Journal of Structural Engineering
Vol.25 No.1 April 1998 pp.1–20

Computational techniques in stationary and non-stationary load combination - A review and some extensions

Rüdiger Rackwitz*

Whereas the methods to compute time invariant reliabilities are well-known, the methods for time variant reliabilities including their combination under vectorial process loading have not been developed beyond some simple but practically important cases. One of the first proposals for stationary load combination is due to Turkstra[1] which still is the basis of many codified combination rules despite of its limitations. Ferry Borges and Castanheta[2] developed a scheme for random sequences for which a suitable computational algorithm was designed by Rackwitz and Fiessler[3]. All those proposals still set out from the assumption that the load combination is a problem of finding the maximum of a function in several time-variant variables. It turned out that solutions along this line are limited to some special cases as regards the type of combination and the load models. Also, only stationary cases can be dealt with. However, Veneziano et al.[4] for Gaussian processes, Breitung and Rackwitz[5], for rectangular wave renewal processes, Madsen[6] for renewal processes with arbitrarily shaped pulses and Wen[7] for intermittent processes with rectangular waves, adopted the more powerful outcrossing approach. Non-stationary load combination is dealt with only occasionally[8,9]. It appears that for quasi-static loading, the only general and computationally feasible approach is through the outcrossing approach which, unfortunately, provides only asymptotically valid results or probability bounds.

The methods to compute large scale time-invariant reliabilities are well known. Let $X = (X_1, ..., X_n)^T$ be a vector of random variables with continuously differentiable joint distribution function $F_X(x)$ and $g(x)$, a twice differentiable state (performance) or failure function such that $g(x) > 0$ denotes the safe state, $g(x) = 0$, the limit state and $g(x) < 0$ the failure state. $g(x) = 0$ will also be denoted by failure surface. The failure probability is:

$$P_f = \int_{g(x) \le 0} dF(x) = \int_{g(x) \le 0} f_X(x)\, dx \qquad (1)$$

where the second formulation is valid if the probability density exists. The simplest problem of this kind is when failure occurs if a demand S on a system exceeds its capacity R. Then, the performance or state function is given by $g(x) = r - s$. If capacity and/or demand are random, the probability of failure is simply a volume integral extended over the failure domain. Especially for large and compled state function an exact evaluation by numerical integration will require considerable computational effort. Therefore, some special methods have been devised which can do the integration efficiently and accurately. For all computations a probability distribution transformation into the standard normal space will be performed so that with $x = T(u)$, there is:

$$P_f = \int_{g(x) \le 0} f_X(x)\, dx = \int_{g(T(u)) \le 0} \varphi_U(u)\, du \qquad (2)$$

where u is an independent standard normal vector and $\varphi(u)$ its probability density. Approximations to this integral have been proposed by Hasofer and Lind[10] for normal vectors X and later extended to arbitrary distributions by Rackwitz and Fiessler[3]. More general transformation methods exist[13-14]. A first order probability estimate is then obtained from:

$$P_f \approx \Phi(-\beta) \qquad (3)$$

where, $\beta = \|u^*\|$

and, $u^* = \min \|u^*\|$,for $\{u : g(u) \le 0\}$ (4)

The main computational task is the location of the u^*-point (or β-point) by a suitable search algorithm[15] known as FORM. Usually, the probability estimate is sufficiently accurate for many practical purposes.

* Technische Universität München, München, Federal Republic of Germany

Breitung[16] extended FORM and gave it a sound theoretical basis. By making use of asymptotic considerations, put forward by Laplace in 1821, he arrived at

$$P_f \simeq \Phi(-\beta) \prod_{i=1}^{n-1} (1-\beta\kappa_i)^{-1/2} \; ; \quad \beta \to \infty \tag{5}$$

The κ_i are the main curvatures in the solution point u^*. The main curvatures are found from the Hessian matrix of second derivatives of $g(u) = 0$ in u^* after solving an eigenvalue problem. It corresponds to fitting an approximating parabola in the β - point to $g(u)$. The two formuale given for the compution of β break down for $|\beta| \leq 1$ and $\beta\kappa_i \geq 1$. For small β's one can nevertheless retain a quadratic approximation of the failure surface. An exact formula for the exact probability content of a parabola[17] can be used. However, a fitting of the limit state function in this β-range by a paraboloid has no theoretical justification. It is seen that the second order result (SORM) differs from the first order result (FORM) by a factor involving the curvatures of the failure surface which generally is close to unity. A full account of the underlying theory and some extensions is given by Hohenbichler et al.[18].

THE NATURE OF TIME-VARIANT COMPONENT RELIABILITY

Time-variant reliability is more difficult to compute than time-invariant component reliability. In fact, one is hardly interested in the time dependent failure probability function $P_f(t)$ where t is treated as a parameter except in quantities like the probability of first passage into the failure domain, the total duration of exceedances into the failure domain, the duration fo individual exceedances and other related criteria. The quantity

$$N(t) = \int_{g(x,t) \leq 0} f_x(x, t)\, dx \tag{6}$$

will rather be denoted by non–availability so that $A(t) = 1 - N(t)$ is the availability. Both quantities are easily determined by fixing the time in a time–invariant approach. In principle, the basic formulation for first passage problems then is:

$$P_f(t) = P(T \leq t) \tag{7}$$

where T is the random time of exit into the failure domain and [0,t] is the considered time interval. If the component does not fail at time $\tau = 0$, failure occurs at a random time and the distribution function of T must be known. Unfortunately, this is rarely the case. Exceptions are the failure times of non–structural components, where often rich statistical data is available. Then, it is also possible to use the time–invariant computation schemes because the limit state function then simply is $g(x) = T - t < 0$. In all other cases T must be inferred from the characteristics of the random processes affecting the performance of the compontent. T must be considered as a first passage time, i.e. is the time when the component enters the failure domain for the first time given that the component was in the safe state at time $\tau = 0$.

The outcrossing approach is rather mandatory for practical application as there are only five exact solutions for the first passage time of scalar random processes all belonging to the class of Markov processes (Cox and Miller[19], for the first passage time of Brownian motion, Darling and Siegert[20], for a stationary Gaussian process with exponential auto–correlation function, Slepian,[21] for a stationary Gaussian process with triangular auto–correlation function and for the first passage times of simple linear oscillators by Lennox and Fraser[22], and Ariaratnam and Tam[23], where the Fokker–Planck equation is used). Very few other results are available for scalar processes and almost none for vector processes.

In the following, we briefly define a limited but for many practical purposes already sufficiently rich set of random process models for which a solution is possible via the so–called outcrossing approach. Some of the available results on the outcrossing approach will be reviewed and some others will be further developed and combined. In particular, upper and lower probability bounds will be given and time dependent failure probabilities will be computed based on so–called FORM/SORM[18] . It will be shown how results for rectangular wave renewal processes and for Gaussian and Non–Gaussian differentiable processes can be obtained in good approximation, making use of isolated results in the literature. Emphasis is on non–stationary processes and/or time–variant componential state functions as they arise, for example, from structural deterioration or ageing. Concepts of combining these two types of processes will be outlined. A simple model for intermittent loading will be described and applied to stationary and non–stationary combinations.

The results presented are primarily meant for load combinations in quasi–static, not necessarily linear elastic structural systems, i.e. where dynamic effects can be neglected. The random variables or random process variables introduced may be interpreted either as basic variables or as variables describing the state of a component in load–effect space.

RANDOM PROCESS MODELS

Scalar Differentiable Normal (Gaussian) Processes

A scalar normal or Gaussian process with continuously differentiable sample paths is completely defined by its mean value and covariance functions, i.e. by $m_S(t)$ and $C_S(t_1, t_2)$ or, alternatively, by $m_S(t)$, its variance function $\sigma_S^2(t)$ and its autocorrelation coefficient function $\rho_S(t_1 t_2) = C_S(t_1 t_2)/(\sigma_S(t_1)\, \sigma_S(t_2))$ .The covariance function must be positive definite. The marginal distribution function is Gaussian (normal) at any time and it is bivariate normal for any two different points in time. If the process is stationary

the argument in the covariance or the autocorrelation function may be replaced by $\tau = |t_1 - t_2|$. Differentiability of the sample paths implies that the crossings of a given level, form a regular point process. It is assured if the covariance (or autocorelation) function is twice differentiable at $t = t_1 = t_2$ (Fig.1).

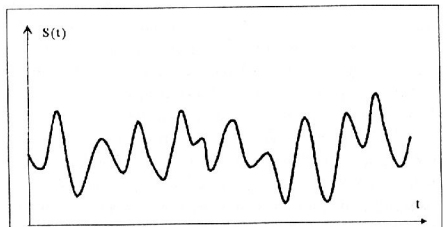

FIG.1 CONTINUOUSLY DIFFERENTIABLE PROCESS

Scalar Nataf and Hermite Processes

A special but important class of non–normal, scalar and differentiable processes can be built by a memoryless transformation from a normal process, i.e.

$$S(t) = h(U(t)) \tag{8}$$

where $U(t)$ is a standard normal process and $h(u)$ is an arbitrary function. For $S(t)$ any admissible unimodal distribution function can be chosen thus defining a certain class of functions $h(u)$. In addition the autocorrelation function $\rho_S(t_1 , t_2)$ has to be specified. However, there are some restrictions on the type of autocorrelation function, i.e. the transformed autocorrelation function $\rho_U(t_1 , t_2)$ must be positive definite. Many additional results for Nataf processes are given by Grigoriu[24].

The Hermite process is a special case of the Nataf process. All marginal distributions must be of Hermite type. For this process the solution of the integral equation occurring for the autocorrelation function of the equivalent (or better generating) standard normal process is analytic. The standard Hermite process has the representation, i.e. a special case of the function $h(u)$.

$$S(t) = \kappa\,(U(t) + \tilde{h}_{3,i}\,(U(t)^2 - 1) + \tilde{h}_{4,i}\,(U(t)^3 - 3U(t))) \tag{9}$$

For the coefficients depending on the first four moments of the marginal disribution of the non–normal process we refer to Winterstein and Bjerager[14]. In addition, the Hermite process requires specification of the autocorrelation function of $S(t)$. Again, there are certain restrictions on the moments of the marginal distributions as well as on the autocorrelation function.

Rectangular Wave Renewal Processes

Scalar rectangular wave renewal processes are useful models for processes changing their amplitude at random renewal points in a random fashion. A scalar rectangular wave renewal process is characterised by the jump rate λ and the distribution function of the amplitude. The renewals occur independently of each other. No specific distribution is assigned to the interarrival times in the following. Thus, it is clear that such a model makes use of asymptotic and ergodicity concepts. For example, a renewal process may have arbitrary distribution function for the interarrival times with existing expectation $E[T]$. Then, there is[19]

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{E[t]} = \lambda \tag{10}$$

where $H(t)$ is the so–called renewal function (=mean number of renewals and λ the renewal rate. Hence, we are considering a renewal process in its asymptotic state meaning that the number of renewals in a large time interval is just inversely proportional to the mean interarrival time. Therefore, the renewal process characterized only by a jump rate captures only long term statistics. The mean duration of pulses is asymptotically equal to $1/\lambda$. For the special cas of a Poisson rectangular wave process the interarrival times and so the durations of the pulses are exponentially distributed with parameter $1/\lambda$. In the special case of a Ferry Borges–Castanheta process[2], the durations are constant and the repetition number $r = (t_2 - t_1)/\Delta$ with Δ the duration of pulses is equal to $\lambda(t_2 - t_1)$. Also, the sequence of amplitudes is an independent sequence. The jump rate can be a function of time as well as the parameters of the distribution function of the amplitudes (Fig.2).

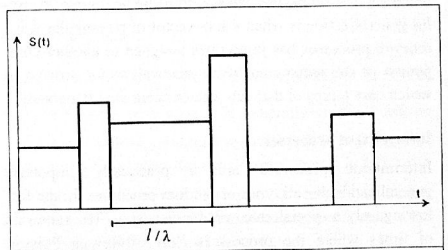

FIG.2 RECTABGULAR WAVE RENEWAL PROCESS

It is assumed that rectangular wave processes jump from a random value $S(t)$ to a new value $S^+(t+\delta)$ with $\delta \to 0$ at a renewal without returning to zero. Rectangular wave renewal processes must be regular processes, i.e. the occurrence of any two or more renewals in a small time interval must be negligible. Non–stationary rectangular wave renewal processes are processes which have either time–dependent parameters of the amplitude distributions or time–dependent jump rates.

Differentiable Vector Processes

The differentiable Gaussian vector processes $S(t)$ are completely specified by the mean vector $m_S(t)$ and symmetric, positive definite matrix of covariance functions.

$$C_S(t_1,t_2) = \{\sigma_{ij}(t_1,t_2)\ ;\ i,j = 1, \ldots n\} \qquad (11)$$

or alternatively, by the variance functions $\sigma_i^2(t)$ and the autocorrelation coefficient functions $\rho_{ij}(t_1, t_2) = \sigma_{ij}(t_1, t_2)/(\sigma_i(t_1)\ \sigma_j(t_2))$. If $m_S = m_S(t), \sigma_i^2 = \sigma_i^2(t)$ and $\rho_{ij}(\tau) = \rho_{ij}(t_1, t_2)$ with $\tau = |t_1 - t_2|$, the process is said to be stationary. Differentiability of the sample paths is assured if the covariance (or autocorrelation) functions are twice differentiable at $t = t_1 = t_2$.

For vector processes of Nataf–and Hermite–type, the same representation is valid. Additionally, the marginal distributions need to be specified. Again certain restriction hold. Even if the inputted matrix of cross–correlation functions is positiive definite, the resulting equivalent matrix of Gaussian cross–correlation functions can be non–positive definite. The model then is no more a valid model.

Rectangular Wave Renewal Vector Processes

A vector rectangular wave renewal process has marginally exactly the same properties as the scalar rectangular wave renewal process. Additionally, the renewals of each components of the vector are independent and all amplitudes are independent. Non–stationary rectangular wave renewal processes are processes which have either time–dependent parameters of the amplitude distributions or time–dependent jump rates. Rectangular wave renewal processes must be regular processes, i.e. the occurrence of any two or more renewals in a small time interval must be negligible. A useful generalization is when a sub–vector of rectangular wave renewal processes has jump rates assigned to another component of the rectangular wave renewal vector process in which case jumps of that sub–vector occur simultaneously.

Intermittent processes

Intermittent processes are a practically important generalization for all types of random processes. In the following only a special case will be discussed. The renewals of times where the process is ''on'' follow a Poisson renewal process with rate κ (or mean inter arrival time $1/\kappa$). At a renewal the process activates an ''on''–state (state ''1''). The ''off''–states are denoted by ''0''. Consequently, the mean number of ''on''–states in the interval $[t_1,t_2]$ is

$$E[M(t_1,t_2)] = \kappa(t_2 - t_1) \qquad (12)$$

The initial durations of ''on''–states will have exponential distribution with mean $1/\mu$ independent of the arrival times. However, we will assume that a ''on''–time is also finished if a next renewal occurs so that the durations have a truncated exponential distribution. By assuming random initial conditions the probabilities of the ''on/off''–states can then be determined by making use of a standard result from renewal theory.

$$p_{off}(t) = \frac{\mu}{\kappa+\mu} + \frac{\kappa-\mu}{\kappa+\mu}\exp[-(\kappa+\mu)t]$$

$$p_{on}(t) = \frac{\kappa}{\kappa+\mu} - \frac{\kappa-\mu}{\kappa+\mu}\exp[-(\kappa+\mu)t] \qquad (13)$$

We will assume that the ''on/off''–process is already in its stationary state where the last terms in these equations vanish. In contrast to normal rectangular wave renewal processes where the duration of the rectangular pulse is exactly same until the next renewal and the duration of the rectangular pulse is exponentially distributed with mean $1/\lambda$, for a Poissonian renewal process the ''on''–times are now truncated at the next renewal. It is easily shown that the effective duration of the ''on''–times is also exponential with mean $1/(\kappa+\mu)$. The so–called interarrival–duration intensity is defined by $\rho = \kappa/\mu$. For $\rho = \kappa/\mu \to \infty$, the processes are almost always active. For $\rho = \kappa/\mu \to 0$, one obtains spike–like processes. Figure 3 shows a simple ''on/off'' process. Figure 4 shows an example of an intermittent rectangular wave renewal process with multiple jumps during an ''on''–time where the jump rate λ of the rectangular wave renewal process is larger than the interarrival rate κ for the ''on''–times.

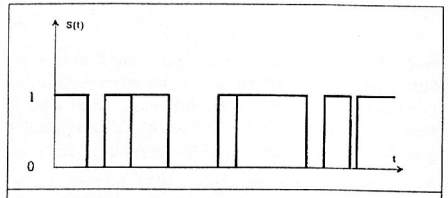

FIG.3 POISSONIAN ''ON–OFF''–PROCESS WITH TRUNCATED EXPONENTIAL DURATIONS

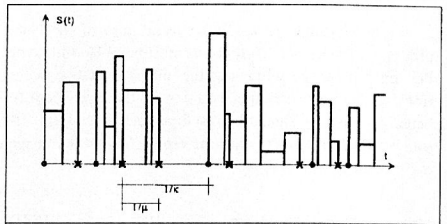

FIG.4 INTERMITTENT RECTANGULAR WAVE RENEWAL PROCESS WITH SEVERAL JUMPS DURING ''ON''–TIMES

Intermittencies can also be defined for differentiable processes. If this is a dependent vector process then the entire vector process must have a common ρ, that is all components of the vector must have the same κ and μ. Independent differentiable vector processes, however, can have different ρ's.

Ergodic Random Vector Sequences

The description of slowly varying random process variables, for example, for the modeling of the sequence of subsequent 10–minute mean wind velocities, can be done by stationary and ergodic random vector sequences of arbitrary joint distribution function. They will be dented by Q–variables. The Q–variables can have arbitrary cross–dependencies. In general, those sequences are auto–dependent to some degree. However, it is not necessary to specify their dependence structure which is a consequence of the ergodicity assumption and the way how they are treated in the reliability analysis. Usually, Q–variables define the parameters of superimposed rapidly fluctuating random process variables denoted by S–variables.

THE OUTCROSSING APPROACH FOR A WIDE CLASS OF RANDOM PROCESSES

The outcrossing approach rests on a few basic assumption which, however, are fulfilled in almost all practical cases. According to Schall et al[24] one can distinguish between three types of variables, R, Q, S, respectively.

- R is a vector of random variables as in time–invariant reliability. Its distribution parameters can be deterministic functions of time. This vector is used to model resistance variables and/or uncertain parameters of Q– or S–variables. The most important characteristics of this type of variable is that they are non–ergodic.
- Q is a vector of stationary and ergodic sequences. Usually, it is used to model long term variations in time (traffic states, sea states, wind velocity regimes, etc.). These variables usually determine the fluctuating parameters of the random process variables.
- S is a vector of (sufficiently mixing) not necessarily stationary random process variables whose parameters can depend on Q and/or R. S can include a rectangular wave renewal vector process and differentiable vector processes denoted by SR and SD, respectively.
- The safe state of the component is defined for $g(r,q,s\,(t),t) > 0$, the limit state for $g(r,q,s\,(t),t) = 0$ and the failure state for $g(r,q,s\,(t),t) \le 0$, respectively. The state function can also contain time as a parameter, and must be twice differentiable.

The presence of R– and Q–variables in the parameters of the S–vector makes it stochastically dependent even if the components of S are initially defined as independent.

The conditional (on $R = r$ and $Q = q$, respectively) rate of outcrossings into the failure domain conditional on q and r can be defined as:

$$\nu^+(F,\tau \mid r.q) = \lim_{\Delta\to 0}\frac{1}{\Delta}P(\{g(S(\tau),\tau) > 0 \mid r,q,\} \cap \{g(S(\tau+\Delta),\tau+\Delta) \le 0 \mid r,\, q\}) \qquad (14)$$

F denotes the failure domain defined by $\{g(r,q,s\,(t),t) \le 0\}$. In order to compute outcrossing rate it is necessary that the limiting operation can be performed. This excludes certain processes which fluctuate too rapidly in time, that is: in a small time interval there is at most one crossing. The probability of more than one crossing is negligibly small. Then, the point process of crossings is a regular point process[19,25]. As a consequence the mean number of crossings in the time interval $[t_1,t_2]$ conditional on q and r can be determined from

$$E\left[N^+(t_1,t_2)|q,r\right] = \int_{t_1}^{t_2}\nu^+(F,\tau|q,r)\,d\tau \qquad (15)$$

If crossings are rare and asymptotically independent, an asymptotic result for the failure probability (or the first passage time)[25,26] is:

$$P_f(t_1,t_2) \approx 1 - E_R\left[\exp\left[-E_Q\,E[N^+(t_1,t_2)|q,r]\right]\right] \qquad (16)$$

An upper bound to the time dependent probability of failure in $[t_1,t_2]$ $(t_2 > t_1)$ can be derived as follows. Failure occurs if there is failure at t_1 or if there is at least one outcrossing from the safe set in $[t_{1+},t_2]$.

$$P_f(t_1,t_2) = P(g(R,Q,S,t_1) \le 0) + P(N^+(t_{1+},t_2) > 0) - P\left(\{g(R,Q,S,t_1) \le 0\} \cap \{N^+(t_{1+},t_2) > 0\}\right) \qquad (17)$$

where $N^+(t_{1+},t_2)$ is the random number of crossings from the safe into the unsafe domain. The last term is smaller than any of the two first terms. Neglecting it provides an upper bound $P_f(t_1,t_2) \le P(g(R,Q,S,t_1) \le 0) + P(N^+(t_{1+},t_2) > 0)$ which is at most twice as large as the exact result. Further, there is $P(N^+(t_{1+},t_2) > 0) \le E_{R,Q}\left[E[N^+(t_{1+},t_2)]\right]$ with $E_{RQ}\left[E[N^+(t_{1+},t_2)]\right]$ the mean number of outcrossings[26,27] and, hence

$$P_f(t_1,t_2) \le P(g(R,Q,S,t_1) \le 0) + E_{RQ}\left[E[N^+(t_1,t_2)\mid r,q]\right] \qquad (18)$$

$P_f(t_1) = P_f(g(R,Q,S,t_1) \le 0)$ is generally small. It will be computed as in time–invariant reliability. If it is guaranteed that the processes start from the origin, then it is $P_f(t_1) = 0$. This upper bound cannot be improved easily[28] . A simple lower bound[29] is also available indicating that it equals the largest instantaneous failure probability.

$$P_f(t_1,t_2) \ge \max_\tau\{P_f(\tau)\} \text{ for } t_1 \le \tau \le t_2 \qquad (19)$$

In the stationary case there is $P_f(t_1,t_2) \ge P_f(t_1)$. It is possible to improve this bound by taking the probability of a union over a discrete set of failure events. If the union includes the event with maximum point–in–time failure probability, then, the following relationship holds:

$$P_f(t_1,t_2) \ge P\left(\bigcup_{i=1}^{n}\{g(R,Q,S(\tau_i),\tau_i) \le 0\}\right) \ge \max_\tau\{P_f(\tau)\} \ge P_f(t_1) \qquad (20)$$

Unfortunately, simple schemes to choose the inner time instants $t_1 \le \tau_1 < \tau_2 \ldots < t^* < \ldots < \tau_n \le t_2$ with $P(g(R,Q,S(t^*),t^*) \le 0)$

the largest local failure probability are only available[30,31], for stationary processes. The lower bound is found to be better in non–stationary cases as compared to the stationary case.

Equations (18) and (19) are the basis for the considerations to come. The quality of the upper and lower bound solutions depends on the reliability level and on the dependence structure of the outcrossings. It is assumed that for all computations a probability distribution transformation into standard space as in time–invariant reliability is performed. It is further assumed that random processes start with random initial conditions. Solutions will be sought such that second order reliability methods can be employed as far as possible. Since it has been found that asymptotic solutions are too inaccurate under non–asymptotic conditions and pure first–order solutions are also not satisfying, a compromise between asymptotic solutions and somewhat more exact solutions will generally be sought. This paticularly concerns the time integration required in Eq.(16). In all cases it is assured that the solutions sought, converge to the asymptotic solutions under appropriate conditions.

Mean Number of Outcrossings and Failure Probabilities for Rectangular Wave Renewal Vector Processes

Breitung and Rackwitz[5] have shown that under stationary conditions the outcrossing rate can be calculated as the product of the jump rate and the probability that a component of the rectangular wave jumps from the safe domain into the failure domain minus the probability that the jump starts in the failure domain and ends in the failure domain, summed up over all components of the vector of rectangular wave renewal process. Ignoring for the moment all R–and S–variables the mean outcrossing rate is:

$$\nu^+(F,\tau) = \sum_{i=1}^{n_s} \lambda_i \left[P\left(\left\{ S_i^- \in \overline{F} \right\} \cap \left\{ S_i^+ \in F \right\} \right) \right]$$
$$= \sum_{i=1}^{n_s} \lambda_i \left[P(S_i^+ \in F) - P\left(\left\{ S_i^+ \in F \right\} \cap \left\{ S_i^- \in F \right\} \right) \right] \quad (21)$$

where $\hat{S}_i$ is the vector of jumping components just before a jump of the i–th component and S_i^+ after a jump. It is assumed that at a jump the component S_i changes its position from a random value to a new random value. Alternatively, one could assume that the component S_i returns to zero (or mean or any other predefined fixed value) before jumping to a new value. In this case the formula has to be modififed appropriately because outcrossing can also occur when the component returns to that predefined value. For linear failure surfaces $\partial F = \alpha_s^T s + \beta = 0$ the outcrossing rate can be given explicitly as:

$$\nu^+(F,\tau) = \sum_{i=1}^{n_s} \lambda_i \left[\Phi(-\beta) - \Phi_2(-\beta, -\beta; \rho_i) \right] \quad (22)$$

with $\Phi_2(.,.,.)$ the two–dimensional normal integral which can be easily evaluated. The correlation coefficient of the two state variables before and after a jump equals $\rho_i = 1 - \alpha_{si}^2$. It is seen that the error in neglecting the probability of having jumps from one failure domain into other can be studied in terms of the general parameters β and ρ_i. Breitung[22] proved that in the stationary case the asymptotic outcrossing rate can be approximated by:

$$\nu^+(F,\tau) \approx \sum_{i=1}^{n_s} \lambda_i \Phi(-\beta) \prod_{j=1}^{n_s-1} (1-\beta\kappa_j)^{-1/2} \quad (23)$$

Hence, the probability that jumps from one failure domain into other can be neglected asymptotically and the failure surface is approximated by a quadratic surface in the β–point. As before the κ_j's are the main curvatures in the β–point. For small β–values it is slightly conservative[33] and it can be shown that the following formula represents an improvement for small β–values.

$$\nu^+(F,\tau) \approx \sum_{i=1}^{n_s} \lambda_i \left[\Phi(-\beta) \prod_{j=1}^{n_s-1} (1-\beta\kappa_j)^{-1/2} - \Phi(-\beta, -\beta; \rho_i) \right]$$
$$= \Phi(-\beta) \prod_{j=1}^{n_s-1} (1-\beta\kappa_j)^{-1/2} - \sum_{i=1}^{n_s} \lambda_i \left[1 - \frac{\Phi(-\beta, -\beta; \rho_i)}{\Phi(-\beta) \prod_{j=1}^{n_s-1} (1-\beta\kappa_j)^{-1/2}} \right] \quad (24)$$

The last factor can be interpreted as a first–order correction to the jump rates. Figure 5 shows the conservatism by neglecting the probabilities of having jumps between the failure domain versus β for various ρ. It displays the ratio of the exact result and the asymptotic result. It is seen that this conservatism is quite considerable for large β and moderate ρ (large α–values).

For brevity of notation we shall write only the asymptotic result (Eqn.23) in the following but understand that the improvement by Eqn. 24 is included. The expected mean number of *stationary outcrossings* is $(t_2 > t_1)$:

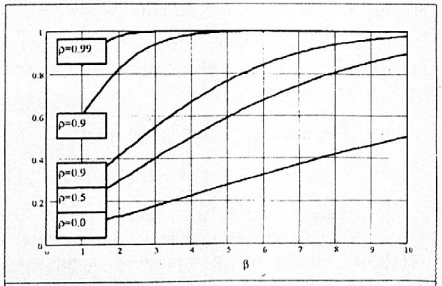

FIG.5 RATIO OF EXACT TO APPROXIMATE RESULT VERSUS SCALE PARAMETER β

$$E\left[N^+(t_1,t_2)\right] = \sum_{i=1}^{n_s} \lambda_i \Phi(-\beta) \prod_{j=1}^{n-1} (1-\beta\kappa_j)^{-1/2} (t_2 - t_1) \quad (25)$$

Note that β is now determined in the entire R–Q–S–space and similarly the curvature correction term.

The *non–stationary case* requires some additional considerations[9]. Non–stationarity can occur if the jump rates are time–dependent, the distribution parameters of the R–and S–variables depend deterministically on time and/or the failure surface depends deterministically on time.

◊ If non–stationarity is only in the jump rates $\lambda_i(\tau)$, the mean value of outcrossings is determined from:

$$E\left[N^+(t_1,t_2)\right] = \int_{t_1}^{t_2} \sum_{i=1}^{n_s} \lambda_i(\tau)\, d\tau\, \Phi(-\beta) \prod_{j=1}^{n-1} (1-\beta\kappa_j)^{-1/2}$$
$$= \bar{\lambda} \Phi(-\beta) \prod_{j=1}^{n-1} (1-\beta\kappa_j)^{-1/2} (t_2 - t_1) \quad (26)$$

where the integral can be evaluated numerically. $\bar{\lambda} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \sum_{i=1}^{n_s} \lambda_i(\tau)\, d\tau$ is a mean jump rate.

For more general forms of non–stationarity the mean number of outcrossings must be determined from:

$$E\left[N^+(t_1,t_2)\right] = \int_{t_1}^{t_2} \nu^+(F,\tau)\, d\tau \quad (27)$$

where

$$\nu^+(F,\tau) \approx \sum_{i=1}^{n_s} \lambda_i(\tau)\, \Phi(-\beta(\tau)) \prod_{j=1}^{n-1} \left(1-\beta(\tau)\kappa_j(\tau)\right)^{-1/2} \quad (28)$$

We now introduce the following additional approximations:

- There is a critical point t^* in time where the outcrossing rate becomes maximum, or, more precisely, where the term $\Phi(-\beta(\tau))$ becomes maximum because it dominates the time variations of the outcrossing rate.
- The variations in time of the jump rates are smooth. Therefore, it is sufficient to take the jump rates at the critical point, i.e. $\lambda_i(t^*)$.
- The second order correction is essentially independent of time and can be taken as $\prod_{j=1}^{n-1} \left(1-\beta(t^*)\kappa_j(t^*)\right)^{-1/2}$
- The integration of the outcrossing rate with respect to time can be approximated by applying the asymptotic Laplace integration method and can be performed independent of probability integration with respect to the other variables. Therefore, any interaction between time and the other variables is neglected. Two cases need to be considered, one where the critical point is an interior point and the other where the critical point is a bounary point.
- The integration with respect to the Q–variables can be performed simultaneously with the integration for the S–variables26. Also, it is admissible to integrate over the R–variables together with the other variables.

The expected number of outcrossing then must be determined[9] from:

$$E_{R,Q}\left[E\left[N^+(t_1,t_2) \mid r,q\right]\right] = \int_{t_1}^{t_2} \sum_{i=1}^{n_s} \lambda_i(\tau)$$
$$\Phi(-\beta(\tau)) \prod_{j=1}^{n-1} \left(1-\beta(\tau)\kappa_j(\tau)\right)^{-1/2} d\tau$$
$$\approx \sum_{i=1}^{n_s} \lambda_i(t^*) \prod_{j=1}^{n-1} \left(1-\beta(t^*)\kappa_j(t^*)\right)^{-1/2} \int_{t_1}^{t_2} \exp\left[\ln \Phi(-\beta(\tau))\right] d\tau \quad (29)$$

where the remaining integral over time is approximated by Laplace's integration method, i.e. the function $f(\tau) = \ln\Phi(-\beta(\tau))$ is expanded to first or second order depending on the location of the critical point. Here, two cases are distinguished:

◊ The critical point is an *interior point* of the considered time interval:

$$t_1 < t^* < t_2 \text{ or } \frac{\partial\beta(t^*)}{\partial\tau} = 0 \text{ and } \frac{\partial^2\beta(t^*)}{\partial\tau^2} > 0$$

$$E\left[N^+(t_1,t_2)\right] \approx \sum_{i=1}^{n_s} \lambda_i(t^*)\, \Phi(-\beta(t^*)) \prod_{j=1}^{n-1} \left(1-\beta(t^*)\kappa_j(t^*)\right)^{-1/2}$$
$$\left(\frac{2\pi}{f''(t^*)}\right)^{1/2} \left(\Phi\left(\sqrt{f''(t^*)}\,(t_2-t^*)\right) - \Phi\left(\sqrt{f''(t^*)}\,(t_1-t^*)\right) \right) \quad (30a)$$

◊ The critical point is the right boundary point of the considered time interval:

$$t^* = t_2 \text{ or } \frac{\partial\beta(t^*)}{\partial\tau} < 0:$$

$$E\left[N^+(t_1,t_2)\right] \approx \sum_{i=1}^{n_s} \lambda_i(t^*)\, \Phi(-\beta(t^*)) \prod_{j=1}^{n-1} \left(1-\beta(t^*)\kappa_j(\tau^*)\right)^{-1/2}$$
$$\frac{1-\exp\left(\left|f'(t_2)\right|(t_1-t_2)\right)}{\left|f'(t_2)\right|} \quad (30b)$$

with $f'(\tau)$ and $f''(\tau)$ the first and second derivative of $f(\tau)$ with respect to time:

$$f'(\tau) \approx -\beta(\tau) \frac{\partial\beta(\tau)}{\partial\tau}$$

$$f''(\tau) \approx \beta(\tau) \frac{\partial^2\beta(\tau)}{\partial\tau^2}$$

where use is made of the well known asymptotic relation-

ship $x \approx \varphi(x)/\Phi(-x)$. If at a boundary point $\partial\beta(\tau)/\partial\tau = 0$ and $\partial^2\beta(\tau)/\partial\tau^2 > 0$, the expected number of outcrossings is one half of the interior point solution.

It should be observed that these approximations are all of asymptotic nature [34]. These equations are significantly better than the true asymptotic relationships where the time correction factor would be $|1/f'(t^*)|$ for boundary points and $(2\pi/f''(t^*))^{1/2}$ for interior points. The solutions break down numerically if $f'(\tau)$ and $f''(\tau)$ are numerically too small. In those cases the stationary results should be used.

The time integration schemes used can be shown to be accurate enough for all practical purposes. For example, for a problem in three dimensions with two time variant jump loading and a limit state surface of the form $g(u) = b\,[u_R\,\sigma_R + (m_R + b_\tau\,\tau + b_{\tau\tau}\,(\tau - t^*)^2) - (u_{s1}\,\sigma_{s1} + m_{s1} + u_{s2}\,\sigma_{s2} + m_{s2})] = 0$ where we modify the reliability level by multiplying the limit state function by a factor $b > 1$ and obtain the interior point solution with t^* being in the mid point of the time interval [0, 1]. Displayed is the ratio of the proposed approximate solution and the exact result obtained by numerical integration versus the parameters b for various values of $b_{\tau\tau}$ and $b_\tau = 0$.

A little less accurate are the results for the boundary point solution (right boundary point) with limit state function as before, now with $b_{\tau\tau} = 0$. It should be noted that our approximations are always conservative.

Outcrossing Rates and Failure Probabilities for Scalar Gaussian Processes

The determination of outcrossing rates for Gaussian processes is well known by Rice's formula[25,35] and extensions to vector processes exist. The stationary case is especially simple. The vector case is complicated. Asymptotic concepts[36] help in both cases. The numerical effort can be considerable. Therefore, the simple scalar case is investigated first and then the vector case is reduced to the scalar case as explained below. The expected number of crossings of the threshold function conditional on R canbe written as:

$$E_{R,Q}\Big[E\,[N(t_1,t_2)]\mid r,q\Big] = \int_{R^R}\int_{R^Q}\int_{\dot s > b_\tau(\tau,q,r)}\int_{t_1}^{t_2} (\dot s - b_\tau(\tau,q,r))\,\varphi_{\dot S}\,(\dot s \mid s)\,\varphi_S(s)\,\varphi_Q(q)\,d\tau\,d\dot s\,dq\,dr \qquad (31)$$

if the process $S(\tau)$ is suitably standardized as well as the threshold function

$$S(\tau) = (X(\tau) - m(\tau))/\sigma(\tau);\quad b(\tau,q,r) = (a(\tau,q,r) - m(\tau))/\sigma(\tau) \qquad (32)$$

so that $$g(\tau,s,q,r) = b(\tau,q,r) - S(\tau) \qquad (33)$$

$b_\tau(\tau,q,r)$ is the time derivative of the threshold function.

The usual probability distribution transformation is performed for all Q–variables (and also for R–variables). These integrals are solved by Laplace's method after the time dependence of the threshold function has been expanded into first order (if the critical point is a boundary point) or to second order (if the critical point is an interior point). The critical point is the usual β–point which has to be located by an appropriate search algorithm. The conditional mean number of outcrossings[25] for the differentiable Gaussian process is:

$$E\Big[N^+(t_1,t_2)\mid r,q\Big] = \int_{t_1}^{t^2} \omega_0\,(\tau\mid r,q)\;\varphi\,(b(\tau,\mid r,q))\;\psi\left(\frac{b_\tau(\tau\mid r,q)}{\omega_0(\tau\mid r,q)}\right) d\tau \qquad (34)$$

where $\psi(x) = \varphi(x) - x\Phi(-x)$ from which the subsequent approximations for scalar Gaussian processes are derived.

We now introduce the following approximations:

- There is a critical point t^* in time where the outcrossing rate becomes maximum.
- The variations in time of the cycle rate $\omega_0(\tau)$ are smooth. Therefore, it is sufficient to take the cycling rate at the critical point, i.e. $\omega_0(t^*)$.
- The second order correction is essentially independent of time.
- The integration of the outcrossing rate with respect to time can be approximated by applying the asymptotic Laplace integration method and can be performed independently of probability integration with respect to the other variables. Therefore, no interaction between time and the other variables is considered.
- The integration with respect to the Q–variables can be performed simultaneously with the integration for the S–variables. It is admissible to integrate over the R–variables together with the other variables.

These assumptions are similar to those for rectangular wave renewal processes. The *stationary case* is relatively easy:

$$E_{R,Q}\Big[E\,[N^+(t_1,t_2)]\Big] = \frac{\varphi(\beta)}{\sqrt{2\pi}}\left(\frac{1}{|\det(H)|}\right)^{1/2}\omega_0\,(t_2 - t_1) \qquad (35)$$

with $\omega_0^2 = \dfrac{\partial^2}{\partial\tau_1\,\partial\tau_2}\rho_S(\tau_1,\tau_2)|_{\tau=\tau_1=\tau_2}$ the variance of the derivative of the standardized process $S(\tau)$ and H collecting the second derivatives of the failure surface with respect to the Q–and R–variables. If the upper bound solution is used the R–variables are also involved in H and treated like Q–variables. If no R–Q–variables are present the last factor is unity. The stationary upcrossing rate for a scalar process without presence of Q– or R–variables is $\nu^+(b) = \varphi(b)\,\omega_0/\sqrt{2\pi}$.

For the *non–stationary case* three possibilities have to be considered:

◊ Non–stationarity arises purely from auto–correlation non–stationarity. Mean value, variance and threshold functions of $S(t)$ do not depend on time. Then,

$$E_Q\Big[E\,[N^+(t_1,t_2)]\Big] = \frac{\varphi(\beta)}{\sqrt{2\pi}}\left(\frac{1}{|\det(H)|}\right)^{1/2}\int_{t_1}^{t_2}\omega_0\,(\tau)\,d\tau = \frac{\varphi(\beta)}{\sqrt{2\pi}}\left(\frac{1}{|\det(H)|}\right)^{1/2}\bar\omega_0\,(t_2 - t_1) \qquad (36)$$

where the integral over the time-dependent zero frequency is evaluated numerically and $\bar\omega_0 = \dfrac{1}{t_2 - t_1}\displaystyle\int_{t_1}^{t_2}\omega_0\,(\tau)\,d\tau$ is a mean zero frequency.

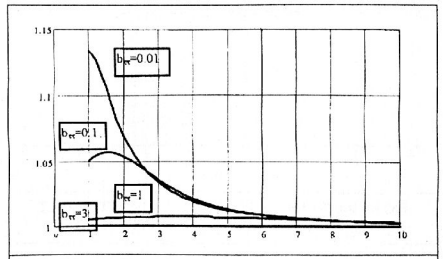

FIG.6 RATIO OF APPROXIMATE TO EXACT RESULT VERSUS SCALE PARAMETER b (INTERIOR POINT)

◊ The critical point is an *interior point.* In this case the second time derivative of the failure surface must be positive while the first time derivative is zero. Time integration is performed independently of the other probability integrations.[37,38] The corrected and extended version of Hagen's result is:

$$E_Q\Big[E\,[N^+(t_1,t_2)]\Big] = \frac{\varphi(\beta)}{\sqrt{2\pi}}\left(\frac{1}{|\det(H)|}\right)^{1/2}\omega_0 \left\{\sqrt{2\pi}\;\frac{1+\gamma^2}{b\,b_{\tau\tau}}\left[\begin{array}{c}\left\{\Phi(\tau_2\;\sqrt{1+\gamma^2}) - \Phi(\tau_1\;\sqrt{1+\gamma^2})\right\} - \dfrac{\sqrt{2\pi}}{\omega_0}\dfrac{1}{\sqrt{1+\gamma^2}}\dfrac{b_{\tau\tau}}{b} \\ \left\{\varphi(\tau_1)\,\Phi(-\tau_1\;\sqrt{1+\gamma^2}) - \varphi(\tau_2)\,\Phi(-\tau_2\;\sqrt{1+\gamma^2})\right\}\end{array}\right]\right\} \qquad (37a)$$

where $b = b(t^*)$ the normalized threshold function and $b_{\tau\tau} = b_{\tau\tau}(t^*)$ the second derivative with respect to time in the critical point, and

$$\gamma^2 = \gamma^2\,(t^*,r^*,q^*) = \frac{b_{\tau\tau}\,(t^*,r^*,q^*)}{b(t^*,r^*,q^*)}\;\frac{1}{\omega_0^2(t^*,r^*,q^*)}$$

$$\tau_{1,2} = \sqrt{b(t^*,r^*,q^*)\,b_{\tau\tau}\,(t^*,r^*,q^*)}\;\;(t_{1,2} - t^*)$$

H collects the second derivatives with respect to Q and R.

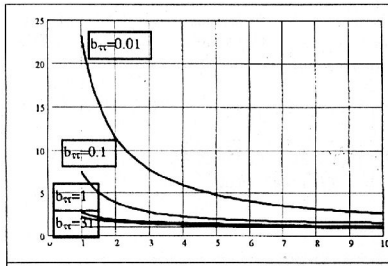

FIG.7 RATIO OF ASYMPTOTIC TO EXACT RESULT VERSUS SCALE PARAMETER b (INERIOR POINT)

All quantities are evaluated in the critical point τ^*. Under asymptotic conditions (large $b(t^*)$) the last factor in braces tends to $(2\pi/(b(t^*)\,b_{\tau\tau}\,(t^*)))^{1/2}$

◊ The critical point is a *boundary point.* In this case the first time derivative must be known. Time integration is performed independently of other probability integrations. The formula is

$$E_Q\Big[E\,[N^+(t_1,t_2)]\Big] = \frac{\varphi(\beta)}{\sqrt{2\pi}}\left(\frac{1}{|\det(K)|}\right)^{1/2}(\omega_0\sqrt{2\pi}\;\psi(a))\left[\frac{1 - \exp(-b(t^*)\mid b_\tau(t^*)\mid(t_2 - t_1))}{b(t^*)\mid b_\tau(t^*)\mid}\right] \qquad (37b)$$

where $b(t^*)$ the normalized threshold function in t^* and $b_\tau(t^*)$ the derivative with respect to time in the critical point. K contains the second order derivatives of the Q– and R–variables. The other quantities lik ω_0 and K are also evaluated in the critical point t^* and there is $a = b_\tau(t^*)\,/\,\omega_0$ and $\psi(x) = \varphi(x) - x\Phi(-x)$. The time integration is performed separately. Under asymptotic conditions the last factor tends to $1/(b(t^*)\mid b_\tau(t^*)\mid)$.

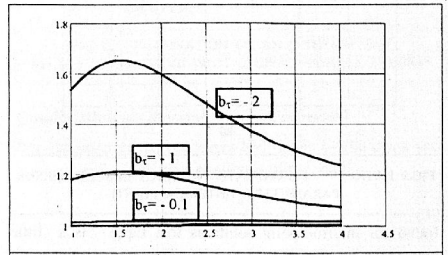

FIG.8 RATIO OF APPROXIMATE TO EXACT RESULT VERSUS SCALE PARAMETER b (BOUNDARY POINT)

If at a boundary point $\partial b(\tau)/\partial\tau = 0$ and $\partial^2 b(\tau)/\partial\tau^2 > 0$, the expected number of outcrossings is one half of the interior point solution.

Theoretically, the asymptotic results[8] which are given below are recovered for large threshold levels. If the critical point is an *interior point.*

$$E\left[N^{+}(t_1, t_2 \mid r, q)\right] \approx \omega_0(t^*, r, q)\, \Phi(-b(t^*, r, q))$$

$$\sqrt{\frac{1}{\omega_0^2(t^*, r, q)} + \frac{b(t^*, r, q)}{b_{\tau\tau}(t^*, r, q)}}$$

resulting in

$$P_f(t_1, t_2 \mid r, q) \le \Phi(-b(0, r, q)) + \omega_0(t^*, r, q)\, \Phi(-b(t^*, r, q))$$

$$\sqrt{\frac{1}{\omega_0^2(t^*, r, q)} + \frac{b(t^*, r, q)}{b_{\tau\tau}(t^*, r, q)}} \qquad (38a)$$

and if it is a *boundary point*

$$P_f(t_1, t_2 \mid r, q) \le \Phi(-b(t^*, r, q)) \qquad (38b)$$

The threshold function[36] which is used to test the accuracy of the approximate formulae is $b(\tau) = b_0 + a(t^* - t)^k$. For large $b(t^*, r, q)$, a and $k > 1$ both the approximate as well as the asymptotic formulae are acceptable. It is generally found for interior point solutions that Eq.(37) is a little better than Eq.(38) (Figs. 9 and 10). The interior point result is also acceptable for small $b(t^*, r, q)$ and a and is slightly conservative for larger $b(t^*, r, q)$ and a as can be seen from Figs. 11 and 12. In these equations $\varphi(b(t^*, r, q))$ should only be replaced by $\Phi(-b(t^*, r, q))$ $b(t^*, r, q)$ as is required for the asymptotic formula when asymptotic conditions are really approached. If, on the other hand, $\Phi(-b(t^*, r, q))$ $b(t^*, r, q)$ is used throughout the results are slightly unconservative for small $b(t^*, r, q)$ and a (Fig.13 where those two options are implemented). The asymptotic formulae by Breitung appear to reflect the effect of varying $\omega_0(t^*, r, q)$ only insufficiently, especially in the boundary point case (Fig.14).

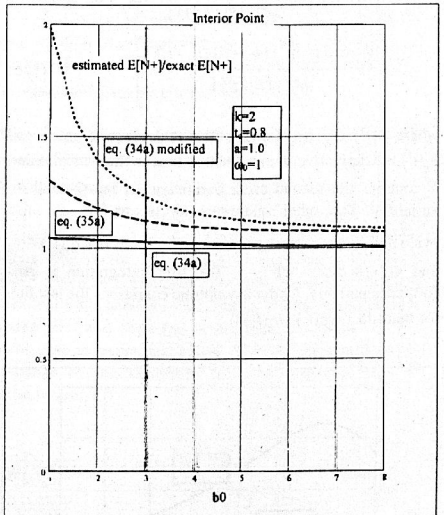

FIG.9 RATIO OF APPROXIMATE TO EXACT RESULT VERSUS PARAMETER b_0 (INTERIOR POINT)

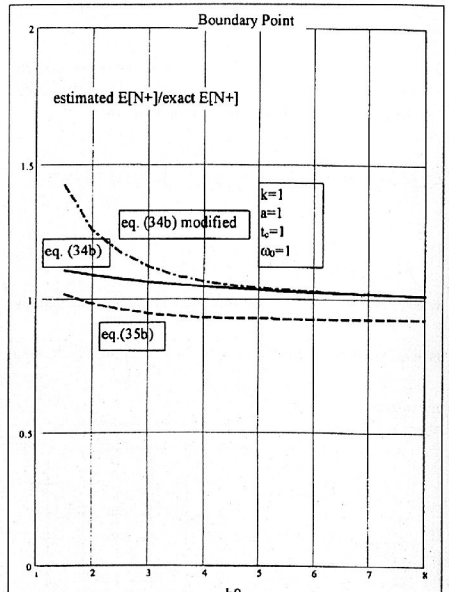

FIG.10 RATIO OF APPROXIMATE TO EXACT RESULT VERSUS PARAMETER b_0 (BOUNDARY POINT)

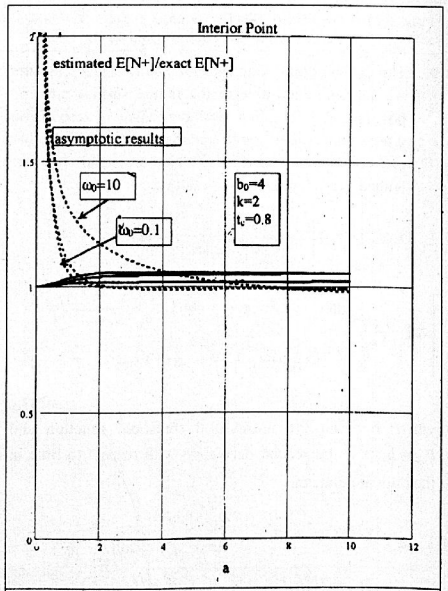

FIG.11 ESTIMATED $E[N^+]$ / EXACT $E[N^+]$ VERSUS CURVATURE PARAMETER FOR DIFFERENT ω_0 (INTERIOR POINT)

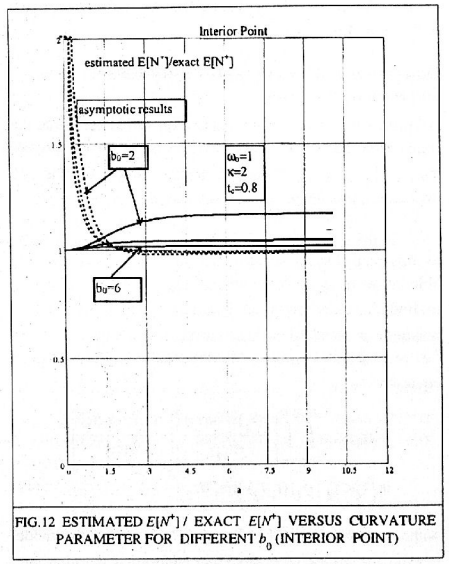

FIG.12 ESTIMATED $E[N^+]$ / EXACT $E[N^+]$ VERSUS CURVATURE PARAMETER FOR DIFFERENT b_0 (INTERIOR POINT)

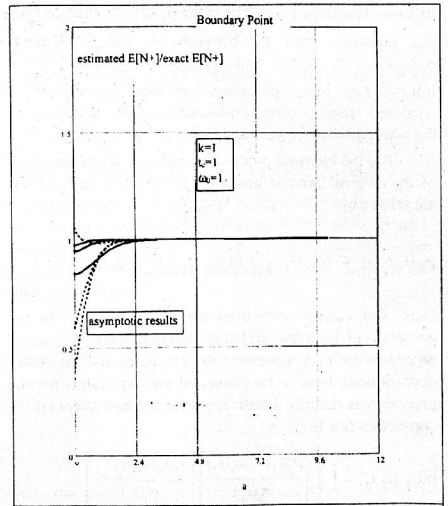

FIG.13 ESTIMATED $E[N^+]$ / EXACT $E[N^+]$ VERSUS CURVATURE PARAMETER a FOR DIFFERENT b_0 (BOUNDARY POINT)

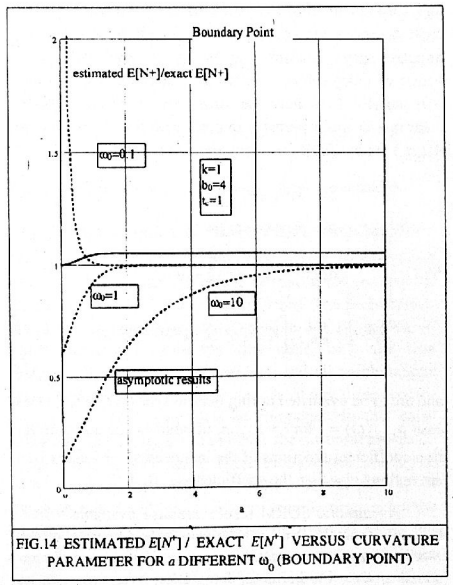

FIG.14 ESTIMATED $E[N^+]$ / EXACT $E[N^+]$ VERSUS CURVATURE PARAMETER FOR a DIFFERENT ω_0 (BOUNDARY POINT)

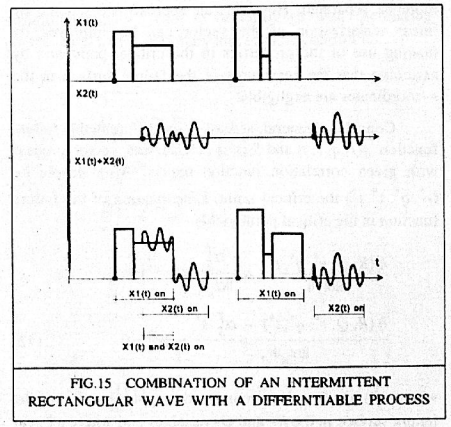

FIG.15 COMBINATION OF AN INTERMITTENT RECTANGULAR WAVE WITH A DIFFERNTIABLE PROCESS

Combination of Gaussian Vector Processes

For Gaussian vector processes $S(t)$ given by the mean vector $m_S(t)$ and symmetric, positive definite matrix

$$C_S(t_1, t_2) = \left\{\sigma_{ij}(t_1, t_2);\ i, j = 1, \ldots n\right\} \qquad (39)$$

of covariance functions the crossing rates are computed according to the generalization[39] of Rice's formula. The corresponding FORM results[4] are nicely compact especially after standardization by:

$$V_i(t) = (S_i(t) - m_i(t))/\sigma_{ii}(t)\ (i = 1, \ldots, n) \qquad (40)$$

and subsequent diagonalization of the correlation coefficient matrix of $V(t)$ by $V(t) = A(t)\,U(t)$ where $A(t)$ is a triangular matrix, possibly depending on time, and $U(t)$ a vector of independent standard normal processes such that $V(t)$ and $A(t)\,U(t)$ have the same cross correlation coefficient matrix for any time t. In particular, the coefficients in $A(t_1\ t_2)$ are by Choleski's decomposition scheme

$$A(t_1,t_2)=\begin{cases} a_{11}(t_1,t_2)=\rho_{U,11}(t_1,t_2) & \\ a_{i1}(t_1,t_2)=\rho_{U,i1}(t_1,t_2) & 2\le i\le n \\ a_{ii}(t_1,t_2)=\left(\rho_{U,ii}(t_1,t_2)-\sum_{k=1}^{i-1}a_{ik}^2(t_1,t_2)\right)^{1/2} & 2\le i\le n \\ a_{ij}(t_1,t_2)=\dfrac{1}{a_{jj}(t_1,t_2)}\left(\rho_{U,ii}(t_1,t_2)-\sum_{k=1}^{j-1}a_{ik}(t_1,t_2)\,a_{jk}(t_1,t_2)\right) & 1<j<i\le n \end{cases} \tag{41}$$

and are to be evaluated during iteration for $t = t_1 = t_2$. In this case $\rho_{U,ii}(t,t) = 1$ for $i = 1,\ldots,n$, of course. The autocorrelation coefficient functions of the independent processes $U(t)$ are nothing else than the coefficients $a_{ii}(t_1,t_2)$.

Asymptotic SORM results are also available[8, 36, 40] but, unfortunately, involve relatively large numerical effort, especially in the non–stationary case and in higher dimensional spaces. Furthermore, integration over time can no more be separated. However, an approximate scheme by linear scalarization of the vector can be employed by making use of the properties in the critical point and by assuming that the curvatures of the failure surface in the s–coordinates are negligible.

Consider a general, at least twice differentiable failure function $g(r, q, s, t)$ and $S(t)$ is a Gaussian vector process with given correlation function matrix. Also, denote by (r^*, q^*, s^*, t^*) the critical point. Linearization of the failure function in the critical point yields

$$\frac{h(R, Q, r^*, q^*, t^*)}{\|\alpha_s\|}+\frac{\alpha_S^T}{\|\alpha_s\|}(S-s^*)= \frac{h(R, Q. r^*, q^*, t^*)-\alpha_S^T\,s^*}{\|\alpha_s\|}+W^L=0 \tag{42}$$

where $h(.)$ is a first– or second–order approximation of the failure surface in the R– and Q–variables and where a linear approximation is used for the S–variables. α_s is the gradient vector of the s–variables normalized with respect to all variables (r,q,s), i.e. by $\partial g(r, q, s, t^*)/\partial s_i \,/\|\nabla_{r,q,s}\, g(r, q, s, t^*)\|$. Then, W^L is a new zero mean and unit variance Gaussian process variable.

$$W^L=\frac{\alpha_S^T}{\|\alpha_S\|}S \tag{43}$$

If $\rho_S(t_1,t_2)$ is the matrix of correlation functions of the original vector process $S(t)$ the autocorrelation function $\rho_{W^L}(t_1,t_2)$ of the new scalar Gaussian process $W(t)$ is:

$$\rho_{W^L}(t_1,t_2)=\left(\frac{\alpha_S}{\|\alpha_S\|}\right)^T\rho_S(t_1,t_2)\left(\frac{\alpha_S}{\|\alpha_S\|}\right) \tag{44}$$

With this new autocorrelation function all results for the scalar Gaussian case are applicable. The new autocorrelation function $\rho_{W^L}(t_1,t_2)$ needs to be determined only in the critical point t^*. The central frequency of outcrossings is obtained by twice differentiating the resulting autocorrelation function with respect to time.

Outcrossing Rates and Failure Probabilities for Nataf and Hermite Processes

All results for normal processes remain valid except that the transformation $X_i(t) + h_i(U_i(t))$ needs to be performed. This involves a transformation of the amplitudes but also the computation of the equivalent autocorrelation function.

In the scalar case, the Nataf process $S(t)$ is defined by its marginal distribution function and its twice–differentiable autocorrelation function $\rho_S(t_1,t_2)$. It is then necessary to find the autocorrelation function $\rho_U(t_1,t_2)$ of the corresponding standard normal process. the autocorrelation function $\rho_U(t_1,t_2)$ must be obtained from the integral equation

$$\rho_S(t_1,t_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\frac{s(t_2)-m(t_2)}{\sigma(t_1)}\right]\left[\frac{s(t_2)-m(t_2)}{\sigma(t_2)}\right] \varphi\left(z_1<z_2:\rho_U(t_1,t_2)\right)dz_1\,dz_2 \tag{45}$$

with $z_1 = \Phi^{-1}[F(s,t_1)]$ and $z_2 = \Phi^{-1}[F(s,t_2)]$. This model is valid for strictly increasing and continuous marginal distribution functions $F_S(s)$. The value $\rho_U(t_1,t_2)$ obtained from this equation must lie between −1 and 1. There is $\rho_S(t_1,t_2) \le \rho_U(t_1,t_2)$ and $\rho_S(t_1,t_2) = \rho_U(t_1,t_2) = 0$. It follows that Nataf processes can only be used for a restricted range of autocorrelation functions depending on the marginal distributions and their parameters.

For the Hermite process the autocorrelation functions of the original process and the equivalent normal process are related by:

$$\rho_S(t_1,t_2)=\kappa^2\left(\rho_U(t_1,t_2)+\tilde{h}_3^2\,\rho_U(t_1,t_2)^2+\tilde{h}_4^2\,\rho_U(t_1,t_2)^3\right) \tag{46}$$

Here also certain restrictions about the moments of the process and its autocorrelation function hold. For vector processes each component of the processes and the cross–dependencies have to be converted into equivalent normal processes as described before, i.e. for the equivalent cross–correlation functions by:

$$\rho_{S_i,S_j}(t_1,t_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\frac{s_i(t_1)-m_i(t_2)}{\sigma_i(t_1)}\right]\left[\frac{s_j(t_2)-m_j(t_2)}{\sigma_j(t_2)}\right] \varphi\left(z_1, z_2:\rho_{U_i,U_j}(t_1,t_2)\right)dz_1\,dz_2 \tag{47}$$

for Nataf–processes and by

$$\rho_{S_i,S_j}(t_1,t_2)=\kappa_{i,j}^2\left(\rho_{U_i,U_j}(t_1,t_2)+\tilde{h}_{3,i}\,\tilde{h}_{3,j}\,\rho_{U_i,U_j}(t_1,t_2)^2+\tilde{h}_{4,i}\,\tilde{h}_{4,j}\,\rho_{U_i,U_j}(t_1,t_2)^3\right) \tag{48}$$

for Hermite processes. Outcrossing rates will be determined on the basis that the resulting scalar process $S(t)$ will have an outcrossing of level $b(t)$ when $U(t)$ has an outcrossing of level $g^{-1}(b(t))$.

Gaussian, Nataf and Hermite processes can be combined. It is only necessary to define the Gaussian process as a special case of the Nataf or the Hermite process.

Combination of Differentiable Processes and Rectangular Wave Renewal Processes

Differentiable and rectangular wave renewal vector processes can be combined easily if they are independent of each other. Due to regularity of the outcrossing process an outcrossing can occur if a renewal process jumps or in between the jumps if the differentiable process has an outcrossing[41,42]. When one of the processes has an outcrossing, the other processes have values at their random point–in–time distribution. Let $S_{R,i}(t)$ the i–th component of the rectangular wave renewal jump process and $S_D(t)$ the (scalarized) differentiable process. Then, the conditional mean number of outcrossings due to jumps of the rectangular wave renewal process is:

$$E\left[N_R^+(t_1,t_2)\mid r,q,s_D\right]\approx\int_{t_1}^{t_2}\sum_{i=1}^{m_j}\lambda_i(\tau)\,P(g(R,Q,S_R(\tau), S_D(\tau))\le 0)\,d\tau \tag{49}$$

$$\approx\int_{t_1}^{t_2}\sum_{i=1}^{m_j}\lambda_i(\tau)\,\Phi(-\beta(\tau))\prod_{j=1}^{n-1}\left(1-\beta(\tau)\,\kappa_j(\tau)\right)^{-1/2}d\tau$$

where $S_R(t)$ and $S_D(t)$ take on the values from their point–in–time distribution. The $S_D(t)$ –variables are treated like Q–variables. The time integral is approximately solved as described previously for rectangular wave renewal processes. The conditional mean number of outcrossings for the (scalarized) differentiable processes for the times in between the jumps is:

$$E\left[N_D^+(t_1,t_2)\mid r,q,s_R\right]=\int_{t_1}^{t_2}\omega_0(\tau\mid r,q,s_R)\,\varphi(b(\tau\mid r,q,s_R))\,\psi\left(\frac{b_\tau(\tau\mid r,q,s_R)}{\omega_0(\tau\mid r,q,s_R)}\right)d\tau \tag{50}$$

Here, the jump processes are treated as Q–variables. The time integral is approximately solved as described previously for scalar Gaussian processes. Making use of the regularity of the stream of outcrossings, the sum of these mean numbers of outcrossings is the exact result for the mean number of outcrossings due to both types of processes.

$$E\left[N^+(t_1,t_2)\mid r,q\right]\le E\left[N_R^+(t_1,t_2)\mid r,q,s_D\right]+E\left[N_D^+(t_1,t_2)\mid r,q,s_R\right] \tag{51}$$

The conditions can be removed as previously described.

COMBINATION OF INTERMITTENT PROCESSES

The combination of intermittent processes can be performed by the so–called load coincidence method. Only the stationary case of ''on/off''–states is considered. The probabilities of the exhaustive, mutually exclusive set of events of ''off''–and ''on''–states, i.e. that none, only the first process, only the second process, ...,. and all combinations of any two processes, all combinations of any three processes, etc. is ''on'', must first be computed. In this coincidence model the sequence in which the different processes occur makes no difference. These probabilities sum up to unity. They are also the probabilities of the set $\{k\}$ of processes being 'on'' at an arbitrary point in time. using the intermittency model of exponential interarrival times and exponential durations truncated at the next renewal, these probabilities are for $\rho\,I = \kappa_i/\mu_I$ being the inter arrival duration intensity[43] (not to be confused with correlation coefficients) under stationary and ergodic conditions.

$$p_0=\frac{1}{\prod_{m=1}^{n}(1+\rho_m)}$$

$$p_1^i=\frac{\rho_i}{\prod_{m=1}^{n}(1+\rho_m)}$$

$$p_2^{i,j}=\frac{\rho_i\,\rho_j}{\prod_{m=1}^{n}(1+\rho_m)}\qquad i>j$$

$$p_3^{i,j,k}=\frac{\rho_i\,\rho_j\,\rho_k}{\prod_{m=1}^{n}(1+\rho_m)}\qquad i>j>k$$

$$\vdots$$

$$p_n=\frac{\prod_{r=1}^{n}\rho_r}{\prod_{m=1}^{n}(1+\rho_m)} \tag{52}$$

Secondly, the failure probability for any combination of processes is determined. The asymptotic mean duration of the total ''on''-time of the selection $\{k\}$ of processes ''on'' is $p_{\{k\}}^{|k|}(t_2-t_1)\le t_2-t_1, t_2>t_1$. Outcrossings can only occur during this shorter time interval. The sum of the

individual time intervals $p^{|k|}_{|k|}\ (t_2 - t_1)$ covers the total interval $[t_1, t_2]$.

For *stationary processes* and *stationary intermittencies*, the final failure probability is determined from

$$P_f(t_1, t_2) \le P_f(t_1) + E[N^+(t_1, t_2)]$$

with

$$P_f(t_1) = \sum_{i=1}^{n} p_1^i P_{f,1}^i(t_1) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_2^{i,j} P_{f,2}^{i,j}(t_1) + \ldots + p_n^{1,2,\ldots,n} P_{f,n}^{1,2,\ldots,n}(t_1)$$

$$E\left[N^+(t_1,t_2)\right] = \sum_{i=1}^{n} \nu_i^+ \left\{p_1^i (t_2 - t_1)\right\} + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \nu_{i+j}^+ \left\{p_2^{ij}(t_2 - t_1)\right\} + \ldots + \nu_{1+2+\ldots+n}^+ \left\{p_n^{1,2,\ldots,n}(t_2 - t_1)\right\} \quad (53)$$

Here, $\nu^+_{|k|}$ denotes the outcrossing rate if the set $\{k\}$ of loads is "on".

The case of *non-stationary processes but stationary and ergodic* "on/off" times is more difficult. Great care is needed when modeling non-stationary behavior. The "on" probabilities remain as before, also at the critical time t*. Therefore, in the *interior point* case

$$E\left[N^+(t_1,t_2)\right] \le \sum_{i=1}^{n} E\left[N_i^+\left(\max\left\{t_1, t_i^* - \frac{p_1^i}{2}(t_2 - t_1)\right\}, \min\left\{t_i^* + \frac{p_1^i}{2}(t_2 - t_1), t_2\right\}\right)\right] + \sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left[N_{i+j}^+\left(\max\left\{t_1, t_{i,j}^* - \frac{p_2^{i,j}}{2}(t_2 - t_1)\right\}, \min\left\{t_{i,j}^* + \frac{p_2^{i,j}}{2}(t_2 - t_1), t_2\right\}\right)\right] + \ldots + E\left[N_{1+2+\ldots+n}^+\left(\max\left\{t_1, t_{1,2,\ldots,n}^* - \frac{p_n^{1,2,\ldots,n}}{2}(t_2 - t_1)\right\}, \min\left\{t_{1,2,\ldots,n}^* + \frac{p_n^{1,2,\ldots,n}}{2}(t_2 - t_1), t_2\right\}\right)\right] \quad (54)$$

Note that each combination may have its own critical time under general conditions. Further, it is assumed that the total length of the "on"-time of the set $\{k\}$ is symmetric around t^* but, of course, truncated at the two boundary times. For the boundary point case we have for the *right boundary point* case ($t^* = t_2$).

$$E\left[N^+(t_1,t_2)\right] \le \sum_{i=1}^{n} E\left[N_i^+(t^* - p_1^i(t_2 - t_1), t^*)\right] + \sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left[N_{i+j}^+ (t^* - p_2^{ij}(t_2 - t_1), t^*)\right] + \ldots + E\left[N_{1+2+\ldots+n}^+ (t^* - p_n^{1,2,\ldots,n}(t_2 - t_1), t^*)\right] \quad (55)$$

For the left *boundary point* case ($t^* = t_1$)

$$E\left[N^+(t_1,t_2)\right] \le \sum_{i=1}^{n} E\left[N_i^+(t^*, t^* + p_1^i(t_2 - t_1))\right] + \sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left[N_{i+j}^+ (t^*, t^* + p_2^{ij}(t_2 - t_1))\right] + \ldots + E\left[N_{1+2+\ldots+n}^+ (t^*, t^* + p_n^{1,2,\ldots,n}(t_2 - t_1))\right] \quad (56)$$

These results are upper bounds because the total "on" time for the selection $\{k\}$ of simultaneously acting processes is taken into account. A lower bound clearly would be obtained if the individual mean "on"-times of the selection $\{k\}$, i.e. the total "on"-time divided by the mean number of individual "on"-times, were taken. Asymptotically, however the influence of the length of the time interval vanishes. Therefore, one can conservatively compute only the upper bound. Under general conditions with respect to the type of non-stationarity it may happen that certain combinations produce interior point solutions while others simultaneously produce boundary point solutions.

The initial failure probability is computed as in the stationary case.

The lower bound is: $P_f(t_1, t_2) \ge$

$$\max_\tau \left\{\sum_{i=1}^{n} p_1^i P_{f,1}^i(\tau) + \sum_{i=1}^{n} \sum_{d_{ij}}^{n} p_2^{i,j} P_{f,2}^{i,j}(\tau) + \ldots + p_n P_{f,n}^{1,2,\ldots,n}(\tau)\right\} \ge \max_\tau \sum_{i=1}^{n} p_1^i P_{f,1}^i(t_i^*) + \sum_{i=1}^{n} \sum_{j=i}^{n} p_2^{i,j} P_{f,2}^{i,j}(t_{i,j}^*) + \ldots + p_n P_{f,n}^{1,2,\ldots,n}(t_{1,2,\ldots,n}^*) \quad (57)$$

A rigorous computation of the lower bound in the non-stationary case requires considerable numerical effort. Therefore, only a lower bound is estimated with $t^*_{|k|}$ the critical times when the sets $\{k\}$ of processes are "on". In the stationary case $P_f(t_1, t_2) \ge P_f(t_1)$.

For a larger number of loads this implies a very large number of β - point determinations (precisely, $2^n - 1$). For all ρ_i's small only the few first terms contribute significantly. For all ρ_i's only the last terms contribute significantly. However, it is difficult to derive a general rule because usually the coincidence probabilities decrease with the number of processes but the conditional failure probabilities (=expected number of outcrossings during "on"-times) increase.

Combination of intermittent differentiable and intermittent rectangular wave renewal processes

The same formulae are valid as for non-intermittent processes except that care is taken of the shorter time interval for the "on"times. (Fig.16).

Combination of intermittent processes with non-intermittent processes

This is a generalization of the foregoing options. Almost all non-intermittent processes can be produced by choosing the interarrival-duration intensity ρ large. More convenient and involving less computational effort is to define certain processes as non-intermittent, i.e. they are always "on". In this case the initial probability $P_f(t_1)$ involves an additional term $p_0\ P_f(t_1)$whereas all other terms have to be computed assuming that the non-intermittent processes are always "on". Similarly, the lower bound involves an additional term.

Concluding Remarks

The theory and computational methods outlined before are presented here for the first time in a consistent manner. The considerations are essentially based on asymptotic SORM concepts but modifications are introduced in order to obtain sufficiently accurate results also in non-asymptotic cases. This requires that a unique critical point can be found in the *r-q-s-t*-space[14]. If this is not the case other methods must be applied. As mentioned upper and lower bound are relatively close to each other for non-stationary cases. The asymptotic result (Eqn.16) is usually close to the upper bound. Both types of solutions should only be used if failure probabilities are less than 0.01, say, and local β-values are larger than about 2.

For the non-stationary case, methods are developed where any interaction of time and componential states is neglected. The overall effect of this and certain other assumptions is not easy to determine. However, for both rectangular wave renewal processes and for Gaussian vector processes importance sampling schemes can be devised which perform the necessary integrations quite efficiently and arbitrarily exact, but nevertheless with considerably more numerical effort. Suitable importance sampling schemes are given in the appendix. More details can be found in Rackwitz[38,44]. Example calculations show that the simplifications introduced above have negligible effect as long as the failure surfaces are sufficiently smooth and the time variations are not too large. Then, the interactions between time and states are in fact, small. In general, the systematic error produced by those simplifications remain well within the error bounds implied by importance sampling. The largest systematic error appears to be produced by the scalarization of differentiable processes when larger curvatures exist in the critical point. Further work is necessary. Here, the importance sampling alternative for vectorial processes is especially efficient.

Although it is believed that the load models and computational methods are already relatively rich, further practically important issues of interest are:

- Combination of non–rectangular pulse shapes
- More general intermittency models
- Occurrence clustering
- Trigger models
- Cross–and auto–dependencies of rectangular wave models.

Almost all of the mentioned problems have already found special solutions. For example, Madsen[6] studied special cases of sums of independent non–rectangular waves by the so–called point crossing approach yielding upper bounds under certain special conditions. But it appears very difficult to extend the results to the general combination framework outlined above. Clipped normal processes as a model for loads has been proposed[45] involving intermittencies determined directly by the amplitude model. Other models are possible for the "on/off" times, in particular, with Erlang distributed renewals and truncated Erlang durations for the individual processes, which differ slightly from the exponential one[43,46]

The general case is presently under study. It will then be possible to replace the simple model for coincidence probabilities in Eq.(52) by more general and more flexible models including the case with almost deterministic occurrences and deterministic pulse durations. A theoretically quite different model for intermittencies has been put forward by Madsen and Ditlevsen[30]. In this model independent, alternating "on" and "off" times are defined. Occurrence clustering models have been proposed by Wen[47] and Schrupp and Rackwitz[48]. They become computationally rather involved unless pulse durations are rather short as demonstrated[49]. The special case of trigger models, i.e. where the occurrence of a load triggers another load, possibly after some delay time, has also been studied[49]. Again, general trigger models are difficult to compute. Various amplitude dependencies have been investigated, for example by Madsen[6], Rackwitz[50] and Wen[49]. Here again no general and practical method is available as of yet. In fact, the methods proposed are either complicated, not very accurate or can require enormous numerical effort. While serial dependencies have to significant effect on the results, at least not for high reliability problems, cross-dependencies are known to influence the final results to a large extent. An indirect way to model such cross-dependencies, however, is by making use of the proposed hierarchical load model.

The methods described above are of limited use for dynamic systems. It is not only difficult to assess the auto- and cross-correlation structure of the interesting output quantities. Also the combination of rectangular waves as well as of intermittencies as described above make little sense if passed through a dynamic structure. The transient phases of the system generally lead to an oscillatory behavior of system states. Therefore, the time integration schemes with respect to time have to be modified.

REFERENCES

1. Tukstra, C., "Theory of Structural Design Decisions," University of Waterloo, *SM-Study, No.2*,170
2. Ferry Borges, J., Castanheta, M., "Structural Safety" Laboratorio Nacional de Engenharia Civil, Lisbon, 1971
3. Rackwitz, R.,Z Fiessler, B., "Structural Riliability under Combined Random Load Sequences", *Computers & Structures, 1978, 9, 484-494*
4. Veneziano, D., Grigoriu, M., Cornell, C.A., Vector-Process Models for System Reliability "Eng. Mech Div., ASCE, 103, EMS, 1977, pp441-460
5. Breitung K., Rackwitz., R., "Nonlinear Combination of Load processes, "*Journ. Struct Mech.*, 10, 2, 1982, pp.145-166
6. Madsen, H.O., Load Models and Load Combinations, *DTH, Res Rep. R-113*, Lyngby, 1979
7. Wen, Y.K., "A Clustering Model for Correlated Load Processes, "*Struct. Div., ASCE*, Vol. 107 ST5, 1981, pp. 965-983

8. Breitung K., "The Extreme Value Distribution of Non-stationary Vector Processes," *Proc.5-th ICOSSAR'89*, San Francisco, (ed. A.H.-S. Ang, M. Shinoxuka, G.I Schueller) *ASCE*, II, 1989, pp. 1327-1332.

9. Rackwitz, R., "On the Combination of Non-Stationary Rectangular Wave Renewal Processes" *Struct. Safety*, 13, 1+2, 1993, pp.21-28

10. Hasofer, A.M., and Lind, N.C., "An Exact and Invariant First-Order Reliability Format" *Eng. Mech., ASCE*, 1974, 100, EMI, 111-121

11. Parkinson, D.B., "First-Order Reliability Analysis Employing Tanslation Systems" *Eng. Mech., ASCE*, Vol. 111, No. 9, 1985, pp.115-1184

12. Hohenbichler, M. and Rackwitz., R., "Non-Normal Dependent Vectors in Structural Safety" *Eng. Mech., ASCE*, 1981, 107, 6, 1227-1249

13. Der Kiureghian, A., Liu, P.-L., "Structural Reliability under Incomplete Probability Information", *Eng. Mech. , ASCE*, 1986, Vol. 112, No.1,85-104

14. Winterstein, S.R., Bjerager, P., "The use of Higher Moments in Reliability Estimation". *Proc. ICASP 5, Int. Conf. on Appl. of Statistics and Prob. in Soil and Struct.*, Vol. 2, Vancouver 1987, pp. 1027-1036

15. Abdo, T., Rackwitz, R., "A new β–Point Algorithm for Large Time-Invariant and Time-Variant Reliability Problems", *3rd WG 7.5 IFIP Working Conf.*, Berkey, March 26-28, 1990, (eds. A. Der Kiureghian, P. Thoft Christensen), Springer, 1991.

16. Breitung, K., "Asymptotic Approximations for Multinormal Integrals," *J. Rng. Mech.*, ASCE, 110,3, 1984, 357-366

17. Tvedt, L., "Two Second-Order Approximations to the Failure Probability", *Det Norske Veritas*, RDIV/20-00-43, 1983

18. Hohenbichler, M., Gollwitzer, S., Kruse, W., and Rackwitz, R., "New Light on First- and Second-Order Reliability Methods" *Structural Safety*, 1987, 4, 267-284

19. Cox, D.R., Miller, H.D., "*The Theory of Stochastic Processes*", Chapman & Hall, London, 1965

20. Darling, D.A., Siegert, A.J.F., "*The First Passage Problem for a Continuous Markov Process*", 1953

21. Slepian, D., "First Passage Time for a Particular Gaussian Process", *Ann Math. Statist.*, 32, pp. 610-912, 1961

22. Lennox. W.C., Fraser, D.A., "On the First Passage Distribution for the Envelope of a Non-stationary Narrow Band Stochastic Process" *Appl. Mech. ASME*, 41,3,1974,pp. 793-797

23. Ariaratnam. S.T., Tam, D.S.F., "Random Vibration and Stability of a Linear Parametrically Excited Oscillator", Z. *Angew. Math. Mech.*, 59, 2, pp. 79-84, 1979

24. Grigoriu, M., "*Applied Non Gaussian Processes*", Prentice Hall, Englewood Cliffs, NY, 1995

25. Cramer, H., Leadbetter, M.R., "*Stationary and Related Processes*", Wiley &Sons, New-York, 1967

26. Scgall, G., Faber, M., Rackwitz, R., "The Ergodicity Assumption for Sea States in the Relibiligy Assessment of Offshore Structures", *Offshore Mech. and Arctic Eng., ASME*, 113, 3, 1991, pp.241-246

27. Bolotin, V.V., "*Wahrscheinlichkeitsmethoden zur Berchnung von Konstruktionen*", VEB Verlag Für Bauwesen Berlin, 1981

28. Engelund, S., Rackwitz, R., Lange, C., "Approximations of First Passage Times for Differentiable a-Processes Based on Higher Order Threshold Crossing" *Prob. Eng. Mech.*, 10, 1, 1995, pp. 53-60

29. Shinozuka, M., Probability of Failure under Random Loading", *Eng. Mech. Div., ASCE*, 90, EM5, 1964, pp. 147-170

30. Madsen, H.O., Ditlesen, O.,"Transient Load Modelling: Markov On off Rectangular Pulse Processes," *Struct. Safety*, Vol. 2, 1985, pp. 253-271

31. Madsen H.O. Krenk, S., Lind., N.C., "*Methods of Structural Safety*" Prentice-Hall, Englewood-Cliffs, 1986

32. Breitung, K., "Asymptotic Approximations for the Crossing Rates of Poisson Square Waves", *Proc. Extreme Value Theory and Applications*, NIST Spec. Publ. 866, 3, 1993, pp. 75-80

33. Rackwitz, R., "Importance sampling for Gaussian Vector Processes-A Review", *RCP-GmbH, Technical Note*, October, 1997

34. Ruben, H., "An Asymptotic Expansion for the Multivariate Normal Distribution and Mill's Ratio", *Res of the National Bureau of Standards*, Vol. 68B, 1, 1964

35. Rice, S.O., "Mathematical Analysis of Random Noise", *Bell System Tech. J.*, 32, 1944, pp. 282 and 25, 1945, pp. 46

36. Bryla, P., Faber, M., Rackwitz, R., "Seconf Order Methods in Time Variant Reliability", *Proc. OMAE 91, II* 1991, pp. 143-150

37. Hagen, O., "Threshould Up-Crossing by Second Order Methods", *Prob. Eng. Mech.*, 7 1992, pp. 235-241

38. Rackwitz., "Importance Sampling update for Rectangular Wave Renewal Processes, *RCP-GmbH, Technical Note*, October, 1997

39. Belyaev, Y.K., "On the Number of Exits across the Boundary of a Region by a Vector Stochastic Process", *Theor. Probab. Appl.*, 1968, 13, pp. 320-324

40. Breitung K., "Asymptotic, Approximations for the Outcrossing Rates of Stationary Vector Processes" *Stochastic Processes and their Applications*, 29, 1988, pp.195-207

41. Madsen, H.O., Kilcup, R., Cornell, A.C.," Mean Upcrossing Rates for Sums of Pulse-Type Stochastic Load Processes," *Prob. Mech. and Struct. Rel., Proc. ASCE Spec. Conf.*, Tuscon, Arizona, 1979

42. Ditlevser, O., Madsen H.O., "Probabilistic Modelling: Man-made Load Process and their Individual and combined Effects," *Proc.* 3rd *ICOSSAR* 1981. Structural Safety and Reliability (eds. T. Moan and M.Shinozuka), Elsevier, Amsterdam, 1981.

43. Shinozuka, M., "Stochastic Characterization of Loads and Load Combinations", *Proc. 3rd ICOSSAR* 1981, Structural Safety and Reliability (eds. T. Moan and M.Shionozula), Elsevier, Amsterdam, 1981, pp. 57-76

44. Rackwitz, R., "Importance sampling for Gaussian Vector Processes", *RPC-GmbH, Technical Note*, November 1997.

45. Ditlevser, O., Madsen H.O., "Transient Load Modelling: Clipped Normal Processes", *Eng. Mech., ASCE*, Vol. 109, EM2, 1983, pp. 94-515

46. Iwankiewixa R. Rackwitz, R., "Coincidence Probabilities for Intermittent Pulse Load Processes with Erlang Arrivals, Reliability and Optimization of Structural Systems", *Proc. 7th IFIP WG7.5 Working Conference*, April 2-4, 1996, Boulder, Co., 1996, pp. 189-196

47. Wen, Y.K., "A Clustering Model for Correlated Load Processes", *Struct. Div., ASCE*, Vol. 107, ST5, 1977, pp.1079-1093

48. Schrupp, K., Rackwitz, R., "Outcrossing Rates of Marked Poisson Cluster Processes in Structural Reliability", *Appl. Math. Modelling*, 12, 1988, Oct., 482-490

49. Wen, Y-K., "*Structural Load Modeling and Combination for Performance and Safety Evalution*", Elsevier, Book Pub. Co., Amsterdam, 1990

50. Rackwitz, R., "Reliability of Systems under Renewal Pulse Loading" *Eng. Mech., ASCE*, Vol. 111, No. 9, 1985, pp. 1175-1184

51. Fujita, M., Schall, G., Rackwiz, R., "Time-Variant Component Reliabilities by FORM/SORM and Updating by Importance Sampling", *Proc. ICASP 5*, Vancouver, May 1987, Vol. I, pp.520–527.

52. Ditlevsen, O., Hasofer, A.M., Bjerager, P., Olesen, R., "Directional Simulation in Gaussian Processes", *Probabilistic Engineering Mechanics*, Vol.3, No.4, 1988, pp. 207-217

53. Rackwitz, R., Discussion to Harbitz, A., "An Efficient Sampling Method for Probability of Failure Caluclation", *Struct. Safety*, 4,4, 1987, pp. 313-314

54. Hohenbichler, M., Gollwitzer., "Improvement of Second-order Reliability Estimates by Importance Sampling", *Eng. Mech., ASCE*, 114 12, 1988, pp. 2195-2199.

55. Fujita, M., Rackwitz, R., "Updating First–and Second–Order Reliability Estimates by Importance Sampling" *Structural Eng./Earthquake Eng.* Vol.5, No.1, pp.53–59, 1998, Japan Society of Civil Engineers (Proc. of JSCE No. 392/I–9).

APPENDIX A

Importance sampling for Gaussian Vector Processes

Several importance sampling schemes have been proposed[51,52]. It is possible to estimate the mean number of outcrossings directly and use the information that a critical point and thus an important region is given by (r^*, q^*, s^*, t^*). Following Belyaev[39] it is:

$$E\left[N^+(t_1,t_2)\right] = \int_{R^{n_r}}\int_{R^{n_q}}\int_{t_1}^{t_2}\int_{\partial G}\int_{\dot{s}\geq\partial\dot{G}} (\dot{s}-\partial\dot{G})\,\varphi\,(\dot{s}\mid S=s)\,\varphi_{n_s}(\dot{s})\,\varphi_{n_s}(s)\varphi_{n_q}(q)\varphi_{n_r}(r)d\,\dot{s}\,ds(s)\,d\tau\,dq\,dr$$

$$= \int_{R^{n_r}}\int_{R^{n_q}}\int_{t_1}^{t_2}\int_{\partial G}\int_{\dot{s}_N\geq\partial\dot{G}} (\dot{s}-\partial\dot{G})\,\varphi\,(\dot{s}_N\mid S=s)\,\varphi_{n_s}(s)\varphi_{n_q}(q)\varphi_{n_r}(r)d\dot{s}_N\,ds(s)\,d\tau\,dq\,dr$$

$$= \int_{R^{n_r}}\int_{R^{n_q}}\int_{t_1}^{t_2}\int_{\partial G} E\left[\max\left\{0,\dot{s}_N-\partial\dot{G}\right\}\mid S=s\right]\varphi_{n_s}(s)\,\varphi_{n_q}(q)\,\varphi_{n_r}(r)\,ds(s)\,Tr\,d\tau\,dq\,dr$$

where $ds(s)$ means surface integration $\dot{s}_N = n_S^T(r,q,s,\tau)\,\dot{\delta}$ is the velocity of the scalarized vector process in the outwards direction prependicular to the failure surface ∂G and $\partial\dot{G}$ is the time derivative of the failure surface. Note than $n = -\alpha$ where is α the normalized gradient of the failure surface. By simple regression the parameters of the scalar velocity process are

$$m_0 = E\left[\dot{S}_N\mid S=s\right] = n^T\dot{R}^T R^{-1}\,s$$

$$\sigma_0^2 = Var\left[\dot{S}_N \mid S=s\right] = n^T\left[\ddot{R} - \dot{R}^T R^{-1} \dot{R}\right] n$$

possibly depending on time. It is noted that the variance of the velocity process does not depend on the position $S = s$ but only on the normal in s. The expectation in the integral is analytic

$$E\left[\max\left\{0, \dot{s}_N - \partial\dot{G}\right\} \mid S=s\right] = \sigma_0\,\varphi\left(\frac{\partial\dot{G} - m_0}{\sigma_0}\right) + (m_0 - \partial\dot{G})\,\Phi\left(-\frac{\partial\dot{G} - m_0}{\sigma_0}\right)$$

The surface integral can be converted into a volume integral using a suitable parameterization of the failure surface. The mean number of outcrossings (or the numerator of the correction factor) is then given by:

$$N = E\left[N_E^+(t_1,t_2)\right] = \int_{R^{n_r}} \int_{R^{n_q}} \int_{t_1}^{t_2} \int_{\partial G} E\left[\max\left\{0, \dot{s}_N - \partial\dot{G}\right\} \mid S=s\right] \varphi_{n_s}(s)\, \varphi_{n_q}(q) \varphi_{n_r}(r)\, ds(s)\, d\tau\, dq\, dr$$

$$= \int_{R^{n_r}} \int_{R^{n_q}} \int_{t_1}^{t_2} \int_{\partial G} E\left[\max\left\{0, \dot{s}_N - \partial\dot{G}\right\} \mid S=s\right] \frac{\varphi_{n_s}(s)\varphi_{n_q}(q)\varphi_{n_r}(r)}{h_{n_s-1}(s)\, h_{n_q}(q)\, h_{n_r}(r)} h_{n_s-1}(s)\, h_{n_q}(q)\, h_{n_r}(r)\, ds(s) d\tau\, dq\, dr$$

$$\approx \frac{1}{K} \sum_{j=1}^{K} (t_2 - t_1)\, E\left[\max\left\{0, \dot{s}_N - \partial\dot{G}\right\} \mid S=s_j\right] \frac{\varphi_{n_s}(s_j)\, \varphi_{n_q}(q_j)\, \varphi_{n_r}(r_j)}{h_{n_s-1}(s_j)\, h_{n_q}(q_j)\, h_{n_r}(r_j)} Tr \frac{1}{\frac{t_2 - t_1}{h_j(\tau_j)}}$$

The functions $h_j(x)$ are appropriately chosen sampling densities centered at the critical point. At first, the variables R, Q and the time τ are simulated followed by $n_s - 1$ values of the S-vector. The last component of S is determined by solving $g(r,q,s,t) = 0$, i.e. $s_{n_s} = g^{-1}(r, q, s_1, \ldots, s_{n_s-1})$. Tr is the transformation determinant for the chosen parameterization. In the non-stationary case the sampling density for the random times τ_j have to be chosen as efficient as posible. It should be different depending on whether the critical point is an interior or one of the two boundary points. Unfortunately, the process or the limit state surface usually depend on time. Therefore, it is not possible to determine an optimum sampling distribution for the specified times in advance so that it is proposed to sample the random times from a uniform distribution in the reference time interval. When choosing the sampling distribution for the vector S one can take advantage of the fact that in the critical point the curvatures can easily be computed.

APPENDIX B

Importance Sampling Appendix Update for Rectangular Wave Renewal Processes

We start from the assumption that the critical point (r^*, q^*, s^*, t^*) already signifies the most important region. This then suggests to use importance sampling. Following Rackwitz[53] and Hohenbichler and Rackwitz[54] we only determine a correction factor to the semi-analytical result making use of

$$E\left[N^+(t_1,t_2)\right] = E\left[N_A^+(t_1,t_2)\right] \frac{E\left[N_E^+(t_1,t_2)\right]}{E\left[N_A^+(t_1,t_2)\right]} = E\left[N_A^+(t_1,t_2)\right] C$$

where C is a correction factor close to one and which is to be evaluated by importance simulation. Fujita and Rackwitz[55] studied various importance sampling schemes. They came to the conclusion that axis-parallel sampling with line searches in the direction of the critical points most efficient in smaller dimensions, especially if one sets out from quadratic approximations of the limit state surface. But these advantages are lost in higher dimensions of the uncertainty space. Therefore, the general, simple sampling around the critical point will be used. More specifically, the correction factor is determined from:

$$C = \frac{E\left[N_{E,s}^+(t_1,t_2)\right]}{E\left[N_{A,s}^+(t_1,t_2)\right]} = \frac{\int_{R^{r\times n}\times R^r} \sum_{i=1}^{m} \lambda_i(\tau)\, I(\{g(Y,X,\tau) > 0\} \cap \{g(Y,X^+,\tau) \le 0\})\, dy\, dx\, d\tau}{\int_{R^{r\times n}} \sum_{i=1}^{m} \lambda_i(\tau^*)\, I(g_A(Y,X,\tau) \le 0)\, C_i(\tau^*)\, dy\, dx}$$

$$\approx \frac{\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{m} \lambda_i(\tau_j)\, I(\{g(y_j,x_{ji},\tau_j) > 0\} \cap \{g(y_j,x_{ji}^+,\tau_j) \le 0\})\, \chi_N(y_j,x_{ji},\tau_j)}{\int_{R^{r\times n}} \sum_{i=1}^{m} \lambda_i(\tau^*)\, I(g_A(Y,X,\tau) \le 0)\, C_i(\tau^*)\, dy\, dx} = \frac{\hat{N}}{D}$$

if the denominator is calculated semi-analytically and

$$C = \frac{E\left[N_{E,s}^+(t_1,t_2)\right]}{E\left[N_{A,s}^+(t_1,t_2)\right]} = \frac{\int_{R^{r\times n}\times R^r} \sum_{i=1}^{m} \lambda_i(\tau)\, I(\{g(Y,X,\tau) > 0\} \cap \{g(Y,X^+,\tau) \le 0\})\, dy\, dx\, d\tau}{\int_{R^{r\times n}} \sum_{i=1}^{m} \lambda_i(\tau^*)\, I(g_A(Y,X,\tau) \le 0)\, C_i(\tau^*)\, dy\, dx}$$

$$\approx \frac{\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{m} \lambda_i(\tau_j)\, I(\{g(y_j,x_{ji},\tau_j) > 0\} \cap \{g(y_j,x_{ji}^+,\tau_j) \le 0\})\, \chi_N(y_j,x_{ji},\tau_j)}{\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{m} \lambda_i(\tau^*)\, I(g_A(y_j,x_{ji},\tau^*) \le 0)\, C_i(\tau^*)\, \chi_D(y_j,x_{ji},\tau^*)} = \frac{\hat{N}}{\hat{D}}$$

if the denominator is calculated by Monte Carlo. In those formulae there is:

$$\chi_N(y_j,x_{ji},\tau_j) = \frac{\varphi_n(y_j,x_{ji},0,I)\, \frac{1}{t_2-t_1}}{\varphi_n(y_j,x_{ij};y^*,x^*,I)\, h(\tau_j,\tau^*)}$$

$$\chi_D(y_j,x_{ji},\tau_j = \tau^*) = (t_2 - t_1)\, \frac{\varphi_n(y_j,x_{ji},0,I)\, \frac{1}{t_2-t_1}}{\varphi_n(y_j,x_{ij};y^*,x^*,I)\, h(\tau_j,\tau^*)}$$

$I(.)$	Indicator function being 1 if the event in (.) is true and 0 otherwise
I	Identity matrix
r	Length of the vector of R—and Q–variables
n	Length of the vector of S–variables. $m \le n$ so that some components of S jump simultaneously
y_j	Vevtor of (R,Q)–variables with sampling density $\varphi_r(y;y^*,I) = \prod_{i=1}^{r} \varphi(y_i;y_i^*,1)$
x_{ji}	The sampled value according to the sampling density $\varphi_n(x;x^*,I) = \prod_{i=1}^{r} \varphi(x_i;x_i^*,1)$
x_{ji}^+	The sampled value according to the sampling density $\varphi_n(x;x^*,I)$ if the i–th component jmps from x_{ji} to x_{ji}^+.
$g(\tau)$	True state function
$g_A(y,x,\tau)$	Approximate parabolic state function
$(y_j^*,x_{ji}^*,\tau_j^*)$	Critical point
$C_T(\tau^*)$	Analytical time correction factor
$h(\tau_j,\tau^*)$	Importance sampling density in time space. It should be different depending on whether the critical point is an interior or one of the two boundary points. For the interior point one may experiment with a truncated normal distribution with density.

$$h(\tau,t^*) = \varphi(\tau;t^*), \sqrt{1/f''(t^*)} \Big/ \left[\varphi(t_2,t^*), \sqrt{1/f''(t^*)} - \varphi(t_1,t^*), \sqrt{1/f''(t^*)}\right]$$

For boundary points suitable sampling densities are:

$$h(\tau,t^* = t_1) = f'(t_1)\, \exp(-f'(t_1)(\tau - t_1) \Big/ \left[1 - \exp\left[-f'(t_1)(t_2 - t_1)\right]\right]$$

$$h(\tau,t^* = t_2) f'(t_2)\, \exp(-f'(t_2)(\tau - t_2) \Big/ \left[\left[-f'(t_2)(t_1 - t_2)\right]\right]$$

K sample size for very large K.

Usually, we are interested only in the first corection factor. For simplicity, it is assumed that all calculations are performed in standard space and, thereofre, the standard deviation of each component of the sampling density is close to 1. Very little improvemet can be achieved if the standard deviation is adjusted according to the curvature of the failure surface at the critical point. Thus, the vectors y_i and x_i are first generated together with τ_j according to the sampling densities proposed before, for both numerator and denominator. Next, for each component of the S-process m additional variables are generated and added to the initial x_{ij}-vector in the numerator as well as in the denominator. Then the indicator functions are evaluated, multiplied with the relevant λ s and summed up after which whole term is multiplied with the bias correction factor, separately for numerator and denominator. Formally, $\tau_j = \tau^*$ in the denominator. This is repeated until the coefficient of variation of the correction factor is sufficiently small. It must, however, be mentioned that the importance sampling scheme is probably not the most efficient one. In fact, dample sized of around 1000 are needed in order to keep the coefficient of variation of the eatimate below 0.2.