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BOOK REVIEW

S. Kondo: “K3 Surfaces”

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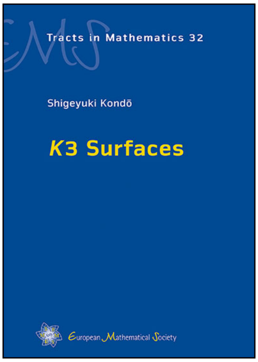

A *K3 surface* is a connected compact 2-dimensional complex manifold that is simply connected and whose canonical line bundle is trivial. Given this definition, it looks like a rather special class of objects. Why should one be interested in them and even read a whole book about them?

Since the 19th century, K3 surfaces showed up in many very different contexts in complex geometry, algebraic geometry, arithmetic geometry, and they continue to show up in sometimes quite surprising and unexpected places, such as in spacetime compactifications in mathematical physics. As such, K3 surfaces connect very different fields and provide stimulation for conjectures and further research. From a modern perspective, they form an important part of the so-called Enriques-Kodaira classification of compact 2-dimensional complex manifolds. They are non-trivial, yet still accessible, which is why they are also an important test class for conjectures. Their name goes back to André Weil (1958):

> *... il s’agit des variétés kählériennes dites K3, ainsi nommées en l’honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire.*

To have an explicit example at hand, we note that the Fermat quartic F_4

$$x^4 + y^4 + z^4 + w^4 = 0$$

in complex projective 3-space $\mathbb{P}^3$ is an example of a K3 surface. Due to length restrictions, I will only discuss the complex analytic side of K3 surfaces, that is, I

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cannot delve into arithmetic aspects, such as K3 surfaces over number fields or function fields or K3 surfaces in positive characteristic. More precisely, I will now give a guided tour through the field of K3 surfaces with an emphasis on Shigeyuki Kondō's book [6].

Kummer. We start in the 19th century: let C be a smooth complex projective curve, or, equivalently, a compact 1-dimensional complex manifold, or, equivalently, a compact Riemann surface. Assume that C is of genus 2. Associated to C, there is its Jacobian $J(C)$, which is a 2-dimensional complex manifold that parametrises degree zero line bundles on C. Tensor product of line bundles turns $J(C)$ into an abelian group. More precisely, $J(C)$ is a compact 2-dimensional complex torus and even algebraisable, that is, an abelian variety. The quotient $J(C)/\pm$ of $J(C)$ by the sign involution $x \mapsto -x$ is a 2-dimensional complex variety: it is not smooth, that is, it is not a manifold, since the 16 fixed points of the sign involution on $J(C)$ give rise to 16 singularities (rational double points of type A_1) in the quotient $J(C)/\pm$. Using the geometry of C, in particular the hyperelliptic involution, theta divisors, and clever observations, one can embed this quotient $J(C)/\pm$ into 3-dimensional projective space $\mathbb{P}^3$ as a quartic surface, the *Kummer surface* associated to C. The minimal resolution of the 16 singularities is an example of a K3 surface. In fact, the Fermat quartic surface F_4 is isomorphic to such a surface, but this is not obvious. For details and explicit equations, see Chap. 4 of [6]. Many examples, techniques, and theorems in Kondō's book are illustrated using Kummer surfaces.

Kähler. A compact complex manifold X is said to be *Kähler* if it admits a real closed 2-form of type $(1, 1)$, that is, a *Kähler form*. The Fubini-Study metric makes complex projective space $\mathbb{P}^d$ a Kähler manifold. In particular, every compact complex submanifold of $\mathbb{P}^d$ becomes a Kähler manifold by restricting the Kähler form of the ambient space. Also, every K3 surface (even a non-algebraic one) is a Kähler manifold by a theorem of Yum-Tong Siu (1983). Moreover, by a theorem of Shing-Tung Yau (1978) (a former conjecture of Eugenio Calabi from 1957), a K3 surface even admits a *Kähler-Einstein metric*. For K3 surfaces, Kähler-Einstein metrics have Ricci curvature zero. For background on Kähler manifolds and a proof of Yau's theorem, I would like to mention [1].

Quite generally, compact complex manifolds that are Kähler-Einstein with zero Ricci curvature are called *Calabi-Yau manifolds*. Important examples are compact complex tori, in particular abelian varieties, and in particular, elliptic curves. In these cases, it is easy to write down an explicit Kähler-Einstein metric. For K3 surfaces, such metrics have not yet been constructed explicitly. We note that Calabi-Yau manifolds play a central rôle in superstring theory in mathematical and theoretical physics, where there are 6 real extra dimensions of spacetime, which carry the structure of 3-dimensional complex Calabi-Yau manifolds. Since K3 surfaces are 2-dimensional Calabi-Yau manifolds, they are an important test ground for such theories.

Kodaira. Smooth and projective algebraic curves, that is, compact 1-dimensional complex manifolds, that is, compact Riemann surfaces are classified according to their genus. A major achievement of the Italian school in algebraic geometry in the first half of the 20th century was the classification of complex projective algebraic surfaces. Later, this classification was extended to compact complex 2-dimensional manifolds, which now goes under the name *Kodaira-Enriques classification*. It classifies surfaces according to their Kodaira dimension $\kappa \in \{-\infty, 0, 1, 2\}$. More precisely, let ω_X be the canonical line bundle, that is, the determinant of the cotangent bundle. Let $p_m(X) := \dim H^0(X, \omega_X^{\otimes m})$ be the dimension of the space of pluricanonical forms. Then, either $p_m(X) = 0$ for all $m > 0$ (Kodaira dimension $\kappa := -\infty$) or it grows like m^κ for some $\kappa \in \{0, \ldots, \dim(X)\}$ as m tends to infinity. For background in dimension 2, I would like to mention the by-now classic textbook [2].

A particular interesting case is Kodaira dimension zero, which means that the line bundle $\omega_X^{\otimes m}$ is trivial for some $m > 0$. In dimension 1, these are precisely curves of genus 1, that is, elliptic curves. These are characterised by carrying the structure of a group and projectivity forces the group law to be abelian. In dimension 2, there are four classes: tori (which are abelian surfaces if algebraisable and the most 'obvious' generalisation of elliptic curves to dimension 2), bielliptic surfaces (quotients of the former by finite group actions), and then, there are two more classes, which is quite remarkable: K3 surfaces and Enriques surfaces (quotients of the former by the finite group of order 2). In particular, one might think of the latter as the 'unexpected' generalisation of elliptic curves to dimension 2. This is discussed in Chap. 3 of [6].

Some - but by no means all - K3 surfaces are quite directly related to elliptic curves: a K3 surface is called *elliptic* if it admits a fibration X to a curve. In this case, the base curve is necessarily $\mathbb{P}^1$ and a general fibre of such a fibration is an elliptic curve, so that one may think of X as a 1-dimensional family of elliptic curves. The Fermat quartic surface $F_4 \subset \mathbb{P}^3$ is an example: it contains 48 lines and if $\ell \subset \mathbb{P}^3$ is such a line and if $H \subset \mathbb{P}^3$ is a plane containing ℓ, then $H \cap F_4$ is the union of ℓ and a curve of degree 3 in $H \cong \mathbb{P}^2$. Varying H, but keeping ℓ, we obtain a family of degree 3 curves on F_4, which can be put into a fibration. The general member of this family is smooth, that is, an elliptic curve: the pencil of planes through ℓ cuts out an elliptic fibration on F_4. Elliptic K3 surfaces are quite accessible for explicit computations, see Chap. 3 of [6]. In this context, I would also like to mention the recent book [7] on elliptic surfaces.

Torelli. Given a K3 surface X, there is the second cohomology group $\Lambda_X := H^2(X, \mathbb{Z})$. Poincaré duality equips it with a symmetric non-degenerate and bilinear pairing. This makes Λ_X an even unimodular lattice of rank 22, which is isometric to $U^3 \perp E_8^2$, where U denotes the hyperbolic plane and E_8 denotes the E_8-lattice. Next, there is the Hodge decomposition

$$\Lambda_X \otimes_{\mathbb{Z}} \mathbb{C} \cong H^2(X, \mathbb{C}) \cong H^2(X, \mathcal{O}_X) \oplus H^1(X, \Omega_X^1) \oplus H^0(X, \Omega_X^2)$$

Since the canonical line bundle is trivial, there is a unique (up to scaling) holomorphic 2-form $\omega \in H^0(X, \Omega_X^2)$ and under the Hodge decomposition it gives rise to an element in $\Lambda_X \otimes_{\mathbb{Z}} \mathbb{C}$. We note that $\omega^2 = 0$ and $\omega \cdot \bar{\omega} > 0$. Thus, associated to X, one has the data of a lattice Λ_X and an element in $\Lambda_X \otimes_{\mathbb{Z}} \mathbb{C}$. Roughly speaking, the Torelli theorems for K3 surfaces state that a K3 surface is uniquely determined by this (bi-)linear algebra data. These theorems are a central topic of Kondō's book [6].

The *period domain* for K3 surfaces is

$$\Omega := \{\omega \in \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C}) : \omega \cdot \omega = 0, \ \omega \cdot \overline{\omega} > 0\},$$

where $\Lambda := U^3 \perp E_8^2$ as above. We note that Ω is a bounded symmetric domain of type IV. A K3 surface X defines a point $\phi(X) \in \Omega$ by associating to X the (bi)-linear algebra data just defined. ϕ is called the *period map* and the *local Torelli theorem* states that ϕ identifies the Kuranishi space, that is, the local deformation space of X, with a small neighbourhood of $\phi(X)$ in Ω. From there, one studies the surjectivity of ϕ and establishes a *global Torelli theorem*. Understanding the situation for K3 surfaces of Kummer type is an important step in the proofs. These results are due to Ilya Piatetski-Shapiro and Igor Shafarevich (1971) and Dan Burns and Michael Rapoport (1975). The proof of these results takes up Chaps. 5, 6, and 7 in [6] and the background in lattice theory and reflection groups is discussed in Chaps. 1 and 2 of [6]. In particular, the Torelli theorem including a complete proof takes up a large part of this book.

Mathieu. An important application of the Torelli theorem is that it translates the study of the automorphism group $\mathrm{Aut}(X)$ of a K3 surface X into questions about isometries of the lattice Λ_X. For example, if X has Picard rank one (which holds true for a very general algebraic K3 surface), then $\mathrm{Aut}(X)$ must be trivial. On the other extreme, there exist K3 surfaces with infinite $\mathrm{Aut}(X)$. In any case, $\mathrm{Aut}(X)$ is discrete. To give a non-trivial example, we note that for the Fermat quartic surface F_4, permutation of the four variables x, y, z, w gives rise to an action of the symmetric group S_4 on this surface and thus, $S_4 \subset \mathrm{Aut}(F_4)$, see also Example 8.20 of [6]. For details, see Chap. 11 of [6].

In Chap. 12 of [6], the automorphism groups of K3 surfaces of Kummer type are studied in detail: here, the *Leech lattice*, a certain 24-dimensional unimodular and even lattice that was discovered by John Leech in 1967, plays a rôle. For explicit realisations of such automorphisms, one can use *Cremona transformations* of $\mathbb{P}^3$ (certain birational and rational self-maps introduced by Luigi Cremona in the 1860's) that induce birational automorphisms on quartic surfaces in $\mathbb{P}^3$.

A completely surprising result is the following theorem of Shigeru Mukai (1988): let M_{23} be the Mathieu group of degree 23, which is a finite simple sporadic group of order 10, 200, 960. One can realise M_{23} as a permutation group acting on a set Ω of order 23. Given a finite group G, there exists a K3 surface X with a faithful G-action and such that G acts trivially on $H^0(X, \Omega_X)$ (such actions are called *symplectic*) if and only if G can be embedded into M_{23}, such that its order of orbits on Ω is greater than or equal to 5. The question whether this is more of a coincidence, whether there is a deeper reason, and what precisely the rôle of M_{23} is in this context (note that M_{23} itself does not occur as automorphism group of a K3 surface!), is still subject to current research. For details, see Chap. 11 of [6].

Enriques. Given a K3 surface X and a fixed point free involution ι on it, the quotient X/ι is a compact complex 2-dimensional manifold that is called an *Enriques surface*. Federigo Enriques (1896) constructed the first examples of such surfaces in order to obtain counter-examples to a question of Guido Castelnuovo about a cohomological characterisation of rational surfaces. (This led Castelnuovo to the 'correct' cohomological rationality criterion and such a criterion in dimension at least three is still unknown.) Using the K3 double cover of an Enriques surface, one can construct period maps, establish Torelli theorems, and study their automorphism groups, which leads to a theory that is parallel to that of K3 surfaces. This is discussed in Chap. 9 of [6]. I would also like to mention the forthcoming books [3, 4] on Enriques surfaces.

Kondō. In Chap. 10 of [6], an application to moduli spaces of curves is given: let $f(x, y, w)$ be a homogenous quartic, such that $f(x, y, z) = 0$ defines a smooth complex curve C of degree 4 in $\mathbb{P}^2$. Then, $z^4 - f(x, y, w) = 0$ is a K3 surface in $\mathbb{P}^3$. This gives rise to an interesting interplay between plane curves of degree 4 and certain K3 surfaces. As an application, Shigeyuki Kondō (2000) showed that the moduli space of non-hyperelliptic curves of genus 3 is a 6-dimension complex ball quotient.

Summing up, Shigeyuki Kondō's book treats the complex analytic side of K3 surfaces with an emphasis on the Torelli theorem, on automorphism groups, and on special classes of K3 surfaces. It does not try to cover as many topics as possible, but rather discusses the chosen topics - especially the Torelli theorems - in detail and including proofs. For a different perspective on this subject, I would like to mention the recent book by Daniel Huybrechts [5], which also focuses on K3 surfaces over the complex numbers, but which treats slightly different topics, such as the Kuga-Satake correspondence, moduli spaces of sheaves, derived categories, rational curves, and Brauer groups.

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