LABORATORIUM FÜR DEN KONSTRUKTIVEN INGENIEURBAU (LKI)
TECHNISCHE UNIVERSITÄT MÜNCHEN

BERICHTE
zur
ZUVERLÄSSIGKEITSTHEORIE DER BAUWERKE

STRUCTURAL RELIABILITY UNDER NON-STATIONARY GAUSSIAN VECTOR PROCESS LOADS

J.-Y. Plantec and R. Rackwitz

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ABSTRACT: Breitung's asymptotic result for the crossing rate of stationary Gaussian vector processes out of smooth, curved failure surfaces is generalized to non–stationary processes and/or time–dependent failure surfaces. A suitable special solution for the corresponding Laplace integral is derived. It is further extended to also perform the necessary time integration for the mean number of outcrossings in a given time interval. Thereby, one has to distinguish between the case where the maximum outcrossing rate occurs in the interior of the time integration interval and the case where it is at one of its boundaries . The results are applied to two illustrating examples.

1. INTRODUCTION

Structural reliability problems with Gaussian vector process loading usually are solved by the well–known outcrossing approach. Let $F = \{g(\mathbf{x},\tau) \leq 0\}$ be a failure set with boundary ∂F and $\mathbf{X}(\tau)$ a vector process with continuously differentiable sample paths. Also denote by $N^{+}(\partial F;t)$ the number of outcrossings in a time interval $[0,t]$ and by $\nu^{+}(\partial F;\tau)$ the outcrossing rate. Under suitable regularity conditions, the following reliability estimate can be derived [1]

$$R(t) \geq R(0) - E[N^{+}(\partial F;t)]\, R(0) \tag{1}$$

with

$$E[N^{+}(\partial F;t)] = \int_{0}^{t} \nu^{+}(\partial F;\tau)\, d\tau \tag{2}$$

the mean number of outcrossings. $R(0)$ is the probability that $\mathbf{X}(0) \in \bar{F}$ with $\bar{F}$ being the complement of F, i.e. the safe set. If $\mathbf{X}(\tau)$ is a sufficiently mixing process, then the asymptotic approximation

$$R(t) \sim \exp[-E[N^+(\partial F;t)]] \tag{3}$$

can be proved for rare crossing events or high reliability R(t) [2]. On heuristic grounds a slight improvement of this formula has been proposed by setting

$$R(t) \sim R(0)\exp[-E[N^+(\partial F;t)]/R(0)] \tag{4}$$

where the modified term in the exponent must be interpreted as a conservative approximation to $E[N(\partial F;t) \mid R(0)]$. It improves the results for small t.

In general, the failure set additionally depends on a random vector **Z** which collects time–invariant strength or geometry parameters but can also include uncertain process parameters such as the mean value vector. This implies that the mixing condition for the crossing events is violated and eq. (3) or (4) are no more valid while eq. (1) still holds. Since it can be shown that eq. (3) provides a certain numerical improvement over eq. (1) and is a non–degenerate complementary distribution for the time to first failure eq. (3) is preferred. In this case the reliability in eq. (3) must first be calculated conditional on $\mathbf{Z} = \mathbf{z}$. The total reliability is obtained upon integration, i.e.

$$R(t) = \int R(t \mid \mathbf{Z} = \mathbf{z})\, dF_{\mathbf{Z}}(\mathbf{z}) \tag{5}$$

where $dF_{\mathbf{Z}}(\mathbf{z})$ is the joint distribution of **Z**. This complication in practical applications will require some thought and we will deal with it in the last section. For the following we temporarily assume that all conditions are valid for eq. (3) to hold.

If $X(\tau)$ is a scalar process and, thus, the safe set is bounded by a simple threshold function, exact formulae are available for the non–stationary "upcrossing" rate in eq. (2) but an analytical solution for $E[N^+(\partial F;t)]$ is known only for a very special case (for a covariance stationary process with linearly varying mean and/or threshold [2]). Otherwise, one usually has to integrate numerically over time. Guers/Rackwitz [3] proposed an approximation for the time integration in eq. (2) by using the asymptotic concepts for so–called Laplace integrals to which the integral in eq. (2) can be reduced.

Exact formulae for the outcrossing rate for vector processes have only been given for some special cases with respect to the shape of the failure domain and the correlation structure of the process and, with one exception, only for stationary processes [4]. However, for smoothly bounded failure domains, Breitung [5,6] derived a second–order approximation for the stationary outcrossing rate by using asymptotic concepts for Laplace integrals which later was generalized to non–differentiable failure domains in [7]. This approximation can be updated to an arbitrary degree of accuracy even under non–asymptotic conditions by certain importance sampling procedures [8]. The only analytical solution for the outcrossing rate for non–stationary vector processes known to the authors is due to Ditlevsen [9] for linearly bounded failure domains. It can be used as a first–order approximation to more general cases. It makes use of the fact that a linear combination of Gaussian processes is again Gaussian so that all results for the scalar case can be used. But non–stationary crossing rates for vector processes out of arbitrarily bounded failure domains are also of interest in several areas. For example, the excitation of structures by earthquake–induced ground motions are highly non–stationary and failure criteria of structural members frequently are given as so–called interaction curves for the multidimensional response quantities. The non–stationary case is of particular interest if the resistance properties are degrading in time due to fatigue, aging, corrosion or other wear–out phenomena and the multidimensional stress response, e.g. in a ship in a random sea during a storm, must be checked against v. Mises' yield criterion.

In this paper the methods of the asymptotic analysis of Laplace integrals are applied to formula (2) by assuming non–stationarity of the vector process and time–varying smoothly bounded failure domains thus generalizing the results in [5] and [6]. Formulae for the mean number of outcrossings of the non–stationary vector process into a time–variant failure domain are derived. A numerical scheme for the integration in eq. (5) is proposed. The results are applied to some examples.

2. OUTCROSSING RATE AND MEAN NUMBER OF CROSSINGS

Let $\mathbf{X}(\tau) = (X_1(\tau),\dots,X_n(\tau))^T$ be a n–dimensional non–stationary Gaussian process with differentiable sample paths and derivative process $\dot{\mathbf{X}}(\tau) = (\dot{X}_1(\tau),\dots,\dot{X}_n(\tau))^T$. Without loss of generality, we suppose that the process is standardized in the sense that there is:

$$\begin{aligned} &E[X_i(\tau)] = E[\dot{X}_i(\tau)] = 0 && (i=1,\dots,n)\\ &COV[X_i(\tau),X_j(\tau)] = 0 && (i \neq j)\\ &COV[X_i(\tau),X_i(\tau)] = 1 && (i=1,\dots,n) \end{aligned}$$

Due to the standardization, $X_i(\tau)$ and $\dot{X}_i(\tau)$ become uncorrelated but not necessarily $X_i(\tau)$ and $\dot{X}_j(\tau)$. For convenience of notations we introduce:

$$\Sigma_{XX}(\tau) = R = \{r_{ij}\} = \{COV[X_i(\tau), X_j(\tau)]\}_{i,j=1,\dots,n} = I$$
$$\Sigma_{X\dot{X}}(\tau) = \dot{R} = \{\dot{r}_{ij}\} = \{COV[X_i(\tau), \dot{X}_j(\tau)]\}_{i,j=1,\dots,n}$$
$$\Sigma_{\dot{X}\dot{X}}(\tau) = \ddot{R} = \{\ddot{r}_{ij}\} = \{COV[\dot{X}_i(\tau), \dot{X}_j(\tau)]\}_{i,j=1,\dots,n}$$

Also, let F be a failure set varying in time with ∂F its boundary which is at least locally twice differentiable in $\mathbf{x}$ and τ. It is described by:

$$\partial F = \partial F(\tau) = \{\mathbf{x}; g(\mathbf{x};\tau) = 0\}$$

The outcrossing rate of the process $\mathbf{X}(\tau)$ through the hypersurface ∂F during the time interval $\Delta\tau$ is defined as [1,12,13]:

$$\nu^{+}(\partial F;\tau) = \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} P_1(\partial F;\Delta\tau) \tag{6}$$

In this formula, $P_1(\partial F;\Delta\tau)$ is the probability of one crossing of ∂F by the process $\mathbf{X}(\tau)$ from the safe domain $S = \{g(\mathbf{x},\tau) > 0\}$ into the failure domain $F = \{g(\mathbf{x},\tau) \le 0\}$ during $\Delta\tau$. As usual, regularity of the point process counting the number of crossings is assumed, i.e. there is $P(N(\Delta\tau) > 1) = o(\Delta\tau)$. $P_1(\partial F;\Delta\tau)$ can be given by

$$P_1(\partial F;\Delta\tau) = P\left[\begin{array}{l} \mathbf{X}(\theta) \in \Delta(\partial F(\theta)) \\ \dot{X}_N(\theta) > \partial\dot{F}(\theta) \\ \tau \le \theta \le \tau+\Delta\tau \end{array}\right] \tag{7}$$

where $\dot{X}_N(\theta) = \mathbf{n}^T(\mathbf{x})\,\dot{\mathbf{X}}(\theta)$ is the projection of $\dot{\mathbf{X}}(\theta)$ on the normal $\mathbf{n}^T(\mathbf{x})$ of $\partial F(\theta)$ in the point $\mathbf{x}$, $\partial\dot{F}(\theta)$ the time–variation of the surface ∂F at $\mathbf{x}$ and $\Delta(\partial F)$ a thin layer enveloping ∂F with height $(\dot{x}_N(\theta) - \partial\dot{F}(\theta))\,\Delta\tau$. Introducing the joint density function $\varphi_{n+1}(\mathbf{x},\dot{x}_N;\theta)$ of $\mathbf{X}$ and $\dot{X}_N$ allows to express P_1 by the following integral:

$$P_1(\partial F;\Delta\tau) = \int_{\Delta(\partial F)} \int_{\dot{x}_N(\theta) > \partial\dot{F}(\theta)} \varphi_{n+1}(\mathbf{x},\dot{x}_N;\theta)\, d\mathbf{x}\, d\dot{x}_N \tag{8}$$

The integral over $\Delta(\partial F)$ can be transformed into a surface integral over ∂F. The layer $\Delta(\partial F)$ is understood as the sum of infinitely small cylinders with height $(\dot{x}_N(\theta) - \partial\dot{F}(\mathbf{x};\theta))\,\Delta\tau$ and basis $ds(\mathbf{x})$, where $ds(\mathbf{x})$ is a surface neighborhood of the crossing point and $\partial\dot{F}(\mathbf{x};\theta)$ is made explicitly dependent in $\mathbf{x}$. Integrating over $\mathbf{x}$ yields:

$$P_1(\partial F;\Delta\tau) = \int_{\partial F} \int_{\dot{x}_N > \partial\dot{F}(\mathbf{x};\tau)} (\dot{x}_N - \partial\dot{F}(\mathbf{x};\tau))\, \varphi_{n+1}(\mathbf{x},\dot{x}_N;\tau)\, \Delta\tau\, d\dot{x}_N\, ds(\mathbf{x}) \tag{9}$$

Introducing now the density function of $\dot{X}_N$ conditional on $\mathbf{X} = \mathbf{x}$ and taking the integral over τ in $[0,t]$ leads to:

$$E[N^{+}(\partial F;t)] = \int_0^t \int_{\partial F} \int_{\dot{x}_N(\tau) > \partial\dot{F}(\mathbf{x};\tau)} (\dot{x}_N - \partial\dot{F}(\mathbf{x};\tau))\, \varphi_1(\dot{x}_N;\tau \mid \mathbf{X}(\tau)=\mathbf{x})\, \varphi_n(\mathbf{x})\, d\dot{x}_N\, ds(\mathbf{x})\, d\tau \tag{10a}$$

By considering the fact that the above reasoning is achieved by fixing the time τ, it is obvious that the time–variation $\partial\dot{F}(\mathbf{x};\tau)$ of the surface corresponds to the time–variation of the function $g(\mathbf{x};\tau)$ and does not involve the time variation of its gradients.

A similar reasoning leads to the expected number of incrossings during $[0,t]$:

$$E[N^{-}(\partial F;t)] = -\int_0^t \int_{\partial F} \int_{\dot{x}_N(\tau) < \partial\dot{F}(\mathbf{x};\tau)} (\dot{x}_N - \partial\dot{F}(\mathbf{x};\tau))\, \varphi_1(\dot{x}_N;\tau \mid \mathbf{X}(\tau)=\mathbf{x})\, \varphi_n(\mathbf{x})\, d\dot{x}_N\, ds(\mathbf{x})\, d\tau \tag{10b}$$

Combination of the two contributions leads to the expected number of crossings :

$$E[N(\partial F;t)] = E[N^{+}(\partial F;t)] + E[N^{-}(\partial F;t)] =$$
$$\int_0^t \int_{\partial F(\tau)} \int_{\mathbb{R}^1} |\dot{x}_N - \partial\dot{F}(\mathbf{x};\tau)|\, \varphi_1(\dot{x}_N;\tau \mid \mathbf{X}(\tau)=\mathbf{x})\, \varphi_n(\mathbf{x})\, d\dot{x}_N\, ds(\mathbf{x})\, d\tau \tag{10c}$$

The mixture of a volume integral over time and $\dot{x}_N$ and a surface integral can be transformed into a simple volume integral by using a suitable parameterization involving the time and the n–1 first coordinates of $\mathbf{x}$ as parameters and the n–th coordinate x_n as a function $p(\tilde{\mathbf{x}},\tau)$, so that for all $(\tilde{\mathbf{x}},\tau) \in \partial F \times T$, the integration domain is $W \times T$. Define:

$$\mathbf{x} = (\tilde{\mathbf{x}}, p(\tilde{\mathbf{x}},\tau))^T \text{ with } \tilde{\mathbf{x}} = (x_1,\dots,x_{n-1})^T \in W \text{ and } T = [0,t]$$

The formula for the expected number of outcrossings then takes the form

$$E[N^{+}(\partial F;t)] =$$

$$\int_T \int_W \int_{\dot{x}_N(\tau) > \partial\dot{F}(\mathbf{x};\tau)} (\dot{x}_N - \partial\dot{F}(\mathbf{x};\tau))\,\varphi_1(\dot{x}_N;\tau \mid \mathbf{X}(\tau)=\mathbf{x})\,\varphi_n(\tilde{\mathbf{x}},p(\tilde{\mathbf{x}},\tau))\,Tr(\tilde{\mathbf{x}},\tau)\,d\dot{x}_N\,d\tilde{\mathbf{x}}\,d\tau \quad (11a)$$

where $Tr(\tilde{\mathbf{x}},\tau)$ is the absolute value of the transformation determinant:

$$Tr(\tilde{\mathbf{x}},\tau) = \left\| \frac{\nabla g(\mathbf{x};\tau)}{g_n(\mathbf{x};\tau)} \right\|$$

In a similar way the expected number of incrossings and the expected number of crossings are defined:

$$E[N^{-}(\partial F;t)] =$$

$$-\int_T \int_W \int_{\dot{x}_N(\tau) < \partial\dot{F}(\mathbf{x};\tau)} (\dot{x}_N - \partial\dot{F}(\mathbf{x};\tau))\,\varphi_1(\dot{x}_N;\tau \mid \mathbf{X}(\tau)=\mathbf{x})\,\varphi_n(\tilde{\mathbf{x}},p(\tilde{\mathbf{x}},\tau))\,Tr(\tilde{\mathbf{x}},\tau)\,d\dot{x}_N\,d\tilde{\mathbf{x}}\,d\tau \quad (11b)$$

$$E[N(\partial F;t)] =$$

$$\int_T \int_W \int_{\mathbb{R}^1} \left| \dot{x}_N - \partial\dot{F}(\mathbf{x};\tau) \right| \varphi_1(\dot{x}_N;\tau \mid \mathbf{X}(\tau)=\mathbf{x})\,\varphi_n(\tilde{\mathbf{x}},p(\tilde{\mathbf{x}},\tau))\,Tr(\tilde{\mathbf{x}},\tau)\,d\dot{x}_N\,d\tilde{\mathbf{x}}\,d\tau \quad (11c)$$

In the following two somewhat different approaches for the calculation of the expected number of outcrossings which is of most interest will be outlined. In the direct method A, integration with respect to $\dot{x}_N$, which is analytic in eqs. (11), will be performed first. Then, the concepts of asymptotic analysis will be applied to the remaining integral (see appendix 1). The alternative method will first determine the expected number of crossings and then, indirectly, the expected number of outcrossings by using an equation which relates the expected number of out– and incrossings to the expected number of crossings. . The necessary supplementary formulae for this method B are derived below.

In noting the obvious algebraic identity

$$a = (a + b)/2 + (a - b)/2$$

one can connect the expected numbers of in– and outcrossings with the expected number of crossings as follows:

$$E[N^{+}(\partial F;t)] = \frac{1}{2}\,E[N(\partial F;t)] + \frac{1}{2}\,(E[N^{+}(\partial F;t)] - E[N^{-}(\partial F;t)]) \quad (12)$$

Now, in accordance with a result given in Cramer/Leadbetter [2] the second term can be given by:

$$E[N^{+}(\partial F;t)] - E[N^{-}(\partial F;t)] =$$
$$P[\{g(X;0)>0\} \cap \{g(X;t)\le 0\}] - P[\{g(X;0)\le 0\} \cap \{g(X;t)>0\}] \quad (13)$$

This formula can easily be interpreted. The actual numbers of out– and incrossings can differ at most by one. If the process starts and finishes in the safe domain the number of out– and incrossings are equal. The same is true if the process starts and finishes in the failure domain. Therefore, the difference $N^{+}(\partial F;t) - N^{-}(\partial F;t)$ must be zero. The difference is +1 if the process starts in the safe domain and finishes in the failure domain and is –1 in the opposite case. The probability of the first event (start in $\bar{F}$ and finish in F) is $P[\{g(X;0) > 0\} \cap \{g(X;t) \le 0\}]$ whereas for the second event (start in F and finish in $\bar{F}$) it is $P[\{g(X;0) \le 0\} \cap \{g(X;t) > 0\}]$. According to our assumption of a sufficiently mixing process the two events at time 0 and at time t, respectively, may be assumed independent. Therefore,

$$E[N^{+}(\partial F;t)] - E[N^{-}(\partial F;t)] \sim$$
$$[1-P\{g(X;0) \le 0\}]\,P\{g(X;t) \le 0\} - P\{g(X;0) \le 0\}\,[1-P\{g(X;t) \le 0\}]$$
$$= P\{g(X;t) \le 0\} - P\{g(X;0) \le 0\} \quad (14)$$

for a high–reliability problem. But these two probabilities can be calculated by using classical SORM techniques. The corresponding results are given in appendix 2 for easy reference.

3. INTEGRATION OF EQS. (11) BY THE THEORY OF LAPLACE –INTEGRALS

The integrals in eqs. (11) have no analytical solution for arbitrary non–stationary processes and failure surfaces. If the vector $X(\tau)$ has six dimensions, for example, the stresses in a three–dimensional body, the dimension of the integral is eight. But even in smaller dimensions numerical integration would be time–consuming. Therefore, a semi–analytical, approximate solution is proposed by applying the concepts of asymptotic analysis to this integral which after some manipulation is, in fact, a special type of the so–called Laplace–integrals. These integrals have the form

$$I(\lambda) = \int_D h(\mathbf{y})\, |k(\mathbf{y})|\, \exp[-\lambda f(\mathbf{y})]\, d\mathbf{y} \tag{15}$$

where D is a simply connected domain in $\mathbb{R}^n$ containing the origin. $f(\mathbf{y})$ is assumed to be at least twice differentiable and has a minimum at the origin $\mathbf{y} = \mathbf{y}^* = \mathbf{0}$. $h(\mathbf{y})$ is a smoothly varying function around the origin and $h(\mathbf{0}) \neq 0$. $k(\mathbf{y})$ is a function which can be zero at the origin and has a linear expansion $k(\mathbf{y}) \approx \mathbf{c}^T\mathbf{y} + c_0$ around the origin. c_0 is admitted to become zero. For this integral analytical results can be derived for $\lambda \to \infty$. They rest on the fact that the dominating part of the integrand for larger λ clearly comes from $f(\mathbf{y})$ in the exponent. Analytical results are possible because the function $f(\mathbf{y})$ and, if necessary, $k(\mathbf{y})$ need to be represented only by their first– or second order Taylor–expansion. Appendix 1 collects a few basic facts about Laplace integrals in terms of a lemma and five theorems, the last of which has been derived for the purpose of our subject. In order to apply those results to eqs. (11) a number of intermediate steps have to be taken. As the remainder of this section contains rather technical details, the reader may omit it at first reading and proceed directly to the results.

Introducing the well–known scaling by a factor $\beta > 1$ according to Breitung [6] and using the transformations

$$\mathbf{x} = \beta\, \mathbf{y} \tag{16a}$$

$$\tau = \beta\, \vartheta \tag{16b}$$

or, for the parameterized domain W×T,

$$x_i = \beta\, y_i \quad (i=1,\dots,n-1)$$
$$x_n = p(\tilde{\mathbf{x}},\tau) = \beta\, p(\tilde{\mathbf{y}},\vartheta), \tag{16c}$$

the original integration domain W×T is scaled by β into the domain $W_1 \times T_1$. Hence, for the expected number of outcrossings:

$$E[N^+(\partial F;t)] = \beta^n \int_{W_1\times T_1} \int_{\dot{x}_N(\beta\vartheta) > \partial\dot{F}(\beta\mathbf{y};\beta\vartheta)} (\dot{x}_N(\beta\vartheta) - \partial\dot{F}(\beta\mathbf{y};\beta\vartheta))\, \varphi_1(\dot{x}_N;\beta\vartheta \,|\, \mathbf{X}(\beta\vartheta) = \beta\mathbf{y})$$
$$\times\, \varphi_n(\beta\tilde{\mathbf{y}}, \beta p(\tilde{\mathbf{y}},\vartheta))\, \mathrm{Tr}(\beta\tilde{\mathbf{y}},\beta\vartheta)\, d\dot{x}_N\, d\tilde{\mathbf{y}}\, d\vartheta \tag{17a}$$

By a similar procedure we obtain

$$E[N^-(\partial F;t)] = -\beta^n \int_{W_1\times T_1} \int_{\dot{x}_N(\beta\vartheta) < \partial\dot{F}(\beta\mathbf{y};\beta\vartheta)} (\dot{x}_N(\beta\vartheta) - \partial\dot{F}(\beta\mathbf{y};\beta\vartheta))\, \varphi_1(\dot{x}_N;\beta\vartheta \,|\, \mathbf{X}(\beta\vartheta) = \beta\mathbf{y})$$
$$\times\, \varphi_n(\beta\tilde{\mathbf{y}}, \beta p(\tilde{\mathbf{y}},\vartheta))\, \mathrm{Tr}(\beta\tilde{\mathbf{y}},\beta\vartheta)\, d\dot{x}_N\, d\tilde{\mathbf{y}}\, d\vartheta \tag{17b}$$

and:

$$E[N(\partial F;t)] = \beta^n \int_{W_1\times T_1} \int_{\mathbb{R}} |\dot{x}_N(\beta\vartheta) - \partial\dot{F}(\beta\mathbf{y};\beta\vartheta)|\, \varphi_1(\dot{x}_N;\beta\vartheta \,|\, \mathbf{X}(\beta\vartheta) = \beta\mathbf{y})$$
$$\times\, \varphi_n(\beta\tilde{\mathbf{y}}, \beta p(\tilde{\mathbf{y}},\vartheta))\, \mathrm{Tr}(\beta\tilde{\mathbf{y}},\beta\vartheta)\, d\dot{x}_N\, d\tilde{\mathbf{y}}\, d\vartheta \tag{17c}$$

Furthermore, the conditional density function of $\dot{X}_N$ can be given explicitly by using the following formulae for the mean and variance of $\dot{X}_N(\tau)$ under the condition that $\mathbf{X}(\tau) = \beta\,\mathbf{y}$ (see [14])

$$E[\dot{X}_N(\tau) \,|\, \mathbf{X}(\tau) = \beta\,\mathbf{y}] = \mathbf{n}^T(\mathbf{y})\, [\mathbf{m}_{\dot{X}}(\tau) + \Sigma_{X\dot{X}}^T(\tau)\, \Sigma_{XX}^{-1}(\tau)\, (\beta\,\mathbf{y} - \mathbf{m}_X(\tau))] \tag{18}$$

and, due to the standardization and according to the foregoing notations:

$$E[\dot{X}_N(\tau) \,|\, \mathbf{X}(\tau) = \beta\,\mathbf{y}] = \mathbf{n}^T(\mathbf{y})\, \dot{R}^T\, \beta\,\mathbf{y} = m(\beta\,\mathbf{y};\, \beta\,\vartheta) = \beta\, m(\mathbf{y};\vartheta) \tag{19}$$

$$VAR[\dot{X}_N(\tau) \,|\, \mathbf{X}(\tau) = \beta\,\mathbf{y}(\tau)] = \mathbf{n}^T(\mathbf{y})\, [\Sigma_{\dot{X}\dot{X}}(\tau) - \Sigma_{X\dot{X}}^T(\tau)\, \Sigma_{XX}^{-1}(\tau)\, \Sigma_{X\dot{X}}(\tau)]\, \mathbf{n}(\mathbf{y})$$
$$= \mathbf{n}^T(\mathbf{y})\, [\ddot{R} - \dot{R}^T\, \dot{R}]\, \mathbf{n}(\mathbf{y}) = \sigma^2(\mathbf{y};\vartheta) \tag{20}$$

In the sense of the foregoing parameterization we can write $m(\mathbf{y};\vartheta)$ and $\sigma(\mathbf{y};\vartheta)$, respectively, as $\tilde{m}(\tilde{\mathbf{y}},\vartheta)$ and $\tilde{\sigma}(\tilde{\mathbf{y}},\vartheta)$ which will further be abbreviated as $\tilde{m} = \tilde{m}(\tilde{\mathbf{y}},\vartheta)$ and $\tilde{\sigma} = \tilde{\sigma}(\tilde{\mathbf{y}},\vartheta)$ except where stated otherwise.

Suppose now in following [6] that in the time interval $T = [0,t]$ there exists a critical time t^* for which the distance β between the hypersurface $\partial F(\tau)$ and the coordinate origin $\mathbf{x} = \mathbf{0}$ is smallest. Let further $\mathbf{x}^*$ be the corresponding critical point on ∂F. The point $(\mathbf{x}^*;t^*)^T$ can be found by solving the following optimization problem which is equivalent to minimize the function f(.) in eq. (15):

$$\beta = \min_{\mathbf{x},\tau}\{\|\mathbf{x}\|\} \text{ for } \begin{cases} \tau \in [0,t] \\ \mathbf{x},\tau \in \{g(\mathbf{x};\tau) = 0\} \end{cases} \tag{21}$$

For further notational convenience a suitable orthogonal transformation (rotation) of the reference base $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ can be performed such that $\mathbf{x}^* = \beta\,\mathbf{e}_n$. The scaling of $\mathbf{x}^*$ implies that $\mathbf{y}^*$ has unit distance to the origin. By a simple translation it is also always possible to assume $\vartheta^* = 0$. Hence, $p(\tilde{\mathbf{y}},\vartheta)$ has the following properties:

a) $p(\breve{0},0) = \beta$ implying $(\tilde{y}, p(\tilde{y},\vartheta))^T = \beta e_n$ for $\tilde{y} = (0,\dots,0)^T$ and $\vartheta = \vartheta^* = 0$

b) $g((\tilde{y}, p(\tilde{y},\vartheta))^T;\vartheta) = 0$ for all $(\tilde{y},\vartheta) \in W_1 \times T_1$

Since only $\tilde{m}$ depends explicitly on $\tilde{y}$ and since its partial derivatives at the critical point $(\breve{0},0)$ with respect to the y_i's and to ϑ exist, we obtain the following linear expansion

$$\tilde{m} = \tilde{m}_0 + \vartheta\, \tilde{m}_{\tau,0} + \sum_{i=1}^{n-1} y_i\, \tilde{m}_{i,0} \tag{22}$$

with:

$$\tilde{m}_0 = \tilde{m}(\breve{0},0) = 0 \tag{23a}$$

$$\tilde{m}_{\tau,0} = \tilde{m}_{\tau}(\breve{0},0) = \frac{\partial}{\partial\vartheta}\tilde{m}(\breve{0},0) = \sum_{j=1}^{n-1} |g_n(e_n;0)|^{-1} g_{j\tau}(e_n;0)\, \dot{r}_{jn} \tag{23b}$$

$$\tilde{m}_{i,0} = \tilde{m}_i(\breve{0},0) = \frac{\partial}{\partial y_i}\tilde{m}(\breve{0},0) = \sum_{j=1}^{n-1} (|g_n(e_n;0)|^{-1} g_{ij}(e_n;0) - \delta_{ij})\, \dot{r}_{jn} \tag{23c}$$

Herein, $g_n(e_n;0)$ denotes the first derivative of the failure surface with respect to the n–th variable and $g_{i\tau}(e_n;0)$ and $g_{ij}(e_n;0)$ denote the second–order derivatives with respect to the i–th variable and to τ and the j–th variable, respectively. δ_{ij} denotes Kronecker's δ–symbol.

At the critical point, the variance $\tilde{\sigma}^2(\tilde{y},\vartheta)$ takes the value:

$$\tilde{\sigma}_0^2 = \ddot{r}_n - \sum_{j=1}^{n-1} \dot{r}_{jn}^2 \tag{24}$$

Using of lemma 1 in following closely the arguments in [6] allows to reduce the integration domain $W_1 \times T_1$ to a surface and time neighborhood $V(\epsilon)$ of the critical point $(\tilde{y},\vartheta)^T = (\breve{0},0)^T$ (see appendix 3 for proof). Hence the expected number of outcrossings takes the form:

$$E[N^+(\partial F;t)] = \beta^n \int_{V(\epsilon)} \int_{\dot{x}_N(\beta\vartheta) > \partial\dot{F}(\beta y;\beta\vartheta)} (\dot{x}_N(\beta\vartheta) - \partial\dot{F}(\beta y;\beta\vartheta))\, \varphi_n(\dot{x}_N;\beta\vartheta \,|\, X(\beta\vartheta) = \beta y)$$
$$\times\, \varphi_n(\beta\tilde{y}, \beta p(\tilde{y},\vartheta))\, Tr(\beta\tilde{y},\beta\vartheta)\, d\dot{x}_N\, d\tilde{y}\, d\vartheta \tag{25}$$

With the substitution

$$\beta z = \left(\frac{\dot{x}_N - \beta\tilde{m}}{\tilde{\sigma}}\right)$$

and, since

$$\partial\dot{F}(\beta y;\beta\vartheta) = \beta\partial\dot{F}(y;\vartheta) \text{ with } \partial\dot{F}(y;\vartheta) = \frac{\partial g(y;\vartheta)}{\partial\tau} = \partial\dot{F}$$

one obtains

$$E[N^+(\partial F;t)] = (2\pi)^{-1/2} \beta^n \int_{V(\epsilon)} \int_{z > \beta^{-1}\tilde{a}} \beta(\tilde{\sigma}\beta z - \beta\partial\dot{F} + \beta\tilde{m}) \exp[-\tfrac{1}{2}(\beta z)^2]$$
$$\times\, \varphi_n(\beta\tilde{y}, \beta p(\tilde{y},\vartheta))\, Tr(\beta\tilde{y},\beta\vartheta)\, dz\, d\tilde{y}\, d\vartheta \tag{26a}$$

where:

$$\tilde{a} = \tilde{a}(\tilde{y},\vartheta) = \frac{\beta\partial\dot{F}(y;\vartheta) - \beta\tilde{m}(\tilde{y},\vartheta)}{\tilde{\sigma}(\tilde{y},\vartheta)}$$

In the same manner

$$E[N^-(\partial F;t)] = -(2\pi)^{-1/2} \beta^n \int_{V(\epsilon)} \int_{z < \beta^{-1}\tilde{a}} \beta(\tilde{\sigma}\beta z - \beta\partial\dot{F} + \beta\tilde{m}) \exp[-\tfrac{1}{2}(\beta z)^2]$$
$$\times\, \varphi_n(\beta\tilde{y}, \beta p(\tilde{y},\vartheta))\, Tr(\beta\tilde{y},\beta\vartheta)\, dz\, d\tilde{y}\, d\vartheta \tag{26b}$$

and:

$$E[N(\partial F;t)] = (2\pi)^{-1/2} \beta^n \int_{V(\epsilon)} \int_{\mathbb{R}^1} \beta\, |\tilde{\sigma}\beta z - \beta\partial\dot{F} + \beta\tilde{m}|\, \exp[-\tfrac{1}{2}(\beta z)^2]$$
$$\times\, \varphi_n(\beta\tilde{y}, \beta p(\tilde{y},\vartheta))\, Tr(\beta\tilde{y},\beta\vartheta)\, dz\, d\tilde{y}\, d\vartheta \tag{26c}$$

The multinormal density function $\varphi_n(\beta\tilde{y}, \beta p(\tilde{y},\vartheta))$ as a function of $\tilde{y}$ and ϑ in these equations is:

$$\varphi_n(\beta\tilde{y}, \beta p(\tilde{y},\vartheta)) = (2\pi)^{-n/2} \exp[-\frac{\beta^2}{2}(\sum_{i=1}^{n-1} y_i^2 + p^2(\tilde{y},\vartheta))] \tag{27}$$

Apart from constants, the normal densities in eqs. (26) can then be written as:

$$\exp[-\frac{\beta^2}{2}\{z^2 + \sum_{i=1}^{n-1} y_i^2 + p^2(\tilde{y},\vartheta)\}]$$

If the function $p(\tilde{y},\vartheta)$ is taken as an approximating paraboloid to ∂F in the critical point and if the paraboloid is expanded to first order so that only squared variables remain the term in braces in the exponent can be taken as the function f(.) in Laplace's theorems. For eqs. (26a) and (26b) the term $\tilde{\sigma}\beta z - \beta\dot{F} + \beta\tilde{m}$ is in brackets and can be taken with the other terms as the function h(.). In eq. (26c), however, where it is in absolute signs it can be taken as the function $|k(.)|$ whereas the other terms are collected in the function h(.). After some algebra concerning this term $\tilde{\sigma}\beta z - \beta\dot{F} + \beta\tilde{m}$, it remains to set $\lambda = \beta^2$ in order to obtain a form to which the theorems in appendix 1 are applicable. Two cases need to be considered. The critical point can be an interior point or a boundary point of $W_1 \times T_1$. The function p(...) depends on the position of the critical time t^* in T (or of ϑ^* in T_1) and on the geometry of the hypersurface at this point. If ϑ^* is an interior point of T_1, the function p(...) can be chosen as a complete quadratic form depending only on the second derivatives of the surface with respect to y_i (i=1,....,n) and ϑ. All first derivatives of $g(y;\vartheta)$ vanish at the critical point

$$p(\tilde{y},\vartheta) = 1-|g_n(e_n;0)|^{-1}[\frac{1}{2}\sum_{i,j=1}^{n-1} g_{ij}(e_n;0)\, y_i y_j + \sum_{j=1}^{n-1} g_{\vartheta j}(e_n;0)\,\vartheta y_j + \frac{1}{2} g_{\vartheta\vartheta}(e_n;0)\vartheta^2] \tag{28a}$$

with:

$$g_i(e_n;0) = \frac{\partial}{\partial y_i} g(e_n;0) \text{ and } g_{\vartheta}(e_n;0) = \frac{\partial}{\partial \vartheta} g(e_n;0)$$

In the case where ϑ^* is a boundary point of the time interval, the first derivative of $g(y;\vartheta)$ with respect to time has to be taken into account:

$$p(\tilde{y},\vartheta) = 1-|g_n(e_n;0)^2+g_{\vartheta}(e_n;0)^2|^{-1/2}$$

$$[\frac{1}{2}\sum_{i,j=1}^{n-1} g_{ij}(e_n;0) y_i y_j + \sum_{j=1}^{n-1} g_{\vartheta j}(e_n;0)\,\vartheta y_i + g_{\vartheta}(e_n;0)\,\vartheta] \tag{28b}$$

The results thus obtained can, of course, be given in terms of the quantities in the original space by performing the inverse of the orthogonal transformation which transformed x^* into $\beta\, e_n$. This yields (see also appendix (6)) the following approximations for the mean number of crossings of $X(\tau)$ out of $F = \{g(x;\tau) \le 0\}$ given that the critical point $(x^*;t^*)$ has been found according to eq. (24) and the statistical properties of $X(\tau)$ are defined as in the

formulae just above eq. (6). The two cases are illustrated in figure 1 and figure 2 for a two–dimensional stationary process and a time–variant failure surface. For simplicity of presentation, $t^* = 0$ is assumed in the following.

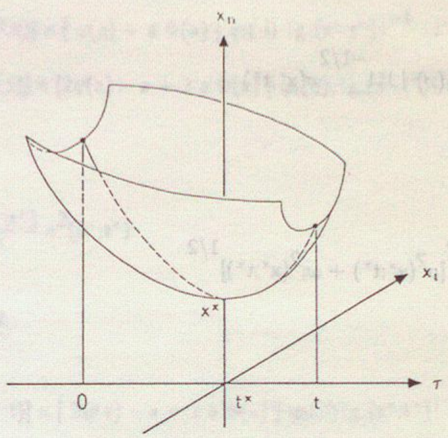

Figure 1: Critical point is an interior point

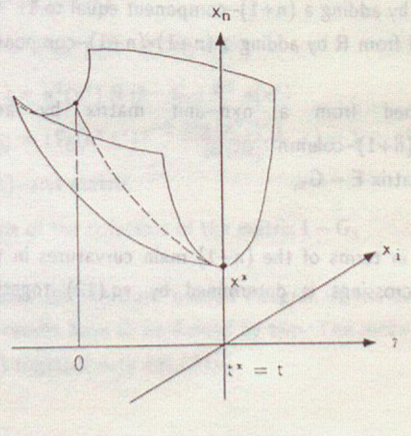

Figure 2: Critical point is a boundary point

4. RESULTS

4.1 GENERAL NON STATIONARY CASE

4.1.1 The critical point is an interior point:

Method A:

$$E[N^{+}(\partial F;t)] = E[N^{-}(\partial F;t)] \approx \varphi_1(\beta)\ |J^{*}|^{-1/2}\ \sigma(x^{*};t^{*}) \qquad (29a)$$

$$E[N(\partial F;t)] = 2\ E[N^{+}(\partial F;t)] \qquad (29b)$$

Method B:

$$E[N(\partial F;t)] \approx 2\ \varphi_1(\beta)\ |J^{*}|^{-1/2}\ [\sigma^2(x^{*};t^{*}) + \omega_1^{\,2}(x^{*};t^{*})]^{1/2} \qquad (30)$$

where

$J^{*} = \mathbf{n}^T(x^{*})\ C_{x_\tau}\ \mathbf{n}(x^{*})$

$\sigma^2(x^{*};t^{*}) = \mathbf{n}^T(x^{*})\ (\ddot{R} - \dot{R}^T \dot{R})\ \mathbf{n}(x^{*})$

$\omega_1^{\,2}(x^{*};t^{*}) = \mathbf{n}_{n+1}^{\ T}(x^{*})\ \dot{R}_{n+1}\ (E - G_{x_\tau})\ \dot{R}_{n+1}^{\ T}\ \mathbf{n}_{n+1}(x^{*})$

$$G_{x_\tau} = \Big\{ g_{ij} = |\nabla g(x^{*};t^{*})|^{-1} \frac{\partial^2 g(x^{*};t^{*})}{\partial x_i \partial x_j};\ i,j = 1,\dots,n;$$
$$g_{in+1} = |\nabla g(x^{*};t^{*})|^{-1} \frac{\partial^2 g(x^{*};t^{*})}{\partial x_i \partial \tau};\ i = 1,\dots,n;$$
$$g_{n+1n+1} = |\nabla g(x^{*};t^{*})|^{-1} \frac{\partial^2 g(x^{*};t^{*})}{\partial \tau \partial \tau} \Big\}$$

$\mathbf{n}_{n+1}$ = (n+1)–vector obtained from $\mathbf{n}$ by adding a (n+1)–component equal to 1

$\dot{R}_{n+1}$ = (n+1)×(n+1)–matrix obtained from $\dot{R}$ by adding a (n+1)×(n+1)–component equal to $-|g_n(x^{*};t^{*})|$

E = (n+1)×(n+1)–matrix obtained from a n×n–unit matrix by adding a zero–(n+1)–row and a zero–(n+1)–column

C_{x_τ}= matrix of the cofactors of the matrix $E - G_{x_\tau}$

The factor J^{*} can also be expressed in terms of the (n+1) main curvatures in the point $(x^{*};t^{*})$. The mean number of outcrossings is determined by eq.(12) together with eqs. (A16).

4.1.2 The critical point is a boundary point (non–linear failure surface)

Method A:

$$E[N^{+}(\partial F;t)] \approx [\ \varphi(a) - a\ \Phi(-a)\ ]\ \varphi_1(\beta)\ |g_\tau(x^{*};t^{*})|^{-1}\ |J^{*}|^{-1/2}\ \sigma(x^{*};t^{*}) \qquad (31a)$$

$$E[N^{-}(\partial F;t)] \approx [\ \varphi(a) + a\ \Phi(a)\ ]\ \varphi_1(\beta)\ |g_\tau(x^{*};t^{*})|^{-1}\ |J^{*}|^{-1/2}\ \sigma(x^{*};t^{*}) \qquad (31b)$$

$$E[N(\partial F;t)] \approx [2\varphi(a) - a + 2\ a\ \Phi(a)\ ]\ \varphi_1(\beta)\ |g_\tau(x^{*};t^{*})|^{-1}\ |J^{*}|^{-1/2}\ \sigma(x^{*};t^{*}) \qquad (31c)$$

with

$$a = \frac{\partial g(x^{*};t^{*})}{\partial \tau}\ \sigma^2(x^{*};t^{*})$$

Method B:

$$E[N(\partial F;t)] \approx [\ 2\varphi(a) - a + 2\ a\ \Phi(a)\ ]\ \varphi_1(\beta)|g_\tau(x^{*};t^{*})|^{-1}\ |J^{*}|^{-1/2}\ [\sigma^2(x^{*};t^{*}) + \omega_b^{\,2}(x^{*};t^{*})]^{1/2} \qquad (32)$$

with

$$a = \frac{\partial g(x^{*};t^{*})}{\partial \tau}\ [\sigma^2(x^{*};t^{*}) + \omega_b^{\,2}(x^{*};t^{*})]^{-1/2}$$

$J^{*} = \mathbf{n}^T(x^{*})\ C_x\ \mathbf{n}(x^{*})$

$\sigma^2(x^{*};t^{*}) = \mathbf{n}^T(x^{*})\ (\ddot{R} - \dot{R}^T \dot{R})\ \mathbf{n}(x^{*})$

$\omega_b^{\,2}(x^{*};t^{*}) = \mathbf{n}^T(x^{*})\ \dot{R}\ (I - G_x)\ \dot{R}^T\ \mathbf{n}(x^{*})$

$G_x = \{\ g_{ij} = |\nabla g(x^{*};t^{*})|^{-1} \frac{\partial^2 g(x^{*};t^{*})}{\partial x_i \partial x_j};\ \ i,j = 1,\dots,n\}$

I = (n×n)–unit matrix,

C_x= matrix of the cofactors of the matrix $I - G_x$

Φ(.) denotes the standard normal integral. If the time derivative vanishes at the critical point the results have to be divided by two. The mean number of outcrossings is determines by eq.(12) together with eqs.(A16).

4.2 NON–STATIONARY SCALAR CASE

In the one dimensional case the results can be deduced from the vectorial case. Alternatively, one can derive them by making use of chapter 3.

4.2.1 The critical point is an interior point (scalar process):

Method A:

$$E[N^{+}(\partial F;t)] = E[N^{-}(\partial F;t)] \approx \varphi_1(\beta)\ |J^{x}|^{-1/2}\ \sigma(x^{x};t^{x}) \tag{33a}$$
$$E[N(\partial F;t)] = 2\ E[N^{+}(\partial F;t)] \tag{33b}$$

Method B:

$$E[N(\partial F;t)] \approx 2\ \varphi_1(\beta)\ |J^{x}|^{-1/2} [\sigma^2(x^{x};t^{x}) + \omega_1^2(x^{x};t^{x})]^{1/2} \tag{34}$$

with:

$J^{x} = 1^T\ C_{x_\tau}\ 1$

$\sigma^2(x^{x};t^{x}) = 1^T\ (\ddot{R} - \dot{R}^T\ \dot{R})\ 1$

$\omega_1^2(x^{x};t^{x}) = 1^T\ \dot{R}_{n+1}\ (E - G_{x_\tau})\ \dot{R}_{n+1}{}^T\ 1$

$G_{x_\tau} = \{g_{11} = |\nabla g(x^{x};t^{x})|^{-1} \frac{\partial^2 g(x^{x};t^{x})}{\partial x^2};$

$\qquad g_{12} = |\nabla g(x^{x};t^{x})|^{-1} \frac{\partial^2 g(x^{x};t^{x})}{\partial x_1 \partial\tau};$

$\qquad g_{22} = |\nabla g(x^{x};t^{x})|^{-1} \frac{\partial^2 g(x^{x};t^{x})}{\partial\tau\partial\tau}\}$

1 = vector $(1,1)^T$

$\dot{R}_2$ = 2^2–matrix obtained from $\dot{r}$ by adding a $(n+1)\times(n+1)$–component equal to $-|g_n(x^{x};t^{x})|$

E = 2^2–matrix where only the component e_{11} is not zero and equal to 1

C_{x_τ} = matrix of the cofactors of the matrix $E - G_{x_\tau}$.

The factor J^{x} can also be expressed in terms of the two main curvatures in the point $(x^{x};t^{x})$. The mean number of outcrossings is determined by eq. (12) together with eqs. (A16).

4.2.2 The critical point is a boundary point (scalar process)

Method A:

$$E[N^{+}(\partial F;t)] \approx [\ \varphi(a) - a\ \Phi(-a)\ ]\ \varphi_1(\beta)\ |g_\tau(x^{x};t^{x})|^{-1}\ |J^{x}|^{-1/2}\ \sigma(x^{x};t^{x}) \tag{35a}$$
$$E[N^{-}(\partial F;t)] \approx [\ \varphi(a) + a\ \Phi(a)\ ]\ \varphi_1(\beta)\ |g_\tau(x^{x};t^{x})|^{-1}\ |J^{x}|^{-1/2}\ \sigma(x^{x};t^{x}) \tag{35b}$$
$$E[N(\partial F;t)] \approx [\ 2\varphi(a) - a + 2\ a\ \Phi(a)\ ]\ \varphi_1(\beta)\ |g_\tau(x^{x};t^{x})|^{-1}\ |J^{x}|^{-1/2}\ \sigma(x^{x};t^{x}) \tag{35c}$$

with

$$a = \frac{\partial g(x^{x};t^{x})}{\partial\tau}\ \sigma^2(x^{x};t^{x})$$

This is essentially the result in [3].

Method B:

$$E[N(\partial F;t)] \approx [\ 2\varphi(a) - a + 2\ a\ \Phi(a)\ ]\ \varphi_1(\beta)|g_\tau(x^{x};t^{x})|^{-1}\ |J^{x}|^{-1/2} \times [\sigma^2(x^{x};t^{x}) + \omega_b^2(x^{x};t^{x})]^{1/2} \tag{36}$$

with

$a = \frac{\partial g(x^{x};t^{x})}{\partial\tau} [\sigma^2(x^{x};t^{x}) + \omega_b^2(x^{x};t^{x})]^{-1/2}$

$J^{x} = 1 - \frac{\partial^2 g(x^{x};t^{x})}{\partial x^2}$

$\sigma^2(x^{x};t^{x}) = \ddot{r} - \dot{r}^2$

$\omega_b^2(x^{x};t^{x}) = \dot{r}^2(1 - \frac{\partial^2 g(x^{x};t^{x})}{\partial x^2})$

If the time derivative vanishes at the critical point the results have to be divided by two. The mean number of outcrossings is determined by eq. (12) together with eqs. (A16).

4.3 NON–STATIONARY CASE WITH LINEAR FAILURE SURFACE

When the failure surface is a hyperplane the above formulae reduce to much simpler forms.

4.3.1 The critical point is an interior point (linear failure surface)

Method A:

$$E[N^+(\partial F;t)] = E[N^-(\partial F;t)] \approx \varphi_1(\beta)\, \sigma^2(\mathbf{x}^*;t^*) \qquad (37a)$$

$$E[N(\partial F;t)] \approx 2\, \varphi_1(\beta)\, \sigma^2(\mathbf{x}^*;t^*) \qquad (37b)$$

with

$$\sigma^2(\mathbf{x}^*;t^*) = \mathbf{n}^T(\mathbf{x}^*)\,(\ddot{R} - \dot{R}^T \dot{R})\,\mathbf{n}(\mathbf{x}^*)$$

Method B:

$$E[N(\partial F;t)] \approx 2\, \varphi_1(\beta)\, [\sigma^2(\mathbf{x}^*;t^*) + \omega_1^{\,2}(\mathbf{x}^*;t^*)]^{1/2} \qquad (38)$$

with

$$\sigma^2(\mathbf{x}^*;t^*) = \mathbf{n}^T(\mathbf{x}^*)\,(\ddot{R} - \dot{R}^T \dot{R})\,\mathbf{n}(\mathbf{x}^*)$$

$$\omega_1^{\,2}(\mathbf{x}^*;t^*) = \mathbf{n}_{n+1}^T(\mathbf{x}^*)\, \dot{R}_{n+1}\, \dot{R}_{n+1}^{\;T}\, \mathbf{n}_{n+1}(\mathbf{x}^*)$$

The mean number of outcrossings is determined by eq. (12) together with eqs. (A16).

4.3.2 The critical point is a boundary point (linear failure surface)

Method A:

$$E[N^+(\partial F;t)] \approx [\,\varphi(a) - a\,\Phi(-a)\,]\, \varphi_1(\beta)\, |g_\tau(\mathbf{x}^*;t^*)|^{-1} \sigma^2(\mathbf{x}^*;t^*) \qquad (39a)$$

$$E[N^-(\partial F;t)] \approx [\,\varphi(a) + a\,\Phi(a)\,]\, \varphi_1(\beta)\, |g_\tau(\mathbf{x}^*;t^*)|^{-1} \sigma^2(\mathbf{x}^*;t^*) \qquad (39b)$$

$$E[N(\partial F;t)] \approx [\,2\varphi(a) - a + 2\,a\,\Phi(a)\,]\, \varphi_1(\beta)\, |g_\tau(\mathbf{x}^*;t^*)|^{-1} \sigma^2(\mathbf{x}^*;t^*) \qquad (39c)$$

with

$$a = \frac{\partial g(\mathbf{x}^*;t^*)}{\partial \tau}\, \sigma^2(\mathbf{x}^*;t^*)$$

$$\sigma^2(\mathbf{x}^*;t^*) = \mathbf{n}^T(\mathbf{x}^*)\,(\ddot{R} - \dot{R}^T \dot{R})\,\mathbf{n}(\mathbf{x}^*)$$

Method B:

$$E[N^+(\partial F;t)] \approx [\,\varphi(a) - a\,\Phi(-a)\,]\, \varphi_1(\beta)\, |g_\tau(\mathbf{x}^*;t^*)|^{-1} [\sigma^2(\mathbf{x}^*;t^*) + \omega_b^{\,2}(\mathbf{x}^*;t^*)]^{1/2} \qquad (40a)$$

$$E[N^-(\partial F;t)] \approx [\,\varphi(a) + a\,\Phi(a)\,]\, \varphi_1(\beta)\, |g_\tau(\mathbf{x}^*;t^*)|^{-1} [\sigma^2(\mathbf{x}^*;t^*) + \omega_b^{\,2}(\mathbf{x}^*;t^*)]^{1/2} \qquad (40b)$$

$$E[N(\partial F;t)] \approx [\,2\varphi(a) - a + 2\,a\,\Phi(a)\,]\, \varphi_1(\beta)\, |g_\tau(\mathbf{x}^*;t^*)|^{-1} [\sigma^2(\mathbf{x}^*;t^*) + \omega_b^{\,2}(\mathbf{x}^*;t^*)]^{1/2} \qquad (40c)$$

with

$$a = \frac{\partial g(\mathbf{x}^*;t^*)}{\partial \tau}\, [\sigma^2(\mathbf{x}^*;t^*) + \omega_b^{\,2}(\mathbf{x}^*;t^*)]^{-1/2}$$

$$\sigma^2(\mathbf{x}^*;t^*) = \mathbf{n}^T(\mathbf{x}^*)\,(\ddot{R} - \dot{R}^T \dot{R})\,\mathbf{n}(\mathbf{x}^*)$$

$$\omega_b^{\,2}(\mathbf{x}^*;t^*) = \mathbf{n}^T(\mathbf{x}^*)\, \dot{R}\, \dot{R}^T\, \mathbf{n}(\mathbf{x}^*)$$

The mean number of outcrossings is determined by eq. (12) together with eqs. (A16). The foregoing formula and in part the previous formulae can further be reduced but we will not do so herein.

4.4 STATIONARY PROCESSES AND TIME–INVARIANT FAILURE DOMAINS

In this case further significant simplifications occur. First of all, since the integration over time is now an integration over a constant crossing rate it is

$$E[N^+(\partial F;t)] = \nu^+(\partial F)\, t \qquad (41)$$

and is suffices to give the outcrossing rate. Secondly, there is no distinction between interior and boundary points and the special version of theorem 2 in appendix 2 with $k(0) = 0$ already obtained by Breitung [6] applies

$$\nu^+(\partial F) \sim \varphi_1(\beta) \prod_{i=1}^{n-1} (1-\kappa_i\beta)^{-1/2} (2\pi)^{-1/2} [\sigma^2(\mathbf{x}^*) + \omega^2(\mathbf{x}^*)]^{1/2} \quad \text{for } \beta \to \infty \tag{42}$$

where

$$\sigma^2(\mathbf{x}^*) = \mathbf{n}^T(\mathbf{x}^*)\,(\ddot{R} - \dot{R}^T\dot{R})\,\mathbf{n}(\mathbf{x}^*)$$

$$\omega^2(\mathbf{x}^*) = \mathbf{n}^T(\mathbf{x}^*)\,\dot{R}^T\,G\,\dot{R}\,\mathbf{n}(\mathbf{x}^*)$$

$$G = \{\delta_{jk} + \|\nabla g(\mathbf{x}^*)\|^{-1} \frac{\partial^2 g(\mathbf{x}^*)}{\partial x_i \partial x_j}\ ;\ i,j = 1,\ldots,n\}$$

As before, β is the minimal distance between the failure surface ∂F and the origin and the κ_i's are the main curvatures of ∂F in that point.

If, furthermore, X and $\dot{X}$ are independent one can simplify this result to:

$$\nu^+(\partial F) \sim \varphi_1(\beta) \prod_{i=1}^{n-1} (1-\kappa_i\beta)^{-1/2} \left[\frac{(\mathbf{n}(\mathbf{x}^*)^T\ddot{R}\mathbf{n}(\mathbf{x}^*))}{2\pi}\right]^{1/2} \quad \text{for } \beta \to \infty \tag{43}$$

If, finally, the failure domain is a half–space the well–known result referred to in Veneziano et al. [4] is obtained by omitting the product term involving the main curvatures κ_i of ∂F from eq. (42) or (43).

4.5 DISCUSSION

According to Breitung [6] the term $\sigma^2(\mathbf{x}^*;t^*)$ is the variance of the random variable $\mathbf{n}^T(\mathbf{x}^*)\,\dot{X}(\tau)$ conditional on $X(\tau) = \mathbf{x}^*$. The term $\omega^2(\mathbf{x}^*;t^*)$ ($\omega_i^2(\mathbf{x}^*;t^*)$ in the case of an interior point and $\omega_b^2(\mathbf{x}^*;t^*)$ in the case of a boundary point) is approximately the variance of the mean in eq. (17a) of the same variable conditional on $X(\tau) = \mathbf{x}^*$ if $\mathbf{x}$ varies around $\mathbf{x}^*$. It is this second additional term why we first computed $E[N(\partial F;t)]$ by eq. (10c) via its asymptotic approximations and then used eq. (12) for the mean number of outcrossings. Direct application of theorem 1 in appendix 1 to eq. (10a) yields similar results but with $\omega^2(\mathbf{x}^*;t^*) = 0$. This term, therefore, must be considered as a higher order but nevertheless asymptotically non–negligible correction. Even if application of theorem 3 would yield numerically close results the term $\omega^2(\mathbf{x}^*;t^*)$ may be retained for reasons of consistency with the stationary case in which application of theorem 4 is mandatory as can easily be verified. Geometrically, this term takes account of those additional crossings through the failure surface somewhat away from $\mathbf{x}^*$ which are possible because there is already $\mathbf{n}(\mathbf{x}) \neq \mathbf{n}(\mathbf{x}^*)$.

It is also worth noting that the term $\omega^2(\mathbf{x}^*)$ in eq. (42) is not present in eq. (43) and the same applies to the terms $\omega^2(\mathbf{x}^*;t^*)$ in the previous formulae under the same circumstances. Therefore, if $X(t)$ and $\dot{X}(t)$ are independent the formulae simplify greatly. Since the matrices $\dot{R}$ in general have entries which are small as compared to the entries in $\ddot{R}$ it is even concluded that neglecting the terms $\omega^2(\mathbf{x}^*;t^*)$ in eqs. (30), (32), (34) (38) and (40) frequently is an acceptable approximation. A cruder approximation is achieved by also neglecting the term involving the $\dot{R}$– or $\dot{R}_{n+1}$–matrix in the expression for $\sigma^2(\mathbf{x}^*;t^*)$. A greater error usually is introduced if the curvature information for the failure surface in the critical point is not used in higher dimensional problems.

5. NUMERICAL ANALYSIS OF EQ. (5)

Suppose that part of the random variables are Gaussian processes with continuously differentiable sample paths. Let X be this part of the random variables. The rest of the random variables is collected in a random vector Z which is represented by the Rosenblatt–transformation of its joint distribution function $F_Z(z)$. Let T be the random time to a first passage of the process $X(\tau)$ from the safe set to the failure set. $T = T(z)$ is a random variable which is conditional on a given set z of time invariant variables Z. Failure occurs when the time $T(z)$ is smaller than the intended service time t. Hence, the conditional probability of failure can be written as

$$P(T(z) - t \leq 0 \,|\, Z = z) = F_T(t \,|\, Z = z) \tag{44}$$

where $F_T(t|Z = z)$ is the distribution of the first passage time $T(z)$. The failure probability takes the form:

$$P_f(t) = \int \{T(z) - t\}\, dF_Z(z) \tag{45}$$

The last expression is well suited for using FORM–SORM methods. Therefore, the distribution of the time T to failure conditional on z has first to be defined in terms of the process characteristics. Under certain circumstances (rare events or high reliability) it can be well approximated by an exponential distribution (see eq.(3)).

$$F_T(t \,|\, Z = z) \approx 1 - \exp[-\int_0^t \nu^+(\partial F;\tau\,|\,z)\, d\tau] \tag{46}$$

where $\nu^+(\partial F;\tau|z)$ is the conditional outcrossing rate of the process $X(\tau)$ out of the safe set at time τ. Introducing an auxiliary standard variable U_T such that

$$F_T(t \mid Z = z) = \Phi(U_T) \tag{47}$$

allows to express the variable $T(z)$ as a function of $\nu^+(\partial F;\tau \mid z)$. In the non–stationary case, eq. (47) does not reduce to a simple expression. Denote by $I(z)$ the integral

$$I(z) = \int_0^t \nu^+(\partial F;\tau \mid z)\, d\tau = E[N^+(\partial F;t \mid z)] \tag{48}$$

which is the conditional expected number of outcrossings of the process $X(\tau)$ through the conditional failure surface $\partial F(z)$ calculated by the previously derived formulas. In these formulas the β–point, the gradient and the curvatures at the minimum point are determined by using the failure surface $\partial F(z)$ as limit–state function in a FORM–SORM algorithm.

Hence, having calculated $E[N^+(\partial F;t \mid z)]$ conditional on z, the time to failure $T(z)$ can be written as

$$T(z) = I^{-1}[-\ln(-\Phi(U_T))] \tag{49}$$

where $I^{-1}[.]$ represents the inversion of the integral $I(z)$ with respect to the upper integration limit. The failure criterion (eq. 44) now is

$$I^{-1}[-\ln(-\Phi(U_t))] - t \leq 0 \tag{50}$$

which is used in a second FORM–SORM algorithm dealing with the time invariant random variables. Note that in order to avoid numerical problems this limit state function may be written on a logarithmic scale.

6. NUMERICAL EXAMPLES

Example 1:

Consider the following limit–state function

$$g(x;\tau) = \sigma_r(\tau) - [\sigma_x(\tau)^2 + 3\sigma_t(\tau)^2]^{1/2}$$

which corresponds to a two–dimensional v. Mises' yield criterion where the resistance $\sigma_r(\tau)$

is a function of time τ and where $\sigma_x(\tau)$ and $\sigma_t(\tau)$ are the components of a two–dimensional process $X(\tau)$ having the following parameters:

	MEAN	COVARIANCE MATRIX			
$\sigma_x(\tau)$	$E[\sigma_x(\tau)]$	1940	−7940	0	−22.4
$\sigma_t(\tau)$	$E[\sigma_t(\tau)]$	−7940	32566	22.4	0
$\dot\sigma_x(\tau)$	$E[\dot\sigma_x(\tau)]$	0	22.4	114000	−403000
$\dot\sigma_t(\tau)$	$E[\dot\sigma_t(\tau)]$	−22.4	0	−403000	1780000

$E[\sigma_x(\tau)]$ and $E[\sigma_x(\tau)]$ can be functions of the time τ to be defined later. The failure domain F is defined as usual by:

$$F = \{g(x;\tau) \leq 0\} \qquad \tau \in [0,t]$$

The variation of $\sigma_r(\tau)$ is described by the very simple yield stress function

$$\sigma_r(\tau) = \sigma_0(1 + c\,(\tau - \tau_0)^2) \qquad \tau \in [0,t]$$

When $X(\tau)$ is stationary the minimum of this function and, therefore, the maximum of the outcrossing rate is readily determined as $\tau = \tau_0$. We first assume $E[\sigma_x(\tau)] = E[\sigma_t(\tau)] = 0$ Due to the fact that the mean–value vector and the correlation matrix of $X(\tau)$ do not depend on time, the critical time t^* corresponds to the time where the function $\sigma_r(\tau)$ takes its minimal value, i.e. $t^* = \tau_0$. We define the following constants in appropriate units: $\sigma_0 = \tau_0 = 50$ and $c = 2\ 10^{-4}$.

- If $t = \tau_0$, the critical point t^* is a boundary point of the interval $[0,t]$. $\sigma_r(\tau)$ is a degrading function of τ within $[0,t]$. The time derivative of the function $g(x;\tau)$ is zero at t^*. Formula (14) and the remark below eq. (15) furnishes:

  $E[N(\partial F;t)] = 1.37\ 10^{-6}$

- If $t > \tau_0$, the critical time t^* is an interior point of the interval $[0,t]$, i.e. $t^* = \tau_0$. The time derivative of the function $g(x;\tau)$ at the critical point also vanishes. One obtains

  $E[N(\partial F;t)] = 2.74\ 10^{-6}$

  which is twice the foregoing value.

– If $t < \tau_0$, the critical point $t^* = t$ is certainly a boundary point with non–vanishing time derivative. With eq. (15) and $t = 45$ one determines:

$$E[N(\partial F;t)] = 1.15\ 10^{-6}$$

Next, $E[\sigma_x(\tau)]$ and $E[\sigma_t(\tau)]$ are defined as the following functions of time:

$$E[\sigma_x(\tau)] = \sigma_{mx} \exp[-c_x (\tau - \tau_m)^2]$$
$$E[\sigma_x(\tau)] = \sigma_{mt} \exp[-c_t (\tau - \tau_m)^2]$$

The critical time must now be determined numerically according to eq. (13). Depending on the parameters of the mean value functions and the yield stress function the critical point can be either an interior or a boundary point. With $\sigma_{mx} = \sigma_{mt} = 2 \cdot 0^2$, $c_x = c_t = .2$, $\tau_m = 40$ and $t > \tau_0$ the critical point is an interior point with $t^* = 40.03$. Eq. (14) gives

$$E[N(\partial F;t)] = 3.5\ 10^{-3}$$

In all cases the first–order and asymptotic second–order results coincide which is readily explained by the fact that the radii of curvatures of the failure surfaces are several orders of magnitude larger than the β–values which are around 5.

Example 2:

As a second example let us consider the failure criterion for a brace–chord joint in a jacket–platform submitted to axial load Z_1 and in–plane (ipb) Z_2 and out–of–plane (opb) Z_3 bending moments:

$$\{g(x;\tau) \le 0\} = \{r(\tau) - |\frac{Z_1}{P_a}| - \arcsin [(\frac{Z_2}{M_{aipb}})^2 + (\frac{Z_3}{M_{aopb}})^2]^{1/2} \le 0\}$$

The arcsin term is in radians. The terms with an index a represent the corresponding allowable capacities in the brace and are given by

$$P_a = Q_u \frac{F_y\ T^2}{1.7 \sin\vartheta} \quad \text{and} \quad M_a = Q_u \frac{F_y\ T^2}{1.7 \sin\vartheta} (0.8\ d)$$

where Q_u is the ultimate strength factor which varies with the joint and load type. For convenience Q_u will be set to 1. F_y is the allowable stress in the brace. All other parameters are defined in fig. 3. Assume that P_a, M_{aipb} and M_{aopb} are the components of a

three–dimensional process $Y(\tau)$ which is a linear combination of three independent processes $X(\tau)$ as

$$Y(\tau) = A\ X(\tau)$$

where A is a three–dimensional constant matrix. The processes $X_i(\tau)$ have constant mean vector m_i and auto–covariance function:

$$C_{i_{xx}}(t_1,t_2) = \sigma(t_1)\ \sigma(t_2) \exp[-\alpha\ |t_1-t_2|]\ [\cos \beta(t_1-t_2) + \frac{\alpha}{\beta} \sin \beta(t_1-t_2)] + c_0$$

with $\sigma(t) = b\ t \exp[-c\ t^2]$. At a point $t_1 = t_2 = \tau$, we obtain:

$$C_{i_{xx}}(\tau) = \sigma(\tau)^2 + c_0$$

Moreover, the velocities $\dot{X}_i(\tau)$ are also supposed to be independent. The cross–covariance functions of the processes $X_i(\tau)$ are given by

$$C_{i_{x\dot{x}}}(\tau) = \sigma(\tau)\ (1 - 2\ c\ \tau^2)\ b \exp(-c\ \tau^2)$$

$$C_{i_{\dot{x}\dot{x}}}(\tau) = b^2 \exp(-c\ \tau^2)\ [(1 - 2\ c\ \tau^2)^2 - (\alpha^2 + \beta^2)\ \tau^2]$$

after taking the derivatives at the point τ. Hence the mean vector and the covariance matrices of the vector $Y(\tau)$ take the form

$$m_y(\tau) = A\ m_x(\tau)$$
$$R_y(\tau) = \{COV[Y_i(\tau),Y_j(\tau)]\}_{i=1,\dots,3} = A^T\ R_x(\tau)\ A$$
$$\dot{R}_y(\tau) = \{COV[Y_i(\tau),\dot{Y}_j(\tau)]\}_{i=1,\dots,3} = A^T\ \dot{R}_x(\tau)\ A$$
$$\ddot{R}_y(\tau) = \{COV[\dot{Y}_i(\tau),\dot{Y}_j(\tau)]\}_{i=1,\dots,3} = A^T\ \ddot{R}_x(\tau)\ A$$

where the matrices $R_x(\tau)$, $\dot{R}_x(\tau)$ and $\ddot{R}_x(\tau)$ are respectively the diagonal matrices collecting the covariance functions $C_{i_{xx}}(\tau)$, $C_{i_{x\dot{x}}}(\tau)$ and $C_{i_{\dot{x}\dot{x}}}(\tau)$ of the vector $X(\tau)$. $r(\tau)$ is taken as a decreasing function of time $r(\tau) = 1 - \gamma\ \tau$. By varying the parameter b in the function $\sigma(t)$ which determines the strength of the excitation, one obtains the expected number of outcrossings in $T = [0,.06]$ according to eq. (11c). In Figure 3 where the parameters are given in appropriate units the first–(lower curve) and second–order (upper curve) results are given for comparison.

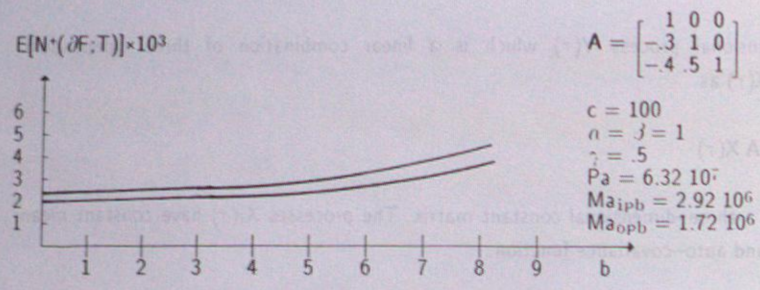

Figure 3: Mean number of outcrossings versus parameter b

7.CONCLUSIONS

Breitung's asymptotic result for the crossing rate of stationary Gaussian vector out of smooth, curved failure surfaces is generalized to non–stationary processes and/or time–dependent failure surfaces. A suitable special solution for the corresponding Laplace integral is derived. It is further extended to also perform the necessary time integration for the mean number of outcrossings in a given time interval. Thereby, one has to distinguish between the case where the maximum outcrossing rate occurs in the interior of the time integration interval and the case where it is at one of its boundaries. Multidimensional integration is reduced to a simple single constraint non–linear optimization problem and some simple algebra. The results are applied to two illustrating examples. The derivations suggest certain simplifications in practical applications, e.g. to neglect the possible cross–correlations between the process and its derivative and to discard curvature information about the failure surface in the critical point in the sense of a first–order approximation.

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APPENDICES

Appendix 1:

In the following we shall summarize certain results for integrals of the type

$$I(\lambda) = \int_D h(y)\ |k(y)|\ \exp[-\lambda f(y)]\ dy \qquad (A1)$$

for $\lambda \to \infty$ and D a simply connected domain in $\mathbb{R}^n$ containing the origin. f(y) is assumed to be at least twice differentiable and has a minimum at the origin $y = y^* = 0$. h(y) is a smoothly varying function around the origin and $h(0) \neq 0$. k(y) is a function which can be zero at the origin and has a linear expansion $k(y) \approx c^T y + c_0$ around the origin. Hence, c_0 is admitted to become zero. The basic results are given in terms of a lemma and a few theorems without proof, applicable to the various forms of k(y) and D. Instead of formally presenting the proofs hopefully helpful interpretations will be given. The complete derivations and proofs are available on request.

Lemma 1 is a generalization of a corresponding lemma for the simpler integral (see [10])

$$I(\lambda) = \int_D h(y) \exp[-\lambda f(y)]\ dy \qquad (A2)$$

or even:

$$I(\lambda) = \int_D \exp[-\lambda f(y)]\ dy \qquad (A3)$$

Theorems 1 and 2 are classical results. Certain generalizations turned out to be crucial when evaluating probability integrals over standard normal variables. Those are contained in theorem 3. Theorem 4 has been applied to vector outrossings for stationary processes in [6] and only theorem 5, which is a generalization of theorem 4, is new.

LEMMA 1:

Define a small neighborhood V of $y^* = 0$ such that the following conditions are fulfilled [10]:

$$f(0) > \sup\{f(y) : y \in D \backslash V\} \qquad (A4a)$$

$$\int_{D \cap V} h(y)\ |k(y)|\ \exp[-\lambda f(y)]\ dy > 0 \qquad (A4b)$$

$$\int_D |h(y)|\ |k(y)|\ \exp[-\lambda f(y)]\ dy < \infty \qquad (A4c)$$

Then, the following statement can be derived (see [6]):

$$\int_D h(y)\ |k(y)| \exp[-\lambda f(y)]\ dy \sim \int_{D \cap V} h(y)\ |k(y)| \exp[-\lambda f(y)]\ dy \quad \text{for } \lambda \to \infty \qquad (A5)$$

Lemma 1 says that only a (small) neighborhood V of y^* essentially contributes to $I(\lambda)$ when $\lambda \geq \infty$. Usually it is proven by assuming a neighborhood, for example, in the form of $V(\epsilon) = \{\|y\| < \epsilon\}$. It is proved by showing that:

$$\lim_{\lambda \to \infty} \frac{I(\lambda, D \backslash V(\epsilon))}{I(\lambda, D \cap V(\epsilon))} = 0 \text{ for } \epsilon > 0$$

The important implication of lemma 1 is that only the local behavior of the functions h(y), k(y), f(y) and possibly the bounding functions $g_i(y)$ of D, i.e. when D is given as $D = \{\cap g_i(y) \leq 0\}$ has to be taken into account. For this a Taylor expansion of those functions up to and including the first non-vanishing term is sufficient. On the other hand this lemma allows to extend the integration domain even outside D, i.e. into the domain $D \backslash V(\epsilon)$.

#

THEOREM 1:

Assume that the origin, where the function $f(\mathbf{y})$ obtains a minimum, is in the interior of D. $f(\mathbf{y})$ has the quadratic Taylor–expansion

$$f(\mathbf{y}) \approx f(\mathbf{0}) + \frac{1}{2}\,\mathbf{y}^T\, S(\mathbf{0})\, \mathbf{y}$$

and $k(\mathbf{0}) = 1$. The Hessian matrix $S(\mathbf{0}) = \{\frac{\partial^2 f(\mathbf{0})}{\partial y_i\, \partial y_j};\ i,j=1,\ldots,n\}$ of second–order derivatives of $f(\mathbf{y})$ is positive definite and, of course, the gradient of $f(\mathbf{y})$ vanishes in $\mathbf{y}^* = \mathbf{0}$, i.e. $\nabla f(\mathbf{0}) = \mathbf{0}$. Then, a classical result is [11], p. 338:

$$I(\lambda) = \int_D h(\mathbf{y}) \exp[-\lambda\, f(\mathbf{y})]\, d\mathbf{y} \sim h(\mathbf{0}) \exp[-\lambda\, f(\mathbf{0})]\, \left(\frac{2\pi}{\lambda}\right)^{n/2} |\mathrm{Det}(S(\mathbf{0}))|^{-1/2} \qquad \text{for } \lambda \to \infty \qquad \text{(A6)}$$

This result is best appreciated if it is assumed that by a suitable rotation of the coordinate system the off–diagonal terms of $S(\mathbf{0})$ vanish. The term $h(\mathbf{0})$ is drawn in front of the integral. The exponent term assumes the form $\exp[-\lambda\, (f(\mathbf{0}) + \frac{1}{2} \sum_{i=1}^{n} s_{ii}\, y_i^2)] = \exp[-\lambda\, f(\mathbf{0})]\ \prod_{i=1}^{n} \exp[-\lambda/2\, s_{ii}\, y_i^2]$. Therefore, $I(\lambda) = h(\mathbf{0}) \exp[-\lambda\, f(\mathbf{0})]\, (2\pi/\lambda)^{n/2} \prod_{i=1}^{n} (s_{ii})^{-1/2}$ in noting that $\int \exp[-\lambda/2\, s_{ii}\, y_i^2]\, dy_i = (2\pi/\lambda)^{1/2}\, (s_{ii})^{-1/2}$ if integration is from $-\infty$ to $+\infty$.

#

THEOREM 1a:

If, however, the gradient $\nabla f(\mathbf{0})$ vanishes at a boundary minimum point $\mathbf{y}^* = \mathbf{0}$ and integration is extended only in the positive domain for $i = 1,\ldots,k$ $(k \in \{0,\ldots,n\})$ and otherwise the conditions specified for theorem 1 hold, we have

$$I(\lambda) = \int_{D \cap \mathbb{R}_+^k} h(\mathbf{y}) \exp[-\lambda\, f(\mathbf{y})]\, d\mathbf{y}$$

$$\sim h(\mathbf{0}) \exp[-\lambda\, f(\mathbf{0})]\, \left(\frac{2\pi}{\lambda}\right)^{n/2} 2^{-k}\, \mathrm{Det}(S(\mathbf{0}))|^{-1/2} \qquad \text{for } \lambda \to \infty \qquad \text{(A7)}$$

as can easily be shown by use of theorem 1 and simple geometric considerations in generalizing a result in [11], p. 339.

#

THEOREM 2:

Consider now the case where there is $a_i = \frac{\partial f(\mathbf{0})}{\partial y_i} > 0$ for $i = 1,\ldots,n$, $k(\mathbf{0}) = 1$, the function $f(\mathbf{y})$ has a minimum at the origin and the integration domain is $\mathbb{R}_+^n$. The function $f(\mathbf{y})$ now has the expansion:

$$f(\mathbf{y}) \approx f(\mathbf{0}) + \mathbf{a}^T\, \mathbf{y}$$

Similar to theorem 1 $h(\mathbf{0})$ and the term $\exp[-\lambda\, f(\mathbf{0})]$ are drawn in front of the integral. If integration is extended to the whole positive quadrant it is easy to see that $\int \exp[-\lambda\, a_i\, y_i]\, dy_i = |-\lambda\, a_i|^{-1}$. Therefore, it follows:

$$I(\lambda) = \int_{D \cap \mathbb{R}_+^n} h(\mathbf{y}) \exp[-\lambda\, f(\mathbf{y})]\, d\mathbf{y}$$

$$\sim h(\mathbf{0}) \exp[-\lambda\, f(\mathbf{0})]\, \lambda^{-n} \prod_{i=1}^{n} \left|\frac{\partial f(\mathbf{0})}{\partial y_i}\right|^{-1} \qquad \text{for } \lambda \to \infty \qquad \text{(A8)}$$

#

THEOREM 3:

Assume still that $k(\mathbf{0}) = 1$ and $k \in \{0,\ldots,n\}$. The function $f(\mathbf{y})$ obtains its minimum at the origin but there is $\frac{\partial f(\mathbf{0})}{\partial y_i} > 0$ for $i = 1,\ldots,k$ and the Hessian matrix $S(\mathbf{0}) = \{\frac{\partial^2 f(\mathbf{y}^*)}{\partial y_i\, \partial y_j};$ $i,j = k+1,\ldots,n\}$ of second derivatives of $f(\mathbf{y})$ is positive definite. The integration domain is now extended to $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$. Then (see [10]):

$$I(\lambda) = \int_{D \cap (\mathbb{R}_+^k \times \mathbb{R}^{n-k})} h(\mathbf{y}) \exp[-\lambda\, f(\mathbf{y})]\, d\mathbf{y} \sim$$

$$\sim h(\mathbf{0}) \exp[-\lambda\, f(\mathbf{0})]\, \frac{(2\pi)^{\frac{n-k}{2}}}{\lambda^{\frac{n+k}{2}}} \left(\prod_{i=1}^{k} \left|\frac{\partial f(\mathbf{0})}{\partial y_i}\right|^{-1}\right) |\mathrm{Det}(S(\mathbf{0}))|^{-\frac{1}{2}} \qquad \text{for } \lambda \to \infty \qquad \text{(A9)}$$

This theorem combines the results of theorem 1 and theorem 2. An important aspect in this theorem is that it can be shown that the contribution of the mixed derivatives for $i = 1, \ldots, k$, $j = k + 1,\ldots,n$ in the expansion of $f(\mathbf{y})$ is negligible [10].

With $h(0) = k(0) = 1$ it is the basis for probability integrals over independent standard normal variables and domains given as $D = \{\cap g_i(y) \le 0\}$ for $i = 1,2,\ldots,k$ $(k \le n)$. For $k = 1$, Breitung's original result is obtained [5]. For $k = n$ a formula due to Ruben for the multinormal integral can be derived [15]. The general case, $1 \le k \le n$, is given in [10,16].

#

THEOREM 4:

Under the conditions of theorem 1 but now with $k(\mathbf{0}) = 0$ and the approximation $k(\mathbf{y}) \approx \mathbf{c}^T\mathbf{y}$ around $\mathbf{y}^* = \mathbf{0}$ it can be shown that [6]

$$I(\lambda) = \int_D h(\mathbf{y})\, |k(\mathbf{y})| \exp[-\lambda f(\mathbf{y})]\, d\mathbf{y} \sim$$

$$\sim h(\mathbf{0}) \exp[-\lambda f(\mathbf{0})]\, \lambda^{-\frac{n+1}{2}}\, 2\, (2\pi)^{\frac{n-1}{2}}\, |\det(\mathbf{S}(\mathbf{0}))|^{-\frac{1}{2}}\, \sigma_c \quad \text{for } \lambda \to \infty \qquad \text{(A10)}$$

with

$$\sigma_c = (\mathbf{c}^T \mathbf{S}^{-1}(\mathbf{0})\, \mathbf{c})^{1/2} = \Big(\sum_{i,j=1}^{n} c_i c_j \overset{*}{s}_{ij}\Big)^{1/2}$$

and where the $\overset{*}{s}_{ij}$'s are the elements of the matrix $\mathbf{S}^{-1}(\mathbf{0})$. This result is obtained by expanding $f(\mathbf{y})$ into a quadratic form and by calculating the resulting integral as the expectation of the absolute value of the normal variable $\mathbf{c}^T\mathbf{y}$ with $\mathbf{y}$ a zero mean normal vector with covariance matrix $\mathbf{S}^{-1}(\mathbf{0})$.

#

THEOREM 5:

Assume that there is $\frac{\partial f(\mathbf{0})}{\partial y_i} > 0$ for $i = 1,\ldots,k$ and $k \in \{0,\ldots,n\}$. The function $f(\mathbf{y})$ obtains its minimum at the origin which is a boundary point and the matrix $\tilde{\mathbf{S}}(\mathbf{0}) = \{\frac{\partial^2 f(\mathbf{0})}{\partial y_i\, \partial y_j};\ i,j = k+1,\ldots,n\}$ of second derivatives of $f(\mathbf{y})$ in $\mathbf{y}^* = \mathbf{0}$ is positive definite. Further, the function $k(\mathbf{y})$ has the following approximate representation around $\mathbf{y}^* = \mathbf{0}$

$$k(\mathbf{y}) \approx c(\overline{\mathbf{y}}) + \sum_{i=k+1}^{n} \frac{\partial k(\mathbf{0})}{\partial y_i} y_i = c(\overline{\mathbf{y}}) + \tilde{\mathbf{c}}^T \tilde{\mathbf{y}} \qquad \text{(A11)}$$

where $\overline{\mathbf{y}} = (y_1,\ldots,y_k)^T$ and $\tilde{\mathbf{y}} = (y_{k+1},\ldots,y_n)^T$. We integrate first over the y_i's for $i = k+1,\ldots,n$. Using lemma 1, which allows to take only account of a small neighborhood V of $\tilde{\mathbf{y}}^* = \tilde{\mathbf{0}}$ and, therefore, replacing the function $h(\mathbf{y}) = h(\overline{\mathbf{y}},\tilde{\mathbf{y}})$ by its value at this point, the integral takes the form

$$I(\lambda) = \int_{D\cap\mathbb{R}_+^k} h(\overline{\mathbf{y}},\tilde{\mathbf{0}}) \int_{D\cap V\cap\mathbb{R}_+^{n-k}} |c(\overline{\mathbf{y}}) + \mathbf{c}^T\tilde{\mathbf{y}}|\, \exp[-\lambda f(\mathbf{y})]\, d\tilde{\mathbf{y}}\, d\overline{\mathbf{y}} \qquad \text{(A12)}$$

$f(\mathbf{y})$ now has the following expansion at the point $\mathbf{y}^* = \mathbf{0}$:

$$f(\mathbf{y}) \approx f(\mathbf{0}) + \frac{1}{2} \tilde{\mathbf{y}}^T\, \tilde{\mathbf{S}}(\mathbf{0})\, \tilde{\mathbf{y}} + \sum_{i=1}^{k} \frac{\partial f(\mathbf{0})}{\partial y_i} y_i \qquad \text{(A13)}$$

Making the transformation $\mathbf{x} = \lambda^{1/2}\, \tilde{\mathbf{y}}$, the integral over $\tilde{\mathbf{y}}$ can now be understood as the multinormal expectation of the absolute value of $\mathbf{c}^T\mathbf{x} + \bar{c}\lambda^{1/2}$, with $\mathbf{x}$ a zero mean normal $(n-k)$–dimensional vector with covariance matrix $\mathbf{S}^{-1}(\mathbf{0})$ and $\bar{c} = c(\overline{\mathbf{y}})$. One obtains

$$I(\lambda) \sim \exp[-\lambda f(\mathbf{0})]\, \frac{(2\pi)^{\frac{n-k}{2}}}{\lambda^{\frac{n-k+1}{2}}}\, |\mathrm{Det}(\mathbf{S}(\mathbf{0}))|^{-1/2}$$

$$\int_{D\cap\mathbb{R}_+^k} h(\overline{\mathbf{y}},\mathbf{0})\, \Big[2\sigma_c\, \varphi\Big(\frac{\bar{b}\lambda^{1/2}}{\sigma_c}\Big) + \bar{c}\lambda^{1/2} - \bar{c}\lambda^{1/2}\, \Phi\Big(\frac{\bar{c}\lambda^{1/2}}{\sigma_c}\Big)\Big] \exp\Big[-\lambda \sum_{i=1}^{k} \frac{\partial f(\mathbf{0})}{\partial y_i} y_i\Big]\, d\overline{\mathbf{y}} \qquad \text{(A14)}$$

for $\lambda \to \infty$

with

$$\sigma_c = \Big(\sum_{i,j=k+1}^{n} c_i c_j \overset{*}{s}_{ij}\Big)^{1/2}$$

and where the $\overset{*}{s}_{ij}$'s are the elements of the matrix $\mathbf{S}^{-1}(\mathbf{0})$. $\varphi_1(.)$ is the univariate standard normal density and $\Phi(.)$ the standard normal integral. Observing now that the function in brackets which arises from the foregoing integration is always positive, it is possible to apply the result of theorem 3 in the special case where $n = k$. This leads to

$$I(\lambda) = \int_{D\cap(\mathbb{R}_+^k \times \mathbb{R}^{n-k})} h(y)\,|k(y)|\,\exp[-\lambda f(y)]\,dy$$

$$\sim h(0)\exp[-\lambda f(0)]\frac{(2\pi)^{\frac{n-k}{2}}}{\lambda^{\frac{n+k+1}{2}}}\left(\prod_{i=1}^{k}\left|\frac{\partial f(0)}{\partial y_i}\right|^{-1}\right)|\mathrm{Det}(S(0))|^{-1/2}$$

$$\times\ \tilde{\sigma}_c[2\,\varphi(a) + a - 2\,a\,\Phi(-a)] \qquad \text{for } \lambda \to \infty \qquad \text{(A15)}$$

with

$$a = \frac{c_0\lambda^{1/2}}{\tilde{\sigma}_c}$$

$$\tilde{\sigma}_c = (\tilde{c}^T S^{-1}(0)\,\tilde{c})^{1/2} = \left(\sum_{i,j=k+1}^{n} c_i c_j s_{ij}^{*}\right)^{1/2}$$

$$c_0 = c(\bar{y} = \bar{0})$$

Again, the last term is the result of an expectation operation as in theorem 4 instead of simple integration as in theorem 3 for the first k integration variables. Similar to theorem 3 it can be proven that the mixed derivation for $i = 1,\dots,k$; $j = k+1,\dots, n$ can be omitted from the expansion of $f(y)$ and for the expansion of $k(y)$ only the terms for $i = k+1,\dots,n$ need to be retained.

Appendix 2: Asymptotic Probability Integrals In Eq. (14)

Following [6] it is asymptotically for each event in equation (14)

$$P\{g(X;0) \le 0\} \sim \Phi(-\beta(0))\prod_{i=1}^{n-1}[1-\kappa_i(0)\beta(0)]^{-1/2} \qquad \text{(A16a)}$$

$$P\{g(X;t) \le 0\} \sim \Phi(-\beta(t))\prod_{i=1}^{n-1}[1-\kappa_i(t)\beta(t)]^{-1/2} \qquad \text{(A16b)}$$

with $\beta(0)$ and $\beta(t)$ the so–called geometric safety indices defined by $\beta(.) = \min \|x(.)\|$ for $x(.)$ out of $\{g(x;.) = 0\}$ at the times 0 and t, respectively, and with $\kappa_i(0)$ and $\kappa_i(t)$ the i–th main curvatures of the surface in the β–points at the two times. These formulae can ber derived from theorem 3 for the special case $k = 1$ [6].

Appendix 3: Application of Lemma 1 of Appendix 1 to Eqs. (11)

We are going to apply lemma 1 in following closely the arguments in [6]. Define a surface and time neighborhood $V(\epsilon)$ of the critical point $(\tilde{y},\vartheta)^T = (\tilde{0},0)^T$ so that, for each $(\tilde{y},\vartheta)^T \in V(\epsilon)$:

$$\|(y - y^*; \vartheta - \vartheta^*)\| < \zeta(\epsilon) \text{ with } \zeta(\epsilon) \to 0 \text{ when } \epsilon \to 0$$

This permits to split up the integral over $W_1 \times T_1$ into two parts

$$E[N^+(\partial F;t)] = I^+(\beta) + J^+(\beta) \qquad \text{(A17)}$$

with

$$I^+(\beta) = \beta^n \int_{V(\epsilon)} \int_{\dot{X}_N(\beta\vartheta) > \partial\dot{F}(\beta y;\beta\vartheta)} (\dot{x}_N(\beta\vartheta) - \partial\dot{F}(\beta y;\beta\vartheta))\,\varphi_1(\dot{x}_N;\beta\vartheta|X(\beta\vartheta) = \beta y)$$

$$\times\ \varphi_n(\beta\tilde{y},\beta p(\tilde{y},\vartheta))\,Tr(\beta\tilde{y},\beta\vartheta)\,d\dot{x}_N\,d\tilde{y}\,d\vartheta \qquad \text{(A18)}$$

and:

$$J^+(\beta) = \beta^n \int_{W_1\times T_1\setminus V(\epsilon)} \int_{\dot{X}_N(\beta\vartheta) > \partial\dot{F}(\beta y;\beta\vartheta)} (\dot{x}_N(\beta\vartheta) - \partial\dot{F}(\beta y;\beta\vartheta))\,\varphi_1(\dot{x}_N;\beta\vartheta|X(\beta\vartheta) = \beta y)$$

$$\times\ \varphi_n(\beta\tilde{y},\beta p(\tilde{y},\vartheta))\,Tr(\beta\tilde{y},\beta\vartheta)\,d\dot{x}_N\,d\tilde{y}\,d\vartheta \qquad \text{(A19)}$$

Similar equations are obtained for $E[N^-(\partial F;t)]$:

$$E[N^-(\partial F;t)] = I^-(\beta) + J^-(\beta) \qquad \text{(A20)}$$

Consider the domain $D_0 = W_1 \times T_1 \times [\partial\dot{F}(\beta y;\beta\vartheta), \infty]$ of equation (24) and the point $(\tilde{0}, 0, \dot{x}_N^*)$ of D_0, where $\dot{x}_N^*$ is the value which maximizes $\varphi_1(\dot{x}_N)$ within $[\partial\dot{F}(\beta y;\beta\vartheta), \infty]$. Due to the form of the multinormal Gaussian density function, the function $\varphi_n(\beta\tilde{y}, \beta p(\tilde{y},\vartheta))$ takes its maximal value at the critical point $(\tilde{0},0)$, so that for a small neighborhood $N = V(\epsilon)\times[\partial\dot{F}(\beta y;\beta\vartheta), \infty]$ of $(\tilde{0}, 0, \dot{x}_N^*)$:

$$\varphi_1(\dot{x}_N^*;0|X(\beta\vartheta) = \beta y^*)\,\varphi_n(\tilde{0}, \beta p(\tilde{0},0)) >$$

$$\mathrm{Sup}\{\varphi_1(\dot{x}_N;\beta\vartheta|X(\beta\vartheta) = \beta y)\,\varphi_n(\beta\tilde{y}, \beta p(\tilde{y},\vartheta))\ ;\ (\tilde{y},\vartheta,\dot{x}_N) \in D_0\setminus N\}$$

According to lemma 1 the left–hand term even becomes dominating for $\beta \to \infty$ and any $\epsilon > 0$ so that:

$$E[N^+(\partial F;t)] \sim I^+(\beta) \qquad (A21)$$

Similarly, we have:

$$E[N^-(\partial F;t)] \sim I^-(\beta) \qquad (A22)$$

Therefore, we only need to evaluate $I^+(\beta)$ and $I^-(\beta)$.

Appendix 4: Derivation of Main Results (Interior Point)

Denote by f_i the function:

$$f_i(\tilde{y},\vartheta) = \frac{1}{2}\left[\sum_{i=1}^{n-1} y_i^2 + p^2(\tilde{y},\vartheta)\right] \qquad (A23)$$

If the critical time $t^* = 0$ is an interior point of T, the point $(\tilde{0},0)$ is an interior point of $V(\epsilon)$. Hence, according to the parameterization defined by eqs. (28) and taking a linear expansion of the term $p^2(\tilde{y},\vartheta)$, the function $f_i(\tilde{y},\vartheta)$ has the following expansion at the critical point

$$\begin{aligned} f_i(\tilde{y},\vartheta) &= \frac{1}{2}\Big[1 + \sum_{i=1}^{n-1} y_i^2 - |g_n(e_n;0)|^{-1}\Big[\sum_{i,j=1}^{n-1} g_{ij}(e_n;0)\, y_i y_j \\ &\quad + 2\sum_{j=1}^{n-1} g_{\tau j}(e_n;0)\,\vartheta y_j + g_{\tau\tau}(e_n;0)\,\vartheta^2\Big]\Big] \\ &= \frac{1}{2}\Big[1 + \sum_{i,j=1}^{n-1} (\delta_{ij} - |g_n(e_n;0)|^{-1} g_{ij}(e_n;0)) y_i y_j \\ &\quad - 2\sum_{j=1}^{n-1} |g_n(e_n;0)|^{-1} g_{\tau j}(e_n;0)\,\vartheta y_j - |g_n(e_n;0)|^{-1} g_{\tau\tau}(e_n;0)\,\vartheta^2\Big] \end{aligned} \qquad (A24)$$

or in condensed form:

$$f_i(\tilde{y},\vartheta) = \frac{1}{2}(1 + \tilde{\tilde{y}}^T \tilde{\tilde{Q}}\, \tilde{\tilde{y}})$$

$\tilde{\tilde{Q}}$ is the positive definite Hessian matrix of the function $\|\tilde{y}\|^2 + p^2(\tilde{y},\vartheta)$ at the critical point $(\tilde{0},0)$ and $\tilde{\tilde{y}}$ is a n–dimensional vector $\tilde{\tilde{y}} = (\tilde{y},\vartheta) = (y_1,\ldots y_{n-1},\vartheta)^T$. Hence, the density function

φ_n can be written as:

$$\varphi_n(\beta\tilde{y},\beta p(\tilde{y},\vartheta)) = (2\pi)^{-n/2} \exp(-\beta^2 f_i(\tilde{y},\vartheta)) \qquad (A25)$$

Denoting by $f(z,\tilde{y},\vartheta)$ the function $f_i(\tilde{y},\vartheta) + \frac{1}{2}z^2$ and by $k_i(z,\tilde{y},\vartheta)$ the function

$$k_i(z,\tilde{y},\vartheta) = \tilde{\sigma} z - \partial\dot{F} + \tilde{m} \qquad (A26)$$

involving three terms, the whole integral takes the form:

$$I^+(\beta) = (2\pi)^{-\frac{n+1}{2}} \beta^{n+2} \int_{V(\epsilon)} \int_{z > \beta^{-1}\tilde{a}} Tr(\beta\tilde{y},\beta\vartheta) k_i(z,\tilde{y},\vartheta) \exp[-\beta^2 f(z,\tilde{y},\vartheta)]\,dz\, d\tilde{y}\, d\vartheta \qquad (A27)$$

The time derivative $\partial\dot{F}_0 = \partial\dot{F}(e_n;0)$ of the surface $\{g(y;\vartheta) = 0\}$ at the critical point vanishes and $\tilde{m}$ is equal to zero.

We ignore this observation temporarily. A first method consists in performing the integration over z. This leads to

$$E[N^+(\partial F;t)] = (2\pi)^{-n/2} \beta^n \int_{V(\epsilon)} \tilde{\sigma}\, Tr(\beta\tilde{y},\beta\vartheta)[\varphi(\tilde{a}) - \tilde{a}\Phi(-\tilde{a})] \exp(-\beta^2 f_i(\tilde{y},\vartheta))\,d\tilde{y}\, d\vartheta \qquad (A28)$$

with:

$$\tilde{a} = \frac{\beta\partial\dot{F} - \beta\tilde{m}}{\tilde{\sigma}}$$

Define now as $h(\mathbf{y})$ in theorem 1a the function

$$h_i(\tilde{y},\vartheta) = \tilde{\sigma}\, Tr(\beta\tilde{y},\beta\vartheta)\,[\varphi(\tilde{a}) - \tilde{a}\Phi(-\tilde{a})] \qquad (A29)$$

which is positive within the integration domain since $\tilde{\sigma}$ is bounded away from zero, $Tr(\beta\tilde{y},\beta\vartheta)$ is positive due to the choice of the parameterization and the function $\psi(\tilde{a}) = \varphi(\tilde{a}) - \tilde{a}\Phi(-\tilde{a})$ is always positive. Extending the integration domain $V(\epsilon)$ to $\mathbb{R}^n$ according to lemma 1 allows to use theorem 1a. This yields

$$E[N^+(\partial F;t)] \approx \varphi_1(\beta)\,|\mathrm{Det}(\tilde{\tilde{Q}})|^{-1/2}\,\tilde{\sigma}_0 \tag{A30}$$

with $\tilde{\sigma}_0$ in eq. (38). We further have $E[N^+(\partial F;t)] = E[N^-(\partial F;t)]$ and $E[N^+(\partial F;t)] + E[N^-(\partial F;t)] = 2\,E[N^+(\partial F;t)]$ so that:

$$E[N(\partial F;t)] = 2\,E[N^+(\partial F;t)] \tag{A31}$$

On the other hand, if we combine both upcrossings and incrossings in the same integral as in eq. (11c), one has (method B):

$$E[N(\partial F;t)] = E[N^+(\partial F;t)] + E[N^-(\partial F;t)] =$$

$$(2\pi)^{-\frac{n+1}{2}}\beta^{n+2}\int_{V(\epsilon)}\int_{\mathbb{R}^1} Tr(\beta\tilde{y},\beta\vartheta)\,|k_i(z,\tilde{y},\vartheta)|\exp[-\beta^2 f(z,\tilde{y},\vartheta)]\,dz\,d\tilde{y}\,d\vartheta \tag{A32}$$

Using the development of $\tilde{m}$ defined by eq. (22) and taking the linear expansion of $\partial\dot{F}$ at the critical point $(e_n;0)$,

$$\partial\dot{F}(y;\vartheta) = \partial\dot{F}_0 + \sum_{i=1}^{n-1} g_{\tau i}(e_n;0)\,y_i + g_{\tau\tau}(e_n;0)\,\vartheta \tag{A33}$$

we can show that the function $k_i(z,\tilde{y},\vartheta)$ can be written as:

$$k_i(z,\tilde{y},\vartheta) = \tilde{\sigma}z - \tilde{\tilde{r}}^T\,\tilde{\tilde{Q}}\,\tilde{\tilde{y}}$$

with $\tilde{\tilde{y}}$ and $\tilde{\tilde{Q}}$ as previously defined and $\tilde{\tilde{r}}$ the n–dimensional vector:

$$\tilde{\tilde{r}} = (\dot{r}_{1n},\ldots,\dot{r}_{n-1n},-|g_n(e_n;0)|^{-1})^T$$

All conditions for application of theorem 4 are now fulfilled. One obtains

$$E[N(\partial F;t)] \approx 2\,\varphi_1(\beta)\,|\mathrm{Det}(\tilde{\tilde{Q}})|^{-1/2}\,\tilde{\tilde{\sigma}}_1 \tag{A34}$$

where:

$$\tilde{\tilde{\sigma}}_1 = (\tilde{\sigma}_0^2 + \tilde{\tilde{r}}^T\,\tilde{\tilde{Q}}\,\tilde{\tilde{r}})^{1/2}$$

This result is a direct generalization of the result in [6]. Together with eq. (12) one can determine the expected number of outcrossings.

The foregoing procedure also holds in the case of a boundary point with vanishing time derivative of the function g at the critical point. However, according to theorem 1a, the expected number of crossings is just half the number calculated by eq. (12).

Appendix 5: Derivation of Main Results (Boundary Point)

Consider now the case where t^x is a boundary point and suppose that the first time derivative of $g(y;\vartheta)$ does not vanish at the corresponding critical point $(y^x,0)$ and is positive. With equation (28), the following expansion can be derived:

$$f_b(\tilde{y},\vartheta) = \tfrac{1}{2}[1 + \sum_{i=1}^{n-1} y_i^2 - |g_n(e_n;0)^2 + g_\tau(e_n;0)^2|^{-1/2}\,[\sum_{i,j=1}^{n-1} g_{ij}(e_n;0)\,y_i y_j$$
$$+ 2\sum_{j=1}^{n-1} g_{\tau j}(e_n;0)\,\vartheta y_j + 2g_\tau(e_n;0)\,\vartheta]] \tag{A35}$$

Denote by $\tilde{Q}$ the matrix

$$\tilde{Q} = \{\frac{\partial^2}{\partial y_i\,\partial y_j} f_b(\tilde{0},0)\}_{i,j=1,\cdots,n-1} =$$
$$= \{\delta_{ij} - (|g_n(e_n;0)|^2 + |g_\tau(e_n;0)|^2)^{-1/2}\,g_{ij}(e_n;0))\}_{i,j=1,\cdots,n-1} \tag{A36}$$

which is positive definite since it is the Hessian of the positive function $f_b(\tilde{y},0)$ at the point $\tilde{0}$. Let further define the functions A and $A(y_i)$ as follows:

$$A = 2g_\tau(e_n;0) \tag{A37}$$

$$A(y_i) = 2\sum_{j=1}^{n-1} g_{\tau j}(e_n;0)\,y_j \tag{A38}$$

Then, f_b can be written as:

$$f_b(\tilde{y},\vartheta) = \tfrac{1}{2}[1 + \tilde{y}^T\,\tilde{Q}\,\tilde{y} + (A(y_i) + A)\,\vartheta] \tag{A39}$$

Hence, with

$$\varphi_n(\beta\tilde{y},\beta p(\tilde{y},\vartheta)) = (2\pi)^{-1/2}\exp[-\beta^2 f_b(\tilde{y},\vartheta)]$$

the integral takes the form:

$$E[N^+(\partial F;t)] = (2\pi)^{-\frac{n+1}{2}}\beta^{n+2}\int_{V(\epsilon)}\int_{z>\beta^{-1}\tilde{a}}(\tilde{\sigma}z-\partial\dot{F}+\tilde{m})Tr(\beta\tilde{y},\beta\vartheta)$$

$$\exp[-\beta^2[\frac{z^2}{2}+f_b(\tilde{y},\vartheta)]]dzd\tilde{y}d\vartheta \qquad (A40)$$

Note that the term $\partial\dot{F}$ which represents the time derivative of the function $g(y;\vartheta)$ at the crossing point does not vanish at the critical point $(\tilde{0},0)$. This means $k_b(0,\tilde{0},0) \neq 0$ where $k_b(z,\tilde{y},\vartheta) = \tilde{\sigma}z - \partial\dot{F} + \tilde{m}$.

As in the preceding case of an interior critical point we try to integrate first separately over z and consider the same function $h(\tilde{y},\vartheta)$ as in eq. (A29). Then, theorem 3 applies and one obtains

$$E[N^+(\partial F;t)] \approx [\varphi(\tilde{a}_0)-\tilde{a}_0\,\Phi(-\tilde{a}_0)]\beta^{-1}\varphi_1(\beta)\,|g_{,r}(e_n;0)|^{-1}|Det(\tilde{Q})|^{-1/2}\,\tilde{\sigma}_0 \qquad (A41)$$

with:

$$\tilde{a}_0 = \frac{\beta\partial\dot{F}_0}{\tilde{\sigma}_0}$$

In a similar way

$$E[N^-(\partial F;t)] \approx [\varphi(\tilde{a}_0)+\tilde{a}_0\,\Phi(\tilde{a}_0)]\beta^{-1}\varphi_1(\beta)\,|g_{,r}(e_n;0)|^{-1}|Det(\tilde{Q})|^{-1/2}\,\tilde{\sigma}_0 \qquad (A42)$$

and, therefore:

$$E[N(\partial F;t)] \approx [2\varphi(\tilde{a}_0) - \tilde{a}_0 + 2\tilde{a}_0\,\Phi(\tilde{a}_0)]\,\beta^{-1}\varphi_1(\beta)\,|g_{,r}(e_n;0)|^{-1}|Det(\tilde{Q})|^{-1/2}\,\tilde{\sigma}_0 \qquad (A43)$$

However, combining both upcrossings and incrossings in the same integral as before yields

$$E[N(\partial F;t)] = E[N^+(\partial F;t)] + E[N^-(\partial F;t)] =$$

$$(2\pi)^{-\frac{n+1}{2}}\beta^{n+2}\int_{V(\epsilon)}\int_{\mathbb{R}^1}Tr(\beta\tilde{y},\beta\vartheta)\,|k_b(z,\tilde{y},\vartheta)|\exp(-\beta^2 f(z,\tilde{y},\vartheta))dzd\tilde{y}d\vartheta \qquad (A44)$$

with:

$$k_b(z,\tilde{y},\vartheta) = \tilde{\sigma}z - \partial\dot{F} + \tilde{m} \qquad (A45)$$

In this case, the function $k_b(z,\tilde{y},\vartheta)$ is not zero at the critical point because of the non–vanishing term $\partial\dot{F}_0$. Considering a linear expansions of $\tilde{m}$ only with respect to the y_i's but not with respect to time and ignoring in this case the linear expansion of $\partial\dot{F}$, we observe again that the function $k_b(z,\tilde{y},\vartheta)$ can be written as

$$k_b(z,\tilde{y},\vartheta) = \tilde{\sigma}z - \tilde{r}^T\,\tilde{Q}\,\tilde{y} - \partial\dot{F}_0$$

with $\tilde{Q}$ and $\partial\dot{F}_0$ as previously defined and $\tilde{r}$ the (n–1)–dimensional vector:

$$\tilde{r} = (\dot{r}_{1n},\dots,\dot{r}_{n-1n})^T$$

We are now going to apply theorem 5 with k = 1 . One obtains

$$E[N(\partial F;t)] \approx [\,2\varphi(a_{b0}) - a_{b0} + 2a_{b0}\,\Phi(a_{b0})\,]\beta^{-1}\varphi_1(\beta)\,|g_{,r}(e_n;0)|^{-1}|Det(\tilde{Q})|^{-1/2}\,\tilde{\sigma}_b \qquad (A46)$$

with:

$$\tilde{\sigma}_b = (\tilde{\sigma}_0^2 + \tilde{r}^T\,\tilde{Q}\,\tilde{r})^{1/2}$$

$$a_{b0} = \frac{\beta\partial\dot{F}_0}{\tilde{\sigma}_b}$$

Together with eq. (12) this yields the expected number of outcrossings.

It is interesting to compare the results obtained according to theorem 1 to 3 and according to theorems 4 and 5, respectively. Inspection shows that for an interior critical point as well as a boundary critical point the only difference is in the calculation of the conditional expectation of $\dot{X}_N$.

Appendix 6: Results in Original Space

The results can also be given in the original space by performing the inverse of the orthogonal transformation T which transformed x^* into $\beta\, e_n$.

Critical point is an interior point

Due to the definition of $T = \{t_{ij}\}_{i,j=1,\dots,n}$, we have:

$$T\, y^* = e_n \qquad \text{(A47)}$$

Therefore, since $n(y^*) = y^* = t_n$ where t_n is the vector $(t_{1n},\dots,t_{nn})^T$, we can write

$$(\dot{\rho}_{1n},\dots,\dot{\rho}_{nn}) = T\, \dot{R}^T\, n(y^*) \qquad \text{(A48)}$$

and, setting the last component to $-|g_\tau(e_n;0)|$ which is written in the original space as $-|g_\tau(x^*;0)|$:

$$\overset{\approx}{\rho} = T\, \dot{R}^T\, n(y^*) \qquad \text{(A49)}$$

Furthermore, the matrix $\overset{\approx}{Q}$ takes the form

$$\overset{\approx}{Q} = \overset{\approx}{T}\,(E - G_{x_\tau})\, \overset{\approx}{T}^T \qquad \text{(A50)}$$

where $\overset{\approx}{T}$ is a $n\times(n+1)$ matrix obtained from T by adding a $t_{n+1\cdot n+1}$ component equal to 1 and by deleting the (n−1)–row, G_{x_τ} the (n+1)–dimensional matrix

$$G_{x_\tau} = \quad \Big\{ g_{jk} = |\nabla g(x^*;t^*)|^{-1} \frac{\partial^2 g(x^*;t^*)}{\partial x_j\, \partial x_k} \;;\; j,k = 1,\dots,n$$

$$g_{j\,n+1} = g_{n+1\,j} = |\nabla g(x^*;t^*)|^{-1} \frac{\partial^2 g(x^*;t^*)}{\partial x_j\, \partial \tau} \;;\; j = 1,\dots,n$$

$$g_{n+1\,n+1} = |\nabla g(x^*;t^*)|^{-1} \frac{\partial^2 g(x^*;t^*)}{\partial \tau^2} \;;\; j,k = 1,\dots,n \Big\} \qquad \text{(A51)}$$

and where E is the (n+1)–dimensional matrix obtained from a n–dimensional unity natrix by adding a zero–(n+1)–row and a zero–(n+1)–column.

Hence

$$\overset{\approx}{\rho}^T\, \overset{\approx}{Q}\, \overset{\approx}{\rho} = n^T(x^*)\, \dot{R}\, (E - G_{x_\tau})\, \dot{R}^T\, n(x^*) \qquad \text{(A52)}$$

and

$$\tilde{\sigma}_0 = \sigma(x^*;t^*) = (n^T(x^*)\, (\ddot{R} - \dot{R}^T \dot{R})\, n^T(x^*))^{1/2} \qquad \text{(A53)}$$

The value of the determinant of the matrix $\overset{\approx}{Q}$ is given by:

$$\mathrm{Det}(\overset{\approx}{Q}) = n^T(x^*)\, C_{x_\tau}\, n(x^*) \qquad \text{(A54)}$$

where C_{x_τ} is the matrix of cofactors of the matrix $E - G_{x_\tau}$.

Finally, the expected number of outcrossings of the process $x(\tau)$ through the hypersurface $\partial F(x;\tau)$ is given by

$$E[N^+(\partial F;t)] \approx \varphi_1(\beta)\, |J^*|^{-1/2} [\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{1/2}$$

$$+ \frac{1}{2}\, [P(g(x;0) \le 0) - P(g(x;t)\;;\;0)] \qquad \text{(A55)}$$

with

$$J^* = n^T(x^*)\, C_{x_\tau}\, n(x^*)$$
$$\sigma^2(x^*;t^*) = n^T(x^*)\, (\ddot{R} - \dot{R}^T \dot{R})\, n^T(x^*)$$
$$\omega^2(x^*;t^*) = n^T(x^*)\, \dot{R}\, (E - G_{x_\tau})\, \dot{R}^T\, n(x^*)$$

and $P(g(x;0) \le 0)$ and $P(g(x;t) \le 0)$ as defined in appendix 2.

Critical point is a boundary point

A similar reasoning can be achieved for the case of a boundary point. Again we write

$$(\dot{\rho}_{1n},\ldots,\dot{\rho}_{nn}) = T\,\dot{R}^T\,n(x^*) \tag{A56}$$

and, setting the last component to zero:

$$(\dot{\tilde{\rho}},0) = T\,\dot{R}^T\,n(x^*) \tag{A57}$$

Moreover, the matrix $\tilde{Q}$ is again written as

$$\tilde{Q} = \tilde{T}\,(I - G_x)\,\tilde{T}^T \tag{A58}$$

where $\tilde{T}$ is the (n–1)×n–matrix obtained from T by deleting the last row and G_x defined by:

$$G = \{\,|\nabla g(x^*;t^*)|^{-1}\,\frac{\partial^2 g(x^*;t^*)}{\partial x_j\,\partial x_k}\ ;\ j,k = 1,\ldots,n\} \tag{A59}$$

Hence:

$$\dot{\tilde{\rho}}^T\,\tilde{Q}\,\dot{\tilde{\rho}} = n^T(x^*)\,\dot{R}\,(I - G)\,\dot{R}^T\,n(x^*) \tag{A60}$$

$\tilde{\sigma}_0 = \sigma(x^*;t^*)$ remains as previously defined.

The value of the determinant of the matrix $\tilde{Q}$ is given by

$$\mathrm{Det}(\tilde{Q}) = n^T(x^*)\,C_x\,n(x^*) \tag{A61}$$

where C_x is the matrix of cofactors of the matrix $I + G_x$. Since the time derivative $\beta\,g_{,\tau}(e_n;0)$ can be written in the original space as $g_{,\tau}(x^*;0)$, the expected number of outcrossings of the process $x(\tau)$ through the hypersurface $\partial F(x;\tau)$ in the case of a time boundary point is given by

$$E[N^+(\partial F;t)] \approx \frac{1}{2}\,[\,2\varphi(a) - a + 2a\,\Phi(a)\,]\,\varphi_1(\beta)\,|g_{,\tau}(x^*;t^*)|^{-1}\,|J^*|^{-1/2} \times [\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{1/2} + \frac{1}{2}\,[P(g(x;0) \le 0) - P(g(x;t) \le 0)] \tag{A62}$$

with

$$a = g_{,\tau}(x^*;t^*)\,[\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{-1/2}$$

$$J^* = n^T(x^*)\,C_x\,n(x^*)$$

$$\sigma^2(x^*;t^*) = n^T(x^*)\,(\ddot{R} - \dot{R}^T\,\dot{R})\,n(x^*)$$

$$\omega^2(x^*;t^*) = n^T(x^*)\,\dot{R}\,(I - G_x)\,\dot{R}^T\,n(x^*)$$

and $P(g(x;0) \le 0)$ and $P(g(x;t) \le 0)$ as defined by eq. (16).

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