Eclectic flavor scheme from ten-dimensional string theory - II detailed technical analysis

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Abstract

String theory leads to a flavor scheme where modular symmetries play a crucial role. Together with the traditional flavor symmetries they combine to an eclectic flavor group, which we determine via outer automorphisms of the Narain space group. Unbroken flavor symmetries are subgroups of this eclectic group and their size depends on the location in moduli space. This mechanism of local flavor unification allows a different flavor structure for different sectors of the theory (such as quarks and leptons) and also explains the spontaneous breakdown of flavor- and $\mathcal{CP}$-symmetries (via a motion in moduli space). We derive the modular groups, including $\mathcal{CP}$ and R-symmetries, for different sub-sectors of six-dimensional string compactifications and determine the general properties of the allowed flavor groups from this top-down perspective. It leads to a very predictive flavor scheme that should be confronted with the variety of existing bottom-up constructions of flavor symmetry in order to clarify which of them could have a consistent top-down completion.

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1. Introduction

The consideration of finite modular flavor groups has been suggested in the pioneering work of Feruglio [1]. A combination of these modular symmetries and traditional flavor symmetries leads to the eclectic flavor scheme [2,3]. Here, we present a study of the eclectic flavor approach towards ten-dimensional string theory with $D = 6$ compactified spatial dimensions.[1] This generalizes previous work with $D = 2$ compact dimensions which is shown to fail to capture all the relevant properties of the eclectic picture. A few of the highlights of this discussion have been presented in an earlier paper [12] without explaining the full technical details. These highlights include

- a further enhancement of the eclectic group,
- a new interpretation of discrete R-symmetries originating from modular transformations of the complex structure modulus,
- a relation between the R-charges and the modular weights of matter fields, and
- new insights in the nature of $\mathcal{CP}$-symmetry and its spontaneous breakdown.

These results have been illustrated previously [3,12] in an example of a $D = 2$ sublattice based on the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold. Here we shall now present the complete results in the generic situation and provide the full technical details of the derivation. In addition, we shall illustrate an interesting novel feature: the existence of accidental continuous (gauge) symmetries of the effective theory at special loci in moduli space, which turn out to be further enhancements of the traditional flavor symmetries.

In section 2 we shall discuss basic background material concerning the $D = 2$ Narain lattice formulation [13–16] and its outer automorphisms relevant for (discrete) modular symmetries [17,18]. These include the groups $\mathrm{SL}(2, \mathbb{Z})_T$ and $\mathrm{SL}(2, \mathbb{Z})_U$ for the Kähler modulus T and the complex structure modulus U, respectively, mirror symmetry as well as a $\mathcal{CP}$-like transformation. Results for the generators are summarized in Table 1 for various $\mathbb{T}^2/\mathbb{Z}_K$ orbifolds. We include a discussion of the modular properties of the superpotential and Kähler potential in the modular invariant field theory, including modular weights and automorphy factors.

Section 3 concentrates on the discussion of those aspects of the modular group $\mathrm{SL}(2, \mathbb{Z})_U$ of the complex structure modulus which will become relevant for the analysis in $D = 6$ compact dimensions. In contrast to $\mathrm{SL}(2, \mathbb{Z})_T$ (which exchanges windings and momenta), $\mathrm{SL}(2, \mathbb{Z})_U$ can be given a fully geometrical interpretation. In previous discussions of $D = 2$ orbifolds $\mathbb{T}^2/\mathbb{Z}_K$, the subtleties of this analysis have not yet been completely considered, because the modulus U is frozen as a result of the orbifold twist (except for the case $\mathbb{T}^2/\mathbb{Z}_2$ which has been discussed elsewhere [19]). We show here that, even with a frozen value $\langle U \rangle$ of the complex structure U, there are nontrivial elements of $\mathrm{SL}(2, \mathbb{Z})_U$ that remain unbroken. These lead to additional discrete R-symmetries according to the stabilizer subgroups given in formula (82). These symmetries will be relevant for the $D = 6$ case in the form of sublattice rotations.

In section 4, we consider the $\mathbb{T}^6/P$ orbifold (with point group P). We start with the discussion in full generality (including the restrictions of $\mathcal{N} = 1$ supersymmetry in four space-time dimensions) and then illustrate the results for the $\mathbb{Z}_6$-II orbifold. We shall identify the sublattice rotations of $\mathrm{SL}(2, \mathbb{Z})_U$, their action on matter fields and determine which (fractional) modular

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[1] For further work on modular flavor symmetries in string theory, see e.g. refs. [4–11].

weights are relevant for the emerging R-symmetries. Section 4.3 includes a discussion of $\mathcal{CP}$-transformations, including the full $D = 6$ picture (summarized in Table 3).

Section 5 presents the complete description for a sublattice $\mathbb{T}^2/\mathbb{Z}_3$ of the $\mathbb{Z}_6$-II orbifold. This includes the traditional flavor symmetry $\Delta(54)$, the finite modular symmetry T' from $\mathrm{SL}(2, \mathbb{Z})_T$, a discrete R-symmetry $\mathbb{Z}_9^R$ from $\mathrm{SL}(2, \mathbb{Z})_U$ as well as a $\mathcal{CP}$-like transformation (which is separately discussed in subsections 5.6 and 5.7). The maximal eclectic group is shown to have 3, 888 elements, extending $\Omega(2) \cong [1944, 3448]$ (see Table 5) by the $\mathcal{CP}$-like transformation. We discuss restrictions of this symmetry on the form of the superpotential and Kähler potential of the $\mathcal{N} = 1$ supersymmetric field theory.

Only part of the eclectic flavor symmetry is realized linearly. For generic values of the moduli only the traditional flavor symmetry is unbroken. This symmetry will be enhanced at special points (or sub-loci) of the moduli space. In section 6 we present this phenomenon of *local flavor unification* for the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold and identify these enhanced groups (see Table 6 and Fig. 3) as well as the specific forms of the superpotential. Moving away from these sub-loci in moduli space will correspond to a spontaneous breakdown of the enhanced flavor groups (in particular also for the violation of $\mathcal{CP}$-symmetry). A novel observation concerns the emergence of (accidental) continuous gauge symmetries at special loci in moduli space, here discussed in section 6.4.

Section 7 will present conclusions and an outlook for future applications of our results.

Note to the reader: As our discussion is partially very technical we include a short specific summary at the end of each section 2–6 to make the results of the discussion as transparent as possible.

2. Modular symmetries in string theory

In general, the modular group $\mathrm{SL}(2, \mathbb{Z})$ can be defined as the group generated by two abstract generators, S and T, that satisfy the defining relations

$$\mathrm{S}^4 = (\mathrm{S\,T})^3 = \mathbb{1} \qquad \text{and} \qquad \mathrm{S}^2\,\mathrm{T} = \mathrm{T\,S}^2\,. \tag{1}$$

It turns out convenient to represent these generators by the matrices

$$\mathrm{S} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{T} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,. \tag{2}$$

In this context, the elements $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ can be expressed by 2×2 matrices with integer entries and determinant one, i.e.

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{with} \qquad ad - bc = 1,\ a, b, c, d \in \mathbb{Z}\,. \tag{3}$$

Particularly relevant in string models (and in bottom-up flavor model building) are the so-called finite modular groups. Their generators are also denoted by S and T, which fulfill the following defining relations

$$\Gamma'_N : \qquad \mathrm{S}^4 = (\mathrm{S\,T})^3 = \mathrm{T}^N = \mathbb{1} \qquad \text{and} \qquad \mathrm{S}^2\,\mathrm{T} = \mathrm{T\,S}^2\,, \tag{4a}$$

$$\Gamma_N : \qquad \mathrm{S}^2 = (\mathrm{S\,T})^3 = \mathrm{T}^N = \mathbb{1}\,, \tag{4b}$$

where Γ'_N is the double cover of Γ_N. The finite modular groups Γ'_N and Γ_N can be obtained from the modular groups $\mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{PSL}(2, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}_2$ as the quotients $\Gamma'_N \cong \mathrm{SL}(2, \mathbb{Z})/\Gamma(N)$ and $\Gamma_N \cong \mathrm{PSL}(2, \mathbb{Z})/\tilde{\Gamma}(N)$, where $\Gamma(N) \cong \langle \gamma \in \mathrm{SL}(2, \mathbb{Z}) \,|\, \gamma = \mathbb{1} \mod N \rangle$ and $\tilde{\Gamma}(N) \cong \langle \gamma \in \mathrm{PSL}(2, \mathbb{Z}) \,|\, \gamma = \mathbb{1} \mod N \rangle$ are modular congruence subgroups.

2.1. Narain lattice of torus compactifications

In string theory, target-space modular symmetries can arise from the two-dimensional world-sheet CFT. To see this, let us first consider string theory compactified on a D-dimensional torus $\mathbb{T}^D$, which is most naturally described in the Narain formulation [13–15]. There, one considers right- and left-moving string coordinates y_R and y_L, which are related to the ordinary internal coordinates y and their dual coordinates $\tilde{y}$ via

$$y := \sqrt{\frac{\alpha'}{2}}\,(y_L + y_R) \qquad \text{and} \qquad \tilde{y} := \sqrt{\frac{\alpha'}{2}}\,(y_L - y_R)\,. \tag{5}$$

Then, (y_R, y_L) are compactified on a $2D$-dimensional auxiliary torus. This torus is defined by the so-called Narain lattice Γ that is spanned by the so-called Narain vielbein E. In more detail, the Narain torus boundary condition in this formalism reads

$$\begin{pmatrix} y_R \\ y_L \end{pmatrix} \sim \begin{pmatrix} y_R \\ y_L \end{pmatrix} + \lambda\,, \quad \text{where} \quad \lambda \in \Gamma := \left\{ E\,\hat{N} \,\middle|\, \hat{N} = \begin{pmatrix} n \\ m \end{pmatrix} \in \mathbb{Z}^{2D} \right\}, \tag{6}$$

and $n, m \in \mathbb{Z}^D$ are known as winding and Kaluza–Klein (KK) numbers, respectively. Due to worldsheet modular invariance of the one-loop string partition function, the Narain lattice Γ spanned by E has to be an even, integer, self-dual lattice with metric of signature (D, D). This translates into the following condition on the Narain vielbein E:

$$E^T \eta\, E = \hat{\eta} := \begin{pmatrix} 0 & \mathbb{1}_D \\ \mathbb{1}_D & 0 \end{pmatrix}, \quad \text{where} \quad \eta := \begin{pmatrix} -\mathbb{1}_D & 0 \\ 0 & \mathbb{1}_D \end{pmatrix} \tag{7}$$

denotes the metric of signature (D, D) and $\mathbb{1}_D$ is the $(D \times D)$-dimensional identity matrix. Then, the Narain scalar product of two Narain vectors $\lambda_i = E\,\hat{N}_i \in \Gamma$ with $\hat{N}_i = (n_i, m_i) \in \mathbb{Z}^{2D}$ for $i \in \{1, 2\}$ reads

$$\lambda_1^T \eta\, \lambda_2 = \hat{N}_1^T \hat{\eta}\, \hat{N}_2 = n_1^T m_2 + m_1^T n_2 \in \mathbb{Z}\,. \tag{8}$$

This confirms that the lattice Γ is integer and even, as the scalar product is integer and, if $\hat{N}_1 = \hat{N}_2$, even. Also, as expected from eq. (7), we note that $\hat{\eta}$ is the Narain metric in the lattice basis.

In the absence of (discrete) Wilson lines [20], one can choose the $(2D \times 2D)$-dimensional Narain vielbein E fulfilling eq. (7) as

$$E := \frac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{e^{-T}}{\sqrt{\alpha'}}\,(G - B) & -\sqrt{\alpha'}\,e^{-T} \\[2ex] \dfrac{e^{-T}}{\sqrt{\alpha'}}\,(G + B) & \sqrt{\alpha'}\,e^{-T} \end{pmatrix}, \tag{9}$$

see refs. [16–18]. Here, α' is the Regge slope that enters the Narain vielbein E, such that E is dimensionless. Then, E is parameterized by the $(D \times D)$-dimensional geometrical vielbein e, yielding the metric $G := e^T e$ of the geometrical torus $\mathbb{T}^D$, and the $(D \times D)$-dimensional anti-symmetric B-field B.

Using the explicit parameterization (9), we can translate the Narain boundary condition (6) to the coordinates y and their duals. We obtain

$$\begin{pmatrix} y \\ \tilde{y} \end{pmatrix} \sim \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} e\,n \\ e^{-T}\,(B\,n + \alpha'\,m) \end{pmatrix}. \tag{10}$$

Note that the boundary condition $y \sim y + e\,n$ for the geometrical coordinates y of closed strings motivates the name of $n \in \mathbb{Z}^D$ as winding numbers of the geometrical torus $\mathbb{T}^D$ spanned by e.

The Narain lattice Γ is beneficial as it incorporates the winding numbers $n \in \mathbb{Z}^D$ and the KK numbers $m \in \mathbb{Z}^D$ on equal footing. Hence, it allows for a natural formulation of T-duality in string theory, where KK and winding numbers get interchanged.

2.2. Outer automorphisms of the Narain lattice

In the Narain lattice basis, the Narain metric $\hat{\eta}$ can be used to define the group of outer automorphisms[2] of the Narain lattice Γ that preserve the Narain metric $\hat{\eta}$ as

$$\mathrm{O}_{\hat{\eta}}(D, D, \mathbb{Z}) := \left\langle \hat{\Sigma} \,\middle|\, \hat{\Sigma} \in \mathrm{GL}(2D, \mathbb{Z}) \quad \text{with} \quad \hat{\Sigma}^{\mathrm{T}} \hat{\eta}\, \hat{\Sigma} = \hat{\eta} \right\rangle, \tag{11}$$

such that each Narain lattice vector $\lambda = E\,\hat{N} \in \Gamma$ is transformed by an outer automorphism $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(D, D, \mathbb{Z})$ according to

$$\lambda = E\,\hat{N} \;\xrightarrow{\hat{\Sigma}}\; E\,\hat{\Sigma}^{-1}\hat{N} \in \Gamma\,. \tag{12}$$

Here, we use $\hat{\Sigma}^{-1}$ instead of $\hat{\Sigma}$ for later convenience, see section 2.5. Then, $\hat{\Sigma}^{-1}\hat{N} \in \mathbb{Z}^{2D}$ and the Narain scalar product $\lambda^{\mathrm{T}}\eta\,\lambda$ from eq. (8) is invariant under the transformation (12). Note that the transformation of the Narain lattice vector eq. (12) can be interpreted as induced by the transformation of the Narain vielbein under outer automorphisms given by

$$E \;\xrightarrow{\hat{\Sigma}}\; E\,\hat{\Sigma}^{-1} \qquad \text{for} \qquad \hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(D, D, \mathbb{Z})\,. \tag{13}$$

The group of outer automorphisms of the Narain lattice $\mathrm{O}_{\hat{\eta}}(D, D, \mathbb{Z})$ with these properties contains the modular group, as we discuss next.

For concreteness, let us focus on a geometrical two-torus $\mathbb{T}^2$ with $D = 2$. One can easily verify that $\mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z})$ contains two copies of $\mathrm{SL}(2, \mathbb{Z})$ as follows: First, we can define two $\mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z})$ elements

$$\hat{K}_{\mathrm{S}} := \begin{pmatrix} 0 & -\epsilon \\ -\epsilon & 0 \end{pmatrix} \quad, \quad \hat{K}_{\mathrm{T}} := \begin{pmatrix} \mathbb{1}_2 & 0 \\ -\epsilon & \mathbb{1}_2 \end{pmatrix}, \quad \text{where} \quad \epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{14}$$

which satisfy the defining relations (1) of $\mathrm{SL}(2, \mathbb{Z})$,

$$\left(\hat{K}_{\mathrm{S}}\right)^4 = \left(\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}\right)^3 = \mathbb{1}_4 \qquad \text{and} \qquad \left(\hat{K}_{\mathrm{S}}\right)^2 \hat{K}_{\mathrm{T}} = \hat{K}_{\mathrm{T}}\left(\hat{K}_{\mathrm{S}}\right)^2. \tag{15}$$

We denote this modular group by $\mathrm{SL}(2, \mathbb{Z})_T$. Moreover, we identify the $\mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z})$ elements

$$\hat{C}_{\mathrm{S}} := \begin{pmatrix} -\epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \quad, \quad \hat{C}_{\mathrm{T}} := \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-\mathrm{T}} \end{pmatrix}, \quad \text{where} \quad \gamma := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \tag{16}$$

which also satisfy the $\mathrm{SL}(2, \mathbb{Z})$ defining relations,

$$\left(\hat{C}_{\mathrm{S}}\right)^4 = \left(\hat{C}_{\mathrm{S}}\,\hat{C}_{\mathrm{T}}\right)^3 = \mathbb{1}_4 \qquad \text{and} \qquad \left(\hat{C}_{\mathrm{S}}\right)^2 \hat{C}_{\mathrm{T}} = \hat{C}_{\mathrm{T}}\left(\hat{C}_{\mathrm{S}}\right)^2. \tag{17}$$

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[2] Here, $\mathrm{O}_{\hat{\eta}}(D, D, \mathbb{Z})$ gives the discrete "rotational" outer automorphisms of the Narain lattice. Furthermore, there are continuous translations that act by conjugation on Narain lattice vectors and, therefore, correspond to the trivial automorphism of Γ.

Hence, $\hat{C}_\mathrm{S}$ and $\hat{C}_\mathrm{T}$ give rise to another factor of the modular group, denoted by $\mathrm{SL}(2,\mathbb{Z})_U$. Even though all elements of $\mathrm{SL}(2,\mathbb{Z})_T$ commute with those of $\mathrm{SL}(2,\mathbb{Z})_U$, these two factors are not fully independent since they share a common element, given by

$$\left(\hat{K}_\mathrm{S}\right)^2 \;=\; \left(\hat{C}_\mathrm{S}\right)^2 \;=\; -\mathbb{1}_4 \,. \tag{18}$$

The group $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ of outer automorphisms of the Narain lattice Γ contains two additional generators, which we can choose for example as

$$\hat{\Sigma}_* \;:=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{M} \;:=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,. \tag{19}$$

They give rise to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Motivated by the relations

$$\hat{\Sigma}_* \,\hat{K}_\mathrm{S}\, \hat{\Sigma}_*^{-1} \;=\; \left(\hat{K}_\mathrm{S}\right)^{-1} \quad , \quad \hat{\Sigma}_* \,\hat{K}_\mathrm{T}\, \hat{\Sigma}_*^{-1} \;=\; \left(\hat{K}_\mathrm{T}\right)^{-1} \,, \tag{20a}$$

$$\hat{\Sigma}_* \,\hat{C}_\mathrm{S}\, \hat{\Sigma}_*^{-1} \;=\; \left(\hat{C}_\mathrm{S}\right)^{-1} \quad , \quad \hat{\Sigma}_* \,\hat{C}_\mathrm{T}\, \hat{\Sigma}_*^{-1} \;=\; \left(\hat{C}_\mathrm{T}\right)^{-1} \,, \tag{20b}$$

we call $\hat{\Sigma}_*$ a $\mathcal{CP}$-like transformation. Furthermore, we call $\hat{M}$ a mirror transformation as it interchanges $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$, i.e.

$$\hat{M}\,\hat{K}_\mathrm{S}\,\hat{M}^{-1} \;=\; \hat{C}_\mathrm{S} \quad \text{and} \quad \hat{M}\,\hat{K}_\mathrm{T}\,\hat{M}^{-1} \;=\; \hat{C}_\mathrm{T} \,. \tag{21}$$

2.3. Moduli and modular symmetries of $\mathbb{T}^2$

The modular subgroups $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$ of $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ are associated with the transformations of the moduli of a $\mathbb{T}^2$ torus, as we now discuss.

The Narain vielbein E of the geometrical two-torus $\mathbb{T}^2$ with background B-field allows for certain deformations while satisfying the conditions eq. (7) required for E to span a Narain lattice. These deformations correspond to two moduli fields:

$$\text{Kähler modulus} \qquad T := \frac{1}{\alpha'}\left(B_{12} + \mathrm{i}\sqrt{\det G}\right) \quad \text{and} \tag{22a}$$

$$\text{complex structure modulus} \qquad U := \frac{1}{G_{11}}\left(G_{12} + \mathrm{i}\sqrt{\det G}\right) \,. \tag{22b}$$

Comparing these definitions with eq. (9), we realize that the Narain vielbein can be expressed in terms of the $\mathbb{T}^2$ moduli, $E = E(T, U)$, up to some unphysical transformations, see for example ref. [16]. Hence, the action of $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ on the Narain vielbein given by eq. (13) is equivalent to a transformation of the T and U moduli. To determine explicitly the transformation properties of the moduli, it is convenient to define the generalized metric (valid for an arbitrary dimension D)

$$\mathcal{H} \;:=\; E^\mathrm{T} E \;=\; \begin{pmatrix} \frac{1}{\alpha'}\left(G - B\,G^{-1}B\right) & -B\,G^{-1} \\ G^{-1}B & \alpha'\,G^{-1} \end{pmatrix} \,. \tag{23}$$

For a two-torus $\mathbb{T}^2$, the generalized metric in terms of the torus moduli reads

$$\mathcal{H}(T,U) = \frac{1}{\operatorname{Im} T \operatorname{Im} U} \begin{pmatrix} |T|^2 & |T|^2 \operatorname{Re} U & \operatorname{Re} T \operatorname{Re} U & -\operatorname{Re} T \\ |T|^2 \operatorname{Re} U & |T U|^2 & |U|^2 \operatorname{Re} T & -\operatorname{Re} T \operatorname{Re} U \\ \operatorname{Re} T \operatorname{Re} U & |U|^2 \operatorname{Re} T & |U|^2 & -\operatorname{Re} U \\ -\operatorname{Re} T & -\operatorname{Re} T \operatorname{Re} U & -\operatorname{Re} U & 1 \end{pmatrix}. \quad (24)$$

Due to the transformation of the vielbein eq. (13), the action of $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ on the generalized metric is given by

$$\mathcal{H}(T,U) \overset{\hat{\Sigma}}{\longmapsto} \hat{\Sigma}^{-\mathrm{T}} \mathcal{H}(T,U) \hat{\Sigma}^{-1} =: \mathcal{H}(T',U'). \quad (25)$$

From this equation and the expression (24), given a transformation $\hat{\Sigma}$, one can readily obtain the forms of the transformed moduli T' and U'. For example, considering as $\hat{\Sigma}$ the $\mathrm{SL}(2,\mathbb{Z})_T$ modular generators $\hat{K}_\mathrm{S}$ and $\hat{K}_\mathrm{T}$ in eq. (14), we observe that the associated moduli transformations are

$$T \overset{\hat{K}_\mathrm{S}}{\longrightarrow} T' = -\frac{1}{T}, \quad U \overset{\hat{K}_\mathrm{S}}{\longrightarrow} U' = U, \qquad \text{and} \qquad T \overset{\hat{K}_\mathrm{T}}{\longrightarrow} T' = T+1, \quad U \overset{\hat{K}_\mathrm{T}}{\longrightarrow} U' = U, \quad (26)$$

which explains the index chosen for the modular group $\mathrm{SL}(2,\mathbb{Z})_T$. From this observation, as shown e.g. in ref. [18], we know that under a modular transformation $\gamma_T \in \mathrm{SL}(2,\mathbb{Z})_T$ the moduli T and U transform as

$$T \overset{\gamma_T}{\longrightarrow} \gamma_T T := \frac{a_T T + b_T}{c_T T + d_T} \quad \text{and} \quad U \overset{\gamma_T}{\longrightarrow} U \quad \text{for} \quad \gamma_T = \begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_T, \quad (27)$$

in the 2×2 matrix notation used in eq. (3).

Analogously, letting the generalized metric (24) transform under the $\mathrm{SL}(2,\mathbb{Z})_U$ generators defined in eq. (16), one finds that the T and U moduli transform according to

$$T \overset{\hat{C}_\mathrm{S}}{\longrightarrow} T' = T, \quad U \overset{\hat{C}_\mathrm{S}}{\longrightarrow} U' = -\frac{1}{U}, \qquad \text{and} \qquad T \overset{\hat{C}_\mathrm{T}}{\longrightarrow} T' = T, \quad U \overset{\hat{C}_\mathrm{T}}{\longrightarrow} U' = U+1. \quad (28)$$

Hence, a general modular element $\gamma_U \in \mathrm{SL}(2,\mathbb{Z})_U$ acts on the moduli as

$$T \overset{\gamma_U}{\longrightarrow} T \quad \text{and} \quad U \overset{\gamma_U}{\longrightarrow} \gamma_U U := \frac{a_U U + b_U}{c_U U + d_U} \quad \text{for} \quad \gamma_U = \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_U, \quad (29)$$

as we will also re-derive later in section 3.1 using an alternative approach. Moreover, repeating the previous steps, the mirror transformation $\hat{M}$ interchanges T and U,

$$T \overset{\hat{M}}{\longrightarrow} T' = U \quad \text{and} \quad U \overset{\hat{M}}{\longrightarrow} U' = T, \quad (30)$$

while the $\mathcal{CP}$-like transformation $\hat{\Sigma}_*$ acts as

$$T \overset{\hat{\Sigma}_*}{\longrightarrow} T' = -\bar{T} \quad \text{and} \quad U \overset{\hat{\Sigma}_*}{\longrightarrow} U' = -\bar{U}. \quad (31)$$

A brief remark is in order: The modular transformation $(\hat{K}_\mathrm{S})^2$, which equals $(\hat{C}_\mathrm{S})^2$, acts trivially on both moduli. Hence, the modular groups restricted to their action only on the moduli

is PSL(2, $\mathbb{Z}$)$_T$ and PSL(2, $\mathbb{Z}$)$_U$, where for example $\gamma_T \in$ SL(2, $\mathbb{Z}$)$_T$ and $-\gamma_T \in$ SL(2, $\mathbb{Z}$)$_T$ are identified in PSL(2, $\mathbb{Z}$)$_T$. However, in full string theory $(\hat{K}_S)^2$ acts nontrivially, for example, on massive winding strings.

2.4. Orbifold compactifications in the Narain formulation

Defining $Y := (y_R, y_L)^T$ for the Narain coordinates, a $\mathbb{T}^D/\mathbb{Z}_K$ orbifold in the Narain formulation is obtained by extending the boundary conditions of closed strings eq. (6) to

$$Y \sim \Theta^k Y + E \hat{N}, \quad \text{where} \quad \Theta^K = \mathbb{1}_{2D} \quad \text{and} \quad k = 0, \dots, K - 1. \tag{32}$$

Moreover, the so-called Narain twist Θ shall not interchange right- and left-movers. Thus, it is given by

$$\Theta := \begin{pmatrix} \theta_R & 0 \\ 0 & \theta_L \end{pmatrix} \in \mathrm{O}(D) \times \mathrm{O}(D). \tag{33}$$

Hence, Θ generates a $\mathbb{Z}_K$ rotation that is modded out from the Narain torus in the orbifold. The Narain twist has to be a (rotational) symmetry of the Narain lattice of the torus, i.e. $\Theta \Gamma = \Gamma$. Thus, Θ has to be an outer automorphism of the Narain lattice Γ. This translates into the following condition on the so-called Narain twist in the Narain lattice basis

$$\hat{\Theta} := E^{-1}\Theta E \overset{!}{\in} \mathrm{GL}(2D, \mathbb{Z}). \tag{34}$$

Furthermore, from eq. (33) we find that the Narain metric (7) and the generalized metric (23) are orbifold invariant,

$$\hat{\Theta}^T \hat{\eta} \hat{\Theta} = \hat{\eta} \quad \text{and} \quad \hat{\Theta}^T \mathcal{H} \hat{\Theta} = \mathcal{H}. \tag{35}$$

In summary, the Narain twist in the Narain lattice basis $\hat{\Theta} := E^{-1}\Theta E$ has to be a Narain-metric preserving outer automorphism of the Narain lattice, which, in addition, leaves the generalized metric (and therefore all torus moduli) invariant, i.e.

$$\hat{\Theta} \overset{!}{\in} \mathrm{O}_{\hat{\eta}}(D, D, \mathbb{Z}), \quad T_a \overset{\hat{\Theta}}{\longrightarrow} T_a' \overset{!}{=} T_a \quad \text{and} \quad U_b \overset{\hat{\Theta}}{\longrightarrow} U_b' \overset{!}{=} U_b, \tag{36}$$

where T_a and U_b denote all Kähler and complex structure moduli of a $\mathbb{T}^D/\mathbb{Z}_K$ orbifold, respectively. This invariance condition of the moduli results in a stabilization of some of them: One says that some moduli are frozen geometrically in order to satisfy eq. (36).

2.5. Modular symmetries of orbifold compactifications

We shall restrict ourselves here to symmetric orbifolds $\mathbb{T}^D/\mathbb{Z}_K$, for which right- and left-movers are equally affected by the orbifold twist, which is granted if we additionally impose in eq. (33) the condition $\theta := \theta_R = \theta_L \in \mathrm{O}(D)$. Then, the orbifold twist θ generates the so-called geometrical point group $P \cong \mathbb{Z}_K \subset \mathrm{O}(D)$.

The group structure and the symmetries of the orbifold can be realized by expressing the boundary conditions eq. (32) of the Narain orbifold in terms of the so-called Narain space group S_{Narain} as

$$Y \sim g Y := \Theta^k Y + E \hat{N}, \quad \text{where} \quad g = (\Theta^k, E \hat{N}) \in S_{\text{Narain}}. \tag{37}$$

The Narain space group is equipped with the product defined by

$$g' g \ = \ (\Theta^{k'}, E\,\hat{N}')(\Theta^k, E\,\hat{N}) \ := \ (\Theta^{k'+k}, E\,\hat{N}' + \Theta^{k'} E\,\hat{N})\,, \quad g, g' \in S_{\mathrm{Narain}}\,. \tag{38}$$

An arbitrary element $g \in S_{\mathrm{Narain}}$ can be expressed in the Narain lattice basis by conjugating with $h := (E, 0)$ as

$$\hat{g} \ := \ h^{-1}\,g\,h \ = \ (E^{-1}, 0)(\Theta^k, E\,\hat{N})(E, 0) \ = \ (E^{-1}\Theta^k E, \hat{N}) \ = \ (\hat{\Theta}^k, \hat{N})\,, \tag{39}$$

where we have used the Narain twist in the Narain lattice basis defined in eq. (34). In our notation, the Narain space group and its elements are indicated by a hat in the Narain lattice basis, i.e. $\hat{g} \in \hat{S}_{\mathrm{Narain}}$.

To identify the modular symmetries of the four-dimensional effective theory after orbifolding amounts to determining the outer automorphisms of the Narain space group $\hat{S}_{\mathrm{Narain}}$. An outer automorphism of the Narain space group is given by a transformation

$$\hat{h} \ := \ (\hat{\Sigma}, \hat{T}) \ \notin \ \hat{S}_{\mathrm{Narain}}\,, \tag{40}$$

such that $\hat{h}^{-1}\hat{g}\,\hat{h} \in \hat{S}_{\mathrm{Narain}}$ for all $\hat{g} = (\hat{\Theta}^k, \hat{N}) \in \hat{S}_{\mathrm{Narain}}$ is satisfied. Explicitly, the condition for $\hat{h}$ to be an outer automorphism of $\hat{S}_{\mathrm{Narain}}$ reads

$$(\hat{\Sigma}, \hat{T})^{-1}(\hat{\Theta}^k, \hat{N})(\hat{\Sigma}, \hat{T}) \ = \ (\hat{\Sigma}^{-1}\hat{\Theta}^k\hat{\Sigma}, \ \hat{\Sigma}^{-1}(\hat{\Theta}^k - \mathbb{1}_{2D})\hat{T} + \hat{\Sigma}^{-1}\hat{N}) \ \overset{!}{\in} \ \hat{S}_{\mathrm{Narain}}\,, \tag{41}$$

see refs. [19,21] for an algorithm to classify the outer automorphisms of a (Narain) space group. If $\hat{S}_{\mathrm{Narain}}$ is generated by pure rotations $(\hat{\Theta}, 0)$ and pure translations $(\mathbb{1}_{2D}, \hat{N})$ with $\hat{N} \in \mathbb{Z}^{2D}$, the condition (41) is equivalent to demanding that

$$\hat{\Sigma}^{-1}\hat{\Theta}^k\hat{\Sigma} \ \overset{!}{=} \ \hat{\Theta}^{k'}\,, \quad \hat{\Sigma}^{-1}\hat{N} \ \overset{!}{\in} \ \mathbb{Z}^{2D} \quad \text{and} \quad \left(\mathbb{1}_{2D} - \hat{\Theta}^k\right)\hat{T} \ \overset{!}{\in} \ \mathbb{Z}^{2D}\,, \tag{42}$$

for all $k = 0, \ldots, K-1$ and $\hat{N} \in \mathbb{Z}^{2D}$. Then, also the group of outer automorphism is generated by pure rotations $(\hat{\Sigma}, 0)$ and pure translations $(\mathbb{1}_{2D}, \hat{T})$, such that $\hat{\Sigma} \neq \hat{\Theta}^k$ and $\hat{T} \notin \mathbb{Z}^{2D}$. The translational outer automorphisms contribute to the so-called traditional flavor symmetry of the theory, as we will review in section 5.2 in the case of a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. Let us focus now on the rotational elements $(\hat{\Sigma}, 0)$. Note that choosing $k = 0$ in eq. (42) delivers precisely the outer automorphisms of the Narain lattice, studied in section 2.2. In particular, we find the condition $\hat{\Sigma}^{-1}\hat{N} \in \mathbb{Z}^{2D}$, which is satisfied only if $\hat{\Sigma} \in O_{\hat{\eta}}(D, D, \mathbb{Z})$, cf. eqs. (11) and (12). However, for an orbifold compactification, the outer automorphism $\hat{\Sigma}$ is further constrained by the first condition in eq. (42) for $k \neq 0$.

Let us focus now on $D = 2$. As we have seen in section 2.2, in this case the outer automorphisms of $\mathbb{T}^2$ build the group of modular symmetries of a torus compactification, generated by $\hat{C}_{\mathrm{S}}, \hat{C}_{\mathrm{T}}, \hat{K}_{\mathrm{S}}, \hat{K}_{\mathrm{T}}, \hat{M}$ and $\hat{\Sigma}_*$, given in eqs. (14), (16) and (19). Thus, the outer automorphisms $\hat{\Sigma}$ of the orbifold must be a modular subgroup thereof, depending on the orbifold geometry $\mathbb{T}^2/\mathbb{Z}_K$, with $K \in \{2, 3, 4, 6\}$. Table 1 presents the Narain twists $\hat{\Theta}$ and a choice of generators of the modular symmetries arising from the outer automorphisms of all symmetric $\mathbb{T}^2/\mathbb{Z}_K$ orbifolds. As we shall discuss explicitly in section 3.2, even though the Narain twist $\hat{\Theta}$ is an inner automorphism of the orbifold $\mathbb{T}^2/\mathbb{Z}_K$, it becomes an outer automorphism if the $\mathbb{T}^2/\mathbb{Z}_K$ orbifold is only a subsector of a six-dimensional factorized orbifold.

2.6. Modular invariant field theory from strings

In string models, matter fields are associated with the excitation modes of strings. In a two-dimensional compactification of heterotic strings on a $\mathbb{T}^2/\mathbb{Z}_K$ orbifold, matter fields arise from

Table 1

(Rotational) Outer automorphisms $\hat{\Sigma}$ of all consistent symmetric orbifolds $\mathbb{T}^2/\mathbb{Z}_K$ with Narain twist $\hat{\Theta}$.

K	$\mathbb{Z}_K$ Narain twist $\hat{\Theta} \in O_{\hat{\eta}}(2,2,\mathbb{Z})$	Outer automorphisms $\hat{\Sigma}$
2	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\hat{C}_S, \hat{C}_T, \hat{K}_S, \hat{K}_T, \hat{M}, \hat{\Sigma}_*$
3	$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\hat{K}_S, \hat{K}_T, \hat{C}_S \hat{\Sigma}_*$
4	$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\hat{K}_S, \hat{K}_T, \hat{\Sigma}_*$
6	$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	$\hat{K}_S, \hat{K}_T, \hat{C}_S \hat{\Sigma}_*$

strings that close on the orbifold. They can be arranged in three categories:

(i) Untwisted strings, which are trivially closed strings, even before compactification;

(ii) Winding strings, which close only after winding along the directions e_1 and/or e_2 of the $\mathbb{T}^2$ torus. At a generic point in moduli space, winding strings have masses at or beyond the string scale. Thus, they do not lead to matter fields that appear in the low-energy effective field theory; and

(iii) Twisted strings, which close only due to the action of the twist.

This classification can be stated in terms of elements of the orbifold space group, which set the boundary conditions for the strings to close. This relation is revealed by regarding eq. (37) as the boundary conditions of closed strings in the orbifold, according to

$$Y(\tau, \sigma + 1) = g\, Y(\tau, \sigma) := \Theta^k Y(\tau, \sigma) + E\,\hat{N}, \quad \text{where} \quad g = (\Theta^k, E\,\hat{N}) \in S_{\text{Narain}}.$$
(43)

$Y = Y(\tau, \sigma)$ corresponds to the (bosonic) worldsheet string field in terms of the worldsheet coordinates τ and σ. Hence, we note that untwisted strings are associated with the trivial space group element $(\mathbb{1}_4, 0)$, winding strings are connected with translational space group elements $(\mathbb{1}_4, E\,\hat{N})$, and twisted strings are related to more general space group elements $(\Theta^k, E\,\hat{N})$, with $\Theta^k \neq \mathbb{1}_4$.

The properties and dynamics of string matter fields are inextricably linked to the attributes of the compact space on which these strings live. In particular, if two of the extra dimensions are associated with a $\mathbb{T}^2/\mathbb{Z}_K$ orbifold sector, matter fields inherit target space modular symmetries, whose generators are listed in Table 1 for each twist order K. Matter fields of heterotic orbifold compactifications are endowed with a modular weight for each SL(2, $\mathbb{Z}$) modular symmetry, and we denote the set of all modular weights by $\mathfrak{n}$. Let us discuss the example of two modular symmetries SL(2, $\mathbb{Z}$)$_T$ and SL(2, $\mathbb{Z}$)$_U$ of the Kähler modulus T and the complex structure

modulus U, where we have $\mathfrak{n} = (n_T, n_U) \in \mathbb{Q}^2$ (see e.g. refs. [22,23]). We denote untwisted and twisted matter fields of the orbifold theory as $\Phi_\mathfrak{n}$ given that matter fields can be distinguished in general by their modular weights $\mathfrak{n}$. In the case of twisted matter fields, they build multiplets $\Phi_\mathfrak{n} = (\phi_1, \phi_2, \ldots)^\mathrm{T}$, where the fields ϕ_i are associated with different closed strings, whose boundary conditions are produced by inequivalent space group elements with the same power k of Θ (i.e. by space group elements from the same twisted sector but different conjugacy classes). Consequently, in the absence of discrete Wilson lines [20] in the $\mathbb{T}^2/\mathbb{Z}_K$ orbifold sector, the twisted fields ϕ_i in the multiplet $\Phi_\mathfrak{n}$ share identical gauge quantum numbers and modular weights.

Ignoring $\mathcal{CP}$-like and mirror modular transformations that shall be addressed in the following section, a general matter field $\Phi_\mathfrak{n}$ transforms under general modular transformations $\gamma_T \in \mathrm{SL}(2, \mathbb{Z})_T$ and $\gamma_U \in \mathrm{SL}(2, \mathbb{Z})_U$, as defined by eqs. (27) and (29), according to

$$\Phi_\mathfrak{n} \xrightarrow{\gamma_T} \Phi'_\mathfrak{n} := (c_T\, T + d_T)^{n_T}\, \rho_s(\gamma_T)\, \Phi_\mathfrak{n}\,, \tag{44a}$$

$$\Phi_\mathfrak{n} \xrightarrow{\gamma_U} \Phi'_\mathfrak{n} := (c_U\, U + d_U)^{n_U}\, \rho_s(\gamma_U)\, \Phi_\mathfrak{n}\,, \tag{44b}$$

where $(c_T\, T + d_T)^{n_T}$ and $(c_U\, U + d_U)^{n_U}$ are the so-called automorphy factors with modular weights $\mathfrak{n} = (n_T, n_U) \in \mathbb{Q}^2$. The matrices $\rho_s(\gamma_T)$ and $\rho_s(\gamma_U)$ build a (reducible or irreducible) representation s of some finite modular group (for example $\Gamma'_{N_T} \times \Gamma'_{N_U}$, where in terms of the 2×2 matrices S, T given in eq. (2), the pair of transformation matrices $\rho_s(\gamma_T = \mathrm{S})$ and $\rho_s(\gamma_T = \mathrm{T})$ satisfy the defining relations eq. (4a) for some integer N_T and analogously for γ_U).

As we can infer from the modular symmetry generators listed in Table 1 and shall be discussed in detail in section 3, in all $\mathbb{T}^2/\mathbb{Z}_K$ orbifold compactifications but those with $K = 2$, the complex structure U is not dynamic as it is fixed at a value $\langle U \rangle$. We will see that this fact together with the transformation (44b) leads to new insights about the effective nature of these symmetry transformations: they give rise to the well-known R-symmetries in heterotic orbifold compactifications. If the complex structure modulus U is fixed, the strength of a superpotential coupling is given by a modular form $\hat{Y}^{(n_Y)}_{s_Y}(T)$ of the Kähler modulus T. Then, under a general modular transformation $\gamma_T \in \mathrm{SL}(2, \mathbb{Z})_T$ such a modular form transforms as

$$\hat{Y}^{(n_Y)}_{s_Y}(T) \xrightarrow{\gamma_T} \hat{Y}^{(n_Y)}_{s_Y}\left(\frac{a_T\, T + b_T}{c_T\, T + d_T}\right) = (c_T\, T + d_T)^{n_Y}\, \rho_{s_Y}(\gamma_T)\, \hat{Y}^{(n_Y)}_{s_Y}(T)\,. \tag{45}$$

Here, s_Y denotes an s_Y-dimensional representation of the finite modular group (e.g. Γ_{N_T} or Γ'_{N_T}) and $(c_T\, T + d_T)^{n_Y}$ is the automorphy factor of γ_T with modular weight n_Y. If the finite modular group is Γ_N, n_Y must be even. However, for the double cover groups Γ'_N that appear in string orbifold compactifications also odd n_Y are allowed [24].

String models based on orbifold compactifications yield typically an $\mathcal{N} = 1$ effective field theory. Here, we are mainly interested in its Kähler potential K and superpotential $\mathcal{W}$. The superpotential is a holomorphic function of moduli and matter fields $\Phi_\mathfrak{n}$. Under $\gamma_T \in \mathrm{SL}(2, \mathbb{Z})_T$ and $\gamma_U \in \mathrm{SL}(2, \mathbb{Z})_U$ modular transformations, the superpotential becomes

$$\mathcal{W}(T, U, \Phi_\mathfrak{n}) \xrightarrow{\gamma_T} \mathcal{W}\left(\frac{a_T\, T + b_T}{c_T\, T + d_T}, U, \Phi'_\mathfrak{n}\right) = (c_T\, T + d_T)^{-1}\, \mathcal{W}(T, U, \Phi_\mathfrak{n})\,, \tag{46a}$$

$$\mathcal{W}(T, U, \Phi_\mathfrak{n}) \xrightarrow{\gamma_U} \mathcal{W}\left(T, \frac{a_U\, U + b_U}{c_U\, U + d_U}, \Phi'_\mathfrak{n}\right) = (c_U\, U + d_U)^{-1}\, \mathcal{W}(T, U, \Phi_\mathfrak{n})\,, \tag{46b}$$

i.e. $\mathcal{W}$ behaves similar to a matter field with modular weights $n_U = n_T = -1$ and which is invariant under the finite modular group of the theory. Consequently, each allowed superpotential coupling of matter fields $\Phi_\mathfrak{n}$ has to satisfy the conditions

$$n_Y + \sum_{\text{matter fields } \Phi_{\mathfrak{n}}} n_T = -1 \quad \text{and} \quad \rho_{s_Y}(\gamma_T) \otimes \bigotimes_{\text{matter fields } \Phi_{\mathfrak{n}}} \rho_s(\gamma_T) \supset \mathbb{1}, \quad (47)$$

for $\gamma_T \in \mathrm{SL}(2, \mathbb{Z})_T$ with $\mathfrak{n} = (n_T, n_U)$, ρ_s is the representation s associated with $\Phi_{\mathfrak{n}}$, and the coupling $\hat{Y}_{s_Y}^{(n_Y)}$ of weight n_Y transforms in the representation s_Y. An analogous result holds for $\gamma_U \in \mathrm{SL}(2, \mathbb{Z})_U$.

On the other hand, the general Φ-independent contribution to the Kähler potential is given by [25]

$$K \supset K_0(T, U) := -\ln\left(-\mathrm{i}\, T + \mathrm{i}\, \bar{T}\right) - \ln\left(-\mathrm{i}\, U + \mathrm{i}\, \bar{U}\right). \quad (48)$$

This universal Kähler contribution transforms under $\gamma_T \in \mathrm{SL}(2, \mathbb{Z})_T$ and $\gamma_U \in \mathrm{SL}(2, \mathbb{Z})_U$ as

$$K_0(T, U) \xrightarrow{\gamma_T} K_0\left(\frac{a_T\, T + b_T}{c_T\, T + d_T}, U\right) = K_0(T, U) + f_T(T) + \overline{f_T(T)}, \quad (49\mathrm{a})$$

$$K_0(T, U) \xrightarrow{\gamma_U} K_0\left(T, \frac{a_U\, U + b_U}{c_U\, U + d_U}\right) = K_0(T, U) + f_U(U) + \overline{f_U(U)}, \quad (49\mathrm{b})$$

where $f_T(T) = \ln(c_T\, T + d_T)$ and $f_U(U) = \ln(c_U\, U + d_U)$. Note that the terms $f_T(T) + \overline{f_T(T)}$ and $f_U(U) + \overline{f_U(U)}$ can be removed by performing a Kähler transformation after each modular transformation, rendering the Kähler potential modular invariant. A general Kähler transformation is defined as [26, ch. 23]

$$K \to K + \mathcal{F} + \overline{\mathcal{F}}, \qquad \mathcal{W} \to e^{-\mathcal{F}}\,\mathcal{W}, \quad (50)$$

where $\mathcal{F}$ is a holomorphic function of chiral superfields. Therefore, any additional Φ-dependent contribution to K must be invariant under modular transformations, cf. ref. [27]. On the other hand, it turns out that the superpotential is also modular invariant since e.g. the modular transformation eq. (46a) followed by the Kähler transformation applied for achieving invariance of K yields

$$\mathcal{W} \xrightarrow{\gamma_T} (c_T\, T + d_T)^{-1}\, \mathcal{W} \xrightarrow{\text{Kähler}} (c_T\, T + d_T)^{-1}\, e^{f_T(T)}\,\mathcal{W} = \mathcal{W}, \quad (51)$$

with $f_T(T) = \ln(c_T\, T + d_T)$.

2.7. Summary

The modular symmetry groups $\mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{PSL}(2, \mathbb{Z})$ as well as their respective finite modular groups Γ'_N and Γ_N are natural to string compactifications. For example, as explained in section 2.3 strings on an internal two-torus yield two moduli: a complex structure modulus U as well as a Kähler modulus T. This is in contrast to bottom-up models of flavor that typically consider only one modulus. Using the Narain description of a torus compactification introduced in section 2.1, one can find the modular transformations acting on both moduli by computing the outer automorphisms of the Narain lattice associated with the compact space (see section 2.2). These modular transformations build in general a large group that includes $\mathrm{SL}(2, \mathbb{Z})_T$ and $\mathrm{SL}(2, \mathbb{Z})_U$ for the standard modular transformations of T and U, as well as two additional special transformations: a $\mathbb{Z}_2$ mirror duality that exchanges $\mathrm{SL}(2, \mathbb{Z})_T$ and $\mathrm{SL}(2, \mathbb{Z})_U$, and a $\mathbb{Z}_2$ $\mathcal{CP}$-like transformation. From this result, one can explore how this changes for all admissible $\mathbb{T}^2/\mathbb{Z}_K$ toroidal orbifolds, whose Narain formulation is introduced in section 2.4. As displayed in Table 1, we find that only $\mathbb{T}^2/\mathbb{Z}_2$ orbifolds preserve all the modular symmetries of the torus,

while for $K \in \{3, 4, 6\}$ the modular groups are subgroups of the group of modular symmetries of the torus. Interestingly, under modular transformations string matter fields $\Phi_\mathbf{n}$ and the couplings among them transform as representations of finite modular groups Γ'_N. In these terms, as detailed in section 2.6, we review the modular properties of the superpotential and Kähler potential that yield a modular invariant effective field theory.

3. $\mathrm{SL}(2, \mathbb{Z})_U$ of the complex structure

$\mathrm{SL}(2, \mathbb{Z})_T$ of the Kähler modulus T is of stringy nature as it relates, for example, compactifications on compact spaces with small and large volumes. In contrast, in this section we will show explicitly using the Narain formulation of $D = 2$ extra dimensions that the $\mathrm{SL}(2, \mathbb{Z})_U$ factor of the complex structure modulus U allows for a pure geometrical interpretation: On the one hand, $\mathrm{SL}(2, \mathbb{Z})_U$ modular transformations are defined in terms of a special class of outer automorphisms of the Narain lattice. On the other hand, $\mathrm{SL}(2, \mathbb{Z})_U$ only affects the geometrical vielbein e. Hence, we will see in a second step that rotational $\mathbb{Z}_N$ symmetries of the two-torus are described by those $\mathrm{SL}(2, \mathbb{Z})_U$ modular transformations that leave the complex structure modulus U invariant. Moreover, the automorphy factors of $\mathrm{SL}(2, \mathbb{Z})_U$ modular transformations turn out to be related to discrete $\mathbb{Z}_N$ charges. Furthermore, they promote the two-dimensional rotational $\mathbb{Z}_N$ symmetry to a discrete R-symmetry.

3.1. Geometrical interpretation of $\mathrm{SL}(2, \mathbb{Z})_U$ of the complex structure

In the Narain lattice basis, the generators $\hat{C}_\mathrm{S}$ and $\hat{C}_\mathrm{T}$ of $\mathrm{SL}(2, \mathbb{Z})_U$ (and also the $\mathcal{CP}$-like transformation $\hat{\Sigma}_*$) as defined in section 2.2 yield elements of the form

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma}^{-\mathrm{T}} \end{pmatrix} \in \langle \hat{C}_\mathrm{S}, \hat{C}_\mathrm{T}, \hat{\Sigma}_* \rangle \subset \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z}), \tag{52}$$

where $\hat{\sigma} \in \mathrm{GL}(2, \mathbb{Z})$. Note that, in contrast to $\mathrm{SL}(2, \mathbb{Z})_T$, modular transformations of this kind do not interchange winding and KK numbers. Next, we can change the basis to right- and left-moving coordinates (in analogy to eq. (39))

$$\Sigma := E \hat{\Sigma} E^{-1}. \tag{53}$$

Hence, Σ acts on right- and left-moving string coordinates as

$$\begin{pmatrix} y_\mathrm{R} \\ y_\mathrm{L} \end{pmatrix} \xrightarrow{\hat{\Sigma}} \Sigma \begin{pmatrix} y_\mathrm{R} \\ y_\mathrm{L} \end{pmatrix}, \tag{54}$$

see section 2.5. Consequently, the coordinates y and their dual coordinates $\tilde{y}$, as defined in eq. (5), transform as

$$\begin{pmatrix} y \\ \tilde{y} \end{pmatrix} \xrightarrow{\hat{\Sigma}} \begin{pmatrix} \sigma & 0 \\ e^{-\mathrm{T}}\hat{\sigma}^{-\mathrm{T}}\left(\hat{\sigma}^{\mathrm{T}} B \hat{\sigma} - B\right) e^{-1} & \sigma^{-\mathrm{T}} \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix}, \tag{55}$$

where $\sigma := e\,\hat{\sigma}\,e^{-1}$. It is easy to see that

$$\hat{\sigma}^{\mathrm{T}} B \hat{\sigma} = \det(\hat{\sigma})\,B \qquad \text{for} \qquad B = B_{12} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{56}$$

and $\hat{\sigma} \in \mathrm{GL}(2, \mathbb{Z})$.

Now let us discuss the case $\det(\hat{\sigma}) = +1$, i.e. when $\hat{\sigma} \in \mathrm{SL}(2, \mathbb{Z})$ corresponds to a modular transformation without $\mathcal{CP}$. Then, $\hat{\sigma} \in \mathrm{SL}(2, \mathbb{Z})$ is compatible with the B-field in the two-torus $\mathbb{T}^2$, i.e. $\hat{\sigma}^{\mathrm{T}} B \hat{\sigma} = B$, and we obtain

$$\Sigma = E \hat{\Sigma} E^{-1} = \frac{1}{2} \begin{pmatrix} \sigma + \sigma^{-\mathrm{T}} & \sigma - \sigma^{-\mathrm{T}} \\ \sigma - \sigma^{-\mathrm{T}} & \sigma + \sigma^{-\mathrm{T}} \end{pmatrix}. \tag{57}$$

Furthermore, eq. (55) yields in the case $\det(\hat{\sigma}) = +1$ the simple transformations

$$y \xrightarrow{\hat{\Sigma}} \sigma y \qquad \text{and} \qquad \tilde{y} \xrightarrow{\hat{\Sigma}} \sigma^{-\mathrm{T}} \tilde{y}. \tag{58}$$

These transformations can be absorbed completely into a redefinition of the geometrical torus vielbein e. Explicitly, the torus boundary condition

$$y \sim y + e \, n \qquad \text{with} \quad n \in \mathbb{Z}^2 \tag{59}$$

is mapped under $y \to \sigma y$ to

$$\sigma y \sim \sigma y + e n \qquad \Leftrightarrow \qquad y \sim y + \sigma^{-1} e n = y + \left(e \hat{\sigma}^{-1} \right) n. \tag{60}$$

Hence, an $\mathrm{SL}(2, \mathbb{Z})_U$ transformation $\hat{\Sigma}$ can be performed alternatively by a transformation of only the vielbein e, which then induces a change of the torus metric $G = e^{\mathrm{T}} e$, i.e.

$$e \xrightarrow{\hat{\Sigma}} \sigma^{-1} e = e \hat{\sigma}^{-1} \qquad \Rightarrow \qquad G \xrightarrow{\hat{\Sigma}} \hat{\sigma}^{-\mathrm{T}} G \hat{\sigma}^{-1}, \tag{61}$$

while the B-field is invariant. Note that the boundary condition

$$\tilde{y} \sim \tilde{y} + e^{-\mathrm{T}} \left(B n + \alpha' m \right) \qquad \text{with} \quad n, m \in \mathbb{Z}^2 \tag{62}$$

of the dual coordinates $\tilde{y}$ (as given in eq. (10)) transforms analogously under $\tilde{y} \to \sigma^{-\mathrm{T}} \tilde{y}$ from eq. (58). Since $\hat{\sigma} \in \mathrm{SL}(2, \mathbb{Z})$, the two-dimensional lattice spanned by e is mapped to itself under $\hat{\sigma}^{-1}$ in eq. (61): in other words, $\hat{\sigma} \in \mathrm{SL}(2, \mathbb{Z})$ is an outer automorphism of the two-dimensional lattice spanned by the geometrical vielbein e. For example, under modular S and T transformations of the complex structure modulus U, the geometrical lattice transforms as

$$e_1 \xrightarrow{\hat{C}_{\mathrm{S}}} -e_2 \qquad \text{and} \qquad e_2 \xrightarrow{\hat{C}_{\mathrm{S}}} e_1 \qquad \qquad \text{for S transformation}, \tag{63a}$$

$$e_1 \xrightarrow{\hat{C}_{\mathrm{T}}} e_1 \qquad \text{and} \qquad e_2 \xrightarrow{\hat{C}_{\mathrm{T}}} e_1 + e_2 \qquad \text{for T transformation}, \tag{63b}$$

using eq. (61) with $\hat{\sigma} = -\epsilon$ for $\hat{C}_{\mathrm{S}}$ and $\hat{\sigma} = \gamma$ for $\hat{C}_{\mathrm{T}}$ as given in eq. (16). Moreover, using eq. (63) the complex structure modulus U defined in eq. (22b) transforms as

$$U \xrightarrow{\hat{C}_{\mathrm{S}}} -\frac{1}{U} \qquad \qquad \text{for S transformation}, \tag{64a}$$

$$U \xrightarrow{\hat{C}_{\mathrm{T}}} U + 1 \qquad \qquad \text{for T transformation}. \tag{64b}$$

Note that these transformations (64) can also be obtained directly following ref. [9] if one takes the torus lattice vectors e_1 and e_2 to be complex numbers, i.e. $e_1, e_2 \in \mathbb{C}$, and rewrites the complex structure modulus as $U = e_2 / e_1$. Then, the transformations (63) of e_1 and e_2 imply eqs. (64).

In addition, we take the transformation eq. (61) of the metric G under $\hat{\Sigma}$ given in eq. (52) in order to obtain the general transformation property of the complex structure modulus U, i.e. (see also ref. [28])

$$U \xrightarrow{\hat{\Sigma}} \frac{a_U\, U + b_U}{c_U\, U + d_U} \quad \text{for} \quad \hat{\Sigma} = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma}^{-\mathrm{T}} \end{pmatrix} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})\,, \quad \text{where}$$

$$\hat{\sigma} =: \begin{pmatrix} a_U & -b_U \\ -c_U & d_U \end{pmatrix}\,, \tag{65}$$

for $a_U, b_U, c_U, d_U \in \mathbb{Z}$ satisfying $a_U\, d_U - b_U\, c_U = +1$, while the Kähler modulus T is invariant.

To summarize and to be specific, on the one hand we have identified a special class of outer automorphisms $\hat{\Sigma}$ of the Narain lattice Γ for $D = 2$, given by

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma}^{-\mathrm{T}} \end{pmatrix} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) \quad \text{with} \quad \hat{\sigma} =: \begin{pmatrix} a_U & -b_U \\ -c_U & d_U \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{66}$$

On the other hand, we have shown that an outer automorphism of this class acts geometrically on the vielbein e that defines the two-torus as

$$e \xrightarrow{\hat{\Sigma}} e\,\hat{\sigma}^{-1} \quad \Leftrightarrow \quad \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \xrightarrow{\hat{\Sigma}} \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}\,. \tag{67}$$

Hence, it gives rise to a modular transformation of the complex structure modulus

$$U \xrightarrow{\gamma_U} \frac{a_U\, U + b_U}{c_U\, U + d_U}\,, \quad \text{where} \quad \gamma_U := \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_U\,, \tag{68}$$

while T is invariant. In other words, there exist outer automorphisms of the Narain lattice Γ for $D = 2$ that are specified by their geometrical action $\hat{\sigma} \in \mathrm{SL}(2,\mathbb{Z})$ on the torus vielbein e and translate to modular transformations $\gamma_U \in \mathrm{SL}(2,\mathbb{Z})_U$ of the complex structure modulus U, where the dictionary between these two reads explicitly

$$\gamma_U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\sigma}^{-\mathrm{T}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_U\,. \tag{69}$$

3.2. Geometrical rotations from $\mathrm{SL}(2,\mathbb{Z})_U$

Based on the discussion from the last section, we now analyze a special class of outer automorphisms of the Narain lattice that corresponds to geometrical rotations in the extra-dimensional space, see also ref. [9] for a related discussion. We focus on two extra dimensions ($D = 2$), but the generalization to six extra dimensions is straightforward, since a rotation in six dimensions can be decomposed into three rotations in three orthogonal two-dimensional planes. Thus, the results of this section will be important for both: i) to define an orbifold twist and ii) to perform a two-dimensional "sublattice rotation" of a six-dimensional orbifold.

We begin with a discrete rotation $\theta_{(K)} \in \mathbb{Z}_K \subset \mathrm{SO}(2)$ that acts as

$$y \xrightarrow{\theta_{(K)}} \theta_{(K)}\, y\,, \quad \text{where} \quad \theta_{(K)} := \begin{pmatrix} \cos(2\pi\, v_{(K)}) & -\sin(2\pi\, v_{(K)}) \\ \sin(2\pi\, v_{(K)}) & \cos(2\pi\, v_{(K)}) \end{pmatrix} \in \mathrm{SO}(2)\,, \tag{70}$$

while it leaves all orthogonal coordinates inert. The geometrical rotation angle $v_{(N)}$ is defined as

$$v_{(K)} := \frac{1}{K}\,, \quad \text{such that} \quad \left(\theta_{(K)}\right)^K = \mathbb{1}_2\,. \tag{71}$$

In order to be a rotational symmetry of the geometrical torus, $\theta_{(K)}$ has to map the lattice spanned by the torus vielbein e to itself. Hence, we have to impose the condition

$$\hat{\sigma}_{(K)} := e^{-1}\theta_{(K)}\,e \overset{!}{\in} \mathrm{SL}(2,\mathbb{Z}) \qquad \text{for} \qquad \theta_{(K)} \in \mathrm{SO}(2)\,. \tag{72}$$

This constrains the order K to the allowed orders of the $\mathbb{Z}_K$ wallpaper groups, being $K \in \{2, 3, 4, 6\}$, see e.g. ref. [29].

Now, we translate this discrete geometrical rotation into the Narain formulation of string theory. To do so, we split y into right- and left-moving string coordinates using eq. (5) and define the action on these coordinates as

$$\begin{pmatrix} y_{\mathrm{R}} \\ y_{\mathrm{L}} \end{pmatrix} \xrightarrow{\Theta_{(K)}} \Theta_{(K)} \begin{pmatrix} y_{\mathrm{R}} \\ y_{\mathrm{L}} \end{pmatrix}, \qquad \text{where} \qquad \Theta_{(K)} := \begin{pmatrix} \theta_{(K),\,\mathrm{R}} & 0 \\ 0 & \theta_{(K),\,\mathrm{L}} \end{pmatrix}, \tag{73}$$

and $\theta_{(K),\,\mathrm{R}}, \theta_{(K),\,\mathrm{L}} \in \mathbb{Z}_K \subset \mathrm{SO}(2)$ are of order K. Using eq. (70) we obtain

$$y := \sqrt{\frac{\alpha'}{2}}\,(y_{\mathrm{R}} + y_{\mathrm{L}}) \xrightarrow{\Theta_{(K)}} \sqrt{\frac{\alpha'}{2}}\,\big(\theta_{(K),\,\mathrm{R}}\,y_{\mathrm{R}} + \theta_{(K),\,\mathrm{L}}\,y_{\mathrm{L}}\big) \overset{!}{=} \theta_{(K)}\,y\,, \tag{74}$$

such that the rotation has to be left-right symmetric

$$\theta_{(K)} := \theta_{(K),\,\mathrm{R}} = \theta_{(K),\,\mathrm{L}}\,. \tag{75}$$

Thus, the left-right-symmetric rotation $\Theta_{(K)}$ defined in eq. (73) together with eq. (75) corresponds to the transformation Σ from eq. (57) with $\sigma = \theta_{(K)} \in \mathrm{SO}(2)$, i.e. $\sigma^{-\mathrm{T}} = \sigma$. Consequently, we can follow the discussion around eq. (57) and write the rotation in terms of an outer automorphism of the Narain lattice as

$$\hat{\Theta}_{(K)} := E^{-1}\Theta_{(K)}\,E = \begin{pmatrix} \hat{\sigma}_{(K)} & 0 \\ 0 & \hat{\sigma}_{(K)}^{-\mathrm{T}} \end{pmatrix} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})\,. \tag{76}$$

Due to this block-structure together with $\hat{\sigma}_{(K)} := e^{-1}\theta_{(K)}\,e \in \mathrm{SL}(2,\mathbb{Z})$, the Narain twist $\hat{\Theta}_{(K)}$ can be decomposed in terms of the generators $\hat{C}_{\mathrm{S}}$ and $\hat{C}_{\mathrm{T}}$, given in eq. (16). Then, using eq. (69) we can translate the action of the outer automorphism $\hat{\Theta}_{(K)}$ to a corresponding modular transformation from $\mathrm{SL}(2,\mathbb{Z})_U$ as

$$e^{-1}\theta_{(K)}\,e = \hat{\sigma}_{(K)} =: \begin{pmatrix} a_U & -b_U \\ -c_U & d_U \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \quad \Leftrightarrow$$

$$\gamma_{(K)} := \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_U\,, \tag{77}$$

where the modular transformation $\gamma_{(K)}$ is of order K, i.e. $\big(\gamma_{(K)}\big)^K = \mathbb{1}_2$.

3.3. Stabilizing the complex structure modulus by geometrical rotations

Now, we can use the transformation property of the torus metric $G = e^{\mathrm{T}}e$ given in eq. (61) for a rotational symmetry eq. (72) in order to find

$$G \xrightarrow{\hat{\Theta}_{(K)}} \hat{\sigma}_{(K)}^{-\mathrm{T}}\,G\,\hat{\sigma}_{(K)}^{-1} = G\,. \tag{78}$$

Hence, the metric G and, therefore, the complex structure modulus U (as defined in eq. (22b)) must be invariant under a Narain twist $\hat{\Theta}_{(K)}$. In other words, the vev of the complex structure

Table 2

Geometrical $\mathbb{Z}_K$ symmetries $\hat{\Theta}_{(K)} \in O_{\hat{\eta}}(2, 2, \mathbb{Z})$ of the Narain lattice, translated to modular transformations $\gamma_{(K)} \in \mathrm{SL}(2, \mathbb{Z})_U$ that leave the vev $\langle U \rangle$ of the associated complex structure modulus invariant. As a consequence, $\hat{\Theta}_{(K)}$ acts left-right symmetrically and rotates the geometrical coordinate y by an angle $v_{(K)}$. At the same time, the superpotential picks up a phase $\exp\left(2\pi i\, v_{(K)}\right)$ such that the geometrical rotation $\hat{\Theta}_{(K)}$ generates an R-symmetry, as shown in section 3.4.

Order K	Rotational symmetry $\hat{\Theta}_{(K)} \in O_{\hat{\eta}}(2, 2, \mathbb{Z})$	$\gamma_{(K)} \in \mathrm{SL}(2, \mathbb{Z})_U$	$\langle U \rangle$	$v_{(K)}$
2	$\hat{\Theta}_{(2)} := \left(\hat{C}_{\mathrm{S}}\right)^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\mathrm{S}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	arb.	1/2
3	$\hat{\Theta}_{(3)} := \hat{C}_{\mathrm{S}}\, \hat{C}_{\mathrm{T}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\mathrm{S}\,\mathrm{T} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	ω	1/3
4	$\hat{\Theta}_{(4)} := \hat{C}_{\mathrm{S}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	i	1/4
6	$\hat{\Theta}_{(6)} := \left(\hat{C}_{\mathrm{S}}^3 \hat{C}_{\mathrm{T}}\right)^5 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	$(\mathrm{S}^3\mathrm{T})^5 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	ω	1/6

modulus $\langle U \rangle$ is a fixed point of those modular transformations $\gamma_{(K)} \in \mathrm{SL}(2, \mathbb{Z})_U$ that correspond to rotational symmetries of the geometrical torus,

$$U \xrightarrow{\gamma_{(K)}} \gamma_{(K)}\, U \;=\; \frac{a_U\, U + b_U}{c_U\, U + d_U} \;=\; U \qquad \text{at} \qquad U \;=\; \langle U \rangle, \tag{79}$$

where the vacuum expectation value $\langle U \rangle$ of the complex structure modulus parameterizes the torus vielbein e, except for the overall size of its two-torus.

In our case of two-dimensional rotational symmetries of order K, the Kähler modulus T is invariant but the complex structure modulus U is frozen geometrically to, for example,

$$
\begin{aligned}
U &= \text{unstabilized} && \text{if } K = 2, \\
\langle U \rangle &= \omega && \text{if } K = 3 \text{ or } 6, \\
\langle U \rangle &= \mathrm{i} && \text{if } K = 4.
\end{aligned}
\tag{80}
$$

The detailed results for $\mathbb{Z}_K$ rotational symmetries of the Narain lattice in $D = 2$ for $K \in \{2, 3, 4, 6\}$ are listed in Table 2. Stabilizing the complex structure modulus by geometrical rotations has two important effects:

i) Even if the complex structure modulus is stabilized, some modular transformations from $\mathrm{SL}(2, \mathbb{Z})_U$ will remain unbroken, i.e. there are elements in $\mathrm{SL}(2, \mathbb{Z})_U$ that leave $\langle U \rangle$ invariant. In order to analyze this, we define the so-called stabilizer subgroup $H_{\langle U \rangle}$ of $\mathrm{SL}(2, \mathbb{Z})_U$ modular transformations that leave $\langle U \rangle$ invariant,

$$H_{\langle U \rangle} \;=\; \left\{ \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})_U \;\middle|\; \frac{a_U\, \langle U \rangle + b_U}{c_U\, \langle U \rangle + d_U} \;=\; \langle U \rangle \right\}, \tag{81}$$

i.e. $\langle U \rangle$ is a fixed point of the elements of $H_{\langle U \rangle}$. Hence, the vev $\langle U \rangle$ of the stabilized complex structure modulus spontaneously breaks $\mathrm{SL}(2, \mathbb{Z})_U$ to an unbroken symmetry group, being

$$
\begin{aligned}
H_U &\cong \mathbb{Z}_2 & &\text{if } K = 2 \,, \\
H_{\langle U \rangle = \omega} &\cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3 & &\text{if } K = 3 \text{ or } 6 \,, \\
H_{\langle U \rangle = \mathrm{i}} &\cong \mathbb{Z}_4 & &\text{if } K = 4 \,,
\end{aligned}
\tag{82}
$$

where the $\mathbb{Z}_2$ factors are generated by $\mathrm{S}^2 \in \mathrm{SL}(2, \mathbb{Z})_U$ and $\mathbb{Z}_3$ is generated by $\mathrm{S}\,\mathrm{T} \in \mathrm{SL}(2, \mathbb{Z})_U$. These unbroken modular transformations will be of importance for the symmetries after orbifolding.

ii) As we will analyze next, stabilizing the complex structure modulus U promotes the automorphy factor $(c_U\, U + d_U)$ to a phase.

3.4. R-symmetries from $\mathrm{SL}(2, \mathbb{Z})_U$

In addition, we can compute the automorphy factor $(c_U\, \langle U \rangle + d_U)^{-1}$ of the superpotential $\mathcal{W}$, see eq. (46), for a rotational $\mathbb{Z}_K$ symmetry that is described by a modular transformation $\gamma_{(K)} \in \mathrm{SL}(2, \mathbb{Z})_U$. It has to be a phase of order K since $(\gamma_{(K)})^K = \mathbb{1}_2$. Indeed, we obtain

$$
c_U\, \langle U \rangle + d_U \;=\; \exp\!\left(-2\pi \mathrm{i}\, v_{(K)} \right) \,,
\tag{83}
$$

where c_U and d_U are given by $\gamma_{(K)} \in \mathrm{SL}(2, \mathbb{Z})_U$ and $v_{(K)} = 1/K$ turns out to coincide with the geometrical rotation angle corresponding to $\gamma_{(K)}$ as defined in section 3.2 and listed in Table 2. This yields

$$
\mathcal{W} \;\xrightarrow{\;\gamma_{(K)}\;}\; (c_U\, \langle U \rangle + d_U)^{-1}\, \mathcal{W} \;=\; \exp\!\left(2\pi \mathrm{i}\, v_{(K)} \right) \mathcal{W} \,.
\tag{84}
$$

Since the modular transformation $\gamma_{(K)} \in \mathrm{SL}(2, \mathbb{Z})_U$ leaves the (stabilized) complex structure modulus $\langle U \rangle$ invariant (cf. eq. (79)), the Kähler potential eq. (48) is invariant under $\gamma_{(K)}$. Hence, the phase in eq. (84) has to be compensated by a transformation

$$
\vartheta \;\xrightarrow{\;\gamma_{(K)}\;}\; \exp\!\left(2\pi \mathrm{i}\, R_\vartheta \right) \vartheta \,, \qquad \text{where} \qquad R_\vartheta := \frac{1}{2K}
\tag{85}
$$

of the Grassmann number ϑ of $\mathcal{N} = 1$ superspace. Thus, the rotational modular transformation $\gamma_{(K)} \in \mathrm{SL}(2, \mathbb{Z})_U$ of the complex structure modulus is an R-transformation. In more detail, using $\int \mathrm{d}\vartheta\, \vartheta = 1$ we know that the R-charge of $\int \mathrm{d}\vartheta$ is $-R_\vartheta$, such that $\mathcal{L} \supset \int \mathrm{d}^2\vartheta\, \mathcal{W}$ is invariant.

3.5. Summary

In summary, a two-dimensional rotational $\mathbb{Z}_K$ symmetry of the extra dimensional space is an outer automorphism of the Narain lattice that leaves the torus metric G and the B-field invariant. Because of its geometric nature, it corresponds to a modular transformation $\gamma_{(K)}$ from $\mathrm{SL}(2, \mathbb{Z})_U$ of the complex structure modulus U that leaves U evaluated at its vev and T invariant, as might have been expected from geometrical intuition. Moreover, a geometric rotation in two dimensions acts as an R-transformation, where the R-charge of the superpotential originates from the automorphy factor $(c_U\, \langle U \rangle + d_U)^{-1}$ of the modular transformation $\gamma_{(K)} \in \mathrm{SL}(2, \mathbb{Z})_U$.

4. *R*-symmetries and $\mathcal{CP}$ in six-dimensional orbifolds

It is known that toroidal orbifold compactifications of string theory lead to discrete *R*-symmetries of the four-dimensional effective theory. In the traditional approach, *R*-symmetries are explained to originate from rotational isometries of the six-dimensional orbifold geometry [30–34]. Here, we give a novel interpretation for the origin of *R*-symmetries as a special class of outer automorphisms of the Narain space group: *R*-symmetries correspond to elements from the modular group $\mathrm{SL}(2, \mathbb{Z})_U$ that leave the vev of the (stabilized) complex structure modulus U invariant.

4.1. Six-dimensional orbifolds $\mathbb{T}^6/P$

In order to specify a geometrical point group $P \cong \mathbb{Z}_K$ of a six-dimensional orbifold, we define a six-dimensional orbifold twist $\theta \in \mathrm{SO}(6)$ by combining three two-dimensional $\mathbb{Z}_{K_i}$ rotations $\theta_{(K_i)} \in \mathrm{SO}(2)$ given in eq. (70) for $i \in \{1, 2, 3\}$. In detail, the orbifold twist is chosen as

$$\theta := \left(\theta_{(K_1)}\right)^{k_1} \oplus \left(\theta_{(K_2)}\right)^{k_2} \oplus \left(\theta_{(K_3)}\right)^{k_3} \quad \text{such that} \quad y \xrightarrow{\theta} \theta\, y\,, \tag{86}$$

for $k_i \in \{0, 1, \dots, K_i - 1\}$ and $y \in \mathbb{R}^6$. One can choose complex coordinates $z^i := y^{2i-1} + \mathrm{i}\, y^{2i}$ for $i \in \{1, 2, 3\}$. Then, each two-dimensional rotation $\theta_{(K_i)}$ acts only in the i-th complex plane z^i. The orbifold twist θ generates a $\mathbb{Z}_K$ rotation group, whose order K is given by the least common multiple of K_1, K_2 and K_3. By combining the three rotation angles $k_i\, v_{(K_i)} = k_i/K_i$ of $(\theta_{(K_i)})^{k_i}$ we obtain the $\mathbb{Z}_K$ orbifold twist vector

$$v := \left(0, \frac{k_1}{K_1}, \frac{k_2}{K_2}, \frac{k_3}{K_3}\right)\,. \tag{87}$$

We have extended v by an additional null entry for its action in string light-cone coordinates, including those of the uncompactified space. Using the results from Table 2, the orbifold twist vector v corresponds to a modular transformation

$$\gamma := \left(\gamma_{(K_1)}\right)^{k_1} \oplus \left(\gamma_{(K_2)}\right)^{k_2} \oplus \left(\gamma_{(K_3)}\right)^{k_3} \in \mathrm{SL}(2, \mathbb{Z})_{U_1} \times \mathrm{SL}(2, \mathbb{Z})_{U_2} \times \mathrm{SL}(2, \mathbb{Z})_{U_3}\,. \tag{88}$$

This transformation leaves the three complex structure moduli U_i invariant, where $i \in \{1, 2, 3\}$ corresponds to the three complex planes, i.e.

$$U_i \xrightarrow{\gamma} \left(\gamma_{(K_i)}\right)^{k_i} U_i \overset{!}{=} U_i \qquad \text{for} \qquad U_i = \langle U_i \rangle \tag{89}$$

and $\langle U_i \rangle$ is given in eq. (80) for the cases $K_i \in \{2, 3, 4, 6\}$. Hence, a complex structure modulus U_i is either stabilized if $K_i \neq 2$ or unstabilized if $K_i = 2$.

In addition, using the three automorphy factors arising from the modular transformations $\gamma_{(K_i)} \in \mathrm{SL}(2, \mathbb{Z})_{U_i}$ for $i \in \{1, 2, 3\}$ in combination with eqs. (84) and (85), we confirm that the superpotential $\mathcal{W}$ and the Grassmann number ϑ are invariant under the combined modular transformation γ from eq. (88),

$$\mathcal{W} \xrightarrow{\gamma} \exp\left(2\pi\mathrm{i}\left(k_1\, v_{(K_1)} + k_2\, v_{(K_2)} + k_3\, v_{(K_3)}\right)\right) \mathcal{W} = \mathcal{W}\,, \tag{90a}$$

$$\vartheta \xrightarrow{\gamma} \exp\left(\frac{2\pi\mathrm{i}}{2}\left(k_1\, v_{(K_1)} + k_2\, v_{(K_2)} + k_3\, v_{(K_3)}\right)\right) \vartheta = \vartheta\,, \tag{90b}$$

if the associated orbifold twist vector v preserves $\mathcal{N} = 1$ supersymmetry, i.e. if

$$\frac{k_1}{K_1} + \frac{k_2}{K_2} + \frac{k_3}{K_3} \;=\; 0 \bmod 2 \;. \tag{91}$$

On the other hand, the Kähler potential is invariant under the modular transformation γ given in eq. (88): For $K_i \in \{3, 4, 6\}$, the complex structure modulus U_i is fixed to the values $\langle U_i \rangle$ provided in eq. (80). Hence, the Kähler potential of the theory does not exhibit any field-dependence on U_i. For $K_i = 2$, the Kähler potential does depend on the (unstabilized) modulus U_i explicitly, but is left invariant by the rotational modular transformation γ because neither the complex structure modulus itself nor the Φ-dependent terms of the form $|\Phi|^2$ are altered under these transformations.

Factorized orbifolds   In the following we concentrate on factorized $\mathbb{T}^6/\mathbb{Z}_K$ orbifold geometries, where the six-torus factorizes as $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ and each two-dimensional $\mathbb{Z}_{K_i}$ rotation $\theta_{(K_i)}$ is a symmetry of the respective two-torus $\mathbb{T}^2$. In this case, the orbifold twist $\hat{\Theta}$ in the Narain lattice basis reads

$$\hat{\Theta} := \hat{\Theta}_{(K_1)} \oplus \hat{\Theta}_{(K_2)} \oplus \hat{\Theta}_{(K_3)} \in O_{\hat{\eta}}(2, 2, \mathbb{Z}) \times O_{\hat{\eta}}(2, 2, \mathbb{Z}) \times O_{\hat{\eta}}(2, 2, \mathbb{Z}) \subset O_{\hat{\eta}}(6, 6, \mathbb{Z}) \;,$$
$$\tag{92}$$

where the $\mathbb{Z}_{K_i}$ Narain twists $\hat{\Theta}_{(K_i)} \in O_{\hat{\eta}}(2, 2, \mathbb{Z})$ of order $K_i \in \{2, 3, 4, 6\}$ are given in Table 2.

For example, for the phenomenological promising $\mathbb{Z}_6$-II $(1, 1)$ orbifold geometry [30,35,36], we have an orbifold twist vector

$$v \;=\; (0, {}^1\!/_6, {}^1\!/_3, -{}^1\!/_2) \;, \tag{93}$$

which preserves $\mathcal{N} = 1$ supersymmetry since $1/6 + 1/3 - 1/2 = 0$. For the $\mathbb{Z}_6$-II $(1, 1)$ orbifold geometry [29] this twist vector corresponds to an orbifold twist $\hat{\Theta}$ in the Narain lattice basis

$$\hat{\Theta} := \hat{\Theta}_{(6)} \oplus \hat{\Theta}_{(3)} \oplus \hat{\Theta}_{(2)} \in O_{\hat{\eta}}(2, 2, \mathbb{Z}) \times O_{\hat{\eta}}(2, 2, \mathbb{Z}) \times O_{\hat{\eta}}(2, 2, \mathbb{Z}) \subset O_{\hat{\eta}}(6, 6, \mathbb{Z}) \;. \tag{94}$$

Note that we have chosen the rotation angle of $\hat{\Theta}_{(2)}$ as $-1/2$ in order to fix the action on spacetime spinors, see for example eq. (90b). Using the results from Table 2, the $\mathbb{Z}_6$-II orbifold twist vector v corresponds to a modular transformation

$$\gamma := \gamma_{(6)} \oplus \gamma_{(3)} \oplus \gamma_{(2)} \in \mathrm{SL}(2, \mathbb{Z})_{U_1} \times \mathrm{SL}(2, \mathbb{Z})_{U_2} \times \mathrm{SL}(2, \mathbb{Z})_{U_3} \;. \tag{95}$$

This transformation leaves the three complex structure moduli U_i of the three complex planes $i \in \{1, 2, 3\}$ invariant,

$$(U_1, U_2, U_3) \xrightarrow{\;\gamma\;} \big(\gamma_{(6)} U_1, \gamma_{(3)} U_2, \gamma_{(2)} U_3\big) \;=\; (U_1, U_2, U_3) \;, \tag{96}$$

if the moduli are evaluated at

$$(U_1, U_2, U_3) \;=\; (\omega, \omega, U_3) \;, \tag{97}$$

see Table 2. Hence, the complex structure moduli U_1 and U_2 of the $\mathbb{Z}_6$-II orbifold geometry are stabilized (for example) at $\langle U_1 \rangle = \langle U_2 \rangle = \omega$, while U_3 in the $\mathbb{Z}_2$ orbifold plane is unstabilized.

4.2. Sublattice rotations from $\mathrm{SL}(2, \mathbb{Z})_U$ of the complex structure modulus

For simplicity, we have chosen the six-torus $\mathbb{T}^6$ that underlies the $\mathbb{T}^6/\mathbb{Z}_K$ orbifold geometry to be factorized as $\mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$. Then, the six-dimensional orbifold has three independent $\mathbb{Z}_{K_i}$

rotational isometries, whose orders $K_i \in \{2, 3, 4, 6\}$ for $i \in \{1, 2, 3\}$ are given in general by the components of the $\mathbb{Z}_K$ orbifold twist vector defined in eq. (87): there is one rotational isometry per two-torus $\mathbb{T}^2$. These $\mathbb{Z}_{K_i}$ rotations are also called sublattice rotations.

In detail, as the $\mathbb{T}^6/\mathbb{Z}_K$ orbifold geometry (with $\mathbb{Z}_K$ orbifold twist vector given in eq. (87)) is assumed to be factorized, it contains three two-dimensional subsectors, which we call $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold *sectors* for $i \in \{1, 2, 3\}$. Then, a sublattice rotation $\theta_{(K_i)}$ acts on the geometrical coordinate y of the $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sector as

$$y \xrightarrow{\theta_{(K_i)}} \theta_{(K_i)}\, y\,, \qquad \text{where} \qquad \theta_{(K_i)} \in \mathbb{Z}_{K_i} \subset \mathrm{SO}(2)\,, \tag{98}$$

while it leaves the orthogonal coordinates invariant. Following section 3.2, these three geometrical transformations for $i \in \{1, 2, 3\}$ correspond to three outer automorphisms $\hat{\mathrm{R}}_i$ of the full six-dimensional Narain space group, given by

$$\hat{\mathrm{R}}_1 := \hat{\Theta}_{(K_1)} \oplus \mathbb{1}_4 \oplus \mathbb{1}_4\,, \tag{99a}$$
$$\hat{\mathrm{R}}_2 := \mathbb{1}_4 \oplus \hat{\Theta}_{(K_2)} \oplus \mathbb{1}_4\,, \tag{99b}$$
$$\hat{\mathrm{R}}_3 := \mathbb{1}_4 \oplus \mathbb{1}_4 \oplus \hat{\Theta}_{(K_3)}\,, \tag{99c}$$

where $\hat{\mathrm{R}}_i \in \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z}) \times \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z}) \times \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z}) \subset \mathrm{O}_{\hat{\eta}}(6, 6, \mathbb{Z})$ and

$$\hat{\Theta}_{(K_i)} = \begin{pmatrix} \hat{\sigma}_{(K_i)} & 0 \\ 0 & \hat{\sigma}_{(K_i)}^{-\mathrm{T}} \end{pmatrix} \in \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z}) \quad \text{and} \quad \hat{\sigma}_{(K_i)} =: \begin{pmatrix} a_i & -b_i \\ -c_i & d_i \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})\,, \tag{100}$$

for $i \in \{1, 2, 3\}$. Each of them is associated with a modular transformation

$$\gamma_{(K_i)} := \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})_{U_i} \tag{101}$$

corresponding to the complex structure modulus U_i, which leaves *all* moduli invariant. Note that the product $\hat{\mathrm{R}}_1 \hat{\mathrm{R}}_2 \hat{\mathrm{R}}_3$ from eq. (99) equals the Narain twist $\hat{\Theta}$ given in eq. (94). Hence, $\hat{\mathrm{R}}_1 \hat{\mathrm{R}}_2 \hat{\mathrm{R}}_3$ is an inner automorphism of the Narain space group.

Next, we analyze the action of the modular transformation $\gamma_{(K_i)} \in \mathrm{SL}(2, \mathbb{Z})_{U_i}$ on matter fields $\Phi_{\mathfrak{n}}$, where $\mathfrak{n}$ collectively denotes the set of all modular weights, containing the modular weights n_{U_i} of $\mathrm{SL}(2, \mathbb{Z})_{U_i}$ for $i \in \{1, 2, 3\}$, see e.g. ref. [22,23] for the definition of the (fractional) modular weights for various orbifold geometries. In this case, the modular transformation (44) of matter fields reads

$$\Phi_{\mathfrak{n}} \xrightarrow{\gamma_{(K_i)}} (c_i \langle U_i \rangle + d_i)^{n_{U_i}}\, \rho_s(\gamma_{(K_i)})\, \Phi_{\mathfrak{n}} = \exp\left(-2\pi \mathrm{i}\, v_{(K_i)}\, n_{U_i}\right) \rho_s(\gamma_{(K_i)})\, \Phi_{\mathfrak{n}}\,, \tag{102}$$

such that the superpotential picks up a phase

$$\mathcal{W} \xrightarrow{\gamma_{(K_i)}} (c_i \langle U_i \rangle + d_i)^{-1}\, \mathcal{W} = \exp\left(2\pi \mathrm{i}\, v_{(K_i)}\right) \mathcal{W}\,. \tag{103}$$

Here, we used that the automorphy factor evaluated at $\langle U_i \rangle$ becomes a modulus-independent phase, see eq. (83). In order to ensure that $(\gamma_{(K_i)})^{K_i} = \mathbb{1}_2$, the matrix $\rho_s(\gamma_{(K_i)})$ of the modular transformation has to be of order K_i. Furthermore, note that $\rho_s(\gamma_{(K_i)})$ can be diagonal or non-diagonal: On the one hand, it is diagonal for example for $K_i \in \{2, 3\}$ since the corresponding $\mathrm{O}_{\hat{\eta}}(6, 6, \mathbb{Z})$ sublattice rotation maps each string (i.e. each conjugacy class of the constructing element) to itself, possibly multiplied by a phase. On the other hand, $\rho_s(\gamma_{(K_i)})$ is non-diagonal for example for a $K_i = 4$ sublattice rotation in a $\mathbb{T}^6/\mathbb{Z}_4$ orbifold geometry, see ref. [33].

Eq. (102) defines the discrete R-charges of matter fields $\Phi_{\mathfrak{n}}$ for modular transformations $\gamma_{(K_i)} \in \mathrm{SL}(2, \mathbb{Z})_{U_i}$ that correspond to geometrical sublattice rotations. On the other hand, it is known [30–34] that a matter field $\Phi_{\mathfrak{n}}$ transforms under a geometrical sublattice rotation $\theta_{(K_i)}$ as

$$\Phi_{\mathfrak{n}} \xrightarrow{\theta_{(K_i)}} \exp\left(\frac{2\pi \mathrm{i}\, R_i}{K_i^2}\right) \Phi_{\mathfrak{n}} . \tag{104}$$

Using K_i^2 as denominator, the R-charges R_i of superfields $\Phi_{\mathfrak{n}}$ become integer, as defined in ref. [32] for $K_i \in \{2, 3, 6\}$. Consequently, R-charges R_i and modular weights n_{U_i} are related. For example, if the representation matrix $\rho_s(\gamma_{(K_i)})$ from eq. (102) is diagonal, we find

$$R_i = \left(-n_{U_i} + \alpha_i\right) K_i , \tag{105}$$

where α_i gives the order K_i phase of the i-th entry on the diagonal of $\rho_s(\gamma_{(K_i)})$, i.e. $\rho_s(\gamma_{(K_i)}) = \mathrm{diag}(\exp(2\pi \mathrm{i}\alpha_i(\phi_1)/K_i), \ldots, \exp(2\pi \mathrm{i}\alpha_i(\phi_s)/K_i))$.

Couplings $\hat{Y}_{s_Y}^{(n_Y)}$ do not depend on the complex structure modulus U_i if $K_i \in \{3, 4, 6\}$. In contrast, for $K_i = 2$ they do depend on U_i but are invariant under the "rotational" modular transformation $\gamma_{(2)}$. Hence, we can set $n_Y = 0$ and $\rho_{s_Y}(\gamma_{(2)}) = 1$. Then, we obtain from eq. (47) the invariance conditions

$$\sum_{\text{matter fields } \Phi_{\mathfrak{n}}} n_{U_i} = -1 \qquad \text{and} \qquad \bigotimes_{\text{matter fields } \Phi_{\mathfrak{n}}} \rho_s(\gamma_{(K_i)}) \supset \mathbb{1} . \tag{106}$$

The later condition can be rewritten (for diagonalized representation matrices $\rho_s(\gamma_{(K_i)})$) as

$$\sum_{\text{matter fields } \Phi_{\mathfrak{n}}} \alpha_i = 0 \bmod K_i . \tag{107}$$

Consequently, we see that using eqs. (106) we have to impose the constraint

$$\sum_{\text{matter fields } \Phi_{\mathfrak{n}}} R_i = K_i \bmod (K_i)^2 \tag{108}$$

on the R-charges R_i for each superpotential coupling of matter fields $\Phi_{\mathfrak{n}}$ to be allowed, cf. refs. [30–34]. Hence, the order K_i sublattice rotation yields an R-symmetry that is in general $\mathbb{Z}^R_{(K_i)^2}$, where the nontrivial right-hand side of eq. (108) follows from eq. (103). As we have shown, this R-symmetry originates from unbroken modular transformations $\mathrm{SL}(2, \mathbb{Z})_{U_i}$ (that leave the complex structure modulus U_i invariant), even in the case when U_i is frozen geometrically by the orbifolding.

Finally, let us note that for special points in moduli space (e.g. for special values of a complex structure modulus or for vanishing discrete Wilson lines) there can be additional R-symmetries originating from the modular group $\mathrm{SL}(2, \mathbb{Z})_U$ of a complex structure modulus *and* from $\mathrm{SL}(2, \mathbb{Z})_T$ of the Kähler modulus. A detailed example of sublattice rotations in a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector will be discussed later in section 5.4.

4.3. $\mathcal{CP}$ for six-dimensional orbifolds

A $\mathcal{CP}$-like transformation has to map a field to a $\mathcal{CP}$-conjugate partner with inverse (discrete and gauge) charges. For heterotic orbifolds, a string state with constructing element $\hat{g} \in \hat{S}_{\mathrm{Narain}}$ finds its $\mathcal{CP}$-partner in a string state with constructing element $\hat{g}^{-1} \in \hat{S}_{\mathrm{Narain}}$. Hence, a $\mathcal{CP}$-like transformation is an outer automorphism of the Narain space group $\hat{S}_{\mathrm{Narain}}$, such that

$$\hat{g} \xrightarrow{\mathcal{CP}} \left(\hat{\mathcal{CP}},0\right)^{-1} \hat{g}\left(\hat{\mathcal{CP}},0\right) \stackrel{!}{=} \hat{g}^{-1} \,, \tag{109}$$

for all constructing elements $\hat{g} \in \hat{S}_{\text{Narain}}$. This means in particular that the Narain twist is mapped to its inverse under a $\mathcal{CP}$-like transformation. In detail, in the Narain lattice basis we have to impose the conditions

$$\hat{\Theta} \xrightarrow{\mathcal{CP}} \left(\hat{\mathcal{CP}}\right)^{-1} \hat{\Theta}\left(\hat{\mathcal{CP}}\right) \stackrel{!}{=} \hat{\Theta}^{-1} \quad \text{and} \quad \hat{\mathcal{CP}} \in \mathrm{O}_{\hat{\eta}}(6,6,\mathbb{Z}) \,. \tag{110}$$

As in section 4.1, we now assume our six-dimensional orbifold geometry $\mathbb{T}^6/\mathbb{Z}_K$ under consideration to be factorized, i.e. $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$. In more detail, we take the $\mathbb{Z}_K$ Narain twist $\hat{\Theta}$ to be built out of three two-dimensional orbifold sectors

$$\hat{\Theta} := \hat{\Theta}_{(K_1)} \oplus \hat{\Theta}_{(K_2)} \oplus \hat{\Theta}_{(K_3)} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) \times \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) \times \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) \subset \mathrm{O}_{\hat{\eta}}(6,6,\mathbb{Z}) \,, \tag{111}$$

see eq. (92). Then, eq. (110) is solved by

$$\hat{\Theta}_{(K_i)} \xrightarrow{\mathcal{CP}} \left(\hat{\mathcal{CP}}_{(K_i)}\right)^{-1} \hat{\Theta}_{(K_i)}\left(\hat{\mathcal{CP}}_{(K_i)}\right) \stackrel{!}{=} \left(\hat{\Theta}_{(K_i)}\right)^{-1} \,, \tag{112}$$

where

$$\hat{\mathcal{CP}} := \hat{\mathcal{CP}}_{(K_1)} \oplus \hat{\mathcal{CP}}_{(K_2)} \oplus \hat{\mathcal{CP}}_{(K_3)} \quad \text{and} \quad \hat{\mathcal{CP}}_{(K_i)} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) \,. \tag{113}$$

This means that for each complex plane $i \in \{1,2,3\}$ (or each $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sector) on which a Narain twist $\hat{\Theta}_{(K_i)}$ of order $K_i \in \{2,3,4,6\}$ acts, we look for a corresponding $\mathcal{CP}$-like transformation that we call $\hat{\mathcal{CP}}_{(K_i)}$. In addition to eq. (112), a two-dimensional $\mathcal{CP}$-like transformation has to act on the Kähler modulus T_i of the $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sector as

$$T_i \xrightarrow{\mathcal{CP}} -\bar{T}_i \,, \tag{114}$$

and in addition, if $K_i = 2$, on the complex structure modulus as $U_i \to -\bar{U}_i$, see ref. [17,37,38]. The results are given in Table 3. Hence, the six-dimensional $\mathcal{CP}$-like transformation given by eq. (113) acts simultaneously on all six dimensions of the compactified space, see also ref. [39].

At first sight, there seems to be a special case if $K_i = 2$, i.e. if the six-dimensional orbifold geometry contains a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector. Let us discuss an example, where the first complex plane contains a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector ($K_1 = 2$), i.e.

$$\hat{\Theta} := \hat{\Theta}_{(2)} \oplus \hat{\Theta}_{(K_2)} \oplus \hat{\Theta}_{(K_3)} \,, \tag{115}$$

and $K_i \neq 2$ for $i \in \{2,3\}$. Then, we can define two independent transformations

$$\hat{\mathcal{CP}} := \hat{\mathcal{CP}}_{(2)} \oplus \hat{\mathcal{CP}}_{(K_2)} \oplus \hat{\mathcal{CP}}_{(K_3)} \quad \text{and} \quad \hat{\mathcal{CP}}' := \hat{\mathcal{CP}}_{(2)} \oplus \mathbb{1}_4 \oplus \mathbb{1}_4 \,, \tag{116}$$

with $\hat{\mathcal{CP}}_{(2)} = \hat{\Sigma}_*$ given in Table 3. Both transformations, $\hat{\mathcal{CP}}$ and $\hat{\mathcal{CP}}'$, satisfy eq. (110) since $(\hat{\Theta}_{(2)})^{-1} = \hat{\Theta}_{(2)}$ for $\hat{\Theta}_{(2)} := -\mathbb{1}_4$. However, only $\hat{\mathcal{CP}}$ is a consistent transformation but not $\hat{\mathcal{CP}}'$: $\hat{\mathcal{CP}}'$ acts nontrivially on the Kähler modulus T_1 and the complex structure modulus U_1 associated with the first complex plane of the six-dimensional orbifold, but leaves the moduli associated with the four orthogonal compact dimensions invariant. Nonetheless, the superpotential $\mathcal{W}$ has to be a holomorphic function of all fields (matter fields *and* moduli) and after $\mathcal{CP}$-conjugation it needs to

Table 3
$\mathcal{CP}$-like transformations in the Narain formulation of two-dimensional $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sectors for $K_i \in \{2, 3, 4, 6\}$.

Order K_i	Orbifold twist $\hat{\Theta}_{(K_i)} \in O_{\hat{\eta}}(2,2,\mathbb{Z})$	$\hat{\mathcal{CP}}_{(K_i)} \in O_{\hat{\eta}}(2,2,\mathbb{Z})$	Action of $\hat{\mathcal{CP}}_{(K_i)}$ on moduli
2	$\hat{\Theta}_{(2)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\hat{\Sigma}_* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$T_i \to -\bar{T}_i$ $U_i \to -\bar{U}_i$
3	$\hat{\Theta}_{(3)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\hat{K}_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$T_i \to -\bar{T}_i$ $\langle U_i \rangle = \omega$ invariant
4	$\hat{\Theta}_{(4)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\hat{\Sigma}_* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$T_i \to -\bar{T}_i$ $\langle U_i \rangle = i$ invariant
6	$\hat{\Theta}_{(6)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	$\hat{K}_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$T_i \to -\bar{T}_i$ $\langle U_i \rangle = \omega$ invariant

become anti-holomorphic. Therefore, assuming that there are some superpotential couplings that depend on moduli associated with all six compact dimensions, $\mathcal{CP}$ has to act nontrivially in all three complex planes of the six-dimensional orbifold simultaneously. In this case, $\mathcal{CP}$ is broken spontaneously if any modulus departs from one of its $\mathcal{CP}$-conserving points in moduli space.

4.4. Summary

Many properties of factorized six-dimensional orbifolds can be understood in terms of three two-dimensional $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sectors, $i \in \{1, 2, 3\}$, as we show in section 4.1. Nevertheless, these six-dimensional orbifolds display a richer structure compared to two-dimensional ones. In particular, we observe two important new features:

i) Inner automorphisms of a Narain space group of a two-dimensional orbifold sector can become *outer* automorphisms of a Narain space group of a suitable six-dimensional orbifold. This applies especially to two-dimensional sublattice rotations of six-dimensional orbifolds, which are given by (unbroken) $\mathrm{SL}(2, \mathbb{Z})_{U_i}$ modular transformations. Interestingly, this uncovers an unexpected relation, eq. (105), between the R-charge R_i of a matter field associated with a sublattice rotation in the i-th two-torus and its $\mathrm{SL}(2, \mathbb{Z})_{U_i}$ modular weight.

ii) $\mathcal{CP}$-like transformations in six-dimensional factorized orbifolds must act simultaneously in all two-dimensional orbifold sectors, in order to not spoil the holomorphicity of the superpotential. This is based on the assumption that at least one superpotential coupling depends on the moduli from all three two-dimensional orbifold sectors. We list in Table 3 the $\mathcal{CP}$-like transformations for all $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sectors ($K_i \in \{2, 3, 4, 6\}$).

5. R-symmetries and $\mathcal{CP}$ in a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector

The four-dimensional effective field theory (obtained from a heterotic compactification on an orbifold geometry that contains a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector without discrete Wilson lines) is equipped with various types of symmetries:

- a $\Delta(54)$ traditional flavor symmetry,
- an $\mathrm{SL}(2,\mathbb{Z})_T$ modular symmetry that acts as a T' finite modular group on matter fields and couplings,
- a $\mathbb{Z}_9^R$ discrete R-symmetry,
- a $\mathcal{CP}$-like transformation,

see also refs. [30–34,40–44]. These discrete symmetries combine to the eclectic flavor symmetry $\Omega(2) \cong [1944, 3448]$. Furthermore, as we have seen in the previous sections, all of them have a common origin from string theory: they can be defined as outer automorphisms of the Narain space group [12,17,18]. This novel approach has revealed several interconnections between discrete symmetries of the various types, with the ultimate aim to connect bottom-up flavor model building with a consistent top-down approach from strings.

In this section, we first recall the main results from the analysis of the effective field theory obtained from a heterotic string compactification on a $\mathbb{T}^6/\mathbb{Z}_K$ orbifold geometry that contains a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, for example in the first complex plane. The moduli in this plane are denoted for simplicity by T and U (without subindex). In addition, we focus on three copies (labeled by an index $i \in \{1, 2, 3\}$) of twisted matter fields (X_i, Y_i, Z_i) without string oscillator excitations. These matter fields are localized in the extra dimensions at the three fixed points of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. We call them collectively $\Phi^i_{-2/3} = (X_i, Y_i, Z_i)^{\mathrm{T}}$ for $i \in \{1, 2, 3\}$, where $n_T = -2/3$ is the modular weight with respect to $\mathrm{SL}(2,\mathbb{Z})_T$. In this section, we will obtain new results concerning the origin and nature of discrete R-symmetries and $\mathcal{CP}$, related to the modular symmetries of the theory.

5.1. Modular symmetry in a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector

One defines a so-called finite modular group Γ'_N using the relations (1) combined with the additional constraint

$$\mathrm{T}^N = \mathbb{1}, \tag{117}$$

which renders the group finite. For the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, we are interested in the case $N = 3$ that corresponds to the finite modular group

$$T' \cong \Gamma'_3 \cong \mathrm{SL}(2, 3). \tag{118}$$

Then, a modular form of T' with modular weight $n_Y \in \mathbb{N}$ is defined as a (vector of) holomorphic function(s) of the Kähler modulus T that transforms under general modular transformations $\gamma_T \in \mathrm{SL}(2,\mathbb{Z})_T$ as

$$\hat{Y}^{(n_Y)}_{s_Y}(T) \xrightarrow{\gamma_T} \hat{Y}^{(n_Y)}_{s_Y}\left(\frac{a_T\, T + b_T}{c_T\, T + d_T}\right) = (c_T\, T + d_T)^{n_Y} \rho_{s_Y}(\gamma_T)\, \hat{Y}^{(n_Y)}_{s_Y}(T). \tag{119}$$

Here, s_Y denotes an s-dimensional representation of T' and $(c_T\, T + d_T)^{n_Y}$ is the automorphy factor of γ_T with modular weight n_Y.

It turns out that there exists only one (two-component) modular form of weight $n_Y = 1$

$$\hat{Y}_{\mathbf{2}''}^{(1)}(T) := \begin{pmatrix} \hat{Y}_1(T) \\ \hat{Y}_2(T) \end{pmatrix} := \begin{pmatrix} -3\sqrt{2}\frac{\eta^3(3T)}{\eta(T)} \\ 3\frac{\eta^3(3T)}{\eta(T)} + \frac{\eta^3(T/3)}{\eta(T)} \end{pmatrix}. \tag{120}$$

Using eq. (119) it can be checked explicitly that this modular form transforms in a two-dimensional (unitary) representation $\mathbf{2}''$ of T', where

$$\rho_{\mathbf{2}''}(\hat{K}_{\mathrm{S}}) := -\frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \quad \text{and} \quad \rho_{\mathbf{2}''}(\hat{K}_{\mathrm{T}}) := \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}. \tag{121}$$

Note that we use the T' conventions from ref. [45] (with $p = \mathrm{i}$).

In addition, twisted matter fields $\Phi^i_{-2/3} = (X_i, Y_i, Z_i)^{\mathrm{T}}$ carry a modular weight $n_T = -2/3$ and build a representation $\mathbf{s} = \mathbf{2}' \oplus \mathbf{1}$ of T' for $i \in \{1, 2, 3\}$, as shown in ref. [18], i.e.

$$\Phi^i_{-2/3} \xrightarrow{\gamma_T} \Phi^{i\prime}_{-2/3} := (c_T\, T + d_T)^{-2/3}\, \rho_{\mathbf{s}}(\gamma_T)\, \Phi^i_{-2/3}\,, \tag{122}$$

for $\gamma_T \in \mathrm{SL}(2, \mathbb{Z})_T$, where the representation matrices $\rho_{\mathbf{s}}(\gamma_T)$ of $\mathrm{SL}(2, \mathbb{Z})_T$ elements γ_T are generated by [3,18]

$$\rho(\hat{K}_{\mathrm{S}}) := \frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \quad \text{and} \quad \rho(\hat{K}_{\mathrm{T}}) := \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{123}$$

5.2. Traditional flavor symmetry in a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector

It is well-known that a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector yields a $\Delta(54)$ traditional flavor symmetry, where twisted matter fields build triplet-representations [40,46]. New insight was gained using the Narain space group and its outer automorphisms [17,18]: In the absence of discrete Wilson lines, there exist two translational outer automorphisms, A and B, of the $\mathbb{T}^2/\mathbb{Z}_3$ Narain space group,

$$\mathrm{A} := (\mathbb{1}, \hat{T}_1)\,, \ \ \mathrm{B} := (\mathbb{1}, \hat{T}_2)\,, \quad \text{where} \quad \hat{T}_1 := \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \end{pmatrix}, \ \hat{T}_2 := \begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \tag{124}$$

see the discussion around eq. (42) in section 2.5. Similar to eq. (10), one can see that A and B induce the following geometrical translations

$$y \xrightarrow{\ \mathrm{A}\ } y + \frac{1}{3}\,(e_1 + 2e_2) \quad \text{and} \quad y \xrightarrow{\ \mathrm{B}\ } y\,, \tag{125}$$

as illustrated in Fig. 1a. Note that, in addition to eq. (125), there is a nontrivial action on $\tilde{y}$. These Narain-translations leave the Kähler modulus inert. Hence, they belong to the traditional flavor symmetry. Acting on twisted matter fields $\Phi^i_{-2/3}$, they are represented as

$$\Phi^i_{-2/3} \xrightarrow{\ \mathrm{A}\ } \rho(\mathrm{A})\, \Phi^i_{-2/3} \quad \text{and} \quad \Phi^i_{-2/3} \xrightarrow{\ \mathrm{B}\ } \rho(\mathrm{B})\, \Phi^i_{-2/3}\,, \tag{126}$$
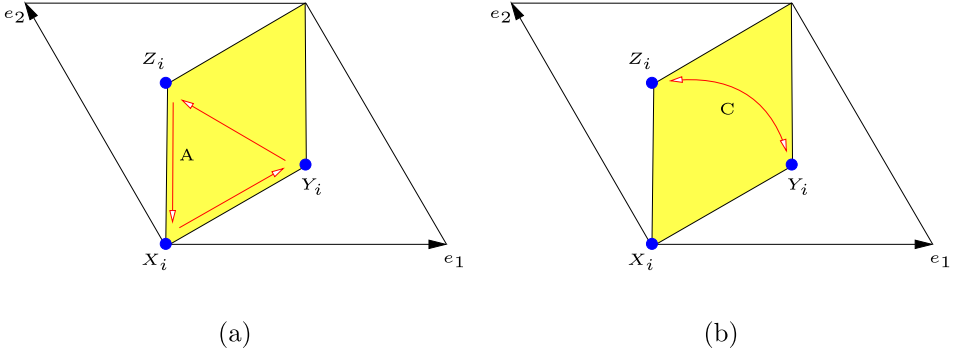
where

Fig. 1. The $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector and the geometrical actions of the Narain outer automorphisms: (a) the $\mathbb{Z}_3$ translation A rotates the three twisted states (X_i, Y_i, Z_i) and (b) the $180°$ rotation C maps X_i to itself and interchanges Y_i and Z_i.

$$\rho(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{127}$$

and $\omega := \exp(2\pi i/3)$. These matrices $\rho(A)$ and $\rho(B)$ generate a $\Delta(27)$ traditional flavor symmetry, where the twisted matter fields $\Phi^i_{-2/3}$ form a triplet. This non-Abelian flavor symmetry includes the well-known $\mathbb{Z}_3^{(\mathrm{PG})} \times \mathbb{Z}_3$ point and space group selection rules of strings that split and join while propagating on the surface of an orbifold [47–49]: In our case of a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, it turns out that the symmetry transformations

$$A^2 B^2 A B \qquad \text{and} \qquad B \tag{128}$$

correspond to the point and space group selection rules, respectively. In detail, for twisted matter fields $\Phi^i_{-2/3} = (X_i, Y_i, Z_i)^{\mathrm{T}}$ (from the first twisted sector), we obtain the transformations

$$\mathbb{Z}_3^{(\mathrm{PG})} \quad : \quad \rho(A)^2 \rho(B)^2 \rho(A) \rho(B) = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{129a}$$

$$\mathbb{Z}_3 \quad : \quad \rho(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{129b}$$

In addition, there is a rotational outer automorphism $C := (-\mathbb{1}_4, 0)$ of the $\mathbb{T}^2/\mathbb{Z}_3$ Narain space group, which corresponds to a geometrical rotation by $180°$ in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. Hence, C generates an R-symmetry. As we have seen in eq. (82), C corresponds to a modular transformation $S^2 \in \mathrm{SL}(2, \mathbb{Z})_U$ from the stabilizer subgroup $H_{\langle U \rangle = \omega}$. For the twisted matter fields, C is represented as

$$\rho(C) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{130}$$

By comparing with eq. (123), we note that $\rho(C) = \rho(\hat{K}_S)^2$, which is indeed a consequence of the more fundamental identity $C = \hat{K}_S^2$, see section 2.2 of ref. [3] for more details. This additional element enhances $\Delta(27)$ to a non-Abelian R-symmetry [50] $\Delta(54)$, where the twisted matter

fields $\Phi^i_{-2/3}$ transform in the representation $\mathbf{3}_2$. Note that the non-diagonal structure of $\rho(\mathrm{C})$ is an example of a non-diagonal representation matrix $\rho_s(\gamma_{(K_i)})$ in eq. (102). Furthermore, one can understand geometrically the fact that Y_i and Z_i get interchanged under the action of C, while X_i is mapped to itself, as illustrated in Fig. 1b.

5.3. Superpotential in a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector

Taking only twisted matter fields from one twisted sector into account, the modular group $\mathrm{SL}(2, \mathbb{Z})_T$, its associated finite modular group T' and the traditional flavor group $\Delta(54)$ constrain the tri-linear superpotential $\mathcal{W} \supset \Phi^1_{-2/3} \Phi^2_{-2/3} \Phi^3_{-2/3}$ of twisted matter fields $\Phi^i_{-2/3}$ for $i \in \{1, 2, 3\}$, such that there is only one free parameter, denoted by $c^{(1)}$. The precise result reads [3]

$$\mathcal{W}(T, X_i, Y_i, Z_i) \supset c^{(1)} \Big[\hat{Y}_2(T)\big(X_1 X_2 X_3 + Y_1 Y_2 Y_3 + Z_1 Z_2 Z_3\big) \tag{131a}$$

$$-\frac{\hat{Y}_1(T)}{\sqrt{2}}\big(X_1 Y_2 Z_3 + X_1 Y_3 Z_2 + X_2 Y_1 Z_3 \tag{131b}$$

$$+ X_3 Y_1 Z_2 + X_2 Y_3 Z_1 + X_3 Y_2 Z_1\big)\Big],$$

As a remark, assuming that $\Phi^1_{-2/3} = (X_1, Y_1, Z_1)^{\mathrm{T}}$ develops a non-trivial vev, this superpotential results in a (symmetric) mass matrix

$$\mathcal{W}(T, X_i, Y_i, Z_i) \supset \begin{pmatrix} X_2 & Y_2 & Z_2 \end{pmatrix} \begin{pmatrix} \hat{Y}_2(T)\langle X_1 \rangle & -\frac{\hat{Y}_1(T)}{\sqrt{2}}\langle Z_1 \rangle & -\frac{\hat{Y}_1(T)}{\sqrt{2}}\langle Y_1 \rangle \\ -\frac{\hat{Y}_1(T)}{\sqrt{2}}\langle Z_1 \rangle & \hat{Y}_2(T)\langle Y_1 \rangle & -\frac{\hat{Y}_1(T)}{\sqrt{2}}\langle X_1 \rangle \\ -\frac{\hat{Y}_1(T)}{\sqrt{2}}\langle Y_1 \rangle & -\frac{\hat{Y}_1(T)}{\sqrt{2}}\langle X_1 \rangle & \hat{Y}_2(T)\langle Z_1 \rangle \end{pmatrix} \begin{pmatrix} X_3 \\ Y_3 \\ Z_3 \end{pmatrix}.$$

$$\tag{132}$$

This observation can be useful in semi-realistic models e.g. to determine the mass textures of quarks and leptons, considering that $\Phi^1_{-2/3}$ corresponds to three Higgs fields coupled to the three generations of quark and lepton superfields.

5.4. $\mathbb{Z}_9{}^R$ R-symmetry in a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector

We consider a Narain space group S_{Narain} of a $\mathbb{T}^6/\mathbb{Z}_K$ orbifold, which contains a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector (for example in the first complex plane). Then, there exists an outer automorphism

$$\hat{R}_1 := \hat{\Theta}_{(3)} \oplus \mathbb{1}_4 \oplus \mathbb{1}_4 \in \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z}) \times \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z}) \times \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z}) \subset \mathrm{O}_{\hat{\eta}}(6, 6, \mathbb{Z}) \tag{133}$$

of the Narain space group S_{Narain}, where

$$\hat{\Theta}_{(3)} := \hat{C}_{\mathrm{S}}\,\hat{C}_{\mathrm{T}} = \begin{pmatrix} \hat{\sigma}_{(3)} & 0 \\ 0 & \hat{\sigma}_{(3)}^{-\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z})\,, \tag{134}$$

acts on the $D = 2$ extra dimensions of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. The columns of the upper 2×2 block $\hat{\sigma}_{(3)} = e^{-1}\theta_{(3)}e$ of $\hat{\Theta}_{(3)}$ can be interpreted as

$$e_1 \xrightarrow{\theta_{(3)}} \theta_{(3)}\,e_1 = e_2 \qquad \text{and} \qquad e_2 \xrightarrow{\theta_{(3)}} \theta_{(3)}\,e_2 = -e_1 - e_2\,, \tag{135}$$

Table 4

Discrete $\mathbb{Z}_9^R$ R-charges R_1 for a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector with trivial Wilson line of: matter fields Φ_{n_T}, the Grassmann number ϑ, the superpotential $\mathcal{W}$, and the Kähler potential K in conventions, similar to those of ref. [32]. The $\mathbb{Z}_9^R$ R-symmetry originates from a $\mathbb{Z}_3$ sublattice rotation of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector in the normalization where all R-charges of bosons (but not of fermions) are integer. Note that the $\mathbb{Z}_3^{(\mathrm{PG})}$ point group symmetry eq. (129a) is a subgroup of $\mathbb{Z}_9^R$ (using that $\vartheta \to -\vartheta$ and fermions $\to -$fermions follows from four-dimensional Lorentz invariance).

	Φ_0	Φ_{-1}	$\Phi_{-2/3}$	$\Phi_{-5/3}$	$\Phi_{-1/3}$	$\Phi_{+2/3}$	ϑ	$\mathcal{W}$	K
R-charges R_1	0	3	1	-2	2	5	$3/2$	3	0

see eqs. (70) with $y = e_i$ and (72). In other words, the two-dimensional sublattice spanned by e_1 and e_2, illustrated for example in Fig. 1, is left invariant by a $120°$ rotation.

As a remark, let us assume that the order three discrete Wilson line in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector is chosen to be non-zero. Then, this $120°$ rotational symmetry remains unbroken. In contrast, the $\mathbb{Z}_2$ outer automorphism $\mathrm{C} := (-\mathbb{1}_4, 0)$, discussed in section 5.2, is broken in this case.

As we have seen in section 3.2, the outer automorphism eq. (133) translates to a modular transformation

$$\gamma_{(3)} := \mathrm{S}\,\mathrm{T} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \in H_{\langle U \rangle = \omega} \subset \mathrm{SL}(2, \mathbb{Z})_U \,, \tag{136}$$

which leaves (by definition of the stabilizer subgroup $H_{\langle U \rangle = \omega}$) the complex structure modulus U of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector invariant,

$$U \xrightarrow{\gamma_{(3)}} \frac{1}{-U - 1} \overset{!}{=} U \quad \text{at} \quad \langle U \rangle = \omega \,. \tag{137}$$

Finally, the superpotential transforms under the modular transformation $\gamma_{(3)} \in \mathrm{SL}(2, \mathbb{Z})_U$ as

$$\mathcal{W} \xrightarrow{\gamma_{(3)}} (-\langle U \rangle - 1)^{-1} \mathcal{W} = \omega \mathcal{W} = \exp\left(2\pi \mathrm{i}\, v_{(3)}\right) \mathcal{W} \,, \qquad \text{where} \qquad v_{(3)} := 1/3 \,, \tag{138}$$

using that the complex structure modulus of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector is frozen at $\langle U \rangle = \omega$.

This sublattice rotation $\hat{\mathrm{R}}_1$ acts on matter fields as

$$\Phi_{n_T} \xrightarrow{\hat{\mathrm{R}}_1} \exp(2\pi \mathrm{i}\, R_1/9) \Phi_{n_T} \,, \tag{139}$$

where the R-charges R_1 are normalized to be integer for superfields, see Table 4. Hence, the $\mathbb{Z}_3$ sublattice rotation $\hat{\mathrm{R}}_1$ leads to a $\mathbb{Z}_9^R$ R-symmetry of the theory, where the superpotential eq. (131) transforms according to eq. (138) with a $\mathbb{Z}_9^R$ R-charge 3. Note that fermions have half-integer R-charges. Hence, this symmetry acts as a $\mathbb{Z}_{18}^R$ symmetry of the fermionic sector [32]. In more detail, one can show that the Grassmann number ϑ of $\mathcal{N} = 1$ superspace has a $\mathbb{Z}_9^R$ R-charge $R(\vartheta) = 3/2$. Furthermore, using $\int \mathrm{d}\vartheta\, \vartheta = 1$ we know that the R-charge of $\int \mathrm{d}\vartheta$ is $-3/2$, such that the Lagrangian $\mathcal{L} \supset \int \mathrm{d}^2\vartheta\, \mathcal{W}$ is invariant.

5.5. The full eclectic flavor group without $\mathcal{CP}$ in a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector

Combining the discrete $\mathbb{Z}_9^R$ R-symmetry with the traditional flavor symmetry $\Delta(54)$ yields the *full* traditional flavor symmetry

$$\Delta'(54, 2, 1) \cong \Delta(54) \cup \mathbb{Z}_9^R \cong [162, 44] \,. \tag{140}$$

Table 5

Table from ref. [12]: eclectic flavor group $\Omega(2)$ for orbifolds $\mathbb{T}^6/P$ that contain a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. In this case, $\mathrm{SL}(2,\mathbb{Z})_U$ of the stabilized complex structure modulus $U = \exp(2\pi i/3)$ is broken, resulting in a remnant $\mathbb{Z}_9^R$ R-symmetry. Including $\mathbb{Z}_9^R$ enhances the traditional flavor group $\Delta(54)$ to $\Delta'(54,2,1) \cong [162,44]$ and, thereby, the eclectic group to $\Omega(2) \cong [1944,3448]$. Note that $\Omega(1) \subset \Omega(2)$.

Nature of symmetry		Outer automorphism of Narain space group	Flavor groups				
eclectic	modular	rotation $S \in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_4$		T'		$\Omega(2)$
		rotation $T \in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_3$				
	traditional flavor	translation A	$\mathbb{Z}_3$	$\Delta(27)$	$\Delta(54)$	$\Delta'(54,2,1)$	
		translation B	$\mathbb{Z}_3$				
		rotation $C = S^2 \in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_2^R$				
		rotation $\hat{R}_1 \Leftrightarrow \gamma_{(3)} \in \mathrm{SL}(2,\mathbb{Z})_U$	$\mathbb{Z}_9^R$				

Hence, the full eclectic flavor group of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector without $\mathcal{CP}$ is given by

$$\Omega(2) \cong \Omega(1) \cup \mathbb{Z}_9^R \cong [1944,3448], \tag{141}$$

where $\Omega(1) \cong \Delta(54) \cup T'$, see ref. [51] for the nomenclature. The various groups and their origins in string theory are summarized in Table 5. Note that $\Omega(2)$ contains the $\Delta(54)$ traditional flavor group and all of its enhancements (without $\mathcal{CP}$) to the various unified flavor groups at special points in T moduli space, as we shall discuss later in section 6.

5.6. $\mathcal{CP}$ in a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector

For heterotic orbifold compactifications, the modular group $\mathrm{SL}(2,\mathbb{Z})_T$ of the Kähler modulus T originates from outer automorphisms of the Narain space group, as we reviewed in section 2. For a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, it can be enhanced naturally by an additional outer automorphism

$$\hat{K}_* := \hat{C}_S\,\hat{C}_T\,\hat{C}_S\,\hat{\Sigma}_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{142}$$

that acts $\mathcal{CP}$-like [17,18], i.e.

$$T \xrightarrow{\hat{K}_*} -\bar{T} \quad \text{and} \quad U \xrightarrow{\hat{K}_*} -\frac{\bar{U}}{1+\bar{U}} = U \quad \text{at} \quad U = \langle U \rangle = \omega, \tag{143}$$

see also refs. [37,39,52]. Following the discussion in section 3.1 we see that $\hat{K}_*$ acts geometrically as

$$y \xrightarrow{\hat{K}_*} \sigma\, y \quad \text{using} \quad \sigma = e\,\hat{\sigma}\,e^{-1} \quad \text{and} \quad \hat{\sigma} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}. \tag{144}$$

This gives rise to a reflection at the line perpendicular to e_2, as illustrated in Fig. 2. Finally, note that $\mathcal{CP}$ has to act in all extra dimensions simultaneously, following our discussion in section 4.3.
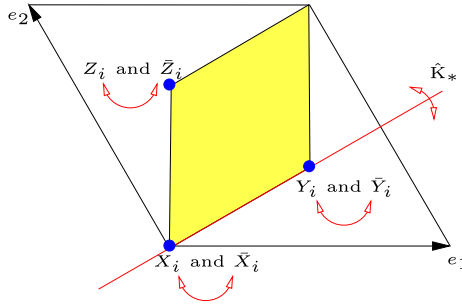
Fig. 2. The fundamental domain of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold is depicted in yellow and the three inequivalent fixed-points are blue. (X, Y, Z) denote three (left-chiral) twisted strings from the first twisted sector $\hat{\Theta}$, while $(\bar{X}, \bar{Y}, \bar{Z})$ are three (right-chiral) twisted strings from the second twisted sector $\hat{\Theta}^2$. The $\mathcal{CP}$-like transformation $\hat{K}_*$ acts as $(\hat{K}_*)^{-1} \hat{\Theta} \hat{K}_* = \hat{\Theta}^2$ in agreement with eq. (110). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

5.7. $\mathcal{CP}$ as a modular symmetry

On the level of 2×2 matrices, the $\mathcal{CP}$-like transformation with $\hat{K}_*$ can be represented by

$$\mathrm{K}_* := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{145}$$

Since $\det(\mathrm{K}_*) = -1$, it enhances the modular group $\mathrm{SL}(2, \mathbb{Z})_T$ to $\mathrm{GL}(2, \mathbb{Z})_T$. Then, the modular transformations in eq. (27) of the Kähler modulus T are complemented by

$$T \xrightarrow{\gamma} \frac{a\,\bar{T} + b}{c\,\bar{T} + d} \quad \text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})_T \quad \text{with} \quad \det(\gamma) = -1 , \tag{146}$$

see also ref. [38] for a bottom-up derivation.

Moreover, the transformation of matter fields eq. (44) can be generalized to $\mathcal{CP}$-like transformations (with $\det(\gamma) = -1$) as

$$\Phi_{n_T} \xrightarrow{\gamma} \bar{\Phi}_{n_T}{}' := (c\,\bar{T} + d)^{n_T} \rho_s(\gamma)\, \bar{\Phi}_{n_T} , \tag{147}$$

where the field $\bar{\Phi}_{n_T}$ on the right-hand side is evaluated at the parity-transformed point in space-time. In addition, as derived in ref. [18] for our concrete case of a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, twisted matter fields transform under the $\mathcal{CP}$-like generator $\hat{K}_*$ as

$$\begin{pmatrix} \Phi_{n_T} \\ \bar{\Phi}_{n_T} \end{pmatrix} \xrightarrow{\hat{K}_*} \rho_s(\hat{K}_*) \begin{pmatrix} \Phi_{n_T} \\ \bar{\Phi}_{n_T} \end{pmatrix} := \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix} \begin{pmatrix} \Phi_{n_T} \\ \bar{\Phi}_{n_T} \end{pmatrix} , \tag{148}$$

where we used that the automorphy factor $(c\,\bar{T} + d)^{n_T}$ in eq. (147) equals 1 for K_* defined in eq. (145).

Consequently, one can show that the finite modular group $T' \cong \mathrm{SL}(2, 3)$ gets enhanced by $\mathcal{CP}$ to $\mathrm{GL}(2, 3)$. Simultaneously, the eclectic flavor group $\Omega(2)$ is enhanced by $\mathcal{CP}$ to a group of order 3,888 (while the $\Omega(1)$ subgroup of $\Omega(2)$ is enhanced to [1296, 2891] by $\mathcal{CP}$).

Modular forms are also affected by the inclusion of $\mathcal{CP}$-like modular transformations. Under a $\mathcal{CP}$-like transformation $T \to -\bar{T}$, we find for the Dedekind η-function $\eta(r\,T) \to \eta(-r\,\bar{T}) = (\eta(r\,T))^*$ if $r \in \mathbb{R}$. Hence, the weight 1 modular forms of T' defined in eq. (120) transform as

$$\hat{Y}_1(T) \xrightarrow{\hat{K}_*} \hat{Y}_1(-\bar{T}) = \left(\hat{Y}_1(T)\right)^* , \tag{149a}$$

$$\hat{Y}_2(T) \xrightarrow{\hat{K}_*} \hat{Y}_2(-\bar{T}) = \left(\hat{Y}_2(T)\right)^* , \tag{149b}$$

cf. ref. [53]. We have chosen here a basis of T' where all Clebsch–Gordan coefficients in T' tensor products are real (see refs. [45,54] with $p = \mathrm{i}$, $p_1 = 1$, and $p_2 = -1$). Therefore, all irreducible T' modular forms of modular weight $n_Y \geq 1$ are complex conjugated by $\hat{K}_*$, analogously to eqs. (149), i.e.

$$\hat{Y}_s^{(n_Y)}(T) \xrightarrow{\hat{K}_*} \hat{Y}_s^{(n_Y)}(-\bar{T}) = \left(\hat{Y}_s^{(n_Y)}(T)\right)^* . \tag{150}$$

This can be used to generalize eq. (119) to all elements $\gamma \in \mathrm{GL}(2, \mathbb{Z})_T$ with $\det(\gamma) = -1$ as

$$\hat{Y}_s^{(n_Y)}(T) \xrightarrow{\gamma} \hat{Y}_s^{(n_Y)}\left(\frac{a\,\bar{T} + b}{c\,\bar{T} + d}\right) = (c\,\bar{T} + d)^{n_Y} \rho_s(\gamma) \left(\hat{Y}_s^{(n_Y)}(T)\right)^* , \tag{151}$$

where we have $\rho_s(\hat{K}_*) = \mathbb{1}$ in our basis of $\mathrm{GL}(2, 3)$ modular forms. Consequently, $T' \cong \mathrm{SL}(2, 3)$ modular forms naturally become modular forms of the $\mathcal{CP}$-extended finite modular group $\mathrm{GL}(2, 3)$.

Finally, we discuss the transformation of the superpotential $\mathcal{W}$ and the Kähler potential K under general $\mathcal{CP}$-like transformations for a $\mathcal{CP}$ invariant theory. Again, we take $\gamma \in \mathrm{GL}(2, \mathbb{Z})_T$ with $\det(\gamma) = -1$. Then, if the superpotential and the Kähler potential transform as

$$\mathcal{W}\left(T, \Phi_{n_T}\right) \xrightarrow{\gamma} \mathcal{W}\left(\frac{a\,\bar{T} + b}{c\,\bar{T} + d}, \bar{\Phi}_{n_T}'\right) = (c\,\bar{T} + d)^{-1} \left(\mathcal{W}(T, \Phi_{n_T})\right)^* , \tag{152a}$$

$$K\left(T, \bar{T}, \Phi_{n_T}, \bar{\Phi}_{n_T}\right) \xrightarrow{\gamma} K\left(\frac{a\,\bar{T} + b}{c\,\bar{T} + d}, \frac{a\,T + b}{c\,T + d}, \bar{\Phi}_{n_T}', \Phi_{n_T}'\right) \tag{152b}$$

$$= K\left(T, \bar{T}, \Phi_{n_T}, \bar{\Phi}_{n_T}\right) + f(T) + f(\bar{T}) , \tag{152c}$$

the theory is $\mathcal{CP}$ invariant, using the transformed matter fields Φ_{n_T}' given in eq. (147) and a Kähler transformation to remove the holomorphic function $f(T) = \ln(c\,T + d)$, see section 2.6. Let us remark that our convention for K_* given in eq. (145) is chosen such that the automorphy factors in eqs. (147), (151) and (152) have a standard form.

Applied to our explicit trilinear superpotential $\mathcal{W}$ eq. (131), we note that $\mathcal{W}$ satisfies the transformation property of a $\mathcal{CP}$ invariant theory. For example, under $\hat{K}_*$

$$\mathcal{W}(T, X_i, Y_i, Z_i) \xrightarrow{\hat{K}_*} \mathcal{W}(-\bar{T}, \bar{X}_i, \bar{Y}_i, \bar{Z}_i) = \left(\mathcal{W}(T, X_i, Y_i, Z_i)\right)^* , \tag{153}$$

provided that the coefficient $c^{(1)}$ is real as can be chosen in our superpotential by appropriate field redefinitions. However, at a generic point in moduli space, the $\mathcal{CP}$-like transformation $\hat{K}_*$ does not leave the vacuum $\langle T \rangle$ invariant, i.e. $\langle T \rangle \xrightarrow{\hat{K}_*} -\langle \bar{T} \rangle \neq \langle T \rangle$, which can indicate spontaneous $\mathcal{CP}$ breaking. Moreover, even though the Clebsch–Gordan coefficients in T' tensor products are real, this is not true for $\Delta(54)$ tensor products: tensor products of $\Delta(54)$ triplets can also yield $\Delta(54)$ doublets with complex Clebsch–Gordan coefficients [45], indicating that $\mathcal{CP}$ can be broken [54]. However, in order to see this effect in our superpotential one has to consider higher order couplings that, from a stringy point of view, originate from the exchange of heavy $\Delta(54)$ doublets, cf. ref. [55].

5.8. Summary

In this section, we illustrate our previous findings on R-symmetries and $\mathcal{CP}$ in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. It is known that this case is endowed with a T' finite modular symmetry and a $\Delta(54)$ traditional flavor symmetry. However, when embedded into a six-dimensional orbifold, sublattice rotations (from $\mathrm{SL}(2,\mathbb{Z})_U$) yield a $\mathbb{Z}_9^R$ R-symmetry, which acts linearly on matter fields, enhancing the traditional flavor symmetry to $\Delta'(54,2,1) \cong [162,44]$. This leads to the largest eclectic symmetry without $\mathcal{CP}$, being $\Omega(2) \cong \Delta'(54,2,1) \cup T' \cong [1944,3448]$. On the other hand, the modular group in $\mathbb{T}^2/\mathbb{Z}_3$ orbifolds is naturally enhanced by a $\mathcal{CP}$-like generator K_* that promotes $\mathrm{SL}(2,\mathbb{Z})_T$ to $\mathrm{GL}(2,\mathbb{Z})_T$. We confirm that a general modular invariant theory is also invariant under $\mathcal{CP}$-like transformations from $\mathrm{GL}(2,\mathbb{Z})_T$ (cf. eq. (152)), provided that matter fields and modular forms transform according to eqs. (147) and (151), respectively. Note that these conditions are expected to be valid also in general bottom-up models involving modular symmetries with $\mathcal{CP}$. Including $\mathcal{CP}$, the full eclectic symmetry of a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector embedded in a higher-dimensional orbifold is $\Omega(2) \cup \mathbb{Z}_2^{\mathcal{CP}}$, which is a symmetry of order 3, 888.

6. Local flavor unification

In this section, we discuss first in general the mechanism of local flavor unification [18] for $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sectors, taking into account the effect of their embedding in six-dimensional orbifolds. We will then focus on the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector as our working example, where we additionally study the implications of this scheme on the superpotential.

Consider a $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sector, whose complex structure and Kähler moduli shall be denoted here as U and T (instead of U_i and T_i), in order to simplify the notation as in section 5. Let us focus on the case $K_i \neq 2$, for which the U modulus is geometrically fixed at a value $\langle U \rangle$ according to Table 2. (Some of the differences appearing in the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, which has been studied in ref. [19], shall be briefly mentioned below.) We focus first on modular transformations that are not $\mathcal{CP}$-like and assume that the corresponding Kähler modulus T has been stabilized [56,57] at a self-dual point in moduli space. In other words, we suppose that T is stabilized at a vacuum expectation value (vev) $\langle T \rangle$, at which there exists a nontrivial finite [58] subgroup $H_{\langle T \rangle} \subset \mathrm{SL}(2,\mathbb{Z})_T$ that leaves $\langle T \rangle$ invariant. Just as $H_{\langle U \rangle} \subset \mathrm{SL}(2,\mathbb{Z})_U$ defined in eq. (81), $H_{\langle T \rangle}$ is also a modular stabilizer subgroup, defined as

$$H_{\langle T \rangle} = \left\{ \begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_T \;\middle|\; \frac{a_T\,\langle T \rangle + b_T}{c_T\,\langle T \rangle + d_T} = \langle T \rangle \right\}, \tag{154}$$

i.e. $\langle T \rangle$ is a fixed point of the stabilizer subgroup $H_{\langle T \rangle}$ of $\mathrm{SL}(2,\mathbb{Z})_T$. Hence, the vev $\langle T \rangle$ of the stabilized Kähler modulus spontaneously breaks $\mathrm{SL}(2,\mathbb{Z})_T$ to an unbroken finite subgroup $H_{\langle T \rangle}$. If this subgroup is nontrivial, there are three main consequences:

i) On the one hand, the vev $\langle T \rangle$ of the Kähler modulus and, hence, all couplings $\hat{Y}_s^{(n_Y)}(\langle T \rangle)$ are invariant under $\gamma_T \in H_{\langle T \rangle}$. Thus, for all $\gamma_T \in H_{\langle T \rangle}$ we have

$$\hat{Y}_s^{(n_Y)}(\langle T \rangle) \xrightarrow{\;\gamma_T\;} \hat{Y}_s^{(n_Y)}\left(\frac{a_T\langle T \rangle + b_T}{c_T\langle T \rangle + d_T}\right) = (c_T\,\langle T \rangle + d_T)^{n_Y} \rho_s(\gamma_T)\,\hat{Y}_s^{(n_Y)}(\langle T \rangle) = \hat{Y}_s^{(n_Y)}(\langle T \rangle), \tag{155}$$

using eqs. (119) and (154). This implies that the couplings $\hat{Y}_s^{(n_Y)}(\langle T \rangle)$ become eigenvectors of $\rho_s(\gamma_T)$ with eigenvalues $(c_T\,\langle T \rangle + d_T)^{-n_Y}$. Since the representation matrices $\rho_s(\gamma_T)$ have finite order, the eigenvalues of $\rho_s(\gamma_T)$ are pure phases, especially

$$\left| (c_T \langle T \rangle + d_T)^{-n_Y} \right|^2 \;=\; 1 \qquad \Rightarrow \qquad |c_T \langle T \rangle + d_T|^2 \;=\; 1 \,. \tag{156}$$

Compared to traditional flavon models, eq. (155) corresponds to the mechanism of flavon vev alignment, see also refs. [53,59,60]. In this scenario, since the vev $\langle T \rangle$ of the Kähler modulus breaks some modular symmetry but leaves the traditional flavor symmetry unbroken, an additional flavon will be needed for its breaking.

ii) On the other hand, matter fields $\Phi_{\mathfrak{n}}$ with $\mathrm{SL}(2, Z)_T$ modular weight $n_T \in \mathfrak{n}$ (where $\mathfrak{n}$ denotes the set of all modular weights of the matter field $\Phi_{\mathfrak{n}}$) transform in general nontrivially under $\gamma_T \in H_{\langle T \rangle}$, according to

$$\Phi_{\mathfrak{n}} \xrightarrow{\;\gamma_T\;} \Phi_{\mathfrak{n}}{}' \;=\; \rho_{t,\langle T \rangle}(\gamma_T)\,\Phi_{\mathfrak{n}} \;:=\; (c_T \langle T \rangle + d_T)^{n_T}\,\rho_s(\gamma_T)\,\Phi_{\mathfrak{n}} \,, \tag{157}$$

see eq. (44). Since the moduli are invariant under a modular transformation $\gamma_T \in H_{\langle T \rangle}$, we realize that γ_T is linearly realized. Thus, the transformation matrix $\rho_{t,\langle T \rangle}(\gamma_T)$ enhances the traditional flavor symmetry (now, also containing the discrete $\mathbb{Z}^R_{(K_i)^2}$ R-symmetry that originates from $\mathrm{SL}(2, \mathbb{Z})_U$ as discussed in section 4.2) to the so-called *unified flavor symmetry* [18] at the point $\langle T \rangle$ in moduli space. Moreover, the transformation matrix $\rho_{t,\langle T \rangle}(\gamma_T)$ builds a t-dimensional representation t of the unified flavor group. It is important to stress the presence of the extra phase in $\rho_{t,\langle T \rangle}(\gamma_T)$ that originates from the automorphy factor $(c_T \langle T \rangle + d_T)^{n_T}$ in eq. (157).

iii) In addition, the superpotential transforms under $\gamma_T \in H_{\langle T \rangle}$, according to eq. (46), as

$$\mathcal{W}(\langle T \rangle, \langle U \rangle, \Phi_{\mathfrak{n}}) \xrightarrow{\;\gamma_T\;} \mathcal{W}\!\left(\langle T \rangle, \langle U \rangle, \Phi_{\mathfrak{n}}'\right) \;=\; (c_T \langle T \rangle + d_T)^{-1}\,\mathcal{W}(\langle T \rangle, \langle U \rangle, \Phi_{\mathfrak{n}}) \,, \tag{158}$$

where $(c_T \langle T \rangle + d_T)^{-1}$ is a complex phase, see eq. (156), and $\langle U \rangle$ is the fixed value of the complex structure U of the $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sector, see Table 2. By definition of $H_{\langle T \rangle}$, the vev $\langle T \rangle$ is invariant under a transformation $\gamma_T \in H_{\langle T \rangle}$. Thus, also the Kähler potential is invariant under $\gamma_T \in H_{\langle T \rangle}$. Since the Lagrangian $\mathcal{L} \supset \int \mathrm{d}^2\vartheta\,\mathcal{W}$ must be invariant under these transformations as well, the Grassmann number ϑ must transform with a phase

$$\vartheta \xrightarrow{\;\gamma_T\;} (c_T \langle T \rangle + d_T)^{-1/2}\,\vartheta \,, \tag{159}$$

cf. section 4. As a consequence, unbroken modular transformations $\gamma_T \in H_{\langle T \rangle}$ generate R-symmetries at self-dual points in moduli space if $c_T \langle T \rangle + d_T \neq 1$. These R-symmetries appear in addition to those arising from $\mathrm{SL}(2, \mathbb{Z})_U$ discussed in section 3.4.

Taking into account the discussion of section 5.7, one can extend these observations to further include $\mathcal{CP}$. In particular, it is possible to define $H_{\langle T \rangle}$ as the subgroup of $\mathrm{GL}(2, \mathbb{Z})_T$ (instead of $\mathrm{SL}(2, \mathbb{Z})_T$) that leaves the stabilized Kähler modulus $\langle T \rangle$ invariant, see eq. (146). In this case, the unified flavor symmetry can include $\mathcal{CP}$-like transformations generated by the representations of the generators shown in Table 3 combined with some $\mathrm{SL}(2, \mathbb{Z})_T$ elements, endowing the theory with a $\mathcal{CP}$ symmetry. Note that in these terms, one could envisage a scenario in which $\mathcal{CP}$ violation occurs as a transition from a point $\langle T \rangle$ in moduli space where $H_{\langle T \rangle}$ contains an unbroken $\mathcal{CP}$-like symmetry to another point where no $\mathcal{CP}$-like transformation stabilizes $\langle T \rangle$.

In summary, taking into account in the $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sector the traditional flavor symmetry G_{fl}, the discrete $\mathbb{Z}^R_{(K_i)^2}$ R-symmetry from $\mathrm{SL}(2, \mathbb{Z})_U$, and the stabilizer subgroup $H_{\langle T \rangle}$ which may also include $\mathcal{CP}$-like transformations, the unified flavor symmetry is given by the multiplicative closure of these groups. Thus, the unified flavor symmetry can be expressed as
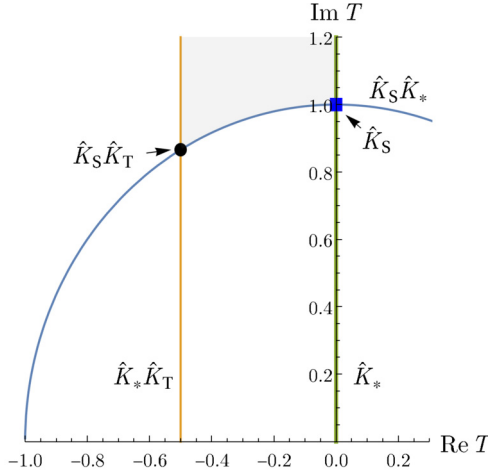
Fig. 3. Fixed points and curves in the moduli space of the Kähler modulus $T = \mathrm{Re}\,T + \mathrm{i}\,\mathrm{Im}\,T$. The point $\langle T \rangle = \mathrm{i}$ (blue square) is left fixed by the modular $\hat{K}_\mathrm{S}$ transformation, whereas $\hat{K}_\mathrm{S}\hat{K}_\mathrm{T}$ leaves $\langle T \rangle = \omega$ (black bullet) invariant. These points lie at the intersections of the curves left invariant by the $\mathrm{GL}(2,\mathbb{Z})$ elements indicated next to them. These curves build the boundary of the fundamental domain (shaded region) of the moduli space in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. We omit the $\hat{K}_\mathrm{S}^2$ modular symmetry, which acts trivially on the whole moduli space and is included in the traditional flavor group.

$$G_{\mathrm{uf},\langle T \rangle} \;\cong\; G_{\mathrm{fl}} \;\cup\; \mathbb{Z}^R_{(K_i)^2} \;\cup\; H_{\langle T \rangle} \;\subset\; G_{\mathrm{eclectic}}\,, \tag{160}$$

where also the transformations from $\mathbb{Z}^R_{(K_i)^2} \cup H_{\langle T \rangle}$ are linearly realized in spite of their modular origin.

Let us now spend a few words on the case of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector. As detailed in ref. [19], in this case both T and U moduli are not fixed. However, one can choose special vevs $(\langle T \rangle, \langle U \rangle)$ that are left invariant by a modular stabilizer subgroup $H_{(\langle T \rangle, \langle U \rangle)}$, which, in contrast to $H_{\langle T \rangle}$ in eq. (154), includes elements not only from $\mathrm{SL}(2, \mathbb{Z})_T$, but from the whole modular group of the orbifold. Hence, the unified flavor symmetry in this case can be expressed analogously to eq. (160), replacing $H_{\langle T \rangle}$ by $H_{(\langle T \rangle, \langle U \rangle)}$.

In the following, we illustrate the scheme of local flavor unification by assuming that the Kähler modulus is stabilized at various different points in moduli space of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. This moduli space is illustrated in Fig. 3. We explore first the resulting unified flavor symmetries at the boundaries of the moduli space, depicted by the curves in the figure, where we realize that a unique purely $\mathcal{CP}$ enhancement of the flavor symmetry occurs. Then, we focus on two self-dual points $\langle T \rangle = \mathrm{i}$ and $\langle T \rangle = \omega$, displayed in the Fig. 3 as a (blue) square and a (black) bullet, respectively. Since we suppose the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector to be embedded in a six-dimensional orbifold, our starting point is the traditional flavor symmetry $\Delta'(54, 2, 1) \cong [162, 44] \cong \Delta(54) \cup \mathbb{Z}_9^R$, cf. Table 5. Depending on the value of $\langle T \rangle$ in moduli space, different unbroken modular transformations contribute to enhance this traditional flavor group to various unified flavor symmetries contained in the full eclectic flavor group, $\Omega(2) \cup \mathbb{Z}_2^{\mathcal{CP}}$, which is of order 3,888. A brief summary of these findings is shown in Table 6. Furthermore, we study the details of the twisted matter fields $\Phi^i_{-2/3} = (X_i, Y_i, Z_i)^\mathrm{T}$ and their superpotential eq. (131) compatible with the unbroken symmetries at the different points in moduli space.

Table 6

Unified flavor symmetries at different Kähler modulus vevs $\langle T \rangle$ of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector embedded in a six-dimensional orbifold. The $\mathrm{GL}(2, \mathbb{Z})_T$ modular transformations that leave $\langle T \rangle$ invariant (generating the so-called stabilizer modular subgroup) enhance the traditional flavor symmetry $\Delta'(54, 2, 1) \cong \Delta(54) \cup \mathbb{Z}_9^R$ of the orbifold sector to various large *unified flavor symmetries*, which generically include $\mathcal{CP}$-like transformations. Note that the unified flavor symmetry is unique at all curves depicted in Fig. 3 that build the boundary of the fundamental domain of the moduli space of T. Furthermore, since $(\hat{K}_S)^2$ already belongs to the traditional flavor symmetry, we do not list it here.

$\langle T \rangle$	Stabilizer generators		Unified flavor symmetries	
	Non $\mathcal{CP}$-like	$\mathcal{CP}$-like	Without $\mathcal{CP}$	With $\mathcal{CP}$
i	$\hat{K}_S$	$\hat{K}_*$	$\Xi(2, 2) \cong [324, 111]$	[648, 548]
ω	$\hat{K}_S \hat{K}_T$	$\hat{K}_* \hat{K}_T$	$H(3, 2, 1) \cong [486, 125]$	[972, 469]
$\mathrm{Re}\,\langle T \rangle = 0$		$\hat{K}_*$	$\Delta'(54, 2, 1) \cong [162, 44]$	[324, 125]
$\mathrm{Re}\,\langle T \rangle = -1/2$		$\hat{K}_* \hat{K}_T$	$\Delta'(54, 2, 1) \cong [162, 44]$	[324, 125]
$|\langle T \rangle| = 1$		$\hat{K}_S \hat{K}_*$	$\Delta'(54, 2, 1) \cong [162, 44]$	[324, 125]

6.1. Flavor enhancement at the boundary of the moduli space of $\mathbb{T}^2/\mathbb{Z}_3$

Let us begin the discussion of local flavor unification in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector by studying the flavor symmetry enhancement at the various curves in Fig. 3, which describe the boundary of the fundamental domain in moduli space for the Kähler modulus T. The three curves are defined by the constraints $\mathrm{Re}\,\langle T \rangle = 0$, $\mathrm{Re}\,\langle T \rangle = -1/2$ and $|\langle T \rangle| = 1$, respectively. We find that the stabilizer elements $H_{\langle T \rangle} \subset \mathrm{GL}(2, \mathbb{Z})_T$ compatible with the previous conditions are generated in each case by

$$\text{at } \mathrm{Re}\,\langle T \rangle = 0 \quad : \quad \langle T \rangle \xrightarrow{\hat{K}_*} -\langle \bar{T} \rangle = \langle T \rangle \,, \tag{161a}$$

$$\text{at } \mathrm{Re}\,\langle T \rangle = -1/2 : \quad \langle T \rangle \xrightarrow{\hat{K}_*} -\langle \bar{T} \rangle \xrightarrow{\hat{K}_T} -\langle \bar{T} \rangle - 1 = \langle T \rangle \,, \tag{161b}$$

$$\text{at } |\langle T \rangle| = 1 \quad : \quad \langle T \rangle \xrightarrow{\hat{K}_S} -\frac{1}{\langle T \rangle} \xrightarrow{\hat{K}_*} \frac{1}{\langle \bar{T} \rangle} = \langle T \rangle \,. \tag{161c}$$

Besides, we find the unbroken symmetry $\hat{K}_S^2$ that leaves the whole moduli space invariant and satisfies $\hat{K}_S^2 = C$, becoming a symmetry generator already included in the traditional flavor group. Therefore, we find that the stabilizer subgroup is $H_{\langle T \rangle} = \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathcal{CP}}$. Note that the stabilizer elements that can enhance the traditional flavor group include the $\mathcal{CP}$-like generator $\hat{K}_*$ in all cases. Hence, there is no flavor enhancement without $\mathcal{CP}$-like transformations at a generic point on the boundary of the fundamental domain in T moduli space. That is, ignoring the possibility of $\mathcal{CP}$, the traditional flavor symmetry $\Delta'(54, 2, 1) \cong \Delta(54) \cup \mathbb{Z}_9^R$ is not enhanced at the boundaries of the fundamental domain.

In order to figure out the flavor enhancements that the $\mathcal{CP}$-like stabilizer elements in eq. (161) induce, we must consider i) the matrix representations of these stabilizer elements when acting on the twisted matter fields $\Phi_{-2/3}$ (and their $\mathcal{CP}$-conjugate $\bar{\Phi}_{-2/3}$), and ii) the automorphy factors associated with these transformations. The representations are obtained as products of $\rho(\hat{K}_S)$ and $\rho(\hat{K}_T)$ given in eq. (123), as well as $\rho(\hat{K}_*)$ defined in eq. (148). Further, by using the 2×2 matrices that represent the $\mathrm{GL}(2, \mathbb{Z})_T$ generators $\hat{K}_S$, $\hat{K}_T$ (see eq. (2)) and $\hat{K}_*$ (see eq. (145)), we find that the automorphy factors are trivial for $\mathrm{Re}\,\langle T \rangle = 0, -1/2$, i.e.

$$\Phi_{-2/3} \xrightarrow{\hat{K}_*} \bar{\Phi}_{-2/3} \, , \tag{162a}$$

$$\Phi_{-2/3} \xrightarrow{\hat{K}_* \hat{K}_T} \rho(\hat{K}_T)^* \bar{\Phi}_{-2/3} \, . \tag{162b}$$

At the points $\langle T \rangle$ satisfying $|\langle T \rangle| = 1$, we have

$$\Phi_{-2/3} \xrightarrow{\hat{K}_S \hat{K}_*} \bar{\Phi}'_{-2/3} = ((\bar{T}))^{-2/3} \rho(\hat{K}_S) \, \bar{\Phi}_{-2/3} \, . \tag{163}$$

In all three cases, one can show easily that $H_{\langle T \rangle} = \mathbb{Z}_2^{\mathcal{CP}}$ acts on the matter fields as a $\mathbb{Z}_2$ symmetry.[3] Thus, the order of the traditional flavor symmetry is enhanced by a factor of two, resulting in the unified flavor symmetry

$$G_{\text{uf}, \langle T \rangle \in \text{boundary}} \cong \Delta'(54, 2, 1) \cup \mathbb{Z}_2^{\mathcal{CP}} \cong [324, 125] \tag{164}$$

at a generic point on the boundary of the fundamental domain of the T moduli space.

6.2. Flavor symmetry at $\langle T \rangle = \mathrm{i}$ in moduli space of $\mathbb{T}^2/\mathbb{Z}_3$

As one can infer from Fig. 3, at $\langle T \rangle = \mathrm{i}$ the stabilizer subgroup includes the $\mathrm{SL}(2, \mathbb{Z})_T$ transformation $\hat{K}_S$,

$$\langle T \rangle \xrightarrow{\hat{K}_S} -\frac{1}{\langle T \rangle} = \langle T \rangle \quad \text{at} \quad \langle T \rangle = \mathrm{i} , \tag{165}$$

and the $\mathcal{CP}$-like generator $\hat{K}_*$ because $\langle T \rangle = \mathrm{i}$ satisfies eq. (161a). Since $\hat{K}_S^4 = \mathbb{1}$, the full stabilizer subgroup is $H_{\mathrm{i}} \cong \mathbb{Z}_4 \times \mathbb{Z}_2^{\mathcal{CP}}$, where the $\mathbb{Z}_2 \subset \mathbb{Z}_4$ subgroup associated with $\hat{K}_S^2 = \mathrm{C}$ (see eq. (130)) already belongs to the traditional flavor symmetry $\Delta(54) \subset \Delta'(54, 2, 1)$.

As the vev $\langle T \rangle = \mathrm{i}$ of the modulus is invariant under H_{i}, the coupling strengths $\hat{Y}_1(\langle T \rangle)$ and $\hat{Y}_2(\langle T \rangle)$ are also invariant. Using the general modular transformations eq. (119), this implies particularly that the $\hat{K}_S$ transformation on the coupling strengths leads to

$$\hat{Y}_{\mathbf{2}''}^{(1)}(\mathrm{i}) = \begin{pmatrix} \hat{Y}_1(\mathrm{i}) \\ \hat{Y}_2(\mathrm{i}) \end{pmatrix} \xrightarrow{\hat{K}_S} \begin{pmatrix} \hat{Y}_1\left(-\frac{1}{\mathrm{i}}\right) \\ \hat{Y}_2\left(-\frac{1}{\mathrm{i}}\right) \end{pmatrix} = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} \hat{Y}_1(\mathrm{i}) \\ \hat{Y}_2(\mathrm{i}) \end{pmatrix} = \begin{pmatrix} \hat{Y}_1(\mathrm{i}) \\ \hat{Y}_2(\mathrm{i}) \end{pmatrix} . \tag{166}$$

Consequently, the doublet of couplings $\hat{Y}_{\mathbf{2}''}^{(1)}(\mathrm{i})$ is an eigenvector of the matrix $\rho_{\mathbf{2}''}(\hat{K}_S)$ with eigenvalue i. Thus, eq. (166) amounts to the vev alignment

$$\hat{Y}_{\mathbf{2}''}^{(1)}(\mathrm{i}) = \hat{Y}_2(\mathrm{i}) \begin{pmatrix} -\frac{\sqrt{2}}{1+\sqrt{3}} \\ 1 \end{pmatrix} , \tag{167}$$

which provides a nontrivial relation between $\hat{Y}_1(\mathrm{i})$ and $\hat{Y}_2(\mathrm{i})$. As a side remark, because of the vev alignment eq. (167), the T' modular forms $\hat{Y}_{\mathbf{1}}^{(4)}(T)$, $\hat{Y}_{\mathbf{1}'}^{(4)}(T)$ and $\hat{Y}_{\mathbf{3}}^{(4)}(T)$ of weight 4 given in ref. [3] get aligned at $\langle T \rangle = \mathrm{i}$ as

$$\hat{Y}_{\mathbf{1}}^{(4)}(\mathrm{i}) = Y_{\mathbf{1}'}^{(4)}(\mathrm{i}) = 3(3 - 2\sqrt{3}) \, \hat{Y}_2(\mathrm{i})^4 \quad \text{and} \quad \hat{Y}_{\mathbf{3}}^{(4)}(\mathrm{i}) = 3(2 - \sqrt{3}) \, \hat{Y}_2(\mathrm{i})^4 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} .$$

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[3] To verify this, one must use $\rho(\hat{K}_T)\rho(\hat{K}_T)^* = \rho(\hat{K}_S)\rho(\hat{K}_S)^* = \mathbb{1}$.

$$(168)$$

Furthermore, both modular forms, $\hat{Y}_1(\mathrm{i})$ and $\hat{Y}_2(\mathrm{i})$, are real at $\langle T \rangle = \mathrm{i}$ (as one can also infer directly from eq. (149)) and we find

$$\hat{Y}_1(\mathrm{i}) = -3\sqrt{2}\frac{\eta^3(3\,\mathrm{i})}{\eta(\mathrm{i})} \approx -0.5234 \quad \text{and} \quad \hat{Y}_2(\mathrm{i}) \approx 1.0112 \,. \tag{169}$$

We can apply the vev alignment eq. (167) also to the trilinear superpotential eq. (131) of twisted matter fields $\Phi^i_{-2/3} = (X_i, Y_i, Z_i)^{\mathrm{T}}$, $i = 1, 2, 3$. At $\langle T \rangle = \mathrm{i}$, it turns out that

$$\mathcal{W}(\mathrm{i}, X_i, Y_i, Z_i) \supset c^{(1)}\,\hat{Y}_2(\mathrm{i})\Big[\big(X_1\,X_2\,X_3 + Y_1\,Y_2\,Y_3 + Z_1\,Z_2\,Z_3\big) \tag{170a}$$

$$+ \frac{1}{1+\sqrt{3}}\big(X_1\,Y_2\,Z_3 + X_1\,Y_3\,Z_2 + X_2\,Y_1\,Z_3 \tag{170b}$$

$$+ X_3\,Y_1\,Z_2 + X_2\,Y_3\,Z_1 + X_3\,Y_2\,Z_1\big)\Big] \,.$$

Under a modular $\hat{K}_{\mathrm{S}}$ transformation with its representation given by eq. (123), twisted matter fields transform, according to eq. (157), as

$$\begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} \xrightarrow{\hat{K}_{\mathrm{S}}} \begin{pmatrix} X'_i \\ Y'_i \\ Z'_i \end{pmatrix} = \underbrace{\frac{\mathrm{i}(-\mathrm{i})^{-2/3}}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}}_{=:\, \rho_{\mathbf{3}_2,\mathrm{i}}(\hat{K}_{\mathrm{S}})} \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} \,, \tag{171}$$

where the multivalued factor $(-\mathrm{i})^{-2/3}$ can be fixed to $-\omega^2 = \exp(\pi\mathrm{i}/3)$ by the $\mathbb{Z}_3$ point group symmetry, and the representation $t = \mathbf{3}_2$ following the notation of eq. (157) corresponds to the triplet that the matter fields build in the resulting unified flavor symmetry at $\langle T \rangle = \mathrm{i}$. It then follows that $\hat{K}_{\mathrm{S}}$ acts on the superpotential evaluated at $\langle T \rangle = \mathrm{i}$ as an R-symmetry,

$$\mathcal{W}(\mathrm{i}, X_i, Y_i, Z_i) \xrightarrow{\hat{K}_{\mathrm{S}}} \mathcal{W}(\mathrm{i}, X'_i, Y'_i, Z'_i) = \mathrm{i}\,\mathcal{W}(\mathrm{i}, X_i, Y_i, Z_i) \,, \tag{172}$$

as one can also see directly from eq. (46a). Moreover, due to $\rho_{\mathbf{3}_2,\mathrm{i}}(\hat{K}_{\mathrm{S}})$, the modular $\hat{K}_{\mathrm{S}}$ transformation at $\langle T \rangle = \mathrm{i}$ generates a $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ traditional flavor symmetry when acting on matter fields, cf. eq. (171), where $\mathbb{Z}_4 \times \mathbb{Z}_3$ can be generated by

$$\big(\rho_{\mathbf{3}_2,\mathrm{i}}(\hat{K}_{\mathrm{S}})\big)^3 = \frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad \text{and} \quad \big(\rho_{\mathbf{3}_2,\mathrm{i}}(\hat{K}_{\mathrm{S}})\big)^4 = \omega^2 \mathbb{1} \,, \tag{173}$$

respectively. It is easy to see that the $\mathbb{Z}_4$ factor generated by $(\rho_{\mathbf{3}_2,\mathrm{i}}(\hat{K}_{\mathrm{S}}))^3$ is an R-symmetry, while the $\mathbb{Z}_3$ factor generated by $(\rho_{\mathbf{3}_2,\mathrm{i}}(\hat{K}_{\mathrm{S}}))^4$ is equivalent to the point group symmetry (129a) of the orbifold sector.[4] Using that $(\rho_{\mathbf{3}_2,\mathrm{i}}(\hat{K}_{\mathrm{S}}))^6 = \rho_{\mathbf{3}_2}(\mathrm{C})$, without including $\mathcal{CP}$-like transformations the traditional flavor symmetry is enhanced from

$$\Delta'(54, 2, 1) \quad \text{to} \quad \Xi(2, 2) \cong [324, 111] \quad \text{at } \langle T \rangle = \mathrm{i} \,. \tag{174}$$

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[4] The transformation $\hat{K}_{\mathrm{S}}^3$ at $\langle T \rangle = \mathrm{i}$ acts as a $\mathbb{Z}_4$ on bosons and on $\mathcal{W}$. However, due to eq. (159), it is actually an $\mathbb{Z}_8$ R-symmetry when acting on fermions. Since discrete R-symmetries are relevant for phenomenology [61,62], it may be interesting to explore the consequences of this symmetry.

This is the largest traditional flavor group without $\mathcal{CP}$ that can be obtained in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector embedded in a higher-dimensional orbifold, and contains the group $\Sigma(36 \times 3) \cong$ [108, 15] that results when the discrete $\mathbb{Z}_9^R$ R-symmetry from $\mathrm{SL}(2, \mathbb{Z})_U$ is not considered [3,17, 18]. Similarly to that case, the $\Xi(2, 2)$ group can be generated by the matrices $\rho(A)$, $\rho(B)$ given in eq. (127), the $\mathbb{Z}_9^R$ generator $\rho(\hat{R}_1)$ eq. (139), and $\rho_{\mathbf{3}_2, i}(\hat{K}_S)$ in the faithful triplet representation $\mathbf{3}_2$ of $\Xi(2, 2)$.

As a side remark, the relation $\rho_{\mathbf{3}_2, i}(\hat{K}_S^4) = (\rho_{\mathbf{3}_2, i}(\hat{K}_S))^4 = \omega^2 \mathbb{1}$ in eq. (173) is a consistent representation of $\hat{K}_S^4 = \mathbb{1}$ (see eq. (1)) because the latter indicates that $\hat{K}_S^4$ must act trivially on the fields up to the action of the traditional flavor symmetry.

So far, we have not yet considered the $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation $\hat{K}_*$ at $\langle T \rangle = \mathrm{i}$. Using the $\mathcal{CP}$-like transformation (162a) of matter fields, one can explicitly show that

$$\mathcal{W}(\mathrm{i}, X_i, Y_i, Z_i) \xrightarrow{\hat{K}_*} \mathcal{W}(\mathrm{i}, \bar{X}_i, \bar{Y}_i, \bar{Z}_i) = \left(\mathcal{W}(\mathrm{i}, X_i, Y_i, Z_i) \right)^*, \tag{175}$$

using the superpotential eq. (170) with $c^{(1)} \hat{Y}_2(\mathrm{i}) \in \mathbb{R}$ and the discussion on $\mathcal{CP}$ from section 5.7. Consequently, if we stabilize the Kähler modulus T at $\langle T \rangle = \mathrm{i}$, the trilinear superpotential eq. (170) respects $\mathcal{CP}$. Including $\mathcal{CP}$, the unified flavor group at $\langle T \rangle = \mathrm{i}$ is given by

$$G_{\mathrm{uf}, \langle T \rangle = \mathrm{i}} \cong [648, 548] \cong \Delta'(54, 2, 1) \cup \mathbb{Z}_4 \cup \mathbb{Z}_2^{\mathcal{CP}}, \tag{176}$$

where $\Xi(2, 2) \cong \Delta'(54, 2, 1) \cup \mathbb{Z}_4$ is the maximal flavor subgroup of $G_{\mathrm{uf}, \langle T \rangle = \mathrm{i}}$ which does not act $\mathcal{CP}$-like.

6.3. Flavor symmetry at $\langle T \rangle = \omega$ in moduli space of $\mathbb{T}^2/\mathbb{Z}_3$

Let us now assume that the Kähler modulus is stabilized at $\langle T \rangle = \omega$ in moduli space. There, as illustrated in Fig. 3, the unbroken $\mathrm{GL}(2, \mathbb{Z})_T$ symmetries can be generated by the $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation $\hat{K}_* \hat{K}_T$, and the $\mathbb{Z}_3$ modular transformation $\hat{K}_S \hat{K}_T$,

$$\langle T \rangle \xrightarrow{\hat{K}_S} -\frac{1}{\langle T \rangle} \xrightarrow{\hat{K}_T} -\frac{1}{\langle T \rangle + 1} = \langle T \rangle. \tag{177}$$

In addition, we have the $\mathbb{Z}_2$ transformation $\hat{K}_S^2 = C$ that is already included in the traditional flavor group. The resulting stabilizer subgroup reads $H_\omega \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\mathcal{CP}}$.

At this point in moduli space, the coupling strengths governed by the modular forms $\hat{Y}_1(\langle T \rangle)$ and $\hat{Y}_2(\langle T \rangle)$ with modular weight $n_Y = 1$ get aligned as

$$\hat{Y}_{\mathbf{2}''}^{(1)}(\omega) = \hat{Y}_2(\omega) \begin{pmatrix} \frac{1}{\sqrt{2}} \omega \\ 1 \end{pmatrix}, \quad \text{where} \quad \hat{Y}_2(\omega) \approx 0.97399899. \tag{178}$$

Incidentally, the vev alignment given by eq. (178) implies that the T' modular forms $\hat{Y}_{\mathbf{1}}^{(4)}(T)$, $\hat{Y}_{\mathbf{1}'}^{(4)}(T)$ and $\hat{Y}_{\mathbf{3}}^{(4)}(T)$ of weight 4 given in ref. [3] are aligned at $\langle T \rangle = \omega$ according to

$$\hat{Y}_{\mathbf{1}}^{(4)}(\omega) = 0, \quad \hat{Y}_{\mathbf{1}'}^{(4)}(\omega) = \frac{9}{4} \omega \hat{Y}_2(\omega)^4 \quad \text{and} \quad \hat{Y}_{\mathbf{3}}^{(4)}(\omega) = \frac{3}{2} \hat{Y}_2(\omega)^4 \begin{pmatrix} 1 \\ -\frac{1}{2} \omega \\ -\omega^2 \end{pmatrix}. \tag{179}$$

On the other hand, in this case the trilinear superpotential eq. (131) simplifies to

$$\mathcal{W}(\omega, X_i, Y_i, Z_i) \supset c^{(1)}\,\hat{Y}_2(\omega)\Big[X_1\,X_2\,X_3 + Y_1\,Y_2\,Y_3 + Z_1\,Z_2\,Z_3 \tag{180a}$$

$$-\frac{\omega}{2}\big(X_1\,Y_2\,Z_3 + X_1\,Y_3\,Z_2 + X_2\,Y_1\,Z_3 \tag{180b}$$

$$+ X_3\,Y_1\,Z_2 + X_2\,Y_3\,Z_1 + X_3\,Y_2\,Z_1\big)\Big]\,.$$

Under a modular $\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}$ transformation at $\langle T \rangle = \omega$, the three triplets of twisted matter fields $\Phi^i_{-2/3}$ transform as

$$\begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} \xrightarrow{\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}} \begin{pmatrix} X_i' \\ Y_i' \\ Z_i' \end{pmatrix} = \underbrace{\frac{\mathrm{i}\,(\omega^2)^{-2/3}}{\sqrt{3}}\begin{pmatrix} \omega^2 & 1 & 1 \\ \omega^2 & \omega^2 & \omega \\ \omega^2 & \omega & \omega^2 \end{pmatrix}}_{=:\,\rho_{\mathbf{3}_2,\omega}(\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}})} \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}\,, \tag{181}$$

where $t = \mathbf{3}_2$ denotes the triplet representation of the twisted matter fields $\Phi^i_{-2/3}$ under the unified flavor symmetry at $\langle T \rangle = \omega$. We used eq. (123) for $\rho(\hat{K}_{\mathrm{S}})$ and $\rho(\hat{K}_{\mathrm{T}})$, and the automorphy factor $(-\langle T \rangle - 1)^{-2/3} = (\omega^2)^{-2/3}$ for $\gamma_T = \hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}$ evaluated at $\langle T \rangle = \omega$ in moduli space, see eq. (157). Note that the multivalued factor $(\omega^2)^{-2/3}$ can be fixed to $\exp(2\pi\mathrm{i}2/9)$ by the $\mathbb{Z}_3^{(PG)}$ point group symmetry of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, see eq. (129a). Due to this phase, $\rho_{\mathbf{3}_2,\omega}(\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}})$ generates a $\mathbb{Z}_9$ symmetry that contains the $\mathbb{Z}_3^{(PG)}$ point group symmetry as a subgroup,

$$(\rho_{\mathbf{3}_2,\omega}(\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}))^3 = \omega^2\mathbb{1}\,. \tag{182}$$

Thus, even though $\rho_{\mathbf{3}_2,\omega}(\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}})$ has order 9, the traditional flavor symmetry (without $\mathcal{CP}$) is only enhanced from

$$\Delta'(54, 2, 1) \cong [162, 44] \qquad \text{to} \qquad H(3, 2, 1) \cong [486, 125] \qquad \text{at } \langle T \rangle = \omega\,, \tag{183}$$

i.e. the order increases by a factor of three (see ref. [51] for the nomenclature).

It follows that the $\mathbb{Z}_9 \subset H(3, 2, 1)$ transformation eq. (181) acts on the superpotential eq. (180) as

$$\mathcal{W}(\omega, X_i, Y_i, Z_i) \xrightarrow{\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}} \mathcal{W}(\omega, X_i', Y_i', Z_i') = \omega\,\mathcal{W}(\omega, X_i, Y_i, Z_i)\,. \tag{184}$$

This is expected due to the automorphy factor $(-\langle T \rangle - 1)^{-1} = \omega$ of the superpotential $\mathcal{W}$ for $\gamma_T = \hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}$ evaluated at the point $\langle T \rangle = \omega$ in moduli space. Thus, the $\mathbb{Z}_9$ symmetry enhancement yields a discrete R-symmetry.

Let us now consider also the $\mathcal{CP}$-like transformation $\hat{K}_*\hat{K}_{\mathrm{T}}$ included in H_ω. Under $\hat{K}_*\hat{K}_{\mathrm{T}}$, twisted matter fields $\Phi^i_{-2/3}$ transform according to eq. (162b), which in terms of the component fields reads $(X_i, Y_i, Z_i)^{\mathrm{T}} \xrightarrow{\hat{K}_*\hat{K}_{\mathrm{T}}} (\omega\bar{X}_i, \bar{Y}_i, \bar{Z}_i)^{\mathrm{T}}$. Then, we find that the superpotential at $\langle T \rangle = \omega$, eq. (180), respects $\mathcal{CP}$-like transformations, i.e.

$$\mathcal{W}(\omega, X_i, Y_i, Z_i) \xrightarrow{\hat{K}_*\hat{K}_{\mathrm{T}}} \mathcal{W}(\omega, \omega\bar{X}_i, \bar{Y}_i, \bar{Z}_i) = \big(\mathcal{W}(\omega, X_i, Y_i, Z_i)\big)^*\,. \tag{185}$$

This can be easily confirmed by applying the identities

$$\hat{Y}_1(\omega)\omega = \big(\hat{Y}_1(\omega)\big)^* \quad \text{and} \quad \hat{Y}_2(\omega) = \big(\hat{Y}_2(\omega)\big)^*\,, \tag{186}$$

which follow from eqs. (119) and (149). Finally, we find that this $\mathcal{CP}$ enhancement leads to the unified flavor symmetry given by

$$G_{\mathrm{uf},\langle T\rangle=\omega} \cong [972, 469] \cong \Delta'(54, 2, 1) \cup \mathbb{Z}_9 \cup \mathbb{Z}_2^{\mathcal{CP}}, \tag{187}$$

where $H(3, 2, 1) \cong \Delta'(54, 2, 1) \cup \mathbb{Z}_9$ is the maximal flavor subgroup of $G_{\mathrm{uf},\langle T\rangle=\omega}$ which does not act $\mathcal{CP}$-like.

6.4. Gauge symmetry enhancement at $\langle T\rangle = \omega$

Let us analyze the "accidental" continuous symmetries of the superpotential eq. (180) at $\langle T\rangle = \omega$ in moduli space. We assume that the twisted matter fields $\Phi^i_{-2/3} = (X_i, Y_i, Z_i)^{\mathrm{T}}$ transform identically for $i \in \{1, 2, 3\}$ (typically the three copies of twisted matter fields $\Phi^i_{-2/3}$ differ in some other gauge charges, such that there is no flavor symmetry that mixes $i \in \{1, 2, 3\}$). To identify continuous symmetries, we define a general infinitesimal U(3) transformation

$$\begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} \xrightarrow{\mathrm{U(3)}} (\mathbb{1}_3 + \mathrm{i}\,\alpha_a\,\mathrm{T}_a) \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}, \tag{188}$$

where summation over $a = 1, \ldots, 9$ is implied and the 3×3 matrices T_a denote the Hermitian generators of the U(3) Lie algebra. By demanding invariance of the superpotential eq. (180) at leading order in the parameters α_a, we obtain two linear independent generators t_a that we choose as

$$\mathrm{t}_1 := \frac{1}{3\sqrt{2}} \begin{pmatrix} 0 & \omega^2 & \omega^2 \\ \omega & 0 & 1 \\ \omega & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{t}_2 := \frac{\mathrm{i}}{3\sqrt{2}} \begin{pmatrix} 0 & \omega^2 & -\omega^2 \\ -\omega & 0 & 1 \\ \omega & -1 & 0 \end{pmatrix}. \tag{189}$$

Note that $\mathrm{t}_1\,\mathrm{t}_2 = \mathrm{t}_2\,\mathrm{t}_1$. Hence, we found a U(1) × U(1) symmetry of the superpotential eq. (180) at $\langle T\rangle = \omega$. In a full string discussion (see e.g. [41]), one can show that this "accidental" symmetry is actually an exact gauge symmetry, where the U(1) gauge bosons correspond to certain winding strings that become massless at the self-dual point $\langle T\rangle = \omega$ in moduli space.

Next, we diagonalize the U(1) generators eq. (189) simultaneously by performing a (unitary) basis change in field space, i.e.

$$\begin{pmatrix} X_i^{\mathrm{g}} \\ Y_i^{\mathrm{g}} \\ Z_i^{\mathrm{g}} \end{pmatrix} := M^{\mathrm{g}} \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} \quad \text{where} \quad M^{\mathrm{g}} := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega^2 & \omega^2 \\ 1 & 1 & \omega \\ 1 & \omega & 1 \end{pmatrix}, \tag{190}$$

where the superscript character "g" labels the "gauge" basis, in which the U(1) generators are diagonal. In this basis, the U(1) generators $\mathrm{t}_a^{\mathrm{g}} := M^{\mathrm{g}}\,\mathrm{t}_a\,(M^{\mathrm{g}})^{-1}$ for $a \in \{1, 2\}$ read

$$\mathrm{t}_1^{\mathrm{g}} = \frac{1}{3\sqrt{2}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathrm{t}_2^{\mathrm{g}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{191}$$

This reproduces the U(1) charges of $\mathbb{Z}_3$ twisted strings (M_1, M_2, M_3) computed using a $\mathbb{Z}_3$ shift orbifold, see ref. [41] and especially Table 2 therein. Next, we change the basis for the generators $\rho(\mathrm{A})$, $\rho(\mathrm{B})$ and $\rho(\mathrm{C})$ of the traditional flavor group $\Delta(54)$ given in eqs. (127) and (130). We obtain

$$\rho^{\mathrm{g}}(\mathrm{A}) := \begin{pmatrix} 0 & 0 & \omega^2 \\ \omega & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho^{\mathrm{g}}(\mathrm{B}) := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$\rho^{\mathrm{g}}(\mathrm{C}) := - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{192}$$

Using this, one can show easily that the $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup of $\Delta(54)$, generated by the diagonal representation matrices

$$\rho^{\mathrm{g}}(\mathrm{A})\,\rho^{\mathrm{g}}(\mathrm{B}) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho^{\mathrm{g}}(\mathrm{B})\,\rho^{\mathrm{g}}(\mathrm{A}) = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{193}$$

is contained in the continuous $U(1) \times U(1)$ symmetry generated by eq. (191),

$$\rho^{\mathrm{g}}(\mathrm{A})\,\rho^{\mathrm{g}}(\mathrm{B}) = \exp\left(2\pi\mathrm{i}\left(+\sqrt{2}\,\mathrm{t}_1^{\mathrm{g}} - \sqrt{\tfrac{2}{3}}\,\mathrm{t}_2^{\mathrm{g}}\right)\right), \tag{194a}$$

$$\rho^{\mathrm{g}}(\mathrm{B})\,\rho^{\mathrm{g}}(\mathrm{A}) = \exp\left(2\pi\mathrm{i}\left(-\sqrt{2}\,\mathrm{t}_1^{\mathrm{g}} - \sqrt{\tfrac{2}{3}}\,\mathrm{t}_2^{\mathrm{g}}\right)\right). \tag{194b}$$

The elements $\rho^{\mathrm{g}}(\mathrm{B})$ and $\rho^{\mathrm{g}}(\mathrm{C})$ generate an S_3 permutation group, which is an R-symmetry since its generator C is defined as a $180°$ rotation in extra dimensions. This S_3 does not commute with the continuous $U(1) \times U(1)$ symmetry. Hence, one can show that the traditional flavor symmetry $\Delta(54)$ gets enhanced at $\langle T \rangle = \omega$ in moduli space to

$$\left(U(1) \times U(1)\right) \rtimes S_3^R, \tag{195}$$

see ref. [41]. This is further enhanced by $\mathbb{Z}_9^R$ (if the orbifold sector is embedded in a higher-dimensional orbifold) and by the unbroken $SL(2,\mathbb{Z})_T$ modular transformation $\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}$ as discussed in section 6.3. In the new field basis, eq. (181) yields

$$\rho^{\mathrm{g}}_{\mathbf{3}_2,\omega}(\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}}) := M^{\mathrm{g}}\,\rho_{\mathbf{3}_2,\omega}(\hat{K}_{\mathrm{S}}\,\hat{K}_{\mathrm{T}})\,(M^{\mathrm{g}})^{-1} = \exp(2\pi\mathrm{i}2/9)\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{196}$$

Finally, we can perform the basis change in field space eq. (190) for the superpotential eq. (131) and obtain

$$\begin{aligned}
\mathcal{W}(T, X_i^{\mathrm{g}}, Y_i^{\mathrm{g}}, Z_i^{\mathrm{g}}) = \;& \frac{c^{(1)}}{\sqrt{3}}\Bigg[\left(\hat{Y}_2(T) - \sqrt{2}\,\omega^2\,\hat{Y}_1(T)\right)\left(X_1^{\mathrm{g}} X_2^{\mathrm{g}} X_3^{\mathrm{g}} + Y_1^{\mathrm{g}} Y_2^{\mathrm{g}} Y_3^{\mathrm{g}} + Z_1^{\mathrm{g}} Z_2^{\mathrm{g}} Z_3^{\mathrm{g}}\right) \\
& + \left(\hat{Y}_2(T) + \frac{\omega^2}{\sqrt{2}}\,\hat{Y}_1(T)\right)\left(X_1^{\mathrm{g}}(Y_2^{\mathrm{g}} Z_3^{\mathrm{g}} + Y_3^{\mathrm{g}} Z_2^{\mathrm{g}}) + X_2^{\mathrm{g}}(Y_1^{\mathrm{g}} Z_3^{\mathrm{g}} + Y_3^{\mathrm{g}} Z_1^{\mathrm{g}})\right. \\
& + \left. X_3^{\mathrm{g}}(Y_1^{\mathrm{g}} Z_2^{\mathrm{g}} + Y_2^{\mathrm{g}} Z_1^{\mathrm{g}})\right)\Bigg].
\end{aligned} \tag{197}$$

By using the $U(1) \times U(1)$ generators defined in eq. (191), we see that the terms from the first line of eq. (197), such as $Y_1^{\mathrm{g}} Y_2^{\mathrm{g}} Y_3^{\mathrm{g}}$, break this symmetry, whereas the terms from the second line preserve it. On the other hand, at the point $\langle T \rangle = \omega$ in moduli space, the couplings align as

$$\hat{Y}_1(\omega) \;=\; \frac{\omega}{\sqrt{2}}\,\hat{Y}_2(\omega)\,, \tag{198}$$

see eq. (178). This alignment leads to a vanishing coefficient in the first line of eq. (197) and we are left with

$$\mathcal{W}(\omega, X_i^{\mathrm{g}}, Y_i^{\mathrm{g}}, Z_i^{\mathrm{g}}) \;=\; c\left(X_1^{\mathrm{g}}(Y_2^{\mathrm{g}}Z_3^{\mathrm{g}} + Y_3^{\mathrm{g}}Z_2^{\mathrm{g}}) + X_2^{\mathrm{g}}(Y_1^{\mathrm{g}}Z_3^{\mathrm{g}} + Y_3^{\mathrm{g}}Z_1^{\mathrm{g}}) + X_3^{\mathrm{g}}(Y_1^{\mathrm{g}}Z_2^{\mathrm{g}} + Y_2^{\mathrm{g}}Z_1^{\mathrm{g}})\right)\,, \tag{199}$$

where $c := \sqrt{3}/2\,c^{(1)}\,\hat{Y}_2(\omega)$. This superpotential at $\langle T\rangle = \omega$ is clearly invariant under the U(1) × U(1) transformations generated by eq. (191).

6.5. Summary

Let us now highlight the main results of this section. Unified flavor symmetries $G_{\mathrm{uf},\langle T\rangle}$ are enhancements of the traditional flavor symmetry that occur at different self-dual points $\langle T\rangle$ and loci in moduli space of $\mathbb{T}^2/\mathbb{Z}_{K_i}$ orbifold sectors, where some modular transformations are linearly realized. The new flavor symmetries include $\mathrm{SL}(2,\mathbb{Z})_T$ (and in the case of $K_i = 2$ $\mathrm{SL}(2,\mathbb{Z})_U$) transformations as well as $\mathcal{CP}$-like symmetries. Thus, in this scenario $G_{\mathrm{uf},\langle T\rangle}$ (and $\mathcal{CP}$) breakdown is triggered by moving away from the self-dual loci in moduli space. We have illustrated this in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. At a generic point in moduli space, this orbifold exhibits a $\Delta'(54, 2, 1) \cong \Delta(54) \cup \mathbb{Z}_9^R$ traditional flavor symmetry, when embedded in a higher-dimensional orbifold. At the points depicted in Fig. 3, we find the following unified flavor symmetries:

$$\Delta'(54,2,1) \xrightarrow{\hat{K}_* \text{ or } \hat{K}_* \hat{K}_\mathrm{T} \text{ or } \hat{K}_\mathrm{S}\hat{K}_*} G_{\mathrm{uf},\langle T\rangle\in\mathrm{boundary}} \cong [324,125]\,, \qquad \text{see section 6.1}$$

$$\Delta'(54,2,1) \xrightarrow{\hat{K}_\mathrm{S}} \Xi(2,2) \xrightarrow{\hat{K}_*} G_{\mathrm{uf},\langle T\rangle=\mathrm{i}} \cong [648,548]\,, \qquad \text{see section 6.2}$$

$$\Delta'(54,2,1) \xrightarrow{\hat{K}_\mathrm{S}\hat{K}_\mathrm{T}} H(3,2,1) \xrightarrow{\hat{K}_*\hat{K}_\mathrm{T}} G_{\mathrm{uf},\langle T\rangle=\omega} \cong [972,469]\,, \qquad \text{see section 6.3}$$

where we indicate the unbroken modular transformations that are linearly realized. In these cases, we have studied how the value of $\langle T\rangle$ induces an alignment mechanism on the couplings, which in turn sets the structure of the superpotential of the theory. Finally, in section 6.4 we observe that the traditional flavor symmetry is further enhanced to (continuous) gauge symmetries at self-dual points of the theory. In particular, by inspecting the (accidental) symmetries of the superpotential at $\langle T\rangle = \omega$, we have shown that the $\Delta(54)$ flavor symmetry is enhanced to $(\mathrm{U}(1)\times\mathrm{U}(1))\rtimes S_3^R$, which is broken spontaneously elsewhere in moduli space.

7. Conclusions and outlook

In this paper we have presented a new step of the eclectic approach towards ten-dimensional string theory. This discussion goes beyond previous studies in $D = 2$ that concentrated on the modular transformations $\mathrm{SL}(2,\mathbb{Z})_T$ of a single Kähler modulus T. We now include a detailed analysis of $\mathrm{SL}(2,\mathbb{Z})_U$ of the complex structure modulus U. In the orbifolds $\mathbb{T}^2/\mathbb{Z}_K$ with $K = 3, 4, 6$ the U-modulus is fixed because of the orbifold twist. We show that even in the case of frozen U we encounter remnant symmetries of $\mathrm{SL}(2,\mathbb{Z})_U$ in form of R-symmetries that extend the eclectic group. When embedded in $D = 6$ compact dimensions these symmetries are relevant

as sublattice rotations. We illustrate the full eclectic picture in the case of $\mathbb{T}^2/\mathbb{Z}_3$, where we obtain $G_{\mathrm{ecl}} = \Delta(54) \cup T' \cup \mathbb{Z}_9^R \cup \mathbb{Z}_2^{\mathcal{CP}}$, a discrete symmetry group of order $3,888$.

In a next step towards $D = 6$ we shall have to consider more than one unconstrained modulus beyond T. The simplest case with two unconstrained moduli (T and U) is the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold that has already been discussed in ref. [19]. There, it was shown that the remnants of the full modular group $O_{\hat{\eta}}(2,2,\mathbb{Z})$ (that contains $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$) lead to the finite modular group $((S_3 \times S_3) \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2^{\mathcal{CP}}$, where mirror symmetry acts as a $\mathbb{Z}_4$ on the level of matter fields but only as a $\mathbb{Z}_2$ permutation of T and U, and $\mathbb{Z}_2^{\mathcal{CP}}$ is a $\mathcal{CP}$-like transformation [19]. Together with the traditional flavor group $(D_8 \times D_8)/\mathbb{Z}_2$ this leads to an eclectic group with $4,608$ elements. The full technical details of the analysis of ref. [19] will be presented in ref. [63]. This includes a discussion of modular forms with more than one unconstrained modulus, such as e.g. Siegel modular forms. Incidentally, in a recent work of Ding, Feruglio and Liu [64] such a situation has been discussed from the bottom-up perspective, including the aforementioned finite modular group $(S_3 \times S_3) \rtimes \mathbb{Z}_2$ (on the level of the moduli and without $\mathcal{CP}$). It will be interesting to see how our top-down approach from string theory can make contact with the bottom-up discussion of ref. [64].

After having control of the $\mathbb{T}^2/\mathbb{Z}_2$ case, all of the relevant building blocks for the $D = 6$ case are at our disposal. A discussion of the generic case with $D = 6$ will be too difficult to be analyzed in detail. Simpler cases can be found in models of elliptic fibrations of $D = 6$ compact space. Sublattice rotations of the various $D = 2$ sectors would be relevant for the eclectic group. Finally, the next step would require the selection of specific string models that could successfully describe the flavor structure of quarks and leptons to make connections to the bottom-up constructions (see e.g. [65] and references therein).

CRediT authorship contribution statement

Hans Peter Nilles: Conceptualization, Methodology, Writing – original draft. **Saúl Ramos–Sánchez:** Conceptualization, Methodology, Writing – original draft. **Patrick K.S. Vaudrevange:** Conceptualization, Methodology, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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