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Standard super-activation for Gaussian channels requires squeezing

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Abstract. The quantum capacity of bosonic Gaussian quantum channels can be non-additive in a particularly striking way: a pair of such optical fiber type channels can individually have zero quantum capacity but super-activate each other such that the combined channel has strictly positive capacity. This has been shown in Smith *et al* (2011 *Nature Photon.* **5** 624) where it was conjectured that squeezing is a necessary resource for this phenomenon. We provide a proof of this conjecture by showing that for gauge covariant channels a Choi matrix with positive partial transpose implies that the channel is entanglement-breaking. In addition, we construct an example which shows that this implication fails to hold for Gaussian channels which arise from passive interactions with a squeezed environment.

Contents

1. Introduction

A question at the heart of information theory—classical as well as quantum—is how to transmit information reliably, given imperfect resources. The means to transmit information are referred to as the (typically noisy) *channel*. Its most important quantitative characteristic is how many units of information can be reliably sent per use in the limit of many uses of the channel. This number is called the *capacity* of the channel [1].

For classical memoryless channels, Shannon [2] posed and answered this question in his groundbreaking work in which he provided a tractable formula for the capacity of any such channel.

One of the fundamental insights of quantum information theory is that for quantum channels several distinct capacities can be defined depending on which kind of information (classical, quantum, etc) is to be sent and which ancillary resources are provided [3]. Moreover, despite considerable progress, no closed general expression is known for the classical or quantum capacity. What complicates matters in the quantum world is that quantum correlations between different channel uses can improve the performance or, mathematically speaking, lead to non-additivity effects. This has two practical consequences: not only is it in general necessary to entangle the channel inputs over many uses to fully exploit the capacity of the channel but also this capacity can in some cases be further enhanced by combining two different channels, so that their joint capacity exceeds the sum of the two parts. One of the most striking examples of these effects is the *super-activation* of the quantum capacity: a pair of channels can individually have zero quantum capacity, but when combined give rise to a channel whose quantum capacity is strictly positive [4].

In [5] it was shown that this effect can even occur within the practically most important class of (bosonic) Gaussian channels. Among others, they describe the transmission of the continuous degrees of freedom of light in free space and in optical fibers [6] (taking into account the most common loss and noise mechanisms) as well as the time evolution of quantum memories which are based on collective excitations in atomic systems [7].

One of the channels used in the construction in [5] can indeed be regarded as a simple model of a lossy single-mode optical fiber. The second channel, however, involves *squeezing*, an experimentally more demanding interaction arising from processes in which, e.g. photons are created or annihilated pairwise [8]. Squeezing can be produced by birefringent materials and selectively reduces the quantum noise associated with certain observables of the electromagnetic field below its standard quantum value. In [5] super-activation was shown employing a high degree of squeezing within the interaction between a two-mode system and its environment—something considerably more difficult to realize than simple loss processes. Smith *et al* [5] write: 'Although an example using only linear optical elements would be desirable, we suspect, but cannot prove, that none exist.' The present paper aims to settle this conjecture in the affirmative.

Currently, there is basically one approach toward super-activation. This is based on the fact that there are only two classes of channels known, which have provably zero quantum capacity: channels with a symmetric extension and so-called positive partial transpose (PPT) channels. Since both classes are closed with respect to parallel composition, the only combination with a chance of successful super-activation is to take one channel from each class. In this work we show that if we restrict ourselves to passive Gaussian channels (i.e. those not involving squeezing), then the set of PPT channels becomes a strict subset of the set of channels with a symmetric extension, therefore rendering super-activation impossible.

In the following we will first briefly review the main mathematical tools to describe Gaussian channels and precisely define 'passive channels', i.e. the class of Gaussian channels without squeezing within which we then show that super-activation is not possible.

2. Prerequisites

We begin with recalling basic notions and results needed for our purpose.

2.1. Gaussian states and channels

We consider a continuous variable system of n bosonic modes whose description involves n pairs of generalized position and momentum operators Q_k, P_k which may correspond to the quadratures of electromagnetic field modes. With the definition $R := (Q_1, P_1, \ldots, Q_n, P_n)$ the canonical commutation relations read

$$[R_k, R_l] = \mathrm{i}\,(\sigma_n)_{kl}\,\mathbb{1} \quad \text{with}\ \sigma_n := \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1}$$

being the symplectic form.

We associate with every density operator ρ its displacement vector d, with $d_k := \mathrm{tr}\,[\rho R_k]$ and its covariance matrix Γ with $\Gamma_{kl} := \mathrm{tr}\,[\rho\{R_k - d_k\mathbb{1}, R_l - d_l\mathbb{1}\}_+]$, $k, l = 1, \ldots, 2n$. d and Γ contain the first and second moments of the corresponding phase space distribution. The significance of d and Γ becomes evident as we restrict our attention from now on to Gaussian states. These are defined as quantum states with a Gaussian Wigner phase space distribution function, see [9]. The Hilbert space of a continuous variable system is of infinite dimensions. The restriction to the set of Gaussian states allows for a much simpler description that requires only a finite number of parameters. In particular a Gaussian state is completely specified by d and Γ, the latter being any real symmetric matrix that satisfies the uncertainty relation

$$\Gamma \geqslant \mathrm{i}\sigma_n. \tag{2}$$

In the following all states are assumed to be centered (i.e. $d = 0$) since displacements in phase space are local unitaries in Hilbert space which are irrelevant for our purpose. Gaussian states form only a small subset of the state space. Yet they provide a good description for many experimentally accessible states, including coherent laser beams and Gibbs states of electromagnetic modes.

Now we turn our attention to state transformations, which are mathematically described by completely positive maps. If such a map is also trace-preserving and preserves the Gaussian nature of states, it is called *Gaussian channel*, see [10]. Again neglecting its effect on d it can be characterized by its action on covariance matrices, which is given by

$$\Gamma \mapsto X\Gamma X^{\mathrm{T}} + Y, \qquad X, Y = Y^{\mathrm{T}} \in \mathcal{M}_{2n}(\mathbb{R}). \tag{3}$$

For a pair of real matrices X and $Y = Y^{\mathrm{T}}$ to describe a bona fide Gaussian channel it is necessary and sufficient that

$$Y + \mathrm{i}\left(\sigma_n - X\sigma_n X^{\mathrm{T}}\right) \geqslant 0. \tag{4}$$

Unitary Gaussian evolutions then correspond to $Y = 0$ and X being real symplectic, i.e. $X \in \mathrm{Sp}(2n, \mathbb{R}) = \{S \in \mathcal{M}_{2n}(\mathbb{R}) |\, S\sigma_n S^{\mathrm{T}} = \sigma_n\}$. As is well known, every channel can be realized by a unitary dilation U describing the evolution of the system coupled to an environment in state ρ_{E}

$$\rho \mapsto \mathrm{tr}_{\mathrm{E}}\left[U(\rho_{\mathrm{E}} \otimes \rho)U^{\dagger}\right]. \tag{5}$$

For a Gaussian n-mode channel, ρ_{E} can be chosen as a Gaussian state of $n_{\mathrm{E}} \leqslant 2n$ environmental modes and U as a $(n + n_{\mathrm{E}})$-mode Gaussian unitary [11]. Gaussian unitaries are generated by Hamiltonians that are quadratic in the generalized position and momentum operators R_k and represent a family of transformations that can be realized, e.g. in quantum optical experiments [12].

On the level of phase space, the evolution of the covariance matrices of environment and input state, Γ_{E} and Γ, is governed by the symplectic transformation S and looks like

$$(\Gamma_{\mathrm{E}} \oplus \Gamma) \mapsto S\,(\Gamma_{\mathrm{E}} \oplus \Gamma)\,S^{\mathrm{T}}. \tag{6}$$

In the notation $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$ that reflects the decomposition of the total system into the environment and the n-modes system, one finds $X = S_4$ and $Y = S_3\Gamma_{\mathrm{E}}S_3^{\mathrm{T}}$. Here, S describes the most general Gaussian coupling between system and environment including, in particular, squeezing. Since our aim in the following is to analyze capabilities of quantum channels in the absence of this (in practice very demanding) ingredient, we now proceed to take a closer look at the simpler set of unitaries generated by *passive* Hamiltonians.

2.2. The symplectic orthogonal group

Passive Hamiltonians are those given by quadratic expressions in Q_k, P_k or, equivalently, in the annihilation and creation operators $a_k := (Q_k + \mathrm{i}P_k)/\sqrt{2}$ and $a_k^{\dagger}$ that commute with the total particle number operator $\sum_k a_k^{\dagger}a_k$. They take the form

$$H = \sum_{k,l=1}^{m} h_{kl}a_k^{\dagger}a_l + \mathrm{h.c.} \tag{7}$$

with $h_{kl} \in \mathbb{C}$. A unitary Gaussian evolution generated by a passive Hamiltonian as in (7) corresponds to a symplectic orthogonal matrix $S \in K(m) := \mathrm{Sp}(2m, \mathbb{R}) \cap O(2m, \mathbb{R})$. Mathematically, $K(m)$ is the largest compact subgroup of the real symplectic group. Physically, it corresponds to the set of operations which can be implemented using beam splitters and phase shifters only [13]. Note that some of the most common channels such as the lossy channel and the thermal noise channel can be described by passive Gaussian dilations.

In the following, it is useful to exploit that the group $K(m)$ is isomorphic to the group $U(m)$ of $m \times m$ unitary matrices [14]. This can be verified easily: first one observes that elements $R \in \mathcal{M}_{2m}(\mathbb{R})$ in the commutant of σ_m have the form

$$[\sigma_m, R] = 0 \Leftrightarrow R = (r_{ij})_{i,j=1}^m \quad \text{with } r_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{pmatrix},\ a_{ij}, b_{ij} \in \mathbb{R}. \tag{8}$$

With this result one verifies that the map

$$\begin{aligned} \Lambda : U(m) &\to K(m) \\ (c_{ij}) \mapsto (C_{ij}), &\quad C_{ij} = \begin{pmatrix} \Re(c_{ij}) & \Im(c_{ij}) \\ -\Im(c_{ij}) & \Re(c_{ij}) \end{pmatrix} \end{aligned} \tag{9}$$

is indeed a group isomorphism.

At this point we add two observations that will help us later to exploit the particular structure of real symplectic orthogonals (9). The set

$$\mathcal{C}_n := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| A, B \in \mathcal{M}_n(\mathbb{R}) \right\} \tag{10}$$

together with the operation of matrix multiplication forms a semigroup with neutral element. As such, it is isomorphic to $\mathcal{M}_n(\mathbb{C})$. An isomorphism is given by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \mathrm{i}B \tag{11}$$

and finally, for complex square matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ one finds the following criterion for positive-semidefiniteness: [3]

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \geqslant 0 \Leftrightarrow A \pm \mathrm{i}B \geqslant 0. \tag{12}$$

Now we combine the passive Gaussian unitaries with

2.3. *Properties of Gaussian channels*

Definition 1. *We call a Gaussian channel 'passive' if it can be generated by a* $m = (n_\mathrm{E} + n)$*-mode passive Hamiltonian* H *(7) that couples the system to an environment in a Gibbs state* ρ_E *of a passive Hamiltonian* H'*:*

$$\rho_\mathrm{E} = \frac{\mathrm{e}^{-\beta H'}}{\mathrm{tr}\left[\mathrm{e}^{-\beta H'}\right]}, \qquad H' = \sum_{k,l=1}^{n_\mathrm{E}} h'_{kl} a_k^\dagger a_l + \mathrm{h.c.} \tag{13}$$

One can show that, as a consequence of the normal mode decomposition of Gaussian states, (13) is equivalent to $\left[\Gamma_\mathrm{E}, \sigma_{n_\mathrm{E}}\right] = 0$, where Γ_E is the covariance matrix of the Gaussian state ρ_E and

[3] This equivalence can be seen from the similarity transformation $\begin{pmatrix} A - \mathrm{i}B & \\ & A + \mathrm{i}B \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathrm{i}\mathbb{1} \\ \mathrm{i}\mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathrm{i}\mathbb{1} \\ -\mathrm{i}\mathbb{1} & \mathbb{1} \end{pmatrix}$.

σ_{n_E} is the corresponding symplectic form. This implies for passive Gaussian channels

$$[Y, \sigma_n] = \left[S_3 \Gamma_E S_3^T, \sigma_n\right] = 0, \tag{14}$$

as one derives from the block structure of S_3 (9) and the analogous structure of the elements in the commutant of σ_n (8). Similarly we find $[X, \sigma_n] = [S_4, \sigma_n] = 0$. This is a useful property; it implies that the channel commutes with the passive unitary generated by the number operator (represented on phase space by $e^{i\phi\sigma_n}$). Gaussian channels with this property are called 'gauge covariant' [15] and are characterized by matrices (X, Y) commuting with σ_n.

Accordingly, all passive Gaussian channels are gauge covariant. Let us add a remark on our definition of passive channels. Clearly, no active (squeezing-type) interaction is needed to generate interaction or environmental state. Coupling the system to a Gibbs state for a passive Hamiltonian includes the typical situation, since both non-equilibrium states and squeezing Hamiltonians usually require active preparation, which do not naturally occur in the uncontrolled environment[4].

Before stating and proving our main result, we finally need to characterize on the level of covariance matrices the two types of channels that have been used in the examples of super-activation. There, two types of noisy channels with restricted capability to transmit entanglement have been studied. On the one hand, *entanglement-breaking channels* that, when acting on part of an entangled state always produce a separable output and *PPT channels*, that may transmit entanglement but only in the form of states that remain positive under partial transposition and thus represent bound entanglement that cannot be locally distilled into pure entangled states [16, 17]. Since a finite quantum capacity requires the ability to transmit (asymptotically) pure entangled states, this capacity vanishes for PPT channels.

A Gaussian channel is entanglement-breaking if and only if Y admits a decomposition into real matrices M and N such that [18]

$$Y = M + N, \quad M \geqslant i\sigma_n, \quad N \geqslant iX\sigma_n X^T. \tag{15}$$

This reflects the fact that any entanglement-breaking quantum channel consists of a measurement, followed by a state preparation depending on the outcome of the measurement [19, 20]. As a consequence, every entanglement-breaking channel has a symmetric extension and therefore zero quantum capacity by the no-cloning lemma [21].

A quantum channel T is called a PPT channel if $\theta \circ T$ is completely positive, where θ denotes time reversal, which in Schrödinger representation corresponds to transposition[5]. A Gaussian channel characterized by (X, Y) is PPT iff

$$Y - i\left(\sigma_n + X\sigma_n X^T\right) \geqslant 0. \tag{16}$$

Now we are in a position to turn to the question of whether super-activation of PPT and entanglement-breaking channels is possible in the Gaussian setting without squeezing. The main technical step is to show that for gauge covariant channels being PPT and being entanglement-breaking are equivalent.

4 Another plausible definition of passive channels would be to require passive coupling to an environment in a state that is not squeezed (i.e. Γ_E has no eigenvalue smaller than 1). This includes our definition but adds high-temperature Gibbs states of squeezing Hamiltonians. We cannot rule out super-activation for these types of channels.

5 In phase space, θ corresponds to the transformation $\Gamma \mapsto \tilde{\theta}\Gamma\tilde{\theta}^T$ with $\tilde{\theta} = \bigoplus_{i=1}^n \mathrm{diag}(1, -1)$.

3. Main result

Proposition 2. *A gauge covariant Gaussian channel T is entanglement-breaking iff it is PPT.*

Proof. Evidently, entanglement-breaking implies PPT, so we have to prove the reverse implication. To this end it is convenient to reorder the canonical coordinates as $(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$. In this notation

$$\sigma_n = \begin{pmatrix} & \mathbb{1}_n \\ -\mathbb{1}_n & \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_1 & Y_2 \\ -Y_2 & Y_1 \end{pmatrix}. \tag{17}$$

The latter follows from (14) together with (8). Below we omit the index of σ_n.

We prove first that we can restrict ourselves to the case $X = \hat{X} \oplus \hat{X}$, where by virtue of the symplectic singular value decomposition [22] the matrix $\hat{X}$ is diagonal and non-negative. Assume, this is not the case. Then we can replace the unitary dilation U associated with T, which describes the interaction between system and environment, by $U' = (\mathbb{1}_E \otimes U_G)U(\mathbb{1}_E \otimes U_F)$. U_F and U_G are passive unitary evolutions that act only on the system. They correspond to symplectic transformations $F, G \in K(n)$ in phase space. We denote the resulting channel by T'. T is PPT iff T' is PPT. The same holds for the entanglement-breaking property. We find

$$X' = GXF = \begin{pmatrix} G_1 & G_2 \\ -G_2 & G_1 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \begin{pmatrix} F_1 & F_2 \\ -F_2 & F_1 \end{pmatrix}, \tag{18}$$

with $F_1 + \mathrm{i}F_2, G_1 + \mathrm{i}G_2 \in U(n)$. Now we can exploit the isomorphism (11) and choose $F_{1,2}$ and $G_{1,2}$ such that $(G_1 + \mathrm{i}G_2)(X_1 + \mathrm{i}X_2)(F_1 + \mathrm{i}F_2) =: \hat{X}$ is the singular value decomposition of $X_1 + \mathrm{i}X_2$. Hence, $X' = \hat{X} \oplus \hat{X}$ with $\hat{X}$ diagonal and non-negative.

We will now exploit criterion (15) by showing that the decomposition of Y into $M := Y - X^2$ and $N := X^2$ obeys the required conditions, which read

$$\begin{aligned} &N - \mathrm{i}X\sigma X^{\mathrm{T}} = X(\mathbb{1} - \mathrm{i}\sigma)X^{\mathrm{T}} \geqslant 0, \\ &M - \mathrm{i}\sigma = \begin{pmatrix} Y_1 - \hat{X}^2 & Y_2 - \mathrm{i}\mathbb{1} \\ -Y_2 + \mathrm{i}\mathbb{1} & Y_1 - \hat{X}^2 \end{pmatrix} \geqslant 0. \end{aligned} \tag{19}$$

The first inequality in (19) follows simply from $(\mathbb{1} - \mathrm{i}\sigma) \geqslant 0$. In order to arrive at the second inequality we use (12) and rewrite the inequality as

$$\begin{aligned} Y_1 - \hat{X}^2 \pm \mathrm{i}(Y_2 - \mathrm{i}\mathbb{1}) \geqslant 0 \quad &\Leftrightarrow \\ Y_1 \pm \mathrm{i}Y_2 - (\hat{X}^2 \mp \mathbb{1}) \geqslant 0 \quad &\Leftrightarrow \\ Y_1 \pm \mathrm{i}Y_2 - (\hat{X}^2 + \mathbb{1}) \geqslant 0 \quad &\Leftrightarrow \\ \begin{cases} Y_1 + \mathrm{i}Y_2 + (\hat{X}^2 + \mathbb{1}) \geqslant 0 \\ Y_1 - \mathrm{i}Y_2 - (\hat{X}^2 + \mathbb{1}) \geqslant 0 \end{cases} \quad &\Leftrightarrow \\ Y_1 \pm \mathrm{i}\left(Y_2 - \mathrm{i}(\mathbb{1} + \hat{X}^2)\right) \geqslant 0.& \end{aligned} \tag{20}$$

Here we used two elementary facts: (i) a matrix is positive iff its complex conjugate is positive; and (ii) the sum of two positive matrices is again positive. In the last line, with (12), we recover the PPT criterion (16)

$$Y - \mathrm{i}(\sigma + X\sigma X^{\mathrm{T}}) = Y - \mathrm{i}\sigma(\mathbb{1} + X^2) = \begin{pmatrix} Y_1 & Y_2 - \mathrm{i}(\mathbb{1} + \hat{X}^2) \\ -Y_2 + \mathrm{i}(\mathbb{1} + \hat{X}^2) & Y_1 \end{pmatrix} \geqslant 0, \tag{21}$$

which concludes the proof.

□

Proposition 3. (No super-activation without squeezing). *Let T_1, T_2 be passive Gaussian quantum channels. If each channel either has a symmetric extension or satisfies the PPT property, then $Q\,(T_1 \otimes T_2) = 0$.*

Proof. Let T_i $(i = 1 \text{ or } 2)$ be PPT. T_i is gauge covariant, because it is passive, and according to Proposition 2 it is thus entanglement-breaking. In particular, it has a symmetric extension, which then also holds for $T_1 \otimes T_2$. Hence, the combined channel has zero quantum capacity.

□

4. Passive interactions with a squeezed environment

We now consider an example of a Gaussian channel T for $n = n_{\mathrm{E}} = 2$. T is generated by a passive interaction, as in (7), but the environment is assumed to be in a mixed squeezed state ρ_{E} (i.e. $\det \Gamma_{\mathrm{E}} \neq 1$, $\Gamma_{\mathrm{E}} \geqslant \mathrm{i}\sigma_2$ and $\Gamma_{\mathrm{E}} \ngeqslant \mathbb{1}_4$). T will be shown to be PPT but not entanglement-breaking. We omit the index of $\mathbb{1}_2$ and σ_2 and choose

$$\Gamma_{\mathrm{E}} = \frac{3+\sqrt{13}}{2} \begin{pmatrix} 5 & & 3 & \\ & 5 & & -3 \\ 3 & & 2 & \\ & -3 & & 2 \end{pmatrix}, \tag{22}$$

$$S = \sqrt{\frac{1}{3}} \begin{pmatrix} -\sqrt{2}\mathbb{1} & & \mathbb{1} & \\ & -\mathbb{1} & & \sqrt{2}\mathbb{1} \\ \mathbb{1} & & \sqrt{2}\mathbb{1} & \\ & \sqrt{2}\mathbb{1} & & \mathbb{1} \end{pmatrix}. \tag{23}$$

S represents two beamsplitters: one of transmittivity $\frac{2}{3}$ that couples the first system mode to the first mode of the environment and a second of transmittivity $\frac{1}{3}$ that acts between the two remaining modes. The corresponding (X, Y), which characterize the Gaussian channel, then read

$$X = \sqrt{\frac{1}{3}} \begin{pmatrix} \sqrt{2}\mathbb{1} & \\ & \mathbb{1} \end{pmatrix}, \tag{24}$$

$$Y = \frac{3+\sqrt{13}}{6} \begin{pmatrix} 5 & & 3\sqrt{2} & \\ & 5 & & -3\sqrt{2} \\ 3\sqrt{2} & & 4 & \\ & -3\sqrt{2} & & 4 \end{pmatrix}. \tag{25}$$

Proposition 4. *The Gaussian channel determined by (24) exhibits the PPT property but it is not entanglement-breaking.*

Proof. Equations (4) and (16) are satisfied as one verifies explicitly.

It remains to show that T is not entanglement-breaking. With the inequalities (15) in mind we observe that this is equivalent to

$$\max_{(\lambda,M)\in\mathcal{D}} \lambda < 1, \quad \text{where } \mathcal{D} = \left\{ \begin{matrix} (\lambda, M) \in \\ \mathbb{R}\times\mathcal{M}_4(\mathbb{R}) \end{matrix} \middle| \begin{matrix} M = M^{\mathrm{T}}, \\ M \geqslant \lambda \mathrm{i}\sigma, \\ Y - M \geqslant \lambda \mathrm{i} X\sigma X^{\mathrm{T}} \end{matrix} \right\}. \tag{26}$$

This is a semi-definite program [23], so that the corresponding dual program can be used to construct a witness which certifies (26). Its specific form is given in the appendix.

□

5. Discussion

Super-additivity of channel capacities is one of the surprises between classical and quantum information theory and its mechanisms and quantitative importance are still poorly understood. Super-activation of the quantum capacity is one of the most extreme examples of such effects.

In the practically relevant Gaussian setting, super-activation can be achieved using squeezing, adding to a long list of quantum effects—entanglement generation, metrology, information coding or continuous-variable key-distribution—whose realization squeezing enables. In the Gaussian regime, we know that it is sometimes even a necessary resource. This is the case for entanglement generation [24] and, as we have proven in this work, also for super-activation of the quantum capacity. In the latter case, however, the proof of necessity relies on the framework—the basic idea behind the construction of all currently known instances of super-activation. In order to make a stronger statement, we would need to know whether there are other types of channels with zero quantum capacity [25].

Another question, which suggests itself, is how much squeezing is necessary within the given framework. Unfortunately, we do not at the moment see an approach toward settling this quantitative question.

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Appendix

In the following we show how to certify (26). Note that with the notation $\tilde{Y} = 0_4 \oplus Y$, $\tilde{X} = \mathrm{i}\sigma \oplus \mathrm{i}X\sigma X^{\mathrm{T}}$, $\tilde{M} = M \oplus -M$, the two inequalities in the definition of $\mathcal{D}$ can be rewritten as

$$\lambda\tilde{X} + \tilde{Y} + \tilde{M} \geqslant 0. \tag{A.1}$$

In the following we confirm (26) by showing that for all $(\lambda, M) \in \mathcal{D}$, $\lambda \leqslant 0.94$. For this purpose, let us define the witness matrix Ω,

$$\Omega = (A + \mathrm{i}B) \oplus (A + \mathrm{i}C). \tag{A.2}$$

$$A = \begin{pmatrix} a_1 & & -a_3 & \\ & a_1 & & a_3 \\ -a_3 & & a_2 & \\ & a_3 & & a_2 \end{pmatrix}, \quad a = \begin{pmatrix} 0.512 \\ 0.722 \\ 0.592 \end{pmatrix},$$

$$B = \begin{pmatrix} & b_1 & & b_3 \\ -b_1 & & b_3 & \\ & -b_3 & & b_2 \\ -b_3 & & -b_2 & \end{pmatrix}, \quad b = \begin{pmatrix} -0.212 \\ 0.552 \\ -0.368 \end{pmatrix},$$

$$C = \begin{pmatrix} & c_1 & & c_3 \\ -c_1 & & c_3 & \\ & -c_3 & & c_2 \\ -c_3 & & -c_2 & \end{pmatrix}, \quad c = \begin{pmatrix} 0.39 \\ -0.3 \\ 0.368 \end{pmatrix}$$

and state some of its properties:

(i) Ω is positive definite;

(ii) $\forall (\lambda, M) \in \mathcal{D}:\ \mathrm{tr}[\Omega\tilde{M}] = \mathrm{i}\,\mathrm{tr}\,[(B - C)M] = 0$, since $(B - C)$ is anti-symmetric and M is symmetric;

(iii) $\mathrm{tr}[\Omega\tilde{X}] = 2(b_1 + b_2 + \frac{2}{3}c_1 + \frac{1}{3}c_2) = 1$; and

(iv) $\mathrm{tr}[\Omega\tilde{Y}] = (1 + \frac{\sqrt{13}}{3})(5a_1 + 4a_2 - 6\sqrt{2}a_3) < 0.94$.

Let now be $(\lambda, M) \in \mathcal{D}$. Applying (ii) and (iii) in the first line and (i) together with (A.1) in the third line leads to

$$\begin{aligned} \lambda - \mathrm{tr}\left[\Omega\tilde{Y}\right] &= -\lambda\,\mathrm{tr}\left[\Omega\tilde{X}\right] - \mathrm{tr}\left[\Omega\tilde{Y}\right] - \mathrm{tr}\left[\Omega\tilde{M}\right] \\ &= -\mathrm{tr}\left[\Omega\left(\lambda\tilde{X} + \tilde{Y} + \tilde{M}\right)\right] \\ &\leqslant 0 \end{aligned} \tag{A.3}$$

with (iv) we obtain $\lambda < 0.94$.

References

[1] Cover T M and Thomas J A 1991 *Elements of Information Theory* (New York: Wiley)

[2] Shannon C E 1948 A mathematical theory of communication *Bell Syst. Tech. J.* **27** 379–423, 623–56

[3] Wilde M M 2013 *Quantum Information Theory* (Cambridge: Cambridge University Press)

[4] Smith G and Yard J 2008 Quantum communication with zero-capacity channels *Science* **321** 1812–5

[5] Smith G, Smolin J A and Yard J 2011 Quantum communication with Gaussian channels of zero quantum capacity *Nature Photon.* **5** 624–7

[6] Weedbrook C, Pirandola S, Garcia-Patron R, Cerf N J, Ralph T C, Shapiro J H and Lloyd S 2012 Gaussian Quantum Information *Rev. Mod. Phys.* **84** 621

[7] Hammerer K, Sørensen A S and Polzik E S 2010 Quantum interface between light and atomic ensembles *Rev. Mod. Phys.* **82** 1041–93

[8] Dodonov V V 2002 'Nonclassical' states in quantum optics: a 'squeezed' review of the first 75 years *J. Opt. B: Quantum Semiclass. Opt.* **4** R1

[9] Holevo A S 2011 Gaussian states *Probabilistic and Statistical Aspects of Quantum Theory* (Pisa: Edizioni della Normale) pp 187–218

[10] Eisert J and Wolf M M 2007 Gaussian quantum channels *Quantum Information with Continuous Variables of Atoms and Light* (London: Imperial College Press) pp 23–42 arXiv:quant-ph/0505151

[11] Caruso F, Eisert J, Giovannetti V and Holevo A S 2008 Multi-mode bosonic Gaussian channels *New J. Phys.* **10** 083030

[12] Braunstein S L and van Loock P 2005 Quantum information with continuous variables *Rev. Mod. Phys.* **77** 513–77

[13] Reck M and Zeilinger A 1994 Experimental realization of any discrete unitary operator *Phys. Rev. Lett.* **73** 58–61

[14] Arvind Dutta B, Mukunda N and Simon R 1995 The real symplectic groups in quantum mechanics and optics *Pramana* **45** 471

[15] Holevo A S 2010 Two mathematical problems in quantum information theory *Mathematical Horizons for Quantum Physics* (Singapore: World Scientific) chapter 5 pp 109–31

[16] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Mixed-state entanglement and quantum error correction *Phys. Rev.* A **54** 3824–51

[17] Horodeckii M, Horodecki P and Horodecki R 1998 Mixed-state entanglement and distillation: is there a 'bound entanglement in' nature? *Phys. Rev. Lett.* **80** 5239–42

[18] Holevo A S 2008 Entanglement-breaking channels in infinite dimensions *Problems Inform. Transm.* **44** 171–84

[19] Horodecki M, Shor P W and Ruskai M B 2003 General entanglement breaking channels *Rev. Math. Phys.* **15** 629–41

[20] Holevo A S, Shirokov M E and Werner R F 2005 Separability and entanglement-breaking in infinite dimensions *Russian Math. Surv.* **60** 359–60 (arXiv:quant-ph/0504204)

[21] Wootters W K and Zurek W H 1982 A single quantum cannot be cloned *Nature* **299** 802

[22] Wolf M M 2008 Not-so-normal mode decomposition *Phys. Rev. Lett.* **100** 070505

[23] Boyd S and Vandenberghe L 2011 *Convex Optimization* (Cambridge: Cambridge University Press) chapters 4.6 and 5.9 pp 167–9, 265–6

[24] Wolf M M, Eisert J and Plenio M B 2003 The entangling power of passive optical elements *Phys. Rev. Lett.* **90** 047904

[25] Smith G and Smolin J A 2012 Detecting incapacity of a quantum channel *Phys. Rev. Lett.* **108** 230507