



New results for the k -secretary problem [☆]

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ARTICLE INFO

Article history:

Received 30 January 2020

Received in revised form 28 January 2021

Accepted 15 February 2021

Available online 17 February 2021

Communicated by G.F. Italiano

Keywords:

Online algorithms

Secretary problem

Random order model

ABSTRACT

Suppose that n items arrive online in random order and the goal is to select k of them such that the expected sum of the selected items is maximized. The decision for any item is irrevocable and must be made on arrival without knowing future items. This problem is known as the k -secretary problem, which includes the classical secretary problem with the special case $k = 1$. It is well-known that the latter problem can be solved by a simple algorithm of competitive ratio $1/e$ which is optimal for $n \rightarrow \infty$. Existing algorithms beating the threshold of $1/e$ either rely on involved selection policies already for $k = 2$, or assume that k is large.

In this paper we present results for the k -secretary problem, considering the interesting and relevant case that k is small. We focus on simple selection algorithms, accompanied by combinatorial analyses. As a main contribution we propose a natural deterministic algorithm designed to have competitive ratios strictly greater than $1/e$ for small $k \geq 2$. This algorithm is hardly more complex than the elegant strategy for the classical secretary problem, optimal for $k = 1$, and works for all $k \geq 1$. We derive its competitive ratios for $k \leq 100$, ranging from 0.41 for $k = 2$ to 0.75 for $k = 100$.

Moreover, we consider an algorithm proposed earlier in the literature, for which no rigorous analysis is known. We show that its competitive ratio is 0.4168 for $k = 2$, implying that the previous analysis was not tight. Our analysis reveals a surprising combinatorial property of this algorithm, which might be helpful to find a tight analysis for all k .

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1. Introduction

The *secretary problem* is a well-known problem in the field of optimal stopping theory and is defined as follows: Given a sequence of n items which arrive online and in random order, select the maximum item. The decision to accept or reject an item must be made immediately and irrevocably upon its arrival, especially without knowing future items. The statement of the problem dates back to the 1960s and the optimal algorithm was published by Lindley [1] and Dynkin [2]. For discussions on the origin of the problem, we refer to the survey by Ferguson [3].

In the past years, generalizations of the secretary problem involving selection of multiple items have become very popular. We consider one of the most canonical generalizations known as the k -secretary problem: Here, the algorithm is allowed to choose k elements and the goal is to maximize the expected sum of accepted elements. Other objective functions, such as maximizing the probability of accepting the k best [4,5] or general submodular functions [6], have been studied as well.

[☆] Work supported by the European Research Council, Grant Agreement No. 691672. An earlier version of this paper has appeared in *Proceedings of 30th International Symposium on Algorithms and Computation (ISAAC 2019)*.

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Maximizing the sum of accepted items is closely related to the *knapsack secretary problem* [7–9]. If all items have unit weight and thus the capacity constraint is a cardinality bound, the k -secretary problem arises. The *matroid secretary problem*, introduced by Babaioff et al. [10], is a generalization where an algorithm must maintain a set of accepted items that form an independent set of a given matroid. We refer the reader to [11–13] for recent work. If the matroid is k -uniform, again, the k -secretary problem occurs. Another closely related problem was introduced by Buchbinder, Jain, and Singh [14]. In the (J, K) -secretary problem, an algorithm has J choices and the objective is to maximize the number of selected items among the K best. It can be shown that any monotone algorithm for the (k, k) -secretary problem corresponds to a k -secretary algorithm of the same competitive ratio [14]. Here, an algorithm is monotone if for any pair of items, it accepts the better item with higher probability. On the other side, any ordinal algorithm for k -secretary can be transformed to an algorithm for the (k, k) -secretary problem while maintaining the competitive ratio [14]. Ordinal algorithms [15] decide based on the total order of items only, rather than on their numeric values. In fact, most known and elegant algorithms for the k -secretary problem are ordinal [1,2,8,16].

The large interest in generalizations of the classical secretary problem is motivated mainly by numerous applications in online market design [10,16,17]. Apart from these applications, the secretary problem is the prototype of an online problem analyzed in the random order model: An adversarial input order often rules out good (or even constant) competitive ratios when considering online optimization problems without further constraints. By contrast, the assumption that the input is ordered randomly improves the competitive ratios in many optimization problems. This includes packing problems [7,9,18], scheduling problems [19], and graph problems [20,21]. Therefore, developing new techniques for secretary problems may, more generally, yield relevant insights for this input model as well.

1.1. Previous work

The k -secretary problem was introduced by Kleinberg [16] in 2005. He presents a randomized algorithm attaining a competitive ratio of $1 - 5/\sqrt{k}$, which approaches 1 for $k \rightarrow \infty$. Moreover, Kleinberg shows that any algorithm has a competitive ratio of $1 - \Omega(\sqrt{1/k})$. Therefore, from an asymptotic point of view, the k -secretary problem is solved by Kleinberg's result. However, the main drawback can be seen in the fact that the competitive ratio is not defined for $k \leq 24$ and breaks the barrier of $1/e$ only if $k \geq 63$ (see Fig. 2, p. 112).

In 2007 the problem was revisited by Babaioff et al. [8]. The authors propose two algorithms called VIRTUAL and OPTIMISTIC and prove that both algorithms have a competitive ratio of at least $1/e$ for any k . While the analysis of VIRTUAL is simple and tight, it takes much more effort to analyze OPTIMISTIC [8,17]. The authors believe that their analysis for OPTIMISTIC is not tight for $k \geq 2$.

Further indications for competitive ratios strictly greater than $1/e$ can be obtained from the framework of Buchbinder, Jain, and Singh [14]. In this framework, optimal algorithms for (J, K) -secretary and other variants of the secretary problem can be obtained using linear programming techniques. By numerical simulations for the (k, k) -secretary problem with $n = 100$, Buchbinder et al. obtained competitive ratios of 0.474, 0.565, and 0.612, for $k = 2, 3$, and 4, respectively. However, deriving an algorithm from their framework requires a formal analysis of the corresponding LP in the limit of $n \rightarrow \infty$, which is not provided in the article [14, p. 192].

Chan, Chen, and Jiang [22] revisited the (J, K) -secretary problem and obtained several fundamental results. Notably, they showed that optimal algorithms for the k -secretary problem require access to the numeric values of the items, which complements the previous line of research in the ordinal model. Chan et al. demonstrate this by providing a 0.4920-competitive algorithm for the 2-secretary problem which is based on an algorithm for the $(2, 2)$ -secretary problem of competitive ratio 0.4886. Still, a rigorous analysis for the general (J, K) -secretary problem revealing the numeric competitive ratios is not known, even for $J = K$. Moreover, the resulting algorithms seem overly involved. This dims the prospect of elegant k -secretary algorithms for $k \geq 3$ obtained from this approach.

1.2. Our contribution

We study the k -secretary problem, the most natural and immediate generalization of the classical secretary problem. While the extreme cases $k = 1$ and $k \rightarrow \infty$ are well studied, hardly any results for small values of $k \geq 2$ exist. We believe that simple selection algorithms, performing well for small k , are interesting both from a theoretical point of view and for practical settings. Moreover, the hope is that existing algorithms for related problems based on k -secretary algorithms can be improved this way [14, p. 191]. We study ordinal, threshold-based algorithms in the style of [1,2].

As main contribution, we propose and analyze a simple deterministic algorithm SINGLE-REF. This algorithm uses a single reference value as threshold for accepting items. To the best of our knowledge, this approach has not been explored for the k -secretary problem so far, although this natural idea arises in algorithms for related problems [23]. As a strength of our algorithm we see its simplicity: It is of plain combinatorial nature and can be fine-tuned using only two parameters. In contrast, the optimal algorithms following theoretically from the (J, K) -secretary approach [22] involve k^2 parameters and the same number of different decision rules.

The analysis of SINGLE-REF crucially depends on the fact that items can be partitioned into two classes, which we will call *dominating* and *non-dominating*. Both have certain properties on which we base our fully parameterized analysis. In Table 1, we list the competitive ratios of SINGLE-REF for $k \leq 20$ assuming $n \rightarrow \infty$. While the competitive ratio for $k = 1$ is optimal,

Table 1
Competitive ratios α of SINGLE-REF for $k \in [1..20]$.

k	1	2	3	4	5	6	7	8	9	10
α	$1/e$	0.41	0.44	0.47	0.49	0.51	0.53	0.54	0.55	0.56
k	11	12	13	14	15	16	17	18	19	20
α	0.57	0.58	0.59	0.59	0.60	0.60	0.61	0.62	0.62	0.63

we obtain a value significantly greater than $1/e$ already for $k = 2$. Furthermore, the competitive ratios are monotonically increasing in the interval $k \in [1..20]$, already breaking the threshold of 0.5 at $k = 6$. Numerical computations suggest that this monotonicity holds for general k . See Fig. 2 (p. 112) for the competitive ratios up to $k = 100$ and a comparison with Kleinberg’s algorithm [16]. Providing a closed formula for the competitive ratio for any value of k is one direction of future work (see Section 6).

Moreover, we investigate the OPTIMISTIC algorithm by Babaioff et al. [8] for the case $k = 2$. Although Chan et al. [22] provided a strong algorithm for $k = 2$, we think studying this elegant algorithm is interesting for two reasons: First, a tight analysis of OPTIMISTIC is stated as open problem in [8]. Article [8] does not provide the proof of the $(1/e)$ -bound and a recent journal publication [24] (evolved from [8] and [10]) does not cover the OPTIMISTIC algorithm at all. We make progress in this problem by proving that its competitive ratio is exactly 0.4168 for $k = 2$, which significantly breaks the $(1/e)$ -barrier. Second, our proof reveals an interesting property of this algorithm, which we show in Lemma 7: The probability that OPTIMISTIC accepts the second best item is exactly the probability that the optimal algorithm for $k = 1$ accepts the best item. A similar property might hold for $k \geq 3$, which could be a key insight into the general case.

From a technical point of view, we analyze the algorithms using basic combinatorial constructs exclusively. This is in contrast to previous approaches [14,22] which can only be analyzed using heavyweight linear programming techniques. The combinatorial parts of our analysis are exact and hold for all n . In order to evaluate the competitive ratios numerically, we find lower bounds that hold for sufficiently large n . Throughout the analyses of both algorithms, we associate probabilities with sets of permutations (see Section 2.2). Hence, probability relations can be shown equivalently by set relations. This is a simple but powerful technique which may be useful in the analysis of other optimization problems with random arrival order as well.

2. Preliminaries

Let $v_1 > v_2 > \dots > v_n$ be the *elements* (also called *items*) of the input. Note that we can assume w.l.o.g. that all items are distinct in the ordinal model. Therefore, we say that i is the *rank* of element v_i . An *input sequence* is any permutation of the list v_1, \dots, v_n . We denote the position of an element v in the input sequence π with $\text{pos}_\pi(v) \in \{1, \dots, n\}$ and write $\text{pos}(v)$ whenever the input sequence is clear from the context.

Given any input sequence, an algorithm can accept up to k items, where the decision whether to accept or reject an item must be made immediately upon its arrival. Let ALG denote the sum of items accepted by the algorithm. The algorithm is α -competitive if $\mathbf{E}[\text{ALG}] \geq (\alpha - o(1)) \cdot \text{OPT}$ holds for all item sets, where the expectation is taken over the uniform distribution of all $n!$ input sequences. Throughout the paper, $o(1)$ terms are asymptotic with respect to the number of items n and $\text{OPT} = \sum_{i=1}^k v_i$ denotes the value of an optimal offline solution.

2.1. Notation

For $a, b \in \mathbb{N}$ with $a \leq b$, we use the notation $[a..b]$ to denote the set of integers $\{a, a + 1, \dots, b\}$ and write $[a]$ for $[1..a]$. The (half-)open integer intervals $(a..b]$, $[a..b)$, and $(a..b)$ are defined accordingly. Further, we use the notation n^k for the falling factorial $\frac{n!}{(n-k)!}$.

2.2. Random permutations

We often use the following process to obtain a permutation drawn uniformly at random: Fix any order u_1, u_2, \dots, u_n of positions. Then, draw the element for position u_1 uniformly at random among all n elements, next the element for position u_2 among the remaining $n - 1$ elements, and so on.

Moreover, the uniform distribution of permutations allows us to prove probability relations using functions: Suppose that p_i is the probability that item v_i is accepted in a random permutation. Then $p_i = |P_i|/n!$, where P_i is the set of all input sequences where v_i is accepted. Thus, we can prove $p_i \leq p_j$ by finding an injective function $f: P_i \rightarrow P_j$ and get $p_i = p_j$ if f is bijective. This technique turns out to be highly useful (e.g. in the proof of Lemma 7, which relates probabilities of different algorithms).

2.3. Combinatorics

When analyzing random permutations, we often need to analyze the probability that K items in a sequence have a certain property. This is described by the following random experiment which is a special case of the hypergeometric distribution.

Fact 1. Suppose there are N balls in an urn from which M are blue and $N - M$ red. The probability of drawing K blue balls without replacement in a sequence of length K is $h(N, M, K) := \binom{M}{K} / \binom{N}{K}$.

To simplify the binomial coefficients arising from $h(N, M, K)$, we make use of three useful identities (see [25]) stated in the following.

For integers l, m, n, w with $l, m \geq 0$ and $n \geq w \geq 0$, it holds that

$$\sum_{k=0}^l \binom{l-k}{m} \binom{w+k}{n} = \binom{l+w+1}{m+n+1}. \quad (\text{R1})$$

The well-known symmetry property for binomial coefficients states that for any integers n, k with $n \geq 0$,

$$\binom{n}{k} = \binom{n}{n-k}. \quad (\text{R2})$$

Finally, by the trinomial revision rule, for any integers m and k , and any real number r , it holds that

$$\binom{r}{m} \binom{m}{k} = \binom{r}{k} \binom{r-k}{m-k}. \quad (\text{R3})$$

2.4. Bounding sums by integrals

To bound sums over monotone increasing summands we make use of the following facts.

Fact 2. Let $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and $a, b \in \mathbb{N}$. If f is

(A) monotonically decreasing, then $\int_a^{b+1} f(i) \, di \leq \sum_{i=a}^b f(i) \leq \int_{a-1}^b f(i) \, di$.

(B) monotonically increasing, then $\int_{a-1}^b f(i) \, di \leq \sum_{i=a}^b f(i) \leq \int_a^{b+1} f(i) \, di$.

3. Algorithms

In this section, we state our proposed algorithm SINGLE-REF and the OPTIMISTIC algorithm by Babaioff et al. [8] and compare both strategies. While both algorithms have an initial sampling phase in which the first $t - 1$ items are rejected, the main difference is the policy for accepting items: Let s_j be the j -th best item from the sampling.

Algorithm 1: SINGLE-REF.

Parameters: $t \in (k..n - k]$ (sampling threshold),
 $r \in [k]$ (reference rank)

- 1 **Sampling phase:** Reject the first $t - 1$ items.
 - 2 Let s_r be the r -th best item from the sampling phase.
 - 3 **Selection phase:** Choose the first k items better than s_r .
-

Algorithm 2: OPTIMISTIC [8].

Parameters: $t \in (k..n - k]$ (sampling threshold)

- 1 **Sampling phase:** Reject the first $t - 1$ items.
 - 2 Let $s_1 > \dots > s_k$ be the k best items from the sampling phase.
 - 3 **Selection phase:** As the j -th accepted item, choose the first item better than s_{k-j+1} .
-

SINGLE-REF uses only item s_r as reference element. In the selection phase, the algorithm accepts the first k elements better than s_r . Despite its simple structure, a challenging part in the analysis of SINGLE-REF is the dependence between both parameters r and t .

OPTIMISTIC uses the k best items from the sampling as reference elements. Right after the sampling phase, the first item better than s_k will be accepted. The following accepted items are chosen similarly, but with $s_{k-1}, s_{k-2}, \dots, s_1$ as reference

items. Note that OPTIMISTIC sticks to this order of reference elements, even if the first item already outperforms s_1 . Hence, it is optimistic in the sense that it always expects high-value items in the future.

Note that in the case $k = 1$, both OPTIMISTIC and SINGLE-REF become the optimal algorithm for the secretary problem [1,2]: After rejecting the first $t - 1$ items, choose the first one better than the best from sampling. This strategy selects the best item with probability $\frac{t-1}{n} \sum_{i=t}^n \frac{1}{i-1}$.

From now on, let

$$p_i := \Pr[\mathcal{A} \text{ accepts item } v_i],$$

where \mathcal{A} is either SINGLE-REF or OPTIMISTIC and $i \in [n]$. We further define

$$p_i^{(j)} := \Pr[\mathcal{A} \text{ accepts item } v_i \text{ as the } j\text{-th item}]$$

for $i \in [n]$ and $j \in [k]$. Clearly, $p_i = \sum_{j=1}^k p_i^{(j)}$. Next, we define monotonicity of k -secretary algorithms and prove this property for both algorithms.

Definition 1. An algorithm is called *monotone* if $p_i \geq p_j$ holds for any two items $v_i > v_j$.

Lemma 1. OPTIMISTIC and SINGLE-REF are monotone.

Proof. We prove that $p_i \geq p_{i+1}$ for all $i \in [n - 1]$. By the concept described in Section 2.2, it is sufficient to show that for each input sequence where v_{i+1} is accepted, there exists a unique input sequence where v_i is accepted.

Consider any input sequence π in which v_{i+1} is accepted. Let $s_j < v_{i+1}$ be the sampling item to which v_{i+1} is compared (in case of SINGLE-REF, we have $j = r$). Since v_{i+1} is accepted, we have $s_j \neq v_i$. By swapping v_i with v_{i+1} , we obtain a new permutation π' with the same reference element s_j . This is obvious if v_i is not in the sampling of π . Otherwise, note that in the ordered sequences of sampling items from π and π' , both v_{i+1} and v_i have the same position. This implies that s_j is the j -th best sampling item in π' . Further, item v_i is at the former position of v_{i+1} in π' , thus both algorithms accept v_i at this position since $v_i > v_{i+1} > s_j$.

The claim follows by applying the inequality $p_i \geq p_{i+1}$ iteratively. \square

Due to the monotonicity property, the competitive ratios of both algorithms can be easily analyzed using the following lemma.

Lemma 2. The competitive ratio of any monotone algorithm is $(1/k) \sum_{i=1}^k p_i$.

Proof. By monotonicity (Lemma 1) and by definition of the item set, both sequences p_1, \dots, p_k and v_1, \dots, v_k are sorted decreasingly. Let $\text{OPT} = \sum_{i=1}^k v_i$ and $\mathbf{E}[\mathcal{A}]$ be the expected sum of the items accepted by the monotone algorithm. Chebyshev's sum inequality [25] states that if $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq \dots \geq b_n$, then $\sum_{i=1}^n a_i b_i \geq (1/n) (\sum_{i=1}^n a_i) (\sum_{i=1}^n b_i)$. Applying this inequality yields

$$\mathbf{E}[\mathcal{A}] = \sum_{i=1}^n p_i v_i \geq \sum_{i=1}^k p_i v_i \geq \frac{1}{k} \left(\sum_{i=1}^k v_i \right) \left(\sum_{i=1}^k p_i \right) = \left(\frac{1}{k} \sum_{i=1}^k p_i \right) \text{OPT}.$$

Note that the above inequalities are tight: Consider a set of items where the top k items are basically identical, and all remaining items are close to zero. More formally, set $v_i = 1 - \varepsilon$ for $i \in [1..k]$ and $v_i = \varepsilon$ for $i \in (k..n]$, where $\varepsilon \rightarrow 0$. Then, the competitive ratio is exactly $(1/k) \sum_{i=1}^k p_i$. \square

The same argument is used in [14] to show that any monotone algorithm for (k, k) -secretary corresponds to an algorithm for k -secretary of the same competitive ratio.

4. Analysis of SINGLE-REF

In this section we analyze our proposed algorithm SINGLE-REF. Recall that this algorithm uses s_r , the r -th best sampling item, as the threshold for accepting items. As implied by Lemma 2, only the k largest items v_1, \dots, v_k contribute to the objective function. One essential idea of our approach is to separate the set of top- k items into two classes according to the following definition.

Definition 2. We say that item v_i is *dominating* if $i \leq r$, and *non-dominating* if $r + 1 \leq i \leq k$.

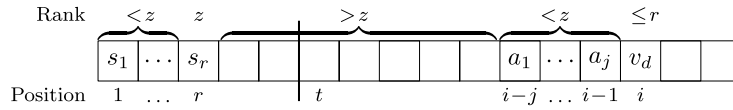


Fig. 1. Event $\tilde{E}_j(z, i)$ considered in the proof of Lemma 3.

The crucial property of dominating items becomes clear in the following scenario: Assume that any dominating item v occurs after the sampling phase. Since s_r is the r -th best item from the sampling phase, it follows that $v > s_r$. That is, each dominating item outside the sampling beats the reference item. Therefore, there are only two situations when dominating items are rejected: Either they appear before position t , or after k accepted items.

4.1. Dominating items

In Lemma 3 we compute the acceptance probability for dominating items. Since the algorithm cannot distinguish any two dominating items, each dominating item has the same probability.

Lemma 3. Let v_d be a dominating item and $j \in [0..k)$. We have

$$p_d^{(j+1)} = \frac{\kappa \tau}{n} \sum_{i=t+j}^n \binom{i-t}{j} \frac{1}{(i-1)^{r+j}},$$

where $\kappa = (r-1+j)^j$ and $\tau = (t-1)^t$.

Proof. Let $E_j(z, i)$ be the event that SINGLE-REF accepts v_d as $(j+1)$ -th item at position $i = \text{pos}(v_d)$ and s_r has rank z (thus $s_r = v_z$) in a random permutation. Note that there must be elements s_1, \dots, s_{r-1} of rank smaller than z in the sampling. Similarly, there must be j accepted elements a_1, \dots, a_j of rank smaller than z outside the sampling, but before position i .

The proof is in several steps. We first consider a stronger event $\tilde{E}_j(z, i)$. Later, we show how the probability of $E_j(z, i)$ can be obtained from $\tilde{E}_j(z, i)$. In the end, the law of total probability yields $p_d^{(j+1)}$.

Analysis of $\tilde{E}_j(z, i)$ Event $\tilde{E}_j(z, i)$ is defined as $E_j(z, i)$ with additional position constraints (see Fig. 1): Elements s_1, \dots, s_r are in this order at the first r positions and elements a_1, \dots, a_j are in this order at the j positions immediately before v_d . Therefore, $\tilde{E}_j(z, i)$ holds if and only if the random input sequence satisfies the following conditions:

- (i) $\text{pos}(s_\ell) = \ell$ for $\ell \in [r]$, $\text{pos}(a_m) = i - j + m - 1$ for $m \in [j]$, and $\text{pos}(v_d) = i$.
- (ii) Elements s_1, \dots, s_{r-1} have rank smaller than z
- (iii) Elements a_1, \dots, a_j have rank smaller than z
- (iv) All remaining items at positions $r+1, \dots, i-j-1$ have rank greater than z .

As described in Section 2.2, we think of sequentially drawing the elements for the positions $1, \dots, r$ first, then $i-j, \dots, i$, and finally $r+1, \dots, i-j-1$. The probability for (i) is

$$\beta := \prod_{\ell=0}^{j+r} \frac{1}{n-\ell} = \frac{1}{n^{j+r+1}},$$

since each item has the same probability to occur at each remaining position. In (ii), the $r-1$ elements can be chosen out of $z-2$ remaining items of rank smaller than z (since v_d is dominating and was already drawn). Therefore, we get a factor of $\binom{z-2}{r-1}$. After this step, there remain $z-2-(r-1) = z-r-1$ elements of rank smaller than z , so we get factor $\binom{z-r-1}{j}$ for step (iii).

Finally, the probability of (iv) can be formulated using Fact 1. Note that at this point, there remain $n-(1+r+j)$ items and no item of rank greater than z has been drawn so far. In terms of the random experiment from Fact 1, we draw $K = i-j-r-1$ balls (items) from an urn of size $N = n-(1+r+j)$ where $M = n-z$ balls are blue (rank greater than z). Hence, the probability for (iv) is $H := h(n-r-j-1, n-z, i-j-r-1)$. Therefore, we obtain

$$\Pr[\tilde{E}_j(z, i)] = \beta \cdot \binom{z-2}{r-1} \binom{z-r-1}{j} \cdot H.$$

This term can be simplified further by applying (R3) and (R2). Let $R = z-2$, $K = r-1$, and $M = j+r-1$. It holds that

$$\binom{z-2}{r-1} \binom{z-r-1}{j} = \binom{R}{K} \binom{R-K}{M-K} \stackrel{(R3)}{=} \binom{R}{M} \binom{M}{K} \stackrel{(R2)}{=} \binom{R}{M} \binom{M}{M-K} = \binom{z-2}{j+r-1} \binom{j+r-1}{j}.$$

Let $\kappa = (j+r-1)^{\underline{j}}$, then $\binom{j+r-1}{j} = \kappa/j!$ and we get

$$\Pr[\tilde{E}_j(z, i)] = \frac{\beta\kappa}{j!} \cdot \binom{z-2}{j+r-1} \cdot H.$$

Relating $\tilde{E}_j(z, i)$ to $E_j(z, i)$ In contrast to $\tilde{E}_j(z, i)$, in the event $E_j(z, i)$, the elements s_1, \dots, s_r can have any positions in $[t-1]$ and a_1, \dots, a_j can have any positions in $[t..i]$. In the random order model, the probability of an event depends linearly on the number of permutations for which the event happens. Hence, we can multiply the probability with corresponding factors $(t-1)^{\underline{t}} =: \tau$ and $(i-t)^{\underline{j}} = \binom{i-t}{j} j!$ and get

$$\Pr[E_j(z, i)] = \binom{i-t}{j} \tau j! \cdot \Pr[\tilde{E}_j(z, i)].$$

Relating $E_j(z, i)$ to $p_d^{(j+1)}$ As the final step, we sum over all possible values for i and z to obtain $p_d^{(j+1)}$. The position $i = \text{pos}(v_d)$ ranges between $t+j$ and n , while the reference rank z is between $r+j+1$ (there are $r-1$ sampling elements and $j+1$ accepted elements of rank less than z) and n . Thus we get:

$$\begin{aligned} p_d^{(j+1)} &= \sum_{i=t+j}^n \sum_{z=r+j+1}^n \Pr[E_j(z, i)] \\ &= \tau j! \sum_{i=t+j}^n \binom{i-t}{j} \sum_{z=r+j+1}^n \Pr[\tilde{E}_j(z, i)] \\ &= \beta\kappa\tau \sum_{i=t+j}^n \binom{i-t}{j} \sum_{z=r+j+1}^n \binom{z-2}{j+r-1} \cdot H \\ &= \beta\kappa\tau \sum_{i=t+j}^n \binom{i-t}{j} \frac{1}{\binom{n-r-j-1}{i-j-r-1}} \sum_{z=r+j+1}^n \binom{z-2}{j+r-1} \binom{n-z}{i-j-r-1}. \end{aligned} \tag{1}$$

The sum over z in (1) can be resolved using (R1). Let $L = n - r - j - 1$, $N = W = r + j - 1$, and $M = i - j - r - 1$. In order to apply (R1) we need to verify $L, M \geq 0$ and $N \geq W \geq 0$. We can assume $k \leq n/2$, since for $k > n/2$, there exists a trivial $(1/2)$ -competitive algorithm. Therefore, we have $L = n - r - j - 1 \geq n - k - (k - 1) - 1 = n - 2k \geq 0$. Further, $i \geq t + j$, thus $i - j \geq t \geq k + 1 \geq r + 1$ which implies $M \geq 0$. The condition $N \geq W \geq 0$ holds trivially. By (R1) we obtain

$$\begin{aligned} &\sum_{z=r+j+1}^n \binom{z-2}{j+r-1} \binom{n-z}{i-j-r-1} \\ &= \sum_{z=0}^{n-r-j-1} \binom{r+j-1+z}{j+r-1} \binom{n-r-j-1-z}{i-j-r-1} \\ &= \sum_{z=0}^L \binom{W+z}{N} \binom{L-z}{M} = \binom{L+W+1}{M+N+1} = \binom{n-1}{i-1}. \end{aligned} \tag{2}$$

By inserting (2) into (1), we obtain the quotient of binomial coefficients $\binom{n-1}{i-1} / \binom{n-r-j-1}{i-j-r-1}$, which can be simplified further using (R3) since

$$\binom{n-1}{i-1} / \binom{n-1-(r+j)}{i-1-(r+j)} = \binom{n-1}{r+j} / \binom{i-1}{r+j} = \frac{(n-1)^{\underline{r+j}}}{(i-1)^{\underline{r+j}}}.$$

Recall $\beta = 1/n^{\underline{j+r+1}}$, thus $(n-1)^{\underline{r+j}} \cdot \beta = 1/n$. Together with (1) we get

$$p_d^{(j+1)} = \beta\kappa\tau \cdot (n-1)^{\underline{r+j}} \sum_{i=t+j}^n \binom{i-t}{j} \frac{1}{(i-1)^{\underline{r+j}}} = \frac{\kappa\tau}{n} \sum_{i=t+j}^n \binom{i-t}{j} \frac{1}{(i-1)^{\underline{r+j}}}, \tag{3}$$

which concludes the proof. \square

4.2. Non-dominating items

It remains to consider the acceptance probabilities of the non-dominating items v_{r+1}, \dots, v_k . Interestingly, these probabilities can be related finally to those for dominating items. First, we obtain the following result.

Lemma 4. *Let v_{r+i} be a non-dominating item with $i \in [k - r]$ and let $j \in [i]$. We have $p_{r+i}^{(j)} = p_{r+i}^{(i+1)}$.*

Proof. We construct a bijective function $f: P \rightarrow Q$ where P (resp. Q) is the set of permutations where v_{r+i} is the j -th (resp. $(i + 1)$ -th) accept.

Let $\pi \in P$. First, we argue that the algorithm accepts at least $i + 1$ elements in π . As v_{r+i} is accepted, $s_r < v_{r+i}$ and thus all elements from the set $S = \{v_1, \dots, v_{r+i}\}$ beat s_r . Since s_r is the r -th best element in the sampling, at most $r - 1$ elements from S can be part of the sampling. Consequently, at least $r + i - (r - 1) = i + 1$ elements from S , including v_{r+i} , are accepted.

Now, let a_1, \dots, a_{i+1} denote the first $i + 1$ accepts, where $a_j = v_{r+i}$. The function f can be defined as follows: Swap the positions of a_1, \dots, a_{i+1} in a cyclic shift, such that $a_j = v_{r+i}$ is at the former position of a_{i+1} . This yields a permutation $f(\pi)$ where v_{r+i} is the $(i + 1)$ -th accept. Note that f is bijective as the cyclic shift can be reversed. \square

By the following lemma, the remaining probabilities can be related to corresponding probabilities for dominating items.

Lemma 5. *Let v_{r+i} be a non-dominating item with $i \in [k - r]$ and let $j \in [k - i]$. Let v_d be any dominating item. It holds that $p_{r+i}^{(i+j)} = p_d^{(i+j)}$.*

Proof. Again, we prove the claim by defining a bijective mapping $f: P \rightarrow Q$ where P is the set of permutations where v_{r+i} is the $(i + j)$ -th accept and Q is the set of permutations where v_d is the $(i + j)$ -th accept. For any $\pi \in P$, let $f(\pi)$ be obtained from π by swapping v_{r+i} with v_d .

We first show that $f: P \rightarrow Q$. To that end, consider any fixed $\pi \in P$. As v_{r+i} is accepted, $s_r < v_{r+i}$. We can argue that s_r is the r -th best sampling element in $f(\pi)$ as well: This holds clearly if no item is moved out of or into the sampling by f . Otherwise, f moves v_{r+i} into the sampling and v_d outwards. But since $s_r < v_{r+i} < v_d$, this does not change the role of s_r as the r -th best sampling element. Therefore, π and $f(\pi)$ have the same reference element s_r . Further, v_d is accepted in $f(\pi)$ at the former position of v_{r+i} . Finally, we observe that f is bijective, since v_{r+i} and v_d have unique ranks. \square

4.3. Competitive ratio

The following main theorem states the exact competitive ratio of SINGLE-REF. By the results from Sections 4.1 and 4.2, the final term only depends on the acceptance probabilities of dominating items.

Theorem 1. *The competitive ratio of SINGLE-REF is*

$$\frac{1}{k} \sum_{j=1}^k \gamma_j \cdot p_1^{(j)} \quad \text{where } \gamma_j = \begin{cases} r + 2 \cdot (j - 1) & \text{if } j \leq k - r + 1 \\ k & \text{else.} \end{cases}$$

Proof. According to Lemma 2, the competitive ratio is given by

$$\frac{1}{k} \sum_{i=1}^k p_i = \frac{1}{k} \left(\sum_{i=1}^r p_i + \sum_{i=1}^{k-r} p_{r+i} \right), \tag{4}$$

where we split the sum according to dominating and non-dominating items. By Lemma 3, $p_i^{(j)} = p_1^{(j)}$ holds for any dominating item v_i and $j \in [k]$. Recall that $p_i = \sum_{j=1}^k p_i^{(j)}$ for all $i \in [k]$. Therefore,

$$\sum_{i=1}^r p_i = r \sum_{j=1}^k p_1^{(j)} = \sum_{j=1}^k r p_1^{(j)}. \tag{5}$$

Now, we consider the sum for non-dominating items. We have

$$\sum_{i=1}^{k-r} p_{r+i} = \sum_{i=1}^{k-r} \sum_{j=1}^k p_{r+i}^{(j)} = \sum_{i=1}^{k-r} \sum_{j=1}^i p_{r+i}^{(j)} + \sum_{i=1}^{k-r} \sum_{j=1}^{k-i} p_{r+i}^{(i+j)}. \tag{6}$$

Next, we simplify the second last sum of Equation (6). From Lemmas 4 and 5 it follows that

$$\sum_{i=1}^{k-r} \sum_{j=1}^i p_{r+i}^{(j)} = \sum_{i=1}^{k-r} i p_{r+i}^{(i+1)} = \sum_{i=1}^{k-r} i p_1^{(i+1)} = \sum_{j=1}^{k-r+1} (j-1) p_1^{(j)}. \tag{7}$$

The last sum in Equation (6) can be simplified using Lemma 5 followed by algebraic manipulations.

$$\sum_{i=1}^{k-r} \sum_{j=1}^{k-i} p_{r+i}^{(i+j)} = \sum_{i=1}^{k-r} \sum_{j=1}^{k-i} p_1^{(i+j)} = \sum_{i=1}^{k-r} \sum_{j=i+1}^k p_1^{(j)} = \sum_{j=1}^{k-r+1} (j-1) p_1^{(j)} + \sum_{j=k-r+2}^k (k-r) p_1^{(j)}. \tag{8}$$

Combining Equations (4) to (8) yields the claim. \square

4.4. Dominating items – asymptotic setting

As we have seen in Theorem 1, the competitive ratio mainly depends on the probabilities $p_1^{(j)}$. However, the term from Equation (3) is cumbersome and hard to optimize over r and t . The goal of this subsection is to derive a lower bound for $p_1^{(j)}$ assuming that n is large enough. For this purpose, we assume $t - 1 = cn$ for $c \in (0, 1)$, i.e., the sampling length is some constant c of the input length. Further, we assume that $k \in o(n)$. We obtain the following lemma.

Lemma 6. Let $j \in [0..k]$. For $\ell \in [0..j]$, define $\beta_\ell := (-1)^\ell \binom{j}{\ell}$ and $\alpha_\ell := \binom{j+r-1}{\ell+r-1}$. Assuming $t - 1 = cn$, it holds that

$$p_1^{(j+1)} \geq \begin{cases} c \cdot \left(\ln \frac{1}{c} - \sum_{\ell=1}^j \beta_\ell \frac{c^\ell - 1}{\ell} \right) - o(1) & \text{if } r = 1 \\ \frac{c}{r-1} \cdot \left(1 - c^{r-1} \cdot \sum_{\ell=0}^j \alpha_\ell (1-c)^{j-\ell} c^\ell \right) - o(1) & \text{if } r \geq 2. \end{cases}$$

Proof of Lemma 6. Let $S := \sum_{i=t+j}^n \binom{i-t}{j} \frac{1}{(i-1)^{r+j}}$ be the sum from Equation (3). First, we obtain a lower bound for S using $(n-k)^k < n^k < n^k$ and similar inequalities:

$$\begin{aligned} S &= \sum_{i=t+j}^n \binom{i-t}{j} \frac{1}{(i-1)^{r+j}} = \frac{1}{j!} \sum_{i=t+j}^n \frac{(i-t)^j}{(i-1)^{r+j}} > \frac{1}{j!} \sum_{i=t+j}^n \frac{(i-t-j+1)^j}{(i-1)^{r+j}} = \frac{1}{j!} \sum_{i=1}^{n-t-j+1} \frac{i^j}{(i+t+j-2)^{r+j}} \\ &= \frac{1}{j!} \sum_{i=1}^m f(i), \end{aligned} \tag{9}$$

where we defined $f(i) := i^j / (i+y)^{r+j}$ with $y := t+j-2$ and $m := n-t-j+1$. Unfortunately, f is not necessarily monotone, hence we cannot apply Fact 2 directly to bound S by a corresponding integral. However, f has a single maximum point $z = \frac{jy}{r}$ and is monotone increasing (resp. monotone decreasing) for $i \leq z$ (resp. $i \geq z$). We prove this property in Lemma 10 in Appendix B. This allows to split S into two monotone parts.

$$\begin{aligned} \sum_{i=1}^m f(i) &= \sum_{i=1}^{\lfloor z \rfloor} f(i) + \sum_{i=\lfloor z \rfloor+1}^m f(i) \\ &\geq \int_0^{\lfloor z \rfloor} f(i) \, di + \int_{\lfloor z \rfloor+1}^{m+1} f(i) \, di && \text{Fact 2, Lemma 10B} \\ &= \int_0^{m+1} f(i) \, di - \int_{\lfloor z \rfloor}^{\lfloor z \rfloor+1} f(i) \, di \\ &= \int_0^{m+1} f(i) \, di - f(z) && \text{Lemma 10A.} \end{aligned} \tag{10}$$

Therefore, if F is a function such that $\int_0^{m+1} f(i) \, di = F(m+1) - F(0)$, we obtain from Lemma 3, Inequality (10), and the previous observation

$$p_1^{(j+1)} \geq \frac{\kappa \tau}{n} S > \frac{\kappa \tau}{nj!} (F(m+1) - F(0)) - \frac{\kappa \tau}{nj!} f(z). \tag{11}$$

We first argue that the term $(\kappa\tau)/(nj!) \cdot f(z)$ from Inequality (11) vanishes for $n \rightarrow \infty$. It holds that $\kappa\tau = (r + j - 1)^{\underline{j}} \cdot (t - 1)^{\underline{r}} < (r + j)^{\underline{j}} \cdot (t - 1)^{\underline{r}}$. Moreover,

$$f(z) = \frac{z^j}{(z + y)^{r+j}} < \frac{z^j}{y^{r+j}} = \left(\frac{zy}{y^2}\right)^j = \left(\frac{j}{r}\right)^j \cdot \frac{1}{y^r} < \frac{j^j}{(t - 1)^r \cdot \left(1 - \frac{1}{t-1}\right)^r},$$

where the last inequality follows from $r \geq 1$ and $y^r \geq (t - 2)^r = (t - 1)^r \cdot \left(1 - \frac{1}{t-1}\right)^r$. Therefore,

$$\frac{\kappa\tau}{nj!} f(z) < \frac{(r + j)^{\underline{j}} \cdot (t - 1)^{\underline{r}}}{nj!} \cdot \frac{j^j}{(t - 1)^r \cdot \left(1 - \frac{1}{t-1}\right)^r} = \frac{(rj + j^2)^{\underline{j}}}{j!} \cdot \frac{1}{n \cdot \left(1 - \frac{1}{cn}\right)^r} = o(1). \tag{12}$$

In the remainder of the proof, we consider the cases $r = 1$ and $r \geq 2$ separately. For both cases we use an appropriate antiderivative F , where we prove $F'(i) = f(i)$ in Lemmas 11 and 12 in Appendix B.

Case $r = 1$ We observe first that in Equation (11), the factor $\frac{\kappa\tau}{nj!}$ resolves to c as

$$\frac{\kappa \cdot \tau}{nj!} = \frac{(j + r - 1)^{\underline{j}} \cdot (t - 1)^{\underline{r}}}{nj!} = \frac{j^{\underline{j}} \cdot (t - 1)^{\underline{1}}}{nj!} = \frac{j! \cdot (t - 1)}{nj!} = c.$$

With $\beta_\ell = (-1)^\ell \binom{j}{\ell}$ for $1 \leq \ell \leq j$ and

$$F(i) = \ln(i + y) - \sum_{\ell=1}^j \frac{\beta_\ell}{\ell} \cdot \left(\frac{y}{i + y}\right)^\ell,$$

it holds that

$$\begin{aligned} & F(m + 1) - F(0) \\ &= \left(\ln(m + 1 + y) - \sum_{\ell=1}^j \frac{\beta_\ell}{\ell} \cdot \left(\frac{y}{m + 1 + y}\right)^\ell \right) - \left(\ln y - \sum_{\ell=1}^j \frac{\beta_\ell}{\ell} \right) \\ &= \ln\left(\frac{m + 1 + y}{y}\right) - \sum_{\ell=1}^j \frac{\beta_\ell}{\ell} \cdot \left(\left(\frac{y}{m + 1 + y}\right)^\ell - 1\right) \\ &= \ln\left(\frac{n}{y}\right) - \sum_{\ell=1}^j \frac{\beta_\ell}{\ell} \cdot \left(\left(\frac{y}{n}\right)^\ell - 1\right) \\ &= \ln\frac{1}{c} - \left(\sum_{\ell=1}^j \frac{\beta_\ell}{\ell} \cdot (c^\ell - 1)\right) - o(1), \end{aligned}$$

where we used $m + 1 + y = n$ and $y = t - j - 2 = cn - j - 1$. Combining with Inequality (11) yields the claim for $r = 1$.

Case $r \geq 2$ Define $\alpha_\ell = \binom{j+r-1}{\ell+r-1}$ for $0 \leq \ell \leq j$ and let

$$F(i) = -\frac{\sum_{\ell=0}^j \alpha_\ell i^{j-\ell} y^\ell}{\alpha_0(r-1)(i+y)^{r+j-1}}.$$

Further, let $G(i) = -\alpha_0 \cdot (r - 1) \cdot F(i)$. Since $\alpha_0 = \binom{j+r-1}{r-1} = \kappa/j!$ and $\tau/n = (t - 1)^{\underline{r}}/n = c \cdot (t - 2)^{\underline{r-1}}$, we have

$$\frac{\kappa\tau}{nj!} (F(m + 1) - F(0)) = \frac{c}{r - 1} \cdot (t - 2)^{\underline{r-1}} \cdot (G(0) - G(m + 1)). \tag{13}$$

Now, it holds that

$$G(0) = \frac{\sum_{\ell=0}^j \alpha_\ell \cdot 0^{j-\ell} \cdot y^\ell}{y^{r+j-1}} = \frac{\alpha_j \cdot y^j}{y^{r+j-1}} = \frac{1}{y^{r-1}} = \frac{1}{(t + j - 2)^{r-1}} \tag{14}$$

and further

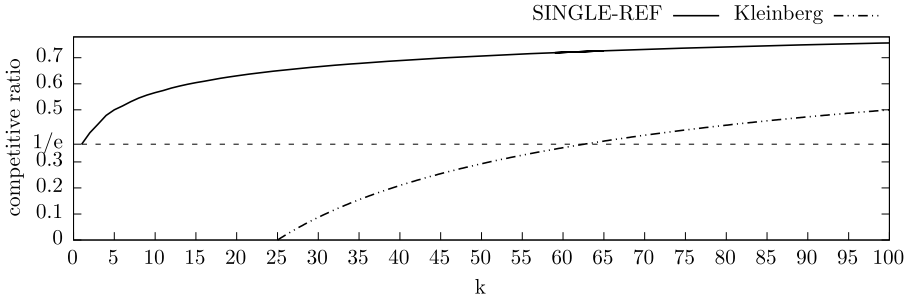


Fig. 2. Comparison of our algorithm SINGLE-REF and the algorithm by Kleinberg [16].

$$(t - 2)^{r-1} \cdot G(0) \geq \frac{(t - r)^{r-1}}{(t + j - 2)^{r-1}} = \left(1 - \frac{j + r - 2}{t + j - 2}\right)^{r-1} = 1 - o(1). \tag{15}$$

Finally, we analyze the term $(t - 2)^{r-1} \cdot G(m + 1)$. Using $m = n - t - j + 1$ and $y = t + j - 2 = cn + j - 1$, we obtain

$$\begin{aligned} (t - 2)^{r-1} \cdot G(m + 1) &= (t - 2)^{r-1} \cdot \frac{\sum_{\ell=0}^j \alpha_{\ell} \cdot (m + 1)^{j-\ell} \cdot y^{\ell}}{(m + 1 + y)^{r+j-1}} \\ &\leq (t - 2)^{r-1} \cdot \frac{\sum_{\ell=0}^j \alpha_{\ell} \cdot (n \cdot (1 - c) + 1)^{j-\ell} \cdot (cn + j - 1)^{\ell}}{n^{r+j-1}} \\ &\leq (t - 2)^{r-1} \cdot \frac{\left(\sum_{\ell=0}^j \alpha_{\ell} \cdot (n \cdot (1 - c))^{j-\ell} \cdot (cn)^{\ell}\right)}{n^{r+j-1}} + o(1) \\ &< (t - 1)^{r-1} \cdot \frac{\left(\sum_{\ell=0}^j \alpha_{\ell} \cdot n^{j-\ell} \cdot (1 - c)^{j-\ell} \cdot c^{\ell} n^{\ell}\right)}{n^{r+j-1}} + o(1) \\ &= c^{r-1} \cdot \left(\sum_{\ell=0}^j \alpha_{\ell} \cdot (1 - c)^{j-\ell} \cdot c^{\ell}\right) + o(1). \end{aligned} \tag{16}$$

Combining Equations (11) to (13), (15) and (16) concludes the proof. \square

4.5. Competitive ratio – asymptotic setting

The competitive ratio of SINGLE-REF in the asymptotic setting $n \rightarrow \infty$ can be evaluated using Lemma 6 in combination with Theorem 1. For $k \in [1..100]$, we optimized the resulting objective function over r and c numerically. As shown in Fig. 2, SINGLE-REF reaches competitive ratios of up to 0.75 and outperforms the algorithm by Kleinberg [16] on this interval. The optimal parameters for $k \in [1..100]$ and the resulting competitive ratios can be found in Table A.2 of Appendix A.

5. OPTIMISTIC for $k = 2$

Let \mathcal{A}_2 denote the OPTIMISTIC algorithm with $k = 2$ and parameters r, t in the following. To analyze the relevant probabilities p_1 and p_2 , we again relate these probabilities to corresponding sets (see Section 2.2). For $i \in \{1, 2\}$, let P_i be the set of permutations in which \mathcal{A}_2 accepts v_i .

5.1. Acceptance probability of v_2

In Lemma 7, we show a surprising relation between OPTIMISTIC for $k = 2$ and SINGLE-REF for $k = r = 1$: Assuming that both algorithms have the same sampling length of $t - 1$, the probability that OPTIMISTIC accepts v_2 is exactly the probability that SINGLE-REF accepts v_1 . The proof uses a sophisticatedly tailored bijection between two respective sets of permutations.

Lemma 7. *Let \mathcal{A}_1 be the SINGLE-REF algorithm with parameters $k = r = 1$ and t . It holds that $\Pr[\mathcal{A}_2 \text{ accepts } v_2] = \Pr[\mathcal{A}_1 \text{ accepts } v_1]$.*

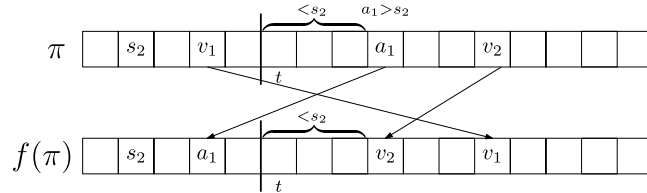
Proof. Let Ω be the set of all $n!$ permutations and $Q_1 \subset \Omega$ be the set of permutations where \mathcal{A}_1 accepts v_1 . We prove the claim by constructing a bijective function $f: \Omega \setminus P_2 \rightarrow \Omega \setminus Q_1$.

We first investigate the two complementary sets $\Omega \setminus P_2$ and $\Omega \setminus Q_1$. The set $\Omega \setminus P_2$ contains the permutations where v_2 is not accepted by \mathcal{A}_2 . This occurs in exactly one of three cases: v_2 is in the sampling, v_2 comes behind the first accept

and v_1 is in the sampling (then \mathcal{A}_2 will reject all following elements), or v_2 comes behind two elements accepted by \mathcal{A}_2 . Similarly, $\Omega \setminus Q_1$ contains the permutations where v_1 is not accepted by \mathcal{A}_1 . In these permutations, v_1 is either in the sampling, or behind the first accepted element.

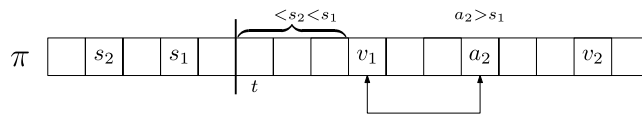
Now, fix any $\pi \in \Omega \setminus P_2$. To define $f(\pi)$, we distinguish five cases (A)-(E) based on the position of v_2 in π and the accepted elements of \mathcal{A}_2 on input π .

- (A) v_2 is in the sampling. We obtain $f(\pi)$ from π by swapping v_1 with v_2 . In $f(\pi)$, item v_1 is in the sampling and cannot be accepted by \mathcal{A}_1 .
- (B) v_2 comes behind the first accept and v_1 is in the sampling. Then, we have an accepted element a_1 with $s_2 < a_1 < v_2$. To obtain $f(\pi)$, we swap v_1 with a_1 and afterwards v_1 with v_2 :

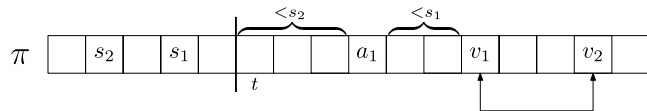


Since $a_1 > s_2$, item a_1 is the best element in the sampling of $f(\pi)$. Particularly, the second best element s_2 of the sampling is maintained in this case. This fact will be important later. Since v_2 is in front of v_1 in $f(\pi)$, algorithm \mathcal{A}_1 accepts v_2 in $f(\pi)$.

- (C) v_2 comes behind two accepts and v_1 is the first accept. Then, there must be another accept a_2 between v_1 and v_2 . We obtain $f(\pi)$ by swapping v_1 with a_2 . Since $a_2 > s_1$, algorithm \mathcal{A}_1 accepts a_2 in $f(\pi)$.



- (D) v_2 comes behind two accepts and v_1 is the second accept. Here, we define $f(\pi)$ such that v_1 is swapped with v_2 . \mathcal{A}_1 accepts v_2 in $f(\pi)$.



- (E) v_2 comes behind two accepts and v_1 is not accepted. Set $f(\pi) = \pi$ in this case. Since there is at least one item better than s_1 before v_1 in $f(\pi)$, algorithm \mathcal{A}_1 cannot select v_1 in $f(\pi)$.

In order to show the bijectivity of f , we have to argue carefully following the definition of f in the different cases. Let $\mathcal{C} = \{A, B, C, D, E\}$ represent the set of cases (A)-(E).

f is injective Let $\pi_1, \pi_2 \in \Omega \setminus P_2$ with $\pi_1 \neq \pi_2$. We have to show $f(\pi_1) \neq f(\pi_2)$. Let $X, Y \in \mathcal{C}$ be such that π_1 and π_2 satisfy the conditions of case X and Y , respectively.

First, we consider the situation where π_1 and π_2 are mapped by f according to the same case $X = Y$. In all cases, the operation defined by f involves v_1, v_2 , and possibly a further accepted element. All of these elements can be retrieved given any permutation $f(\pi)$: Items v_1 and v_2 have a unique rank, item a_1 from case B is the maximum element in the sampling, and item a_2 from case C is the first item after the sampling better than s_1 . Hence, $f(\pi_1) = f(\pi_2)$ implies $\pi_1 = \pi_2$, which is equivalent to the claim we wanted to show.

It remains to consider all pairs of cases where $X \neq Y$.

$X = A, Y \in \{B, C, D, E\}$. Since item v_1 is moved into the sampling in case (A) and to some position after the sampling in all remaining cases (B)-(E), we immediately get $f(\pi_1) \neq f(\pi_2)$.

$X = E, Y \in \{C, D\}$. Assume $f(\pi_1) = f(\pi_2)$ for contradiction. In all cases (C)-(E), the function f maintains the items in the sampling phase. Particularly, π_1 and π_2 must have the same two best elements s_1, s_2 . The construction in case (E) ensures that \mathcal{A}_2 would accept two elements in $f(\pi_1)$ before position $\min\{\text{pos}_{f(\pi_1)}(v_1), \text{pos}_{f(\pi_1)}(v_2)\}$, while there is only one such accept in $f(\pi_2)$. This contradicts $f(\pi_1) = f(\pi_2)$.

$X = B, Y \in \{D, E\}$. Assume $f(\pi_1) = f(\pi_2)$ for contradiction. In all three cases (B),(D),(E), the function f maintains the second best sampling item s_2 . Similar to the previous case, we get a contradiction as there is no element before v_2 better than s_2 in $f(\pi_1)$, while $f(\pi_2)$ has at least one such item. Thus, $f(\pi_1) \neq f(\pi_2)$.

$X = C, Y \in \{B, D\}$. By construction, item v_1 is before v_2 in $f(\pi_1)$, while the relative order of v_1 and v_2 in $f(\pi_2)$ is the other way round. Therefore, $f(\pi_1) \neq f(\pi_2)$.

f is surjective Let $\pi' \in \Omega \setminus Q_1$. We show that there is $\pi \in \Omega \setminus P_2$ with $f(\pi) = \pi'$.

The obvious case is when v_1 is in the sampling of π' , then π can be obtained from case (A). If v_1 is not in the sampling of π' , there must be an element before v_1 that \mathcal{A}_1 accepts. Consequently, \mathcal{A}_2 would accept at least one element before v_1 in π' as well. Furthermore, it follows that v_2 is not in the sampling either, as otherwise v_1 would be accepted by \mathcal{A}_1 . Therefore, considering \mathcal{A}_2 on input π' , item v_2 can be the first or second accept, or can follow two accepts.

If v_2 is the first accept of \mathcal{A}_2 , the desired π can be constructed according to case (B). Similarly, the situation where v_2 is the second accept of \mathcal{A}_2 corresponds to case (D). Finally, consider the case where v_2 follows two accepted elements. If v_1 is the second accept, π' was obtained from case (C), and if v_1 is not accepted from case (E). Note that v_1 cannot be the first accept of \mathcal{A}_2 , as otherwise \mathcal{A}_1 would also accept v_1 which would contradict $\pi' \in \Omega \setminus Q_1$. \square

5.2. Acceptance probability of v_1

In this part, we prove $p_1 = p_2 + \delta$ for $\delta > 0$. First, we observe that P_2 can be related to a set $P'_1 \subset P_1$ of equal cardinality.

Lemma 8. Let $P'_1 = \{\pi \in P_1 \mid \text{pos}_\pi(v_2) < t \Rightarrow \mathcal{A}_2 \text{ accepts } v_1 \text{ as the first item}\}$. It holds that $|P'_1| = |P_2|$.

Proof. Let f be the function that swaps v_1 with v_2 in a given input sequence. We first prove that $f(\pi) \in P'_1$ holds for each $\pi \in P_2$ and thus $f : P_2 \rightarrow P'_1$.

Let $\pi \in P_2$. In $f(\pi)$, algorithm \mathcal{A}_2 accepts v_1 at position $\text{pos}_{f(\pi)}(v_1) = \text{pos}_\pi(v_2)$, as $v_1 > v_2$. So far we showed $f(\pi) \in P_1$. If $\text{pos}_{f(\pi)}(v_2) \geq t$, there is nothing to show. Assuming that $\text{pos}_{f(\pi)}(v_2) < t$, it follows that v_1 is the best element in the sampling of π . Since no item (particularly not v_2) beats v_1 , but v_2 is accepted by \mathcal{A}_2 in π , we get that v_2 is the first accept in π . Hence, v_1 is the first accept in $f(\pi)$ and therefore $f(\pi) \in P'_1$.

Clearly, f is injective. To prove surjectivity, let $\pi' \in P'_1$ and let π the permutation obtained from π' by swapping (back) v_1 with v_2 . If $\text{pos}_{\pi'}(v_2) < t$, by definition of P'_1 we know that v_1 is the first accept in π' , implying that no item before $\text{pos}_{\pi'}(v_1) = \text{pos}_\pi(v_2)$ is accepted by \mathcal{A}_2 . In the case $\text{pos}_{\pi'}(v_2) \geq t$, since $\text{pos}_{\pi'}(v_1) \geq t$, the smallest rank in the sampling of π' is 3 or greater. Therefore, v_2 is accepted if at most one item before v_2 is accepted. This holds for π , as $\text{pos}_\pi(v_2) = \text{pos}_{\pi'}(v_1)$. \square

From $|P_1| = |P'_1| + |P_1 \setminus P'_1| = |P_2| + |P_1 \setminus P'_1|$ we get $p_1 = p_2 + \delta$, where $\delta = |P_1 \setminus P'_1|/n!$. That is, δ is the probability that a random permutation is element of $|P_1 \setminus P'_1|$. We analyze this event in Lemma 9 using a similar counting argument as in the proof of Lemma 3.

Lemma 9. Let $\delta = \Pr[\pi \in P_1 \setminus P'_1]$ where π is drawn uniformly at random from the set of all permutations and P'_1 is defined as in Lemma 8. It holds that $\delta = \frac{t-1}{n} \frac{t-2}{n-1} \sum_{i=t}^{n-1} \frac{n-i}{(i-2)(i-1)}$.

Proof. The set $P_1 \setminus P'_1$ contains exactly those permutations where v_2 is in the sampling and \mathcal{A}_2 accepts v_1 as the second item. Hence, $s_1 = v_2$, and there exists an accepted item v_x before v_1 .

Let $z \geq 3$ be the rank of $s_2 = v_z$ and $x \in [3..z]$ be the rank of the first accepted element v_x . Let π be drawn uniformly at random. Consider the following sequence of random events.

- (i) $\text{pos}(v_x) = i$ for $i \in [t..n]$
- (ii) $\text{pos}(v_1) = \ell$ for $\ell \in (i..n)$
- (iii) $\text{pos}(v_2) \leq t - 1$
- (iv) $\text{pos}(s_2) \leq t - 1$
- (v) All items with positions in $[i - 1] \setminus \{\text{pos}(s_2), \text{pos}(v_2)\}$ have ranks greater than z .

The above conditions exactly characterize the event $\pi \in P_1 \setminus P'_1$ (see also Fig. 3): They ensure that the best two sampling elements are $s_1 = v_2$ and s_2 and fix the positions of v_x and v_1 to be i and ℓ , respectively. Further, all elements before position i except from s_1 and s_2 must have ranks greater than z , such that in fact s_1 and s_2 are the best two elements in the sampling and no item before v_x is selected by \mathcal{A}_2 . Note that we do not need an extra event ensuring that v_1 gets accepted if it comes after the first accept: Since the second accept must beat $s_1 = v_2$, the only item with this property is v_1 .

The probability for the first four parts (i-iv) is $\beta := \frac{1}{n} \frac{1}{n-1} \frac{t-1}{n-2} \frac{t-2}{n-3}$. For event (v), we need the probability that the next $i - 3$ items, drawn from the set of $n - 4$ remaining items, all have rank greater than z . As no item from $\{v_{z+1}, \dots, v_n\}$ was

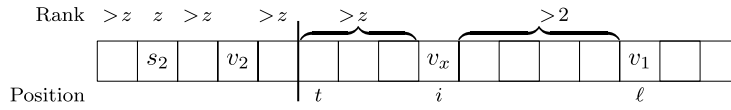


Fig. 3. Event $\pi \in P_1 \setminus P'_1$ considered in Lemma 9.

drawn so far, $n - z$ items of rank greater than z remain. Therefore, by Fact 1 the probability for step (v) is $h(n - 4, n - z, i - 3)$. By the law of total probability we obtain finally

$$\delta = \sum_{i=t}^{n-1} \sum_{\ell=i+1}^n \sum_{z=3}^n \sum_{x=3}^{z-1} \beta \cdot h(n - 4, n - z, i - 3) = \beta \sum_{i=t}^{n-1} (n - i) \frac{1}{\binom{n-4}{i-3}} \sum_{z=3}^n (z - 3) \binom{n - z}{i - 3}. \tag{17}$$

The last term can be simplified further. First, we eliminate the sum over z by applying (R1):

$$\sum_{z=3}^n (z - 3) \binom{n - z}{i - 3} = \sum_{z=0}^{n-3} \binom{z}{1} \binom{n - 3 - z}{i - 3} = \binom{n - 2}{i - 1}. \tag{18}$$

Using Equation (18) in Equation (17) yields

$$\delta = \beta \sum_{i=t}^{n-1} (n - i) \frac{\binom{n-2}{i-1}}{\binom{n-4}{i-3}} = \beta \sum_{i=t}^{n-1} (n - i) \frac{(n - 3)(n - 2)}{(i - 2)(i - 1)}$$

and the claim follows by resubstituting β . \square

5.3. Competitive ratio

Finally we can state the competitive ratio of OPTIMISTIC in the case $k = 2$. Again, we consider the asymptotic setting where $n \rightarrow \infty$ and $t - 1 = cn$ for some constant $c \in (0, 1)$.

Theorem 2. OPTIMISTIC is 0.4168-competitive for $k = 2$ assuming $t - 1 = cn$ for $c = 0.3521$.

Proof. By the relation between OPTIMISTIC and SINGLE-REF proven in Lemma 7 and by Lemma 3, we obtain $p_2 = \frac{t-1}{n} \sum_{i=t}^n \frac{1}{i-1}$. As proven in Lemma 6, this term approaches $c \ln(1/c)$ in the asymptotic setting. Further, in Section 5.2 we showed $p_1 = p_2 + \delta$ where $\delta = \frac{t-1}{n} \frac{t-2}{n-1} \sum_{i=t}^{n-1} \frac{n-i}{(i-2)(i-1)}$. The term $\sum_{i=t}^{n-1} \frac{n-i}{(i-2)(i-1)}$ is bounded asymptotically from above and below by $\frac{1}{c} - \ln \frac{1}{c} - 1$ (see Lemma 13 of Appendix C for a proof). Further, $\frac{t-1}{n} \frac{t-2}{n-1} = c^2 - \frac{1-c}{n-1} = c^2 - o(1)$. According to Lemma 2, the competitive ratio is

$$\frac{1}{2} (p_1 + p_2) = \frac{1}{2} (p_2 + \delta + p_2) = c \ln \frac{1}{c} + \frac{c^2}{2} \left(\frac{1}{c} - \ln \frac{1}{c} - 1 \right).$$

The optimal choice for c is around $c^* = 0.3521 < 1/e$, which gives a competitive ratio of 0.4168. \square

6. Conclusion and future work

In this work, we investigated two algorithms for the k -secretary problem with a focus on small values for $k \geq 2$. We introduced and analyzed the algorithm SINGLE-REF. For any value of k , the competitive ratio of SINGLE-REF can be obtained by numerical optimization. Further, we provided a tight analysis of the OPTIMISTIC algorithm [8] in the case $k = 2$.

We see various directions of future work. For SINGLE-REF, it remains to find the right dependency between the parameters r , c , and k in general and, if possible, to find a closed formula for the competitive ratio for any value of k . OPTIMISTIC seems a promising and elegant algorithm, however no tight analysis for general $k \geq 3$ is known so far. For $k = 2$, we identified a key property in Lemma 7. Similar properties may hold in the general case.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Optimal parameters for SINGLE-REF

Table A.2

Optimal parameters and corresponding competitive ratios (c.r.) of SINGLE-REF for $k \in [1..100]$. For readability, the numeric values are truncated after the fourth decimal place.

k	r	c	c.r.	k	r	c	c.r.	k	r	c	c.r.
1	1	0.3678	0.3678	35	7	0.1662	0.6782	69	12	0.1508	0.7309
2	1	0.2545	0.4119	36	8	0.1830	0.6805	70	12	0.1489	0.7320
3	2	0.3475	0.4449	37	8	0.1788	0.6829	71	12	0.1470	0.7330
4	2	0.2928	0.4785	38	8	0.1748	0.6851	72	12	0.1452	0.7340
5	2	0.2525	0.4999	39	8	0.1710	0.6873	73	12	0.1434	0.7350
6	2	0.2217	0.5148	40	8	0.1673	0.6893	74	12	0.1417	0.7360
7	3	0.2800	0.5308	41	8	0.1638	0.6912	75	12	0.1400	0.7369
8	3	0.2549	0.5453	42	8	0.1605	0.6930	76	12	0.1384	0.7378
9	3	0.2338	0.5567	43	9	0.1750	0.6948	77	13	0.1473	0.7387
10	3	0.2159	0.5660	44	9	0.1716	0.6968	78	13	0.1456	0.7397
11	4	0.2570	0.5740	45	9	0.1683	0.6986	79	13	0.1440	0.7406
12	4	0.2410	0.5834	46	9	0.1651	0.7004	80	13	0.1424	0.7415
13	4	0.2267	0.5914	47	9	0.1621	0.7021	81	13	0.1408	0.7424
14	4	0.2140	0.5983	48	9	0.1592	0.7037	82	13	0.1393	0.7433
15	4	0.2026	0.6043	49	9	0.1563	0.7052	83	13	0.1378	0.7441
16	4	0.1924	0.6096	50	9	0.1536	0.7067	84	13	0.1363	0.7449
17	5	0.2231	0.6155	51	10	0.1662	0.7082	85	13	0.1349	0.7457
18	5	0.2133	0.6211	52	10	0.1635	0.7098	86	14	0.1429	0.7465
19	5	0.2042	0.6261	53	10	0.1608	0.7113	87	14	0.1415	0.7473
20	5	0.1959	0.6306	54	10	0.1582	0.7127	88	14	0.1400	0.7482
21	5	0.1882	0.6347	55	10	0.1557	0.7141	89	14	0.1386	0.7490
22	5	0.1811	0.6384	56	10	0.1532	0.7155	90	14	0.1372	0.7497
23	6	0.2054	0.6426	57	10	0.1509	0.7168	91	14	0.1359	0.7505
24	6	0.1985	0.6465	58	10	0.1486	0.7180	92	14	0.1346	0.7512
25	6	0.1919	0.6502	59	11	0.1597	0.7193	93	14	0.1333	0.7520
26	6	0.1858	0.6535	60	11	0.1574	0.7206	94	14	0.1320	0.7527
27	6	0.1800	0.6566	61	11	0.1551	0.7219	95	14	0.1307	0.7534
28	6	0.1746	0.6595	62	11	0.1529	0.7231	96	15	0.1381	0.7541
29	7	0.1947	0.6625	63	11	0.1508	0.7243	97	15	0.1368	0.7548
30	7	0.1893	0.6655	64	11	0.1487	0.7255	98	15	0.1356	0.7555
31	7	0.1842	0.6684	65	11	0.1467	0.7266	99	15	0.1343	0.7562
32	7	0.1793	0.6711	66	11	0.1447	0.7277	100	15	0.1331	0.7569
33	7	0.1747	0.6736	67	11	0.1428	0.7287				
34	7	0.1703	0.6760	68	12	0.1527	0.7298				

Appendix B. Omitted proofs for SINGLE-REF

Lemma 10. Let $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $f(i) = i^j / (i + y)^{r+j}$ and constants $j \geq 0, r \geq 1$, and $y > 0$. Define $z = \frac{jy}{r}$. The function f has the following properties:

- (A) f has a global maximum point at z ,
- (B) f is monotonically increasing on $[0, z]$ and monotonically decreasing on $[z, \infty)$.

Proof. Let $g(i) = i^j$ and $h(i) = (i + y)^{r+j}$, thus $f(i) = g(i)/h(i)$. For the first derivative of f we obtain $f'(i) = \frac{g'(i)h(i) - g(i)h'(i)}{h(i)^2}$. Since $h(i)^2$ is non-negative for all $i \geq 0$, we have

$$\begin{aligned}
 f'(i) \geq 0 &\Leftrightarrow g'(i)h(i) \geq g(i)h'(i) \\
 &\Leftrightarrow ji^{j-1}(i + y)^{r+j} \geq i^j(r + j)(i + y)^{r+j-1} \\
 &\Leftrightarrow j(i + y) \geq i(r + j) \\
 &\Leftrightarrow z \geq i.
 \end{aligned}$$

Hence, f is monotonically increasing on $[0, z]$ and monotonically decreasing on $[z, \infty)$. An analogous calculus shows $f'(i) = 0$ if and only if $i = z$. Therefore, z is a global maximum point. \square

Lemma 11. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(i) = i^j / (i + y)^{r+j}$ and constants $j \geq 0, r = 1$, and $y > 0$. The following function F fulfills $F'(i) = f(i)$:

$$F(i) = \ln(i + y) - \sum_{\ell=1}^j \frac{\beta_\ell}{\ell} \cdot \left(\frac{y}{i + y}\right)^\ell,$$

where $\beta_\ell = (-1)^\ell \binom{j}{\ell}$ for $1 \leq \ell \leq j$.

Proof. We have

$$\begin{aligned} F'(i) &= \frac{1}{i + y} - \sum_{\ell=1}^j \frac{\beta_\ell}{\ell} \cdot \ell \cdot \left(\frac{y}{i + y}\right)^{\ell-1} \cdot \left(-\frac{y}{(i + y)^2}\right) \\ &= \frac{1}{i + y} + \sum_{\ell=1}^j \beta_\ell \cdot \frac{y^\ell}{(i + y)^{\ell+1}} \\ &= \sum_{\ell=0}^j \beta_\ell \cdot \frac{y^\ell}{(i + y)^{\ell+1}} \\ &= \sum_{\ell=0}^j (-1)^\ell \binom{j}{\ell} \cdot \frac{y^\ell}{(i + y)^{\ell+1}} \\ &= \frac{1}{(i + y)^{j+1}} \cdot \sum_{\ell=0}^j \binom{j}{\ell} \cdot (-y)^\ell \cdot (i + y)^{j-\ell} \\ &= \frac{i^j}{(i + y)^{j+1}}, \end{aligned}$$

where the last inequality follows from the binomial theorem: For all $a, b, n \in \mathbb{Z}$ it holds that $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$. \square

Lemma 12. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(i) = i^j / (i + y)^{r+j}$ and constants $j \geq 0, r \geq 2$, and $y > 0$. The following function F fulfills $F'(i) = f(i)$:

$$F(i) = -\frac{\sum_{\ell=0}^j \alpha_\ell i^{j-\ell} y^\ell}{\alpha_0 (r-1) (i + y)^{r+j-1}},$$

where $\alpha_\ell = \binom{j+r-1}{\ell+r-1}$ for $0 \leq \ell \leq j$.

Proof. Let $G(i) = -\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell} \cdot y^\ell$ and $H(i) = \alpha_0 \cdot (r-1) \cdot (i + y)^{r+j-1}$ be the numerator and denominator of $F(i)$, respectively. We derive the first derivatives $G'(i) = -\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell-1} \cdot y^\ell \cdot (j-\ell)$ and $H'(i) = \alpha_0 \cdot (r-1) \cdot (r+j-1) \cdot (i + y)^{r+j-2} = H(i) \cdot \frac{r+j-1}{i+y}$. Therefore,

$$\begin{aligned} F'(i) &= \frac{G'(i) \cdot H(i) - G(i) \cdot H'(i)}{H(i)^2} \\ &= \frac{G'(i) - G(i) \cdot \frac{r+j-1}{i+y}}{H(i)} \\ &= \frac{G'(i) \cdot (i + y) - G(i) \cdot (r + j - 1)}{\alpha_0 \cdot (r - 1) \cdot (i + y)^{r+j}}. \end{aligned}$$

Hence, the claim follows if we can show

$$G'(i) \cdot (i + y) - G(i) \cdot (r + j - 1) = i^j \cdot \alpha_0 \cdot (r - 1). \tag{B.1}$$

To show Equation (B.1), we observe

$$\begin{aligned} &G'(i) \cdot (i + y) - G(i) \cdot (r + j - 1) \\ &= -\left(\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell-1} \cdot y^\ell \cdot (j - \ell)\right) \cdot (i + y) + \left(\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell} \cdot y^\ell\right) \cdot (r + j - 1) \end{aligned}$$

$$\begin{aligned}
 &= - \left(\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell} \cdot y^\ell \cdot (j-\ell) \right) - \left(\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell-1} \cdot y^{\ell+1} \cdot (j-\ell) \right) \\
 &\quad + \left(\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell} \cdot y^\ell \cdot (r+j-1) \right) \\
 &= \left(\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell} \cdot y^\ell \cdot (r-1+\ell) \right) - \left(\sum_{\ell=0}^j \alpha_\ell \cdot i^{j-\ell-1} \cdot y^{\ell+1} \cdot (j-\ell) \right).
 \end{aligned}$$

Let S_1 and S_2 be the first and the second sum in the last expression, respectively. Note that S_1 can be rewritten as follows

$$\begin{aligned}
 S_1 &= \alpha_0 \cdot i^j \cdot (r-1) + \sum_{\ell=1}^j \alpha_\ell \cdot i^{j-\ell} \cdot y^\ell \cdot (r-1+\ell) \\
 &= \alpha_0 \cdot i^j \cdot (r-1) + \sum_{\ell=1}^j \alpha_{\ell-1} \cdot i^{j-\ell} \cdot y^\ell \cdot (j-\ell+1) \\
 &= \alpha_0 \cdot i^j \cdot (r-1) + S_2,
 \end{aligned}$$

where we used the fact that $\frac{\alpha_\ell}{\alpha_{\ell-1}} = \frac{j-\ell+1}{\ell+r-1}$ for $1 \leq \ell \leq j$ for the second equality. This proves Equation (B.1) and concludes the proof of the lemma. \square

Appendix C. Omitted proofs for OPTIMISTIC

Lemma 13. *Assuming $t - 1 = cn$ for $c \in (0, 1)$, it holds that*

$$\frac{1}{c} - \ln \frac{1}{c} - 1 - o(1) \leq \sum_{i=t}^{n-1} \frac{n-i}{(i-2)(i-1)} \leq \frac{1}{c} - \ln \frac{1}{c} - 1 + o(1).$$

Proof. The lower and upper bounds follow basically from Fact 2A. For the lower bound, note that $(n-i)/i^2$ decreases monotonically in i . Therefore,

$$\begin{aligned}
 \sum_{i=t}^{n-1} \frac{n-i}{(i-2)(i-1)} &> \sum_{i=t}^{n-1} \frac{n-i}{i^2} \geq \int_t^n \frac{n-i}{i^2} di = \ln \frac{t}{n} + \frac{n}{t} - 1 > \ln \frac{t-1}{n} + \frac{n}{t-1} \cdot \frac{t-1}{t} - 1 \\
 &= \frac{1}{c} - \ln \frac{1}{c} - 1 - \underbrace{\frac{1}{ct}}_{=o(1)}.
 \end{aligned}$$

The upper bound follows likewise. Observe that

$$\sum_{i=t}^{n-1} \frac{n-i}{(i-2)(i-1)} < \sum_{i=t}^n \frac{n-i}{(i-2)^2} < \sum_{i=t-2}^n \frac{n-i}{i^2} = \left(\sum_{i=t}^n \frac{n-i}{i^2} \right) + \xi,$$

where

$$\xi = \frac{n-(t-2)}{(t-2)^2} + \frac{n-(t-1)}{(t-1)^2} = o(1).$$

Since $\frac{n+2-i}{i^2}$ decreases monotonically in i , we obtain

$$\left(\sum_{i=t}^n \frac{n-i}{i^2} \right) + \xi \leq \left(\int_{t-1}^n \frac{n-i}{i^2} di \right) + \xi = \ln \frac{t-1}{n} + \frac{n}{t-1} - 1 + \xi = \frac{1}{c} - \ln \frac{1}{c} - 1 + \xi. \quad \square$$

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