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Siegel modular flavor group and $\mathcal{CP}$ from string theory

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A B S T R A C T

We derive the potential modular symmetries of heterotic string theory. For a toroidal compactification with Wilson line modulus, we obtain the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ that includes the modular symmetries $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$ (of the "geometric" moduli T and U) as well as mirror symmetry. In addition, string theory provides a candidate for a $\mathcal{CP}$-like symmetry that enhances the Siegel modular group to $\mathrm{GSp}(4,\mathbb{Z})$.

1. Introduction

Modular symmetries might play an important role for a description of the flavor structure in particle physics [1]. In string theory, modular transformations appear as the exchange of winding and momentum (Kaluza-Klein) modes in compactified extra dimensions, combined with a nontrivial transformation of the moduli. In the application to flavor symmetries, these moduli play the role of flavon fields that are responsible for the spontaneous breakdown of flavor and $\mathcal{CP}$ symmetries. While string theory requires six compact space dimensions with many moduli, the explicit discussion in flavor physics has, up to now, mainly concentrated on two compact extra dimensions and few geometric moduli (see e.g. ref. [2]). In the top-down discussion, this included i) the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold with Kähler modulus T (and frozen complex structure modulus U) [3–5] subject to the modular group $\mathrm{SL}(2,\mathbb{Z})_T$ and ii) the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold with T and U moduli with a corresponding modular group $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ combined with a mirror symmetry that interchanges T and U [6].

The present paper performs a next step towards a more exhaustive discussion of the "many-moduli-case". Our results are based on the observation that string theory includes more moduli beyond the (geometric) T- and U-moduli in form of Wilson lines connected to gauge symmetries in extra dimensions. Modular transformations act nontrivially on these Wilson lines and require a modified geometric interpretation. In the present paper, we illustrate this situation for compactifications on two-tori and the corresponding transformation of the Narain lattice in heterotic string theory. Our main results are:

- Wilson line moduli lead to an enhancement of modular flavor symmetries,
- for the case of two compactified dimensions, this leads to the appearance of the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$, which includes $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ as well as mirror symmetry,
- a generalized geometric interpretation of the origin of these symmetries is given through an auxiliary Riemann surface of genus 2 (see Fig. 1) that combines the metric and gauge moduli in a common setting,[1] and
- a candidate $\mathcal{CP}$-like symmetry naturally appears in string models with two compact dimensions; interestingly, this symmetry also arises in a bottom-up discussion as an outer automorphism of the Siegel modular group, extending it to $\mathrm{GSp}(4,\mathbb{Z})$.

The paper is organized as follows. In section 2, we introduce the Siegel modular group $\mathrm{Sp}(2g,\mathbb{Z})$. Specific properties and subgroups are illustrated for the genus 2 case $\mathrm{Sp}(4,\mathbb{Z})$, where the subgroups include $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ and mirror symmetry. Besides the T- and U-moduli, the Siegel modular group acts on a third modulus Z. In section 3, we relate this third modulus to Wilson lines in heterotic

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[1] This interpretation was first anticipated in the discussion of gauge threshold corrections in heterotic string theory [7].

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string theory. We introduce the $2D+16$-dimensional Narain lattice and its outer automorphism $\mathrm{O}_{\hat{\eta}}(D,D+16,\mathbb{Z})$ and then specialize on $D=2$ with a nontrivial Wilson line. The subgroup $\mathrm{O}_{\hat{\eta}}(2,3,\mathbb{Z})$ of $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ can be mapped to $\mathrm{Sp}(4,\mathbb{Z})$ as given explicitly in Table 1. This allows for a connection to the recent bottom-up approach of Ding, Feruglio and Liu [8]. Their "third" modulus can thus be realized as a Wilson line in heterotic string theory. In addition, string constructions admit a $\mathcal{CP}$-like symmetry, which appears at the same footing as all discrete (traditional and modular) symmetries. In section 4, we show that this $\mathcal{CP}$-like symmetry also appears naturally from a bottom-up perspective: It corresponds to an outer automorphism of the Siegel modular group extending it to the general symplectic group $\mathrm{GSp}(4,\mathbb{Z})$. Conclusions and outlook are given in section 5. Finally, some technical details are discussed in two appendices.

2. The Siegel modular group $\mathrm{Sp}(2g,\mathbb{Z})$

The symplectic group over the integers $\mathrm{Sp}(2g,\mathbb{Z})$ (also called the Siegel modular group of genus g) is the group of linear transformations M which preserve a skew-symmetric bilinear form J, i.e.

$$\mathrm{Sp}(2g,\mathbb{Z}) \;:=\; \left\{M\in\mathbb{Z}^{2g\times 2g} \mid M^{\mathrm{T}}JM=J\right\}\;. \tag{1}$$

Here, J is given as

$$J \;:=\; \begin{pmatrix} 0 & \mathbb{1}_g\\ -\mathbb{1}_g & 0\end{pmatrix}\;, \tag{2}$$

and $\mathbb{1}_g$ is the g-dimensional identity matrix.

As reviewed in appendix A, there exists a natural action of $\mathrm{Sp}(2g,\mathbb{Z})$ on a symmetric $g\times g$-dimensional matrix called $\Omega\in\mathbb{H}_g$, where the Siegel upper half plane $\mathbb{H}_g$ is defined as

$$\mathbb{H}_g \;:=\; \left\{\Omega\in\mathbb{C}^{g\times g}\mid \Omega^{\mathrm{T}}=\Omega\;,\;\mathrm{Im}\,\Omega>0\right\}\;. \tag{3}$$

Hence, Ω contains $g\times(g+1)/2$ complex numbers that are called moduli. In more detail, one splits $M\in\mathrm{Sp}(2g,\mathbb{Z})$ into $g\times g$-dimensional blocks A, B, C, and D as follows

$$M \;=\; \begin{pmatrix} A & B\\ C & D\end{pmatrix} \;\in\; \mathrm{Sp}(2g,\mathbb{Z})\;. \tag{4}$$

Then, M acts on Ω as

$$\Omega \;\xrightarrow{M}\; (A\,\Omega+B)\,(C\,\Omega+D)^{-1}\;. \tag{5}$$

Note that $\pm M\in\mathrm{Sp}(2g,\mathbb{Z})$ yield the same transformation eq. (5) of Ω.

In the following, we focus on $g=2$. In this case, the moduli are encoded in a symmetric 2×2 matrix Ω whose components are denoted as

$$\Omega \;=\; \begin{pmatrix} U & Z\\ Z & T\end{pmatrix}\;. \tag{6}$$

As we will see explicitly in the following, T and U are two moduli associated with the modular group $\mathrm{SL}(2,\mathbb{Z})_T\times\mathrm{SL}(2,\mathbb{Z})_U$, while Z is a new modulus that interrelates the two $\mathrm{SL}(2,\mathbb{Z})$ factors.

2.1. *Subgroups of the Siegel modular group* $\mathrm{Sp}(4,\mathbb{Z})$

The Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ contains two factors of $\mathrm{SL}(2,\mathbb{Z})$, i.e.

$$M_{(\gamma_T,\gamma_U)} \;:=\; \begin{pmatrix} a_U & 0 & b_U & 0\\ 0 & a_T & 0 & b_T\\ c_U & 0 & d_U & 0\\ 0 & c_T & 0 & d_T\end{pmatrix} \;\in\; \mathrm{Sp}(4,\mathbb{Z})\;, \tag{7}$$

where $a_Td_T-b_Tc_T=a_Ud_U-b_Uc_U=1$. Hence,

$$\gamma_T \;:=\; \begin{pmatrix} a_T & b_T\\ c_T & d_T\end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{Z})_T \quad\text{and}\quad \gamma_U \;:=\; \begin{pmatrix} a_U & b_U\\ c_U & d_U\end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{Z})_U\;. \tag{8}$$

Here, $\mathrm{SL}(2,\mathbb{Z})$ denotes the modular group generated by

$$\mathrm{S} \;:=\; \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} \quad\text{and}\quad \mathrm{T} \;:=\; \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\;. \tag{9}$$

In detail, $\mathrm{SL}(2,\mathbb{Z})_U$ is contained in $\mathrm{Sp}(4,\mathbb{Z})$ because

$$M_{(\mathbb{1}_2,\gamma_U)} = \begin{pmatrix} a_U & 0 & b_U & 0\\ 0 & 1 & 0 & 0\\ c_U & 0 & d_U & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \quad\text{as long as}\quad \gamma_U = \begin{pmatrix} a_U & b_U\\ c_U & d_U\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_U\;, \tag{10}$$

due to the defining condition $M^{\mathrm{T}}_{(\mathbb{1}_2,\gamma_U)}\,J\,M_{(\mathbb{1}_2,\gamma_U)} = J$ of $\mathrm{Sp}(4,\mathbb{Z})$, see eq. (1). Then, we use eq. (5) and find that the moduli transform as

$$T \xrightarrow{M_{(\mathbb{1}_2,\gamma_U)}} T - \frac{c_U\,Z^2}{c_U\,U + d_U}\,, \tag{11a}$$

$$U \xrightarrow{M_{(\mathbb{1}_2,\gamma_U)}} \frac{a_U\,U + b_U}{c_U\,U + d_U}\,, \tag{11b}$$

$$Z \xrightarrow{M_{(\mathbb{1}_2,\gamma_U)}} \frac{Z}{c_U\,U + d_U}\,. \tag{11c}$$

Note that for $Z = 0$ we see that T and Z are invariant under $\mathrm{SL}(2,\mathbb{Z})_U$ modular transformations, while U transforms as expected from $\mathrm{SL}(2,\mathbb{Z})_U$. Similarly, we can embed $\mathrm{SL}(2,\mathbb{Z})_T$ into $\mathrm{Sp}(4,\mathbb{Z})$ via

$$M_{(\gamma_T,\mathbb{1}_2)} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a_T & 0 & b_T\\ 0 & 0 & 1 & 0\\ 0 & c_T & 0 & d_T \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \quad\text{while}\quad \gamma_T = \begin{pmatrix} a_T & b_T\\ c_T & d_T \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_T\,, \tag{12}$$

such that the moduli transform as

$$T \xrightarrow{M_{(\gamma_T,\mathbb{1}_2)}} \frac{a_T\,T + b_T}{c_T\,T + d_T}\,, \tag{13a}$$

$$U \xrightarrow{M_{(\gamma_T,\mathbb{1}_2)}} U - \frac{c_T\,Z^2}{c_T\,T + d_T}\,, \tag{13b}$$

$$Z \xrightarrow{M_{(\gamma_T,\mathbb{1}_2)}} \frac{Z}{c_T\,T + d_T}\,, \tag{13c}$$

using eq. (5). Let us remark that the modular S^2 transformations from $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$ are related in $\mathrm{Sp}(4,\mathbb{Z})$, i.e. $M_{(\mathrm{S}^2,\mathbb{1}_2)} = -M_{(\mathbb{1}_2,\mathrm{S}^2)}$ and the moduli transform as

$$T \xrightarrow{M} T\,, \tag{14a}$$

$$U \xrightarrow{M} U\,, \tag{14b}$$

$$Z \xrightarrow{M} -Z\,, \tag{14c}$$

for $M \in \{M_{(\mathrm{S}^2,\mathbb{1}_2)}, M_{(\mathbb{1}_2,\mathrm{S}^2)}\}$.

In addition, $\mathrm{Sp}(4,\mathbb{Z})$ contains a $\mathbb{Z}_2$ mirror transformation

$$M_\times := \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \quad\text{with}\quad (M_\times)^2 = \mathbb{1}_4\,. \tag{15}$$

As the name suggests, a mirror transformation interchanges T and U, i.e. using eq. (5) one can verify easily that

$$T \xrightarrow{M_\times} U\,, \tag{16a}$$

$$U \xrightarrow{M_\times} T\,, \tag{16b}$$

$$Z \xrightarrow{M_\times} Z\,. \tag{16c}$$

Finally, $\mathrm{Sp}(4,\mathbb{Z})$ contains elements $M(\Delta)$ with $\Delta \in \mathbb{Z}^2$. These elements are intrinsically tied to the modulus Z. They can be defined as

$$M(\Delta) := \begin{pmatrix} 1 & 0 & 0 & -\ell\\ m & 1 & -\ell & 0\\ 0 & 0 & 1 & -m\\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \quad\text{for}\quad \Delta := \begin{pmatrix} \ell\\ m \end{pmatrix} \in \mathbb{Z}^2\,. \tag{17}$$

Then, eq. (5) yields

$$T \xrightarrow{M(\Delta)} T + m\,(m\,U + 2\,Z - \ell)\,, \tag{18a}$$

$$U \xrightarrow{M(\Delta)} U\,, \tag{18b}$$

$$Z \xrightarrow{M(\Delta)} Z + m\,U - \ell\,. \tag{18c}$$

3. The origin of the Sp(4, $\mathbb{Z}$) Siegel modular group from strings

It is well-known that compactifications of heterotic string theory on tori (and toroidal orbifolds) are naturally described in the Narain formulation [9–11]. There, one considers D right- and $D+16$ left-moving (bosonic) string modes $(y_\mathrm{R}, y_\mathrm{L})$ to be compactified as

$$Y \sim Y + E\,\hat{N}\,, \quad \text{where} \quad Y := \begin{pmatrix} y_\mathrm{R} \\ y_\mathrm{L} \end{pmatrix}, \tag{19}$$

i.e. on an auxiliary torus of dimension $2D+16$. The 16 extra left-moving degrees of freedom give rise to an $\mathrm{E}_8 \times \mathrm{E}_8$ (or SO(32)) gauge symmetry of the heterotic string. In more detail, the auxiliary torus corresponding to the identification (19) can be defined by the so-called Narain lattice

$$\Gamma := \left\{ E\,\hat{N} \;\middle|\; \hat{N} = \begin{pmatrix} n \\ m \\ p \end{pmatrix} \in \mathbb{Z}^{2D+16} \right\} \tag{20}$$

that is spanned by the Narain vielbein E, a matrix of dimension $(2D+16) \times (2D+16)$. Here, $n \in \mathbb{Z}^D$ gives the winding numbers, $m \in \mathbb{Z}^D$ the Kaluza–Klein numbers and $p \in \mathbb{Z}^{16}$ the gauge quantum numbers. As the one-loop partition function of the string worldsheet has to be modular invariant, the Narain lattice Γ has to be an even, integer and self-dual lattice with a metric η of signature $(D, D+16)$. This condition on Γ holds if the Narain vielbein E satisfies

$$\hat{\eta} := E^\mathrm{T} \eta\, E = \begin{pmatrix} 0 & \mathbb{1}_D & 0 \\ \mathbb{1}_D & 0 & 0 \\ 0 & 0 & g \end{pmatrix}, \quad \text{where} \quad \eta := \begin{pmatrix} -\mathbb{1}_D & 0 & 0 \\ 0 & \mathbb{1}_D & 0 \\ 0 & 0 & \mathbb{1}_{16} \end{pmatrix}. \tag{21}$$

Here, $g := \alpha_\mathrm{g}^\mathrm{T} \alpha_\mathrm{g}$ is the Cartan matrix of the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge symmetry and α_g denotes a matrix whose columns are the simple roots of $\mathrm{E}_8 \times \mathrm{E}_8$ (or in the case of an SO(32) gauge symmetry, α_g is a basis of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ weight lattice).

It is convenient to define the so-called generalized metric of the Narain lattice in terms of the metric $G := e^\mathrm{T} e$ (of the D-dimensional torus spanned by the geometrical vielbein e), the anti-symmetric B-field B and the Wilson lines A,

$$\mathcal{H} := E^\mathrm{T} E := \begin{pmatrix} \frac{1}{\alpha'}\left(G + \alpha' A^\mathrm{T} A + C^\mathrm{T} G^{-1} C\right) & -C^\mathrm{T} G^{-1} & (\mathbb{1}_2 + C^\mathrm{T} G^{-1}) A^\mathrm{T} \alpha_\mathrm{g} \\ -G^{-1} C & \alpha' G^{-1} & -\alpha' G^{-1} A^\mathrm{T} \alpha_\mathrm{g} \\ \alpha_\mathrm{g}^\mathrm{T} A (\mathbb{1}_2 + G^{-1} C) & -\alpha' \alpha_\mathrm{g}^\mathrm{T} A G^{-1} & \alpha_\mathrm{g}^\mathrm{T} \left(\mathbb{1}_{16} + \alpha' A G^{-1} A^\mathrm{T}\right) \alpha_\mathrm{g} \end{pmatrix}, \tag{22}$$

where $C := B + \frac{\alpha'}{2} A^\mathrm{T} A$ and we use conventions similar to those of refs. [12,13], but replacing C by C^T for later convenience. Note that due to eq. (21), the generalized metric $\mathcal{H}$ satisfies the condition

$$\left(\mathcal{H}\,\hat{\eta}^{-1}\right)^2 = \mathbb{1}_{2D+16}\,. \tag{23}$$

The outer automorphisms of the Narain lattice are given by "rotational" transformations

$$\mathrm{O}_{\hat{\eta}}(D, D+16, \mathbb{Z}) := \left\langle \hat{\Sigma} \;\middle|\; \hat{\Sigma} \in \mathrm{GL}(2D+16, \mathbb{Z}) \quad \text{with} \quad \hat{\Sigma}^\mathrm{T} \hat{\eta}\, \hat{\Sigma} = \hat{\eta} \right\rangle. \tag{24}$$

This is the general modular group of a toroidal compactification of the heterotic string. Elements $\hat{\Sigma}$ of $\mathrm{O}_{\hat{\eta}}(D, D+16, \mathbb{Z})$ act on the Narain vielbein E as [4]

$$E \xrightarrow{\hat{\Sigma}} E\,\hat{\Sigma}^{-1}\,, \tag{25}$$

such that the Narain scalar product $\lambda_1^\mathrm{T} \eta \lambda_2$ is invariant for $\lambda_i \in \Gamma$, $i \in \{1, 2\}$.

In the following we take $D = 2$. Moreover, the (continuous) Wilson lines are chosen as $A_i = (a_i, -a_i, 0^{14})^\mathrm{T}$ for $i \in \{1, 2\}$, where A_i denote the two columns of A. Thus, we allow for continuous Wilson lines a_i in the direction of the simple root $(1, -1, 0^{14})^\mathrm{T}$ of $\mathrm{E}_8 \times \mathrm{E}_8$ (or SO(32)) for both directions e_i of the geometrical two-torus. Then, we define moduli (T, U, Z) of the two-torus with B-field

$$B := \alpha' b\, \epsilon\,, \quad \text{where} \quad \epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{26}$$

and Wilson lines background fields A as

$$T := \frac{1}{\alpha'}\left(B_{12} + \mathrm{i}\sqrt{\det G}\right) + a_1\left(-a_2 + U\, a_1\right), \tag{27a}$$

$$U := \frac{1}{G_{11}}\left(G_{12} + \mathrm{i}\sqrt{\det G}\right) = \frac{|e_2|}{|e_1|}\, \mathrm{e}^{\mathrm{i}\phi}\,, \tag{27b}$$

$$Z := -a_2 + U\, a_1\,, \tag{27c}$$

cf. ref. [7]. Moreover, e_1 and e_2 are the two columns of the geometrical vielbein e, and ϕ denotes the angle enclosed by them. Note that the continuous Wilson lines a_1 and a_2 not only yield a new "Wilson line modulus" called Z but they also alter the definition of the Kähler modulus T. In contrast, the complex structure modulus U remains unchanged in the presence of Wilson lines.

In what follows, it will be important to compute the transformation of the moduli (T, U, Z) under general modular transformations from $\mathrm{O}_{\hat{\eta}}(2, 3, \mathbb{Z}) \subset \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$. To do so, the generalized metric $\mathcal{H} = E^\mathrm{T} E$ and eq. (25) can be used to obtain

$$\mathcal{H}(T, U, Z) \xrightarrow{\hat{\Sigma}} \hat{\Sigma}^{-\mathrm{T}} \mathcal{H}(T, U, Z) \hat{\Sigma}^{-1} =: \mathcal{H}(T', U', Z')\,, \tag{28}$$

for a general modular transformation $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2, 3, \mathbb{Z})$.

3.1. Mapping between $\mathrm{O}_{\hat{\eta}}(2,3,\mathbb{Z})$ of the Narain lattice and $\mathrm{Sp}(4,\mathbb{Z})$

In this section, we discuss various subgroups and generators of $\mathrm{O}_{\hat{\eta}}(2,3,\mathbb{Z}) \subset \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$, derive their actions on the moduli (T,U,Z) and compare them to the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$. By doing so, we will show explicitly that the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ appears naturally in toroidal compactifications of the heterotic string, see also refs. [14–18]. The main results are summarized in Table 1 at the end of this section.

Mirror transformation. We define a so-called mirror transformation

$$\hat{M} := \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix} \in \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})\,, \tag{29}$$

where we have to change the conventions compared to refs. [3,4] due to the presence of Wilson lines and the resulting changes in the generalized metric eq. (22). Using eq. (28) we obtain

$$T \leftrightarrow U \quad \text{and} \quad Z \leftrightarrow Z\,, \tag{30}$$

as expected for a mirror transformation, see eq. (16) for the corresponding case in $\mathrm{Sp}(4,\mathbb{Z})$.

Modular group of the complex structure modulus. The general modular group $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ contains a modular group $\mathrm{SL}(2,\mathbb{Z})_U$ associated with the complex structure modulus U. It can be generated by

$$\hat{C}_\mathrm{S} := \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix} \quad \text{and} \quad \hat{C}_\mathrm{T} := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix}\,. \tag{31}$$

Then, we use eq. (28) in order to verify the $\mathrm{Sp}(4,\mathbb{Z})$ transformations of the moduli (T,U,Z) given in eq. (11).

Modular group of the Kähler modulus. In addition to $\mathrm{SL}(2,\mathbb{Z})_U$, due to mirror symmetry eq. (29) there exists a modular group $\mathrm{SL}(2,\mathbb{Z})_T$ associated with the Kähler modulus T. It can be defined by

$$\hat{K}_\mathrm{S} := \hat{M}\,\hat{C}_\mathrm{S}\,\hat{M}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix} \quad \text{and} \tag{32a}$$

$$\hat{K}_\mathrm{T} := \hat{M}\,\hat{C}_\mathrm{T}\,\hat{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix}\,. \tag{32b}$$

These transformations reproduce the $\mathrm{Sp}(4,\mathbb{Z})$ transformations eq. (13) of (T,U,Z), as can be seen explicitly using eq. (28).

Wilson line shifts. Due to the 16 extra left-moving degrees of freedom of the heterotic string, the general modular group $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ has additional elements called "Wilson line shifts". They are defined as

$$\hat{W}(\Delta A) := \begin{pmatrix} \mathbb{1}_2 & 0 & 0 \\ -\frac{1}{2}\Delta A^\mathrm{T} g\,\Delta A & \mathbb{1}_2 & \Delta A^\mathrm{T} g \\ -\Delta A & 0 & \mathbb{1}_{16} \end{pmatrix} \in \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})\,, \tag{33}$$

where ΔA is a 16×2-dimensional matrix with integer entries. Since g is the Cartan matrix of an even lattice (of $\mathrm{E}_8\times\mathrm{E}_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$), the 2×2 matrix $\frac{1}{2}\Delta A^\mathrm{T} g\,\Delta A$ is integer. We focus on shifts ΔA in the directions of a_1 and a_2. Hence, we define

$$\hat{W}\binom{\ell}{m} := \hat{W}(\Delta A) \quad \text{for} \quad \Delta A := \begin{pmatrix} m & \ell \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}\,, \tag{34}$$

for $\ell, m \in \mathbb{Z}$. By doing so, we will focus in what follows on a subgroup $\mathrm{O}_{\hat{\eta}}(2,3,\mathbb{Z})$ of $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$. Then, using the transformation (28) of the generalized metric, we obtain

Table 1
We list the generators of the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ and their corresponding elements in the subgroup $\mathrm{O}_{\hat\eta}(2,3,\mathbb{Z})$ of the general modular group $\mathrm{O}_{\hat\eta}(2,2+16,\mathbb{Z})$ constructed explicitly in section 3 in the Narain formulation of the heterotic string. In the last column we list the transformation of the moduli, computed in two ways: i) using eq. (5) for $\mathrm{Sp}(4,\mathbb{Z})$, and independently ii) using eq. (28) for $\mathrm{O}_{\hat\eta}(2,3,\mathbb{Z})$. The $\mathcal{CP}$-like transformation M_* will be defined in section 4.

Symmetry	$\mathrm{Sp}(4,\mathbb{Z})$	$\mathrm{O}_{\hat\eta}(2,3,\mathbb{Z})$	Transformation of moduli
$\mathrm{SL}(2,\mathbb{Z})_T$	$M_{(\mathrm{S},\mathbb{1}_2)}$	$\hat K_\mathrm{S}$	$T\to-\frac{1}{T}$ $U\to U-\frac{Z^2}{T}$ $Z\to-\frac{Z}{T}$
	$M_{(\mathrm{T},\mathbb{1}_2)}$	$\hat K_\mathrm{T}$	$T\to T+1$ $U\to U$ $Z\to Z$
$\mathrm{SL}(2,\mathbb{Z})_U$	$M_{(\mathbb{1}_2,\mathrm{S})}$	$\hat C_\mathrm{S}$	$T\to T-\frac{Z^2}{U}$ $U\to-\frac{1}{U}$ $Z\to-\frac{Z}{U}$
	$M_{(\mathbb{1}_2,\mathrm{T})}$	$\hat C_\mathrm{T}$	$T\to T$ $U\to U+1$ $Z\to Z$
Mirror	$M_\times$	$\hat M$	$T\to U$ $U\to T$ $Z\to Z$
Wilson line shift	$M\binom{\ell}{m}$	$\hat W\binom{\ell}{m}$	$T\to T+m\,(m\,U+2\,Z-\ell)$ $U\to U$ $Z\to Z+m\,U-\ell$
$\mathcal{CP}$-like	$M_*\in\mathrm{GSp}(4,\mathbb{Z})$	$\hat\Sigma_*$	$T\to-\bar T$ $U\to-\bar U$ $Z\to-\bar Z$

$$a_1 \xrightarrow{\hat W\binom{\ell}{m}} a_1+m\;,\quad a_2 \xrightarrow{\hat W\binom{\ell}{m}} a_2+\ell \quad\text{and}\quad b \xrightarrow{\hat W\binom{\ell}{m}} b+a_1\,\ell-a_2\,m\;, \tag{35}$$

while the metric G is invariant. Translated to the moduli (T,U,Z) defined in eq. (27), this reproduces the $\mathrm{Sp}(4,\mathbb{Z})$ transformations given in eq. (18).

$\mathcal{CP}$-like transformation. Finally, as discussed in ref. [4], a $\mathcal{CP}$-like generator has to act not only on the $(2+2)$-dimensional Narain coordinates of the geometrical two-torus but also on the 16 extra left-moving degrees of freedom, i.e.

$$\hat\Sigma_* := \begin{pmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & -\mathbb{1}_{16}\end{pmatrix} \in \mathrm{O}_{\hat\eta}(2,2+16,\mathbb{Z})\;. \tag{36}$$

Applying eq. (28) to $\hat\Sigma_*$ gives rise to a $\mathcal{CP}$-like transformation

$$T \xrightarrow{\hat\Sigma_*} -\bar T\;,\quad U \xrightarrow{\hat\Sigma_*} -\bar U \quad\text{and}\quad Z \xrightarrow{\hat\Sigma_*} -\bar Z \tag{37}$$

of the moduli. This string result on $\mathcal{CP}$ can also be understood from a bottom-up perspective as we will see in section 4.

As a remark, there exist further $\mathrm{O}_{\hat\eta}(2,2+16,\mathbb{Z})$ transformations not present in $\mathrm{Sp}(4,\mathbb{Z})$: One can perform Weyl reflections in the 16-dimensional lattice of $\mathrm{E}_8\times\mathrm{E}_8$ (or $\mathrm{Spin}(32)/\mathbb{Z}_2$), see for example $\hat M_\mathrm{W}(\Delta W)$ in ref. [12].

4. $\mathcal{CP}$ as an outer automorphism of $\mathrm{Sp}(4,\mathbb{Z})$

We have seen in the previous section that a $\mathcal{CP}$-like transformation appears naturally in (toroidal) string compactifications. As we shall see in this section in a bottom-up discussion, this transformation does not belong to $\mathrm{Sp}(4,\mathbb{Z})$ but corresponds to an outer automorphism of $\mathrm{Sp}(4,\mathbb{Z})$ that, once included, enhances $\mathrm{Sp}(4,\mathbb{Z})$ to the general symplectic group $\mathrm{GSp}(4,\mathbb{Z})$.

We define a transformation

$$M \xrightarrow{M_*} M' := M_*^{-1}M\,M_* \quad\text{for all}\quad M\in\mathrm{Sp}(4,\mathbb{Z})\;, \tag{38}$$

where M_* is given by

$$M_* := \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad\text{satisfying}\quad M_*^\mathrm{T} J\,M_* = -J\;. \tag{39}$$

Hence, $M_*\notin\mathrm{Sp}(4,\mathbb{Z})$. Rather it lies in the general symplectic group

$$\mathrm{GSp}(4,\mathbb{Z}) := \left\{ M \in \mathbb{Z}^{4\times4} \mid M^{\mathrm{T}} J\, M = \pm J \right\} . \tag{40}$$

Then, it is easy to see that M' defined in eq. (38) is an element from $\mathrm{Sp}(4,\mathbb{Z})$ for all $M \in \mathrm{Sp}(4,\mathbb{Z})$, i.e.

$$\left(M'\right)^{\mathrm{T}} J\, M' = M_*^{\mathrm{T}} M^{\mathrm{T}} \underbrace{M_*^{-\mathrm{T}} J\, M_*^{-1}}_{=-J} M\, M_* = -M_*^{\mathrm{T}} \underbrace{M^{\mathrm{T}} J\, M}_{=J} M_* = +J\,. \tag{41}$$

Hence, eq. (38) defines an automorphism of $\mathrm{Sp}(4,\mathbb{Z})$. It is outer because $M_* \notin \mathrm{Sp}(4,\mathbb{Z})$, as seen in eq. (39).

In order to see the physical meaning of M_*, we apply eq. (38) to various elements of $\mathrm{Sp}(4,\mathbb{Z})$:

$$M_{(\mathrm{S},\mathbb{1}_2)} \xrightarrow{M_*} M_*^{-1} M_{(\mathrm{S},\mathbb{1}_2)}\, M_* = \left(M_{(\mathrm{S},\mathbb{1}_2)}\right)^{-1} , \tag{42a}$$

$$M_{(\mathrm{T},\mathbb{1}_2)} \xrightarrow{M_*} M_*^{-1} M_{(\mathrm{T},\mathbb{1}_2)}\, M_* = \left(M_{(\mathrm{T},\mathbb{1}_2)}\right)^{-1} , \tag{42b}$$

$$M_{(\mathbb{1}_2,\mathrm{S})} \xrightarrow{M_*} M_*^{-1} M_{(\mathbb{1}_2,\mathrm{S})}\, M_* = \left(M_{(\mathbb{1}_2,\mathrm{S})}\right)^{-1} , \tag{42c}$$

$$M_{(\mathbb{1}_2,\mathrm{T})} \xrightarrow{M_*} M_*^{-1} M_{(\mathbb{1}_2,\mathrm{T})}\, M_* = \left(M_{(\mathbb{1}_2,\mathrm{T})}\right)^{-1} , \tag{42d}$$

$$M_\times \xrightarrow{M_*} M_*^{-1} M_\times\, M_* = (M_\times)^{-1} = M_\times\,, \tag{42e}$$

$$M\binom{\ell}{m} \xrightarrow{M_*} M_*^{-1} M\binom{\ell}{m} M_* = M\binom{-\ell}{m} . \tag{42f}$$

Let us analyze eq. (42f) in more detail: For each choice of ℓ, m, one can find an $M \in \mathrm{Sp}(4,\mathbb{Z})$, such that $M M\binom{-\ell}{m} M^{-1} = M\binom{-\ell}{-m}$, which implies that $M\binom{\ell}{m}$ is mapped by M_* to the conjugacy class of its inverse $M\binom{-\ell}{-m}$. Motivated by eqs. (42), we consider M_* a $\mathcal{CP}$-like transformation, see refs. [19,20]. Indeed, as explained in appendix A, the action of $\mathrm{GSp}(4,\mathbb{Z})$ on Ω can be defined in analogy to the action of $\mathrm{GL}(2,\mathbb{Z})$ on one modulus, cf. ref. [3,21,22] and ref. [23]. Explicitly, for an element of the general symplectic group

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{Z}) \tag{43}$$

we find the transformation rules

$$\Omega \xrightarrow{M} \left(A\,\bar\Omega + B\right)\left(C\,\bar\Omega + D\right)^{-1} \qquad \text{if} \quad M^{\mathrm{T}} J\, M = -J\,, \tag{44a}$$

$$\Omega \xrightarrow{M} \left(A\,\Omega + B\right)\left(C\,\Omega + D\right)^{-1} \qquad \text{if} \quad M^{\mathrm{T}} J\, M = +J\,, \tag{44b}$$

where $\bar\Omega$ denotes the complex conjugate of Ω. Consequently, the moduli transform under M_* as

$$T \xrightarrow{M_*} -\bar T\,,\quad U \xrightarrow{M_*} -\bar U\,,\quad Z \xrightarrow{M_*} -\bar Z\,, \tag{45}$$

which confirms our expectation for a $\mathcal{CP}$-like transformation.

5. Conclusions and outlook

The potential (traditional *and* modular) flavor symmetries of string theory compactifications are determined through the outer automorphisms of the Narain lattice. For the heterotic string, the modular symmetries are a subgroup of $\mathrm{O}_{\hat\eta}(D, D+16, \mathbb{Z})$, where D is the dimension of the relevant compact space, i.e. $D \le 6$. As a starting point, we have concentrated in this paper on a $D = 2$ sublattice of compact six-dimensional space. Apart from the Kähler and complex structure moduli T and U, we include a Wilson line modulus Z and arrive at the modular symmetry group $\mathrm{O}_{\hat\eta}(2,3,\mathbb{Z})$. We show that this group is closely related to the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$, which has been studied intensively in the mathematical literature. The (complex) three-dimensional moduli space of $\mathrm{Sp}(4,\mathbb{Z})$ can be visualized through an auxiliary Riemann surface of genus 2 (see Fig. 1). Our top-down construction allows for a physical interpretation of the recent bottom-up discussion of ref. [8]: Their "third" modulus τ_3 (apart from $\tau_1 = U$ and $\tau_2 = T$) can be understood as a Wilson line modulus Z of compactified (heterotic) string theory. Furthermore, we have shown in a general study that, in addition to modular symmetries, there is a natural appearance of a $\mathcal{CP}$-like transformation predicted from the group $\mathrm{O}_{\hat\eta}(2,3,\mathbb{Z})$ in string theory. As discussed in section 4, from a bottom-up perspective, this $\mathcal{CP}$-like transformation can be understood as an outer automorphism of the Siegel modular group extending it to $\mathrm{GSp}(4,\mathbb{Z})$.

Beyond these results, an important open task is to make contact with realistic models of "flavor" including chiral matter. With this purpose, it is necessary to alter the $\mathbb{T}^2$ toroidal compactification by a $\mathbb{Z}_K$ orbifolding, i.e. $\mathbb{T}^2/\mathbb{Z}_K$. In string theory, this orbifolding results in the appearance of twisted strings, which in general give rise to chiral matter. Moreover, it is a remarkable fact that modular symmetries act nontrivially on twisted strings, such that twisted strings build nontrivial representations of a *finite* modular flavor group. Thus, the orbifolding in string theory is instrumental to obtain chiral matter that exhibits finite modular flavor symmetries. This general mechanism of string theory has been discussed in detail for the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold without Wilson lines: in this case, the modular symmetry $\mathrm{SL}(2,\mathbb{Z})_T$ of the Kähler modulus T acts as the *finite* modular flavor symmetry T' on the chiral matter from the twisted sectors of the orbifold [3–5].

As we have seen in this work explicitly, the modular symmetry $\mathrm{O}_{\hat\eta}(2,3,\mathbb{Z})$ of a toroidal compactification of string theory *with* Wilson lines corresponds to the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$. Hence, a $\mathbb{Z}_K$ orbifolding can in general give rise to a *finite* Siegel modular flavor group $\Gamma_{g,n}$ (here, of genus $g = 2$), where chiral matter arises from the twisted sectors of the orbifold and builds nontrivial representations of $\Gamma_{2,n}$. For the $\mathbb{Z}_2$ orbifold we have $n = 2$ and $\Gamma_{2,2}$ is isomorphic to S_6, the permutation group of six elements, see also ref. [8]. This S_6 includes the finite modular group $S_3 \times S_3$ as well as mirror symmetry, as obtained in the string theory discussion of ref. [6], where

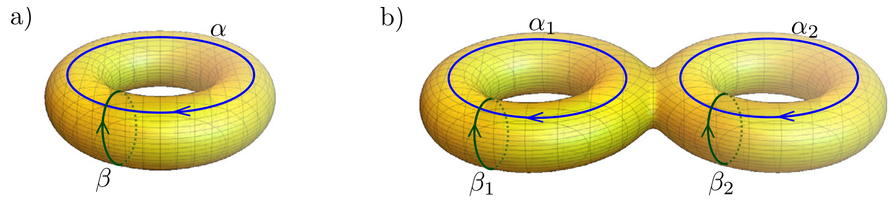

Fig. 1. a) A $\mathcal{T}_1 = \mathbb{T}^2$ torus with the two basis 1-cycles, α and β. Its modular symmetry group is $\mathrm{Sp}(2,\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{Z})$. b) A compact Riemann surface of genus 2 $\mathcal{T}_2$ and its four basis 1-cycles $(\beta_1,\beta_2,\alpha_1,\alpha_2)^{\mathrm{T}}$. The Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ is the modular symmetry group of $\mathcal{T}_2$. As discussed in ref. [7], setting the Wilson line modulus Z defined in eq. (27c) to $Z=0$ splits the genus 2 surface into two separated two-tori. Note that these auxiliary surfaces must not be mistaken as compactification spaces.

only the moduli T and U associated with $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ had been considered and the Wilson line modulus was set to $Z=0$. This indicates the path how to generalize to the case $Z \neq 0$ in string theory. In general, a $\mathbb{Z}_K$ orbifolding can break the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ by discrete Wilson lines: The geometrical $\mathbb{Z}_K$ rotation that acts on the two-torus has to be embedded into the 16 degrees of freedom of the gauge symmetry due to worldsheet modular invariance of the string partition function. It is known that a shift embedding yields discrete Wilson lines [24], such that the Wilson line modulus Z is frozen at some discrete value. In this case, the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ is broken by the fixed Wilson line modulus. For a $\mathbb{Z}_K$ orbifold with $K \neq 2$ the unbroken subgroup from $\mathrm{Sp}(4,\mathbb{Z})$ is at least the modular group $\mathrm{SL}(2,\mathbb{Z})_T$ of the Kähler modulus T, while for $K=2$ one finds at least $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ combined with a mirror symmetry that interchanges T and U, see ref. [6]. On the other hand, a rotational embedding into the 16 gauge degrees of freedom gives rise to continuous Wilson lines [25,26], where the Wilson line modulus Z remains as a free modulus. Hence, one expects that a two-dimensional $\mathbb{Z}_2$ orbifold with rotational embedding yields the full $\mathrm{Sp}(4,\mathbb{Z})$ Siegel modular group, where chiral matter from the twisted sector transforms in representations of the *finite* Siegel modular flavor group $\Gamma_{2,2} \cong S_6$. A full discussion of the symmetries of the $\mathbb{Z}_2$ orbifold, including $\mathcal{CP}$, will be subject of a future publication [27].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Symplectic groups $\mathrm{Sp}(2g,\mathbb{Z})$ and modular transformations

Let us review some aspects of the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ and its relation to modular transformations (see e.g. [28,29] for further details). The symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ can be defined by considering an auxiliary genus-g Riemann surface $\mathcal{T}_g$ and its symmetries as follows: The genus-g surface has $2g$ nontrivial 1-cycles denoted by (β_i,α_j) for $i,j \in \{1,\ldots,g\}$, see Fig. 1 for the cases $g=1$ and $g=2$. These cycles form the canonical basis of the homology group $H_1(\mathcal{T}_g,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. The holomorphic 1-forms ω_i build the dual cohomology basis, which one can choose to satisfy $\int_{\alpha_j}\omega_i = \delta_{ij}$ and $\int_{\beta_j}\omega_i = \int_{\beta_i}\omega_j$. In these terms, the skew-symmetric form J in eq. (1) is interpreted as the intersection numbers $\binom{\beta}{\alpha} \cap \binom{\beta}{\alpha}$ of the $2g$-dimensional vectors of 1-cycles $\binom{\beta}{\alpha} = (\beta_1,\ldots,\beta_g,\alpha_1,\ldots,\alpha_g)^{\mathrm{T}}$, such that $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0$ and $-(\alpha_i \cap \beta_j) = \beta_i \cap \alpha_j = \delta_{ij}$. Now, one transforms the 1-cycles $\binom{\beta}{\alpha}$

$$\begin{pmatrix}\beta\\ \alpha\end{pmatrix} \xrightarrow{M} \begin{pmatrix}\beta'\\ \alpha'\end{pmatrix} := \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \begin{pmatrix}A\,\beta+B\,\alpha\\ C\,\beta+D\,\alpha\end{pmatrix} \quad \text{for} \quad M = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \in \mathbb{Z}^{g\times g}\,. \tag{46}$$

The new 1-cycles $\binom{\beta'}{\alpha'}$ also form a basis of $H_1(\mathcal{T}_g,\mathbb{Z})$ if $M \in \mathrm{GL}(2g,\mathbb{Z})$. Moreover, we have to require that the intersection numbers and, hence, J be invariant under the transformation (46). This amounts to demanding that $M\,J\,M^{\mathrm{T}} = J$. By taking the inverse transpose of this equation we get $M^{-\mathrm{T}}\,J^{-\mathrm{T}}\,M^{-1} = J^{-\mathrm{T}}$. Then, using $J^{-\mathrm{T}} = J$ we obtain the condition $M^{\mathrm{T}}\,J\,M = J$, i.e. $M \in \mathrm{Sp}(2g,\mathbb{Z})$.

A consequence of the Torelli theorem for Riemann surfaces is that the genus-g surface $\mathcal{T}_g$ is determined by the complex g-dimensional torus, which can be defined as the quotient of $\mathbb{C}^g$ divided by a complex lattice. This lattice is given by the columns of the $g \times 2g$ period matrix of $\mathcal{T}_g$,

$$\Pi_g := \begin{pmatrix} \int_{\alpha_1}\omega_1 & \ldots & \int_{\alpha_1}\omega_g & \int_{\beta_1}\omega_1 & \ldots & \int_{\beta_1}\omega_g \\ \vdots & & \vdots & \vdots & & \vdots \\ \int_{\alpha_g}\omega_1 & \ldots & \int_{\alpha_g}\omega_g & \int_{\beta_g}\omega_1 & \ldots & \int_{\beta_g}\omega_g \end{pmatrix}. \tag{47}$$

By choosing a basis in which $\int_{\alpha_j}\omega_i = \delta_{ij}$, one can always rewrite Π_g, such that

$$\Pi_g = (\mathbb{1}_g, \Omega)\,, \tag{48}$$

where we have defined the $g\times g$ complex modulus matrix Ω, such that $\Omega^{\mathrm{T}}=\Omega$. Clearly, the transformations $\binom{\beta}{\alpha}\to M\binom{\beta}{\alpha}$ induce transformations on the modulus matrix Ω in eq. (47). Restricting further to $\operatorname{Im}\Omega>0$, we arrive at the modular space of the genus-g compact surface $\mathcal{T}_g$,

$$\mathbb{H}_g=\left\{\Omega\in\mathbb{C}^{g\times g}\mid\Omega^{\mathrm{T}}=\Omega,\operatorname{Im}\Omega>0\right\}\,. \tag{49}$$

Consider the $g=1$ case. We observe that M are 2×2 integer matrices with unit determinant, i.e. they describe the modular group $\mathrm{Sp}(2,\mathbb{Z})\cong\mathrm{SL}(2,\mathbb{Z})$ of a $\mathbb{T}^2$ torus. Given the holomorphic 1-form $\omega=\mathrm{d}z$ and the nontrivial 1-cycles α and β, shown in Fig. 1a), the period matrix of $\mathbb{T}^2$ is given by

$$\Pi_1\;=\;\left(\int_\alpha\omega,\int_\beta\omega\right)\;=\;(1,\tau)\,,\qquad\tau\in\mathbb{C}\,,\;\operatorname{Im}\tau>0\,. \tag{50}$$

The last equation arises from the choice $\int_\alpha\omega=1$ and the definition of the modulus $\tau:=\int_\beta\omega$. We now let the $\mathrm{SL}(2,\mathbb{Z})$ element $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ act on the 1-cycle vector $(\beta,\alpha)^{\mathrm{T}}$. This implies that the period matrix transforms as

$$\Pi_1\;\to\;\Pi_1'=\left(\int_{c\beta+d\alpha}\omega',\int_{a\beta+b\alpha}\omega'\right)=\left(c\int_\beta\omega'+d\int_\alpha\omega',a\int_\beta\omega'+b\int_\alpha\omega'\right)\,. \tag{51}$$

By demanding that the holomorphic 1-form transforms under M as $\omega'=\omega(c\tau+d)^{-1}$, we normalize the transformed period matrix, which then becomes

$$\Pi_1'\;=\;\left(1,(a\tau+b)(c\tau+d)^{-1}\right)\,. \tag{52}$$

This allows us to identify the standard modular transformation $\tau\to(a\tau+b)(c\tau+d)^{-1}$.

The same discussion can be conducted for the more interesting case $g=2$, which leads to the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$. The Riemann surface $\mathcal{T}_2$ has the holomorphic 1-form basis (ω_1,ω_2) and the nontrivial 1-cycles $\binom{\beta}{\alpha}=(\beta_1,\beta_2,\alpha_1,\alpha_2)^{\mathrm{T}}$, illustrated in Fig. 1b). Thus, its 2×4 period matrix reads

$$\Pi_2\;=\;\begin{pmatrix}\int_{\alpha_1}\omega_1&\int_{\alpha_1}\omega_2&\int_{\beta_1}\omega_1&\int_{\beta_1}\omega_2\\\int_{\alpha_2}\omega_1&\int_{\alpha_2}\omega_2&\int_{\beta_2}\omega_1&\int_{\beta_2}\omega_2\end{pmatrix}\;=\;(\mathbb{1}_2,\,\Omega)\,, \tag{53}$$

where in the last relation we have chosen $\int_{\alpha_j}\omega_i=\delta_{ij}$ and defined the modular matrix Ω, as given in eq. (6), satisfying $\Omega=\Omega^{\mathrm{T}}$ and $\operatorname{Im}\Omega>0$. Next, we perform an $\mathrm{Sp}(4,\mathbb{Z})$ transformation $M\binom{\beta}{\alpha}$, where $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ and A,B,C,D are 2×2 integer matrices. Although the expression for the transformed period matrix Π_2' is more complicated than in the case $g=1$, one can readily show that, by demanding that $\mathrm{Sp}(4,\mathbb{Z})$ transformations on the 1-forms act as $(\omega_1',\omega_2')=(\omega_1,\omega_2)(C\Omega+D)^{-1}$, one arrives at

$$\Pi_2\;\to\;\Pi_2'\;=\;\left(\mathbb{1}_2,\,(A\Omega+B)(C\Omega+D)^{-1}\right)\,. \tag{54}$$

We find thus that $\mathrm{Sp}(4,\mathbb{Z})$ transformations act on the modular matrix Ω as

$$\Omega\;\xrightarrow{M}\;(A\Omega+B)\,(C\Omega+D)^{-1}\,. \tag{55}$$

We can also consider a $\mathrm{GSp}(4,\mathbb{Z})$ transformation, where we are interested in the transformation of Ω under those $\tilde{M}\in\mathrm{GSp}(4,\mathbb{Z})$ with $\tilde{M}^{\mathrm{T}}J\tilde{M}=-J$ (since for the case with $M^{\mathrm{T}}JM=+J$ the result is already given in eq. (55)). In section 4, we have defined a special element $M_*\in\mathrm{GSp}(4,\mathbb{Z})$ with $M_*^{\mathrm{T}}JM_*=-J$. Then, the combined transformation $M:=\tilde{M}\,M_*$ satisfies $M^{\mathrm{T}}JM=+J$, so $M\in\mathrm{Sp}(4,\mathbb{Z})$. Since we know the transformation of Ω for $M\in\mathrm{Sp}(4,\mathbb{Z})$, we only need to know the transformation of Ω for our special element M_* in order to understand the general case with $\tilde{M}=MM_*$. Under M_* the 1-cycles $\binom{\beta}{\alpha}$ transform as

$$\begin{pmatrix}\beta\\\alpha\end{pmatrix}\;\xrightarrow{M_*}\;\begin{pmatrix}-\mathbb{1}_2&0\\0&\mathbb{1}_2\end{pmatrix}\begin{pmatrix}\beta\\\alpha\end{pmatrix}\;=\;\begin{pmatrix}-\beta\\\alpha\end{pmatrix}\,, \tag{56}$$

cf. eq. (46). As can be seen in Fig. 1, this transformation corresponds to a geometrical mirror transformation at a horizontal plane. By choosing appropriate complex coordinates on the surface $\mathcal{T}_2$ (i.e. in each chart) this mirror transformation acts by complex conjugation. Consequently, it is conceivable that M_* has to map the 1-forms ω_i to $\bar{\omega}_i$. For the period matrix, this amounts to

$$\Pi_2\;=\;\left(\mathbb{1}_2,\Omega\right)\;\xrightarrow{M_*}\;\begin{pmatrix}\int_{\alpha_1}\bar{\omega}_1&\int_{\alpha_1}\bar{\omega}_2&\int_{-\beta_1}\bar{\omega}_1&\int_{-\beta_1}\bar{\omega}_2\\\int_{\alpha_2}\bar{\omega}_1&\int_{\alpha_2}\bar{\omega}_2&\int_{-\beta_2}\bar{\omega}_1&\int_{-\beta_2}\bar{\omega}_2\end{pmatrix}\;=\;\left(\mathbb{1}_2,-\bar{\Omega}\right)\,. \tag{57}$$

This proves eq. (45) that we have also found independently in the string setup, see eq. (37). Furthermore, this discussion can be generalized easily to the case $\mathrm{GSp}(2g,\mathbb{Z})$ for $\mathcal{T}_g$ with general genus g. This is particularly easy for $g=1$, where the basis 1-form is $\omega=\mathrm{d}z$. Considering the $\mathcal{CP}$-like transformation $\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ (using e.g. eqs. (66) and (144) of ref. [30]) we get $(\operatorname{Im}\mathrm{d}z,\operatorname{Re}\mathrm{d}z)\xrightarrow{M_*}(-\operatorname{Im}\mathrm{d}z,\operatorname{Re}\mathrm{d}z)$. Consequently, we see that $\mathrm{d}z\xrightarrow{M_*}\mathrm{d}\bar{z}$. It then follows for the period matrix that $\Pi_1=(1,\tau)\xrightarrow{M_*}(\int_\alpha\bar{\omega},\int_{-\beta}\bar{\omega})=(1,-\bar{\tau})$, choosing $\tau=\int_\beta\omega$ and $\int_\alpha\omega=1$, as before. This confirms the well-known $\mathcal{CP}$-like transformation of the modulus τ [3,21,22], which promotes the modular symmetry $\mathrm{Sp}(2,\mathbb{Z})\cong\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{GSp}(2,\mathbb{Z})\cong\mathrm{GL}(2,\mathbb{Z})$.

Appendix B. Relations between elements of Sp(4, $\mathbb{Z}$)

In this appendix, we state several relations between elements of the Siegel modular group Sp(4, $\mathbb{Z}$). We have verified that they also hold in $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ using the dictionary given in Table 1. This gives a further non-trivial proof that Sp(4, $\mathbb{Z}$) and $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ are related.

Elements $M_{(\gamma_T,\gamma_U)}$ of $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U \subset \mathrm{Sp}(4,\mathbb{Z})$ get multiplied as

$$M_{(\gamma_1,\gamma_2)}\, M_{(\delta_1,\delta_2)} \;=\; M_{(\gamma_1\,\delta_1,\gamma_2\,\delta_2)}\;, \tag{58}$$

as might have been expected. Thus, the elements $M_{(\gamma_T,\gamma_U)}$ form a subgroup of Sp(4, $\mathbb{Z}$).

On the other hand, the set of elements $M(\Delta) \in \mathrm{Sp}(4,\mathbb{Z})$ does not form a subgroup of Sp(4, $\mathbb{Z}$) on its own as one can see from the relation

$$M(\Delta_1)\, M(\Delta_2) \;=\; M(\Delta_1 + \Delta_2)\, \left(M_{(\mathrm{T},\mathbb{1}_2)}\right)^{\Delta_1^{\mathrm{T}}\epsilon\,\Delta_2}\;, \tag{59}$$

for $\Delta_1,\ \Delta_2 \in \mathbb{Z}^2$. However, elements of the form $M\binom{\ell}{0}$ and $M\binom{0}{m}$ build two independent subgroups of Sp(4, $\mathbb{Z}$) because

$$M\binom{\ell_1}{0}\, M\binom{\ell_2}{0} = M\binom{\ell_1+\ell_2}{0}\;, \tag{60a}$$

$$M\binom{0}{m_1}\, M\binom{0}{m_2} = M\binom{0}{m_1+m_2}\;. \tag{60b}$$

Next, we consider the action of $\gamma \in \mathrm{SL}(2,\mathbb{Z})_U$ on M_Δ. It is given by

$$M_{(\mathbb{1}_2,\gamma)}\, M(\Delta) \;=\; M(\gamma\,\Delta)\, M_{(\mathbb{1}_2,\gamma)} \tag{61}$$

such that for $\Delta = (\ell, m)^{\mathrm{T}}$ we obtain

$$M_{(\mathbb{1}_2,\mathrm{S})}\, M\binom{\ell}{m} = M\binom{m}{-\ell}\, M_{(\mathbb{1}_2,\mathrm{S})}\;, \tag{62a}$$

$$M_{(\mathbb{1}_2,\mathrm{T})}\, M\binom{\ell}{m} = M\binom{\ell+m}{m}\, M_{(\mathbb{1}_2,\mathrm{T})}\;. \tag{62b}$$

From the point of view of Wilson lines on a two-torus, these equations are not unexpected: ℓ corresponds to the Wilson line A_2 in the e_2 direction, while m corresponds to the Wilson line A_1 in the e_1 direction. Furthermore, under modular S and T transformations from $\mathrm{SL}(2,\mathbb{Z})_U$, the lattice vectors e_1 and e_2 get mapped as

$$e_1 \stackrel{\mathrm{S}}{\to} -e_2\;,\;\; e_2 \stackrel{\mathrm{S}}{\to} e_1\;,\;\; \text{and}\;\; e_1 \stackrel{\mathrm{T}}{\to} e_1\;,\;\; e_2 \stackrel{\mathrm{T}}{\to} e_1 + e_2\;, \tag{63}$$

see e.g. ref. [30]. Hence, eqs. (62) resembles eq. (63) on the level of the associated Wilson lines. In addition, we have checked the following relations, both in Sp(4, $\mathbb{Z}$) and in $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$:

$$M_{(\mathrm{T},\mathbb{1}_2)}\, M\binom{\ell}{m} \left(M_{(\mathrm{T},\mathbb{1}_2)}\right)^{-1} = M\binom{\ell}{m}\;, \tag{64a}$$

$$M_\times\, M_{(\gamma_1,\gamma_2)}\, M_\times = M_{(\gamma_2,\gamma_1)}\;, \tag{64b}$$

$$M_\times\, M\binom{\ell}{0}\, M_\times = M\binom{\ell}{0}\;, \tag{64c}$$

$$M_\times\, M\binom{0}{m}\, M_\times = \left(M_{(\mathrm{S},\mathrm{S})}\right)^{-1} M\binom{0}{-m}\, M_{(\mathrm{S},\mathrm{S})}\;. \tag{64d}$$

Finally, we learn from the relation

$$M\binom{\ell}{m} \;=\; M\binom{\ell}{0}\, \left(M_{(\mathbb{1}_2,\mathrm{S})}\right)^{-1} M\binom{m}{0}\, M_{(\mathbb{1}_2,\mathrm{S})}\, M_\times\, \left(M_{(\mathbb{1}_2,\mathrm{T})}\right)^{-\ell m} M_\times \tag{65}$$

that Sp(4, $\mathbb{Z}$) can be generated by $M_{(\mathbb{1}_2,\mathrm{S})}$, $M_{(\mathbb{1}_2,\mathrm{T})}$, $M_\times$ and $M\binom{1}{0}$ (and its inverse $M\binom{-1}{0}$).

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