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Approximations of first-passage times for differentiable processes based on higher-order threshold crossings

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Methods for calculating approximations of the first-passage probability of differentiable non-narrow band processes based on higher-order threshold crossings are discussed. Two of them use factorial moments of the number of crossings into the failure region, including a new method based on a Gram–Charlier series expansion of the distribution of the number of exits. Several numerical schemes for the evaluation of factorial moments are investigated. The methods are studied for three examples. The examples show that the Gram–Charlier series expansion converges faster towards the exact solution than Rice's "in- and exclusion" series. However, it is difficult to quantify the error made by the proposed method. Further, it is shown that for engineering applications, the Poisson assumption as modified by Ditlevsen such that the initial conditions are taken into account, provides excellent results in almost all cases.

1 INTRODUCTION

In many engineering applications it is necessary to determine the reliability of structural components subject to stochastic process loading. Let failure be defined by $F = \{g(\mathbf{X}(t), t) \leq 0\}$ where $\mathbf{X}(t)$ denotes a vector process, and let $\partial g(t)$ denote the failure surface defined by $g(\mathbf{X}(t), t) = 0$. The probability of failure in the interval $[0; T]$ is then

$$P_f(T) = 1 - P[g(\mathbf{X}(t), t) \leq 0, \quad \forall t \in [0; T]] \tag{1}$$

No simple analytical results exist for this problem. In most cases the computation of the failure probability is based on the outcrossing approach, together with the assumption of independent outcrossings. Then the number of outcrossings follows a Poisson distribution and

$$P_f(T) \approx 1 - \exp[-E[N^+(T)]] \tag{2}$$

where $E[N^+(T)]$ is the mean number of crossings of $X(t)$ into the failure domain F during the interval $[0; T]$ for regular streams of crossings. In the stationary case this is $E[N^+(T)] = \nu^+ T$ where ν^+ denotes the outcrossing intensity [see also eqn (22)]. Cramer and Leadbetter[1] have shown that eqn (2) is asymptotically correct whenever the stream of crossings can be thinned out in a certain manner, e.g. by shifting the failure boundary towards the exterior for sufficiently mixing processes. However, for low levels the error made by assuming independent crossings can be large, depending on the correlation structure of the process. Further, it is not in general possible to quantify the error made by the Poisson assumption. If time-independent variables $\mathbf{R}$ are also present the total failure probability is

$$P_f(T) = \int_\Omega P_f(T\,|\,\mathbf{r}) f_\mathbf{R}(\mathbf{r}) d\mathbf{r} \tag{3}$$

where $\mathbf{R}$ is defined in Ω. Significant contributions to $P_f(T)$ may come from large $P_f(T\,|\,\mathbf{r})$. It is, therefore, important that the method for calculating $P_f(T\,|\,\mathbf{r})$ is sufficiently accurate even for low reliability levels.

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Numerous improvements to eqn (2) exist. For the stationary case Ditlevsen[2] suggests, on rather general grounds, taking account of the initial conditions which then leads to

$$P_f(t) \approx 1 - (1 - P_f(0)) \exp\left[-\frac{\nu^+ T}{1 - P_f(0)}\right] \quad (4)$$

In many cases this simple modification already provides a remarkable improvement over eqn (2).

Particularly for narrow-band scalar processes, a number of other approximations exist. Yang and Shinozuka[3] obtained rather accurate results based on the assumption that the extreme points follow a Markov process. They also pointed out that the outcrossing intensity is reduced because crossings tend to occur in clumps and proposed certain approximations. Another successful method is to consider the outcrossings of the envelope process rather than those of the actual process. This approach also involves considerations about clumps of crossings (see, e.g. Ref. 4 for further discussion and additional references). Madsen and Krenk[5] solve an integral equation which governs the first passage probability function. The equation contains an unknown kernel which can be approximated in different ways but most easily and efficiently for narrow band processes (see also Refs 6 and 7). More recent studies for first-passage times based on other concepts have been performed by Langley[8] and Ditlevsen and Lindgren.[9] For vector processes much less material is available. Veneziano,[10] for example, showed that there is no unique definition of envelope processes and, therefore, this approach is less suitable. Additional more recent work on level crossings for random processes is reviewed by Abrahams[11] indicating that solutions to the specific problem of first-passage time distribution, which are not based on some Markov-like property or on Cramer and Leadbetters asymptotic result[1] are rather rare. If, however, Markovian properties can be assumed for the underlying process, at least in approximation, it is sometimes possible to determine directly the moments of first passage times. Those can then be used to approximate the distribution of first passage times, for example, by maximum entropy distributions. One of the more prominent results in this direction is due to Spencer and Bergman.[12]

The scope of this paper is to review and compare two more general methods for the calculation of approximations for the first-passage probability based on higher-order threshold crossings. It is beyond its scope to compare the results with maximum entropy distributions just mentioned.

2 BOUNDS ON FIRST-PASSAGE TIME

Let p_k denote the probability of exactly k outcrossings in the interval $[0; T]$. It is then evident that the probability of no outcrossings or the complementary first passage probability is

$$\begin{aligned} p_0 &= 1 - \sum_{k=1}^{\infty} p_k \\ &= 1 + \sum_{k=1}^{\infty} p_k \sum_{i=1}^{k} (-1)^i \binom{k}{i} \\ &= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \sum_{k=i}^{\infty} i! \binom{k}{i} p_k \\ &= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \sum_{k=i}^{\infty} k(k-1)\ldots(k-i+1) p_k \\ &= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} m_i \end{aligned} \quad (5)$$

where m_i denotes the ith factorial moment of the number of outcrossings, i.e.

$$m_0 = 1$$

$$m_i = \sum_{k=i}^{\infty} k(k-1)\ldots(k-i+1) p_k \quad \text{for} \quad i \geq 1 \quad (6)$$

and where it is noted that

$$\binom{k}{i} = 0 \quad \text{for} \quad i > k \quad (7)$$

Equation (5) is nothing else than Rice's 'in- and exclusion' series (see Ref. 13) cast into a somewhat different form. Of course, the $m_i (i = 1, 2 \ldots)$ must exist and the series in eqn (5) must converge in order to make eqn (5) a valid representation. The series provides lower and upper bounds for the survival probability upon truncation after an odd or even term, respectively. The computational effort involved in evaluating $P_f(T)$ according to this method, however, is extensive. Further, an increasing number of terms have to be taken into account for increasing $E[N^+(T)]$. Bolotin[14] gave a similar series based on the moments, $E[N^+(T)]$, $E[(N^+(T))^2] \ldots$, of the number of out-crossings. This series can be regarded as a rearrangement of eqn (5) in terms of moments. Bolotin performed a few numerical studies indicating that the bounds are rather wide for large values of $E[N^+(T)]$. Both Rice's original in- and exclusion series and Bolotin's variant have been applied only very rarely due to their computational complexity.

Lange[15] instead considered the distribution function of the discrete random variable N^+ directly. The distribution function $F_N(k)$ of N^+ is generally unknown, but a Gram–Charlier series expansion (type B) can be used to approximate the unknown probabilities of a discrete random variable with known factorial moments m_i by a Poisson distributed variable Y with parameter m_1 (see e.g. Ref. 16)

$$p_k = w_k \sum_{i=0}^{\infty} q_i Q(i, k) \quad (8)$$

where

$$w_k = \frac{m_1^k}{k!} \exp(-m_1) \quad (9)$$

are the probabilities $w_k = P(Y = k)$. The so called Gram–Charlier polynomials can be obtained from

$$Q(0, k) = 1 \quad (10)$$

$$Q(i, k) = \frac{d^i}{dm_1^i} w_k = \frac{i!}{m_1^i} \sum_{j=i-k}^{i} (-1)^j \binom{k}{i-j} \frac{m_1^j}{j!}$$

$$(1 = 1, 2, \ldots) \quad (11)$$

and the coefficients q_i are given by

$$q_i = \frac{m_1^i}{i!} \sum_{k=0}^{\infty} p_k Q(i, k) \quad (12)$$

Equation (12) can be rearranged as

$$q_i = \frac{m_1^i}{i!} \sum_{k=0}^{\infty} (p_k - w_k) Q(i, k) + \frac{m_1^i}{i!} \sum_{k=0}^{\infty} w_k Q(i, k) \quad (13)$$

It can be shown that

$$\frac{m_1^i}{i!} \sum_{k=0}^{\infty} w_k Q(i, k) = \begin{cases} 1 \text{ for } i = 0 \\ 0 \text{ for } i > 0 \end{cases} \quad (14)$$

Then for $i > 0$

$$\begin{aligned} q_i &= \frac{m_1^i}{i!} \sum_{k=0}^{\infty} (p_k - w_k) \frac{i!}{m_1^i} \sum_{j=i-k}^{i} (-1)^j \binom{k}{i-j} \frac{m_1^j}{j!} \\ &= \sum_{j=0}^{i-1} (-1) \frac{m_1^j}{j!} \sum_{k=i-j}^{\infty} (p_k - w_k) \binom{k}{i-j} \\ &= \sum_{j-0}^{i-1} (-1)^j \frac{m_1^i}{j!} (m_{i-j} - m_1^{i-j}) \frac{1}{(i-j)!} \end{aligned} \quad (15)$$

It is further seen that $Q(i, 0) = (-1)^i$. From eqn (8) it then follows

$$p_0 = \exp(-m_1) \sum_{i=0}^{\infty} q_i (-1)^i \quad (16)$$

and the following approximations for the survival probability up to the fourth order are obtained

$$p_{0,1} = \exp(-m_1)$$

$$p_{0,2} = \exp(-m_1)\left(1 + \frac{1}{2}(m_2 - m_1^2)\right)$$

$$p_{0,3} = \exp(-m_1)\left(1 + \frac{1}{2}(m_2 - m_1^2) - \frac{1}{6}(m_3 - m_1^3) + \frac{m_1}{2}(m_2 - m_1^2)\right)$$

$$p_{0,4} = \exp(-m_1)\left(1 + \frac{1}{2}(m_2 - m_1^2) - \frac{1}{6}(m_3 - m_1^3) + \frac{m_1}{2}(m_2 - m_1^2) + \frac{1}{24}(m_4 - m_1^4) - \frac{m_1}{6}(m_3 - m_1^3) + \frac{m_1^2}{4}(m_2 - m_1^2)\right)$$

If the outcrossings are independent, that is, if $p_k = w_k$ for all k then $m_i = m_1^i$ and eqn (16) reduces to $P_0 = exp(-m_1)$. Equation (16), therefore, is correct in the limit according to Cramer and Leadbetter[1] [see eqn (2)]. From eqn (16), it is also seen that the outcrossings approximately follow a Poisson distribution when $1 \gg m_1 \gg m_2 \ldots \gg m_i$, i.e. the Poisson assumption is asymptotically correct for $T \to 0$. Further, under the condition

$$q_{n_0} \geq q_{n_0+1} \geq q_{n_0+2} \ldots > 0 \quad (17)$$

the error of approximation is smaller than $\exp(-m_1)|q_{n_0+1}|$ for $n_0 > 0$ and

$$p_0 \leq \exp(-m_1) \sum_{i=1}^{2n} q_i (-1)^i \qquad (2n > n_0) \quad (18)$$

are upper bounds for p_0 and

$$p_0 \geq \exp(-m_1) \sum_{i=1}^{2n+1} q_i (-1)^i \qquad (2n + 1 > n_0) \quad (19)$$

are lower bounds for p_0. Even if the condition eqn (17) is not fulfilled, convergence of eqn (16) can be established for a wide class of stochastic processes (see Ref. 15).

3 EVALUATION OF FACTORIAL MOMENTS

Belyaev[17] has shown that factorial moments of the number of crossings into a failure region of a stochastic vector process $\mathbf{X}(t)$, of which derivative $\dot{\mathbf{X}}(t)$ exists with probability one or in mean square, are

$$\begin{aligned} m_i = \int_0^T \int_0^T \ldots \int_0^T \int_{\mathbf{x}(t_1) \in \partial g(t_1)} \int_{\mathbf{x}(t_2) \in \partial g(t_2)} \ldots \int_{\mathbf{x}(t_i) \in \partial g(t_i)} \\ E\left[\prod_{j=1}^{i} \max\{0, \dot{x}_{nj} | \mathbf{X}(t_k) = \mathbf{x}(t_k), k = 1, 2 \ldots i\}\right] \\ f(\mathbf{x}(t_1), \mathbf{x}(t_2) \ldots \mathbf{x}(t_i)) \mathrm{d}s(\mathbf{x}(t_1)) \mathrm{d}s(\mathbf{x}(t_2)) \ldots \\ \mathrm{d}s(\mathbf{x}(t_1)) \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_i \end{aligned} \quad (20)$$

where ds indicates surface integration, $E[.]$ denotes the expectation of [.] and

$$\dot{x}_{nj} = [\dot{\mathbf{x}}(t_j) - \mathbf{v}(\mathbf{x}(t_j))]^T \mathbf{n}(\mathbf{x}(t_j)) \quad (21)$$

where $\mathbf{n}$ is a normal to the failure surface and $\mathbf{v}$ is the time-variation of the failure surface. In the general case, analytical solutions of eqn (20) are unknown. However, certain results for the moments of crossings of a constant level up to the 4th order have been studied by Gaganov[18] for special Gaussian scalar processes, and

Miroshin an Zvetkow[19] calculated moments of crossings of a linear function by a Gaussian process. Lange[15] evaluated the factorial moments by simulating the process and simply counting the number of outcrossings. Such a method, however, is not efficient for high levels. Simple numerical integration soon becomes prohibitive because the dimension of the integral in general is too large.

Within the field of reliability analysis, a number of alternative methods for the evaluation of integrals of the type eqn (20) have been suggested and have tentatively been applied to eqn (20). The integral can be solved by nested FORM/SORM (Hohenbichler[20]), as a Laplace integral (Breitung[21]) or as the sensitivity measure of an associated parallel system (Madsen[22], Hagen[23]). All these methods locate an important region where the boundary of the integral can be approximated by a linear or quadratic term. Unfortunately, if the correlation function of the considered process has a periodic term, a large number of such important regions exist (multiple β-points) for $i \geq 2$. Therefore, these methods can, at most, be used in special cases. For further discussion it is convenient to rewrite eqn (20) as

$$m_i = \int_0^T\int_0^T\cdots\int_0^T \nu^+(t_1, t_2, \dots t_i)\,dt_1 dt_2 \dots dt_i \tag{22}$$

where $\nu^+(t_1, t_2, \dots t_i)$ denotes the joint crossing intensity following from eqn (20). The factorial moments can then be evaluated by conditional sampling, where for each random sample of $(t_1, t_2, \dots t_i)$ the crossing intensity is determined by one of the three aforementioned methods (see, for example, Ref. 24). The multiple β-points problem is thus avoided. It is then important that the computational effort involved in the evaluation of ν^+ is small.

Unfortunately, serious problems have been met with both methods. In order to determine the nth order crossing intensity by the sensitivity method, it is necessary to determine the nth order derivatives of a failure probability determined by a FORM/SORM analysis. Derivatives of order higher than one are in general difficult to calculate and are rarely sufficiently accurate. The amount of effort involved in performing a SORM analysis or evaluating a Laplace integral directly, increases rapidly with the dimension of the problem. In conclusion, conditional sampling, even if performed in an adaptive manner, does not offer an efficient alternative. The most efficient numerical method for evaluating the factorial moments is simple importance sampling (see, for example, Ref. 25). This method has the advantage that it is relatively insensitive towards the number of dimensions (see Ref. 26). In order to use importance sampling it is necessary to rearrange the integral eqn (20). We introduce

$$\mathbf{x}_i = r_i \mathbf{a}_i \tag{23}$$

where $\mathbf{a}_i$ is a unit direction vector and r_i the distance between origin and the failure surface in the direction $\mathbf{a}_i$.

It is assumed that the failure region is starshaped, that is every line starting at origin has only one intersection with the failure surface. Further, the relation between the infinitely small area $ds(\mathbf{x}_i)$ on the failure surface and the infinitely small area $ds(\mathbf{a}_i)$ on the unit sphere is

$$ds(\mathbf{x}_i) = \frac{r(\mathbf{a}_i)}{\mathbf{a}_i^T \mathbf{n}[r(\mathbf{a}_i)\mathbf{a}_i]}\,ds(\mathbf{a}_i) \tag{24}$$

The factorial moments can now be evaluated as

$$m_i = \int_0^T\int_0^T\cdots\int_0^T \int_{unit\ sphere}\int_{unit\ sphere}\cdots\int_{unit\ sphere} E\left[\prod_{j=1}^{i} \max\{0, \dot{x}_{nj}\,|\,\mathbf{X}(t_k) = \mathbf{x}(t_k), k = 1, 2 \dots i\}\right] \times \prod_{k=1}^{i} \frac{r(\mathbf{a}_k)}{\mathbf{a}_k^T \mathbf{n}[r(\mathbf{a}_k)\mathbf{a}_k]}\, f(r(\mathbf{a}_1)\mathbf{a}_1, r(\mathbf{a}_2)\mathbf{a}_2, \dots r(\mathbf{a}_i)\mathbf{a}_i)\, ds(\mathbf{a}_1(t_1))ds(\mathbf{a}_2(t_2))\dots ds(\mathbf{a}_i(t_i))dt_1 dt_2 \dots dt_i \tag{25}$$

4 INITIAL CONDITIONS

Following Ditlevsen,[2] it is important to take account of the initial conditions. Consider a n-dimensional stochastic process $\mathbf{X}(t)$. The probability that the process initially belongs to the failure set F can be large. This is taken into account by determining the failure probability as

$$P_f(T) = 1 - p_0(T\,|\,\mathbf{X}(0) \in \{R^n \backslash F\})P(\mathbf{X}(0) \in \{R^n \backslash F\}) \tag{26}$$

where $P(\mathbf{X}(0) \in \{R^n \backslash F\})$ is the probability that the process initially does not belong to the failure region. This is a simple time-invariant reliability problem which can be solved by a FORM/SORM analysis. The probability that no failure occurs given the process initially belongs to the safe region, $p_0(T\,|\,\mathbf{X}(0) \in \{R^n \backslash F\})$ is determined by investigating the conditional process. For $\mathbf{X}(0) = \mathbf{x}(0)$ the conditional probability density of $(\mathbf{X}(t_1), \mathbf{X}(t_2) \dots \mathbf{X}(t_i), \dot{\mathbf{X}}(t_1), \dot{\mathbf{X}}(t_2), \dots \dot{\mathbf{X}}(t_i))$ is

$$f(\mathbf{x}(t_1), \mathbf{x}(t_2) \dots \mathbf{x}(t_i), \dot{\mathbf{x}}(t_1), \dot{\mathbf{x}}(t_2), \dots \dot{\mathbf{x}}(t_i)|\mathbf{x}(0)) = \frac{f(\mathbf{x}(0), \mathbf{x}(t_1), \mathbf{x}(t_2) \dots \mathbf{x}(t_i), \dot{\mathbf{x}}(t_1), \dot{\mathbf{x}}(t_2), \dots \dot{\mathbf{x}}(t_i))}{f(\mathbf{x}(0))} \tag{27}$$

The factorial moments of the number of crossings given $\mathbf{X}(0) \in \{R^n \backslash F\}$ can now be determined as

$$m_i(T\,|\,\mathbf{X}(0) \in \{R^n \backslash F\}) = \int_{\{R^n \backslash F\}} m_i(T\,|\,\mathbf{X}(0) = \mathbf{x}_0)\frac{f(\mathbf{x}(0))}{1 - P_f(0)}\,d\mathbf{x}_0 \tag{28}$$

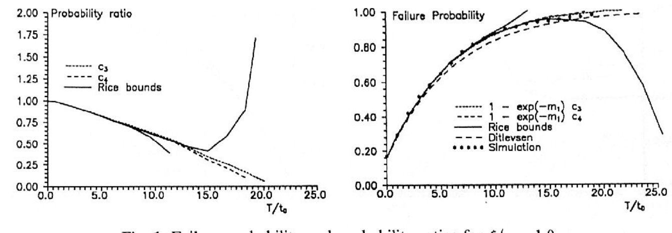

Fig. 1. Failure probability and probability ratios for $\xi/\sigma = 1.0$.

where $m_i(T\,|\,\mathbf{X}(0) = \mathbf{x}_0)$ is evaluated by eqn (20) with the distribution function given by eqn (27). The dimension of the integral which has to be evaluated in order to determine the factorial moment has now increased by n. This, however, does not offer serious problems in view of the method chosen to solve the integral.

5 SIMULATION

An estimate of the failure probability may also be determined by repeatedly simulating the process until failure occurs. On this basis, an empirical lifetime distribution can be obtained. For low levels and not too small failure probabilities simulation might, in fact, provide the most efficient computation scheme for the problem of interest. Although this method would fail for small probability levels, it will be used to check the approximations and bounds described above. The simulation results are obtained by the method suggested by Shinozuka.[12] By this method, the process is simulated by decomposing the prescribed spectral density matrix and performing a summation of a trigonometric series with independent phase angles.

6 EXAMPLES

6.1 Example 1

In the first example, the first passage probability of a scalar Gaussian process over the constant threshold ξ within the time interval $[0, T]$ is studied. The stochastic process has mean value $\mu(t) = 0$ and autocorrelation function

$$\rho(\tau) = \exp(-\tau^2/t_0)$$

The standard deviation is σ. The same process has already been investigated by Bolotin and Lange. It is convenient to introduce

$$c_i = \frac{p_{0,i}}{p_{0,1}}$$

The probability ratio c_i expresses the relative difference between $p_{0,1}$ and the survival probability determined on the basis of the Poisson assumption with the initial conditions taken into account. In Figs 1–3, the approximations of the failure probability and the probability ratios are shown for $\xi/\sigma = 1.0$, 2.0 and 3.0, respectively.

At first it should be noted that the numerical results reported by Bolotin[14] and Lange[15] could only in part be reproduced. In Figs 1–3 it is seen that for small values of T the probability ratios are close to unity for all methods. This implies that for small T and/or small number of exits, the Poisson assumption with the initial conditions taken into account is very accurate. It is further demonstrated that, as expected, the Poisson approximation leads to better results for increasing levels and that consideration of the initial conditions according to eqn (26), in fact, improves the results significantly for larger failure probabilities. As T increases, Rice's bounds become very wide for all levels. For large values of T the estimated failure probability obtained by eqn (16) is

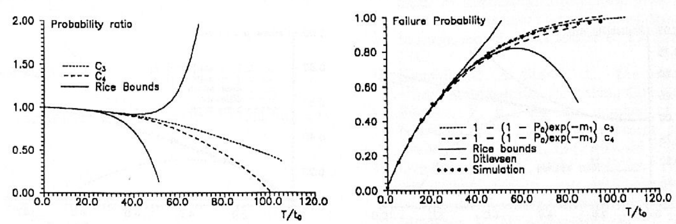

Fig. 2. Failure probability and probability ratios for $\xi/\sigma = 2.0$.

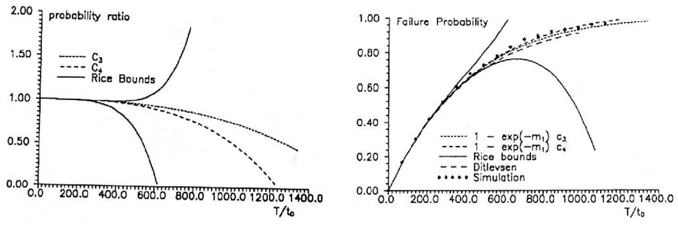

Fig. 3. Failure probability and probability ratios for $\xi/\sigma = 3.0$.

much closer to the 'exact' simulation results than Rice's bounds. For the method in eqn (16), the number of factorial moments which are needed to obtain good estimates of the failure probability increases with increasing T. Further, since the correction factors c_3 and c_4 both become negative for large values of T it is evident that the condition eqn (17) is not always fulfilled. This, however, is only true for very low reliability levels. Nevertheless, it is unfortunate that it is not possible to determine whether c_3 and c_4 are bounds because even higher factorial moments would be required in order to assess the validity of eqn (17). From the simulation, (dots) it can be concluded that eqn (16) with the third or fourth factorial moment included yield rather accurate results, even slightly more accurate than eqn (4) for all levels. It is finally noted that the Poisson assumption becomes unconservative for all levels and times in this example.

6.2 Example 2

Here, we consider a simple linear oscillator with Gaussian white noise excitation. The response $X(t)$ is a stationary Gaussian process with zero mean and autocorrelation function

$$\rho(\tau) = \exp(-\zeta\omega_o|\tau|)\left(\cos\omega_d\tau + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d\tau\right)$$

where ω_0 denotes the eigenfrequency, $\tau = t_1 - t_2$, $\omega_d = \omega_0\sqrt{1-\zeta^2}$ and ζ is the damping. The standard deviation is σ. Failure is defined for $g(x,(t)) = \xi - x(t) \leq 0$ where ξ is a constant. In Fig. 4 the failure probability and the probability ratios are shown for $\xi/\sigma = 2{\cdot}0$, $\zeta = 0{\cdot}05$ and $\omega_0 = 1{\cdot}0$. For this example the outcrossings are highly correlated and occur in clusters. Therefore, $\exp(-m_1)$ is a lower bound for the survival probability. All probability ratios are larger than one. Clearly, this example exactly represents the case where one of the narrow-band approximations would most likely yield good results. For this example only the first three factorial moments have been calculated. The same conclusions as for the first example can be made. However, the example also demonstrates that it is difficult to evaluate higher factorial moments if the correlation function is not very well-behaved. For large values of T in particular, the numerical effort is excessive.

6.3 Example 3

Let $\mathbf{X}(t)$ be a two-dimensional stochastic process with mean values

$$\mu_{X_1} = 4{\cdot}0, \quad \mu_{X_2} = 3{\cdot}0$$

and correlation functions

$$\kappa_{X_1X_1}(\tau) = \exp(-\tau^2/t_0^2)$$

$$\kappa_{X_2X_2}(\tau) = \exp(-\tau^2/t_0^2)$$

$$\kappa_{X_1X_2}(\tau) = 0$$

The failure region F is defined by the failure function

$$g(\mathbf{x}(t)) = 10 - \sqrt{x_1(t)^2 + 3x_2(t)^2}$$

The initial failure probability is $P_f(0) = 0{\cdot}0136$. The failure probability as a function of T is shown in Fig. 5.

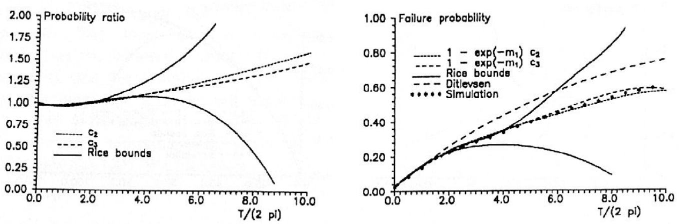

Fig. 4. Failure probability and probability ratios for Example 2.

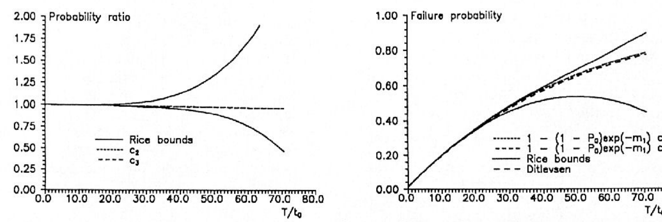

Fig. 5. Failure probabilities and probability ratios for Example 3.

For all investigated values of T there is no difference between $p_{0,2}$ and $p_{0,3}$. Good convergence has thus been obtained even for rather high failure probabilities. It is further seen that the Poisson assumption with the initial conditions taken into account leads to better results than for the one-dimensional case. It is, however, expected that the quality of the Poisson assumption decreases with increasing correlation between X_1 and X_2.

7 SUMMARY AND CONCLUSION

The bounding method by Rice and a Gram–Charlier expansion for calculating approximations of the first-passage probability for stochastic processes are reviewed. Some problems in numerical implementation are discussed. For three examples, the results obtained by these methods have been compared. The examples show that the Gram–Charlier series expansion converges faster towards the exact solution than Rice's in- and exclusion series. However, it is difficult to quantify the error made by this method since this involves the calculation of even higher factorial moments. The Gram–Charlier series expansion works for scalar and vector processes as well as for random fields. It has been verified that for engineering purposes where the failure probability is normally small, the consideration of the initial condition already suggested by Ditlevsen[2] is a substantial improvement over the simple Poissonian approach. Beyond that, any improvement is difficult. Improvements based on higher order moments of the number of crossings are numerically rather involved. This also applies to the suggested Gram–Charlier expansion.

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