\varepsilon^{\alpha,z}$ and $T_z \cap L \neq \emptyset$. Then there exists $T_{z+1} \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2)$ such that $T_{z+1} \subset \Sigma_\varepsilon^{\alpha,z+1}$, $T_{z+1} \cap L \neq \emptyset$, and $T_z \cap T_{z+1} \neq \emptyset$. (The analogous statement with $\Sigma_\varepsilon^{\alpha,z-1}$ in place of $\Sigma_\varepsilon^{\alpha,z+1}$ holds true.)
With the proven claim at hand it is immediate to define a chain of triangles $(T_z)_{z \in \mathbb{Z}}$ which satisfies the properties in (4.54) by initializing the construction from a triangle $T_{z_0} \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2)$ which satisfies $T_{z_0} \cap L \neq \emptyset$ and $T_{z_0} \subset \Sigma_\varepsilon^{\alpha,z_0}$. Such a triangle always exists since the set $\mathbb{R}^2 \setminus \bigcup_{T \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2)} T$ is the union of disjoint open triangles, thus cannot contain L.

To prove the claim let us denote be $\hat{e}_\beta$, $\hat{e}_\gamma$ the remaining two unit vectors connecting points of $\mathcal{L}$ and introduced in Section 2.2 and let us set $\tau_\beta := \mathrm{sign}\langle \hat{e}_\beta, \hat{e}_\alpha^\perp \rangle$, $\tau_\gamma := \mathrm{sign}\langle \hat{e}_\gamma, \hat{e}_\alpha^\perp \rangle$.
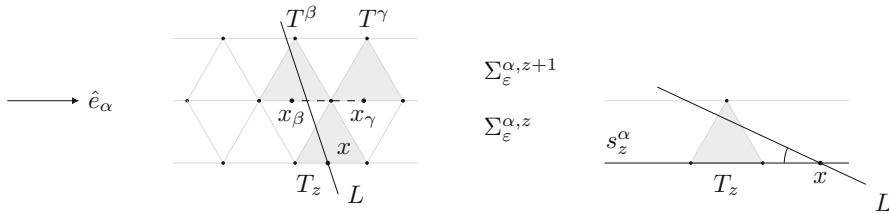
Fig. 10 On the left: the triangles T_z, T^β and T^γ, the line L, and the segment $[x_\beta; x_\gamma]$. On the right: if $x \notin T_z$, the angle between L and s_z^α belongs to $[0, \frac{\pi}{3})$

For later use we observe that $\tau_\beta = \tau_\gamma$ if and only if $\alpha \neq 2$, that is if and only if $\langle \hat{e}_\beta, \hat{e}_\alpha \rangle = -\langle \hat{e}_\gamma, \hat{e}_\alpha \rangle$. In particular, we always have

$$\tau_\beta \langle \hat{e}_\beta, \hat{e}_\alpha \rangle = -\tau_\gamma \langle \hat{e}_\gamma, \hat{e}_\alpha \rangle \quad \text{and} \quad \tau_\beta \langle \hat{e}_\beta, \hat{e}_\alpha \rangle \tau_\gamma \langle \hat{e}_\gamma, \hat{e}_\alpha \rangle = -1/4. \quad (4.55)$$

Suppose now that $T_z \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2)$, $T_z \subset \Sigma_\varepsilon^{\alpha,z}$ with $T_z \cap L \neq \emptyset$. The triangles $T^\beta := T_z + \varepsilon \tau_\beta \hat{e}_\beta \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2)$ and $T^\gamma := T_z + \varepsilon \tau_\gamma \hat{e}_\gamma \in \mathcal{T}_\varepsilon^+(\mathbb{R}^2)$ satisfy $T_z \cap T^\beta \neq \emptyset$ and $T_z \cap T^\gamma \neq \emptyset$. Moreover, they are contained in $\Sigma_\varepsilon^{\alpha,z+1}$. Indeed, for $x \in T_z$ we have $\langle \hat{e}_\alpha^\perp, x + \varepsilon \tau_\beta \hat{e}_\beta \rangle = \langle \hat{e}_\alpha^\perp, x \rangle + \varepsilon \tau_\beta \langle \hat{e}_\alpha^\perp, \hat{e}_\beta \rangle = \langle \hat{e}_\alpha^\perp, x \rangle + \varepsilon \frac{\sqrt{3}}{2} \in [\varepsilon \frac{\sqrt{3}}{2}(z+1), \varepsilon \frac{\sqrt{3}}{2}(z+2)]$, hence $T^\beta \subset \Sigma_\varepsilon^{\alpha,z+1}$ (analogously $T^\gamma \subset \Sigma_\varepsilon^{\alpha,z+1}$).

The triangle T_z has one side contained in $\partial \Sigma_\varepsilon^{\alpha,z}$, i.e., either in $s_z^\alpha := \mathbb{R}\hat{e}_\alpha + \varepsilon \frac{\sqrt{3}}{2} z \hat{e}_\alpha^\perp$ or in $s_{z+1}^\alpha := \mathbb{R}\hat{e}_\alpha + \varepsilon \frac{\sqrt{3}}{2}(z+1)\hat{e}_\alpha^\perp$. Let us assume, without loss of generality, that the side is contained in s_z^α. The line L intersects s_z^α in a point x. We claim that $x \in T_z$. Indeed, if $x \notin T_z$, then the angle in $[0, \frac{\pi}{2}]$ between the lines L and s_z^α belongs to $[0, \frac{\pi}{3})$, since L intersects also T_z, see Fig. 10. Let us fix $y \in T_z \cap L \neq \emptyset$. Then we have $|\langle x - y, \hat{e}_\alpha \rangle| > \frac{1}{2}|x - y|$. This contradicts the fact that $|\langle x - y, \hat{e}_\alpha \rangle| = |x - y||\langle v^\perp, \hat{e}_\alpha \rangle| \leq \frac{1}{2}|x - y|$ since $|\langle v^\perp, \hat{e}_\alpha \rangle| \leq \frac{1}{2}$. In conclusion $x \in T_z \cap s_z^\alpha$. Then $x_\beta := x + \varepsilon \tau_\beta \hat{e}_\beta \in T^\beta \cap s_{z+1}^\alpha$, $x_\gamma := x + \varepsilon \tau_\gamma \hat{e}_\gamma \in T^\gamma \cap s_{z+1}^\alpha$. The line L intersects the segment $[x_\beta; x_\gamma]$, and thus either T^β or T^γ. To see this, we let $y_\lambda := \lambda x_\beta + (1-\lambda)x_\gamma$ for $\lambda \in [0, 1]$. Note that

$$\langle y_0 - x, v \rangle = \varepsilon \tau_\gamma \langle \hat{e}_\gamma, v \rangle = \varepsilon \tau_\gamma \left(\langle \hat{e}_\gamma, \hat{e}_\alpha^\perp \rangle \langle \hat{e}_\alpha^\perp, v \rangle + \langle \hat{e}_\gamma, \hat{e}_\alpha \rangle \langle \hat{e}_\alpha, v \rangle \right)$$

$$= \varepsilon \left(\frac{\sqrt{3}}{2} \langle \hat{e}_\alpha^\perp, v \rangle + \tau_\gamma \langle \hat{e}_\gamma, \hat{e}_\alpha \rangle \langle \hat{e}_\alpha, v \rangle \right),$$

and analogously $\langle y_1 - x, v \rangle = \varepsilon \left(\frac{\sqrt{3}}{2} \langle \hat{e}_\alpha^\perp, v \rangle + \tau_\beta \langle \hat{e}_\beta, \hat{e}_\alpha \rangle \langle \hat{e}_\alpha, v \rangle \right)$. In combination with (4.55), this yields

$$\langle y_0 - x, v \rangle \langle y_1 - x, v \rangle = \varepsilon^2 \left(\frac{3}{4} \langle \hat{e}_\alpha^\perp, v \rangle^2 - \frac{1}{4} \langle \hat{e}_\alpha, v \rangle^2 \right) \leq \frac{3}{8} - \frac{3}{8} = 0, \quad (4.56)$$

where we used that $\langle \hat{e}_\alpha^\perp, v \rangle^2 = \langle \hat{e}_\alpha, v^\perp \rangle^2 \leq \frac{1}{4}$ and $\langle \hat{e}_\alpha, v \rangle^2 \geq \frac{3}{4}$. Now (4.56) together with the continuity of the mapping $\lambda \mapsto \langle y_\lambda - x, v \rangle$ implies that there exists $\lambda \in [0, 1]$ with $\langle y_\lambda - x, v \rangle = 0$, hence $y_\lambda \in (T^\beta \cup T^\gamma) \cap L$.

$\square$

5 Upper bound

It remains to prove the Γ-limsup inequality to complete the proof of Theorem 2.5.

Proposition 5.1 *Let $\mathscr{F}_\varepsilon$ be as in (2.8). Then for every $\chi \in L^1(\Omega)$ we have*

$$\Gamma\text{-}\limsup_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\chi) \le \mathscr{F}(\chi),\tag{5.1}$$

where $\mathscr{F}$ is given by (2.10) and the Γ-$\limsup$ is with respect to the strong topology in $L^1(\Omega)$.

Proof It is not restrictive to assume that $\chi \in BV(\Omega; \{-1, 1\})$. Moreover, thanks to Remark 2.6, the density result [13, Corollary 2.4], and the L^1-lower semicontinuity of the Γ-limsup it suffices to prove (5.1) for $\chi \in BV(\Omega; \{-1, 1\})$ such that J_χ is polygonal, *i.e.*, $J_\chi = \bigcup_{n=1}^N \Gamma_n$, where Γ_n are line segments satisfying $\mathcal{H}^1(\Gamma_n \cap \partial\Omega) = 0$. To simplify the exposition we restrict ourselves to the case $J_\chi = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 = [x_0; x_1]$, $\Gamma_2 = [x_1; x_2]$, $x_0, x_1, x_2 \in \mathbb{R}^2$, *i.e.*, the two segments have one common endpoint. The general case then follows by repeating the construction on each line segment.

Step 1. (Construction of a recovery sequence) Denoting by ℓ_1, ℓ_2 the length of Γ_1, Γ_2 and by ν_1, ν_2 the outer unit normal to the set $\{\chi = 1\}$ on Γ_1, Γ_2, upon relabeling we can assume that $x_1 = x_0 + \ell_1 \nu_1^\perp$, $x_2 = x_1 + \ell_2 \nu_2^\perp$. Moreover, we have

$$\mathscr{F}(\chi) = \ell_1 \varphi(\nu_1) + \ell_2 \varphi(\nu_2),\tag{5.2}$$

where φ is as in (2.12). Let $\rho > 0$ be sufficiently small and let $u_{\varepsilon,\rho}^1 \in \mathcal{SF}_\varepsilon$ and $u_{\varepsilon,\rho}^2 \in \mathcal{SF}_\varepsilon$ be admissible for the minimum problems defining $\varphi(\nu_1), \varphi(\nu_2)$, respectively with

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_{\varepsilon,\rho}^1, Q_\rho^{\nu_1}) = \rho\,\varphi(\nu_1) \quad \text{and} \quad \lim_{\varepsilon \to 0} F_\varepsilon(u_{\varepsilon,\rho}^2, Q_\rho^{\nu_2}) = \rho\,\varphi(\nu_2).\tag{5.3}$$

We now start constructing a recovery sequence for χ by subdividing Γ_1 and Γ_2 into segments of length of order ρ and suitable shifting $u_{\varepsilon,\rho}^1, u_{\varepsilon,\rho}^2$ along these segments. In doing so we need to leave out a small region close to the common endpoint x_1. Namely, denoting by $\theta \in (0, \pi]$ the angle between Γ_1 and Γ_2 we choose $c = c(\nu_1, \nu_2) > 0$ with $c \ge \frac{1}{2} + \frac{1}{2}\cot(\frac{\theta}{2})$ and we only subdivide the smaller segments $[x_0; x_1 - c\rho\nu_1^\perp]$ and $[x_1 + c\rho\nu_2^\perp; x_2]$ as follows. We set $M_{\varepsilon,\rho}^1 := \lfloor \frac{\ell_1 - c\rho}{\rho + 5\varepsilon} \rfloor$, $M_{\varepsilon,\rho}^2 := \lfloor \frac{\ell_2 - c\rho}{\rho + 5\varepsilon} \rfloor$ and we choose lattice points
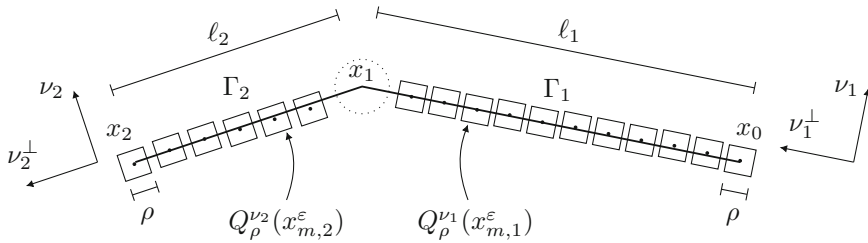
$$x_{m,1}^\varepsilon \in B_{2\varepsilon}\left(x_0 + m(\rho + 5\varepsilon)\nu_1^\perp\right) \cap \mathcal{L}_\varepsilon^1 \quad \text{for } m \in \{0, \dots, M_{\varepsilon,\rho}^1\},$$

$$x_{m,2}^\varepsilon \in B_{2\varepsilon}\left(x_1 + (c\rho + m(\rho + 5\varepsilon))\nu_2^\perp\right) \cap \mathcal{L}_\varepsilon^1 \quad \text{for } m \in \{0, \dots, M_{\varepsilon,\rho}^2\}.$$

Note that the constant c and the lattice points $x_{m,1}^\varepsilon, x_{m,2}^\varepsilon$ are chosen in such a way that, for ε small enough,

$$U_\rho := \bigcup_{m=0}^{M_{\varepsilon,\rho}^1} Q_\rho^{\nu_1}(x_{m,1}^\varepsilon) \cup \bigcup_{m=0}^{M_{\varepsilon,\rho}^2} Q_\rho^{\nu_2}(x_{m,2}^\varepsilon)$$

is a union of pairwise disjoint cubes, see Fig. 11. This allows us to define $u_{\varepsilon,\rho} \in \mathcal{SF}_\varepsilon$ by setting

$$u_{\varepsilon,\rho}(x) := \begin{cases} u_{\varepsilon,\rho}^1(x - x_{m,1}^\varepsilon) & \text{if } x \in Q_\rho^{\nu_1}(x_{m,1}^\varepsilon),\ m \in \{0, \dots, M_{\varepsilon,\rho}^1\}, \\ u_{\varepsilon,\rho}^2(x - x_{m,2}^\varepsilon) & \text{if } x \in Q_\rho^{\nu_2}(x_{m,2}^\varepsilon),\ m \in \{0, \dots, M_{\varepsilon,\rho}^2\}, \\ u_\varepsilon^{\text{pos}}(x) & \text{if } x \in \overline{\{\chi = 1\}} \setminus U_\rho, \\ u_\varepsilon^{\text{neg}}(x) & \text{if } x \in \{\chi = -1\} \setminus U_\rho. \end{cases}$$

Fig. 11 Covering the segments Γ_1 and Γ_2 with cubes of side ρ in the Γ-lim sup construction

We observe that since $x_{m,1}^\varepsilon, x_{m,2}^\varepsilon$ belong to the sublattice $\mathcal{L}_\varepsilon^1$, the boundary conditions satisfied by the shifted functions $u_{\varepsilon,\rho}^1(\cdot - x_{m,1}^\varepsilon), u_{\varepsilon,\rho}^2(\cdot - x_{m,2}^\varepsilon)$ are compatible one with each other and with $u_\varepsilon^{\mathrm{pos}}$ and $u_\varepsilon^{\mathrm{neg}}$ on $\Omega \setminus U_\rho$. In particular, if $x \in \Omega$ is such that $\mathrm{dist}(x, J_\chi) \geq \rho/2$ then $\chi(u_{\varepsilon,\rho})(x) = \chi(x)$, which implies that $\|\chi(u_{\varepsilon,\rho}) - \chi\|_{L^1(\Omega)} \leq C\rho \mathcal{H}^1(J_\chi) \to 0$ as $\rho \to 0$.
Step 2. (Energy estimate) In order to estimate $F_\varepsilon(u_{\varepsilon,\rho})$ we start by rewriting the energy as

$$
F_\varepsilon(u_{\varepsilon,\rho}) = \sum_{m=0}^{M_{\varepsilon,\rho}^1} F_\varepsilon\left(u_{\varepsilon,\rho}, Q_\rho^{\nu_1}(x_{m,1}^\varepsilon)\right) + \sum_{m=0}^{M_{\varepsilon,\rho}^2} F_\varepsilon\left(u_{\varepsilon,\rho}, Q_\rho^{\nu_2}(x_{m,2}^\varepsilon)\right) + \sum_{\substack{T \in \mathcal{T}_\varepsilon(\Omega) \\ T \cap (\Omega \setminus U_\rho) \neq \emptyset}} F_\varepsilon(u_{\varepsilon,\rho}, T),
$$
(5.4)

and we estimate the terms on the right-hand side of (5.4) separately. Let us first consider the energy on triangles $T \in \mathcal{T}_\varepsilon(\Omega)$ with $T \cap (\Omega \setminus U_\rho) \neq \emptyset$. Suppose that $\mathrm{dist}(T, J_\chi) > 5\varepsilon$. Then, if $T \subset \Omega \setminus U_\rho$ we have $u_{\varepsilon,\rho} = u_\varepsilon^{\mathrm{pos}}$ or $u_{\varepsilon,\rho} = u_\varepsilon^{\mathrm{neg}}$ on T, so that $F_\varepsilon(u_{\varepsilon,\rho}, T) = 0$. If instead $T \cap U_\rho \neq \emptyset$, the fact that $\mathrm{dist}(T, J_\chi) > 5\varepsilon$ ensures that T intersects a cube in U_ρ in a region where the boundary conditions are prescribed. Thus, using once more the compatibility of the boundary conditions, we infer that $F_\varepsilon(u_{\varepsilon,\rho}, T) = 0$. This implies that

$$
\sum_{\substack{T \in \mathcal{T}_\varepsilon(\Omega) \\ T \cap (\Omega \setminus U_\rho) \neq \emptyset}} F_\varepsilon(u_{\varepsilon,\rho}, T) \leq 3\varepsilon \#\{T \in \mathcal{T}_\varepsilon(\Omega) : T \cap (\Omega \setminus U_\rho) \neq \emptyset, \ \mathrm{dist}(T, J_\chi) \leq 5\varepsilon\}
$$

$$
\leq C(\rho + \varepsilon/\rho),
$$
(5.5)

where to obtain the first inequality we used $F_\varepsilon(u_{\varepsilon,\rho}, T) \leq 9\varepsilon$, while the second inequality follows by counting triangles contained either in $\left([x_1 - c\rho\nu_1^\perp; x_1] \cup [x_1; x_1 + c\rho\nu_2^\perp]\right) + B_{6\varepsilon}(0)$ or in $\left(\partial U_\rho \cap J_\chi\right) + B_{6\varepsilon}(0)$.

Combining (5.4), (5.5), and (5.3) we deduce that

$$
\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon \to 0} (M_{\varepsilon,\rho}^1 + 1) F_\varepsilon(u_{\varepsilon,\rho}^1, Q_\rho^{\nu_1}) + \limsup_{\varepsilon \to 0} (M_{\varepsilon,\rho}^2 + 1) F_\varepsilon(u_{\varepsilon,\rho}^2, Q_\rho^{\nu_2})
$$

$$
+ C\rho \leq \left(\left\lfloor \frac{\ell_1}{\rho} \right\rfloor + 1\right) \rho \, \varphi(\nu_1) + \left(\left\lfloor \frac{\ell_2}{\rho} \right\rfloor + 1\right) \rho \, \varphi(\nu_2) + C\rho.
$$
(5.6)

Since the latter term converges to $\ell_1 \varphi(\nu_1) + \ell_2 \varphi(\nu_2)$ as $\rho \to 0$, thanks to (5.2) and (5.6), a diagonal argument provides us with a sequence $(u_\varepsilon) = (u_{\varepsilon,\rho(\varepsilon)})$ with $\chi(u_\varepsilon) \to \chi$ in $L^1(\Omega)$ and satisfying $\limsup_\varepsilon F_\varepsilon(u_\varepsilon) \leq \mathscr{F}(\chi)$, from which we finally deduce (5.1). $\qquad\square$

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