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STRUCTURAL RELIABILITY UNDER NON–STATIONARY GAUSSIAN VECTOR PROCESS LOADS

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ABSTRACT

Breitung's asymptotic result for the crossing rate of stationary Gaussian vector processes out of smooth, curved failure surfaces is generalized to non–stationary processes and/or time–dependent failure surfaces. A suitable special solution for the corresponding "Laplace" integral is given. It also performs the necessary time integration for the mean number of outcrossings in a given time interval. Thereby, one has to distinguish between the case where the maximum outcrossing rate occurs in the interior of the time integration interval and the case where it is at one of its boundaries.

NOMENCLATURE

D	= domain in $\mathbb{R}^n$
F	= failure domain
$g(x,\tau)$	= state function depending on time τ
$n(x)$	= normal on ∂F at x
[illegible]	= number of crossings through failure surface in [0,t]
[illegible]	= number of outcrossings through failure surface in [0,t]
[illegible]	= probability of the event A
[illegible]	= reliability for time interval [0,t]
[illegible]	= covariance matrix of $X(\tau)$
[illegible]	= cross–covariance matrix of $X(\tau)$ and $\dot{X}(\tau)$
[illegible]	= covariance matrix of $\dot{X}(\tau)$
[illegible]	= safe domain
[illegible]	= critical time
[illegible]	= reference time
[illegible]	= standardized Gaussian vector process
[illegible]	= derivative process of $X(\tau)$
x^*	= critical point (minimum point)
β	= geometrical safety index
∂F	= failure surface
φ_n	= n–dimensional standard normal density
Φ	= standard normal integral
κ_i	= i–th main curvature of ∂F
ν^+	= outcrossing rate

INTRODUCTION

The computation of the failure probability of structures subjected to sequences of random seas is well–known provided that certain idealizing assumptions are made. The most common approach is based on the assumption of a stationary Gaussian sea in each sea state. Also, load effects are related to the loads in a linear manner. Approximations for the time–variant reliability of structural components are then obtained by determining the mean number of crossings of the load effect process into specified failure regions. Let F be a failure set with boundary ∂F and $X(\tau)$ a Gaussian vector process with continuously differentiable sample paths. Also denote by $N^+(t)$ the number of crossings into F in a time interval [0,t] ... $\nu^+(\tau)$ the outcrossing rate. Under suitable regularity con... following reliability bound can then be derived (C... Leadbetter, 1967)

$$R(t) \geq R(0) - E[N^+(t)]\, R(0)$$

with

$$E[N^+(t)] = \int_0^t \nu^+(\tau)\, d\tau$$

the mean number of outcrossings. R(0) is the proba... $X(0) \in S(0)$ with S being the complement of F, i.e. the ... the stationary case the integral simply is $\nu^+ t$. If $X(\tau)$ is a mixing process, then the asymptotic approximation (3)

$$R(t) \sim \exp[-E[N^+(t)]] \tag{3}$$

is valid for rare crossing events or high reliability R(t) and this is used most frequently in practical applications.

If $X(\tau)$ is a scalar process and, thus, the safe set is bounded by a simple threshold function, exact formulae are available for the non-stationary "upcrossing" rate in eq. (2). But an analytical solution for $E[N^+(t)]$ is known only for a very special case (for a covariance stationary process with linearly varying mean and/or threshold, see Cramer and Leadbetter, 1967). Otherwise, one has to integrate numerically over time. Guers and Rackwitz (1986) proposed a simple approximation for the time integration in eq. (2) by using asymptotic

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concepts for so-called Laplace integrals. They also gave a formulation based on eq. (3) suitable for the numerical analysis when the reliability problem contains non-ergodic variables such as random but time-invariant parameters of the process $X(t)$ or of the threshold function. The latter formulation directly carries over to the vectorial case. Therefore, we shall discuss only sufficiently mixing processes in the sequel.

Non-stationary crossing rates for vector processes out of arbitrarily bounded failure domains are also of interest in several areas. For example, the excitation of structures by earthquake-induced ground motions are highly non-stationary and the failure criteria of structural members frequently are given as so-called interaction curves for the multidimensional response quantities. The non-stationary case is of particular interest if the resistance properties are [illegible]ding in time due to fatigue, aging, corrosion or other wear-out [illegible]omena. The non-stationary case is further of interest for the purpose to quantify the effect of the commonly adopted sta[illegible]rity assumption for the sea states in marine engineering. Even [illegible]tationary vector processes exact formulae for the outcrossing [illegible] are available only for some special cases with respect to the [illegible] of the failure surface and the correlation structure of the [illegible]ss (Veneziano et al., 1977). The only solution for the out[illegible]ng rate of non-stationary vector processes known to the [illegible]rs is due to Ditlevsen (1983) for linearly bounded failure [illegible]ins. It makes use of the fact that a linear combination of [illegible]ian processes is again Gaussian so that all results for the scalar [illegible] can be used. Otherwise serious computational difficulties are [illegible] However, for smoothly bounded failure domains, Breitung [illegible], 1984a) derived a second-order approximation for the station[illegible]utcrossing rate by using asymptotic concepts for Laplace [illegible]als which can be updated by importance sampling procedures to yield numerically exact results (Fujita et al., 1987).

In this paper the concepts of asymptotic analysis are applied to formula (2) by assuming non-stationarity of the vector process and/or time-varying smoothly bounded failure domains. The results will be illustrated at a simple example. Suggestions for simplifications in practical applications are made.

EXPECTED NUMBER OF CROSSINGS OF NON-STATIONARY VECTOR PROCESSES

Let $X(\tau) = (X_1(\tau),\ldots,X_n(\tau))^T$ be a n-dimensional non-stationary Gaussian process with differentiable sample paths and derivative process $\dot{X}(\tau) = (\dot{X}_1(\tau),\ldots,\dot{X}_n(\tau))^T$. Without loss of generality, we suppose that for each $\tau \in [0,t]$ the process can be standardized such that there is:

$$E[X_i(\tau)] = E[\dot{X}_i(\tau)] = 0 \qquad (i=1,\ldots,n)$$
$$COV[X_i(\tau),X_j(\tau)] = 0 \qquad (i \neq j)$$
$$COV[X_i(\tau),X_i(\tau)] = 1 \qquad (i=1,\ldots,n)$$

For later convenience, we also introduce the notations:

$$R = \{r_{ij}\} = \{COV[X_i(\tau),X_j(\tau)]\}_{i,j=1,\cdots,n} = I$$
$$\dot{R} = \{\dot{r}_{ij}\} = \{COV[X_i(\tau),\dot{X}_j(\tau)]\}_{i,j=1,\cdots,n}$$
$$\ddot{R} = \{\ddot{r}_{ij}\} = \{COV[\dot{X}_i(\tau),\dot{X}_j(\tau)]\}_{i,j=1,\cdots,n}$$

Let $F = \{g(x;\tau) \leq 0\}$ be the failure set in the standard space with boundary ∂F varying in time. This is assumed to be at least locally twice differentiable in x and τ.

$$\partial F = \partial F(x;\tau) = \{x,\tau;\, g(x;\tau) = 0\} \tag{4}$$

Following Bolotin (1981) the outcrossing rate of the process $X(\tau)$ through the hypersurface ∂F during the time interval $\Delta\tau$ is defined as:

$$\nu^+(\partial F;\tau) = \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} P_1(\partial F;\Delta\tau) \tag{5}$$

$P_1(\partial F;\Delta\tau)$ is the probability of a crossing of ∂F by the process $X(\tau)$ from the safe domain $S(\tau) = \{g(x,\tau) > 0\}$ into the failure domain $F = \{g(x,\tau) \leq 0\}$ during $\Delta\tau$. As usual, regularity of the point process of crossings is assumed, i.e. there is $P(N(\Delta\tau) > 1) = o(\Delta\tau)$. $P_1(\partial F;\Delta\tau)$ can be given by

$$P_1(\partial F;\Delta\tau) = P\left\{\begin{array}{l} X(t) \in \Delta(\partial F) \\ \dot{X}_n(t) > \partial\dot{F}(x;t) \\ \tau \leq t \leq \tau+\Delta\tau \end{array}\right\} \tag{6}$$

where $\dot{X}_n(t) = n^T(x)\,\dot{X}(t)$ is the projection of $\dot{X}(t)$ on the normal $n^T(x)$ of ∂F at the point x, $\partial\dot{F}(x;t)$ the time-variation of the surface ∂F at x and $\Delta(\partial F)$ a thin layer enveloping ∂F with height $(\dot{X}_n - \partial\dot{F}(x;\tau))\,\Delta\tau$. Introducing the joint density function $\varphi_{n+1}(x,\dot{x}_n;\tau)$ of X and $\dot{X}_n$ allows to express P_1 by the following integral:

$$P_1(\partial F(\tau);\Delta\tau) = \int_{\Delta(\partial F)} \int_{\dot{X}_n(\tau) > \partial\dot{F}(x;\tau)} \varphi_{n+1}(x,\dot{x}_n;\tau)\,dx\,d\dot{x}_n$$

The integral over $\Delta(\partial F)$ can be transformed into a surface integral over ∂F. The layer $\Delta(\partial F)$ can be understood as the sum of infinitely small cylinders with height $(\dot{x}_n(\tau) - \partial\dot{F}(x;\tau))\,\Delta\tau$ and basis $ds(x)$, where $ds(x)$ is a surface neighborhood of the crossing point. Hence, integrating over x yields:

$$P_1(\partial F;\Delta\tau) = \int_{\partial F} \int_{\dot{X}_n > \partial\dot{F}(x;\tau)} (\dot{x}_n - \partial\dot{F}(x;\tau))\,\varphi_{n+1}(x,\dot{x}_n;\tau)\,\Delta\tau\,d\dot{x}_n\,ds(x)$$

Introducing now the density function of $\dot{X}_n$ conditional on $X = x$, proceeding to the limit according to eq. (5) and taking the integral over τ in [0,t] as required in eq. (2) leads to an integral representation for the mean number of outcrossings:

$$E[N^+(\partial F;t)] = \int_0^t \int_{\partial F} \int_{\dot{X}_n > \partial\dot{F}(x;\tau)} (\dot{x}_n - \partial\dot{F}(x;\tau))\,\varphi_1(\dot{x}_n|X(\tau)=x)\,\varphi_n(x)\,d\dot{x}_n\,ds(x)\,d\tau \tag{7a}$$

A similar reasoning leads to the expected number of incrossings:

$$E[N^-(\partial F;t)] = -\int_0^t \int_{\partial F} \int_{\dot{X}_n < \partial\dot{F}(x;\tau)} (\dot{x}_n - \partial\dot{F}(x;\tau))\,\varphi_1(\dot{x}_n|X(\tau)=x)\,\varphi_n(x)\,d\dot{x}_n\,ds(x)\,d\tau \tag{7b}$$

In combining the two contributions the expected number of crossing can be given as:

$$E[N(\partial F;t)] = \int_0^t \int_{\partial F} \int_{\mathbb{R}^1} |\dot{x}_n - \partial\dot{F}(x;\tau)|\;\varphi_1(\dot{x}_n|X(\tau)=x)\;\varphi_n(x)\,d\dot{x}_n\,ds(x)\,d\tau \tag{8}$$

By considering the fact that the above is achieved by fixing the time τ, it is obvious that the time-variation $\partial\dot{F}(x;\tau)$ of the surface ∂F corresponds to the time-variation of the function $g(x;\tau)$ and does not involve the time variation of its gradients. The solution of the integral (8) together with the obvious relation

$$E[N^+(\partial F;t)] = \frac{1}{2}E[N(\partial F;t)] + \frac{1}{2}(E[N^+(\partial F;t)] - E[N^-(\partial F;t)]) \tag{9}$$

where the second term can be evaluated in a simple manner (see appendix 2) yields the reliability estimate eq. (3).

INTEGRATION OF EQ. (8) BY THE THEORY OF LAPLACE-INTEGRALS

The integral eq. (8) has no analytical solution for arbitrary non-stationary processes and failure surfaces. If the vector $X(\tau)$ has six dimensions, for example, the stresses in a three-dimensional body, the dimension of the integral is eight. But even in smaller dimensions numerical integration would be time-consuming. However, a semi-analytical, approximate solution can be found by applying the concepts of asymptotic analysis to this integral which after some manipulation is, in fact, a special type of the so-called Laplace integrals. These integrals have the form

$$I(\lambda) = \int_D h(y)\,|k(y)|\,\exp[-\lambda f(y)]\,dy \tag{10}$$

where D a simply connected domain in $\mathbb{R}^n$ containing the origin. $f(y)$ is assumed to be at least twice differentiable and has a minimum at the origin $y = y^* = 0$. $h(y)$ is a smoothly varying function around the origin and $h(0) \neq 0$. $k(y)$ is a function which can be zero at the origin and has a linear expansion $k(y) \approx c^T y + b$ around the origin. b is admitted to become zero. For this integral analytical results can be derived for $\lambda \to \infty$. They rest on the fact that the dominating part of the integrand for larger λ clearly comes from $f(y)$ in the exponent. Analytical results are possible because the function $f(y)$ and, if necessary, $k(y)$ need to be represented only by their first- or second order Taylor-expansion. Appendix 1 collects a few basic facts about Laplace integrals in terms of a lemma and two theorems the second of which has been derived for the purpose of our subject. In order to apply those results to eq. (8) a number of intermediate steps have to be taken. As the remainder of this section contains rather technical details, the reader may omit it at first reading and proceed directly to the results.

First of all, the mixture of a volume integral over time and $\dot{x}_n$ and a surface integral must be transformed into a simple volume integral by using a suitable parameterization involving the time and the (n-1) first coordinates of x as parameters and the n-th coordinate x_n as a function $p(\tilde{x},\tau)$. Define:

$$x = (\tilde{x},p(\tilde{x},\tau))^T \text{ with } \tilde{x} = (x_1,\ldots,x_{n-1})^T \in W \text{ and } T = [0,t]$$

Eq. (8) is rewritten as:

$$E[N(\partial F;t)] = \int_{W\times T}\int_{\mathbb{R}^1} |\dot{x}_n - \partial\dot{F}(x;\tau)|\;\varphi_1(\dot{x}_n|X(\tau)=x)\,\varphi_n(\tilde{x},p(\tilde{x},\tau))\,Tr(\tilde{x},\tau)\,d\dot{x}_n\,d\tilde{x}\,d\tau$$

where $Tr(\tilde{x},\tau)$ is the absolute value of the transformation determinant. Introducing the well-known scaling by a factor $\beta > 1$ according to Breitung (1984b) and using the transformations

$$x_i = \beta y_i \quad (i=1,\ldots,n-1), \quad x_n = p(\tilde{x},\tau) = p(\beta\tilde{y},\beta\vartheta), \quad \tau = \beta\vartheta$$

leads to the scaled domain $W_1\times T_1$. Hence:

$$E[N(\partial F;t)] = \beta^n \int_{W_1\times T_1}\int_{\mathbb{R}^1} |\dot{x}_n - \partial\dot{F}(\beta y;\beta\vartheta)|\;\varphi_1(\dot{x}_n;\beta\vartheta|X(\beta\vartheta)=\beta y)\;\varphi_n(\beta\tilde{y},p(\beta\tilde{y},\beta\vartheta))\,Tr(\beta\tilde{y},\beta\vartheta)\,d\dot{x}_n\,d\tilde{y}\,d\vartheta \tag{11}$$

Furthermore, the conditional density function of $\dot{X}_n$ can be given explicitly by using the following formulae for the mean and variance. Note that $n(\beta y) = n(y)$.

$$E[\dot{X}_n(\tau)|X(\tau)=\beta y] = n^T(y)\dot{R}^T\beta y = m(\beta y;\beta\vartheta) = \beta\, m(y;\vartheta) = \beta\tilde{m} \tag{12a}$$

$$VAR[\dot{X}_n(\tau)|X(\tau)=\beta y] = n^T(y)[\ddot{R}-\dot{R}^T\dot{R}]n(y) = \sigma^2(y;\vartheta) = \tilde{\sigma}^2 \tag{12b}$$

Suppose now that in the time interval $T = [0,t]$ there exists a critical time t^* for which the distance β between the hypersurface ∂F and the time axis is smallest. Let x^* be the corresponding critical point. The point $(x^*;t^*)^T$ can be found by solving the following optimization problem which is equivalent to minimizing the function $f(.)$ in eq. (10):

$$\beta = \min_{x,\tau}\{\|x\|\} \text{ for } \left\{\begin{array}{l} \tau \in [0,t] \\ x,\tau \in \{g(x;\tau) = 0\} \end{array}\right. \tag{13}$$

For further notational convenience a suitable orthogonal transformation (rotation) $x \to y$ is performed such that $x^* = \beta e_n$. The scaling of x^* implies that y^* has unit distance to the origin. By a simple translation it is also always possible to achieve $\vartheta^* = 0$. Furthermore, the substitution

$$\beta z = \left(\frac{\dot{x}_n - \beta\tilde{m}}{\tilde{\sigma}}\right)$$

is introduced. Apart from some constants the normal densities in eq. (11) can then be written as:

$$\exp[-\frac{\beta^2}{2}\{z^2 + \sum_{i=1}^{n-1} y_i^2 + p^2(\tilde{y},\vartheta)\}]$$

If the function $p(\tilde{y},\vartheta)$ is taken as an approximating paraboloid to ∂F in the critical point and if the paraboloid is expanded to first order the term in braces in the exponent can be taken as the function $f(.)$ in Laplace's theorems, the term in absolute signs in eq. (11) as the function $|k(.)|$ and the other terms are collected in the function $h(.)$. After some algebra concerning the term in absolute signs, it remains to set $\lambda = \beta^2$ in order to obtain a form to which theorem 2 in appendix 1 is applicable. Two cases need to be considered. The critical point can be an interior point or a boundary point of $W_1\times T_1$. The function p depends on the position of the critical time t^* in T (or of ϑ^* in T_1) and on the geometry of the hypersurface at this point. If ϑ^* is an interior point of T_1, the function $p(.)$ can be chosen as a complete quadratic form depending only on the second derivatives of the surface with respect to y_i (i=1,...,n) and ϑ. All first derivatives of $g(y;\vartheta)$ vanish at the critical point. In the case where ϑ^* is a boundary point of the time interval, the first derivative of $g(y;\vartheta)$ with respect to time has to be taken into account.

RESULTS FOR THE GENERAL NON-STATIONARY CASE

The results thus obtained can, of course, be given in terms of the quantities in the original space by performing the inverse of the orthogonal transformation which transformed x^* into βe_n. This yields after some algebra the following approximations for the mean number of crossings of $X(\tau)$ out of $F = \{g(x;\tau) \le 0\}$ given that the critical point $(x^*;t^*)^T$ has been found according to eq. (13) and the statistical properties of $X(\tau)$ are defined as in the formulae just above eq. (4). The two cases are illustrated in figure 1 and figure 2 for a two–dimensional stationary process and a time–variant failure surface. For simplicity of presentation, $t^* = 0$ is assumed in the following.

Case I: t^* is an interior point (non–linear failure surface)

$$E[N(\partial F;t)] \approx 2\, \varphi_1(\beta)\, |J^*|^{-1/2} [\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{1/2} \qquad (14)$$

with

$$J^* = n^T(x^*)\, C_{x_\tau}\, n(x^*)$$
$$\sigma^2(x^*;t^*) = n^T(x^*)\,(\ddot{R} - \dot{R}^T \dot{R})\, n^T(x^*)$$
$$\omega^2(x^*;t^*) = n_{n+1}{}^T(x^*)\, \dot{R}_{n+1}\, (E - G_{x_\tau})\, \dot{R}_{n+1}{}^T\, n_{n+1}(x^*)$$

where G_{x_τ} is the $(n+1)\times(n+1)$–matrix

$$G_{x_\tau} = \Big\{ g_{ij} = |\nabla g(x^*;t^*)|^{-1} \frac{\partial^2 g(x^*;t^*)}{\partial x_i \partial x_j};\ i,j = 1,\ldots,n;$$
$$g_{in+1} = |\nabla g(x^*;t^*)|^{-1} \frac{\partial^2 g(x^*;t^*)}{\partial x_i \partial \tau};\ i = 1,\ldots,n;$$
$$g_{n+1n+1} = |\nabla g(x^*;t^*)|^{-1} \frac{\partial^2 g(x^*;t^*)}{\partial \tau \partial \tau} \Big\}$$

n_{n+1}= $(n+1)$–vector obtained from n by adding a $(n+1)$–component equal to 1

$\dot{R}_{n+1}$= $(n+1)\times(n+1)$–matrix obtained from $\dot{R}$ by adding a $(n+1)\times(n+1)$–component equal to $-|g_n(x^*;t^*)|$

E= $(n+1)\times(n+1)$–matrix obtained from a $n\times n$–unit matrix by adding a zero–$(n+1)$–row and a zero–$(n+1)$–column

C_{x_τ}= matrix of the cofactors of the matrix $E - G_{x_\tau}$.

The factor J^* can also be expressed in terms of the $(n+1)$ main curvatures in the point $(x^*;t^*)$.

Case II: t^* is a boundary point (non–linear failure surface)

$$E[N(\partial F;T)] \approx [\,2\varphi(a) - a + 2\, a\, \Phi(a)\,]\, \varphi_1(\beta)$$
$$|g_\tau(x^*;t^*)|^{-1}\, |J^*|^{-1/2} [\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{1/2} \qquad (15)$$

with

$$a = \frac{\partial g(x^*;t^*)}{\partial \tau} [\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{-1/2}$$
$$J^* = n^T(x^*)\, C_x\, n(x^*)$$
$$\sigma^2(x^*;t^*) = n^T(x^*)\,(\ddot{R} - \dot{R}^T \dot{R})\, n^T(x^*)$$
$$\omega^2(x^*;t^*) = n^T(x^*)\, \dot{R}\, (I - G_x)\, \dot{R}^T\, n(x^*)$$
$$G_x = \Big\{ g_{ij} = |\nabla g(x^*;t^*)|^{-1} \frac{\partial^2 g(x^*;t^*)}{\partial x_i \partial x_j};\ i,j = 1,\ldots,n \Big\}$$

I = $(n\times n)$–unit matrix,

C_x = matrix of the cofactors of the matrix $I - G_x$

If the time derivative vanishes at the critical point the mean number of crossings is just half of the number computed by eq. (14).

NON–STATIONARY CASE WITH LINEAR FAILURE SURFACE

When the failure surface is an hyperplane the above formulae reduce to simpler forms.

Case I: t^* is an interior point (linear failure surface)

$$E[N(\partial F;t)] \approx 2\, \varphi_1(\beta)\, [\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{1/2} \qquad (16)$$

with

$$\sigma^2(x^*;t^*) = n^T(x^*)\,(\ddot{R} - \dot{R}^T \dot{R})\, n^T(x^*)$$
$$\omega^2(x^*;t^*) = n^T_{n+1}(x^*)\, \dot{R}_{n+1}\, \dot{R}_{n+1}{}^T\, n_{n+1}(x^*)$$

Case II: t^* is a boundary point (linear failure surface)

$$E[N(\partial F;T)] \approx [\,2\varphi(a) - a + 2\, a\, \Phi(a)\,]\, \varphi_1(\beta)$$
$$|g_\tau(x^*;t^*)|^{-1} [\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{1/2} \qquad (17)$$

with

$$a = \frac{\partial g(x^*;t^*)}{\partial \tau} [\sigma^2(x^*;t^*) + \omega^2(x^*;t^*)]^{-1/2}$$
$$\sigma^2(x^*;t^*) = n^T(x^*)\,(\ddot{R} - \dot{R}^T \dot{R})\, n^T(x^*)$$
$$\omega^2(x^*;t^*) = n^T(x^*)\, \dot{R}\, \dot{R}^T\, n(x^*)$$

This formula and in part the previous formulae can further be reduced but we will not do so herein.

EXPECTED NUMBER OF OUTCROSSINGS

The expected number of outcrossings can finally be obtained from the expected number of crossings by using the relation (see appendix 2)

$$2E[N^+(\partial F;t)] \sim E[N(\partial F;t)] - [P(g(X;0) \le 0) - P(g(X;t) \le 0)] \qquad (18)$$

with $P(g(X;0) \le 0)$ and $P(g(X;t) \le 0)$ computed as proposed in appendix 2. The results in eq. (14) to (18) are believed to be given here for the first time.

STATIONARY PROCESSES AND TIME–INVARIANT FAILURE DOMAINS

In this case further significant simplifications occur. First of all, since the integration over time is now an integration over a constant crossing rate it is $E[N^+(\partial F;t)] = \nu^+(\partial F)\, t$ and is suffices to give the outcrossing rate. Secondly, there is no distinction between interior and boundary points and the special version of theorem 2 in appendix 2 with $k(0) = 0$ already obtained by Breitung (1984a) applies.

$$\nu^+(\partial F) \sim \varphi_1(\beta) \prod_{i=1}^{n-1} (1 - \kappa_i \beta)^{-1/2} (2\pi)^{-1/2} [\sigma^2(x^*) + \omega^2(x^*)]^{1/2} \quad \text{for } \beta \to \infty \qquad (20)$$

where

$$\sigma^2(x^*) = \alpha(x^*)^T\,(\ddot{R} - \dot{R}^T \dot{R})\, \alpha(x^*)$$
$$\omega^2(x^*) = \alpha(x^*)^T\, \dot{R}^T\, G\, \dot{R}\, \alpha(x^*)$$
$$G = \Big\{ \delta_{jk} + \|\nabla g(x^*)\|^{-1} \frac{\partial^2 g(x^*)}{\partial x_i \partial x_j}\ ;\ i,j = 1,\ldots,n \Big\}.$$

As before, β is the minimal distance between the failure surface ∂F and the origin, $\alpha(x^*) = -n(x)$ the (normalized) gradient in the minimum point x^* and the κ_i's are the main curvatures of ∂F in that point.

If, furthermore, X and $\dot{X}$ are independent one can simplify this result to (Breitung, 1983):

$$\nu^+(\partial F) \sim \varphi_1(\beta) \prod_{i=1}^{n-1} (1 - \kappa_i \beta)^{-1/2} \left[\frac{(\alpha(x^*)^T \ddot{R}\, \alpha(x^*))}{2\pi} \right]^{1/2} \quad \text{for } \beta \to \infty \qquad (21)$$

If, finally, the failure domain is a half–space the well–known result referred to in Veneziano et al. (1977) is obtained by omitting the product term involving the main curvatures κ_i of ∂F from eq. (20) or (21).

DISCUSSION

According to Breitung (1984a) the term $\sigma^2(x^*;t^*)$ is the variance of the random variable $n^T(x^*)\, \dot{X}(\tau)$ conditional on $X(\tau) = x^*$. The term $\omega^2(x^*;t^*)$ is approximately the variance of the mean in eq. (12a) of the same variable conditional on $X(\tau) = x^*$ if x varies around x^*. It is this second additional term why we first computed $E[N(\partial F;t)]$ by eq. (8) via its asymptotic approximations in eqs. (14) to (17) and then used eq. (9) for the mean number of outcrossings. Direct application of theorem 1 in appendix 1 to eq. (7a) would yield similar results but with $\omega^2(x^*;t^*) = 0$. This term, therefore, must be considered as a higher order but nevertheless asymptotically non–negligible correction. Geometrically, this term takes account of those additional crossings through the failure surface somewhat away from x^* which are possible because there is already $n(x) \neq n(x^*)$.

It is also worth noting that the term $\omega^2(x^*)$ in eq. (20) is not present in eq. (21) and the same applies to the terms $\omega^2(x^*;t^*)$ in the previous formulae under the same circumstances. Therefore, if $X(t)$ and $\dot{X}(t)$ are independent the formulae simplify greatly. Since the matrices $\dot{R}$ in general have entries which are small as compared to the entries in $\ddot{R}$ it is even concluded that neglecting the terms $\omega^2(x^*;t^*)$ in eqs. (14) to (17) frequently is an acceptable approximation. A cruder approximation is achieved by also neglecting the term involving the $\dot{R}$– or $\dot{R}_{n+1}$–matrix in the expression for $\sigma^2(x^*;t^*)$. A greater error usually is introduced if the curvature information for the failure surface in the critical point is not used in higher dimensional problems.

NUMERICAL EXAMPLE

Consider the following limit–state function

$$g(x;\tau) = \sigma_r(\tau) - [\sigma_x(\tau)^2 + 3\sigma_t(\tau)^2]^{1/2}$$

which corresponds to a two–dimensional v. Mises yield criterion where the resistance $\sigma_r(\tau)$ is a function of time τ and where $\sigma_x(\tau)$ and $\sigma_t(\tau)$ are the components of a two–dimensional process $X(\tau)$ having the following parameters:

	MEAN	COVARIANCE MATRIX			
$\sigma_x(\tau)$	$E[\sigma_x(\tau)]$	1940	–7940	0	–22.4
$\sigma_t(\tau)$	$E[\sigma_t(\tau)]$	–7940	32566	22.4	0
$\dot{\sigma}_x(\tau)$	$E[\dot{\sigma}_x(\tau)]$	0	22.4	114000	–403000
$\dot{\sigma}_t(\tau)$	$E[\dot{\sigma}_t(\tau)]$	–22.4	0	–403000	1780000

$E[\sigma_x(\tau)]$ and $E[\sigma_x(\tau)]$ can be functions of the time τ to be defined later. Fig. 3 illustrates the non–stationary behavior of the failure domain F defined as usual by:

$$F = \{g(x;\tau) \le 0\} \qquad \tau \in [0,t]$$

The variation of $\sigma_r(\tau)$ is described by a, for the purpose of illustration very simple, yield stress function

$$\sigma_r(\tau) = \sigma_0(1 + c\,(\tau - \tau_0)^2) \qquad \tau \in [0,t]$$

The minimum of this function and, therefore, the maximum of the outcrossing rate when $X(\tau)$ is stationary is readily determined as $\tau = \tau_0$. We first assume $E[\sigma_x(\tau)] = E[\sigma_t(\tau)] = 0$ Due to the fact that the mean–value vector and the correlation matrix of $X(\tau)$ do not depend on time, the critical time t^* corresponds to the time where the function $\sigma_r(\tau)$ takes its minimal value, i.e. $t^* = \tau_0$. We define the following constants in appropriate units: $\sigma_0 = \tau_0 = 50$ and $c = 2\ 10^{-4}$.

If $t = \tau_0$, the critical point t^* is a boundary point of the interval $[0,t]$. $\sigma_r(\tau)$ is a degrading function of τ within $[0,t]$. The time derivative of the function $g(x;\tau)$ is zero at t^*. Formula (14) and the remark below eq. (15) furnishes:

$$E[N(\partial F;t)] = 1.37\ 10^{-6}$$

If $t > \tau_0$, the critical time t^* is an interior point of the interval $[0,t]$, i.e. $t^* = \tau_0$. The time derivative of the function $g(x;\tau)$ at the critical point also vanishes. One obtains

$$E[N(\partial F;t)] = 2.74\ 10^{-6}$$

which is twice as much as the foregoing value.

If $t < \tau_0$, the critical point $t^* = t$ is certainly a boundary point with non–vanishing time derivative. With eq. (15) and $t = 45$ one determines:

$$E[N(\partial F;t)] = 1.15\ 10^{-6}$$

Next, $E[\sigma_x(\tau)]$ and $E[\sigma_t(\tau)]$ are defined as the following functions of time:

$$E[\sigma_x(\tau)] = \sigma_{mx} \exp[-c_x\,(\tau - \tau_m)^2]$$
$$E[\sigma_x(\tau)] = \sigma_{mt} \exp[-c_t\,(\tau - \tau_m)^2]$$

The critical time must now be determined numerically a eq. (13). Depending on the parameters of the mean valu and the yield stress function the critical point can be interior or a boundary point. With $\sigma_{mx} = \sigma_{mt} = 2\ 10^2$, c_x $\tau_m = 40$ and $t > \tau_0$ the critical point is an interior $t^* = 40.03$. Eq. (14) gives

$$E[N(\partial F;t)] = 3.5\ 10^{-3}$$

In all cases the first–order and asymptotic second–o coincide which is readily explained by the fact that the r vatures of the failure surfaces are several orders of magn than the β–values which are around 5.

CONCLUSIONS

The theory of asymptotic Laplace–integrals proved again to be a suitable tool for the derivation of an important, asymptotically exact generalization of the available results for the crossings of vector processes out of given domains. Multidimensional integration is reduced to a simple single constraint non–linear optimization problem and some simple algebra. The results suggest certain simplifications in practical applications, e.g. to neglect the possible cross–correlations between the process and its derivative and to discard curvature information about the failure surface in the critical point in the sense of a first–order approximation.

APPENDIX 1: ASYMPTOTIC LAPLACE INTEGRALS

Under the conditions mentioned for the integral (10) several results can be derived valid for different cases. Here we summarize two versions particularly relevant for the subject under study.he basic results are given in terms of a lemma and two theorems without proof, applicable to various forms of k(y) and D (see Bleistein and Handelsman, 1975, Breitung, 1884a, 1984b, Breitung and Hohenbichler, 1986, and the references therein for proofs and some additional results).

LEMMA 1:

Define a small neighborhood V of $y^* = 0$ such that the following conditions are fulfilled (Breitung and Hohenbichler, 1986):

$$f(0) < \sup\{f(y): y \in D\backslash V\} \qquad (a)$$

$$\int_{D\cap V} h(y)\,|k(y)|\,\exp[-\lambda f(y)]\,dy > 0 \qquad (b)$$

$$\int_{D\cup V} \cdots |k(y)|\,\exp[-\lambda f(y)]\,dy < \infty \qquad (c)$$

$\int \mu(\lambda)$ | following asymptotic relation can be proved:

$k(y)|\exp[-\lambda f(y)]\,dy \sim$

$|k(y)|\exp[-\lambda f(y)]\,dy$

for $\lambda \to \infty$.

says that only a (small) neighborhood V of y^* essentially to $I(\lambda)$ when $\lambda \to \infty$. Usually it is proven by assuming a neighborhood, for example, in the form of $V(\epsilon) = \{\|y\| < \epsilon\}$. Then, it is shown that:

$$\lim_{\lambda\to\infty} \frac{I(\lambda, D\backslash V(\epsilon))}{I(\lambda, D\cap V(\epsilon))} = 0 \text{ for } \epsilon > 0$$

The important implication of lemma 1 is that only the local behavior of the functions h(y), k(y), f(y) and possibly of the bounding functions $g_i(y)$ of D, i.e. when D is given as $D = \{\cap g_i(y) \leq 0\}$ has to be taken into account. For this a Taylor expansion up to and including the first non–vanishing term is sufficient.

#

THEOREM 1:

Assume that $k(0) = 1$ and $k \in \{0,\ldots,n\}$. The function f(y) obtains its minimum within $D\cap(\mathbb{R}_+^k \times \mathbb{R}^{n-k})$ at the origin. Further assume that there is $\frac{\partial f(0)}{\partial y_i} > 0$ for $i = 1,\ldots,k$ and that the Hessian matrix $S(0) = \{\frac{\partial^2 f(y^*)}{\partial y_i\,\partial y_j};\ i,j = k+1,\ldots,n\}$ of second derivatives of f(y) is positive definite. Then (Breitung and Hohenbichler, 1986):

$$I(\lambda) = \int_{D\cap(\mathbb{R}_+^k \times \mathbb{R}^{n-k})} h(y)\exp[-\lambda f(y)]\,dy \sim$$

$$\sim h(0)\exp[-\lambda f(0)]\,\frac{(2\pi)^{(n-k)/2}}{\lambda^{(n+k)/2}}\left(\prod_{i=1}^{k}\left|\frac{\partial f(0)}{\partial y_i}\right|^{-1}\right)|\mathrm{Det}(S(0))|^{-1/2} \qquad (22)$$

for $\lambda \to \infty$. An important aspect in this theorem is that the contribution of the mixed derivatives for $i = 1,\ldots,k$, $j = k+1,\ldots,n$ in the expansion of f(y) can be shown to be negligible.

#

THEOREM 2:

Assume that there is $\frac{\partial f(0)}{\partial y_i} > 0$ for $i = 1,\ldots,k$ and $k \in \{0,\ldots,n\}$. The function f(y) obtains its minimum at the origin which is a boundary point and the Hessian matrix $S(0) = \{\frac{\partial^2 f(y^*)}{\partial y_i\,\partial y_j};\ i,j = k+1,\ldots,n\}$ of second derivatives of f(y) is positive definite. Further, the function k(y) has the following approximate representation around $y^* = 0$

$$k(y) \approx b(\hat{y}) + \sum_{i=k+1}^{n} \frac{\partial k(0)}{\partial y_i} y_i = b(\hat{y}) + c^T\tilde{y}$$

where $\hat{y} = (y_1,\ldots,y_k)^T$ and $\tilde{y} = (y_{k+1},\ldots,y_n)^T$. It can then be shown that (Plantec and Rackwitz, 1988):

$$I(\lambda) = \int_{D\cap(\mathbb{R}_+^k \times \mathbb{R}^{n-k})} h(y)\,|k(y)|\,\exp[-\lambda f(y)]\,dy \sim$$

$$\sim h(0)\exp[-\lambda f(0)]\,\frac{(2\pi)^{(n-k)/2}}{\lambda^{(n+k+1)/2}}\left(\prod_{i=1}^{k}\left|\frac{\partial f(0)}{\partial y_i}\right|^{-1}\right)|\mathrm{Det}(S(0))|^{-1/2}$$

$$\left[2\tilde{\sigma}_c\varphi\left(\frac{b_0\lambda^{1/2}}{\tilde{\sigma}_c}\right) + b_0\lambda^{1/2} - 2b_0\lambda^{1/2}\,\Phi\left(-\frac{b_0\lambda^{1/2}}{\tilde{\sigma}_c}\right)\right] \qquad (23)$$

for $\lambda \to \infty$ with

$$\tilde{\sigma}_c = (\tilde{c}^T S^{-1}(0)\,\tilde{c})^{1/2} = \left(\sum_{i,j=k+1}^{n} c_i c_j s^*_{ij}\right)^{1/2}$$

and

$$b_0 = b(\hat{y} = \hat{0})$$

and where the s^*_{ij}'s are the elements of the matrix $S^{-1}(0)$. $\varphi_1(.)$ is the univariate standard normal density and $\Phi(.)$ the standard normal integral. The last term is the result of an expectation operation for the last n–k integration variables. Similar to theorem 1 it can be proven that the mixed derivatives for $i = 1,\ldots,k$; $j = k+1,\ldots,n$ can be omitted from the expansion of f(y) and for the expansion of k(y) only the terms for $i = k+1,\ldots,n$ need to be retained. The special case with $k = 0$ and $k(0) = 0$ has already been obtained by Breitung (1984a).

APPENDIX 2: RELATION BETWEEN EXPECTED NUMBER OF CROSSINGS AND EXPECTED NUMBER OF OUTCROSSINGS

In the stationary case the expected number of outcrossings can be easily obtained by using the fact that it is just the half of the expected number of crossings (see Cramer and Leadbetter, 1967). In the non–stationary case however the relation is much more involved. Cramer and Leadbetter proposed for the one–dimensional case the following formula:

$$E[N^+(\partial F;t)] - E[N^-(\partial F;t)] =$$
$$P[X(0)<u(0) \cap X(t)>u(t)] - P[X(0)>u(0) \cap X(t)<u(t)]$$

where $u(\tau)$ is a time–variant threshold level. Hence, for the multi–dimensional case as defined previously we have in straightforward generalization:

$$E[N^+(\partial F;t)] - E[N^-(\partial F;t)] =$$
$$P[g(X;0) > 0 \cap g(X;t) \leq 0] - P[g(X;0) \leq 0 \cap g(X;t) > 0]$$
$$\sim P(g(X;0) \leq 0) - P(g(X;t) \leq 0)$$

for a high–reliability problem.

According to our assumption of a sufficiently mixing process the two events at time 0 and at time t, respectively, may be assumed independent for large t. Therefore, following Breitung (1984b) it is asymptotically

$$P(g(X;0) \leq 0) \sim \Phi(-\beta(0))\prod_{i=1}^{n-1}[1-\kappa_i(0)\beta(0)]^{-1/2}$$

$$P(g(X;t) \leq 0) \sim \Phi(-\beta(t))\prod_{i=1}^{n-1}[1-\kappa_i(t)\beta(t)]^{-1/2}$$

with $\beta(0)$ and $\beta(t)$ the so–called geometric safety indices defined by $\beta(.) = \min \|x(.)\|$ for $x(.)$ out of $\{g(x;.) \leq 0\}$ at the times 0 and t respectively and with $\kappa_i(0)$ and $\kappa_i(t)$ the i–th main curvatures of the surface in the β–points at the two times.

Thereafter, it remains to use

$$E[N^+(\partial F;t)] = \tfrac{1}{2}E[N(\partial F;t)] + \tfrac{1}{2}(E[N^+(\partial F;t)] - E[N^-(\partial F;t)])$$

for the mean value of outcrossings provided that $E[N(\partial F;t)]$ is known.

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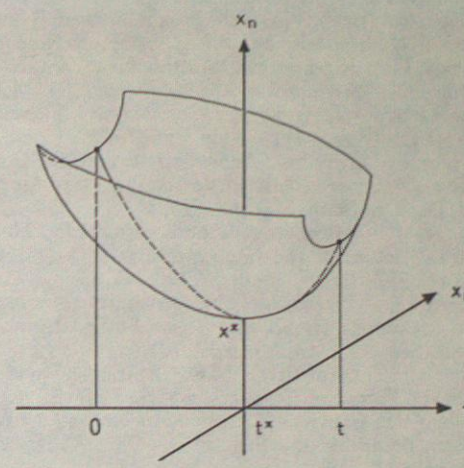

FIGURE 1: CRITICAL POINT IS AN INTERIOR POINT OF [0,t] ON FAILURE SURFACE

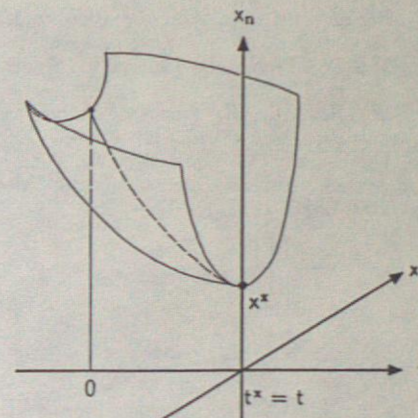

FIGURE 2: CRITICAL POINT IS A BOUNDARY POINT OF [0,t] ON FAILURE SURFACE

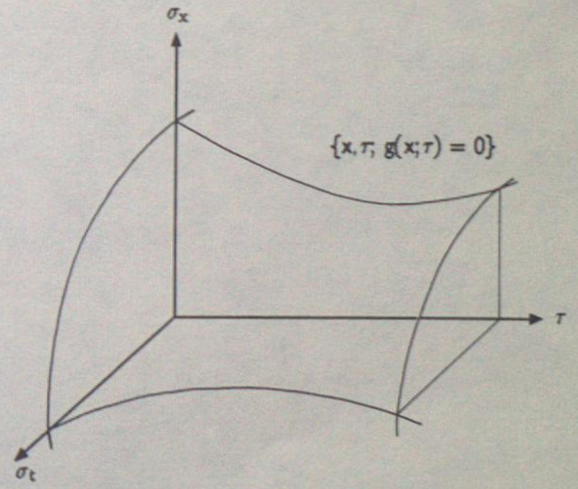

FIGURE 3: FAILURE SURFACE OF EXAMPLE

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