Improvement of Second-Order Reliability Estimates by Importance Sampling

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Introduction

Breitung (1984) has shown that the probability of a domain $D = [\mathbf{u}:g(\mathbf{u}) < 0]$ with $g(\mathbf{u})$ a continuous, at least twice differentiable function near the point $\mathbf{u}^*$ for which $\beta = \min\{\|\mathbf{u}\|\}$ $(\mathbf{u} \in D)$, $g(\mathbf{O}) > 0$ and $U = (U_1, \ldots, U_n)$ an independent, standard normal vector, can be approximated by:

$$P_D = P[g(\mathbf{U}) \leq 0] \approx \Phi(-\beta) \prod_{i=1}^{n-1} (1 - \beta\kappa_i)^{-1/2} \quad \text{(1)}$$

where $\Phi(\cdot)$ is the standard normal integral and $\kappa_1, \ldots, \kappa_{n-1}$ are the main curvatures of $g(\mathbf{u}) = 0$ in $\mathbf{u}^*$. Eq. 1 is asymptotically correct for $\beta \rightarrow \infty$. It rests on a quadratic expansion of $g(\mathbf{u}) = 0$ in $\mathbf{u}^*$ which is assumed to exist. Neglecting the product term in Eq. 1 yields a first-order probability estimate.

Doubts have repeatedly been raised about the accuracy of those probability estimates especially if the asymptotic conditions do not hold. In the cases where β is not large, higher-order derivatives of $g(\mathbf{u})$ in $\mathbf{u}^*$ are of significance and/or a Taylor expansion of $g(\mathbf{u}) = 0$ in $\mathbf{u}^*$ is not representative for the domain D, accurate probability estimates can, nevertheless, be obtained if importance sampling techniques are applied. To the authors' knowledge, these have first been introduced into structural reliability by Shinozuka (1983) and Harbitz (1983), but, in the following, will be derived on different lines especially to update Eq. 1.

Calculations

Apply first a suitable orthogonal transformation such that the point $\mathbf{u}^*$ has coordinates $(0, 0, \ldots, 0, -u_n^*)$ and all mixed second derivatives of $g(\mathbf{u}^*)$ vanish (Fiessler et al. 1979). Let further $u = (v, u_n)$ with $\mathbf{v} = (u_1, \ldots, u^{n-1})$ and $u_n = f(\mathbf{v})$ the root of $g(\mathbf{v}, u_n) = 0$. Also, assume that $u_n < f(\mathbf{v})$ whenever $g(\mathbf{v}, u_n) < 0$. The second-order expansion of $f(\mathbf{v})$ in $\mathbf{u}^*$, then, is

$$t(\mathbf{v}) = -\beta + \frac{1}{2} \sum_{i=1}^{n-1} \kappa_i v_i^2 \quad \text{(2)}$$

The domain probability can be written as

$$P_D = P[g(\mathbf{V}, U_n) < 0] = \int [\int 1_{D(\mathbf{v},u_n)} \phi(u_n) du_n] \phi_{n-1}(\mathbf{v}) d\mathbf{v} \quad \text{(3)}$$

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where $1_{D(\mathbf{v},u_n)}$ is the indicator function of the domain D which has value 1 whenever $(\mathbf{V}, U_n) \in D$ and value 0 otherwise. Introducing

$$p(\mathbf{v}) = \int 1_{D(\mathbf{v},u_n)}\phi(u_n)du_n = \int_{-\infty}^{f(\mathbf{v})} \phi(u_n)du_n = \Phi[f(\mathbf{v})] \quad (4)$$

and its approximation according to Eq. 2

$$q(\mathbf{v}) = \int_{-\infty}^{t(\mathbf{v})} \phi(u_n)du_n = \Phi[t(\mathbf{v})] \quad (5)$$

yields:

$$P_D = \int p(\mathbf{v})\phi_{n-1}(\mathbf{v})d\mathbf{v} = \int \frac{p(\mathbf{v})}{q(\mathbf{v})} q(\mathbf{v})\phi_{n-1}(\mathbf{v})d\mathbf{v} \quad (6)$$

By using the Taylor expansion $\ln \Phi(-\beta + x) = \ln \Phi(-\beta) + x\psi(-\beta) + \ln R(x)$ with $\psi(-\beta) = \phi(-\beta)/k(-\beta)$, $R(x)$ an error term, and $t(\mathbf{v})$ from Eq. 2 in Eq. 5, one can modify Eq. 6 as follows:

$$P_D = \int \frac{p(\mathbf{v})}{q(\mathbf{v})} \exp\,[\ln q(\mathbf{v})]\phi_{n-1}(\mathbf{v})d\mathbf{v}$$

$$= \int \frac{p(\mathbf{v})}{q(\mathbf{v})} \Phi(-\beta) \exp\left[\frac{1}{2}\psi(-\beta)\sum_{i=1}^{n-1}\kappa_i v_i^2\right] R(\mathbf{v})\phi_{n-1}(\mathbf{v})d\mathbf{v}$$

$$= \Phi(-\beta)\int \frac{p(\mathbf{v})}{q(\mathbf{v}} R(\mathbf{v}) \prod_{i=1}^{n-1} \phi\{v_i[1 - \psi(-\beta)\kappa_i]^{1/2}\}d\mathbf{v} = \Phi(-\beta)\prod_{i=1}^{n=1}[1 - \psi(-\beta)\kappa_i]^{1/2}$$

$$\times \int \frac{p(\mathbf{v})}{q(\mathbf{v})} R(\mathbf{v}) \prod_{i=1}^{n-1} \frac{\phi\{v_i[1 - \psi(-\beta)\kappa_i]^{1/2}}{1 - \psi[(-\beta)\kappa_i]^{1/2}} d\mathbf{v}$$

$$= \Phi(-\beta)\prod_{i=1}^{n-1}[1 - \psi(-\beta)\kappa_i]^{1/2} E[Z(\hat{\mathbf{V}})] \quad (7)$$

Herein, it is $Z(\hat{\mathbf{V}}) = R(\hat{\mathbf{V}})p(\hat{\mathbf{V}})/q(\hat{\mathbf{V}})$, $\hat{\mathbf{V}}$ an independent normal vector with means $E(\hat{V}_i) = 0$; and variances var $(\hat{V}_i) = [1 - \psi(-\beta)\kappa_i]^{-1}$ and $R(\hat{\mathbf{V}})$ the error term. Since for large β it is $\psi(-\beta) \sim \beta$, the first two factors in Eq. 7 are asymptotically equivalent to Eq. 1. The third error term is most easily determined by simulation, i.e., by:

$$E[Z(\hat{\mathbf{V}})] \sim \bar{Z}(\hat{\mathbf{v}}) = \frac{1}{N}\sum_{k=1}^{N}\frac{p(\hat{\mathbf{v}}_k)}{q(\mathbf{v}_k)}R(\hat{\mathbf{v}}_k)$$

$$= \frac{1}{N}\sum_{k=1}^{N}\frac{\Phi[f(\mathbf{v}_k)]}{\Phi(-\beta)}\exp\left[-\frac{1}{2}\psi(-\beta)\sum_{i=1}^{n-1}\kappa_i\hat{v}_{ik}^2\right] \quad (8)$$

The $\hat{v}_k$s are the simulation points according to the probability law given before. The correction factor $E(Z(\hat{\mathbf{V}})]$ is the mean of the ratio of the quantities given in Eqs. 4 and 5 times another factor close to one. The variance of $Z(\hat{\mathbf{V}})$ is supposed to be small because $Z(\mathbf{0}) = 1$, $\partial Z(\mathbf{0})/\partial v_i = 0$ for $(1 \leq i \leq n - 1)$ and $\partial^2 Z(\mathbf{0})/(\partial v_i \partial v_j) = 0$ for $(1 \leq i,j \leq n - 1)$. Even smaller is

the variance of $E[\bar{Z}(\hat{\mathbf{V}})]$ approximated by Var $[\bar{Z}(\hat{\mathbf{V}})] \approx$ Var $[Z(\hat{\mathbf{V}})]/N$. Therefore, Eq. 7 with $E[Z(\hat{\mathbf{V}})]$ in Eq. 8 should yield highly accurate probability estimates.

There must be $1 - \psi(-\beta)\kappa_i > 0$. If this is not the case for some $1 \leq i \leq n - 1$, one may replace κ_i by a value which fulfills that condition. It should be noted that the approach also works if the expansion point is only in the vicinity of $\mathbf{u}^*$ and if several other adjacent local minima exist. However, the probability updating then becomes less efficient. Some further observations are given in Hohenbichler (1984).

It is not easy to construct a test example where first- or second-order methods become inadequate so that the proposal scheme can be checked and where an exact solution is possible. One such extreme test example is the probability content of the domain $D = [g(\mathbf{X}) = \pm\Sigma_{i=1}^{n} X_i \mp c \leq 0]$ where

TABLE 1. Equivalent Safety Indices for Various Target Safety Indices β_t and Dimension n of Uncertainty Space

β_t (1)	Indices (2)	$N = 2$ (3)	$N = 5$ (4)	$N = 10$ (5)	$N = 20$ (6)
		(a) $\Sigma_{i=1}^{n} X_i - c(n,\beta_t) \leq 0$			
	β_{I}	2.0703	1.6837	1.3186	0.8411
	β_{II}	2.3305	2.2985	2.2225	2.0474
2.327	r_i	−0.88	−1.96	−3.26	−5.14
	β_{III}	2.3260	2.3292	2.3468	2.3284
	COV	0.43	3.11	4.19	0.572
	β_{I}	3.4692	3.0675	2.6921	2.2087
	β_{II}	3.7313	3.7225	3.6795	3.5667
3.722	r_i	−0.57	−1.45	−2.64	−4.44
	β_{III}	3.7185	3.7223	3.7410	3.7273
	COV	1.54	5.08	4.28	5.85
	β_{I}	4.5138	4.1033	3.7178	3.2275
	β_{II}	4.7675	4.7637	4.7436	4.6621
4.756	r_i	−0.44	−1.17	−2.25	−3.98
	β_{III}	4.7533	4.7567	4.7772	4.7653
	COV	2.09	7.04	4.58	6.07
		(b) $c(n,\beta_t) - \Sigma_{i=1}^{n} X_i \leq 0$			
	β_{I}	2.5413	2.8890	3.2391	3.7088
	β_{II}	2.2419	2.1008	1.8994	1.4845
2.327	r_i	3.56	4.74	6.08	7.97
	β_{III}	2.3393	2.2947	2.3228	2.3025
	COV	6.08	7.77	10.86	14.12
	β_{I}	3.9177	4.2545	4.6007	5.0685
	β_{II}	3.6542	3.5460	3.3998	3.1260
3.722	r_i	4.71	5.81	7.10	8.95
	β_{III}	3.7272	3.6921	3.7145	3.7005
	COV	6.61	8.74	12.58	16.56
	β_{I}	4.9416	5.2697	5.6124	6.0785
	β_{II}	4.6987	4.6045	4.4797	4.2539
4.756	r_i	5.62	6.66	7.90	9.71
	β_{III}	4.7603	4.7292	4.7485	4.7373
	COV	6.88	9.32	13.67	18.29

the X_i are independently and identically exponentially distributed with parameter λ. Applying the necessary probability transformation (Hohenbichler and Rackwitz 1983) produces a highly nonlinear domain boundary, i.e., $D = [g(u) = \mp \sum_{i=1}^{n} \ln \Phi(-U_i)/\lambda \mp c \leq 0]$. The exact result is known to be $P_D = F_{Ga}(c;n,\lambda)$ with F_{Ga} the Gamma (Erlang) distribution function. In Table 1 the resulting probabilities are given in terms of equivalent safety indices $\beta_D = -\Phi^{-1}[P_D]$ for three different probability levels, several n, and $\lambda = 1$. The first number corresponds to the first-order probability estimate, which is seen to become worse with increasing n. The second number is the probability estimate according to Eq. 7 without importance sampling correction. The third number represents the radius of curvature. Its value, not too different from β, explains why the second-order results differ significantly from the first-order results for larger n and larger probabilities. Of course, second-order results are always more accurate. The fourth and fifth number represent the results according to Eq. 7 with $N = 50$ and the sample coefficient of variation in percent, respectively. This estimate is satisfactory for all probabilities and all n. It can be made arbitrarily exact by increasing N. However, the numerical effort increases approximately by a factor of 2 for the second-order results and by a factor of 10 for the simulation results as compared with the first-order estimates. The proposed method, therefore, provides an easy check whether a lower-order method is appropriate for the computational task under question.

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Appendix I. References

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Appendix II. Notation

The following symbols are used in this paper:

D = domain;
$P(\cdot)$ = probability;
$\mathbf{U}$ = $(\mathbf{U}_1, \ldots, \mathbf{U}_n)$ = standard normal vector;
$\hat{\mathbf{V}}$ = $(\hat{\mathbf{V}}_1, \ldots, \hat{\mathbf{V}}_{n-1})$ = normal sampling vector;

$E[\cdot]$ = expectation;
$\|u\|$ = euclidean norm of u;
β = standard safety index;
β_D = equivalent safety index;
κ_i = i-th main curvature;
ϕ = standard normal density;
Φ = standard normal integral; and
$\psi(-\beta)$ = $\phi(-\beta)/\Phi(-\beta)$.