Received: 26 March 2021 | Revised: 11 November 2021 | Accepted: 1 January 2022

DOI: 10.1002/rnc.6015

RESEARCH ARTICLE

WILEY

Model reference adaptive control of piecewise affine systems with state tracking performance guarantees

Tong Liu | **Martin Buss**

Chair of Automatic Control Engineering, Technische Universität München, Munich, Germany

Correspondence
Tong Liu, Chair of Automatic Control Engineering, Technische Universität München, Munich 80333, Germany.
Email: tong.liu@tum.de

Abstract

In this paper, we investigate the model reference adaptive control approach for uncertain piecewise affine systems with state tracking performance guarantees. The proposed approach ensures the error metric, defined as the weighted Euclidean norm of the state tracking error, to be confined within a user-defined time-varying performance bound. We introduce an auxiliary performance bound to construct a barrier Lyapunov function. This auxiliary performance bound is reset at each switching instant, which prevents the barrier transgression caused by the jumps of the error metric at switching instants. The dwell time constraints are derived such that the auxiliary performance bound resides within the user-defined performance bound. We prove that the Lyapunov function is nonincreasing at and in between the switching instants. Therefore, it does not impose extra dwell time constraints and ensures the error metric to fulfill the performance guarantees. Furthermore, we study the robust modification of the adaptive controller for the uncertain piecewise affine systems subject to unmatched disturbances. A numerical example validates the correctness of the proposed approach.

KEYWORDS

barrier Lyapunov function, piecewise affine systems, robust adaptive control, time-varying performance guarantees

1 | INTRODUCTION

The study of piecewise affine (PWA) systems has attracted significant interest due to their capability to approximate nonlinear systems and model hybrid systems. A PWA system consists of several linear subsystems. Each subsystem is associated with a certain region in the state space. Depending on in which region the state vector lies, the PWA system is governed by the associated subsystem dynamics. The switching from one subsystem to another subsystem is triggered, when the state trajectory goes through the boundary of two neighboring regions. Therefore, PWA systems represent a class of state-dependent switched systems. Early studies of PWA systems focus on the controllability and observability,[1,2] convergence analysis,[3] and control synthesis,[4,5] where the system parameters and region partitions are exactly known.

In the physical world, an exact system model is mostly not accessible due to uncertainties and disturbances. Therefore, introducing the adaptive mechanism into the uncertain PWA systems has significant meaning, especially when the uncertainties and disturbances are so large that a single robust controller cannot stabilize the closed-loop system. Due to the hybrid nature of the PWA systems, not only the uncertain parameters need to be estimated by designing adaptation laws, but also the switching behavior of the closed-loop system needs to be carefully considered. In the last decade, model reference adaptive control (MRAC) approaches have been investigated for uncertain PWA systems. The methods proposed in the work of di Bernardo et al.[6-8] rely on common Lyapunov functions, where the closed-loop systems are allowed to switch arbitrarily fast. MRAC for piecewise linear systems, a special version of PWA systems, are investigated in the work of Sang and Tao,[9,10] where dwell time constraints for switches are given to ensure the closed-loop stability. Its extension to PWA systems is reported recently,[11] where the exponential decaying of the state tracking error is proved given that a persistently exciting (PE) condition and some dwell time constraints are fulfilled. To enhance the robustness of the adaptive switched systems against disturbances and time-delay, some robust MRAC approaches have been proposed for switched linear systems, whose formulation is similar to PWA systems but with switching signals given externally. These include robust MRAC with dead zone[12] and leakage,[13] robust H_∞ MRAC[14,15] as well as control approaches with asynchronous switching between subsystems and controllers.[16,17]

Despite the aforementioned advances, the adaptive control for PWA systems fulfilling a user-defined performance guarantee (such as state constraints) is rarely studied. In light of the fact that a lot of systems in practice have state constraints like physical or operational boundaries, saturation, performance and safety specifications, we would like to explore the MRAC of generalized PWA systems with state tracking performance guarantees.

Notable progress has been made in the field of adaptive control with performance guarantees. These include funnel control,[18,19] barrier Lyapunov function-based approach,[20] and prescribed performance control.[21,22] All of these methods are proposed to confine the output tracking error within the predefined constraints. Although some recent barrier Lyapunov function-based controllers achieve the full state constraints,[23-26] they are built upon the backstepping structure, which requires the controlled system to be in strict feedback form or pure feedback form. Thus, they cannot be applied to generalized PWA systems. Recently, a set-theoretic MRAC for linear systems is developed.[27] It uses the barrier Lyapunov function concept to confine the weighted Euclidean norm of the state tracking error within a predefined bound. The controller does not rely on the backstepping-type analysis and therefore does not impose restrictions on the system structure. This method is extended to the cases with time-varying performance bounds,[28] systems with actuator faults,[29] and systems with unstructured uncertainties.[30] However, applying this method to switched systems is nontrivial and challenging. If the barrier Lyapunov function is constructed with the user-defined performance bound being the barrier, as it is done in the linear system case, then the discontinuity of the weighted Euclidean norm of the tracking error at switching instants may cause transgression of the barrier, which makes the barrier Lyapunov function invalid. Besides, only matched uncertainties (uncertainties, which can be compensated with an additional input term) are addressed in the work of set-theoretic MRAC approaches. Since the PWA systems are mostly approximations of nonlinear systems, their approximation errors are not necessarily matched, let alone other kinds of external disturbances. How to enhance the robustness against unmatched uncertainties/disturbances when applying the set-theoretic MRAC to PWA systems is still open.

The main contribution of this paper is twofold. First, a set-theoretic MRAC approach for uncertain PWA systems with state tracking performance guarantees is developed. Second, a robust modification of this method is proposed for PWA systems subject to unmatched disturbances. Specifically, we impose an auxiliary performance bound with a state reset map to construct the barrier Lyapunov function, which bypasses the barrier transgression problem. The dwell time constraints are derived based on the auxiliary performance bound and the user-defined performance bound. The Lyapunov function is nonincreasing, even at switching instants, and therefore, does not impose extra dwell time constraints. Furthermore, a projection-based robust modification of the proposed approach is developed to enhance the robustness against disturbances. Compared with the state-of-the-art set-theoretic MRAC approaches, the disturbances are not required to be matched. This allows broader applications of the proposed method.

The paper is structured as follows. The definition of PWA systems, MRAC, and the performance function are revisited in Section 2. The proposed method is explained in Section 3, in which the stability analysis is also provided. The robust modification is shown in Section 3.4. A numerical example is illustrated in Section 4.

Notations: In this paper, $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ and $\mathbb{N}^+$ denote the set of real numbers, positive real numbers, natural numbers, and positive natural numbers, respectively. $\mathrm{tr}(\cdot)$ represents the trace of a matrix. The Euclidean norm is denoted by $\|\cdot\|_2$. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ represent the maximal and minimal eigenvalues of matrix P, respectively. $\|e\|_P = (e^T P e)^{\frac{1}{2}}$ represents the weighted Euclidean norm of $e \in \mathbb{R}^n$ with the weighting matrix $P \in \mathbb{R}^{n\times n}$.

2 | PRELIMINARIES AND PROBLEM STATEMENT

Let the state space be partitioned into $s \in \mathbb{N}^+$ convex regions $\{\Omega_i\}_{i=1}^s$ without overlaps, that is, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $i,j \in \mathcal{I} \triangleq \{1, 2, \dots, s\}$. The PWA system is of the form

$$\dot{x} = A_i x(t) + B_i u(t) + f_i, \quad x(t) \in \Omega_i, \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times p}$ and $f_i \in \mathbb{R}^n$. $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ denote its state and control input signal. To characterize in which region the state vector locates, we define the following indicator function

$$\chi_i(t) = \begin{cases} 1, & \text{if } x(t) \in \Omega_i \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

Since the regions $\{\Omega_i\}_{i=1}^s$ have no overlaps, we have $\sum_{i=1}^s \chi_i = 1$ and $\prod_{i=1}^s \chi_i = 0$. Thus, the PWA system can be written as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t), \tag{3}$$

with $A(t) = \sum_{i=1}^s \chi_i(t)A_i$, $B(t) = \sum_{i=1}^s \chi_i(t)B_i$ and $f(t) = \sum_{i=1}^s \chi_i(t)f_i$.

In this paper, the reference system is also chosen to be a PWA model, which provides more design flexibility for the user. Without loss of generality, we let the reference PWA system (4) and the controlled PWA system (3) have the same region partitions and therefore, the same indicator functions. The PWA reference system is given by

$$\dot{x}_m(t) = A_m(t)x_m(t) + B_m(t)r(t) + f_m(t), \tag{4}$$

where $x_m \in \mathbb{R}^n$ and $r \in \mathbb{R}^p$ denote the state and input of the reference system, $A_m(t) = \sum_{i=1}^s \chi_i(t)A_{mi}$, $B_m(t) = \sum_{i=1}^s \chi_i(t)B_{mi}$, $f_m(t) = \sum_{i=1}^s \chi_i(t)f_{mi}$ with $A_{mi} \in \mathbb{R}^{n\times n}, B_{mi} \in \mathbb{R}^{n\times p}, f_{mi} \in \mathbb{R}^n, i \in \mathcal{I}$ being the parameters of the reference system. A_{mi} are Hurwitz matrices and there exists a set of positive definite matrices P_i and $Q_i \in \mathbb{R}^{n\times n}, i \in \mathcal{I}$ such that

$$A_{mi}^T P_i + P_i A_{mi} = -Q_i, \quad \forall i \in \mathcal{I}. \tag{5}$$

For each subsystem, a set of controller gains is utilized. Let $K_{xi}^* \in \mathbb{R}^{p\times n}, K_{ri}^* \in \mathbb{R}^{p\times p}, K_{fi}^* \in \mathbb{R}^p, i \in \mathcal{I}$ denote the nominal controller gains for the ith subsystem of (3). The controller gains and the system parameters switch synchronously. Therefore, the controller takes the form

$$u(t) = K_x^* x(t) + K_r^* r(t) + K_f^*, \tag{6}$$

where $K_x^*(t) = \sum_{i=1}^s \chi_i(t)K_{xi}^*$, $K_r^*(t) = \sum_{i=1}^s \chi_i(t)K_{ri}^*$, $K_f^*(t) = \sum_{i=1}^s \chi_i(t)K_{fi}^*$. Taking (6) into (3) yields the closed-loop system. To obtain a closed-loop system having the same behavior as the reference system, we make the usual assumption that following matching equations hold:

$$A_{mi} = A_i + B_i K_{xi}^*, \quad B_{mi} = B_i K_{ri}^*, \quad f_{mi} = f_i + B_i K_{fi}^*, \quad \forall i \in \mathcal{I}. \tag{7}$$

These are typical conditions for state feedback state tracking design. Note that not every system can satisfy such matching equations, some relaxation approaches (state feedback output tracking and output feedback output tracking) can be found in section 4 of Tao's survey.[31] Since A_i, B_i, f_i are unknown, the nominal controller gains $K_{xi}^*, K_{ri}^*, K_{fi}^*$ are not available. Let $K_{xi}(t) \in \mathbb{R}^{p\times n}, K_{ri}(t) \in \mathbb{R}^{p\times p}, K_{fi}(t) \in \mathbb{R}^p$ be the estimates of $K_{xi}^*, K_{ri}^*, K_{fi}^*$. We introduce the following adaptive controller

$$u(t) = K_x(t)x(t) + K_r(t)r(t) + K_f(t), \tag{8}$$

with $K_x(t) = \sum_{i=1}^s \chi_i(t)K_{xi}(t)$, $K_r(t) = \sum_{i=1}^s \chi_i(t)K_{ri}(t)$ and $K_f(t) = \sum_{i=1}^s \chi_i(t)K_{fi}(t)$. Inserting (8) into the controlled PWA system (3) and defining the state tracking error $e(t) = x(t) - x_m(t)$, we have

$$\dot{e} = A_m e + \sum_{i=1}^s \chi_i B_i(\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi}), \tag{9}$$

where $\tilde{K}_{xi} = K_{xi} - K^*_{xi}, \tilde{K}_{ri} = K_{ri} - K^*_{ri}, \tilde{K}_{fi} = K_{fi} - K^*_{fi}$.

We define t_0 to be the initial time instant and the set $\{t_1, t_2, \cdots, t_k, \cdots | k \in \mathbb{N}^+\}$ to be the set of switching time instants.

Definition 1 (Dwell time[32]). The switching of a switched system is said to be with dwell time if there exists a number $\tau_D > 0$ such that the time between every two consecutive switches is no smaller than τ_D, that is, constraint $t_{k+1} - t_k \geq \tau_D$ holds for $\forall k \in \mathbb{N}^+$. Such positive number τ_D is called *dwell time*.

In this paper, we would like to design an adaptive controller for PWA systems such that the norm of the state tracking error e is enforced within a predefined performance bound such that the closed-loop system has performance guarantees. The performance bound can be formulated by a performance function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, a smooth and decreasing function satisfying $\lim_{t\to\infty} \rho(t) = \rho_\infty > 0$. We adopt the following commonly used performance function[21]

$$\rho(t) = (\rho_0 - \rho_\infty)\mathrm{e}^{-l(t-t_0)} + \rho_\infty, \tag{10}$$

where $\rho_0, \rho_\infty, l \in \mathbb{R}^+$ and $\rho_0 > \rho_\infty$. We can see that $\rho(t)$ is smooth and decreasing with the initial value $\rho(t = t_0) = \rho_0$ and the final value $\rho(t \to \infty) = \rho_\infty$. The initial value ρ_0 can be chosen such that the performance guarantee (introduced in (11)) can be satisfied at the initial time instant. The final value ρ_∞ implies a steady-state error bound, while l determines the convergence speed toward ρ_∞. The performance guarantee to be satisfied can be formulated as

$$\|e(t)\|_P < \rho(t). \tag{11}$$

The error metric $\|e(t)\|_P$ serves as a performance measure reflecting the difference between the state of the controlled system and the reference system. P is equal to P_i if subsystem i is activated, that is, $P = \sum_{i=1}^{s} \chi_i(t)P_i$, where the weighting matrices P_i satisfy (5). So the error metric $\|e(t)\|_P$ and the system parameters switch synchronously.

Remark 1. Some questions may arise regarding (11): is it feasible to specify a global weighting matrix for the error metric instead of the switching one? What if the user would like to define a performance guarantee with an arbitrary weighting matrix, which does not necessarily satisfy the Lyapunov equation (5)? In fact, these requirements can be transformed into the formulation (11). We explain this point in the following.

Suppose that a global performance measure, which should hold for every subsystem, is desired by the user, that is, $\|e(t)\|_S < \rho^*(t)$, where $S \in \mathbb{R}^{n\times n}$ is an arbitrary user-defined positive definite matrix and $\rho^*(t)$ represents a user-defined performance function in form of (10). Then, we can choose $P_i, i \in \mathcal{I}$ matrices based on (5). We know $\|e\|_S \leq \frac{1}{\gamma}\|e\|_P$ with $\gamma = \min_{i\in\mathcal{I}} \sqrt{\frac{\lambda_{\min}(P_i)}{\lambda_{\max}(S)}}$. To satisfy $\|e(t)\|_S < \rho^*(t)$, it suffices to let $\|e\|_P < \gamma\rho^*(t)$ hold, which is equivalent to (11) by letting $\rho(t) = \gamma\rho^*(t)$.

The problem to be studied in this paper is formulated as follows:

Problem 1. Given a performance function (10), a reference model (4) and a PWA system (3) with unknown subsystem parameters A_i, B_i, f_i and known regions Ω_i (or equivalently, known indicator functions $\chi_i(t)$), design an adaptive control law $u(t)$ such that the state $x(t)$ of (3) tracks the state $x_m(t)$ of (4) with the tracking error $e(t)$ satisfying the performance guarantee (11).

3 | ADAPTIVE CONTROL DESIGN

In this section, we propose the adaptive controller to solve the given problem in the disturbance-free case and study its robust modification. First, we introduce the auxiliary performance bound and explain the solution concept. Then the proposed adaptation laws are presented, which are followed by the stability analysis of the closed-loop system.

3.1 | Auxiliary performance bound

We define a generalized restricted potential function (barrier Lyapunov function)[28] $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ on the set $\mathcal{D}_\theta \triangleq \{e \mid \|e\|_P \in [0, \theta)\}$

$$\phi(\|e\|_P) = \frac{\|e\|_P^2}{\theta^2(t) - \|e\|_P^2}, \quad \|e\|_P < \theta(t). \tag{12}$$

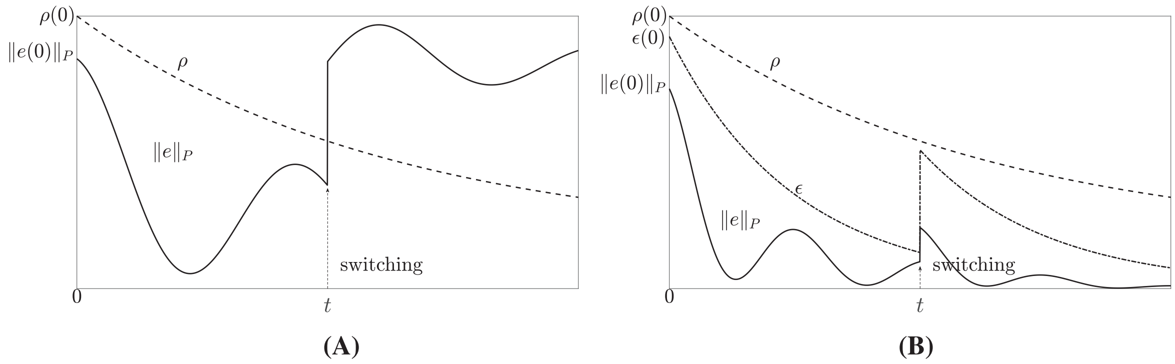

FIGURE 1 Graphical illustration of the barrier transgression problem and the concept to introduce auxiliary performance bound. (A) The barrier transgression problem; (B) Introducing auxiliary performance bound

By properly initializing the reference system or designing the performance function, we can let $\|e(t_0)\|_P < \rho(t_0)$. The set-theoretic MRAC approach for linear systems[28] suggests specifying the barrier θ to be $\rho(t)$ and designing the adaptation laws such that $\phi(\|e\|_P)$ is bounded $\forall t \in [t_0, \infty)$, then it would be obtained that $\|e(t)\|_P < \rho(t), \forall t \in [t_0, \infty)$.

The difficulty in switched systems is that $P = \sum_{i=1}^{s} \chi_i(t)P_i$ leads to the jumps of $\|e(t)\|_P$ at switching instants. Suppose $\chi_i(t) = 1$ for $t \in [t_{k-1}, t_k)$ and $\chi_j(t) = 1$ for $t \in [t_k, t_{k+1})$ for $i \neq j, i, j \in \mathcal{I}$, we have

$$\|e(t_k)\|_P^2 = e^T(t_k)P_j e(t_k) \leq \lambda_{\max}(P_j)\|e(t_k)\|_2^2 \leq \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)}\|e(t_k^-)\|_P^2, \tag{13}$$

which may result in $\|e(t_k)\|_P > \rho(t_k)$ for $\frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)} > 1$ and $\|e(t_k^-)\|_P < \rho(t_k^-)$, as shown in Figure 1A. This further makes the barrier function $\phi(\|e\|_P)$ invalid. We call this *barrier transgression* problem.

To overcome this problem, our idea is to introduce an auxiliary performance bound, denoted by $\epsilon(t)$, which decays faster than the user-defined performance bound $\rho(t)$. $\epsilon(t)$ is reset at each switching instant such that $\|e(t_k)\|_P < \epsilon(t_k)$ for $k \in \mathbb{N}^+$, see Figure 1B. If the adaptive controller ensures $\|e\|_P < \epsilon(t)$ and if $\epsilon(t)$ is designed such that $\epsilon(t) < \rho(t)$ for $t \in [t_0, \infty)$, then the control objective (11) is achieved.

We propose the auxiliary performance bound $\epsilon(t)$ with the following dynamics

$$\dot{\epsilon}(t) = -h\epsilon(t) + g, \quad \epsilon(t_0) \in \left(\frac{g}{h}, \rho_0\right), \quad \epsilon(t_k) = G(\epsilon(t_k^-)), \tag{14}$$

where $h, g \in \mathbb{R}^+$. $G : \mathbb{R}^+ \to \mathbb{R}^+$ is a state reset map. It resets the value of ϵ at each switching instant. Note that ϵ shares the same switching instants with the controlled PWA system $t_k, k \in \mathbb{N}^+$, that is, when the switch of the controlled PWA system occurs, ϵ is reset by the state reset map simultaneously. We specify the state reset map G to be

$$G(\epsilon(t_k^-)) = \sqrt{\mu}\epsilon(t_k^-), \quad \mu \triangleq \max_{i,j \in \mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}. \tag{15}$$

with $\mu > 1$. The parameters h, g, μ control the evolution of the auxiliary performance bound $\epsilon(t)$. Specifically, h represents the decreasing rate of $\epsilon(t)$. g serves as a input for the dynamics of ϵ and affects its minimum value. $\sqrt{\mu}$ reflects the increment of $\epsilon(t)$ at each switching instant. As stated before, $\epsilon(t)$ should be smaller than $\rho(t), \forall t \in [t_0, \infty)$. To achieve this, the state reset of $\epsilon(t)$ needs to satisfy some dwell time constraints, that is, $\min\{t_k - t_{k-1}\} \geq \tau_D, k \in \mathbb{N}^+$ for some $\tau_D \in \mathbb{R}^+$. We have the following lemma:

Lemma 1. *Given the performance function (10) and the auxiliary performance bound (14) with the reset map (15), if $h > l$, $\rho_\infty > \sqrt{\mu}\frac{g}{h}$ and if the dwell time of $\epsilon(t)$ satisfies*

$$\tau_D > \frac{1}{h-l} \ln \frac{\sqrt{\mu}\rho_\infty - \frac{g}{h}\sqrt{\mu}}{\rho_\infty - \frac{g}{h}\sqrt{\mu}}, \tag{16}$$

for $\mu > 1$, then the following inequality holds

$$\frac{g}{h} \leq \epsilon(t) < \rho(t), \quad \forall t \in [t_0, \infty). \tag{17}$$

Proof. The initial value of ϵ satisfies $\epsilon(t_0) > \frac{g}{h}$, meaning that ϵ decreases exponentially toward $\frac{g}{h}$ if no switch occurs. Since $\sqrt{\mu} > 1$, ϵ increases at each switching time instant and $\epsilon(t_k) > \frac{g}{h}$ for $\forall k \in \mathbb{N}^+$. If the switch terminates from some time on, then $\epsilon \to \frac{g}{h}$ for $t \to \infty$, otherwise, $\epsilon > \frac{g}{h}$ for $t \in [t_0, \infty)$. Therefore, we have $\epsilon(t) \geq \frac{g}{h}, \forall t \in [t_0, \infty)$.

Now, we explore the relationship between $\epsilon(t)$ and $\rho(t)$. We have for the time interval $[t_0, t_1)$

$$\epsilon(t) = \epsilon(t_0)\mathrm{e}^{-h(t-t_0)} + g\int_{t_0}^{t} \mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau = \left(\epsilon(t_0) - \frac{g}{h}\right)\mathrm{e}^{-h(t-t_0)} + \frac{g}{h}. \tag{18}$$

Since $\epsilon(t_0) \in \left(\frac{g}{h}, \rho_0\right)$, $h > l$ and $\rho_\infty > \sqrt{\mu}\frac{g}{h}$, we have $\epsilon(t) < \rho(t)$ for $t \in [t_0, t_1)$. For $t = t_1$ it gives

$$\epsilon(t_1) = \sqrt{\mu}\epsilon(t_1^-) = \sqrt{\mu}\left(\epsilon(t_0) - \frac{g}{h}\right)\mathrm{e}^{-h(t_1-t_0)} + \sqrt{\mu}\frac{g}{h}. \tag{19}$$

Let $\Delta t_1 \triangleq t_1 - t_0$, we have

$$\begin{aligned}
\rho(t_1) - \epsilon(t_1) &= (\rho_0 - \rho_\infty)\mathrm{e}^{-l\Delta t_1} - \sqrt{\mu}\left(\epsilon(t_0) - \frac{g}{h}\right)\mathrm{e}^{-h\Delta t_1} + \left(\rho_\infty - \sqrt{\mu}\frac{g}{h}\right) \\
&\geq (\rho_0 - \rho_\infty)\mathrm{e}^{-l\Delta t_1} - \sqrt{\mu}\left(\epsilon(t_0) - \frac{g}{h}\right)\mathrm{e}^{-h\Delta t_1} + \left(\rho_\infty - \sqrt{\mu}\frac{g}{h}\right)\mathrm{e}^{-l\Delta t_1} \\
&= (\rho_0 - \sqrt{\mu}\frac{g}{h})\mathrm{e}^{-l\Delta t_1} - \sqrt{\mu}\left(\epsilon(t_0) - \frac{g}{h}\right)\mathrm{e}^{-h\Delta t_1} \\
&\geq \left(\rho_0 - \sqrt{\mu}\frac{g}{h}\right)\mathrm{e}^{-l\Delta t_1} - \sqrt{\mu}\left(\rho_0 - \frac{g}{h}\right)\mathrm{e}^{-h\Delta t_1}.
\end{aligned} \tag{20}$$

If the inequality

$$\left(\rho_0 - \sqrt{\mu}\frac{g}{h}\right)\mathrm{e}^{-l\Delta t_1} > \sqrt{\mu}\left(\rho_0 - \frac{g}{h}\right)\mathrm{e}^{-h\Delta t_1}, \tag{21}$$

holds, we will immediately have $\rho(t_1) > \epsilon(t_1)$. Since $\rho_0 > \rho_\infty > \sqrt{\mu}\frac{g}{h} > \frac{g}{h}$, we have $\rho_0 - \sqrt{\mu}\frac{g}{h} > 0$ and $\sqrt{\mu}(\rho_0 - \frac{g}{h}) > 0$. Therefore, (21) is equivalent to

$$\frac{\rho_0 - \sqrt{\mu}\frac{g}{h}}{\sqrt{\mu}\left(\rho_0 - \frac{g}{h}\right)} > \mathrm{e}^{-(h-l)\Delta t_1}. \tag{22}$$

Taking the logarithm of both sides we obtain

$$\Delta t_1 > \frac{1}{h-l}\ln\frac{\sqrt{\mu}\rho_0 - \frac{g}{h}\sqrt{\mu}}{\rho_0 - \frac{g}{h}\sqrt{\mu}}. \tag{23}$$

Following the above analysis we can obtain $\epsilon(t) < \rho(t)$ for $t \in [t_{k-1}, t_k)$ and $\epsilon(t_k) < \rho(t_k)$ for $k \in \mathbb{N}^+$ if

$$\Delta t_k > \frac{1}{h-l}\ln\frac{\sqrt{\mu}\rho(t_{k-1}) - \frac{g}{h}\sqrt{\mu}}{\rho(t_{k-1}) - \frac{g}{h}\sqrt{\mu}} = \frac{1}{h-l}\ln\left(\sqrt{\mu} + \frac{(\mu - \sqrt{\mu})\frac{g}{h}}{\rho(t_{k-1}) - \frac{g}{h}\sqrt{\mu}}\right). \tag{24}$$

If the dwell time τ_D is no smaller than the maximal required interval length $\max\{\Delta t_k\}$, then $\epsilon(t) < \rho(t)$ holds for $\cup[t_{k-1}, t_k), k \in \mathbb{N}^+$. Because $\rho(t_{k-1}) \geq \rho_\infty$ for $k \in \mathbb{N}^+$, we have

$$\tau_D \geq \max\{\Delta t_k\} > \frac{1}{h-l}\ln\frac{\sqrt{\mu}\rho_\infty - \frac{g}{h}\sqrt{\mu}}{\rho_\infty - \frac{g}{h}\sqrt{\mu}}. \tag{25}$$

So we can conclude that if (16) holds, then $\epsilon(t) < \rho(t)$ for $t \in [t_0, \infty)$. ■

Lemma 1 tells the dwell time constraint to be fulfilled. We will further discuss how this dwell time constraint can be satisfied later following Remark 4. Since ϵ, the reference system (4) and the closed-loop system share the same switching signal, the first question to ask is, if the reference system is stable with the dwell time constraint (16)? This is answered by the following lemma.

Lemma 2. *The reference system (4) satisfying (5) is stable with the dwell time constraint (16) and h satisfying (28).*

The proof of Lemma 2 can be seen in Appendix A.1.

3.2 | Adaptation laws

Based on the auxiliary performance bound proposed in Section 3.1, we define the following generalized restricted potential function (barrier Lyapunov function) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$

$$\phi(\|e\|_P) = \frac{\|e\|_P^2}{\epsilon^2(t) - \|e\|_P^2}, \quad \|e\|_P < \epsilon(t), \tag{26}$$

with $P = \sum_{i=1}^{s} \chi_i(t) P_i$. Since $\|e\|_P^2$ and $\epsilon^2(t)$ are piecewise continuous and piecewise differentiable, the partial derivative of ϕ with respect to $\|e\|_P^2$ over the time interval $[t_k, t_{k+1}), k \in \mathbb{N}^+$ takes the form $\phi_d \triangleq \partial\phi/\partial\|e\|_P^2 = \epsilon^2(t)/(\epsilon^2(t) - \|e\|_P^2)^2 > 0$. ϕ and ϕ_d have the property that $2\phi_d(\|e\|_P)\|e\|_P^2 - \phi > 0$.

The adaptation laws of the estimated controller gains are given as

$$\dot{K}_{xi} = -\chi_i \Gamma_{xi} \phi_d S_i^T B_{mi}^T P_i e x^T, \quad \dot{K}_{ri} = -\chi_i \Gamma_{ri} \phi_d S_i^T B_{mi}^T P_i e r^T, \quad \dot{K}_{fi} = -\chi_i \Gamma_{fi} \phi_d S_i^T B_{mi}^T P_i e, \tag{27}$$

where $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} \in \mathbb{R}^+$ are positive scaling factors. $S_i \in \mathbb{R}^{p\times p}$ is a matrix such that there exists a symmetric and positive definite matrix $M_i \in \mathbb{R}^{p\times p}$ with $(K_{ri}^* S_i)^{-1} = M_i$. Here we make the usual assumption in multivariable adaptive control[31] that S_i is known. The use of the indicator functions $\chi_i(t)$ in the adaptation laws (27) implies that the controller gains associated with a certain subsystem are updated only when this subsystem is activated. Their adaptation terminates and their values stay unchanged during the inactive phase of the corresponding subsystem. Note that ϕ_d in (27) can also be viewed as an error-dependent gain, whose effect can be weakened or amplified by tuning the constant gains $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi}$. They are chosen by trial and error in the simulation. If $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi}$ are too small, the effect of ϕ_d on the adaptation speeds $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi}$ is weakened. Consequently, ϕ and ϕ_d may have every small denominators and become ill-conditioned. If $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi}$ are too large, the differential equations may become "stiff" and difficult to solve numerically.

3.3 | Stability analysis

The tracking performance and the stability of the closed-loop system are summarized in the following theorem.

Theorem 1. *Given the reference PWA system (4) and the predefined performance function (10), let the PWA system (3) with known regions $\Omega_i, i \in \mathcal{I}$ and unknown subsystem parameters $A_i, B_i, f_i, i \in \mathcal{I}$ be controlled by the feedback controller (8) with the adaptation laws (27). Let the initial state of ϵ satisfy $\|e(t_0)\|_P < \epsilon(t_0)$. The closed-loop system is stable and the state tracking error $e(t)$ fulfills the prescribed performance guarantees (11) if the time constant h in (14) satisfies*

$$h < \frac{1}{2} \min_{i\in\mathcal{I}} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)}, \tag{28}$$

and if the switching signal of the controlled PWA system obeys the dwell time constraint τ_D in (16).

Proof. Without loss of generality, we let the scaling factors in (27) be 1. Consider the following Lyapunov function

$$V = \phi(\|e\|_P) + \underbrace{\sum_{i=1}^{s} \left(\operatorname{tr}\left(\tilde{K}_{xi}^T M_i \tilde{K}_{xi} \right) + \operatorname{tr}\left(\tilde{K}_{ri}^T M_i \tilde{K}_{ri} \right) + \tilde{K}_{fi}^T M_i \tilde{K}_{fi} \right)}_{\triangleq V_\theta}. \tag{29}$$

V is piecewise continuous and piecewise differentiable. In particular, V is continuous and differentiable in between any two consecutive switching instants $[t_{k-1}, t_k), k \in \mathbb{N}^+$, while it is nondifferentiable and (possibly) discontinuous at each switching instant $t_k, k \in \mathbb{N}^+$. The mixture of the continuous evolution and the discontinuous jumps of V constitutes the main challenge of the stability analysis of switched systems. The overall idea is to prove $\dot{V} \leq 0$ in between switches and evaluate the incremental or decremental jumps at each switching instant. First of all, we would like to study the evolution of V in the continuous phase (named *phase 1*), namely, in between two consecutive switches:

phase 1: $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$

V is continuous in the intervals between two successive switches. Without loss of generality, we suppose that the ith subsystem is activated for $t \in [t_{k-1}, t_k)$ and $e(t_{k-1})$ satisfies $\|e(t_{k-1})\|_{P_i} < \epsilon(t_{k-1})$. The time-derivative of V in $[t_{k-1}, t_k)$ is given by

$$\dot{V} = \dot{\phi}(\|e\|_{P_i}) + 2\sum_{i=1}^{s}\left(\operatorname{tr}\left(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}\right) + \operatorname{tr}\left(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}\right) + \tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}\right). \tag{30}$$

First, we simplify the second term of $\dot{V}$. Taking the adaptation laws (27) into the first summand of the second term of $\dot{V}$ gives

$$\operatorname{tr}\left(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}\right) = -\chi_i \phi_d \operatorname{tr}\left(\tilde{K}_{xi}^T M_i S_i^T B_{mi}^T P_i e x^T\right) \tag{31}$$

Since $\left(K_{ri}^* S_i\right)^{-1} = M_i$ and $B_i K_{ri}^* = B_{mi}$, we have $M_i S_i^T B_{mi}^T = M_i S_i^T \left(B_i K_{ri}^*\right)^T = M_i M_i^{-1} B_i^T = B_i^T$, which further gives

$$\operatorname{tr}\left(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}\right) = -\chi_i \phi_d \operatorname{tr}\left(\tilde{K}_{xi}^T B_i^T P_i e x^T\right) = -\chi_i \phi_d \operatorname{tr}\left(x e^T P_i B_i \tilde{K}_{xi}\right) = -\chi_i \phi_d \operatorname{tr}\left(e^T P_i B_i \tilde{K}_{xi} x\right) = -\chi_i \phi_d e^T P_i B_i \tilde{K}_{xi} x. \tag{32}$$

Doing the same simplification for $\operatorname{tr}\left(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}\right)$ and $\tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}$ we have

$$2\sum_{i=1}^{s}\left(\operatorname{tr}\left(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}\right) + \operatorname{tr}\left(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}\right) + \tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}\right) = -2\sum_{i=1}^{s} \chi_i \phi_d e^T P_i B_i \left(\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi}\right), \tag{33}$$

$\dot{\phi}$ can be further simplified as

$$\dot{\phi} = \frac{\partial \phi}{\partial \|e\|_{P_i}^2} \frac{\mathrm{d}\|e\|_{P_i}^2}{\mathrm{d}t} + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} = 2\phi_d(\|e\|_{P_i}) e^T P_i \dot{e} + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon}. \tag{34}$$

Substituting $\dot{e}$ with (9) yields

$$\begin{aligned}
\dot{\phi} &= \phi_d(e^T(A_m^T P_i + P_i A_m)e + 2e^T P_i \sum_{i=1}^{s} \chi_i B_i(\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi})) + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \\
&= -\phi_d e^T Q_i e + 2\sum_{i=1}^{s} \chi_i \phi_d e^T P_i B_i(\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi}) + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon}.
\end{aligned} \tag{35}$$

Therefore, $\dot{V}$ can be simplified as

$$\dot{V} = -\phi_d e^T Q_i e + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon}, \tag{36}$$

with

$$\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} = \frac{-2\epsilon\|e\|_{P_i}^2}{(\epsilon^2 - \|e\|_{P_i}^2)^2}\dot{\epsilon} = -2\phi_d(\|e\|_{P_i})\|e\|_{P_i}^2 \frac{\dot{\epsilon}}{\epsilon} \leq 2\phi_d(\|e\|_{P_i})\|e\|_{P_i}^2 \frac{|\dot{\epsilon}|}{\epsilon}. \tag{37}$$

Invoking Lemma 1, we have $\epsilon(t) \geq \frac{g}{h}, \forall t \in [t_0, \infty)$. Therefore,

$$\frac{|\dot{\epsilon}|}{\epsilon} = \frac{h\epsilon - g}{\epsilon} = h - \frac{g}{\epsilon} \leq h, \tag{38}$$

which leads to

$$\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \le 2h\phi_d(\|e\|_{P_i})\|e\|_{P_i}^2. \tag{39}$$

Taking this into (36) yields

$$\dot{V} \le -\phi_d\|e\|_2^2\lambda_{\min}(Q_i) + 2h\phi_d\|e\|_2^2\lambda_{\max}(P_i) = -\phi_d\|e\|_2^2(\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)). \tag{40}$$

From the condition (28) it follows $\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i) > 0$, which together with the property $2\phi_d(\|e\|_P)\|e\|_P^2 - \phi > 0$ gives

$$\dot{V} \le -\frac{\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)}{2\lambda_{\max}(P_i)}\phi \le 0. \tag{41}$$

The fact $\dot{V} \le 0$ in intervals $[t_{k-1}, t_k), k \in \mathbb{N}^+$ implies that the Lyapunov function decreases between two consecutive switches. ϕ and ϕ_d are bounded in $[t_{k-1}, t_k)$. Since $\|e(t_{k-1})\|_{P_i} < \epsilon(t_{k-1})$, we have $\|e(t)\|_{P_i} < \epsilon(t)$ for $\forall t \in [t_{k-1}, t_k)$.

The property $\dot{V} \le 0$ for each $[t_{k-1}, t_k)$ does not imply the global stability of the closed-loop system over the whole $t \in [t_0, \infty)$. It is necessary to evaluate the discontinuity of V at each switching instant (*phase 2*):

phase 2: jump at switch instant $t_k, k \in \mathbb{N}^+$

Now we analyse the behavior of the Lyapunov function at the switching time instants. Suppose that ith subsystem is activated in $[t_{k-1}, t_k)$ and jth subsystem is activated in $[t_k, t_{k+1})$, where $i, j \in \mathcal{I}, i \ne j$. From the adaptation laws of the estimated controller gains (27), we see that the estimated controller gains are continuous, that is, $\tilde{K}_{xi}(t_k) = \tilde{K}_{xi}(t_k^-), \tilde{K}_{ri}(t_k) = \tilde{K}_{ri}(t_k^-)$ and $\tilde{K}_{fi}(t_k) = \tilde{K}_{fi}(t_k^-)$ for $\forall i \in \mathcal{I}$, from which it follows $V_\theta(t_k^-) = V_\theta(t_k)$. To study the relationship between $V(t_k)$ and $V(t_k^-)$, it remains to analyse $\phi(\|e(t_k)\|_P)$ and $\phi(\|e(t_k^-)\|_P)$. Since $e(t)$ is also continuous, $e(t_k) = e(t_k^-)$. This results in

$$\|e(t_k)\|_P^2 = e^T(t_k)P_je(t_k) \le \lambda_{\max}(P_j)\|e(t_k)\|_2^2 \le \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)}e^T(t_k)P_ie(t_k) = \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)}\|e(t_k^-)\|_P^2 \le \mu\|e(t_k^-)\|_P^2. \tag{42}$$

From the analysis of *phase 1*, we already know that $\|e(t_k^-)\|_P < \epsilon(t_k^-)$. ϵ is reset at t_k and we have

$$\|e(t_k)\|_P \le \sqrt{\mu}\|e(t_k^-)\|_P < \sqrt{\mu}\epsilon(t_k^-) = \epsilon(t_k), \tag{43}$$

which makes the potential function $\phi(\|e(t_k)\|_P)$ also valid at t_k. Recalling the dynamics of ϵ (14) and the above inequalities (42), we have

$$\phi(\|e(t_k)\|_P) = \frac{\|e(t_k)\|_P^2}{\epsilon^2(t_k) - \|e(t_k)\|_P^2} \le \frac{\mu\|e(t_k^-)\|_P^2}{\epsilon^2(t_k) - \mu\|e(t_k^-)\|_P^2} = \frac{\mu\|e(t_k^-)\|_P^2}{\mu\epsilon^2(t_k^-) - \mu\|e(t_k^-)\|_P^2} = \phi(\|e(t_k^-)\|_P). \tag{44}$$

Combining the facts $\phi(\|e(t_k)\|_P) \le \phi(\|e(t_k^-)\|_P)$ and $V_\theta(t_k^-) = V_\theta(t_k)$, we have

$$V(t_k) = \phi(\|e(t_k)\|_P) + V_\theta(t_k) \le \phi(\|e(t_k^-)\|_P) + V_\theta(t_k^-) = V(t_k^-). \tag{45}$$

Therefore, the Lyapunov function is nonincreasing at every switching time instant. This together with the fact $\dot{V} \le 0$ in $[t_k, t_{k+1})$ for $\forall k \in \mathbb{N}$ implies that $V(t)$ is nonincreasing for $\forall t \in [t_0, \infty)$. The discontinuity of the Lyapunov function does not introduce extra dwell time constraints.

Combining the analysis of *phase 1* and *phase 2*, we have $\phi, \tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$, which further leads to $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$. Besides, $\|e(t)\|_P < \epsilon(t) < \rho(t)$ holds for $\forall t \in [t_0, \infty)$. This gives $\phi_d \in \mathcal{L}_\infty$.

Invoking Lemma 2 we have $x_m \in \mathcal{L}_\infty$. This property and $\|e(t)\|_P < \epsilon(t) < \rho(t)$ lead to $x \in \mathcal{L}_\infty$, which together with $r, \phi_d \in \mathcal{L}_\infty$ implies $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. ■

Theorem 1 shows the tracking performance and the stability of the closed-loop system under the dwell time constraint (16). Now we study the case with arbitrary switching. For the PWA reference systems with common Lyapunov matrix P, that is, if positive definite matrices P and $Q_i, i \in \mathcal{I}$ exist such that

$$A_{mi}^TP + PA_{mi} = -Q_i, \quad i \in \mathcal{I}, \tag{46}$$

the error metric $\|e(t)\|_P$ exhibits no jumps at the switching instants. We can construct the potential function with the user-defined performance function directly

$$\phi_0(\|e\|_P) = \frac{\|e\|_P^2}{\rho^2(t) - \|e\|_P^2}, \quad \|e\|_P < \rho(t). \tag{47}$$

Corollary 1. *For the reference PWA system (4) with a common Lyapunov matrix P, if the adaptation laws*

$$\dot{K}_{xi} = -\chi_i \phi_{d0} S_i^T B_{mi}^T P e x^T, \quad \dot{K}_{ri} = -\chi_i \phi_{d0} S_i^T B_{mi}^T P e r^T, \quad \dot{K}_{fi} = -\chi_i \phi_{d0} S_i^T B_{mi}^T P e, \tag{48}$$

are used with $\phi_{d0} \triangleq \frac{\partial \phi_0}{\partial \|e\|_P^2}$, *and if the decaying rate of* ρ *satisfies*

$$l < \frac{1}{2} \min_{i \in \mathcal{I}} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P)}, \tag{49}$$

the closed-loop system is stable under arbitrary switching and the state tracking error $e(t)$ *satisfies the prescribed performance guarantees (11).*

Proof. We propose the following common Lyapunov function

$$V = \phi_0(\|e\|_P) + \sum_{i=1}^{s} \left(\mathrm{tr}\left(\tilde{K}_{xi}^T M_i \tilde{K}_{xi}\right) + \mathrm{tr}\left(\tilde{K}_{ri}^T M_i \tilde{K}_{ri}\right) + \tilde{K}_{fi}^T M_i \tilde{K}_{fi} \right). \tag{50}$$

V is continuous not only within each interval $[t_k, t_{k+1}), k \in \mathbb{N}$ but also at switch instants $t_k, k \in \mathbb{N}^+$. Taking its time derivative and inserting (48) and (9), we obtain

$$\dot{V} = -\phi_{d0} e^T \left(\sum_{i=1}^{s} \chi_i Q_i \right) e + \frac{\partial \phi_0}{\partial \rho} \dot{\rho}. \tag{51}$$

Since $\frac{\partial \phi_0}{\partial \rho} \dot{\rho} \leq 2\phi_{d0}(\|e\|_P)\|e\|_P^2 \frac{|\dot{\rho}|}{\rho}$ and $\frac{|\dot{\rho}|}{\rho} \leq l$, we have

$$\dot{V} \leq -\phi_{d0}\|e\|_2^2 \min_{i \in \mathcal{I}} \lambda_{\min}(Q_i) + 2l\phi_{d0}\|e\|_2^2 \lambda_{\max}(P) \leq -\frac{\min_{i \in \mathcal{I}} \lambda_{\min}(Q_i) - 2l\lambda_{\max}(P)}{2\lambda_{\max}(P)} \phi_0 \leq 0, \tag{52}$$

given that (49) holds. $\dot{V} \leq 0$ is negative semidefinite. Therefore, we have $\phi_0, \tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$ for arbitrary switching. The boundedness of $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}$ implies $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$. Furthermore, $\|e(t)\|_P < \rho(t)$ holds for $\forall t \in [t_0, \infty)$. This leads to $x \in \mathcal{L}_\infty$ and $\phi_{d0} \in \mathcal{L}_\infty$, which together with $r \in \mathcal{L}_\infty$ implies that $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. ∎

It is worth comparing the proposed method with other control approaches for switched systems with performance guarantees. The bang-bang funnel controller[33] enforces the output tracking error of systems, which can be transformed into Byrnes–Isidori normal form, within a predefined funnel. The backstepping-based approaches can achieve output tracking with performance guarantees for systems with special structures (strict-feedback form[34,35] and non-strict-feedback form[36]). In contrast, our approach achieves performance-guaranteed full state tracking without special structural requirements provided that the matching conditions (7) hold. Nevertheless, extra efforts are needed in our case for the design of auxiliary performance bound to bypass the barrier transgression problem. The fault-tolerant approach[37] solves the barrier transgression problem by modifying the performance function when actuator failure occurs. Compared to this concept, our method imposes the auxiliary performance bound with certain dwell time constraints such that the modification of the original performance function $\rho(t)$ is not necessary.

Remark 2. The classical MRAC approaches for switched systems[10,11,14] suggest using $e^T \left(\sum_{i=1}^{s} \chi_i P_i\right) e$ as the error-related term (the first summand) of the Lyapunov function V. This leads to potential increases of V at switching instants. The dwell time constraints are then derived by formulating an inequality in form of $\dot{V} < -\alpha V + \beta$ for some constant $\alpha, \beta > 0$ to keep V exponentially decreasing in between the switches. To achieve this, the projection operator needs to be introduced (see work by Sang and Tao[10] as well as Wu and Zhao[14]) or the input signal must be PE (see work by Kersting and Buss[11]) in the disturbance-free case. One key feature of our approach is that the Lyapunov function V is nonincreasing even at the switching instants and does not impose extra dwell time constraints. This omits the need for introducing projection or the PE condition in the disturbance-free case.

Remark 3. The nonincreasing property at switching instants of Lyapunov functions is also achieved in the recently proposed adaptive control approaches for switched systems,[38,39] which employ time-varying gains for adaptation laws. These time-varying gains are either obtained by interpolating a set of precalculated $P_{i,k}$ matrices satisfying certain linear matrix inequalities[38] or generated by an auxiliary piecewise continuous dynamical system.[39] Compared to these approaches, our method can be viewed as an error-dependent dynamic gain approach (see ϕ_d in adaptation laws (27)) and endows the closed-loop system with a user-defined performance guarantee.

Remark 4. Introducing the auxiliary performance bound ϵ has the advantage that the barrier transgression problem can be avoided. Nevertheless, this imposes one technical challenge: how its parameters are related to the dwell time constraint and the system stability. We resolve this challenge by deriving a novel dwell time constraint in terms of the parameters of ϵ in Lemma 1, which differs from the existing dwell time constraints[40,41] and proving that the resulted Lyapunov function does not impose extra dwell time constraints.

So far, the theoretical results are obtained with the assumption that the reference PWA system (4) and the controlled PWA system (3) switch synchronously, where the switches depend on the state of the controlled PWA system. To show how the dwell time constraint (16) can be satisfied, we consider a more general case, where the reference PWA system switches based on its own state space partitions $x_m \in \{\Omega_i^*\}_{i=1}^{s^*}$. For $x \in \Omega_i$ and $x_m \in \Omega_j^*$, a set of controllers $K_{xij}, K_{rij}, K_{fij}$ is activated for adaptations, whose nominal values $K_{xij}^*, K_{rij}^*, K_{fij}^*$ satisfy the matching equations for $\{A_i, B_i, f_i\}$ and $\{A_{mj}, B_{mj}, f_{mj}\}$. At the switching instants $\{\hat{t}_k\}_{k\in\mathbb{N}^+}$ of the reference PWA system, that is, $x(\hat{t}_k^-)$ and $x(\hat{t}_k) \in \Omega_i, x_m(\hat{t}_k^-) \in \Omega_j, x_m(\hat{t}_k) \in \Omega_l, j \neq l$, we have $P(\hat{t}_k^-) = P_j, P(\hat{t}_k) = P_l$. The reset of ϵ is triggered; At the switching instants $\{\check{t}_k\}_{k\in\mathbb{N}^+}$ of the controlled PWA system, i.e., $x(\check{t}_k^-) \in \Omega_i, x(\check{t}_k) \in \Omega_l, i \neq l, x_m(\check{t}_k^-)$ and $x_m(\check{t}_k) \in \Omega_j$, we have a common $P(\check{t}_k^-) = P(\check{t}_k) = P_j$. ϵ is not reset at $\check{t}_k$. So within each interval $[\hat{t}_{k-1}, \hat{t}_k)$, the analysis follows a common Lyapunov setting shown in Corollary 1; Over the whole time interval $\cup_k[\hat{t}_{k-1}, \hat{t}_k)$, the stability argumentation follows Theorem 1. The above analysis shows that only $\{\hat{t}_k\}_{k\in\mathbb{N}^+}$ of the reference system have to satisfy the dwell time constraint. Since the reference PWA system is designed by the user, the dwell time constraint can be fulfilled by properly designing the reference input and the reference PWA system offline and can be checked in advance.

3.4 | Robust modification

We now study the robust modification of the proposed method to extend it to the case with disturbances and unmodeled dynamics. Consider

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t) + d(x, u, t), \tag{53}$$

where $d(x, u, t) \in \mathbb{R}^n$ can denote the approximation error of the linearization, unmodeled dynamics or external disturbances. d is continuous and its norm is upper bounded, that is, $\|d\|_2 \leq \overline{d}$, where $\overline{d}$ is known.

We propose the following robust adaptation laws

$$\dot{K}_{xi} = -\chi_i\phi_d S_i^T B_{mi}^T P_i e x^T + \chi_i F_{xi}, \quad \dot{K}_{ri} = -\chi_i\phi_d S_i^T B_{mi}^T P_i e r^T + \chi_i F_{ri}, \quad \dot{K}_{fi} = -\chi_i\phi_d S_i^T B_{mi}^T P_i e + \chi_i F_{0i}, \tag{54}$$

where $F_{xi} \in \mathbb{R}^{p\times n}, F_{ri} \in \mathbb{R}^{p\times p}, F_{0i} \in \mathbb{R}^p$ represent the projection terms to confine the estimated controller gains K_{xi}, K_{ri}, K_{fi} within some given bounds. The projection terms have no effect on the adaptation if K_{xi}, K_{ri}, K_{fi} are within their bounds, otherwise, the adaptation terminates. Here we make the assumption that a known matrix $S_i \in \mathbb{R}^{p\times p}$ as well as an unknown diagonal and positive definite matrix $M_i \in \mathbb{R}^{p\times p}$ exist such that $(K_{ri}^* S_i)^{-1} = M_i$.

Remark 5. For the robust adaptive control design, more prior information is required compared with the disturbance-free case. For our projection-based approach, M_i must be diagonal and the element-wise bounds of K_{xi}, K_{ri}, K_{fi} need to be known (see also work by Sang and Tao[9]). The leakage-based approach proposed by Yuan et al.[13] requires M_i to be completely known because they are used in the leakage terms. The follow-up work[42] requires $\lambda_{\max}(M_i^{-1})$ to satisfy some constraints associated with the leakage rates.

Remark 6. Regarding the input matrix, there is another popular formulation $\dot{x} = A_p x + B_p \Lambda u$ for linear systems appearing in many works inspired by aerospace applications,[28,30,43] where B_p is known and Λ is an unknown diagonal matrix with strictly positive diagonal elements. Such arrangement of the input matrix is equivalent to our formulation. Specifically, we have $B = B_m SM$ (we remove the subscript i) in our notations. The unknown diagonal matrix Λ with strictly positive diagonal elements corresponds to the diagonal and positive definite matrix M in our case, while the known control direction B_p corresponds to the multiplication $B_m S$.

Besides, we assume that positive definite matrices $P_i, Q_i, i \in \mathcal{I}$ exist such that

$$A_{mi}^T P_i + P_i A_{mi} + P_i = -Q_i, \quad i \in \mathcal{I}. \tag{55}$$

Before we proceed with the robustness analysis, another property of the potential function, which is useful for the analysis in this paper, is given in the following lemma.

Lemma 3. *For a positive constant $c \in \mathbb{R}^+$ and $c < \min_t \epsilon^2(t)$, the function $\phi(\|e\|_P)$ defined in (26) and its partial derivative ϕ_d with respect to $\|e\|_P^2$ satisfy*

(1) $2\phi_d \cdot (\|e\|_P^2 - c) - \phi > 0$ *for* $\zeta < \|e\|_P^2 < \epsilon^2$
(2) $2\phi_d \cdot (\|e\|_P^2 - c) - \phi \le 0$ *for* $\|e\|_P^2 \le \zeta$

with $\zeta \triangleq \frac{-\epsilon^2 + \sqrt{\epsilon^4 + 8\epsilon^2 c}}{2}$.

The proof of Lemma 3 can be seen in Appendix A.2.

The control performance and the closed-loop stability by using the robust adaptive controller are summarized in the following theorem.

Theorem 2. *Given the reference PWA system (4) and the predefined performance function (10), let the PWA system (3) with known regions $\Omega_i, i \in \mathcal{I}$ and unknown subsystem parameters $A_i, B_i, f_i, i \in \mathcal{I}$ be controlled by the feedback controller (8) with the adaptation laws (54). Let the initial state of ϵ satisfy $\|e(t_0)\|_P < \epsilon(t_0)$. The closed-loop system is stable and the state tracking error $e(t)$ satisfies the prescribed performance guarantees (11) if the time constant h in (14) satisfies*

$$h < \frac{1}{2}\min_{i\in\mathcal{I}} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)}, \qquad \max_{i\in\mathcal{I}} \frac{\lambda_{\max}(P_i)\bar{d}}{\sqrt{\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)}} < \frac{g}{h}, \tag{56}$$

and if the switching signal of the controlled PWA system obeys the dwell time constraint τ_D in (16).

Proof. We propose the same Lyapunov function as (29). The stability analysis can also be divided into two phases as the one in Theorem 1.

phase 1: $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$

Following the same steps from (30) to (35) as in Theorem 1, we have

$$\dot{V} = -\phi_d e^T (A_{mi}^T P_i + P_i A_{mi}) e + \phi_d (e^T P_i d + d^T P_i e) + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} + 2\phi_d (\mathrm{tr}(\tilde{K}_{xi}^T M_i F_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i F_{ri}) + \tilde{K}_{fi}^T M_i F_{0i}). \tag{57}$$

Since M_i is diagonal, we have

$$\phi_d(\mathrm{tr}(\tilde{K}_{xi}^T M_i F_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i F_{ri}) + \tilde{K}_{fi}^T M_i F_{0i}) = \phi_d \left(\sum_{j=1}^{p}\sum_{l=1}^{n} m_i^{(j)} \tilde{k}_{xi}^{(jl)} f_{xi}^{(jl)} + \sum_{j=1}^{p}\sum_{l=1}^{p} m_i^{(j)} \tilde{k}_{ri}^{(jl)} f_{ri}^{(jl)} + \sum_{j=1}^{p} m_i^{(j)} \tilde{k}_{fi}^{(j)} f_{0i}^{(j)} \right) \tag{58}$$

with $\tilde{K}_{xi} = \left[\tilde{k}_{xi}^{(jl)}\right]$, $\tilde{K}_{ri} = \left[\tilde{k}_{ri}^{(jl)}\right]$, $\tilde{K}_{fi} = \left[\tilde{k}_{fi}^{(j)}\right]$, $F_{xi} = \left[f_{xi}^{(jl)}\right]$, $F_{ri} = \left[f_{ri}^{(jl)}\right]$, and $F_{0i} = \left[f_{0i}^{(j)}\right]$. $M_i = \mathrm{diag}\left(m_i^{(1)}, \dots, m_i^{(p)}\right)$. It can be verified that $\tilde{k}_{xi}^{(jl)} f_{xi}^{(jl)} \le 0$, $\tilde{k}_{ri}^{(jl)} f_{ri}^{(jl)} \le 0$ and $\tilde{k}_{fi}^{(jl)} f_{0i}^{(jl)} \le 0$, which together with the fact that $m_i^{(j)} > 0, i \in \mathcal{I}, j = 1, \dots, p$ leads to

$$\dot{V} \le -\phi_d e^T (A_{mi}^T P_i + P_i A_{mi}) e + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} + \phi_d (e^T P_i d + d^T P_i e). \tag{59}$$

Since P_i is positive definite, it can be written as $P_i = H_i H_i^T$ with H_i being a nonsingular matrix. The inequality (59) can be further transformed as

$$\begin{aligned}
\dot{V} &\leq -\phi_d e^T(A_{mi}^T P_i + P_i A_{mi})e + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} + 2\phi_d e^T H_i H_i^T d \\
&\leq -\phi_d e^T(Q_i + P_i)e + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} + \phi_d(e^T H_i H_i^T e + d^T H_i H_i^T d) \\
&= -\phi_d e^T Q_i e + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} + \phi_d d^T H_i H_i^T d \\
&\leq -\phi_d \|e\|_2^2(\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)) + \phi_d d^T P_i d \\
&\leq -\phi_d \|e\|_2^2 \kappa_i + \phi_d \lambda_{\max}(P_i)\overline{d}^2,
\end{aligned} \tag{60}$$

with $\kappa_i \triangleq \lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)$. For P_i, Q_i and h satisfying the condition (56), we have $\kappa_i > 0$. Further analysis can be divided into two cases: $\|e\|_P^2 > \zeta_i$ and $\|e\|_P^2 \leq \zeta_i$, where

$$\zeta_i = \frac{-\epsilon^2 + \sqrt{\epsilon^4 + 8\epsilon^2 c_i}}{2}, \quad i \in \mathcal{I}, \tag{61}$$

with $c_i \triangleq \frac{\lambda_{\max}^2(P_i)}{\kappa_i}\overline{d}^2$. From (56) we obtain

$$\epsilon(t)^2 \geq \frac{g^2}{h^2} > \max_{i\in\mathcal{I}} \frac{\lambda_{\max}^2(P_i)\overline{d}^2}{\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)} = \max_{i\in\mathcal{I}} \left\{ \frac{\lambda_{\max}^2(P_i)}{\kappa_i}\overline{d}^2 \right\} \geq c_i, \tag{62}$$

which further leads to

$$\zeta_i < \frac{-\epsilon^2 + \sqrt{\epsilon^4 + 8\epsilon^2 \cdot \epsilon^2}}{2} = \epsilon^2. \tag{63}$$

Case 1 $\|e\|_P^2 > \zeta_i$: invoking Lemma 3, inequality (60) can be further derived as

$$\dot{V} \leq -\frac{\kappa_i \phi_d}{\lambda_{\max}(P_i)}\left(\|e\|_P^2 - \frac{\lambda_{\max}^2(P_i)}{\kappa_i}\overline{d}^2\right) < -\frac{\kappa_i}{2\lambda_{\max}(P_i)}\phi < 0. \tag{64}$$

Case 2 $\|e\|_P^2 \leq \zeta_i$: defining $\kappa \triangleq \min_{i\in\mathcal{I}}\{\kappa_i\}$, $\alpha = \max_{i\in\mathcal{I}} \lambda_{\max}(P_i)$ and considering the property that $2\phi_d(\|e\|_P)\|e\|_P^2 - \phi > 0$, we have

$$\dot{V} \leq -\frac{\kappa}{2\alpha}\phi + \phi_d \alpha \overline{d}^2 = -\frac{\kappa}{2\alpha}(\phi + V_\theta) + \frac{\kappa}{2\alpha}V_\theta + \phi_d \alpha \overline{d}^2 \leq -\frac{\kappa}{2\alpha}V + \frac{\kappa}{2\alpha}V_\theta + \phi_{d_{\max}} \alpha \overline{d}^2, \tag{65}$$

with $\phi_{d_{\max}} = \max_{\|e\|_P^2 \leq \zeta} \phi_d(\|e\|_P^2) = \phi_d(\max_t \zeta) \in \mathcal{L}_\infty$ for $\zeta = \sum_{i=1}^s \chi_i \zeta_i$. V_θ is defined in (29). $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}$ are bounded due to the utilization of the projection, which leads to $V_\theta \in \mathcal{L}_\infty$. Suppose $\overline{V}_\theta$ to be the maximum of V_θ and let the positive number $\mathcal{B} \in \mathbb{R}^+$ be defined as

$$\mathcal{B} \triangleq \overline{V}_\theta + \frac{2\phi_{d_{\max}}\alpha^2\overline{d}^2}{\kappa}. \tag{66}$$

For $V \leq \mathcal{B}$, V may increase. For $V > \mathcal{B}$, we have $\dot{V} < 0$ and therefore, V is decreasing. Combining *Case 1* and *Case 2*, we know that V is bounded for the interval $[t_{k-1}, t_k)$.

phase 2: jump at switch instant $t_k, k \in \mathbb{N}^+$ Following the same steps as shown in Theorem 1 and we have $V(t_k) \leq V(t_k^-)$.

Based on the analysis of phase 1 and phase 2, we can conclude that

$$V(t) \leq \max\{V(t_0), \mathcal{B}\}, \forall t \in [t_0, \infty), \tag{67}$$

from which we obtain $\phi, \phi_d \in \mathcal{L}_\infty$. The projection leads to $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$, which further leads to $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$. Besides, $\|e(t)\|_P < \epsilon(t) < \rho(t)$ holds for $\forall t \in [t_0, \infty)$. The prescribed performance guarantee (11) is satisfied.

With similar steps in the proof of Lemma 2, one can prove the stability of the reference system with (55), so we have $x_m \in \mathcal{L}_\infty$. This leads to $x \in \mathcal{L}_\infty$, which together with $r, \phi_d \in \mathcal{L}_\infty$ implies $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. ■

Remark 7. The leakage-based robust MRAC approach for switched linear systems[13] obtains the boundedness of the Lyapunov function V by formulating the inequality $\dot{V} \leq -\alpha V + \beta$, where $\alpha > 0$ and β is a disturbance-related term. This, however, does not apply to our approach, because the disturbance-related term in our case has a time-varying coefficient ϕ_d (see the term $\phi_d \lambda_{\max}(P_i)\overline{d}^2$ in (60)). The boundedness of ϕ_d cannot be concluded without proving the boundedness of V, while the boundedness of V requires $\phi_d \lambda_{\max}(P_i)\overline{d}^2$ to be bounded. This potential circular reasoning constitutes one of the main technical challenges of the robust modification. Our solution concept is employing the property of ϕ shown in Lemma 3 to discuss the stability in two separate cases. When $\|e(t)\|_P \leq \zeta$, V may increase with ϕ_d and V upper bounded. V is strictly decreasing if $\|e(t)\|_P > \zeta$ for $\zeta = \sum_{i=1}^{s} \chi_i \zeta_i$.

Remark 8. In work about set-theoretic MRAC by Arabi and Yucelen,[27,28,30] disturbances flow into the system through the same input matrix as the control signal. The fault-tolerant set-theoretic MRAC approach proposed by Xiao and Dong[29] also assumes the actuator fault and external disturbances to be matched, that is, they can be compensated by designing additive terms in the control signal. Compared with these works, a distinctive feature of our approach is that the disturbance term d is also allowed to be unmatched.

Remark 9. According to (16), the length of the dwell time is governed by $\sqrt{\mu}$, the reset map of the auxiliary performance signal $\epsilon(t)$ (see (15)). By reducing $\sqrt{\mu}$, a less-conservative dwell time constraint can be obtained. In both the adaptive controller (27) and the robust adaptive controller (54), the reset map is defined with $\mu = \max_{i,j \in \mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}$, which indicates the maximal possible jump of $\|e(t)\|_P^2$ at each switching instant. Since the current activated subsystem is known (supposed to be p), the maximal jump of $\|e(t)\|_P^2$ at next switching instant is $\mu_p = \max_{i \in \mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_p)} \leq \mu$. For the case where both current subsystem (supposed to be p) and the next subsystem to be switched on (supposed to be q) are known in advance, the maximal jump of $\|e(t)\|_P^2$ at this switching instant is $\mu_{pq} = \frac{\lambda_{\max}(P_q)}{\lambda_{\min}(P_p)} \leq \mu$. Adopting $\sqrt{\mu_p}$ or $\sqrt{\mu_{pq}}$ instead of $\sqrt{\mu}$ as the reset map of ϵ yields a less conservative dwell time constraint. The corresponding stability properties of the reference system (4) and the closed-loop system are still retained. Such dwell time constraints are known as *mode-dependent dwell time*[44] (when μ_p is adopted) and *mode-mode-dependent dwell time*[13] (when μ_{pq} is utilized).

4 | NUMERICAL VALIDATION

In this section, the proposed MRAC approach is validated through a numerical example modified based on the example in the literature,[11] a mass-spring-damper system, which is shown in Figure 2. The displacement of the mass is denoted by p and the force operated on the mass is F, respectively. The mass is $m = 1$ kg and the damping factor is $d = 1$ N s/m. The mass is connected to the static wall with the spring c_x and the damper d. For $|p| \leq \gamma = 1$ m, the spring factor $c_x = 10$ N/m. If it is extended beyond γ, that is, $p > 1$ m, the spring factor c_x is reduced to $c_x = 1$ N/m. The spring $c_y = 90$ N/m is a floating spring with one end connected to the wall. The distance between the mass and the tip of the spring c_y is γ when c_x is in its resting position. The system is equivalent to a classical mass-spring-damper system with the spring exhibiting a PWA stiffness characteristics

$$F_c(p) = \begin{cases} c_1 = 10 \text{ N/m}, & \text{if } |p| \leq 1 \text{ m} \\ c_2 = 1 \text{ N/m}, & \text{if } p > 1 \text{ m} \\ c_3 = 100 \text{ N/m}, & \text{if } p < -1 \text{ m} \end{cases}. \tag{68}$$

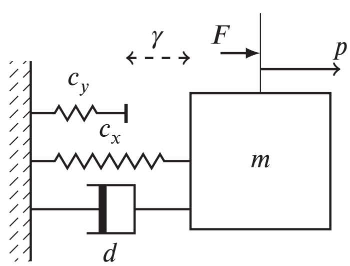

FIGURE 2 The mass-spring-damper system

Let the state $x = [x_1, x_2]^T = [p, \dot{p}]^T$ and the input $u = F$. The system dynamics can be described by a PWA system in form of

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{c_i}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u + \begin{bmatrix} 0 \\ \bar{f}_i \end{bmatrix}, \quad i \in \{1, 2, 3\}, \tag{69}$$

with $\bar{f}_1 = 0, \bar{f}_2 = (c_2 - c_1)/m, \bar{f}_3 = (c_1 - c_3)/m$. The region partitions are given as

$$\Omega_1 = \{x \in \mathbb{R}^2 \,|\, |x_1| \leq 1\}, \quad \Omega_2 = \{x \in \mathbb{R}^2 | x_1 > 1\}, \quad \Omega_3 = \{x \in \mathbb{R}^2 | x_1 < -1\}.$$

The reference system is a PWA system with the following subsystem matrices

$$A_{m1} = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix}, \qquad B_{m1} = \begin{bmatrix} 0 \\ 25 \end{bmatrix}, \qquad f_{m1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{70}$$

$$A_{m2} = \begin{bmatrix} 0 & 1 \\ -16 & -8 \end{bmatrix}, \qquad B_{m2} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}, \qquad f_{m2} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \tag{71}$$

$$A_{m3} = \begin{bmatrix} 0 & 1 \\ -49 & -14 \end{bmatrix}, \qquad B_{m3} = \begin{bmatrix} 0 \\ 49 \end{bmatrix}, \qquad f_{m3} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}. \tag{72}$$

Ideal case:

The adaptive controller in the ideal case with the adaptation laws (27) is tested. The P_i and Q_i matrices satisfying (5) are chosen as

$$P_1 = \begin{bmatrix} 140 & 2 \\ 2 & 5.2 \end{bmatrix}, P_2 = \begin{bmatrix} 121.25 & 3.125 \\ 3.125 & 6.64 \end{bmatrix}, P_3 = \begin{bmatrix} 182.857 & 1.02 \\ 1.02 & 3.644 \end{bmatrix}, Q_i = \begin{bmatrix} 100 & 10 \\ 10 & 100 \end{bmatrix} \quad \text{for } i \in \{1, 2, 3\}, \tag{73}$$

which gives $\sqrt{\mu} = 7.1$. The scaling factors are $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} = 0.1$. The performance function is designed with $\rho_0 = 10$, $\rho_\infty = 1.5, l = 0.02$. We choose $\epsilon(t_0) = 9, h = 0.12$, and $g = 0.01$ such that the condition (28) and further conditions stated in Lemma 1 hold. Let the initial values of the reference system and the controlled PWA system be $[2, 0]^T$. The initial values of the estimated controller gains are specified as $K_{xi}(t_0) = 0.5K^*_{xi}, K_{ri}(t_0) = 0.5K^*_{ri}, K_{fi}(t_0) = 0.5K^*_{fi}, i \in \{1, 2, 3\}$. We use the following input signal r

$$r(t) = \begin{cases} 2 + 0.5\sin(0.2\pi t), & \text{for } 0\text{ s} \leq t < 25\text{ s} \\ -0.08t + 2.8, & \text{for } 25\text{ s} \leq t < 50\text{ s} \\ -2 + 0.8\sin(2t - 100 - \pi)), & \text{for } 50\text{ s} \leq t < 75\text{ s} \\ 0, & \text{for } t \geq 75\text{ s} \end{cases}. \tag{74}$$

The state-space trajectories of the reference system and the closed-loop system in the time interval [23 s, 52 s] are displayed in Figure 3A with black dashed and red solid lines, respectively. The light blue, light green, and light yellow regions refer to Ω_2, Ω_1, and Ω_3. The ellipses centered at the state trajectory of the reference system represent $\|e(t)\|_P = \epsilon(t)$ and indicate the bounds of the state of the closed-loop PWA system. The colors of the ellipses distinguish $\|e(t)\|_{P_1}, \|e(t)\|_{P_2}$, and $\|e(t)\|_{P_3}$. We can observe that the state of the closed-loop system always stays within the auxiliary performance bound. For comparison, the state trajectory of the closed-loop system by using MRAC approach[11] is displayed with blue solid lines in Figure 3B, from which the violation of the performance bound can be observed.

According to Lemma 1, the dwell time of the closed-loop system should satisfy $\tau_D > 24$ s. The small window of Figure 4A shows the mode information of the closed-loop system. We can observe that the dwell time constraint is satisfied. In Figure 4A, the prescribed performance bound $\rho(t)$, the auxiliary performance bound $\epsilon(t)$ and the weighted norm of the state tracking error $\|e(t)\|_P$ are displayed with the black dashed line, the blue solid line, and the red solid line, respectively. We can see that $\|e(t)\|_P < \epsilon(t) < \rho(t)$. The weighted norm of the state tracking error $\|e(t)\|_P$ and the auxiliary

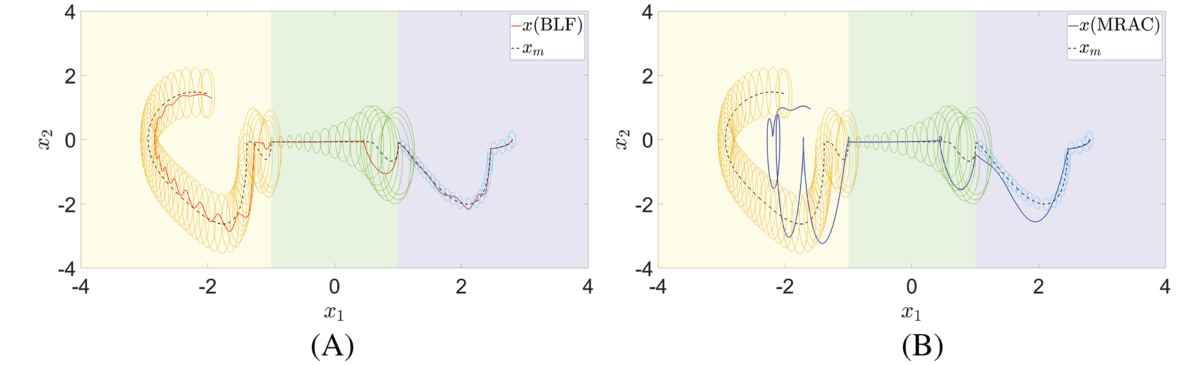

FIGURE 3 Closed-loop system's trajectory by applying proposed method and the classical model reference adaptive control (MRAC) approach. (A) Proposed approach; (B) Classical MRAC approach

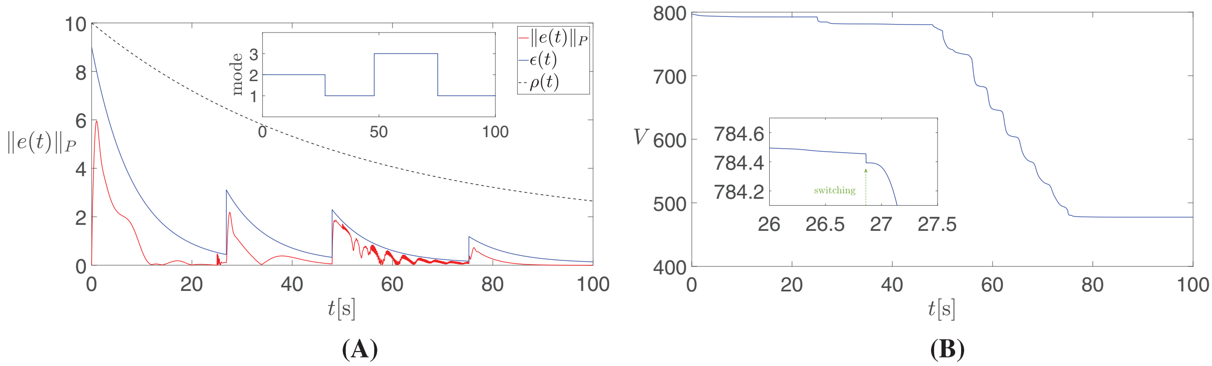

FIGURE 4 Tracking performance of the adaptive controller with adaptation laws (27). (A) State tracking error and performance bound. (B) Lyapunov function

performance bound $\epsilon(t)$ jump at the switching instants, where the relation $\|e(t)\|_P < \epsilon(t)$ is still satisfied. This guarantees the potential function $\phi(t)$ to be valid and the control objective (11) to be fulfilled.

The Lyapunov function V is displayed in Figure 4B. We observe that the Lyapunov function V is nonincreasing, also at the switching instants. This validates the theoretical statement given in Theorem 1.

Robust case:

Now we test the performance of the robust adaptive controller with the adaptation laws (54). The PWA system is subject to an unmatched disturbance term $d = [0.036\cos(0.7t) + 0.072\sin(0.2t) + 0.018\sin(t), 0]^T$. The P_i, Q_i matrices satisfying (55) are chosen as

$$P_1 = \begin{bmatrix} 0.7627 & 0.0353 \\ 0.0353 & 0.0458 \end{bmatrix}, P_2 = \begin{bmatrix} 0.6140 & 0.0504 \\ 0.0504 & 0.0601 \end{bmatrix}, P_3 = \begin{bmatrix} 0.7932 & 0.0183 \\ 0.0183 & 0.0236 \end{bmatrix}, Q_1 = Q_2 = \begin{bmatrix} 1 & 0.7 \\ 0.7 & 0.8 \end{bmatrix}, Q_3 = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.6 \end{bmatrix}, \tag{75}$$

which gives $\sqrt{\mu} = 5.86$. The scaling factors are $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} = 1$. The performance function is designed with $\rho_0 = 10, \rho_\infty = 3.2, l = 0.02$. The auxiliary performance signal is designed with $\epsilon(t_0) = 9, h = 0.08$ and $g = 0.04$ to fulfill the conditions in Lemma 1 and Theorem 2. The dwell time of the closed-loop system must satisfy $\tau_D > 67.7$ s. Let the initial values of the reference system and the controlled PWA system be $[0, 0]^T$. The initial values of the estimated controller gains are specified as $K_{xi}(t_0) = 0.5K_{xi}^*, K_{ri}(t_0) = 0.5K_{ri}^*, K_{fi}(t_0) = 0.5K_{fi}^*, i \in \{1, 2, 3\}$. The input signal r is

$$r(t) = \begin{cases} 0, & \text{for } 0\text{ s} + \frac{KT}{2} \le t < 70\text{ s} + \frac{KT}{2} \\ 2, & \text{for } 70\text{ s} + KT \le t < 140\text{ s} + KT \\ -2, & \text{for } 210\text{ s} + KT \le t < 280\text{ s} + KT \end{cases}, \tag{76}$$

with $K \in \mathbb{N}$ and $T = 280$ s.

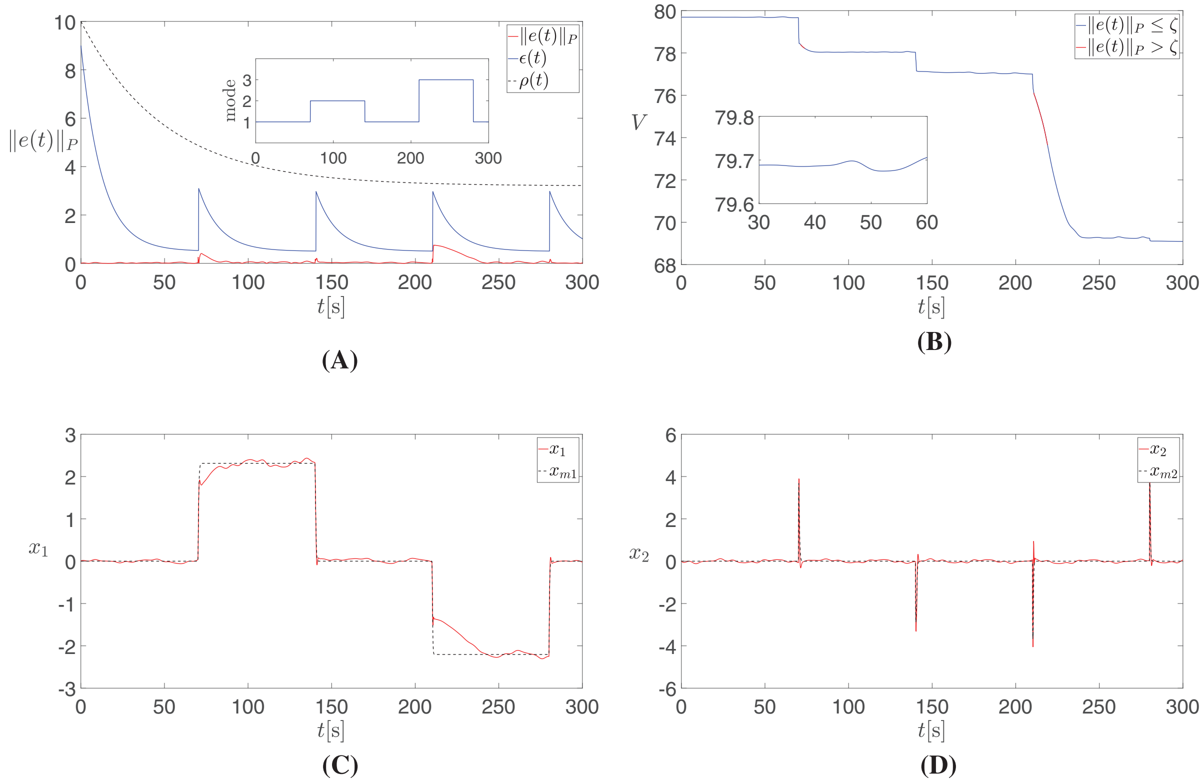

FIGURE 5 Tracking performance of the adaptive controller with adaptation laws (54). (A) State tracking error and performance bound; (B) Lyapunov function; (C) state x_1; (D) state x_2

The small window of Figure 5A shows the switching information of the closed-loop system. It can be observed that the dwell time constraint $\tau_D > 67.7$ s is satisfied. In Figure 5A, the black dashed line, the blue solid line, and the red solid line represent the prescribed performance bound $\rho(t)$, the auxiliary performance bound $\epsilon(t)$ and the weighted norm of the state tracking error $\|e(t)\|_P$, respectively. It can be seen that $\|e(t)\|_P < \epsilon(t) < \rho(t)$ holds. The element-wise tracking performance of the closed-loop system is displayed in Figure 5C,D, where the black dashed lines represent the reference signals and the red solid lines represent the state signals. Despite the existence of the disturbance, the closed-loop state tracks the one of the reference system with the prescribed performance.

The Lyapunov function V is shown in Figure 5B. According to the proof of Theorem 2, V may increase when $\|e\|_P \leq \zeta$. In Figure 5B, V is shown in red for $\|e\|_P > \zeta$ and in blue for $\|e\|_P \leq \zeta$. We observe that V is decreasing for $\|e\|_P > \zeta$ whereas it may increase (as shown in the small window) but remain bounded for $\|e\|_P \leq \zeta$. This validates the theoretical result given in Theorem 2.

5 | CONCLUSION

In this paper, we explored the MRAC approach for PWA systems with time-varying performance guarantees on the state tracking error. The proposed method is based on barrier Lyapunov functions. To solve the barrier transgression problem caused by the discontinuity of the weighted Euclidean norm of the state tracking error, we introduce an auxiliary performance bound with a state reset map at switching instants to construct the barrier Lyapunov function. This auxiliary performance bound resides within the user-defined performance bound if some dwell time constraints are satisfied. The Lyapunov function is nonincreasing at and in between the switching instants, which ensures the weighted Euclidean norm of the state tracking error to fulfill the performance guarantee. To enhance the robustness of the closed-loop system against unmatched disturbances, the projection-based robust modification of the proposed method is presented. Future work may include the extension to indirect MRAC approach and stability analysis when sliding mode on switching hyperplanes occurs. The current approach requires a proper initialization such that the error metric locates within the performance bound at the initial instant. Extending this approach to a global setting is also of interest.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study

ORCID

Tong Liu https://orcid.org/0000-0001-7422-4312

REFERENCES

1. Bemporad A, Ferrari–Trecate G, Morari M. Observability and controllability of piecewise affine and hybrid systems. *IEEE Trans Autom Control*. 2000;45(10):1864–1876.
2. Collins P, Van Schuppen JH. Observability of piecewise–affine hybrid systems. Proceedings of the International Workshop on Hybrid Systems: Computation and Control; 2004:265–279; Springer.
3. Pavlov A, Pogromsky A, Van de Wouw N, Nijmeijer H. On convergence properties of piecewise affine systems. *Int J Control*. 2007;80(8):1233–1247.
4. Rodrigues L, How JP. Observer–based control of piecewise–affine systems. *Int J Control*. 2003;76(5):459–477.
5. Habets LCGJM, Collins PJ, Schuppen JH. Reachability and control synthesis for piecewise–affine hybrid systems on simplices. *IEEE Trans Automat Contr*. 2006;51(6):938–948.
6. Bernardo M, Montanaro U, Santini S. Hybrid model reference adaptive control of piecewise affine systems. *IEEE Trans Automat Contr*. 2013;58(2):304–316.
7. Bernardo MD, Montanaro U, Olm JM, Santini S. Model reference adaptive control of discrete–time piecewise linear systems. *Int J Robust Nonlinear Control*. 2013;23(7):709–730.
8. Bernardo M, Montanaro U, Ortega R, Santini S. Extended hybrid model reference adaptive control of piecewise affine systems. *Nonlinear Anal Hybrid Syst*. 2016;21:11–21.
9. Sang Q, Tao G. Adaptive control of piecewise linear systems with applications to NASA GTM; 2011:1157–1162; IEEE.
10. Sang Q, Tao G. Adaptive control of piecewise linear systems: the state tracking case. *IEEE Trans Automat Contr*. 2012;57(2): 522–528.
11. Kersting S, Buss M. Direct and indirect model reference adaptive control for multivariable piecewise affine systems. *IEEE Trans Automat Contr*. 2017;62(11):5634–5649.
12. Wang Q, Hou Y, Dong C. Model reference robust adaptive control for a class of uncertain switched linear systems. *Int J Robust Nonlinear Control*. 2012;22(9):1019–1035.
13. Yuan S, De Schutter B, Simone B. Robust adaptive tracking control of uncertain slowly switched linear systems. *Nonlinear Anal Hybrid Syst*. 2018;27:1–12.
14. Wu C, Zhao J. H∞ adaptive tracking control for switched systems based on an average dwell–time method. *Int J Syst Sci*. 2015;46(14):2547–2559.
15. Xie J, Zhao J. H∞ model reference adaptive control for switched systems based on the switched closed–loop reference model. *Nonlinear Anal Hybrid Syst*. 2018;27:92–106.
16. Wu C, Zhao J, Sun X–M. Adaptive tracking control for uncertain switched systems under asynchronous switching. *Int J Robust Nonlinear Control*. 2015;25(17):3457–3477.
17. Yuan S, Zhang L, De Schutter B, Baldi S. A novel Lyapunov function for a non–weighted L2 gain of asynchronously switched linear systems. *Automatica*. 2018;87:310–317.
18. Ilchmann A, Schuster H. PI–funnel control for two mass systems. *IEEE Trans Automat Contr*. 2009;54(4):918–923.
19. Hackl CM, Hopfe N, Ilchmann A, Mueller M, Trenn S. Funnel control for systems with relative degree two. *SIAM J Control Optim*. 2013;51(2):965–995.
20. Tee KP, Ge SS, Tay EH. Barrier Lyapunov functions for the control of output–constrained nonlinear systems. *Automatica*. 2009;45(4):918–927.
21. Bechlioulis CP, Rovithakis GA. Robust adaptive control of feedback linearizable MIMO nonlinear systems with prescribed performance. *IEEE Trans Automat Contr*. 2008;53(9):2090–2099.
22. Bechlioulis CP, Rovithakis GA. Prescribed performance adaptive control for multi–input multi–output affine in the control nonlinear systems. *IEEE Trans Automat Contr*. 2010;55(5):1220–1226.
23. Liu Y–J, Li D–J, Tong S. Adaptive output feedback control for a class of nonlinear systems with full–state constraints. *Int J Control*. 2014;87(2):281–290.
24. Liu Y–J, Tong S. Barrier Lyapunov functions–based adaptive control for a class of nonlinear pure–feedback systems with full state constraints. *Automatica*. 2016;64:70–75.

25. Zhao K, Song Y. Removing the feasibility conditions imposed on tracking control designs for state–constrained strict–feedback systems. *IEEE Trans Automat Contr*. 2018;64(3):1265–1272.
26. Niu B, Zhao X, Fan X, Cheng Y. A new control method for state–constrained nonlinear switched systems with application to chemical process. *Int J Control*. 2015;88(9):1693–1701.
27. Arabi E, Gruenwald BC, Yucelen T, Nguyen NT. A set–theoretic model reference adaptive control architecture for disturbance rejection and uncertainty suppression with strict performance guarantees. *Int J Control*. 2018;91(5):1195–1208.
28. Arabi E, Yucelen T. Set–theoretic model reference adaptive control with time–varying performance bounds. *Int J Control*. 2019;92(11):2509–2520.
29. Xiao S, Dong J. Robust adaptive fault–tolerant tracking control for uncertain linear systems with time–varying performance bounds. *Int J Robust Nonlinear Control*. 2019;29(4):849–866.
30. Arabi E, Yucelen T, Gruenwald BC, Fravolini M, Balakrishnan S, Nguyen NT. A neuroadaptive architecture for model reference control of uncertain dynamical systems with performance guarantees. *Syst Control Lett*. 2019;125:37–44.
31. Tao G. Multivariable adaptive control: a survey. *Automatica*. 2014;50(11):2737–2764.
32. Hespanha JP. Uniform stability of switched linear systems: extensions of LaSalle's invariance principle. *IEEE Trans Automat Contr*. 2004;49(4):470–482.
33. Liberzon D, Trenn S. The bang–bang funnel controller for uncertain nonlinear systems with arbitrary relative degree. *IEEE Trans Automat Contr*. 2013;58(12):3126–3141.
34. Bikas LN, Rovithakis GA. Combining prescribed tracking performance and controller simplicity for a class of uncertain MIMO nonlinear systems with input quantization. *IEEE Trans Automat Contr*. 2018;64(3):1228–1235.
35. Zhai D, Xi C, An L, Dong J, Zhang Q. Prescribed performance switched adaptive dynamic surface control of switched nonlinear systems with average dwell time. *IEEE Trans Syst Man Cybern Syst*. 2016;47(7):1257–1269.
36. Li Y, Tong S, Liu L, Feng G. Adaptive output–feedback control design with prescribed performance for switched nonlinear systems. *Automatica*. 2017;80:225–231.
37. Zhang J–X, Yang G–H. Supervisory switching–based prescribed performance control of unknown nonlinear systems against actuator failures. *Int J Robust Nonlinear Control*. 2020;30(6):2367–2385.
38. Yuan S, De Schutter B, Baldi S. Adaptive asymptotic tracking control of uncertain time–driven switched linear systems. *IEEE Trans Automat Contr*. 2016;62(11):5802–5807.
39. Yuan S, Zhang L, Baldi S. Adaptive stabilization of impulsive switched linear time–delay systems: a piecewise dynamic gain approach. *Automatica*. 2019;103:322–329.
40. Müller MA, Liberzon D. Input/output–to–state stability and state–norm estimators for switched nonlinear systems. *Automatica*. 2012;48(9):2029–2039.
41. Long L, Zhao J. Switched–observer–based adaptive neural control of MIMO switched nonlinear systems with unknown control gains. *IEEE Trans Neural Netw Learn Syst*. 2016;28(7):1696–1709.
42. Tao T, Roy S, Baldi S. The issue of transients in leakage–based model reference adaptive control of switched linear systems. *Nonlinear Anal Hybrid Syst*. 2020;36:100885.
43. Lavretsky E. Adaptive output feedback design using asymptotic properties of LQG/LTR controllers. *IEEE Trans Automat Contr*. 2011;57(6):1587–1591.
44. Dehghan M, Ong C–J. Computations of mode–dependent dwell times for discrete–time switching system. *Automatica*. 2013;49(6):1804–1808.
45. Morse AS. Supervisory control of families of linear set–point controllers–part i. exact matching. *IEEE Trans Automat Contr*. 1996;41(10):1413–1431.
46. Hespanha Joao P, Morse AS. Stability of switched systems with average dwell–time; 1999:2655–2660; IEEE.

How to cite this article: Liu T, Buss M. Model reference adaptive control of piecewise affine systems with state tracking performance guarantees. *Int J Robust Nonlinear Control*. 2022;32(7):4129-4148. doi: 10.1002/rnc.6015

APPENDIX. PROOFS

A.1 Proof of Lemma 2

Consider the Lyapunov function $V_m = x_m^T\left(\sum_{i=1}^{s}\chi_i P_i\right)x_m$ for the homogeneous part of (4). The increment of V_m at each switching instant satisfies $V_m(t_k) \leq \mu V_m(t_k^-)$. In the interval $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$, we have $\dot{V}_m \leq -\alpha_m V_m$ with

$$\alpha_m = \min_{i\in\mathcal{I}} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)}. \tag{A1}$$

If the switching satisfies $t_k - t_{k-1} > \frac{\ln\mu}{\alpha_m}, \forall k \in \mathbb{N}^+$, the homogeneous system $\dot{x}_m = A_m x_m$ is exponentially stable and the stability of the reference system (4) can be concluded for bounded input r.[45,46] From (28) we have $h - l < h < \frac{1}{2}\alpha_m$. This together with $\mu > 1$ leads to

$$\tau_D > \frac{2}{\alpha_m} \ln \frac{\sqrt{\mu}\rho_\infty - \frac{g}{h}\sqrt{\mu}}{\rho_\infty - \frac{g}{h}\sqrt{\mu}} > \frac{2}{\alpha_m} \ln \frac{\sqrt{\mu}(\rho_\infty - \frac{g}{h})}{\rho_\infty - \frac{g}{h}} = \frac{\ln\mu}{\alpha_m}. \tag{A2}$$

So this tells that the reference system is stable and $x_m \in \mathcal{L}_\infty$ if the dwell time constraint τ_D in (16) is satisfied.

A.2 Proof of Lemma 3

Proof. From the definition of ϕ given in (26) we have

$$2\phi_d \cdot (\|e\|_P^2 - c) - \phi = \frac{\|e\|_P^4 + \epsilon^2\|e\|_P^2 - 2c\epsilon^2}{(\epsilon^2 - \|e\|_P^2)^2}. \tag{A3}$$

The denominator of (A3) is positive and the sign of $2\phi_d \cdot (\|e\|_P^2 - c) - \phi$ is determined by the numerator, which can be viewed as a quadratic function $f(z) = z^2 + \epsilon^2 z - 2c\epsilon^2$ with $z = \|e\|_P^2$. We have $f(z) \le 0$ for $z \in \left[\frac{-\epsilon^2-\sqrt{\epsilon^4+8\epsilon^2c}}{2}, \frac{-\epsilon^2+\sqrt{\epsilon^4+8\epsilon^2c}}{2}\right]$ and $f(z) > 0$ otherwise. Since ϕ, ϕ_d are defined over $\|e\|_P^2 \in [0, \epsilon^2)$ and $\frac{-\epsilon^2-\sqrt{\epsilon^4+8\epsilon^2c}}{2} < 0$, it can be obtained that $2\phi_d \cdot (\|e\|_P^2 - c) - \phi > 0$ for $\zeta < \|e\|_P^2 < \epsilon^2$ and $2\phi_d \cdot (\|e\|_P^2 - c) - \phi \le 0$ for $\|e\|_P^2 \le \zeta$ with $\zeta = \frac{-\epsilon^2+\sqrt{\epsilon^4+8\epsilon^2c}}{2}$. ■